The Gradient Discretisation Method

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$$\begin{cases} -\Delta \overline{u} = f \text{ in } \Omega, \\ \overline{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a polygonal subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$

weak formulation : $\begin{cases} \text{Find } \overline{u} \in H^1_0(\Omega) \text{ such that, for all } v \in H^1_0(\Omega), \\ \displaystyle\int_\Omega \nabla \overline{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) \mathrm{d}\mathbf{x}. \end{cases}$

simplicial mesh of Ω (e.g. triangles in dimension $d = 2$)
h mesh size used as index of a family of regular discretisations
space V_h = continuous piecewise linear functions on the mesh with zero value on $\partial\Omega$

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that, for all } v_h \in V_h, \\ \displaystyle\int_\Omega \nabla u_h(\mathbf{x}) \cdot \nabla v_h(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) v_h(\mathbf{x}) \mathrm{d}\mathbf{x}. \end{cases}$$

Existence and uniqueness of a discrete solution thanks to $\|\nabla\cdot\|_{L^2(\Omega)}$ norm on V_h

(Cea's lemma)

$$\|\nabla \overline{u} - \nabla u_h\|_{L^2(\Omega)} \leq S_h(\overline{u}) \ \text{ with } S_h(\varphi) = \min_{w_h \in V_h} \|\nabla w_h - \nabla \varphi\|_{L^2(\Omega)}, \ \forall \varphi \in H_0^1(\Omega)$$

Poincaré inequality : There exists $C_P > 0$ such that, $\forall v_h \in V_h$
$\|v_h\|_{L^2(\Omega)} \leq C_P \|\nabla v_h\|_{L^2(\Omega)^d}$.

then $$\|u_h - \overline{u}\|_{L^2(\Omega)} \leq C_P S_h(\overline{u})$$

provides the convergence of the method if

$$S_h(\varphi) \to 0 \text{ as } h \to 0 \text{ for all } \varphi \in H_0^1(\Omega)$$

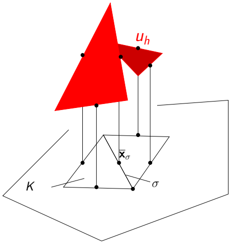
Limitations of conforming finite element method

1. Not in conservation form (no fluxes)
2. Not adapted to heterogeneous anisotropic problems (flow in porous media)
3. Not easily suited to some coupled nonlinear problems

simplicial mesh $\mathcal{M}$ of a domain Ω

For any $u_h = \sum_{\sigma\in\mathcal{F}_{\rm int}} u_\sigma \varphi_\sigma$,

$\forall K \in \mathcal{M}, \forall \mathbf{x} \in K,\ \nabla_{\mathcal{M}} u_h(\mathbf{x}) = \sum_{\sigma\in\mathcal{F}_{\rm int}} u_\sigma \nabla_K \varphi_\sigma$



Approximation method

Find $u_h \in V_h$ such that, $\forall v_h \in V_h$, $\int_\Omega \nabla_{\mathcal{M}} u_h(\mathbf{x}) \cdot \nabla_{\mathcal{M}} v_h(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) v_h(\mathbf{x}) \mathrm{d}\mathbf{x}$.

Existence and uniqueness of a discrete solution thanks to $\|\nabla_{\mathcal{M}} \cdot\|_{L^2(\Omega)}$ norm on V_h

Second Strang lemma :

$$\|\nabla\overline{u} - \nabla_{\mathcal{M}} u_h\|_{L^2(\Omega)} \le S_h(\overline{u}) + W_h(\nabla\overline{u}),$$

where $S_h(\varphi) = \inf_{v_h \in V_h} \|\nabla\varphi - \nabla_{\mathcal{M}} v_h\|_{L^2(\Omega)}, \ \forall \varphi \in H_0^1(\Omega)$

and $W_h(\boldsymbol{\varphi}) = \sup_{w_h \in V_h \setminus \{0\}} \dfrac{\int_\Omega (\boldsymbol{\varphi}(\mathbf{x}) \cdot \nabla_{\mathcal{M}} w_h(\mathbf{x}) + \mathrm{div}\boldsymbol{\varphi}(\mathbf{x}) w_h(\mathbf{x}))\mathrm{d}\mathbf{x}}{\|w_h\|_h}, \ \forall \boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$

Discrete Functional Analysis proves discrete Poincaré inequality :

There exists $C_P > 0$ such that, $\forall v_h \in V_h$, $\|v_h\|_{L^2(\Omega)} \le C_P \|\nabla_h v_h\|_{L^2(\Omega)^d}$.

then $\|u_h - \overline{u}\|_{L^2(\Omega)} \le C_P(S_h(\overline{u}) + W_h(\nabla\overline{u})).$

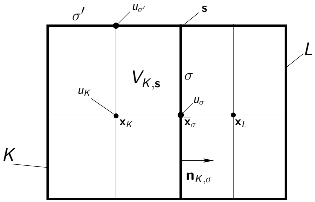
Convergence of the method if

$S_h(\varphi) \to 0$ as $h \to 0$ for all $\varphi \in H_0^1(\Omega)$
$W_h(\boldsymbol{\varphi}) \to 0$ as $h \to 0$ for all $\boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$

limitation : $\nabla_{\mathcal{M}} u_h$ computed from $u_h \in V_h$
$\nabla_{\mathcal{M}} u_h$ cannot be reconstructed independently of u_h (cf. mixed methods)

$$\forall K \in \mathcal{M}\,, \sum_{\sigma \in \mathcal{F}_K} F_{K,\sigma} = \int_K f(\mathbf{x})\mathrm{d}\mathbf{x},$$

and flux conservativity across each interior edge :
$\forall \sigma \in \mathcal{F}_{\rm int}$ common face of K and L
$F_{K,\sigma} + F_{L,\sigma} = 0.$



with

$$F_{K,\sigma} = -|\sigma| \frac{u_\sigma - u_K}{\mathrm{dist}(\bar{\mathbf{x}}_\sigma, \mathbf{x}_K)}.$$
$$u_\sigma = 0 \text{ if } \sigma \subset \partial\Omega,$$

variational formulation

find $((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{F}})$ such that $u_\sigma = 0$ if $\sigma \subset \partial\Omega$

s.t. for all $((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{F}})$ such that $v_\sigma = 0$ if $\sigma \subset \partial\Omega$

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{F}_K} \frac{|\sigma|}{\mathrm{dist}(\bar{\mathbf{x}}_\sigma, \mathbf{x}_K)} (v_\sigma - v_K)(u_\sigma - u_K) = \sum_{K\in\mathcal{M}} v_K \int_K f(\mathbf{x})\mathrm{d}\mathbf{x}.$$

no clear way to see this method as nonconforming finite element method

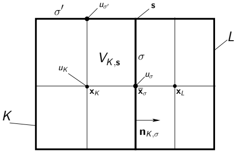
$X_{\mathcal{D},0} = \{((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{F}})$ such that $v_\sigma = 0$ if $\sigma \subset \partial\Omega\}$

$\forall u_\mathcal{D} \in X_{\mathcal{D},0}$, $\Pi_\mathcal{D} u_\mathcal{D}(\mathbf{x}) = u_K$ for a.e. $\mathbf{x} \in K$

$\nabla_{K,\mathbf{s}} u_\mathcal{D} = \nabla^{(\sigma)}_{K,\mathbf{s}} u_\mathcal{D}\ \mathbf{n}_{K,\sigma} + \nabla^{(\sigma')}_{K,\mathbf{s}} u_\mathcal{D}\ \mathbf{n}_{K,\sigma'}$

with $\nabla^{(\sigma)}_{K,\mathbf{s}} u_\mathcal{D} = \dfrac{u_\sigma - u_K}{\mathrm{dist}(\overline{\mathbf{x}}_\sigma, \mathbf{x}_K)}$ and $\nabla^{(\sigma')}_{K,\mathbf{s}} u_\mathcal{D} = \dfrac{u_{\sigma'} - u_K}{\mathrm{dist}(\overline{\mathbf{x}}_{\sigma'}, \mathbf{x}_K)}$

$\forall u_\mathcal{D} \in X_{\mathcal{D},0}$, $\nabla_\mathcal{D} u_\mathcal{D} = \nabla_{K,\mathbf{s}} u_\mathcal{D}$ on $V_{K,\mathbf{s}}$



Then, for $(u_\mathcal{D}, v_\mathcal{D}) \in X^2_{\mathcal{D},0}$, $\displaystyle\sum_{K\in\mathcal{M}} v_K \int_K f(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_\mathcal{D} v_\mathcal{D}(\mathbf{x})\mathrm{d}\mathbf{x}$ and

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{F}_K} \frac{|\sigma|}{\mathrm{dist}(\overline{\mathbf{x}}_\sigma, \mathbf{x}_K)} (v_\sigma - v_K)(u_\sigma - u_K) = \int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\mathbf{x}) \cdot \nabla_\mathcal{D} v_\mathcal{D}(\mathbf{x})\mathrm{d}\mathbf{x}.$$

$$\left\{ \begin{array}{l} \text{Find } u_\mathcal{D} \in X_{\mathcal{D},0} \text{ such that, for all } v_\mathcal{D} \in X_{\mathcal{D},0}, \\ \displaystyle\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\mathbf{x}) \cdot \nabla_\mathcal{D} v_\mathcal{D}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_\mathcal{D} v_\mathcal{D}(\mathbf{x})\mathrm{d}\mathbf{x}. \end{array} \right.$$

Existence and uniqueness of a discrete solution thanks to $\|\nabla_\mathcal{D} \cdot\|_{L^2(\Omega)}$ norm on $X_{\mathcal{D},0}$

Error estimate for the finite difference/volume method

$$\|\nabla\overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\nabla\overline{u})$$

with $S_{\mathcal{D}}(\varphi) := \min_{w_{\mathcal{D}} \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} w_{\mathcal{D}} - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} w_{\mathcal{D}} - \nabla\varphi\|_{L^2(\Omega)^d} \right), \ \forall \varphi \in H_0^1(\Omega)$

and $W_{\mathcal{D}}(\boldsymbol{\varphi}) = \sup_{v_{\mathcal{D}} \in X_{\mathcal{D},0}} \dfrac{\displaystyle\int_\Omega (\boldsymbol{\varphi}(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v_{\mathcal{D}}(\mathbf{x}) + \mathrm{div}\boldsymbol{\varphi}(x)\Pi_{\mathcal{D}} v_{\mathcal{D}}(\mathbf{x})\mathrm{d}\mathbf{x})\mathrm{d}\mathbf{x}}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)^d}}, \ \forall \boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$

Discrete Functional Analysis proves discrete Poincaré inequality :

There exists $C_P > 0$ such that, $\forall v_{\mathcal{D}} \in X_{\mathcal{D},0} \ \|\Pi_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)} \leq C_P \|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)^d}$.

then $\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^2(\Omega)} \leq (C_P + 1)S_{\mathcal{D}}(\overline{u}) + C_P W_{\mathcal{D}}(\nabla\overline{u})$.

Convergence of the method if $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is such that

$S_{\mathcal{D}_m}(\varphi) \to 0$ as $m \to \infty$ for all $\varphi \in H_0^1(\Omega)$
$W_{\mathcal{D}_m}(\boldsymbol{\varphi}) \to 0$ as $m \to \infty$ for all $\boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$

$$\|\nabla \overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \le S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\nabla \overline{u})$$
$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^2(\Omega)} \le (C_P + 1) S_{\mathcal{D}}(\overline{u}) + C_P W_{\mathcal{D}}(\nabla \overline{u}).$$

with

	conforming FE	nonconforming FE	Fin. Diff./Vol.
$X_{\mathcal{D},0}$	$\mathbb{R}^{\mathcal{V}_{\rm int}}$	$\mathbb{R}^{\mathcal{F}_{\rm int}}$	$\mathbb{R}^{\mathcal{M}+\mathcal{F}_{\rm int}}$
$\Pi_{\mathcal{D}}(u)$	$\sum_{\mathsf{s}\in\mathcal{V}_{\rm int}} u_{\mathsf{s}}\varphi_{\mathsf{s}}$	$\sum_{\sigma\in\mathcal{F}_{\rm int}} u_\sigma\varphi_\sigma$	u_K in K
$\nabla_{\mathcal{D}}(u)$	$\sum_{\mathsf{s}\in\mathcal{V}_{\rm int}} u_{\mathsf{s}}\nabla\varphi_{\mathsf{s}}$	$\sum_{\sigma\in\mathcal{F}_K} u_\sigma\nabla\varphi_\sigma$ in K	$\nabla_{K,\mathsf{s}} u$ on $V_{K,\mathsf{s}}$
C_P	Continuous Poincaré	Disc. Funct. Anal.	Disc. Funct. Anal.
$S_{\mathcal{D}}$	$\to 0$ as $h\to 0$	$\to 0$ as $h\to 0$	$\to 0$ as $h\to 0$
$W_{\mathcal{D}}$	0	$\to 0$ as $h\to 0$	$\to 0$ as $h\to 0$

and many other examples (mixed finite element method, DDFV, HMM,...)

included in the Gradient Discretisation Method

Gradient discretization $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$

- $X_{\mathcal{D},0}$ finite dimensional real vector space
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^p(\Omega)$ linear
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^p(\Omega)^d$ linear such that $\| \cdot \|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot \|_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Consistency and stability requested for convergence of the GDM

$(\mathcal{D}_m = (X_{\mathcal{D}_m,0},\ \Pi_{\mathcal{D}_m},\ \nabla_{\mathcal{D}_m}))_{m\in\mathbb{N}}$ family of gradient discretizations

1. **Coercivity** : $$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^p(\Omega)}}{\|v\|_{\mathcal{D}}}$$ $\Rightarrow$ discrete Poincaré inequality

   $C_{\mathcal{D}_m}$ **remains bounded**

2. **Consistency** : $S_{\mathcal{D}_m}(\varphi) \to 0$
$$\forall \varphi \in W_0^{1,p}(\Omega), \quad S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^p(\Omega)^d}\right)$$

3. **Limit-conformity** : $W_{\mathcal{D}_m}(\boldsymbol{\varphi}) \to 0$
$$\forall \boldsymbol{\varphi} \in W_{\mathrm{div}}^{p'}(\Omega),\ W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u \in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|u\|_{\mathcal{D}}} \int_\Omega (\nabla_{\mathcal{D}} u \cdot \boldsymbol{\varphi} + \Pi_{\mathcal{D}} u \,\mathrm{div}\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x}$$

4. For nonlinear problems : **Compactness** : **For any sequence** $(u_m)_{m\in\mathbb{N}}$ **;** $u_m \in X_{\mathcal{D}_m,0}$ $\|u_m\|_{\mathcal{D}_m} \le C$, **then the sequence** $\Pi_{\mathcal{D}_m} u_m$ **relatively compact in** $L^p(\Omega)$ (implies coercivity)

5. **Piecewise constant reconstruction** : There exists a basis $(e_i)_{i\in B}$ of $X_{\mathcal{D},0}$ and disjoint subsets $(\Omega_i)_{i\in B}$ of Ω s.t. $\Pi_{\mathcal{D}} u = \sum_{i\in B} u_i \mathbf{1}_{\Omega_i}$ for all $u = \sum_{i\in B} u_i e_i \in X_{\mathcal{D},0}$

Application to $p-$*Laplace problem* $p \in]1, +\infty[$

$$-\mathrm{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) = f + \mathrm{div}(\mathbf{F}) \text{ in } \Omega,$$
with boundary conditions $\overline{u} = 0$ on $\partial\Omega$,

under the following assumptions :

Ω is an open bounded connected subset of $\mathbb{R}^d$ $(d \in \mathbb{N}^\star)$
$p \in]1, +\infty[$, $f \in L^{p'}(\Omega)$ and $\mathbf{F} \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$

weak solution is given by

$$\overline{u} \in \underset{v \in W_0^{1,p}(\Omega)}{\mathrm{argmin}} \left(\frac{1}{p} \int_\Omega |\nabla v|^p \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) v(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{F}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \mathrm{d}\mathbf{x} \right)$$

$\overline{u} \in W_0^{1,p}(\Omega)$ and, for all $v \in W_0^{1,p}(\Omega)$,
$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{F}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \mathrm{d}\mathbf{x}$$

application of the GDM method

$$u_\mathcal{D} \in \underset{v \in X_{\mathcal{D},0}}{\mathrm{argmin}} \left(\frac{1}{p} \int_\Omega |\nabla_\mathcal{D} v(\mathbf{x})|^p \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) \Pi_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{F}(\mathbf{x}) \cdot \nabla_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x} \right)$$

Find $u \in X_{\mathcal{D},0}$ such that, for any $v \in X_{\mathcal{D},0}$,
$$\int_\Omega |\nabla_\mathcal{D} u(\mathbf{x})|^{p-2} \nabla_\mathcal{D} u(\mathbf{x}) \cdot \nabla_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) \Pi_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{F}(\mathbf{x}) \cdot \nabla_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x}$$

estimates

$$\|\nabla \overline{u}\|_{L^p(\Omega)^d} \leq \left(C_{P,p}\|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d}\right)^{\frac{1}{p-1}}$$

and

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \leq \left(C_{\mathcal{D}}\|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d}\right)^{\frac{1}{p-1}},$$

If $p \in (1,2]$,

$$\|\nabla \overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \leq S_{\mathcal{D}}(\overline{u}) + C_1(p)\left[W_{\mathcal{D}}(|\nabla \overline{u}|^{p-2}\nabla \overline{u} + \mathbf{F}) + S_{\mathcal{D}}(\overline{u})^{p-1}\right] \times \left[S_{\mathcal{D}}(\overline{u})^p + \left[(C_{\mathcal{D}} + C_{P,p})\|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d}\right]^{\frac{p}{p-1}}\right]^{\frac{2-p}{2}}.$$

If $p \in (2,+\infty)$,

$$\|\nabla \overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \leq S_{\mathcal{D}}(\overline{u}) + C_1(p)\Big[W_{\mathcal{D}}(|\nabla \overline{u}|^{p-2}\nabla \overline{u} + \mathbf{F}) + S_{\mathcal{D}}(\overline{u})\left[\left(C_{P,p}\|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d}\right)^{\frac{1}{p-1}} + S_{\mathcal{D}}(\overline{u})\right]^{p-2}\Big]^{\frac{1}{p-1}}.$$

and

$$\|\overline{u} - \Pi_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)} \leq S_{\mathcal{D}}(\overline{u}) + C_{\mathcal{D}}(S_{\mathcal{D}}(\overline{u}) + \|\nabla \overline{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d}).$$

Convergence thanks to coercivity, consistency and limit-conformity

Application to quasilinear elliptic problem $p = 2$

$$-\mathrm{div}(\Lambda(\mathbf{x}, \overline{u}(\mathbf{x}))\nabla\overline{u}) = f \text{ in } \Omega,$$
$$\text{with } \overline{u} = 0 \text{ on } \partial\Omega$$

hypotheses

Ω is an open bounded connected subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^\star$)

Λ is a Caratheodory function from $\Omega \times \mathbb{R}$ to $\mathcal{M}_d(\mathbb{R})$,
$\Lambda(\mathbf{x}, s)$ is measurable w.r.t. $\mathbf{x}$ and continuous w.r.t. s,
there exists $\underline{\lambda}, \overline{\lambda} > 0$ such that, for a.e. $\mathbf{x} \in \Omega$, for all $s \in \mathbb{R}$,
$\Lambda(\mathbf{x}, s)$ is symmetric with eigenvalues in $[\underline{\lambda}, \overline{\lambda}]$,

$f \in L^2(\Omega)$

weak form of the problem

$$\overline{u} \in H_0^1(\Omega),\ \forall v \in H_0^1(\Omega),$$
$$\int_\Omega \Lambda(\mathbf{x}, \overline{u}(\mathbf{x}))\nabla\overline{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}$$

GDM : Find $u \in X_{\mathcal{D},0}$ such that for any $v \in X_{\mathcal{D},0}$,
$$\int_\Omega \Lambda(\mathbf{x}, \Pi_{\mathcal{D}} u(\mathbf{x}))\nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_{\mathcal{D}} v(\mathbf{x})\mathrm{d}\mathbf{x}$$

Properties of the discrete problem

Existence of a discrete solution : mapping $\mathbb{R}^N \to \mathbb{R}^N$, $W \mapsto U$ such that

$$w = \sum_{j=1}^{N} W_j \xi^{(j)},\ u = \sum_{j=1}^{N} U_j \xi^{(j)}$$

Find $u \in X_{\mathcal{D},0}$ such that, $\forall v \in X_{\mathcal{D},0}$,

$$\int_\Omega \Lambda(\mathbf{x}, \Pi_\mathcal{D} w(\mathbf{x})) \nabla_\mathcal{D} u(\mathbf{x}) \cdot \nabla_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) \Pi_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

Estimate letting $v = u$

$$\underline{\lambda} \|\nabla_\mathcal{D} u\|^2_{L^2(\Omega)^d} \leq \|f\|_{L^2(\Omega)} \|\Pi_\mathcal{D} u\|_{L^2(\Omega)} \leq C_\mathcal{D} \|f\|_{L^2(\Omega)} \|\nabla_\mathcal{D} u\|_{L^2(\Omega)^d}$$

implies $\|\nabla_\mathcal{D} u\|_{L^2(\Omega)^d} \leq \dfrac{C_\mathcal{D}}{\underline{\lambda}} \|f\|_{L^2(\Omega)}$.

Proves mapping well defined, and that Brouwer theorem applies.

coercivity hypothesis implies

$$\|\nabla_{\mathcal{D}_m} u_m\|_{L^2(\Omega)^d} \leq \frac{C_P}{\underline{\lambda}} \|f\|_{L^2(\Omega)}.$$

compactness hypothesis implies subsequence such that $\Pi_{\mathcal{D}_m} u_m$ converges in $L^2(\Omega)$ to $\overline{u}$ and $\nabla_{\mathcal{D}_m} u_m$ weakly converges in $L^2(\Omega)$ to G

passing to the limit in limit-conformity relation and prolongement by 0 imply

$$\forall \varphi \in H_{\mathrm{div}}(\mathbb{R}^d),\ \int_{\mathbb{R}^d} \left(G(\mathbf{x}) \cdot \varphi(\mathbf{x}) + u(\mathbf{x}) \mathrm{div} \varphi(\mathbf{x}) \right) \mathrm{d}\mathbf{x} = 0.$$

$\overline{u} = 0$ outside Ω implies $G = \nabla \overline{u}$ and $\overline{u} \in H^1_0(\Omega)$

consistent interpolation $P_{\mathcal{D}} : H_0^1(\Omega) \to X_{\mathcal{D},0}$ defined by

$$P_{\mathcal{D}}\varphi \in \underset{v \in X_{\mathcal{D},0}}{\operatorname{argmin}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2(\Omega)^d} \right).$$

We have

$$\|\Pi_{\mathcal{D}}(P_{\mathcal{D}}\varphi) - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}}(P_{\mathcal{D}}\varphi) - \nabla\varphi\|_{L^2(\Omega)^d} \leq S_{\mathcal{D}}(\varphi)$$

$$\int_\Omega \Lambda(\mathbf{x}, \Pi_{\mathcal{D}_m} u_m(\mathbf{x}))\nabla_{\mathcal{D}_m} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}\varphi)(\mathbf{x})\mathrm{d}\mathbf{x} \to \int_\Omega \Lambda(\mathbf{x}, \overline{u}(\mathbf{x}))\nabla\overline{u}(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x})\mathrm{d}\mathbf{x}$$
$$\text{and } \int_\Omega f(\mathbf{x})\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}\varphi)(\mathbf{x})\mathrm{d}\mathbf{x} \to \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mathbf{x} \quad \text{as } m \to \infty.$$

implies $\overline{u}$ is a solution to the continuous problem

thanks to

$$\lim_{m\to\infty} \int_\Omega \Lambda(\mathbf{x}, \Pi_{\mathcal{D}_m} u_m(\mathbf{x}))\nabla_{\mathcal{D}_m} u_m(\mathbf{x}) \cdot \nabla_{\mathcal{D}_m} u_m(\mathbf{x})\mathrm{d}\mathbf{x}$$
$$= \int_\Omega f(\mathbf{x})\overline{u}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega \Lambda(\mathbf{x}, \overline{u}(\mathbf{x}))\nabla\overline{u}(\mathbf{x}) \cdot \nabla\overline{u}(\mathbf{x})\mathrm{d}\mathbf{x}$$

$$\int_\Omega \Lambda(\mathbf{x}, \Pi_{\mathcal{D}_m} u_m(\mathbf{x}))(\nabla_{\mathcal{D}_m} u_m(\mathbf{x}) - \nabla\overline{u}(\mathbf{x})) \cdot (\nabla_{\mathcal{D}_m} u_m(\mathbf{x}) - \nabla\overline{u}(\mathbf{x}))\mathrm{d}\mathbf{x} \to 0 \quad \text{as } m \to \infty$$

gives strong convergence of the gradient space–time gradient discretisation for $T > 0$

$\mathcal{D}_T = (\mathcal{D}, \mathcal{I}_\mathcal{D}, (t^{(n)})_{n=0,\ldots,N})$ with
$\mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D})$ GD such that $\Pi_\mathcal{D}(X_{\mathcal{D},0}) \subset L^{\max(p,2)}(\Omega)$,
$\mathcal{I}_\mathcal{D} \;:\; L^2(\Omega) \to X_{\mathcal{D},0}$
$t^{(0)} = 0 < t^{(1)} \ldots < t^{(N)} = T$, $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$, $\delta t_\mathcal{D} = \max_{n=0,\ldots,N-1} \delta t^{(n+\frac{1}{2})}$

definition of space-time reconstructions for $\theta \in [0,1]$ and all $t \in [0, T]$

$$\forall v = (v^{(n)})_{n=0,\ldots,N} \in X_{\mathcal{D},0}^{N+1},\ \forall n = 0,\ldots,N-1,\ \forall t \in (t^{(n)}, t^{(n+1)}],$$
$$v_\theta(t) = v^{(n+\theta)} := \theta v^{(n+1)} + (1-\theta) v^{(n)} \text{ and, for a.e. } \mathbf{x} \in \Omega,$$
$$\Pi_\mathcal{D}^{(\theta)} v(\mathbf{x}, t) = \Pi_\mathcal{D}[v_\theta(t)](\mathbf{x}),\ \nabla_\mathcal{D}^{(\theta)} v(\mathbf{x}, t) = \nabla_\mathcal{D}[v_\theta(t)](\mathbf{x})$$
$$\delta_\mathcal{D} v(t) = \delta_\mathcal{D}^{(n+\frac{1}{2})} v := \frac{\Pi_\mathcal{D} v^{(n+1)} - \Pi_\mathcal{D} v^{(n)}}{\delta t^{(n+\frac{1}{2})}}$$
and $\Pi_\mathcal{D}^{(\theta)} v(\cdot, 0) = \Pi_\mathcal{D} v^{(0)}$

definition of dual norm

$$\forall w \in \Pi_\mathcal{D}(X_{\mathcal{D},0}), \|w\|_{\star,\mathcal{D}} = \sup \left\{ \int_\Omega w(\mathbf{x}) \Pi_\mathcal{D} v(\mathbf{x}) \mathrm{d}\mathbf{x} \;:\; v \in X_{\mathcal{D},0}\,,\ \|v\|_\mathcal{D} = 1 \right\}$$

B Banach space, $T > 0$, $\theta \in [0,1]$, $(f_m)_{m\in\mathbb{N}}$ sequence of $L^p(0,T;B)$ such that

1. $(X_m)_{m\in\mathbb{N}}$ "compactly embedded in B" : any sequence $(u_m)_{m\in\mathbb{N}}$ such that $u_m \in X_m$ for all $m \in \mathbb{N}$ and $(\|u_m\|_{X_m})_{m\in\mathbb{N}}$ is bounded is relatively compact in B
2. $X_m \subset Y_m$ for all $m \in \mathbb{N}$ and, for any sequence $(u_m)_{m\in\mathbb{N}}$ such that
   1. $u_m \in X_m$ for all $m \in \mathbb{N}$ and $(\|u_m\|_{X_m})_{m\in\mathbb{N}}$ bounded,
   2. $\|u_m\|_{Y_m} \to 0$ as $n \to +\infty$,
   3. $(u_m)_{m\in\mathbb{N}}$ converges in B,

   it holds $u_m \to 0$ in B.

1. For all $m \in \mathbb{N}$, there exists $N \in \mathbb{N}^\star$, $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} = T$, and $(v^{(n)})_{n=0,\dots,N} \in X_m^{N+1}$ such that, for all $n \in \{0,\dots,N-1\}$ and a.e. $t \in (t^{(n)}, t^{(n+1)})$, $f_m(t) = \theta v^{(n+1)} + (1-\theta)v^{(n)}$, $\delta_m f_m(t) = \dfrac{v^{(n+1)} - v^{(n)}}{t^{(n+1)} - t^{(n)}}$
2. $(f_m)_{m\in\mathbb{N}}$ bounded in $L^p(0,T;B)$
3. $(\|f_m\|_{L^p(0,T;X_m)})_{m\in\mathbb{N}}$ bounded
4. $(\|\delta_m f_m\|_{L^1(0,T;Y_m)})_{m\in\mathbb{N}}$ bounded.

Then $(f_m)_{m\in\mathbb{N}}$ relatively compact in $L^p(0,T;B)$

space-time consistency for a sequence $((\mathcal{D}_T)_m)_{m\in\mathbb{N}}$:

- $\forall\varphi \in W_0^{1,p}(\Omega)\cap L^2(\Omega)\,,\ \lim_{m\to\infty} \widehat{S}_{\mathcal{D}_m}(\varphi) = 0.$

where $\widehat{S}_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},0}} \Big(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^{\max(p,2)}(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^p(\Omega)^d}\Big)$

- $\forall u \in L^2(\Omega),\ \lim_{m\to\infty} \|u - \Pi_{\mathcal{D}_m}\mathcal{I}_{\mathcal{D}_m} u\|_{L^2(\Omega)} = 0.$
- $\delta t_{\mathcal{D}_m} \to 0$ as $m \to \infty$.

$T > 0$, $p \in (1, +\infty)$ and $\theta \in [0, 1]$

$((\mathcal{D}_T)_m = (\mathcal{D}_m, \mathcal{I}_{\mathcal{D}_m}, (t_m^{(n)})_{n=0,\ldots,N_m}))_{m\in\mathbb{N}}$ sequence of space–time-consistent and compact space-time GD

For any $m \in \mathbb{N}$, $v_m \in X_{\mathcal{D}_m,0}^{N_m+1}$ s.t. $\exists C > 0$ with

$$\forall m \in \mathbb{N},\ \int_0^T \|(v_m)_\theta(t)\|_{\mathcal{D}_m}^p \,\mathrm{d}t \le C$$

and

$$\forall m \in \mathbb{N},\ \int_0^T \|\delta_{\mathcal{D}_m} v_m(t)\|_{\star,\mathcal{D}_m} \,\mathrm{d}t \le C$$

Then sequence $(\Pi_{\mathcal{D}_m}^{(\theta)} v_m)_{m\in\mathbb{N}}$ relatively compact in $L^p(\Omega \times (0, T))$

in $\Omega \times (0,T)$
$\partial_t \overline{u} - \mathrm{div}\,(\Lambda(\mathbf{x},\overline{u})\nabla\overline{u}) = f$
$\overline{u}(\cdot,0) = u_{\rm ini}$, on Ω,
$\overline{u} = 0$ on $\partial\Omega$

- Ω is an open bounded connected subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$ and $T > 0$,
- $\Lambda : \Omega \times (0,T) \to \mathcal{M}_d(\mathbb{R})$ is a Caratheodory function $\exists \underline{\lambda}, \overline{\lambda} > 0$ s.t., for a.e. $\mathbf{x} \in \Omega$, for all $s \in \mathbb{R}$ $\Lambda(\mathbf{x}, s)$ symmetric, eigenvalues $\in [\underline{\lambda}, \overline{\lambda}]$,
- $f \in L^2(\Omega \times (0,T))$,
- $u_{\rm ini} \in L^2(\Omega)$.

weak solution

$\overline{u} \in L^2(0,T;H_0^1(\Omega))$ and, for all $\overline{v} \in L^2(0,T;H_0^1(\Omega))$ such that $\partial_t \overline{v} \in L^2(\Omega \times (0,T))$ and $\overline{v}(\cdot,T) = 0$,

$$-\int_0^T \int_\Omega \overline{u}\partial_t\overline{v}\mathrm{d}\mathbf{x}\mathrm{d}t - \int_\Omega u_{\rm ini}(\mathbf{x})\overline{v}(\mathbf{x},0)\mathrm{d}\mathbf{x} + \int_0^T \int_\Omega \Lambda(\mathbf{x},\overline{u})\nabla\overline{u}\cdot\nabla\overline{v}\mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Omega f\overline{v}\mathrm{d}\mathbf{x}\mathrm{d}t$$

gradient scheme

$u^{(0)} = \mathcal{I}_{\mathcal{D}} u_{\rm ini}$ and, for all $n = 0,\ldots,N-1$, $u^{(n+1)}$ satisfies $\forall v \in X_{\mathcal{D},0}$,

$$\int_\Omega \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u \Pi_{\mathcal{D}} v \mathrm{d}\mathbf{x} + \int_\Omega \Lambda(\mathbf{x}, \Pi_{\mathcal{D}} u^{(n+\theta)})\nabla_{\mathcal{D}} u^{(n+\theta)} \cdot \nabla_{\mathcal{D}} v \mathrm{d}\mathbf{x} = \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \Pi_{\mathcal{D}} v \mathrm{d}\mathbf{x}\mathrm{d}t$$

Error estimate in the linear case $p = 2$

$\mathbf{F} = 0$ and $\Lambda(\cdot, s) = \mathrm{Id}$, $\exists h_{\mathcal{D}} > 0$ such that
$\forall \varphi \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega),\ S_{\mathcal{D}}(\varphi) \le h_{\mathcal{D}} \|\varphi\|_{W^{2,\infty}(\Omega)}$,
$\forall \boldsymbol{\varphi} \in W^{1,\infty}(\Omega)^d,\ W_{\mathcal{D}}(\boldsymbol{\varphi}) \le h_{\mathcal{D}} \|\boldsymbol{\varphi}\|_{W^{1,\infty}(\Omega)^d}$,
$\forall \varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega),\ \|\Pi_{\mathcal{D}}\mathcal{I}_{\mathcal{D}}\varphi - \varphi\|_{L^2(\Omega)} \le h_{\mathcal{D}} \|\varphi\|_{W^{1,\infty}(\Omega)}$
$\overline{u} \in C^3(\overline{\Omega} \times [0, T])$

then

$$\max_{t\in[0,T]} \left\| \Pi_{\mathcal{D}}^{(1)} u(\cdot, t) - \overline{u}(\cdot, t) \right\|_{L^2(\Omega)} \le C(\delta t_{\mathcal{D}} + h_{\mathcal{D}})$$
and $\left\| \nabla_{\mathcal{D}}^{(1)} u - \nabla \overline{u} \right\|_{L^2(\Omega\times(0,T))^d} \le C(\delta t_{\mathcal{D}} + h_{\mathcal{D}})$

proof : follow elliptic error estimate, and sum on n using L^2 linear interpolation

Hypotheses : compactness (implies coercivity), space-time consistency, limit-conformity

Estimate (take $v = u$ as test function)

for any $k = 0, \ldots, N$,

$$\frac{1}{2}\int_\Omega (\Pi_\mathcal{D} u^{(k)})^2 \mathrm{d}\mathbf{x} + \int_0^{t^{(k)}} \int_\Omega \Lambda(\mathbf{x}, \Pi_\mathcal{D}^{(\theta)} u) \nabla_\mathcal{D}^{(\theta)} u \cdot \nabla_\mathcal{D}^{(\theta)} u \mathrm{d}\mathbf{x}\mathrm{d}t$$

$$\leq \frac{1}{2}\int_\Omega (\Pi_\mathcal{D} \mathcal{I}_\mathcal{D} u_{\mathrm{ini}}(\mathbf{x}))^2 \mathrm{d}\mathbf{x} + \int_0^{t^{(k)}} \int_\Omega f \Pi_\mathcal{D}^{(\theta)} u \mathrm{d}\mathbf{x}\mathrm{d}t$$

$$\sup_{t\in[0,T]} \left\| \Pi_\mathcal{D}^{(\theta)} u(t) \right\|_{L^2(\Omega)} \leq C_2 \quad \text{and} \quad \left\| \nabla_\mathcal{D}^{(\theta)} u \right\|_{L^2(\Omega\times(0,T))^d} \leq C_2.$$

consequence : existence of a discrete solution (Brouwer or topological degree)

estimate on dual norm $\displaystyle \int_0^T \|\delta_\mathcal{D} u(t)\|_{\star,\mathcal{D}}^2 \, \mathrm{d}t \leq C_3$

Application of discrete Aubin Simon theorem provides strong convergence
pass to the limit on the scheme
uniform-in-time convergence (i.e. in $L^\infty(0, T; L^2(\Omega))$
strong convergence of gradient thanks to uniform-in-time convergence

$\partial_t \overline{u} - \mathrm{div}\,(\Lambda(\mathbf{x})\nabla\zeta(\overline{u})) = f$
in $\Omega \times (0, T)$,

$\overline{u}(\mathbf{x}, 0) = u_{\mathrm{ini}}(\mathbf{x})$
in Ω,

$\zeta(\overline{u}) = 0$
on $\partial\Omega \times (0, T)$

- Ω is an open bounded connected polytopal subset of $\mathbb{R}^d$ $(d \in \mathbb{N}^\star)$ and $T > 0$,
- $\zeta : \mathbb{R} \to \mathbb{R}$ is non-decreasing, Lipschitz continuous with Lipschitz constant $L_\zeta > 0$, $\zeta(0) = 0$ and, for some $M_0, M_1 > 0$, $|\zeta(s)| \geq M_0|s| - M_1$ for all $s \in \mathbb{R}$,
- $\Lambda : \Omega \to \mathcal{M}_d(\mathbb{R})$ is measurable and there exists $\overline{\lambda} \geq \underline{\lambda} > 0$ such that, for a.e. $\mathbf{x} \in \Omega$, $\Lambda(\mathbf{x})$ is symmetric with eigenvalues in $[\underline{\lambda}, \overline{\lambda}]$.
- $u_{\mathrm{ini}} \in L^2(\Omega)$, $f \in L^2(\Omega \times (0, T))$.

weak sense

$$
\begin{aligned}
&\zeta(\overline{u}) \in L^2(0, T; H^1_0(\Omega)),\\
&-\int_0^T \int_\Omega \overline{u}\partial_t \overline{v} \mathrm{d}\mathbf{x}\mathrm{d}t - \int_\Omega u_{\mathrm{ini}}(\mathbf{x})\overline{v}(\mathbf{x}, 0)\mathrm{d}\mathbf{x}\\
&\qquad + \int_0^T \int_\Omega \Lambda(\mathbf{x})\nabla\zeta(\overline{u}) \cdot \nabla\overline{v} \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Omega f\overline{v}\mathrm{d}\mathbf{x}\mathrm{d}t,\\
&\qquad\quad \forall \overline{v} \in L^2(0; T; H^1_0(\Omega)) \text{ such that } \partial_t \overline{v} \in L^2((0, T) \times \Omega) \text{ and } \overline{v}(\cdot, T) = 0.
\end{aligned}
$$

Alt&Luckhaus : $\partial_t u_m - \Delta\zeta(u_m) = f_m$

multiplication by $\overline{u}$ provides $\|\zeta(u_m)\|_{L^2(0,T;H^1_0(\Omega))} \leq C$ and $\|u_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C$
one naturally gets $\|\partial_t u_m\|_{L^2(0,T;H^{-1}(\Omega))} \leq C$ does not lead compactness on $\zeta(u_m)$ in $L^2(0,T;L^2(\Omega))$

Problem : identify limit $\overline{\zeta}$ of $\zeta(u_m)$ as $\zeta(\overline{u})$

remarkable ideas :

$$\begin{aligned}
&\int_0^{T-\tau}\int_\Omega (\zeta(u_m(\mathbf{x},t+\tau)) - \zeta(u_m(\mathbf{x},t)))^2 \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\leq C\int_0^{T-\tau}\int_\Omega (\zeta(u_m(\mathbf{x},t+\tau)) - \zeta(u_m(\mathbf{x},t))(u_m(\mathbf{x},t+\tau) - u_m(\mathbf{x},t))\mathrm{d}\mathbf{x}\mathrm{d}t \\
&= C\int_0^{T-\tau}\int_\Omega (\zeta(u_m(\mathbf{x},t+\tau)) - \zeta(u_m(\mathbf{x},t))\int_0^\tau \partial_t u_m(\mathbf{x},t+s)\mathrm{d}s\mathrm{d}\mathbf{x}\mathrm{d}t \\
&= -C\int_0^{T-\tau}\int_\Omega (\nabla\zeta(u_m(\mathbf{x},t+\tau)) - \nabla\zeta(u_m(\mathbf{x},t))\nabla\zeta(u_m(\mathbf{x},t+s))\mathrm{d}s\mathrm{d}\mathbf{x}\mathrm{d}t + \text{ Term in } f_m \\
&\leq C\tau(\|\zeta(u_m)\|^2_{L^2(0,T;H^1(\Omega))} + \ldots)
\end{aligned}$$

provides strong convergence of $\zeta(u_m)$

allows application of Minty trick : $\lim_{m\to\infty}\int_0^T\int_\Omega (\zeta(u_m) - \zeta(\varphi)(u_m - \varphi)\mathrm{d}\mathbf{x}\mathrm{d}t \geq 0$

thanks to the strong-weak limit in $\int_0^T\int_\Omega \zeta(u_m)u_m\mathrm{d}\mathbf{x}\mathrm{d}t$,

implies $\int_0^T\int_\Omega (\overline{\zeta} - \zeta(\varphi))(\overline{u} - \varphi)\mathrm{d}\mathbf{x}\mathrm{d}t \geq 0$ implies $\overline{\zeta} = \zeta(\overline{u})$

Use of Aubin-Simon in the continuous setting

$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} \le C$ and $\|\partial_t u_m\|_{L^2(0,T;H^{-1}(\Omega))} \le C$
imply compactness on u_m in $L^2(0,T;H^{-1}(\Omega))$
weak convergence of $\zeta(u_m)$ in $L^2(0,T;H^1(\Omega))$ to $\overline{\zeta}$
thanks to the weak-strong limit in $\int_0^T\int_\Omega \zeta(u_m)u_m \mathrm{d}\mathbf{x}\mathrm{d}t$, apply Minty's trick

$$\int_0^T\int_\Omega (\overline{\zeta} - \zeta(\varphi))(\overline{u} - \varphi)\mathrm{d}\mathbf{x}\mathrm{d}t \ge 0$$

implies $\overline{\zeta} = \zeta(\overline{u})$

Conclude compactness on $\zeta(u_m)$ in $L^2(0,T;L^2(\Omega))$ with

$$\int_0^T\int_\Omega (\zeta(\overline{u}) - \zeta(u_m(\mathbf{x},t)))^2\mathrm{d}\mathbf{x}\mathrm{d}t \le C\int_0^T\int_\Omega (\zeta(\overline{u}) - \zeta(u_m(\mathbf{x},t))(\overline{u} - u_m(\mathbf{x},t))\mathrm{d}\mathbf{x}\mathrm{d}t \to 0$$

use of time translates no longer necessary...
application to the GDM

Scheme and estimate

$\Pi_{\mathcal{D}} u = \sum_{i\in I} u_i \chi_{\Omega_i}$ piecewise constant reconstruction for managing time term

$$\left\{\begin{array}{l} u^{(0)} = \mathcal{I}_{\mathcal{D}} u_{\mathrm{ini}} \text{ and, for all } v = (v^{(n)})_{n=1,\ldots,N} \subset X_{\mathcal{D},0}, \\ \displaystyle\int_0^T \int_\Omega \Big[\delta_{\mathcal{D}} u(\mathbf{x},t) \Pi_{\mathcal{D}}^{(1)} v(\mathbf{x},t) + \Lambda(\mathbf{x}) \nabla_{\mathcal{D}}^{(1)} \zeta(u)(\mathbf{x},t) \cdot \nabla_{\mathcal{D}}^{(1)} v(\mathbf{x},t)\Big] \mathrm{d}\mathbf{x}\mathrm{d}t \\ \qquad = \displaystyle\int_0^T \int_\Omega f(\mathbf{x},t) \Pi_{\mathcal{D}}^{(1)} v(\mathbf{x},t) \mathrm{d}\mathbf{x}\mathrm{d}t. \end{array}\right.$$

$\eta : \mathbb{R} \to \mathbb{R}$ defined by $\forall s \in \mathbb{R}, \quad \eta(s) = \displaystyle\int_0^s \zeta(q)\mathrm{d}q \geq \frac{M_0}{4} s^2 - \frac{M_1^2}{M_0}$

for $T_0 \in (0, T]$ and $k = 1, \ldots, N$ s.t $T_0 \in (t^{(k-1)}, t^{(k)}]$

$$\int_\Omega \Pi_{\mathcal{D}}^{(1)} \eta(u)(\mathbf{x}, T_0)\mathrm{d}\mathbf{x} + \int_0^{T_0} \int_\Omega \Lambda(\mathbf{x}) \nabla_{\mathcal{D}}^{(1)} \zeta(u)(\mathbf{x},t) \cdot \nabla_{\mathcal{D}}^{(1)} \zeta(u)(\mathbf{x},t) \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$\leq \int_\Omega \Pi_{\mathcal{D}} \eta(\mathcal{I}_{\mathcal{D}} u_{\mathrm{ini}})(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^{t^{(k)}} \int_\Omega f(\mathbf{x},t) \Pi_{\mathcal{D}}^{(1)} \zeta(u)(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t.$$

Estimates (take $\zeta(u)$ as test function) use coercivity and space-time consistency

$$\sup_{t\in[0,T]} \left\| \Pi_{\mathcal{D}}^{(1)} \eta(u)(t) \right\|_{L^1(\Omega)} \leq C_4, \quad \left\| \nabla_{\mathcal{D}}^{(1)} \zeta(u) \right\|_{L^2(\Omega\times(0,T))^d} \leq C_4$$

$$\text{and } \sup_{t\in[0,T]} \left\| \Pi_{\mathcal{D}}^{(1)} u(t) \right\|_{L^2(\Omega)} \leq C_4 \text{ and } \int_0^T \|\delta_{\mathcal{D}} u(t)\|_{\star,\mathcal{D}}^2 \mathrm{d}t \leq C_5$$

leads to existence and uniqueness of the discrete solution

Discrete H^{-1} Aubin-Simon theorem (use compactness hypothesis) :

if $\forall m \in \mathbb{N}$, $\int_0^T \left\| \Pi^{(\theta)}_{\mathcal{D}_m} v_m(t) \right\|^2_{L^2(\Omega)} \mathrm{d}t \le C$and $\forall m \in \mathbb{N}$, $\int_0^T \|\delta_{\mathcal{D}_m} v_m(t)\|^q_{\star,\mathcal{D}_m} \mathrm{d}t \le C$.
and $\Pi^{(\theta)}_{\mathcal{D}_m} v_m$ cv weakly in $L^2(0,T;L^2(\Omega))$ as $m \to \infty$ to some $\overline{v} \in L^2(0,T;L^2(\Omega))$

Then, as $m \to \infty$, defining reciprocal discrete and continuous Laplace operators

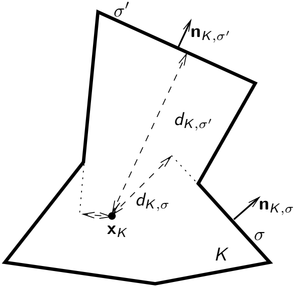
$\Pi^{(\theta)}_{\mathcal{D}_m}(\Delta^i_{\mathcal{D}_m} v_m) \to \Delta^i v$ and $\nabla^{(\theta)}_{\mathcal{D}_m}(\Delta^i_{\mathcal{D}_m} v_m) \to \nabla(\Delta^i \overline{v})$ in L^2
and, if $q > 1$ then $\delta_{\mathcal{D}_m}(\Delta^i_{\mathcal{D}_m} v_m) \to \partial_t(\Delta^i \overline{v})$ weakly in $L^q(0,T;L^2(\Omega))$

Consequence : $\lim_{m\to\infty} \int_0^T \int_\Omega \Pi^{(1)}_{\mathcal{D}_m} u_m \, \zeta\left(\Pi^{(1)}_{\mathcal{D}_m} u_m\right) \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Omega \overline{u}\, \overline{\zeta} \mathrm{d}\mathbf{x}\mathrm{d}t$

Minty trick implies $\overline{\zeta} = \overline{u}$

Consequences : convergence (use space-time consistency and limit-conformity)

$\Pi^{(1)}_{\mathcal{D}_m} u_m \to \overline{u}$	weakly in $L^2(\Omega \times (0,T]$
$\Pi^{(1)}_{\mathcal{D}_m} \zeta(u_m) \to \zeta(\overline{u})$	in $L^2(\Omega \times (0,T))$,
$\nabla^{(1)}_{\mathcal{D}_m} \zeta(u_m) \to \nabla\zeta(\overline{u})$	in $L^2(\Omega \times (0,T))^d$



A cell K of a polytopal mesh

$$X_{\mathfrak{T},0} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{F}})\\ v_K \in \mathbb{R}, v_\sigma \in \mathbb{R},\ v_\sigma = 0 \text{ for all } \sigma \in \mathcal{F}_{\rm ext}\}$$

$\forall v \in X_{\mathfrak{T},0},\ \forall K \in \mathcal{M},$ for a.e. $\mathbf{x} \in K$,

$$\Pi_{\mathfrak{T}} v(\mathbf{x}) = v_K$$

$$\overline{\nabla}_{\mathfrak{T}} v(\mathbf{x}) = \overline{\nabla}_K v := \frac{1}{|K|}\sum_{\sigma\in\mathcal{F}_K} |\sigma|(v_\sigma - v_K)\mathbf{n}_{K,\sigma} = \frac{1}{|K|}\sum_{\sigma\in\mathcal{F}_K} |\sigma| v_\sigma \mathbf{n}_{K,\sigma}.$$

$$\forall v \in X_{\mathfrak{T},0},\ |v|_{\mathfrak{T},p}^p = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K} |\sigma| d_{K,\sigma} \left|\frac{v_\sigma - v_K}{d_{K,\sigma}}\right|^p$$

polytopal toolbox $(X_{\mathfrak{T},0}, \Pi_{\mathfrak{T}}, \overline{\nabla}_{\mathfrak{T}}, |\ |_{\mathfrak{T},p})$

not a GD, since norm is not that of the discrete gradient control of $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ by $\mathfrak{T}$: $\mathbf{\Phi} \ : \ X_{\mathcal{D},0} \longrightarrow X_{\mathfrak{T},0}$,

$$\|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}} = \max_{v \in X_{\mathcal{D},0}\setminus\{0\}} \frac{|\mathbf{\Phi}(v)|_{\mathfrak{T},p}}{\|v\|_{\mathcal{D}}}, \ \omega^{\Pi}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) = \max_{v \in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v - \Pi_{\mathfrak{T}}\mathbf{\Phi}(v)\|_{L^p(\Omega)}}{\|v\|_{\mathcal{D}}},$$

$$\omega^{\nabla}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) = \max_{v \in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|v\|_{\mathcal{D}}} \left(\sum_{K\in\mathcal{M}} |K|^{1-p} \left| \int_K \left[\nabla_{\mathcal{D}} v(\mathbf{x}) - \overline{\nabla}_{\mathfrak{T}}\mathbf{\Phi}(v)(\mathbf{x})\right] \mathrm{d}\mathbf{x} \right|^p \right)^{\frac{1}{p}}.$$

then

C_6 depending only on Ω, p and regularity factors of toolbox

$C_{\mathcal{D}} \le \omega^{\Pi}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) + C_6 \|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}}$

for all $\varphi \in W^{1,p'}(\Omega)^d$

$$W_{\mathcal{D}}(\varphi) \le \|\varphi\|_{W^{1,p'}(\Omega)^d} \left[C_6 h_{\mathcal{M}}(1 + \|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}}) + \omega^{\Pi}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) + \omega^{\nabla}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) \right].$$

consequence :

sequences $(\mathcal{D}_m)_{m\in\mathbb{N}}$, $(\mathfrak{T}_m)_{m\in\mathbb{N}}$, $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$ under regularity conditions

$$\text{if} \begin{cases} \sup_{m\in\mathbb{N}} \|\mathbf{\Phi}_m\|_{\mathcal{D}_m,\mathfrak{T}_m} < +\infty, \\ \lim_{m\to\infty} \omega^{\Pi}(\mathcal{D}_m,\mathfrak{T}_m,\mathbf{\Phi}_m) = 0, \text{ and} \\ \lim_{m\to\infty} \omega^{\nabla}(\mathcal{D}_m,\mathfrak{T}_m,\mathbf{\Phi}_m) = 0. \end{cases}$$

Then $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive, limit-conforming and compact

Local linearly exact (LLE) GD

LLE gradient discretisation $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ defined by

- d.o.f. $S = (\mathbf{x}_i)_{i\in I} \subset \mathbb{R}^d$, $(\alpha_i)_{i\in I} \subset L^\infty(\Omega)$,
- $X_{\mathcal{D},0} = \{u \in \mathcal{F}(I,\mathbb{R}) \;:\; u(i) = 0,\ \forall i \in I_\partial\}$, where $I = I_\Omega \cup I_\partial$ and $\emptyset = I_\Omega \cap I_\partial$
- $\mathcal{M}$ finite family of open disjoint subsets of Ω such that $\bigcup_{K\in\mathcal{M}} \overline{K} = \overline{\Omega}$
- for all $K \in \mathcal{M}$, $I_K \subset I$ such that
  for all $i \in I$ and all $K \in \mathcal{M}$, if $i \notin I_K$ then $\alpha_i = 0$ on K
  for a.e. $\mathbf{x} \in \Omega$ and all $v \in X_{\mathcal{D},0}$, $\displaystyle\sum_{i\in I} \alpha_i(\mathbf{x}) = 1$ $\quad \Pi_{\mathcal{D}} v = \sum_{i\in I} v_i \alpha_i$
- for all $K \in \mathcal{M}$, $\mathcal{G}_K : \mathcal{F}(I_K,\mathbb{R}) \to L^\infty(K)^d$
  for any affine function $L : \mathbb{R}^d \to \mathbb{R}$, if $\xi = (L(\mathbf{x}_i))_{i\in I_K}$ then $\mathcal{G}_K \xi = \nabla L$ on K
  $\nabla_{\mathcal{D}} v = \mathcal{G}_K\left[(v_i)_{i\in I_K}\right]$ on K for all $v \in X_{\mathcal{D},0}$
- $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

$$h_{\mathcal{M}} = \max_{K\in\mathcal{M}} \mathrm{diam}(K)$$

LLE regularity of $\mathcal{D}$ defined by

$$\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}) = \max_{K\in\mathcal{M}} \left(\|\mathcal{G}_K\|_p + \max_{i\in I_K} \frac{\mathrm{dist}(\mathbf{x}_i, K)}{\mathrm{diam}(K)} \right) + \operatorname*{esssup}_{\mathbf{x}\in\Omega} \sum_{i\in I} |\alpha_i(\mathbf{x})|,$$

LLE GDs are consistent

Non conforming $\mathbb{P}_1$ GD

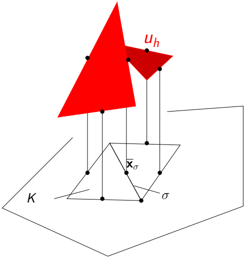
locally linearly exact $I = \mathcal{F} = \mathcal{F}_{\rm int} \cup \mathcal{F}_{\rm ext}$
$S = (\overline{\mathbf{x}}_\sigma)_{\sigma\in\mathcal{F}}$
$\mathcal{M}$ simplicial mesh of Ω

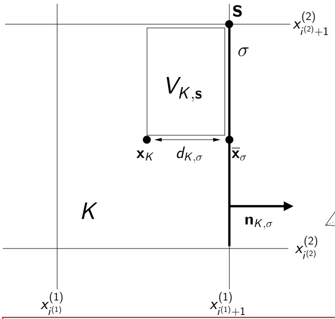
$$\mathbf{x}_K = \overline{\mathbf{x}}_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{F}_K}\overline{\mathbf{x}}_\sigma$$

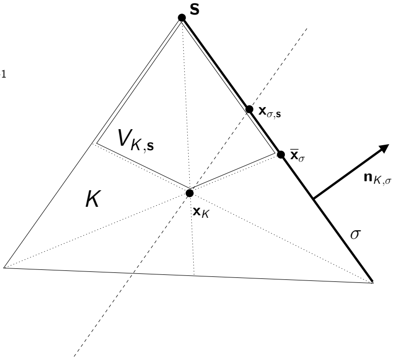
$$\boldsymbol{\Phi}(v)_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{F}_K} v_\sigma = \Pi_{\mathcal{D}} v(\mathbf{x}_K)$$
$$\text{and } \boldsymbol{\Phi}(v)_\sigma = v_\sigma = \Pi_{\mathcal{D}} v(\overline{\mathbf{x}}_\sigma).$$

Then exists C_7 s.t.

$$||\boldsymbol{\Phi}||_{\mathcal{D},\mathfrak{T}} \le \theta_{\mathfrak{T}} d^{1/p},$$
$$\omega^{\Pi}(\mathcal{D},\mathfrak{T},\boldsymbol{\Phi}) \le h_{\mathcal{M}},$$
$$\omega^{\nabla}(\mathcal{D},\mathfrak{T},\boldsymbol{\Phi}) = 0.$$



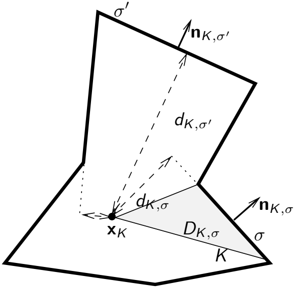




$I = \mathcal{M} \cup \{\tau_{\sigma,\mathbf{s}} \,:\, \sigma \in \mathcal{F},\ \mathbf{s} \in \mathcal{V}_\sigma\}$ and $S = ((\mathbf{x}_K)_{K\in\mathcal{M}}, (\mathbf{x}_{\sigma,\mathbf{s}})_{\sigma\in\mathcal{F},\,\mathbf{s}\in\mathcal{V}_\sigma})$
$I_K = \{K\} \cup \{\sigma, \mathbf{s} \,:\, \sigma \in \mathcal{F}_K,\ \mathbf{s} \in \mathcal{V}_\sigma\}$
$\alpha_i = 1$ for $i = K$ and $\alpha_i = 0$ for $i = \sigma, \mathbf{s}$, implies
$\forall v \in X_{\mathcal{D},0}\,,\ \forall K \in \mathcal{M}\,,\ \forall \mathbf{x} \in K\,,\ \Pi_{\mathcal{D}} v(\mathbf{x}) = v_K.$

$$\text{for a.e. } \mathbf{x} \in V_{K,\mathbf{s}}\,,\ \mathcal{G}_K v(\mathbf{x}) = \frac{1}{|V_{K,\mathbf{s}}|} \sum_{\sigma\in\mathcal{F}_{K,\mathbf{s}}} |\tau_{\sigma,\mathbf{s}}|(v_{\sigma,\mathbf{s}} - v_K)\mathbf{n}_{K,\sigma}.$$

polytopal toolbox $\mathfrak{T}'$ built with cells and subfaces
$\boldsymbol{\Phi} : X_{\mathcal{D},0} \longrightarrow X_{\mathfrak{T}',0}$ s.t. $\boldsymbol{\Phi}(u)_K = u_K$ and $\boldsymbol{\Phi}(u)_{\tau_{\sigma,\mathbf{s}}} = u_{\sigma,\mathbf{s}}$



A cell K of a polytopal mesh

polytopal mesh $\mathfrak{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P}, \mathcal{V})$
$I = \mathcal{M} \cup \mathcal{F}$, $S = ((\mathbf{x}_K)_{K\in\mathcal{M}}, (\overline{\mathbf{x}}_\sigma)_{\sigma\in\mathcal{F}})$
$(\alpha_K)_{K\in\mathcal{M}}$ and $(\alpha_\sigma)_{\sigma\in\mathcal{F}}$ such that
$\alpha_K = 1$ on K, $\alpha_K = 0$ outside K, and $\alpha_\sigma \equiv 0$
implies $\forall v \in X_{\mathcal{D}}$, $\forall K \in \mathcal{M}$, for a.e. $\mathbf{x} \in K$,
$\Pi_{\mathcal{D}} v(\mathbf{x}) = \Pi_{\mathfrak{T}} v(\mathbf{x}) = v_K$.

recall polytopal gradient $\overline{\nabla}_K v = \frac{1}{|K|} \sum_{\sigma\in\mathcal{F}_K} |\sigma| v_\sigma \mathbf{n}_{K,\sigma}$

$\forall K \in \mathcal{M}$, $\forall v \in \boldsymbol{\mathcal{F}}(I_K, \mathbb{R})$, $\forall \sigma \in \mathcal{F}_K$, for a.e. $\mathbf{x} \in D_{K,\sigma}$,

$$\mathcal{G}_K v(\mathbf{x}) = \overline{\nabla}_K v + \frac{\sqrt{d}}{d_{K,\sigma}}(v_\sigma - v_K - \overline{\nabla}_K v \cdot (\overline{\mathbf{x}}_\sigma - \mathbf{x}_K))\mathbf{n}_{K,\sigma}$$

$\mathbf{\Phi} = \mathrm{Id} : X_{\mathcal{D},0} \longrightarrow X_{\mathfrak{T},0}$

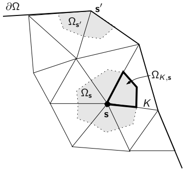
$\mathcal{A} = (\varphi_i)_{i\in I}$ linearly independent family of elements of $W_0^{1,p}(\Omega)$
$X_{\mathcal{D},0} = \mathcal{F}(I,\mathbb{R})$ and, for all $u = (u_i)_{i\in I} \in X_{\mathcal{D},0}$,

$$\Pi_{\mathcal{D}} u = \sum_{i\in I} u_i\varphi_i \text{ and } \nabla_{\mathcal{D}} u = \nabla(\Pi_{\mathcal{D}} u) = \sum_{i\in I} u_i \nabla\varphi_i.$$

coercivity continuous Poincaré's inequality
consistency for convenient choice of shape functions
limit-conformity $W_{\mathcal{D}_m}(\boldsymbol{\varphi}) = 0$ for all $\boldsymbol{\varphi} \in W^{\mathrm{div},p'}(\Omega)$
compactness Rellich's theorem

Mass-lumped $\mathbb{P}_1$ GD

$\forall v \in X_{\mathcal{D},0}$, $\forall \mathbf{s} \in \mathcal{V}$,
$\Pi_{\mathcal{D}}^{\mathrm{ML}} v = v_{\mathbf{s}}$ on $\Omega_{\mathbf{s}}$.

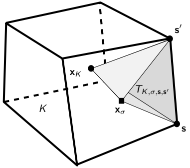


Partitions for the mass-lumping of the $\mathbb{P}_1$ GD

The Vertex Average Gradient method

Elimination of the face degrees of freedom

Mass lumping at centres and vertices



$$H_{\mathrm{div}}(\Omega) = \{\varphi \in L^2(\Omega)^d, \mathrm{div}\varphi \in L^2(\Omega)\},$$
$$\mathbf{V}_h = \{\mathbf{v} \in (L^2(\Omega))^d; \mathbf{v}|_K \in \mathbb{RT}_k(K),\ \forall K \in \mathcal{M}\},$$
$$\mathbf{V}_h^{\mathrm{div}} = \mathbf{V}_h \cap H_{\mathrm{div}}(\Omega),$$
$$W_h = \{p \in L^2(\Omega)\ ;\ p|_K \in \mathbb{P}_k(K),\ \forall K \in \mathcal{M}\},\ \text{spanned by } (\chi_i)_{i\in I}$$
$$M_h^0 = \{\mu : \bigcup_{\sigma\in\mathcal{F}} \overline{\sigma} \to \mathbb{R}, \mu|_\sigma \in \mathbb{P}_k(\sigma), \mu|_{\partial\Omega} = 0\},\ \text{spanned by } (\xi_j)_{j\in J}$$

primal formulation

$$X_{\mathcal{D},0} = \mathcal{F}(I,\mathbb{R}), \Pi_{\mathcal{D}} u = \sum_{i\in I} u_i\chi_i, \forall u \in X_{\mathcal{D},0},$$
$$\forall u \in X_{\mathcal{D},0},\ \Lambda\nabla_{\mathcal{D}} u \in \mathbf{V}_h^{\mathrm{div}},$$
$$\int_\Omega \mathbf{v}(\mathbf{x})\cdot\nabla_{\mathcal{D}} u(\mathbf{x})\mathrm{d}\mathbf{x} + \int_\Omega \Pi_{\mathcal{D}} u(\mathbf{x})\mathrm{div}\mathbf{v}(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h^{\mathrm{div}}.$$

dual formulation

$$X_{\widetilde{\mathcal{D}},0} = \mathcal{F}(I\cup J,\mathbb{R}),\ \Pi_{\widetilde{\mathcal{D}}} u = \sum_{i\in I} u_i\chi_i \text{ and } \Gamma_{\widetilde{\mathcal{D}}} u = \sum_{j\in J} u_j\xi_j,\ \forall u \in X_{\widetilde{\mathcal{D}},0},$$
$$\forall u \in X_{\widetilde{\mathcal{D}},0}, \Lambda\nabla_{\widetilde{\mathcal{D}}} u \in \mathbf{V}_h \text{ and } \int_K \mathbf{w}(\mathbf{x})\cdot\nabla_{\widetilde{\mathcal{D}}} u(\mathbf{x})\mathrm{d}\mathbf{x} + \int_K \Pi_{\widetilde{\mathcal{D}}} u(\mathbf{x})\mathrm{div}\mathbf{w}(\mathbf{x})\mathrm{d}\mathbf{x}$$
$$- \sum_{\sigma\in\mathcal{F}_K}\int_\sigma \Gamma_{\widetilde{\mathcal{D}}} u(\mathbf{x})\ \mathbf{w}|_K(\mathbf{x})\cdot\mathbf{n}_{K,\sigma}\mathrm{d}\gamma(\mathbf{x}) = 0,\ \forall \mathbf{w} \in \mathbf{V}_h.$$

Other examples of problems handled by GDM

$$\nu(\mathbf{x},t,u(\mathbf{x},t),\nabla u(\mathbf{x},t))\partial_t u(\mathbf{x},t) - \mathrm{div}(\mu(|\nabla u(\mathbf{x},t)|)\nabla u(\mathbf{x},t)) = f(\mathbf{x},t), \text{ for a.e. } (\mathbf{x},t)\in\Omega\times(0,T)$$

with the initial condition $u(\mathbf{x},0) = u_{\mathrm{ini}}(\mathbf{x})$, for a.e. $\mathbf{x}\in\Omega$,
and boundary conditions $u(\mathbf{x},t) = 0$, for a.e. $(\mathbf{x},t)\in\partial\Omega\times(0,T)$.

for image processing, approximation of level-set equation

$$\left\{\begin{array}{l} u^{(0)} = \mathcal{I}_{\mathcal{D}}u_{\mathrm{ini}} \text{ and, for } n=0,\ldots,N-1,\ u^{(n+1)} \text{ satisfies} \\ \displaystyle\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega \nu(\mathbf{x},t,\Pi_{\mathcal{D}}u^{(n+1)},\nabla_{\mathcal{D}}u^{(n+1)})\delta_{\mathcal{D}}^{(n+\frac{1}{2})}u(\mathbf{x})\Pi_{\mathcal{D}}v(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t \\ \displaystyle\quad +\delta t^{(n+\frac{1}{2})}\int_\Omega \mu(|\nabla_{\mathcal{D}}u^{{\color{red}(n+1)}{\color{blue}(n)}}(\mathbf{x})|)\nabla_{\mathcal{D}}u^{(n+1)}(\mathbf{x})\cdot\nabla_{\mathcal{D}}v(\mathbf{x})\mathrm{d}\mathbf{x} \\ \displaystyle\quad\quad = \int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f(\mathbf{x},t)\Pi_{\mathcal{D}}v(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t\,, \quad \forall v\in X_{\mathcal{D},0}. \end{array}\right.$$

$$\begin{aligned}&\partial_t \overline{u} - \mathrm{div}\ \mathbf{a}(\mathbf{x}, \overline{u}, \nabla \overline{u}) = f \text{ in } \Omega \times (0, T),\\&\overline{u}(\mathbf{x}, 0) = u_{\mathrm{ini}}(\mathbf{x}) \text{ in } \Omega,\\&\mathbf{a}(\mathbf{x}, \overline{u}, \nabla \overline{u}) \cdot \mathbf{n} = g \text{ on } \partial\Omega \times (0, T).\end{aligned}$$

$$\left\{\begin{aligned}&u^{(0)} = \mathcal{I}_{\mathcal{D}} u_{\mathrm{ini}} \in X_{\mathcal{D}} \text{ and, for all } n = 0, \ldots, N-1,\ u^{(n+1)} \text{ satisfies}\\&\int_\Omega \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) \mathrm{d}\mathbf{x}\\&+ \int_\Omega \mathbf{a}\left(\mathbf{x}, \Pi_{\mathcal{D}} u^{(n+\theta)}, \nabla_{\mathcal{D}} u^{(n+\theta)}(\mathbf{x})\right) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) \mathrm{d}\mathbf{x}\\&\qquad = \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(\mathbf{x}, t) \Pi_{\mathcal{D}} v(\mathbf{x}) \mathrm{d}\mathbf{x} \mathrm{d}t\\&\qquad\quad + \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_{\partial\Omega} g(\mathbf{x}, t) \mathbb{T}_{\mathcal{D}} v(\mathbf{x}) \mathrm{d}s(\mathbf{x}) \mathrm{d}t\,, \quad \forall v \in X_{\mathcal{D}}.\end{aligned}\right.$$

$$\Phi(\mathbf{x})\partial_t S(\mathbf{x},p) - \mathrm{div}(k_1(\mathbf{x},S(\mathbf{x},p))\Lambda(\mathbf{x})\nabla u) = f_1,$$
$$\Phi(\mathbf{x})\partial_t(1 - S(\mathbf{x},p)) - \mathrm{div}(k_2(\mathbf{x},S(\mathbf{x},p))\Lambda(\mathbf{x})\nabla v) = f_2,$$
$$p = u - v, \text{ for } \in \Omega \times (0,T),$$

Initialization : $s_{\mathcal{D}}^{(0)}(\mathbf{x}) = S(\mathbf{x}, \Pi_{\mathcal{D}} p^{(0)}(\mathbf{x}))$,

For $n = 0, \ldots, N-1$:

$$u^{(n+1)} - \bar{u}_{\mathcal{D}} \in X_{\mathcal{D},0}, \;\; v^{(n+1)} - \bar{v}_{\mathcal{D}} \in X_{\mathcal{D},0},$$
$$p^{(n+1)} = u^{(n+1)} - v^{(n+1)}, \;\; s_{\mathcal{D}}^{(n+1)}(\mathbf{x}) = S(\mathbf{x}, \Pi_{\mathcal{D}} p^{(n+1)}(\mathbf{x})),$$
$$\delta_{\mathcal{D}}^{(n+\frac{1}{2})} s_{\mathcal{D}}(\mathbf{x}) = \frac{s_{\mathcal{D}}^{(n+1)}(\mathbf{x}) - s_{\mathcal{D}}^{(n)}(\mathbf{x})}{\delta t^{(n+\frac{1}{2})}},$$
$$\int_\Omega \left(\Phi(\mathbf{x})\delta_{\mathcal{D}}^{(n+\frac{1}{2})} s_{\mathcal{D}}(\mathbf{x})\Pi_{\mathcal{D}} w(\mathbf{x}) + k_1(\mathbf{x}, s_{\mathcal{D}}^{(n+1)}(\mathbf{x}))\Lambda(\mathbf{x})\nabla_{\mathcal{D}} u^{(n+1)}(\mathbf{x}) \cdot \nabla_{\mathcal{D}} w(\mathbf{x}) \right) \mathrm{d}\mathbf{x} =$$
$$\frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f_1(\mathbf{x},t)\Pi_{\mathcal{D}} w(\mathbf{x}) \mathrm{d}\mathbf{x}\mathrm{d}t, \;\; \forall w \in X_{\mathcal{D},0},$$
$$\int_\Omega \left(-\Phi(\mathbf{x})\delta_{\mathcal{D}}^{(n+\frac{1}{2})} s_{\mathcal{D}}(\mathbf{x})\Pi_{\mathcal{D}} w(\mathbf{x}) + k_2(\mathbf{x}, s_{\mathcal{D}}^{(n+1)}(\mathbf{x}))\Lambda(\mathbf{x})\nabla_{\mathcal{D}} v^{(n+1)}(\mathbf{x}) \cdot \nabla_{\mathcal{D}} w(\mathbf{x}) \right) \mathrm{d}\mathbf{x} =$$
$$\frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f_2(\mathbf{x},t)\Pi_{\mathcal{D}} w(\mathbf{x}) \mathrm{d}\mathbf{x}\mathrm{d}t, \;\; \forall w \in X_{\mathcal{D},0}.$$

Stokes problem

$$\left\{\begin{array}{l}\overline{u}\in H_0^1(\Omega)^d,\ \overline{p}\in L_0^2(\Omega),\\ \eta\displaystyle\int_\Omega \overline{u}\cdot\overline{v}\mathrm{d}\mathbf{x}+\int_\Omega \nabla\overline{u}:\nabla\overline{v}\mathrm{d}\mathbf{x}-\int_\Omega \overline{p}\mathrm{div}\overline{v}\mathrm{d}\mathbf{x}=\int_\Omega (f\cdot\overline{v}+G:\nabla\overline{v})\,\mathrm{d}\mathbf{x}\\ \forall\overline{v}\in H_0^1(\Omega)^d,\\ \\ \displaystyle\int_\Omega q\mathrm{div}\overline{u}\mathrm{d}\mathbf{x}=0\,,\quad \forall q\in L_0^2(\Omega),\end{array}\right.$$

$$\left\{\begin{array}{l}u\in X_{\mathcal{D},0}\,,\ p\in Y_{\mathcal{D},0},\\ \eta\displaystyle\int_\Omega \Pi_{\mathcal{D}}u\cdot\Pi_{\mathcal{D}}v\mathrm{d}\mathbf{x}+\int_\Omega \nabla_{\mathcal{D}}u:\nabla_{\mathcal{D}}v\mathrm{d}\mathbf{x}-\int_\Omega \chi_{\mathcal{D}}p\ \mathrm{div}_{\mathcal{D}}v\mathrm{d}\mathbf{x}\\ \qquad\qquad=\displaystyle\int_\Omega (f\cdot\Pi_{\mathcal{D}}v+G:\nabla_{\mathcal{D}}v)\,\mathrm{d}\mathbf{x}\,,\quad \forall v\in X_{\mathcal{D},0},\\ \displaystyle\int_\Omega \chi_{\mathcal{D}}q\mathrm{div}_{\mathcal{D}}u\mathrm{d}\mathbf{x}=0\,,\quad \forall q\in Y_{\mathcal{D},0}.\end{array}\right.$$