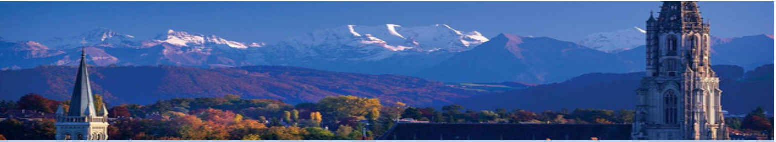

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Discontinuous Galerkin Time Discretizations for Evolution Problems

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Outline

Part I The dG time stepping scheme for IVP

Part II A dG calculus for time-dependent problems

Part III Application to IVP:
Discrete Peano results and blow-up problems

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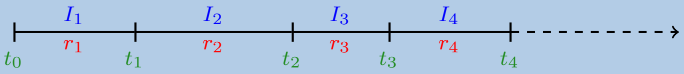
Part I

The dG time stepping scheme

Temporal hp framework



- Time nodes: $0 = t_0 < t_1 < t_2 < \ldots$, and time steps

$$I_m = (t_{m-1}, t_m), \qquad k_m = t_m - t_{m-1}, \qquad m \geq 1.$$

- Finite element temporal mesh: $\mathcal{M} = \{I_m\}_{m \geq 1}$.
- Polynomial degree $r_m \geq 0$ associated with each time step $m \geq 1$.
- Local polynomial spaces in a (real) Hilbert space H:

$$\mathbb{P}^r(I_m; H) = \left\{ v \in C^0(I_m; H) : v(t) = \sum_{k=0}^{r} a_k t^k, a_k \in H \right\}.$$

Why hp Galerkin time stepping?

> Why Galerkin schemes?

— Arbitrary time steps and approximation orders

— Natural framework for existence and uniqueness results, stability and error analysis (weak and pointwise formulations, function spaces)

— Applicability of analysis techniques

> Why hp methods?

— Flexibility in adjusting the time steps and the orders to the local behaviour of the solution (possibly leading to high-order algebraic or even exponential convergence rates)

— Allowing for possibly large time steps
[Matache, Schwab & W, 2005/2006]

The dG scheme for IVP

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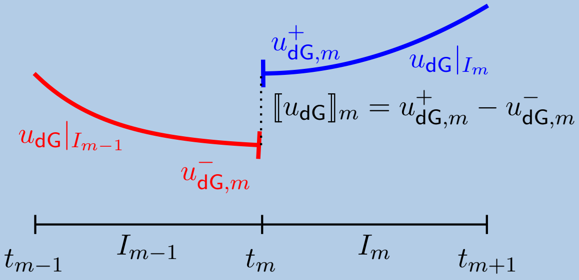
- Given $u_0 \in H$ solve the nonlinear initial value problem

$$u'(t) = \mathcal{F}(t, u(t)), \qquad t > 0$$
$$u(0) = u_0,$$

  for an unknown solution $u = u(t)$.

- DG time stepping—schematically:



The dG scheme for IVP

m^{th} *time step:* Given an initial value

$$u^-_{\mathsf{dG},m-1} := u_{\mathsf{dG}}|_{I_{m-1}}(t_{m-1})$$

find $u_{\mathsf{dG}}|_{I_m} \in \mathbb{P}^{r_m}(I_m; H)$ such that

$$\int_{t_{m-1}}^{t_m} (u'_{\mathsf{dG}}, v)_H \, \mathrm{d}t = \int_{t_{m-1}}^{t_m} (\mathcal{F}(u_{\mathsf{dG}}), v)_H \, \mathrm{d}t$$

for all $v \in \mathbb{P}^{r_m}(I_m; H)$.

The dG scheme for IVP

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$$u^-_{\mathsf{dG},m-1} := u_{\mathsf{dG}}|_{I_{m-1}}(t_{m-1})$$

find $u_{\mathsf{dG}}|_{I_m} \in \mathbb{P}^{r_m}(I_m;H)$ such that

$$\int_{t_{m-1}}^{t_m} (u'_{\mathsf{dG}}, v)_H \, \mathrm{d}t + (u^+_{\mathsf{dG},m-1}, v(t_{m-1}))_H = \int_{t_{m-1}}^{t_m} (\mathcal{F}(u_{\mathsf{dG}}), v)_H \, \mathrm{d}t + (u^-_{\mathsf{dG},m-1}, v(t_{m-1}))_H$$

for all $v \in \mathbb{P}^{r_m}(I_m;H)$.

The dG scheme for IVP

m^{th} *time step:* Given an initial value

$$u^-_{\mathsf{dG},m-1} := u_{\mathsf{dG}}|_{I_{m-1}}(t_{m-1})$$

find $u_{\mathsf{dG}}|_{I_m} \in \mathbb{P}^{r_m}(I_m; H)$ such that

$$\int_{t_{m-1}}^{t_m} (u'_{\mathsf{dG}}, v)_H \, \mathrm{d}t + (u^+_{\mathsf{dG},m-1} - u^-_{\mathsf{dG},m-1}, v^+_{m-1})_H = \int_{t_{m-1}}^{t_m} (\mathcal{F}(u_{\mathsf{dG}}), v)_H \, \mathrm{d}t$$

for all $v \in \mathbb{P}^{r_m}(I_m; H)$.

The dG scheme for IVP

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find $u_{\mathsf{dG}}|_{I_m} \in \mathbb{P}^{r_m}(I_m;H)$ such that

$$\int_{t_{m-1}}^{t_m} (u'_{\mathsf{dG}}, v)_H \, \mathrm{d}t + (\llbracket u_{\mathsf{dG}} \rrbracket_{m-1}, v^+_{m-1})_H = \int_{t_{m-1}}^{t_m} (\mathcal{F}(u_{\mathsf{dG}}), v)_H \, \mathrm{d}t$$

for all $v \in \mathbb{P}^{r_m}(I_m;H)$.

[Reed & Hill, 1973], [Lesaint & Raviart, 1974], [Delfour, Hager & Trochu, 1981], [Delfour & Dubeau, 1986], [Johnson, 1988], [Thomée, 1997], [Schötzau & Schwab, 2000], [Walkington, 2010].

Remarks

- The dG time stepping scheme is an implicit (Radau-type collocation) method.
- Analysis is possible time step by time step.
  [Schötzau & Schwab, 2000]
- Some well-known time stepping schemes appear as special cases. (e.g., $r = 0 \sim$ backward Euler).
- Close relation to other implicit methods.
  [Akrivis, Makridakis & Nochetto, 2009/2011]

Part II

A dG calculus for time-dependent problems

DG time operator

On a time interval I_m define the dG time operator

$$\chi_m^{r_m} : \mathbb{P}^{r_m}(I_m; H) \to \mathbb{P}^{r_m}(I_m; H)$$

by

$$\int_{I_m} (\chi_m^{r_m}(U), V)_H \, \mathrm{d}t = \underbrace{\int_{I_m} (U', V)_H \, \mathrm{d}t}_{\text{derivative}} + \underbrace{(U_{m-1}^+, V_{m-1}^+)_H}_{\text{initial value}}$$

for any $V \in \mathbb{P}^{r_m}(I_m; H)$.

DG time operator

Lifting operator (discrete δ-function):

$$\mathsf{L}_m^{r_m} : H \to \mathbb{P}^{r_m}(I_m; H), \qquad \zeta \mapsto \mathsf{L}_m^{r_m}(\zeta),$$

defined by

$$\int_{I_m} (\mathsf{L}_m^{r_m}(\zeta), V)_H \, \mathsf{d}t = (\zeta, V_{m-1}^{+})_H \qquad \forall V \in \mathbb{P}^{r_m}(I_m; H), \zeta \in H.$$

Incidentally, there holds

$$\mathsf{L}_m^{r_m}(\zeta) = \frac{\zeta}{2}(-1)^{r_m}(r_m + 1)P_m^{(0,1)}.$$

DG time operator

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Then for any $V \in \mathbb{P}^{r_m}(I_m; H)$:

$$\int_{I_m} (\chi_m^{r_m}(U), V)_H \, \mathrm{d}t = \int_{I_m} (U', V)_H \, \mathrm{d}t + (U_{m-1}^+, V_{m-1}^+)_H$$
$$= \int_{I_m} (U' + \mathsf{L}_m^{r_m}(U_{m-1}^+), V)_H \, \mathrm{d}t.$$

Hence there holds the representation:

$$\chi_m^{r_m}(U) \equiv U' + \mathsf{L}_m^{r_m}(U_{m-1}^+) \qquad \forall U \in \mathbb{P}^{r_m}(I_m; H).$$

DG time operator

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The operator

$$\chi_m^{r_m} : \mathbb{P}^{r_m}(I_m; H) \to \mathbb{P}^{r_m}(I_m; H)$$

is an isomorphism.

There holds the discrete Poincaré inequality

$$\|u\|_{L^\infty(I_m;H)} \leq 2k_m^{1-1/p} \|\chi_m^{r_m}(u)\|_{L^p(I_m;H)},$$

for any $u \in \mathbb{P}^{r_m}(I_m; H)$.

[Holm & W, 2016]

“DG integral”

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The “dG integral”

$$(\chi_m^{r_m})^{-1} : \mathbb{P}^{r_m}(I_m; H) \to \mathbb{P}^{r_m}(I_m; H)$$

is uniformly stable: For any $p \in [1, \infty]$ there holds that

$$\|(\chi_m^{r_m})^{-1}(u)\|_{L^\infty(I_m;H)} \leq 2k_m^{1-1/p}\|u\|_{L^p(I_m;H)},$$

for any $u \in \mathbb{P}^{r_m}(I_m; H)$.

[Holm & W, 2016]

Temporal reconstruction

A function

$$U : [0, \infty) \to H, \qquad U|_{I_m} \in \mathbb{P}^{r_m}(I_m; H), \qquad m \geq 1,$$

can be made continuous by means of a reconstruction technique:

$$\widehat{U} : [0, \infty) \to H, \qquad \widehat{U}|_{I_m} \in \mathbb{P}^{r_m+1}(I_m; H),$$

defined by (”main theorem of calculus”)

$$\widehat{U}(t) := U_{m-1}^{-} + \int_{t_{m-1}}^{t} \chi_m^{r_m}(U - U_{m-1}^{-}) \, \mathrm{d}t, \qquad t \in [t_{m-1}, t_m).$$

for $m \geq 1$.

[Makridakis & Nochetto, 2006]

Temporal reconstruction

- Then, for $m \geq 1$, there holds

$$\widehat{U}(t_{m-1}) = U_{m-1}^-.$$

- Furthermore,

$$\begin{aligned}\lim_{t \to t_m} \widehat{U}(t) &= U_{m-1}^- + \int_{I_m} \chi_m^{r_m}(U - U_{m-1}^-)\,\mathrm{d}t \\ &= U_{m-1}^- + \int_{I_m} \left(U' + \mathsf{L}_m^{r_m}([\![U]\!]_{m-1})\right)\,\mathrm{d}t \\ &= U_{m-1}^- + U_m^- - U_{m-1}^+ + [\![U]\!]_{m-1} = U_m^-.\end{aligned}$$

Hence, $\widehat{U}$ is globally continuous.

Temporal reconstruction

There hold "well-behaved" approximation *identities*:

- L^2 norm:

$$\|U - \widehat{U}\|_{L^2(I_m;H)} = \sqrt{\frac{k_m(r_m+1)}{(2r_m+1)(2r_m+3)}}\|[\![U]\!]_{m-1}\|_H$$

- L^∞ norm:

$$\|U - \widehat{U}\|_{L^\infty(I_m;H)} = \|[\![U]\!]_{m-1}\|_H$$

[Makridakis & Nochetto, 2006], [Schötzau & W, 2010], [Georgoulis, Lakkis & W, 2016].

Part III

Application to IVP: Discrete Peano results and blow-up problems

Back to the dG scheme for IVP

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- Fully discrete scheme: Discretize the space H on each time interval I_m by a finite-dimensional subspace H_m, for $m \geq 1$.
- Let $\pi_m : H \to H_m$ denote the H-orthogonal projection.
- m^{th} *time step:* find $u_{\mathsf{dG}}|_{I_m} \in \mathbb{P}^{r_m}(I_m; H_m)$ such that

$$\int_{t_{m-1}}^{t_m} (u'_{\mathsf{dG}}, v)_H \, \mathrm{d}t + (\quad [\![u_{\mathsf{dG}}]\!]_{m-1}, v^+_{m-1})_H = \int_{t_{m-1}}^{t_m} (\mathcal{F}(u_{\mathsf{dG}}), v)_H \, \mathrm{d}t$$

for all $v \in \mathbb{P}^{r_m}(I_m; H_m)$.

Back to the dG scheme for IVP

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for all $v \in \mathbb{P}^{r_m}(I_m; H_m)$.

Back to the dG scheme for IVP

- Fully discrete scheme: Discretize the space H on each time interval I_m by a finite-dimensional subspace H_m, for $m \geq 1$.
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- m^{th} *time step:* find $u_{\mathsf{dG}}|_{I_m} \in \mathbb{P}^{r_m}(I_m; H_m)$ such that

$$\int_{t_{m-1}}^{t_m} (u_{\mathsf{dG}}', v)_H \, \mathrm{d}t + (\pi_m [\![u_{\mathsf{dG}}]\!]_{m-1}, v_{m-1}^+)_H = \int_{t_{m-1}}^{t_m} (\mathcal{F}(u_{\mathsf{dG}}), v)_H \, \mathrm{d}t$$

for all $v \in \mathbb{P}^{r_m}(I_m; H_m)$.

Back to the dG scheme for IVP

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- Notice that

$$u'_{\mathsf{dG}} = (u_{\mathsf{dG}} - \pi_m u^-_{\mathsf{dG},m-1})'.$$

- Furthermore,

$$\begin{aligned}\pi_m [\![u_{\mathsf{dG}}]\!]_{m-1} &= \pi_m u^+_{\mathsf{dG},m-1} - \pi_m u^-_{\mathsf{dG},m-1} \\ &= u^+_{\mathsf{dG},m-1} - \pi_m u^-_{\mathsf{dG},m-1} \\ &= (u_{\mathsf{dG}} - \pi_m u^-_{\mathsf{dG},m-1})^+_{m-1}.\end{aligned}$$

Back to the dG scheme for IVP

> Therefore, using the dG time operator:

$$\int_{I_m} (\chi_m^{r_m}(u_{\mathsf{dG}} - \pi_m u^-_{\mathsf{dG},m-1}), v)_H \,\mathsf{d}t$$
$$= \int_{t_{m-1}}^{t_m} (\qquad \mathcal{F}(u_{\mathsf{dG}}), v)_H \;\mathrm{d}t \quad \forall v \in \mathbb{P}^{r_m}(I_m; H_m),$$

Back to the dG scheme for IVP

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> Therefore, using the dG time operator:

$$\int_{I_m} (\chi_m^{r_m}(u_{\mathsf{dG}} - \pi_m u^-_{\mathsf{dG},m-1}), v)_H \,\mathrm{d}t$$
$$= \int_{t_{m-1}}^{t_m} (\Pi_m^{r_m} \mathcal{F}(u_{\mathsf{dG}}), v)_H \;\mathrm{d}t \quad \forall v \in \mathbb{P}^{r_m}(I_m; H_m),$$

with the $L^2(I_m; H)$-projection $\Pi_m^{r_m} : L^2(I_m; H) \to \mathbb{P}^{r_m}(I_m; H_m)$.

Pointwise form

> Pointwise form:

$$\chi_m^{r_m}\left(u_{\mathsf{dG}} - \pi_m u_{\mathsf{dG},m-1}^{-}\right) = \Pi_m^{r_m}\left(\mathcal{F}(u_{\mathsf{dG}})\right) \qquad \text{in } \mathbb{P}^{r_m}(I_m; H_m).$$

> Fixed-point form (Picard-type dG integral equation):

$$u_{\mathsf{dG}} = \pi_m u_{\mathsf{dG},m-1}^{-} + \left(\chi_m^{r_m}\right)^{-1}\left(\Pi_m^{r_m}\left(\mathcal{F}(u_{\mathsf{dG}})\right)\right).$$

> Fixed-point operator:

$$\mathsf{T}_m^{\mathsf{dG}} : \mathbb{P}^{r_m}(I_m; H_m) \xrightarrow{\text{cont.}} \mathbb{P}^{r_m}(I_m; H_m)$$

defined by

$$\mathsf{T}_m^{\mathsf{dG}}(Z) := \pi_m u_{\mathsf{dG},m-1}^{-} + \left(\chi_m^{r_m}\right)^{-1}\left(\Pi_m^{r_m}\left(\mathcal{F}(Z)\right)\right).$$

Discrete Peano theorem

Existence of a fixed point follows by:

- Time step k_m sufficiently small $\Rightarrow$ fixed point operator $\mathsf{T}_m^{\mathsf{dG}}$ maps a sufficiently small (closed) ball about $\pi_m u^-_{\mathsf{dG},m-1} \in \mathbb{P}^{r_m}(I_m; H_m)$ into itself.
- Brower's fixed point theorem $\Rightarrow$ existence of fixed-points, and thus of discrete dG solutions (discrete Peano theorem).
- Local Lipschitz continuity condition for $\mathcal{F} \Rightarrow$ (local) uniqueness of solutions based on the contraction mapping theorem.

Theorem ([Holm & W, 2016])

Let $m \geq 1$. If the current time step $k_m > 0$ is chosen sufficiently small (independent of the local polynomial degree r_m), then the hp dG time stepping method possesses a unique solution in $\mathbb{P}^{r_m}(I_m; H_m)$.

Application to blow-up problems

> Algebraic growth nonlinearities: For $\beta > 1$, and $\alpha, \delta > 0$ suppose that

$$\|\mathcal{F}(t,u)\|_H \le \alpha \|u\|_H^\beta, \qquad (\mathcal{F}(t,u),u)_H \ge \delta \|u\|_H^{1+\beta},$$

whenever $\|u\|_H \ge c_{\mathcal{F}}$ is sufficiently large, and for all $t \in [0,\infty)$.

$\exists$ blow-up in finite time $T_\infty < \infty$.

> Local Lipschitz continuity: For $\gamma \ge 0$ assume that

$$\|\mathcal{F}(t,u) - \mathcal{F}(t,v)\|_H \le \gamma \max(\|u\|_H, \|v\|_H)^{\beta-1} \|u-v\|_H,$$

whenever $\|u\|_H, \|v\|_H \ge c_{\mathcal{F}}$, and for all $t \in [0,\infty)$.

Application to blow-up problems

Theorem ([Holm & W, 2016])

- *All discretization spaces are the same:* $H_1 = H_2 = \ldots$
- *Initial value* $\|u_0\|_H > c_{\mathcal{F}}$ *sufficiently large.*
- *For a parameter* $\varrho \in (0, \varrho^\star)$ *choose the time steps to be*

$$k_m(\varrho) := c\varrho \|u^-_{\mathsf{dG},m-1}\|_H^{1-\beta}, \qquad m \geq 1.$$

Then, there holds:

1. *The dG solution exists and is (locally) unique for any time step* $m \geq 1$ *(and for any polynomial degree distribution);*
2. *The dG solution blows up in a finite time* $T_\infty^{\mathsf{dG}}(\varrho)$*;*
3. $\lim_{\varrho \searrow 0} T_\infty^{\mathsf{dG}}(\varrho) = T_\infty$.

Application to blow-up problems

Basic time stepping algorithm:

```
Set m = 0; T̃∞ = 0;
loop
    m ← m + 1;
    Compute k_m(ϱ);
    if T̃∞ + k_m(ϱ) == T̃∞ then                ▷ Stopping criterion
        return T̃∞;
    else
        Compute the dG solution u_dG on I_m;
        T̃∞ ← T̃∞ + k_m(ϱ);                     ▷ Update
    end if
end loop
```

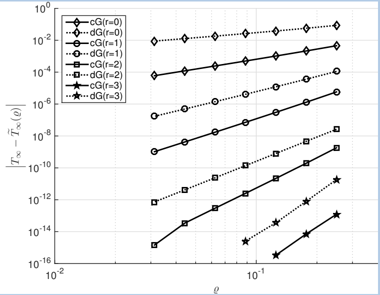
The result, $\widetilde{T}_\infty$, is an approximation of the exact blow-up time.

Numerical Example

Consider

$$u'(t) = \frac{(|u(t)|+1)u(t)}{1+e^{-t}} =: \mathcal{F}(t,u(t)), \qquad u(0) = 3.$$

Exact solution: $u(t) = 3(e^t+1)(5-3e^t)^{-1}$.



dG(r): $|T_\infty - \widetilde{T}_\infty| = \mathcal{O}(\varrho^{2r+1})$
cG($r+1$): $|T_\infty - \widetilde{T}_\infty| = \mathcal{O}(\varrho^{2r+2})$

Constructive proof ongoing. . .

Main Ideas of Proof

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(i) Finite blow-up time: Testing the strong dG form with a suitable test function shows that there exists a constant $C > 0$ independent of the time steps and polynomial degrees such that

$$\|u^-_{\mathsf{dG},m}\|_H \geq (1 + C\varrho)\|u^-_{\mathsf{dG},m-1}\|_H \geq (1 + C\varrho)^m\|u_0\|_H, \quad m \geq 0.$$

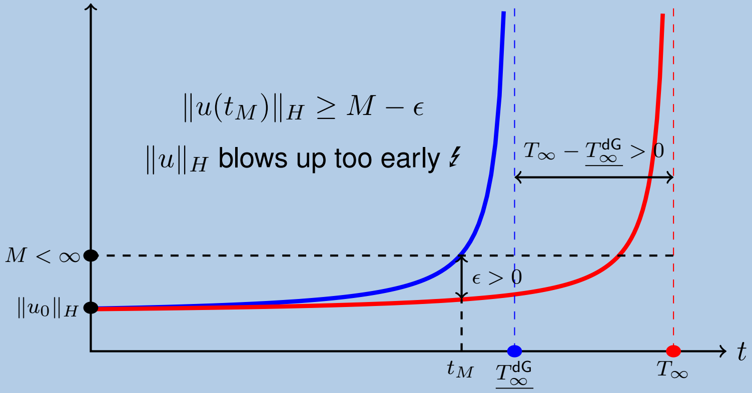
Furthermore,

$$\begin{aligned} T^{\mathsf{dG}}_\infty(\varrho) = \sum_{m=1}^{\infty} k_m(\varrho) &= c\varrho \sum_{m=1}^{\infty} \|u^-_{\mathsf{dG},m-1}\|_H^{1-\beta} \\ &\leq c\varrho\|u_0\|_H^{1-\beta} \sum_{m=1}^{\infty} (1 + C\varrho)^{(m-1)(1-\beta)} < \infty. \end{aligned}$$

Main Ideas of Proof

(ii) Convergence (by contradiction): [Nakagawa, '76]

$$\underline{T^{\mathsf{dG}}_\infty} := \liminf_{\varrho \to 0} T^{\mathsf{dG}}(\varrho) \geq T_\infty \geq \limsup_{\varrho \to 0} T^{\mathsf{dG}}_\infty(\varrho).$$



Concluding remarks

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- Discrete Peano results for IVP and application to blow-up
  - B. Holm & TW
    Continuous and discontinuous Galerkin Time Stepping Methods for Nonlinear Initial Value problems (arXiv:1407.5520)
- Convergence rates for the blow-up time error
- Extension to semi-linear parabolic PDE (with L. Schmutz)
- Blow-up time resolution based on a hp a posteriori error estimates
  - I. Kyza, S. Metcalfe & TW
    hp-Adaptive Galerkin Time Stepping Methods for Nonlinear Initial Value Problems (in preparation)

Concluding remarks

- Semi-discrete hp a posteriori error estimates for parabolic PDE

D. Schötzau & TW
A Posteriori Error Estimation for hp-Version Time-Stepping Methods for Parabolic Partial Differential Equations (Numer. Math., 2010)

- Fully discrete hp a posteriori error estimates for parabolic PDE

E. Georgoulis, O. Lakkis & TW
A posteriori error bounds for fully-discrete hp-discontinuous Galerkin time stepping methods (in preparation)

- Newton-Galerkin approach

M. Amrein & TW
An Adaptive Space-Time Newton-Galerkin Approach for Semilinear Singularly Perturbed Parabolic Evolution Equations (IMA JNA, 2016)

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