Convergent finite difference scheme for compressible viscous isentropic flow

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Introduction

Compressible isentropic Navier-Stokes

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \mathcal{S} \tag{1b}$$

ρ : density
$\mathbf{u}$: velocity
p : pressure, $p = a\rho^{\gamma}$
$\mathcal{S}$: viscous stress, $\mathcal{S} = \mu \nabla \mathbf{u}, \mu > 0$

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Boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{1c}$$

Initial values

$$\rho(\mathbf{x}, 0) = \rho_0 > 0 \tag{1d}$$

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Main goal: convergent scheme.

Weak solution

- A. Valli. 1982. *strong solution for sufficiently smooth initial data, for small time interval.*
- P.-L. Lions. 1998. $\gamma > 9/5$.
- E. Feireisl, A. Novotný, and H. Petzeltová. 2001. $\gamma > 3/2$.

Numerical scheme

Full system FD, FV, FEM, BGK, DG

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Extension to full system

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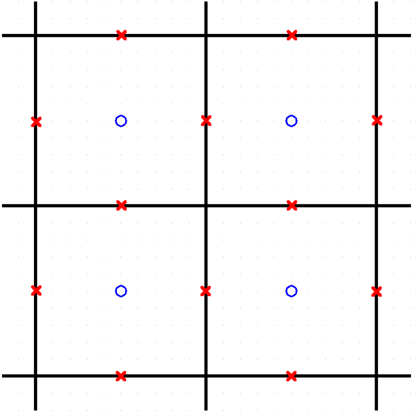
Our motivation simplify; Karper's idea in 1D

- Elements: $\Omega_h = \cup K$
- Faces: $\mathcal{E}$
- Exterior faces: $\mathcal{E}_{ext} = \partial\Omega \cup \mathcal{E}$.
- Interior faces:$\mathcal{E}_{int} = \mathcal{E} \setminus \mathcal{E}_{ext}$
- Faces of element K: $\mathcal{E}(K)$
- Primary grid o :
  density, pressure
- Dual grid × : velocity



$$\mathcal{E}(K) := \left\{\sigma = K \pm \frac{1}{2}\mathbf{e}_s, K \in \Omega_h, s = 1, \cdots, d\right\},$$

$$\sigma,s\pm = K \pm \frac{1}{2}\mathbf{e}_s.$$

Notations II

Between grids

$$\{f\}_\sigma = \frac{1}{2}(f_K + f_L), \quad \forall \sigma = K|L \in \mathcal{E}_{int}.$$

$$\bar{\mathbf{g}}_K = \frac{1}{2}\begin{pmatrix} g^1_{\sigma,1+} + g^1_{\sigma,1-} \\ g^2_{\sigma,2+} + g^2_{\sigma,2-} \\ g^3_{\sigma,3+} + g^3_{\sigma,3-} \end{pmatrix}.$$

The s-th component of vector $\mathbf{g}$ is defined on the face $\sigma \in \mathcal{E}$.

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Functional spaces

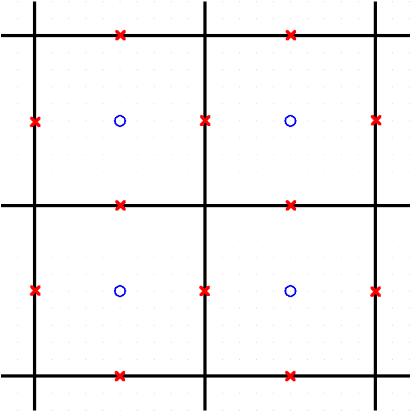
We denote the space for P0 functions with respect to the grid Ω_h by

$$X(\Omega_h) = \{f \in L^\infty(\Omega); f|_K \equiv f_K \in \mathcal{R}\}.$$

The s-th component of $\mathbf{g}$ is constant in the neighbourhood of the edge, not in the element K.

$$X(\mathcal{E}_{int})^3 = \left\{\mathbf{g} \in X(\mathcal{E})^3; \mathbf{g}|_{\mathcal{E}_{ext}} = \mathbf{0}\right\}$$

Discrete differential operators I



Time

$$\partial_h^t \phi^n = \frac{\phi^n - \phi^{n-1}}{\Delta t}$$

Space

Let $\sigma = K|L, L = K + \mathbf{e}_s, s = 1, 2, 3.$

$$(\partial_h^s f)_\sigma = \frac{f_L - f_K}{h}, \ f \in X(\Omega_h),$$

$$(\partial_h^r g^s)_{\sigma + \frac{1}{2}\mathbf{e}_r} = \frac{g^s_{\sigma + \mathbf{e}_r} - g^s_\sigma}{h}, \ \mathbf{g} \in X(\mathcal{E}_{int})^3.$$

When $r = s$, $(\partial_h^s g^s)_K = \frac{g^s_{\sigma,s+} - g^s_{\sigma,s-}}{h}$.

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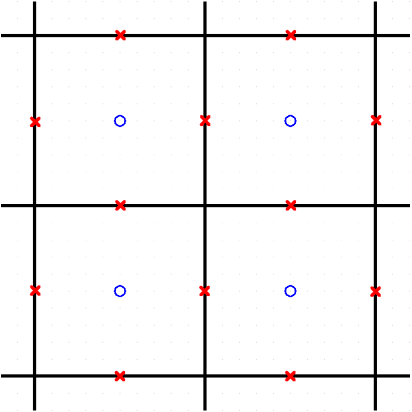

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Laplace

$$(\Delta_h f)_K = \frac{1}{h^2} \sum_{L \in \mathcal{N}(K)} (f_L - f_K),$$

$$(\Delta_h g)_\sigma = \frac{1}{h^2} \sum_{s=1}^{3} (g_{\sigma - \mathbf{e}_s} - 2g_\sigma + g_{\sigma + \mathbf{e}_s}).$$

Upwind flux

Let $f^+ = \max\{0, f\}, \quad f^- = \min\{0, f\}$. The upwind flux is given by

$$\mathrm{Up}[f, \mathbf{u}]_\sigma = f_K (u_\sigma^s)^+ + f_L (u_\sigma^s)^-,$$

Upwind discrete derivative and upwind divergence

$$\partial_s^{\mathrm{Up}}[f, \mathbf{u}]_K = \frac{\mathrm{Up}[f, \mathbf{u}]_{\sigma, s+} - \mathrm{Up}[f, \mathbf{u}]_{\sigma, s-}}{h},$$

$$\mathrm{div}_{\mathrm{Up}}[g, \mathbf{u}]_K = \sum_{s=1}^{3} \partial_s^{\mathrm{Up}}[f, \mathbf{u}]_K.$$

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Let $f \in X(\Omega_h), \mathbf{v} = [v^1, v^2, v^3] \in X(\mathcal{E}_{int})^3$, then $\displaystyle\sum_{K \in \Omega_h} \mathrm{div}_{\mathrm{Up}}[f, \mathbf{v}]_K = 0.$

Numerical Scheme

$$\partial_h^t \rho_K^n + \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K - h^\alpha (\Delta_h \rho^n)_K = 0, \tag{2a}$$

$$\{\partial_h^t (\rho \bar{\mathbf{u}})^n\}_\sigma + \{\mathrm{div}_{\mathrm{Up}}[\rho^n \bar{\mathbf{u}}^n, \mathbf{u}^n]\}_\sigma + \big(\partial_h^s p(\rho^n)\big)_\sigma \mathbf{e}_s - \mu\big(\Delta_h \mathbf{u}^n\big)_\sigma - h^\alpha \sum_{r=1}^{3} \big\{\partial_h^r\big(\{\hat{\mathbf{u}}^n\}\partial_h^r \rho^n\big)\big\}_\sigma = 0, \tag{2b}$$

for all $K \in \Omega_h, \sigma \in \mathcal{E}_{int}$ and $n = \{1, \ldots, N\}$, with boundary conditions.

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for all $K \in \Omega_h, \sigma \in \mathcal{E}_{int}$ and $n = \{1, \dots, N\}$, with boundary conditions.

Stability + Consistency Convergence

Lemma 1 (Existence of num sol)

Let $\varrho_h^{n-1} \in X(\Omega_h)$, $\mathbf{u}_h^{n-1} \in X(\mathcal{E}_{int})^3$ *be given;* $\varrho_K^{n-1} > 0$ *for all* $K \in \Omega_h$*.*
Then the numerical scheme (2a-2b) admits a solution

$$\varrho_h^n \in X(\Omega_h), \varrho_K^n > 0 \textit{ for all } K \in \Omega_h, \mathbf{u}_h^n \in X(\mathcal{E}_{int})^3.$$

Moreover, is satisfies the discrete conservation of mass

$$\sum_{K \in \Omega_h} \varrho_K^n = \sum_{K \in \Omega_h} \varrho_K^{n-1}.$$

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Positivity $\leftarrow$ non-negativity + strictly positivity

Mass conservation $\leftarrow$ upwind flux + artificial diffusion

Lemma 2 (Energy stability)

Let $(\rho_h, \mathbf{u}_h)$ be the numerical solution obtained through the scheme (2a–2b). For any time step $m = 1, \cdots, N$ the following estimate holds,

$$h^3 \sum_{K\in\Omega_h} \left(\rho_K^m \frac{|\bar{\mathbf{u}}_K^m|^2}{2} + \frac{1}{\gamma-1} p(\rho_K^m)\right) + \Delta t h^3 \mu \sum_{n=1}^{m} \sum_{K\in\Omega_h} \sum_{r=1}^{3} \sum_{s=1}^{3} |(\partial_h^r (u^s)^n)_K|^2$$

$$+ \sum_{j=1}^{4} \mathcal{N}_j \leq h^3 \sum_{K\in\Omega_h} \left(\rho_K^0 \frac{|\bar{\mathbf{u}}_K^0|^2}{2} + \frac{1}{\gamma-1} p(\rho_K^0)\right), \quad (3)$$

$$\mathcal{N}_1 = \Delta t h^3 \sum_{n=1}^{m} \sum_{K\in\Omega_h} \sum_{s=1}^{3} \frac{1}{2} \left((h^\alpha + h^2 (u_{\sigma,s\mp}^{s,n})^{\pm}) p''(\rho_{\sigma,s\mp}^{n,\star}) |(\partial_h^s \rho^n)_{\sigma,s\mp}|^2 \right),$$

$$\mathcal{N}_2 = \Delta t^2 h^3 \sum_{n=1}^{m} \sum_{K\in\Omega_h} \frac{p''(\rho_K^\eta)}{2} |\partial_t^h \rho_K^n|^2, \mathcal{N}_3 = \Delta t^2 h^3 \sum_{n=1}^{m} \sum_{K\in\Omega_h} \frac{\rho_K^{n-1}}{2} |\partial_t^h \bar{\mathbf{u}}_K^n|^2,$$

$$\mathcal{N}_4 = \Delta t h^4 \frac{1}{4} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{int}} |Up[\rho^n, \mathbf{u}^n]_\sigma| |(\partial_h^s \bar{\mathbf{u}}^n)_\sigma|^2.$$

Lemma 3 (Uniform bounds)

Let $(\rho_h, \mathbf{u}_h)$ be a numerical solution obtained through the scheme (2a)–(2b) and let the total initial energy D be defined by

$$D = \int_\Omega \frac{1}{2}\rho_0 \mathbf{u}_0^2 + \frac{1}{\gamma - 1} p(\rho_0) dx.$$

Then

$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim D, \quad \|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim D.$$

$$\|\sqrt{\rho_h}\bar{\mathbf{u}}_h\|_{L^\infty(L^2(\Omega))} \lesssim D.$$

$$\|\nabla_h \mathbf{u}_h\|_{L^2(L^2(\Omega))} \lesssim D.$$

$$\|\mathbf{u}_h\|_{L^2(L^6(\Omega))} \lesssim D.$$

Lemma 4 (Consistency for continuity)

Let $\rho_h^n, \hat{\mathbf{u}}_h^n$ *be piecewise constant and piecewise affine representations, respectively, of the solution to the numerical scheme (2a–2b). Then for any* $\phi \in C^1(\Omega)$ *it holds that*

$$\int_\Omega \partial_t^h \rho_h^n \phi dx - \int_\Omega \rho_h^n \hat{\mathbf{u}}_h^n \cdot \nabla_x \phi dx = h^\beta \int_\Omega \mathbf{R}_h \cdot \nabla_x \phi dx, \tag{4}$$

where $\beta > 0$ *and* $\|\mathbf{R}_h\|_{L^1(0,T;L^{3/2}(\Omega))} \lesssim 1$.

Idea of proof

Multiply (2a) with $(\Pi^P \phi)_K$.

Lemma 5 (Consistency for momentum)

Let $(\rho_h^n, \mathbf{u}_h^n)$ be piecewise constant representations of the solution to numerical scheme (2a–2b). Then for any $\mathbf{v} \in C_0^1(\Omega) \cap W^{2,q}(\Omega), q > 1$ it holds that

$$\begin{aligned} &\int_\Omega \partial_h^t (\rho_h \bar{\mathbf{u}}_h)^n \cdot \mathbf{v} dx - \int_\Omega \rho_h^n \bar{\mathbf{u}}_h^n \otimes \bar{\mathbf{u}}_h^n : \nabla_x \mathbf{v} dx - \int_\Omega p(\rho_h^n) div_x \mathbf{v} dx \\ &+ \mu \int_\Omega (\nabla_h \mathbf{u}_h^n) : \nabla_x \mathbf{v} dx \quad = h^{\theta_1} \langle \mathbf{r_h^1}, \nabla_x \mathbf{v} \rangle + h^{\theta_2} \langle \mathbf{r}_h^2, \nabla_x^2 \mathbf{v} \rangle, \end{aligned} \tag{5}$$

where $\mathbf{r_h^1} \in L^1(0, T; L^{\frac{6\gamma}{5\gamma+6}}(\Omega))$, $\mathbf{r_h^2} \in L^1(0, T; L^q(\Omega))$, $\theta_1, \theta_2 > 0$.

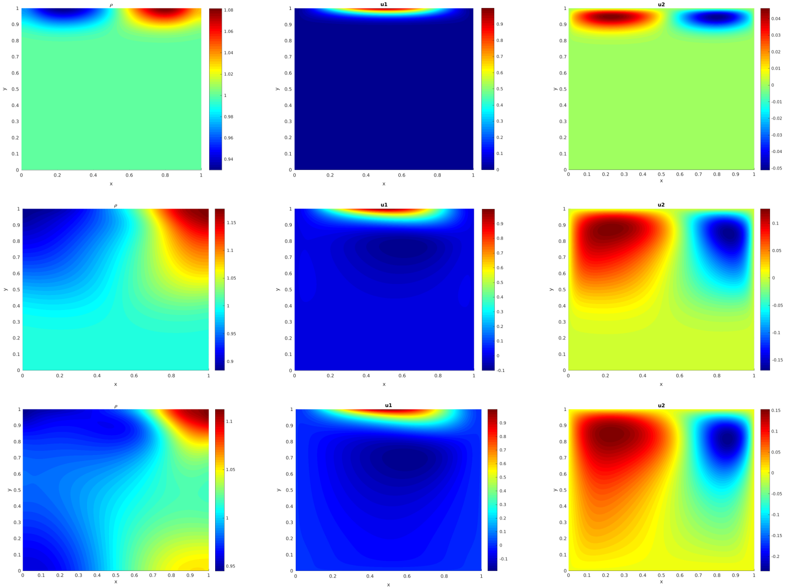
Idea of proof

Multiply momentum scheme (2b) with $\Pi^D \mathbf{v}$.

Numerical test

$$\Omega = [0,1]^2, \mu = 0.01, \ a = 1.0, \gamma = 1.4, \alpha = 0.83.$$

$$\Delta t = \text{CFL} \frac{h}{|u|_{\max}}$$

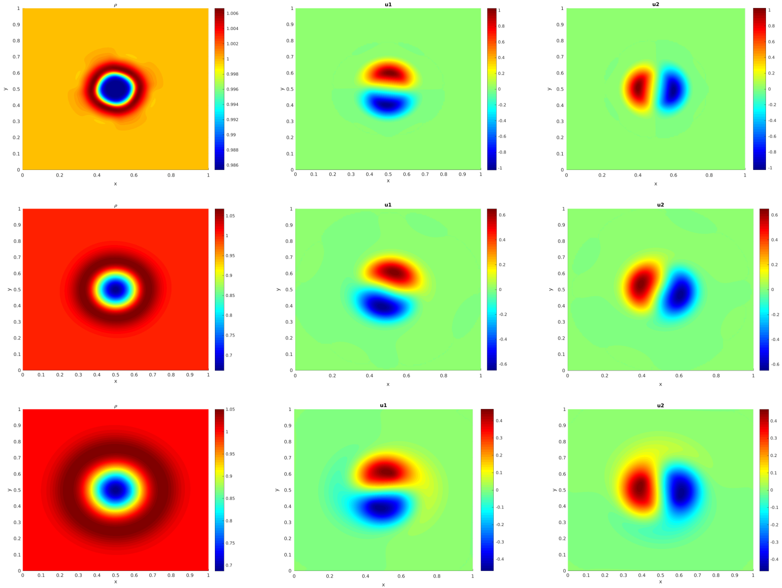


(a) density ρ **(b)** velocity U **(c)** velocity V

Cavity flow, upper boundary $\mathbf{u} = (16x^2(1-x)^2, 0)^T$.

Table: Convergence results of cavity flow

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	6.17e-01	–	4.65e-02	–	7.74e-03	–	4.94e-02	–
1/32	3.08e-01	1.00	2.32e-02	1.00	4.23e-03	0.87	3.19e-02	0.63
1/64	1.51e-01	1.03	1.12e-02	1.05	2.15e-03	0.97	1.96e-02	0.70
1/128	6.60e-02	1.19	4.75e-03	1.23	8.45e-04	1.35	9.97e-03	0.97



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$$U(0,x,y) = u_r(r) * (y - 0.5)/r,$$
$$V(0,x,y) = u_r(r) * (0.5 - x)/r.$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ and

$$u_r(r) = \sqrt{\gamma} \begin{cases} 2r/R & \text{if } 0 \le r < R/2, \\ 2(1 - r/R) & \text{if } R/2 \le r < R, \\ 0 & \text{if } r \ge R, \end{cases}$$

Table: Convergence results of Gresho vortex test

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	2.23e-01	–	7.84e-03	–	3.19e-06	–	6.66e-03	–
1/32	1.19e-01	0.91	4.09e-03	0.94	1.63e-06	0.97	4.27e-03	0.64
1/64	6.04e-02	0.97	2.01e-03	1.03	5.92e-07	1.46	2.27e-03	0.91
1/128	2.66e-02	1.18	8.98e-03	1.16	2.24e-07	1.40	1.17e-03	0.96

Conclusion

FD scheme

Stability

Consistency

Convergence

Error estimate $\cdots$

Higher order ?

Thank you for your attention!

Positivity Recall the renormalized continuity equation with test function

$$B(z) = \max(-z, 0)$$

$$\rightarrow B'(z)z - B(z) = 0. \sum_{K\in\Omega_h} B(\rho_K^n) = \sum_{K\in\Omega_h} \left(B(\rho_K^{n-1}) - P_K\right) \leq 0.$$

$$\sum_{K\in\Omega_h} \max\{-\rho_K^n, 0\} \leq 0. \rightarrow \rho_K^n \geq 0.$$

Choose $K \in \Omega_h$ such that $\rho_K^n \leq \rho_L^n$ for all $L \in \Omega_h$. Then we have

$$\rho_K^n - \rho_K^{n-1} = -\Delta t \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K + \Delta t h^\alpha(\Delta_h \rho^n)$$

$$\geq -\frac{\Delta t}{h}\sum_{s=1}^{3}\left(\rho_K^n u_{\sigma_{s,+}}^s - \rho_K^n u_{\sigma_{s,-}}^s + (\rho_{K+\mathbf{e}_s}^n - \rho_K^n) u_{\sigma_{s,+}}^{s\ -} + (\rho_K^n - \rho_{K-\mathbf{e}_s}^n) u_{\sigma_{s,-}}^{s\ +}\right)$$

$$\geq -\Delta t \rho_K^n (\mathrm{div}_h \mathbf{u}^n)_K \geq -\Delta t \rho_K^n |(\mathrm{div}_h \mathbf{u}^n)_K|.$$

$$\rho_L^n \geq \rho_K^n \geq \frac{1}{1 + \Delta t |(\mathrm{div}_h \mathbf{u}^n)_K|} \rho_K^{n-1} > 0, \qquad \text{for any } L \in \Omega_h,$$