Variational Formulation of Problems Involving Fractional-Order Differential Operators

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$$-D_0^\alpha u + qu = f, \quad \text{in } D = (0,1),$$
$$u(0) = u(1) = 0.$$

- D_0^α: a Riemann-Liouville or Djrbashian-Caputo fractional derivative
- $1 < \alpha < 2$: order of fractional derivative
- $f \in L^2(D)$ or suitable Sobolev space $\widetilde{H}^\beta(D)$

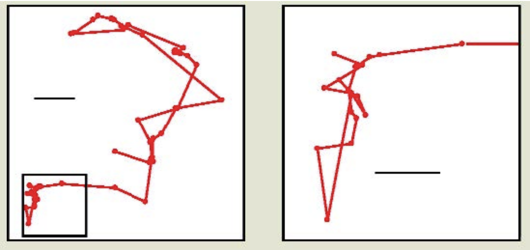
applications

- superdiffusion, e.g., subsurface flow, magnetized plasma
- microscopic: asymmetric Levy flights Chaves 1998, Benson et al 2000
- pure math: biorthogonal basis for space of analytic functions

  Djrbashian 1950s

Caution: D_0^α is different from "fractional Laplacian" / "peridynamics"

R. Nochetto, A. Salgado, Y. Huang, J. P. Borthagaray, S. Acosta, Q. Du, M. Gunzburger, ...



left: size (100m) right: size (50m)

During foraging, the movement of spider monkeys follows a Lévy walk. While the actual reason for this anomalous behaviour is not clear, it has been found that Lévy walks outperform normal Brownian random walks as a strategy for finding randomly located objects. ©: Wikipedia, Klafter-Sokolov, 2005, Phys. Today

$$\textbf{superdiffusion: } \langle x^2 \rangle \propto t^{\alpha},\ \alpha \in (1, 2)$$

Edwards et al **Nature**(2007), Humphries et al **Nature**(2014); Sims et al **PNAS**(2012), Humbries et al **PNAS**(2012) existing works on the problem

- finite-difference methods of time-dependent case Tadjeran et al 2006 ....
- Ervin-Roop (2006) studied the Riemann-Liouville case, *assumed full regularity* on the forward & adjoint solutions
- a few follow-up works, but all under the same assumption ...

goal of the talk

- to develop a proper variational formulation of the problem left-sided (R.-L.) fractional integral

$$({}_0I_x^{\beta} f)(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)\, dt.$$

The right-sided version:

$$({}_xI_1^{\beta} f)(x) = \frac{1}{\Gamma(\beta)} \int_x^1 (t-x)^{\beta-1} f(t)\, dt.$$

- semi-group property: for $\beta, \gamma > 0$,

$$ {}_0I_x^{\beta}\, {}_0I_x^{\gamma} = {}_0I_x^{\beta+\gamma} \text{ and } {}_xI_1^{\beta}\, {}_xI_1^{\gamma} = {}_xI_1^{\beta+\gamma}.$$

- change of integration order formula

$$({}_0I_x^{\beta}\phi, \varphi) = (\phi, {}_xI_1^{\beta}\varphi) \quad \forall \phi, \varphi \in L^2(D).$$

Leibniz (1695), Euler, Riemann, Liouville, Hardy, ...

let $n-1<\beta<n$, $f\in H^n(\bar{D})$

- left-sided Riemann-Liouville derivative of order β:

$$ {}_0^R D_x^\beta f = ({}_0 I_x^{n-\beta} f)^{(n)}. $$

- left-sided Djrbashian-Caputo derivative of order β:

$$ {}_0^C D_x^\beta f = {}_0 I_x^{n-\beta} f^{(n)} $$

  M. Djrbashian: *harmonic analyst*; M. Caputo: *geophysicist*

- right-sided version of fractional derivatives

$$ {}_x^R D_1^\beta f = (-1)^n ({}_x I_1^{n-\beta} f)^{(n)} \text{ and } {}_x^C D_1^\beta f = (-1)^n {}_x I_1^{n-\beta} f^{(n)}. $$

R.-L. derivative is analytically more convenient, but D.-C. derivative is practically more useful

function spaces

- $\widetilde{H}^s(D)$, functions whose zero extension belongs to $H^s(\mathbb{R})$
- $\widetilde{H}^s_L(D)$, $\widetilde{H}^s_R(D)$: whose zero extension to the left, right are in H^s

Jin-Lazarov-Pasciak-Rundell 2015

- ${}^R_0D^\beta_x$ is continuous from $\widetilde{H}^\beta_L(D)$ to $L^2(D)$
- ${}_0I^\beta_x$ is bounded from $\widetilde{H}^s_L(D)$ to $\widetilde{H}^{s+\beta}_L(D)$

strong solution ($q = 0$)

Riemann-Liouville case

For $f \in L^2(D)$ and $\alpha \in (1,2)$, find u such that

$$-{}_0^R D_x^\alpha u = f, \quad \text{in } D \quad \text{with } u(0) = u(1) = 0.$$

Set $g = {}_0I_x^\alpha f$, then

$${}_0I_x^{2-\alpha} g = {}_0I_x^{2-\alpha}\,{}_0I_x^\alpha f = {}_0I_x^2 f \in \widetilde{H}_L^2(D).$$

and thus

$${}_0^R D_x^\alpha g = f.$$

Further, ${}_0^R D_x^\alpha x^{\alpha-1} = 0$. Hence u is given by [with $\beta \in (0, \frac{1}{2})$]

$$u = -{}_0I_x^\alpha f + ({}_0I_x^\alpha f)(1) x^{\alpha-1} \in \widetilde{H}_L^{\alpha-1+\beta}(D)$$

Djrbashian-Caputo case

For $\alpha \in (1,2)$ and $f \in \widetilde{H}^{\beta}(D)$, $\alpha+\beta > \frac{3}{2}$, find u

$$-{}_0^C D_x^{\alpha} u = f, \quad \text{in } D \quad \text{with } u(0) = u(1) = 0$$

if $v \in H^s(D)$, $s > \frac{3}{2}$, then

$${}_0^C D_x^{\alpha} v = {}_0^R D_x^{\alpha} v - \frac{v(0)}{\Gamma(1-\alpha)} - \frac{v'(0)}{\Gamma(2-\alpha)} x^{1-\alpha}$$

With $g = {}_0I_x^{\alpha} f \in \widetilde{H}_L^{\alpha+\beta}(D)$, and thus $g(0) = g'(0) = 0$, and

$${}_0^C D_x^{\alpha} g = {}_0^R D_x^{\alpha} g = f$$

together with ${}_0^C D_x^{\alpha} x = 0 \Rightarrow$

$$u = -{}_0I_x^{\alpha} f + ({}_0I_x^{\alpha} f)(1) x \in H^{\alpha+\beta}(D)$$

If $\alpha+\beta \leq \frac{3}{2}$, no known representation formula !!

UCL

variational formulation

strategy: multiply by a test function, integrate over the domain and move derivatives around.
Fractional derivatives move differently than standard ones.

- ${}^{C}_{x}D^{\beta}_{1} u = {}^{R}_{x}D^{\beta}_{1} u, \quad \forall u \in H^1_0(D), \beta \in (0,1).$
- $({}_0I^{\beta}_{x} \phi, \varphi) = (\phi, {}_xI^{\beta}_{1} \varphi) \quad \forall \phi, \varphi \in L^2(D).$
- ${}_0I^{\beta}_{x}\, {}_0I^{\gamma}_{x} = {}_0I^{\beta+\gamma}_{x}$

UCL CENTRE FOR INVERSE PROBLEMS

Riemann-Liouville case

With $g = {}_0I_x^\alpha f$ and $v \in H_0^1(D)$, and $u = -{}_0I_x^\alpha f + ({}_0I_x^\alpha f)(1)x^{\alpha-1}$ satisfies

$$
\begin{aligned}
-({}_0^RD_x^\alpha u,\, v) &= -(({}_0I_x^{2-\alpha} g)'',\, v) && = (({}_0I_x^{2-\alpha} g)',\, v') \\
&= ({}_0I_x^{2-\alpha} g',\, v') && = ({}_0I_x^{1-\frac{\alpha}{2}} g',\, {}_xI_1^{1-\frac{\alpha}{2}} v') \\
&= -({}_0^RD_x^{\frac{\alpha}{2}} g,\, {}_x^RD_1^{\frac{\alpha}{2}} v)
\end{aligned}
$$

$+ \ ({}_0^RD_x^{\frac{\alpha}{2}} x^{\alpha-1},\, {}_x^RD_1^{\frac{\alpha}{2}} v) = 0 \Rightarrow$ find $u \in U \equiv \widetilde{H}^{\frac{\alpha}{2}}(D)$ s.t.

$$A(u,v) := -({}_0^RD_x^{\frac{\alpha}{2}} u,\, {}_x^RD_1^{\frac{\alpha}{2}} v) = (f, v) \quad \forall v \in U,$$

Ervin-Roop 2006; Jin-Lazarov-Pasciak-Rundell 2015

$$A(u,u) \geq c\|u\|^2_{\widetilde{H}^{\frac{\alpha}{2}}(D)}, \quad \forall u \in \widetilde{H}^{\frac{\alpha}{2}}(D).$$

general case $q \neq 0$: $a(u,v) = A(u,v) + (qu, v)$

Assumption on $a(\cdot,\cdot)$

- $a(u,v) = 0, \ \forall v \in V \quad \Rightarrow u = 0.$
- $a(u,v) = 0, \ \forall u \in U \quad \Rightarrow v = 0.$

- $\exists! u \in U$ s.t. $a(u,v) = \langle F, v \rangle, \forall v \in V$, $F \in V^*$
- further, $u \in \widetilde{H}^{\frac{\alpha}{2}}(D) \cap \widetilde{H}_L^{\alpha-1+\beta}(D)$, $\beta \in (0, \frac{1}{2})$, with

$$\|u\|_{H^{\alpha-1+\beta}(D)} \leq c\|f\|_{L^2(D)}.$$

Djrbashian-Caputo case

rewrite the representation u as $u = -{}_0I_x^\alpha f + xu'(0) \Rightarrow$

$$-({}_0^C D_x^\alpha u, v) = A(u,v) - u'(0)A(x,v) \quad \forall v \in \widetilde{H}_R^1(D)$$

with

$$A(x,v) = c_\alpha(x^{1-\alpha}, v).$$

weak formulation: find $u \in U \equiv \widetilde{H}^{\frac{\alpha}{2}}(D)$ s.t.

$$A(u,v) = (f,v), \quad \forall v \in V,$$

with

$$V = \left\{ \phi \in \widetilde{H}_R^{\frac{\alpha}{2}}(D) : \ (x^{1-\alpha}, \phi) = 0 \right\}$$

it differs from the R.-L. case only in test space V

the inf-sup condition:

$$\sup_{v\in V}\frac{A(u,v)}{\|v\|_{\widetilde{H}_R^{\frac{\alpha}{2}}(D)}}\geq c\|u\|_{\widetilde{H}^{\frac{\alpha}{2}}(D)}$$

general case $q\neq 0$:

$$a(u,v)=A(u,v)+(qu,v)$$

Under the assumption on $a(\cdot,\cdot)$:

- $\exists! u\in U$ s.t. $a(u,v)=\langle F,v\rangle$, $\forall v\in V$, $F\in V^*$
- if $f\in\widetilde{H}^{\beta}(D)$, with $\alpha+\beta>\frac{3}{2}$, and $q\in L^{\infty}(D)\cap\widetilde{H}^{\beta}(D)$, then $u\in\widetilde{H}^{\frac{\alpha}{2}}(D)\cap H^{\alpha+\beta}(D)$ and

$$\|u\|_{H^{\alpha+\beta}(D)}\leq c\|f\|_{\widetilde{H}^{\beta}(D)}.$$

summary on variational formulation

- both cases lead to nonsymmetric bilinear form, with the only difference in the test space V
- both cases have some regularity pickup: the R.-L. case has limited smoothing property due to the presence of $x^{\alpha-1}$
- if $f \in L^2(D)$ only, the proper variational formulation for D.-C. case with $\alpha \in (1, \frac{3}{2}]$ is still unclear.

Finite element method

conforming continuous piecewise linear FEM: find $u_h \in U_h$ s.t.

$$a(u_h, v) = (f, v), \quad \forall v \in V_h$$

under the assumption on $a(\cdot,\cdot)$

- R.-L. case. Let $f \in L^2(D)$, $q \in L^\infty(D)$. Then for $\beta \in (1-\frac{\alpha}{2}, \frac{1}{2})$

$$\|u - u_h\|_{L^2(D)} + h^{\frac{\alpha}{2}-1+\beta}\|u - u_h\|_{\widetilde{H}^{\frac{\alpha}{2}}(D)} \leq c h^{\alpha-2+2\beta}\|f\|_{L^2(D)}.$$

- D.-C. case. Let $f \in \widetilde{H}^\beta(D)$, $q \in L^\infty(D) \cap \widetilde{H}^\beta(D)$ for some $\beta \in [0, \frac{1}{2})$ s.t. $\alpha + \beta > \frac{3}{2}$. Then for any $\delta \in [0, \frac{1}{2})$,

$$\|u - u_h\|_{\widetilde{H}^{\frac{\alpha}{2}}(D)} \leq c h^{\min(\alpha+\beta,2)-\frac{\alpha}{2}}\|f\|_{\widetilde{H}^\beta(D)}.$$

$$\|u - u_h\|_{L^2(D)} \leq c h^{\min(\alpha+\beta,2)-1+\delta}\|f\|_{\widetilde{H}^\beta(D)}.$$

observations

- the $\widetilde{H}^{\frac{\alpha}{2}}(D)$-estimate is optimal
- the $L^2(D)$-estimate is suboptimal by one-half order
  $\Leftarrow$ adjoint problem is R.-L. type, and does not have full regularity
  numerically, the FEM has optimal $L^2(D)$ convergence ...

Table: Numerical results for Riemann-Liouville fractional derivative, $q = 0$, $f = 1$, mesh size $h = 1/(2^k \times 10)$.

α	k	3	4	5	6	7	rate
7/4	L^2	1.77e-4	7.37e-5	3.08e-5	1.29e-5	5.43e-6	1.27 (0.75)
	$H^{\frac{\alpha}{2}}$	2.94e-2	2.24e-2	1.72e-2	1.32e-2	1.01e-2	0.40 (0.38)
3/2	L^2	1.58e-3	7.89e-4	3.94e-4	1.97e-4	9.84e-5	1.01 (0.50)
	$H^{\frac{\alpha}{2}}$	1.17e-1	9.82e-2	8.22e-2	6.87e-2	5.73e-2	0.26 (0.25)
4/3	L^2	6.42e-3	3.60e-3	2.02e-3	1.13e-3	6.35e-4	0.84 (0.33)
	$H^{\frac{\alpha}{2}}$	2.48e-1	2.20e-1	1.94e-1	1.71e-1	1.50e-1	0.18 (0.17)

- the $H^{\frac{\alpha}{2}}$-estimate is sharp
- the L^2-estimate is suboptimal by one half order
- the larger is α, the better the convergence

Table: Numerical results for with a Djrbashian-Caputo fractional derivative, $q = 0$, $f = 1$, mesh size $h = 1/(2^k \times 10)$.

α	k	3	4	5	6	7	rate
$7/4$	L^2	1.03e-5	2.51e-6	6.16e-7	1.51e-7	3.74e-8	2.00 (1.50)
	$H^{\frac{\alpha}{2}}$	1.72e-3	7.91e-4	3.63e-4	1.67e-4	7.65e-5	1.12 (1.13)
$3/2$	L^2	1.24e-5	3.17e-6	8.12e-7	2.07e-7	5.29e-8	1.97 (1.50)
	$H^{\frac{\alpha}{2}}$	9.33e-4	4.08e-4	1.78e-4	7.76e-5	3.37e-5	1.20 (1.25)
$4/3$	L^2	1.97e-5	5.53e-6	1.55e-6	4.36e-7	1.22e-7	1.83 (1.33)
	$H^{\frac{\alpha}{2}}$	6.96e-4	3.12e-4	1.40e-4	6.26e-5	2.80e-5	1.16 (1.17)

- sharp $H^{\frac{\alpha}{2}}$-estimate, suboptimal L^2-estimate
- The convergence rate is better than in the R.-L. case

What are beyond the talk ?

- eigenvalue problem ? Yes but *vastly suboptimal* estimates
- alternative FEM with *optimal* L^2 estimate? Yes Jin-Lazarov-Zhou SINUM 2016
- better FEM techniques for Riemann-Liouville case? Yes ! Jin-Zhou M2NA 2016
- time dependent problem ? Yes for R.-L. derivative, Jin-Lazarov-Pasciak-Zhou SINUM 2014 but unclear for D.-C. derivative ?
- mixed left- and right-sided derivatives ? Yes, but complicated ...
- ...