Structure-preserving spectral elements

Institut d'etudes scientifiques de Cargèse: Advanced Numerical Methods for Partial Differential Equations

Organized by
Daniele Di Pietro, Alexandre Ern, Lucca Formaggia & Donatella Marini

Marc Gerritsma
M.I.Gerritsma@TUDelft.nl

Department of Aerospace Engineering
Delft University of Technology

5 September 2016

Outline of these lectures

Spectral element methods are based on orthogonal polynomials for the basis functions.

Instead of h-refinement (a finer mesh), spectral methods use p-enrichment, i.e. leave the elements unaltered, but increase the polynomial degree.

For sufficiently smooth problem, spectral methods display exponential convergence.

This short course on mimetic spectral elements consists of 2 lectures:

5 September, Lecture 1: Incidence matrices and dual grids

6 September, Lecture 2: Spectral basis functions and scalar Laplace equations

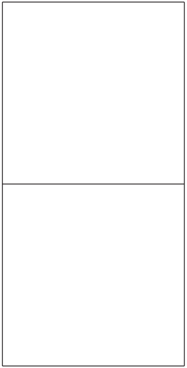
For more detail: *Mimetic framework on curvilinear quadrilaterals of arbitrary order*,

+https://arxiv.org/abs/1111.4304+

Just a little game I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

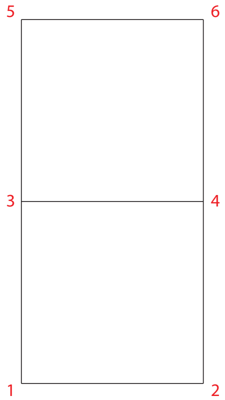
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



Just a little game I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

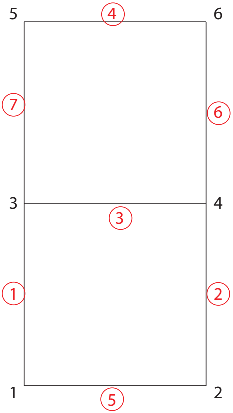
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



Just a little game I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

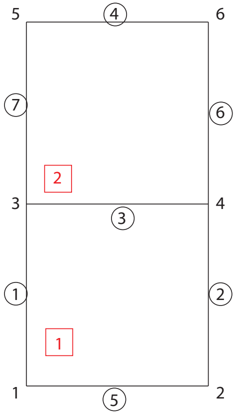
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



Just a little game I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

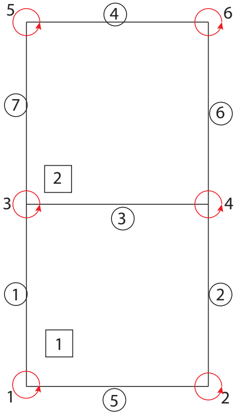
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



Just a little game I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

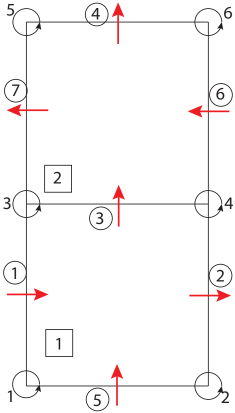
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



Just a little game I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

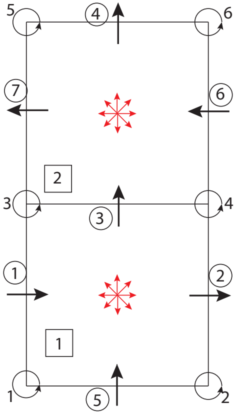
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



Just a little game I

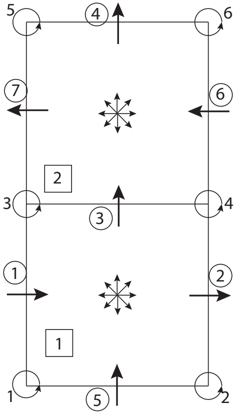
- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh

- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes



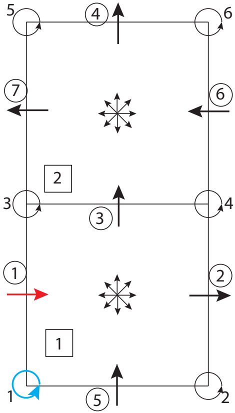
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



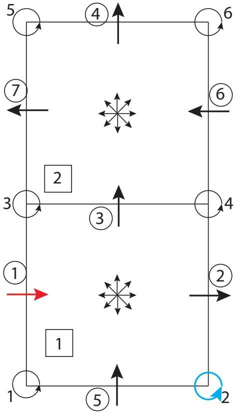
Just a little game I

$$
\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}
$$



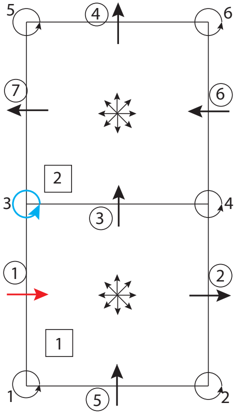
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



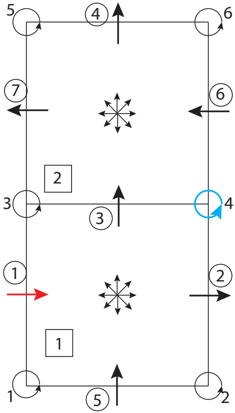
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



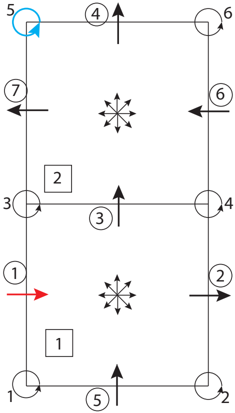
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



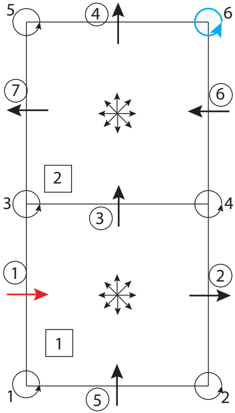
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



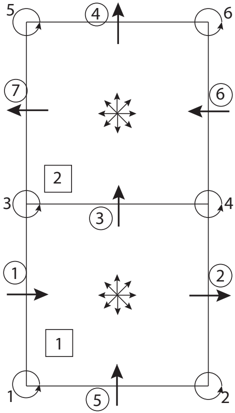
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



Just a little game I

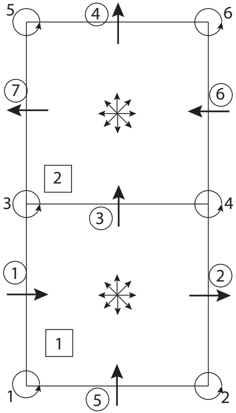
$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

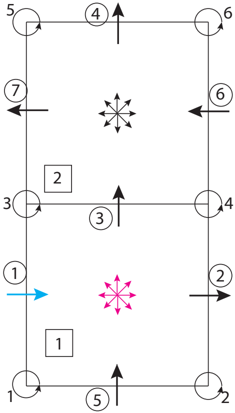
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

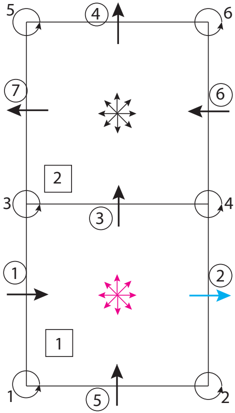
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

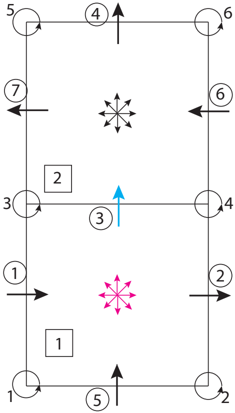
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

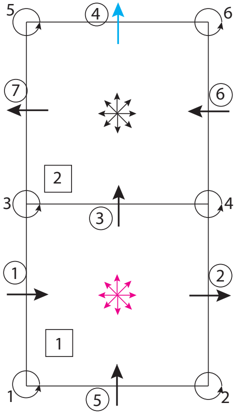
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

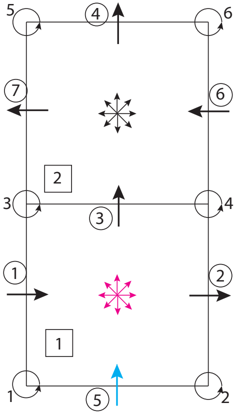
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

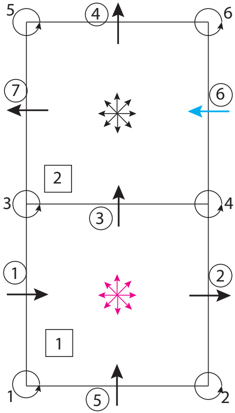
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

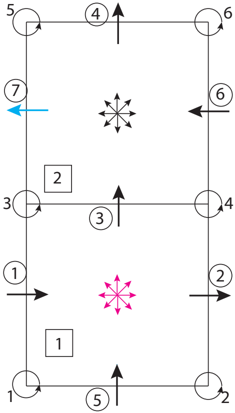
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

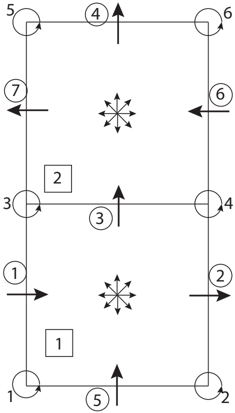
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$



Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

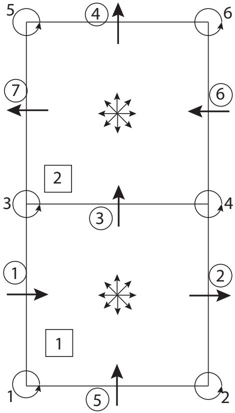


Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The matrices $\mathbb{E}^{1,0}$ and $\mathbb{E}^{2,1}$ are called incidence matrices. The incidence matrices only contain entries -1, 0 and 1.



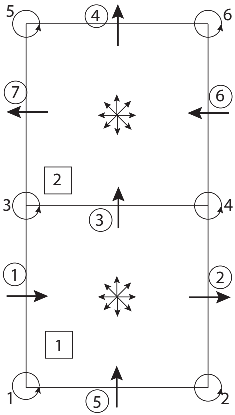
Just a little game I

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The matrices $\mathbb{E}^{1,0}$ and $\mathbb{E}^{2,1}$ are called incidence matrices. The incidence matrices only contain entries -1, 0 and 1.

The incidence matrices are independent of the shape and coarseness of the grid



Just a little game II

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \qquad \mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

.

Just a little game II

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \qquad \mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Exercise: Calculate the product $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0}$

.

AUDIENCE PARTICIPATION

Just a little game II

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \qquad \mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Exercise: Calculate the product $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0}$

$$\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0} = 0$$

This implies that $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \subseteq \mathcal{N}\left(\mathbb{E}^{2,1}\right)$
In this particular case $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \equiv \mathcal{N}\left(\mathbb{E}^{2,1}\right)$.

Just a little game II

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \qquad \mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Exercise: Calculate the product $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0}$

.

$$\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0} = 0$$

This implies that $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \subseteq \mathcal{N}\left(\mathbb{E}^{2,1}\right)$
In this particular case $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \equiv \mathcal{N}\left(\mathbb{E}^{2,1}\right)$.

We also have that ${\mathbb{E}^{1,0}}^T \cdot {\mathbb{E}^{2,1}}^T = 0 \Longrightarrow \mathcal{R}\left({\mathbb{E}^{2,1}}^T\right) \equiv \mathcal{N}\left({\mathbb{E}^{1,0}}^T\right)$

TUDelft

Just a little game III

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \qquad \mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Just a little game III

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \qquad \mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The property $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0} = 0$ also implies that $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \perp \mathcal{R}\left({\mathbb{E}^{2,1}}^T\right)$

Just a little game III

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \qquad \mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The property $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0} = 0$ also implies that $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \perp \mathcal{R}\left({\mathbb{E}^{2,1}}^T\right)$

Because for arbitrary $\mathbf{a} \in \mathbb{R}^6$ and $\mathbf{b} \in \mathbb{R}^2$ we have

$$\left(\mathbb{E}^{1,0}\mathbf{a}, {\mathbb{E}^{2,1}}^T\mathbf{b}\right) = \left(\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0}\mathbf{a}, \mathbf{b}\right) = 0$$

Just a little game III

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \qquad \mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The property $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0} = 0$ also implies that $\mathcal{R}\left(\mathbb{E}^{1,0}\right) \perp \mathcal{R}\left({\mathbb{E}^{2,1}}^T\right)$

Because for arbitrary $\mathbf{a} \in \mathbb{R}^6$ and $\mathbf{b} \in \mathbb{R}^2$ we have

$$\left(\mathbb{E}^{1,0}\mathbf{a}, {\mathbb{E}^{2,1}}^T\mathbf{b}\right) = \left(\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0}\mathbf{a}, \mathbf{b}\right) = 0$$

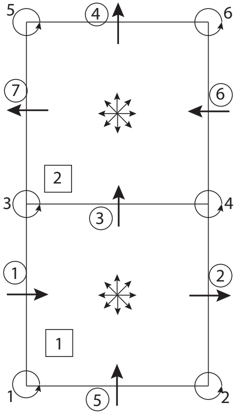
Let $H = \mathcal{N}\left(\mathbb{E}^{2,1}\right) / \mathcal{R}\left(\mathbb{E}^{1,0}\right)$ then any element $\mathbf{x} \in \mathbb{R}^7$ can be uniquely written as

$$\mathbf{x} = \mathbb{E}^{1,0}\mathbf{a} + {\mathbb{E}^{2,1}}^T\mathbf{b} + \mathbf{c}\,, \quad \mathbf{a} \in \mathbb{R}^6\,, \quad \mathbf{b} \in \mathbb{R}^2 \text{ and } \mathbf{c} \in H$$

"Hodge decomposition"

Just a little game IV

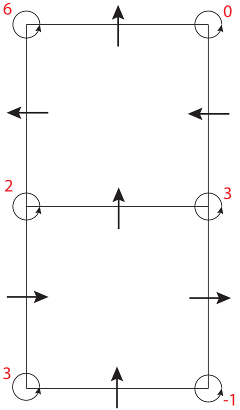
What do the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?



Just a little game IV

What do the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?

Suppose I assign to the vertices the value $\psi = (3, -1, 2, 3, 6, 0)^T$. Physically this could represent the stream function in the points.

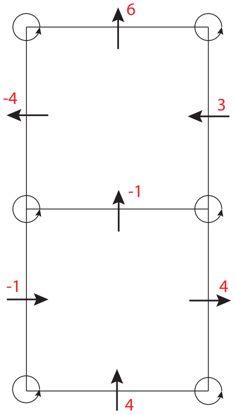


Just a little game IV

What do the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?

Suppose I assign to the vertices the value $\psi = (3, -1, 2, 3, 6, 0)^T$. Physically this could represent the stream function in the points.

Then $\mathbb{E}^{1,0}\psi = \dot{\mathbf{m}} = (-1, 4, -1, 6, 4, 3, -4)^T$ denotes the mass flux over the edges in the direction of the default positive orientation.



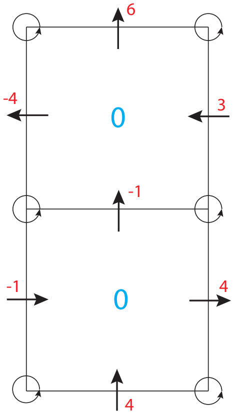
Just a little game IV

What do the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?

Suppose I assign to the vertices the value $\psi = (3, -1, 2, 3, 6, 0)^T$. Physically this could represent the stream function in the points.

Then $\mathbb{E}^{1,0}\psi = \dot{\mathbf{m}} = (-1, 4, -1, 6, 4, 3, -4)^T$ denotes the mass flux over the edges in the direction of the default positive orientation.

Then $\mathbb{E}^{2,1}\dot{\mathbf{m}} = \mathbb{E}^{2,1}\mathbb{E}^{1,0}\psi = (0, 0)^T$ denotes conservation of mass in each cell.



Just a little game IV

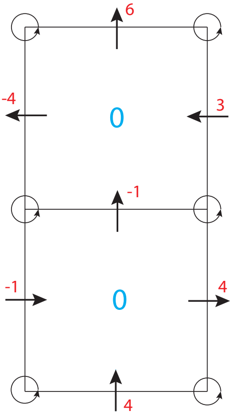
What do the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?

Suppose I assign to the vertices the value $\psi = (3, -1, 2, 3, 6, 0)^T$. Physically this could represent the stream function in the points.

Then $\mathbb{E}^{1,0}\psi = \dot{\mathbf{m}} = (-1, 4, -1, 6, 4, 3, -4)^T$ denotes the mass flux over the edges in the direction of the default positive orientation.

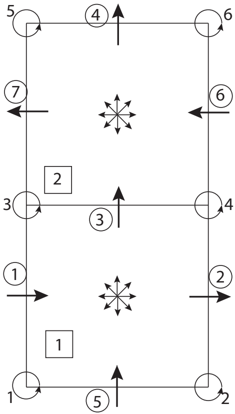
Then $\mathbb{E}^{2,1}\dot{\mathbf{m}} = \mathbb{E}^{2,1}\mathbb{E}^{1,0}\psi = (0, 0)^T$ denotes conservation of mass in each cell.

So $\mathbf{a}$ could represent the stream function and $\mathbb{E}^{1,0}\mathbf{a}$ its associated divergence-free velocity field.



Just a little game V

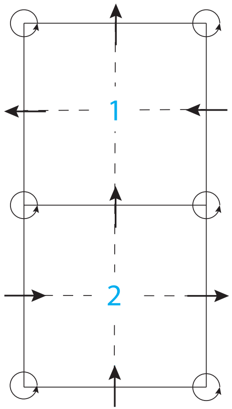
What does the vectors $\mathbf{b} \in \mathbb{R}^2$ then represent?



Just a little game V

What does the vectors $\mathbf{b} \in \mathbb{R}^2$ then represent?

Suppose I assign to the cells the value $\phi = (2,1)^T$. Physically this could represent the velocity potential in the cells.

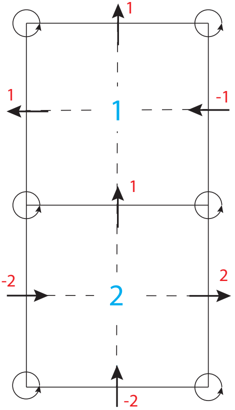


Just a little game V

What does the vectors $\mathbf{b} \in \mathbb{R}^2$ then represent?

Suppose I assign to the cells the value $\phi = (2,1)^T$. Physically this could represent the velocity potential in the cells.

Then $\mathbb{E}^{2,1\,T}\phi = \dot{\mathbf{u}} = (-2,2,1,1,-2,-1,1)^T$ denotes the circulation along the edges.



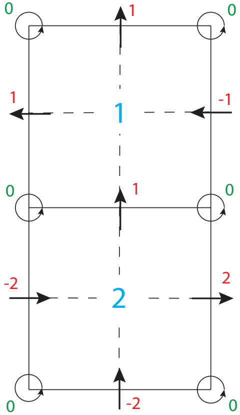
Just a little game V

What does the vectors $\mathbf{b} \in \mathbb{R}^2$ then represent?

Suppose I assign to the cells the value $\phi = (2, 1)^T$. Physically this could represent the velocity potential in the cells.

Then $\mathbb{E}^{2,1\,T}\phi = \dot{\mathbf{u}} = (-2, 2, 1, 1, -2, -1, 1)^T$ denotes the circulation along the edges.

Then $\mathbb{E}^{1,0\,T}\dot{\mathbf{u}} = \mathbb{E}^{1,0\,T}\mathbb{E}^{2,1\,T}\phi = (0, 0, 0, 0, 0, 0)^T$ denotes vorticity in the vertices.



Just a little game V

What does the vectors **b** $\in \mathbb{R}^2$ then represent?

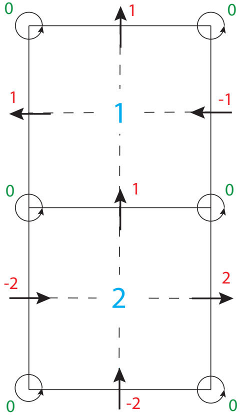
Suppose I assign to the cells the value $\phi = (2,1)^T$. Physically this could represent the velocity potential in the cells.

Then $\mathbb{E}^{2,1\,T}\phi = \dot{\mathbf{u}} = (-2,2,1,1,-2,-1,1)^T$ denotes the circulation along the edges.

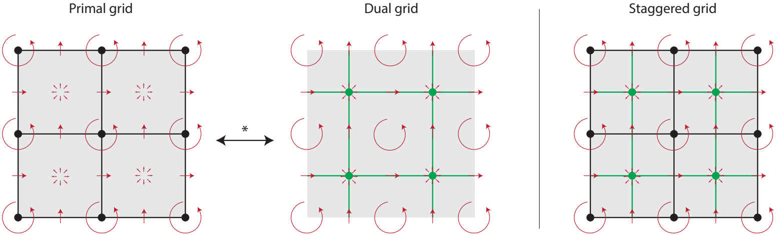
Then $\mathbb{E}^{1,0\,T}\dot{\mathbf{u}} = \mathbb{E}^{1,0\,T}\mathbb{E}^{2,1\,T}\phi = (0,0,0,0,0,0)^T$ denotes vorticity in the vertices.

So **b** could represent the velocity potential and $\mathbb{E}^{2,1\,T}\mathbf{b}$ its associated irrotational flow.

I tacitly drawn dashed lines in the figure. This a dual mesh. The combination of primal and dual grids leads to staggered schemes.

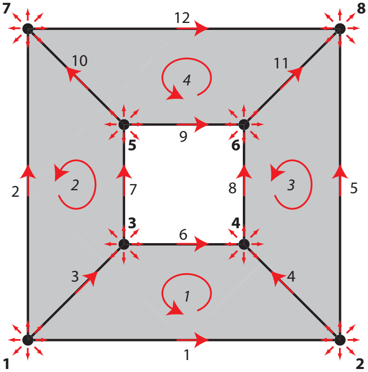


Just a little game VI



$$H(curl;\Omega) \qquad H_0(div;\Omega)$$
$$\downarrow \qquad \uparrow$$
$$H(div;\Omega) \qquad H_0(curl;\Omega)$$
$$\downarrow \qquad \uparrow$$
$$L^2(\Omega) \qquad H_0^1(\Omega)$$

Just a little game VII



Note that $\mathcal{R}(\mathbb{E}^{1,0}) \subseteq \mathcal{N}(\mathbb{E}^{2,1})$ is always true, but $\mathcal{R}(\mathbb{E}^{1,0}) = \mathcal{N}(\mathbb{E}^{2,1})$ only on contractible domains, (Poincaré Lemma). In the above example there is a curl-free flow field which is not the gradient of a potential. This is a harmonic solution and we need an additional boundary condition to fix this harmonic solution (Kutta condition).

Just a little game VIII

We have in 3D

$$\mathbb{R} \longrightarrow H_P \xrightarrow{\mathbb{E}^{1,0}} H_E \xrightarrow{\mathbb{E}^{2,1}} H_S \xrightarrow{\mathbb{E}^{3,2}} H_V \longrightarrow 0$$

Just a little game VIII

We have in 3D

$$\mathbb{R} \longrightarrow H_P \xrightarrow{\mathbb{E}^{1,0}} H_E \xrightarrow{\mathbb{E}^{2,1}} H_S \xrightarrow{\mathbb{E}^{3,2}} H_V \longrightarrow 0$$

$$0 \longleftarrow H_{\tilde{V}} \xleftarrow{{\mathbb{E}^{1,0}}^T} H_{\tilde{S}} \xleftarrow{{\mathbb{E}^{2,1}}^T} H_{\tilde{E}} \xleftarrow{{\mathbb{E}^{3,2}}^T} H_{\tilde{P}} \longleftarrow \mathbb{R}$$

and on the dual grid

Just a little game VIII

We have in 3D

$$\mathbb{R} \longrightarrow H_P \stackrel{\mathbb{E}^{1,0}}{\longrightarrow} H_E \stackrel{\mathbb{E}^{2,1}}{\longrightarrow} H_S \stackrel{\mathbb{E}^{3,2}}{\longrightarrow} H_V \longrightarrow 0$$

$$0 \longleftarrow H_{\tilde{V}} \stackrel{{\mathbb{E}^{1,0}}^T}{\longleftarrow} H_{\tilde{S}} \stackrel{{\mathbb{E}^{2,1}}^T}{\longleftarrow} H_{\tilde{E}} \stackrel{{\mathbb{E}^{3,2}}^T}{\longleftarrow} H_{\tilde{P}} \longleftarrow \mathbb{R}$$

and on the dual grid How do we discretize the Poisson/steady diffusion equation?

Just a little game IX

The physical interpretation can be depicted as

$$\begin{array}{ccccc}
\psi \in P & \xrightarrow[\nabla^{\perp}]{\mathbb{E}^{1,0}} & \dot{\mathbf{m}} \in E & \xrightarrow[\nabla\cdot]{\mathbb{E}^{2,1}} & \phi_S,\ d\dot{\mathbf{m}}/dt \in S \\
T_{\tilde{S}V} \Big\| & & T_{\tilde{E}E} \Big\| & & T_{\tilde{V}S} \Big\| \\
\xi \in \tilde{S} & \xleftarrow[\nabla\times]{\mathbb{E}^{1,0\,T}} & \mathbf{u} \in \tilde{E} & \xleftarrow[\nabla]{\mathbb{E}^{2,1\,T}} & \phi_{\tilde{V}} \in \tilde{V}
\end{array}$$

Just a little game IX

The physical interpretation can be depicted as

$$
\begin{array}{ccccc}
\psi \in P & \xrightarrow[\nabla^{\perp}]{\mathbb{E}^{1,0}} & \dot{\mathbf{m}} \in E & \xrightarrow[\nabla\cdot]{\mathbb{E}^{2,1}} & \phi_S,\ d\dot{\mathbf{m}}/dt \in S \\
T_{\tilde{S}V}\Big\| & & T_{\tilde{E}E}\Big\| & & T_{\tilde{V}S}\Big\| \\
\xi \in \tilde{S} & \xleftarrow[\nabla\times]{{\mathbb{E}^{1,0}}^{T}} & \mathbf{u} \in \tilde{E} & \xleftarrow[\nabla]{{\mathbb{E}^{2,1}}^{T}} & \phi_{\tilde{V}} \in \tilde{V}
\end{array}
$$

This sequence is called the double DeRham sequence

Just a little game IX

The physical interpretation can be depicted as

$$
\begin{array}{ccccc}
\psi \in P & \xrightarrow[\nabla^{\perp}]{\mathbb{E}^{1,0}} & \dot{\mathbf{m}} \in E & \xrightarrow[\nabla\cdot]{\mathbb{E}^{2,1}} & \phi_S,\ d\dot{\mathbf{m}}/dt \in S \\
T_{\tilde{S}V} \Big\| & & T_{\tilde{E}E} \Big\| & & T_{\tilde{V}S} \Big\| \\
\xi \in \tilde{S} & \xleftarrow[\nabla\times]{{\mathbb{E}^{1,0}}^T} & \mathbf{u} \in \tilde{E} & \xleftarrow[\nabla]{{\mathbb{E}^{2,1}}^T} & \phi_{\tilde{V}} \in \tilde{V}
\end{array}
$$

This sequence is called the double DeRham sequence

The particular numerical scheme that we use determines the transformations $T_{\tilde{S}V}$, $T_{\tilde{E}E}$, $T_{\tilde{V}S}$. This is where the basis functions enter the scene.

Just a little game IX

The physical interpretation can be depicted as

$$\begin{array}{ccccc} \psi \in P & \xrightarrow[\nabla^{\perp}]{\mathbb{E}^{1,0}} & \dot{\mathbf{m}} \in E & \xrightarrow[\nabla\cdot]{\mathbb{E}^{2,1}} & \phi_S,\ d\dot{\mathbf{m}}/dt \in S \\ T_{\tilde{S}V}\Big\| & & T_{\tilde{E}E}\Big\| & & T_{\tilde{V}S}\Big\| \\ \xi \in \tilde{S} & \xleftarrow[\nabla\times]{{\mathbb{E}^{1,0}}^T} & \mathbf{u} \in \tilde{E} & \xleftarrow[\nabla]{{\mathbb{E}^{2,1}}^T} & \phi_{\tilde{V}} \in \tilde{V} \end{array}$$

This sequence is called the double DeRham sequence

The particular numerical scheme that we use determines the transformations $T_{\tilde{S}V}$, $T_{\tilde{E}E}$, $T_{\tilde{V}S}$. This is where the basis functions enter the scene.

$$\text{Staggered finite volume: } \dot{\mathbf{m}} = \mathbb{E}^{1,0}\psi + T_{\tilde{E}E}^{-1}{\mathbb{E}^{2,1}}^T T_{\tilde{V}S}\phi_S$$

Just a little game IX

The physical interpretation can be depicted as

$$
\begin{array}{ccccc}
\psi \in P & \xrightarrow[\nabla^{\perp}]{\mathbb{E}^{1,0}} & \dot{\mathbf{m}} \in E & \xrightarrow[\nabla\cdot]{\mathbb{E}^{2,1}} & \phi_S,\ d\dot{\mathbf{m}}/dt \in S \\
T_{\tilde{S}V}\Big\| & & T_{\tilde{E}E}\Big\| & & T_{\tilde{V}S}\Big\| \\
\xi \in \tilde{S} & \xleftarrow[\nabla\times]{{\mathbb{E}^{1,0}}^T} & \mathbf{u} \in \tilde{E} & \xleftarrow[\nabla]{{\mathbb{E}^{2,1}}^T} & \phi_{\tilde{V}} \in \tilde{V}
\end{array}
$$

This sequence is called the double DeRham sequence

The particular numerical scheme that we use determines the transformations $T_{\tilde{S}V}$, $T_{\tilde{E}E}$, $T_{\tilde{V}S}$. This is where the basis functions enter the scene.

$$\text{Finite element methods: } T_{\tilde{E}E}\dot{\mathbf{m}} = T_{\tilde{E}E}\mathbb{E}^{1,0}\boldsymbol{\psi} + {\mathbb{E}^{2,1}}^T T_{\tilde{V}S}\phi_S$$

One dimensional nodal basis functions

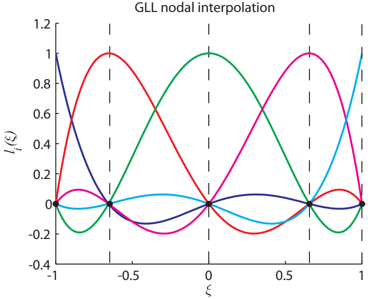
Consider the interval $[-1, 1] \subset \mathbb{R}$ and the partitioning $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$. Let $h_i(\xi)$ be the Lagrange polynomials through these nodes

$$h_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

One dimensional nodal basis functions

Consider the interval $[-1,1] \subset \mathbb{R}$ and the partitioning $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$. Let $h_i(\xi)$ be the Lagrange polynomials through these nodes

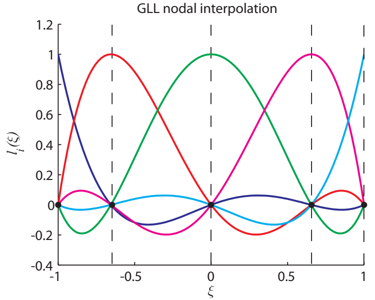
$$h_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



One dimensional nodal basis functions

Consider the interval $[-1, 1] \subset \mathbb{R}$ and the partitioning $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$. Let $h_i(\xi)$ be the Lagrange polynomials through these nodes

$$h_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

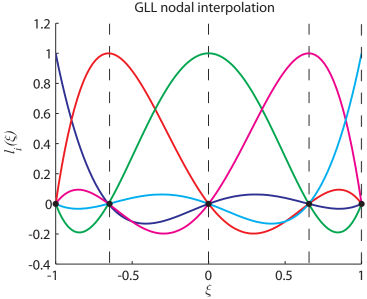


$$f(\xi) = \sum_{i=0}^{N} a_i h_i(\xi)$$

One dimensional nodal basis functions

Consider the interval $[-1,1] \subset \mathbb{R}$ and the partitioning $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$. Let $h_i(\xi)$ be the Lagrange polynomials through these nodes

$$h_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

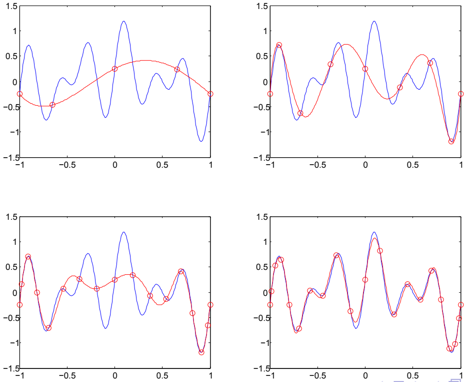


$$f(\xi) = \sum_{i=0}^{N} a_i h_i(\xi) \implies f(\xi_j) = a_j$$

One dimensional nodal basis functions

Consider the interval $[-1,1] \subset \mathbb{R}$ and the partitioning $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$. Let $h_i(\xi)$ be the Lagrange polynomials through these nodes

$$h_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



One dimensional edge basis functions

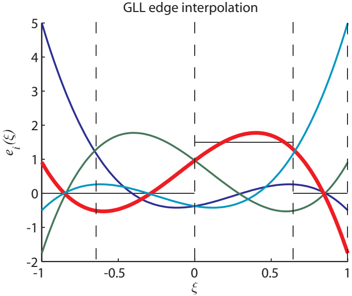
Next consider the edge polynomials, $e_i(\xi)$, defined by

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{dh_k(\xi)}{d\xi}\,\mathrm{d}\xi = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi) \qquad \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

One dimensional edge basis functions

Next consider the edge polynomials, $e_i(\xi)$, defined by

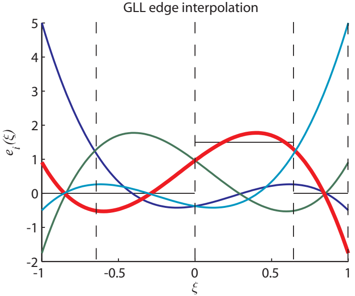
$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{dh_k(\xi)}{d\xi}\, \mathrm{d}\xi = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi) \qquad \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



One dimensional edge basis functions

Next consider the edge polynomials, $e_i(\xi)$, defined by

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{dh_k(\xi)}{d\xi}\,\mathrm{d}\xi = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi) \qquad \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



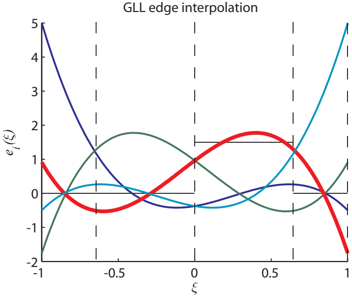
$$f(\xi) = \sum_{i=1}^{N} b_i e_i(\xi)$$

TUDelft

One dimensional edge basis functions

Next consider the edge polynomials, $e_i(\xi)$, defined by

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{dh_k(\xi)}{d\xi}\,\mathrm{d}\xi = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi) \qquad \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



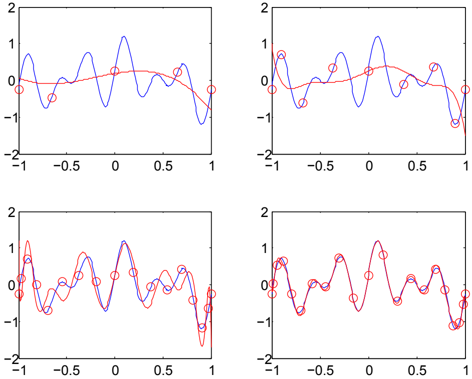
$$f(\xi) = \sum_{i=1}^{N} b_i e_i(\xi) \implies \int_{\xi_{j-1}}^{\xi_j} f(\xi)\,\mathrm{d}\xi = b_j$$

TU Delft

One dimensional edge basis functions

Next consider the edge polynomials, $e_i(\xi)$, defined by

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{dh_k(\xi)}{d\xi}\,\mathrm{d}\xi = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi) \qquad \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



Derivative of nodal function

Let $f(\xi)$ be expanded as

$$f(\xi) = \sum_{i=0}^{N} f_i h_i(\xi)$$

Derivative of nodal function

Let $f(\xi)$ be expanded as

$$f(\xi) = \sum_{i=0}^{N} f_i h_i(\xi)$$

Then the derivative is given by

$$f'(\xi) = \sum_{i=0}^{N} f_i h_i'(\xi)$$

Derivative of nodal function

Let $f(\xi)$ be expanded as

$$f(\xi) = \sum_{i=0}^{N} f_i h_i(\xi)$$

Then the derivative is given by

$$f'(\xi) = \sum_{i=0}^{N} f_i h_i'(\xi) = \sum_{i=1}^{N} (f_i - f_{i-1})\, e_i(\xi)$$

Derivative of nodal function

Let $f(\xi)$ be expanded as

$$f(\xi) = \sum_{i=0}^{N} f_i h_i(\xi) = [h_0 \; h_1 \; \dots \; h_N] \begin{bmatrix} f_0 \\ \vdots \\ f_N \end{bmatrix}$$

Then the derivative is given by

$$f'(\xi) = \sum_{i=0}^{N} f_i h_i'(\xi) = \sum_{i=1}^{N} (f_i - f_{i-1})\, e_i(\xi)$$

TUDelft

Derivative of nodal function

Let $f(\xi)$ be expanded as

$$f(\xi) = \sum_{i=0}^{N} f_i h_i(\xi) = [h_0\ h_1\ \ldots\ h_N] \begin{bmatrix} f_0 \\ \vdots \\ f_N \end{bmatrix}$$

Then the derivative is given by

$$\begin{aligned} f'(\xi) &= \sum_{i=0}^{N} f_i h_i'(\xi) = \sum_{i=1}^{N} (f_i - f_{i-1})\, e_i(\xi) \\ &= [e_1\ \ldots\ e_N] \begin{pmatrix} -1 & 1 & 0 & \ldots & & 0 \\ & \ddots & \ddots & & & 0 \\ & & -1 & 1 & & 0 \\ & & & \ddots & \ddots & \\ 0 & & \ldots & 0 & -1 & 1 \end{pmatrix} \begin{bmatrix} f_0 \\ f \\ \vdots \\ N \end{bmatrix} = [e_1\ \ldots\ e_N]\mathbb{E}^{1,0} \begin{bmatrix} f_0 \\ \vdots \\ f_N \end{bmatrix} \end{aligned}$$

TUDelft

Two dimensional expansion – nodal

Consider $[-1,1]^2 \subset \mathbb{R}^2$. We will use tensor products of nodal expansions to construct a finite dimensional subspace.
Consider of the span of $\{h_i(\xi)h_j(\eta)\}$, $i,j = 0,\ldots,N$. Any element $\psi^h(\xi,\eta)$ can be written as

$$\psi^h(\xi,\eta) = \sum_{i=0}^{N}\sum_{j=0}^{N} \psi_{i,j} h_i(\xi) h_j(\eta) \,.$$

with $\psi_{i,j} = \psi(\xi_i,\eta_j)$.
Let φ^h in the same space, then the L^2-inner product is given by

$$\begin{aligned}\left(\varphi^h,\psi^h\right) &= \int_{-1}^{1}\int_{-1}^{1} \varphi^h\psi^h \,\mathrm{d}\xi\mathrm{d}\eta \\ &= \left[\varphi_{0,0} \;\cdots\; \varphi_{N,N}\right] \mathbb{M}^{(0)} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix} ,\end{aligned}$$

where $\mathbb{M}^{(0)}$ is the nodal mass matrix given by

$$\mathbb{M}^{(0)} = \int_{-1}^{1}\int_{-1}^{1} h_i(\xi)h_j(\eta)h_k(\xi)h_l(\eta)\,\mathrm{d}\xi\mathrm{d}\eta \,.$$

Two dimensional expansion – edge

If we apply the perpendicular gradient $\nabla^\perp$ to this nodal expansion of ψ^h we obtain

$$\nabla^\perp \psi^h = \begin{pmatrix} \sum_{i=0}^N \sum_{j=1}^N (\psi_{i,j} - \psi_{i,j-1}) h_i(\xi) e_j(\eta) \\ \sum_{i=1}^N \sum_{j=0}^N (\psi_{i-1,j} - \psi_{i,j}) e_i(\xi) h_j(\eta) \end{pmatrix}$$

$$= \begin{bmatrix} h_0(\xi)e_1(\eta) & \dots & h_N(\xi)e_N(\eta) & 0 & \dots & 0 \\ 0 & \dots & 0 & e_1(\xi)h_0(\eta) & \dots & e_N(\xi)h_N(\eta) \end{bmatrix} \mathbb{E}^{1,0} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix}$$

Two dimensional expansion – edge

If we apply the perpendicular gradient $\nabla^{\perp}$ to this nodal expansion of ψ^h we obtain

$$\nabla^{\perp}\psi^h = \begin{pmatrix} \sum_{i=0}^{N}\sum_{j=1}^{N}(\psi_{i,j}-\psi_{i,j-1})h_i(\xi)e_j(\eta) \\ \sum_{i=1}^{N}\sum_{j=0}^{N}(\psi_{i-1,j}-\psi_{i,j})e_i(\xi)h_j(\eta) \end{pmatrix}$$

$$= \begin{bmatrix} h_0(\xi)e_1(\eta) & \dots & h_N(\xi)e_N(\eta) & 0 & \dots & 0 \\ 0 & \dots & 0 & e_1(\xi)h_0(\eta) & \dots & e_N(\xi)h_N(\eta) \end{bmatrix} \mathbb{E}^{1,0} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix}$$

If $\mathbf{v}^h$ can be expanded as

$$\mathbf{v}^h = \begin{pmatrix} \sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j}h_i(\xi)e_j(\eta) \\ \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j}e_i(\xi)h_j(\eta) \end{pmatrix}$$

$$= \begin{bmatrix} h_0(\xi)e_1(\eta) & \dots & h_N(\xi)e_N(\eta) & 0 & \dots & 0 \\ 0 & \dots & 0 & e_1(\xi)h_0(\eta) & \dots & e_N(\xi)h_N(\eta) \end{bmatrix} \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix}$$

TU Delft

Two dimensional expansion – edge

If we apply the perpendicular gradient $\nabla^\perp$ to this nodal expansion of ψ^h we obtain

$$\nabla^\perp \psi^h = \begin{pmatrix} \sum_{i=0}^N \sum_{j=1}^N (\psi_{i,j} - \psi_{i,j-1}) h_i(\xi) e_j(\eta) \\ \sum_{i=1}^N \sum_{j=0}^N (\psi_{i-1,j} - \psi_{i,j}) e_i(\xi) h_j(\eta) \end{pmatrix}$$

$$= \begin{bmatrix} h_0(\xi)e_1(\eta) & \dots & h_N(\xi)e_N(\eta) & 0 & \dots & 0 \\ 0 & \dots & 0 & e_1(\xi)h_0(\eta) & \dots & e_N(\xi)h_N(\eta) \end{bmatrix} \mathbb{E}^{1,0} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix}$$

Then the equation $\mathbf{v}^h = \nabla^\perp \psi^h$ implies

$$\mathbb{E}^{1,0} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix} = \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix} \longleftarrow \quad \text{Same as Slide 6}$$

So we preserve the discrete structure between stream function and mass fluxes! This result is independent of the basis functions, shape and coarseness of the grid!

Two dimensional expansion – edge

If we take an arbitrary $\mathbf{a}^h$ expanded as

$$\mathbf{a}^h = \begin{pmatrix} \sum_{k=0}^{N} \sum_{l=1}^{N} a_{k,l} h_k(\xi) e_l(\eta) \\ \sum_{k=1}^{N} \sum_{l=0}^{N} b_{k,l} e_k(\xi) h_l(\eta) \end{pmatrix} ,$$

and take the L^2 inner product with $\nabla^\perp \psi^h$ we obtain

$$\begin{aligned} \left(\mathbf{a}^h, \nabla^\perp \psi^h\right)_{\mathbf{D}^h} &= \int_{-1}^{1} \int_{-1}^{1} (\mathbf{a}^h, \nabla^\perp \psi^h) \, \mathrm{d}\xi \mathrm{d}\eta \\ &= [a_{0,1} \; \dots \; a_{N,N} \; b_{1,0} \; \dots \; b_{N,N}] \mathbb{M}^{(1)} \mathbb{E}^{1,0} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix} \end{aligned}$$

where the mass matrix $\mathbb{M}^{(1)}$ is given by

$$\mathbb{M}^{(1)} = \begin{pmatrix} \int_{-1}^{1} \int_{-1}^{1} h_i(\xi) e_j(\eta) h_k(\xi) e_l(\eta) \, \mathrm{d}\xi \mathrm{d}\eta & 0 \\ 0 & \int_{-1}^{1} \int_{-1}^{1} e_p(\xi) h_q(\eta) e_r(\xi) h_s(\eta) \, \mathrm{d}\xi \mathrm{d}\eta \end{pmatrix} .$$

Two dimensional expansion – surface

Let $\mathbf{u}^h$ be expanded as

$$\mathbf{u}^h = \left(\begin{array}{c} \sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \end{array} \right)$$

Then $\nabla \cdot \mathbf{u}^h$ is given by

$$\begin{aligned} \nabla \cdot \mathbf{u}^h &= \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta) \\ &= \left[e_1(\xi)e_1(\eta) \ \dots \ e_N(\xi)e_N(\eta) \right] \mathbb{E}^{2,1} \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix} . \end{aligned}$$

Two dimensional expansion – surface

Then $\nabla \cdot \mathbf{u}^h$ is given by

$$\begin{aligned}\nabla \cdot \mathbf{u}^h &= \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta) \\ &= \left[e_1(\xi)e_1(\eta) \ \ldots \ e_N(\xi)e_N(\eta) \right] \mathbb{E}^{2,1} \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix} .\end{aligned}$$

Since the basis functions $e_i(\xi)e_j(\eta)$ are linearly independent, $\nabla \cdot \mathbf{u}^h = 0$ reduces to

$$\mathbb{E}^{2,1} \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix} = 0 \quad \longleftarrow \quad \text{Same as Slide 6}$$

Two dimensional expansion – surface

Let q^h can be expanded as

$$q^h(\xi,\eta) = \sum_{k=1}^{N}\sum_{l=1}^{N} q_{k,l} e_k(\xi) e_l(\eta) \ .$$

With this expansion we can write $(q^h, \nabla\cdot\mathbf{u}^h)_{S^h}$ as

$$(q^h, \nabla\cdot\mathbf{u}^h)_{S^h} = \begin{bmatrix} q_{1,1} & \dots & q_{N,N} \end{bmatrix} \mathbb{M}^{(2)}\mathbb{E}^{2,1} \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix} ,$$

where $\mathbb{M}^{(2)}$ is the mass matrix on S^h given by

$$\mathbb{M}^{(2)} := \int_{-1}^{1}\int_{-1}^{1} e_i(\xi)e_j(\eta)e_k(\xi)e_l(\eta)\,\mathrm{d}\xi\mathrm{d}\eta$$

Two dimensional Hodge decomposition

With these basis functions we can write for the Hodge decomposition of the velocity field $\mathbf{u}^h \in H_0(\text{div})$

$$\mathbf{u}^h = \nabla^\perp \psi^h + \nabla \phi^h$$

Multiplying with any finite dimensional $\mathbf{v}^h \in H_0(\text{div})$ and integrating over the domain gives

$$\begin{aligned} \left(\mathbf{v}^h, \mathbf{u}^h\right) &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(\mathbf{v}^h, \nabla \phi^h\right) \\ &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(-\nabla \cdot \mathbf{v}^h, \phi^h\right) \end{aligned}$$

Two dimensional Hodge decomposition

With these basis functions we can write for the Hodge decomposition of the velocity field $\mathbf{u}^h \in H_0(\text{div})$

$$\mathbf{u}^h = \nabla^\perp \psi^h + \nabla \phi^h$$

Multiplying with any finite dimensional $\mathbf{v}^h \in H_0(\text{div})$ and integrating over the domain gives

$$\begin{aligned}\left(\mathbf{v}^h, \mathbf{u}^h\right) &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(\mathbf{v}^h, \nabla \phi^h\right) \\ &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(-\nabla \cdot \mathbf{v}^h, \phi^h\right)\end{aligned}$$

$$\vec{v}^T \mathbb{M}^{(1)} \vec{u} = \vec{v}^T \mathbb{M}^{(1)} \mathbb{E}^{1,0} \vec{\psi} + \vec{v}^T \mathbb{E}^{2,1\,T} \mathbb{M}^{(2)} \vec{\phi}$$

Two dimensional Hodge decomposition

With these basis functions we can write for the Hodge decomposition of the velocity field $\mathbf{u}^h \in H_0(\text{div})$

$$\mathbf{u}^h = \nabla^\perp \psi^h + \nabla \phi^h$$

Multiplying with any finite dimensional $\mathbf{v}^h \in H_0(\text{div})$ and integrating over the domain gives

$$\begin{aligned} \left(\mathbf{v}^h, \mathbf{u}^h\right) &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(\mathbf{v}^h, \nabla \phi^h\right) \\ &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(-\nabla \cdot \mathbf{v}^h, \phi^h\right) \end{aligned}$$

$$\vec{v}^T \mathbb{M}^{(1)} \vec{u} = \vec{v}^T \mathbb{M}^{(1)} \mathbb{E}^{1,0} \vec{\psi} + \vec{v}^T \mathbb{E}^{2,1\,T} \mathbb{M}^{(2)} \vec{\phi}$$

Two dimensional Hodge decomposition

With these basis functions we can write for the Hodge decomposition of the velocity field $\mathbf{u}^h \in H_0(\mathrm{div})$

$$\mathbf{u}^h = \nabla^\perp \psi^h + \nabla \phi^h$$

Multiplying with any finite dimensional $\mathbf{v}^h \in H_0(\mathrm{div})$ and integrating over the domain gives

$$\begin{aligned}\left(\mathbf{v}^h, \mathbf{u}^h\right) &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(\mathbf{v}^h, \nabla \phi^h\right) \\ &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(-\nabla \cdot \mathbf{v}^h, \phi^h\right)\end{aligned}$$

$$\mathbb{M}^{(1)} \vec{u} = \quad \mathbb{M}^{(1)} \mathbb{E}^{1,0} \vec{\psi} + \quad {\mathbb{E}^{2,1}}^T \mathbb{M}^{(2)} \vec{\phi}$$

Two dimensional Hodge decomposition

With these basis functions we can write for the Hodge decomposition of the velocity field $\mathbf{u}^h \in H_0(\text{div})$

$$\mathbf{u}^h = \nabla^\perp \psi^h + \nabla \phi^h$$

Multiplying with any finite dimensional $\mathbf{v}^h \in H_0(\text{div})$ and integrating over the domain gives

$$\begin{aligned}\left(\mathbf{v}^h, \mathbf{u}^h\right) &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(\mathbf{v}^h, \nabla \phi^h\right) \\ &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(-\nabla \cdot \mathbf{v}^h, \phi^h\right)\end{aligned}$$

$$\mathbb{M}^{(1)} \vec{u} = \quad \mathbb{M}^{(1)} \mathbb{E}^{1,0} \vec{\psi} + \quad {\mathbb{E}^{2,1}}^T \mathbb{M}^{(2)} \vec{\phi}$$

$$\vec{u} = \mathbb{E}^{1,0} \vec{\psi} + {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \mathbb{M}^{(2)} \vec{\phi}$$

Two dimensional Hodge decomposition

With these basis functions we can write for the Hodge decomposition of the velocity field $\mathbf{u}^h \in H_0(\text{div})$

$$\mathbf{u}^h = \nabla^\perp \psi^h + \nabla \phi^h$$

Multiplying with any finite dimensional $\mathbf{v}^h \in H_0(\text{div})$ and integrating over the domain gives

$$\begin{aligned}\left(\mathbf{v}^h, \mathbf{u}^h\right) &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(\mathbf{v}^h, \nabla \phi^h\right) \\ &= \left(\mathbf{v}^h, \nabla^\perp \psi^h\right) + \left(-\nabla \cdot \mathbf{v}^h, \phi^h\right)\end{aligned}$$

$$\mathbb{M}^{(1)} \vec{u} = \mathbb{M}^{(1)} \mathbb{E}^{1,0} \vec{\psi} + \mathbb{E}^{2,1\,T} \mathbb{M}^{(2)} \vec{\phi}$$

$$\vec{u} = \mathbb{E}^{1,0} \vec{\psi} + \mathbb{M}^{(1)\,-1} \mathbb{E}^{2,1\,T} \mathbb{M}^{(2)} \vec{\phi}$$

$$\|\mathbf{u}^h\|_0^2 = \vec{u}^T \mathbb{M}^{(1)} \vec{u} = \vec{\psi}^T \underbrace{\mathbb{E}^{1,0\,T} \mathbb{M}^{(1)} \mathbb{E}^{1,0}}_{-\Delta} \vec{\psi} + \vec{\phi}^T \underbrace{\mathbb{M}^{(2)} \mathbb{E}^{2,1} \mathbb{M}^{(1)\,-1} \mathbb{E}^{2,1\,T} \mathbb{M}^{(2)}}_{-\Delta} \vec{\phi}$$

Nodal Laplace problem

Consider the scalar Laplace problem for $\psi \in H_0^1(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta\psi = f \qquad \text{in } \Omega$$

Nodal Laplace problem

Consider the scalar Laplace problem for $\psi \in H_0^1(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta\psi = f \qquad \text{in } \Omega$$

Multiply this equation with any $\tilde{\psi} \in H_0^1(\Omega)$ and integrate over the domain Ω

$$\int_\Omega -\Delta\psi\,\tilde{\psi}\,\mathrm{d}\Omega = \int_\Omega \left(\nabla\psi, \nabla\tilde{\psi}\right)\,\mathrm{d}\Omega = \int_\Omega f\,\tilde{\psi}\,\mathrm{d}\Omega$$

Nodal Laplace problem

Consider the scalar Laplace problem for $\psi \in H_0^1(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta\psi = f \qquad \text{in } \Omega$$

Multiply this equation with any $\tilde{\psi} \in H_0^1(\Omega)$ and integrate over the domain Ω

$$\int_\Omega -\Delta\psi\, \tilde{\psi}\, \mathrm{d}\Omega = \int_\Omega \left(\nabla\psi, \nabla\tilde{\psi}\right)\, \mathrm{d}\Omega = \int_\Omega f\, \tilde{\psi}\, \mathrm{d}\Omega$$

If we restrict ourselves to the conforming subspace spanned by the nodal functions

$$\psi^h = \sum_{i=0}^{N} \sum_{j=0}^{N} \psi_{i,j} h_i(\xi) h_j(\eta)$$

we obtain the discrete Laplace equation

$$\mathbb{E}^{1,0\,T} \mathbb{M}^{(1)} \mathbb{E}^{1,0} \vec{\psi} = \mathbb{M}^{(0)} \vec{f}$$

Nodal Laplace problem

Consider the scalar Laplace problem for $\psi \in H_0^1(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta\psi = f \qquad \text{in } \Omega$$

Multiply this equation with any $\tilde{\psi} \in H_0^1(\Omega)$ and integrate over the domain Ω

$$\int_\Omega -\Delta\psi\,\tilde{\psi}\,\mathrm{d}\Omega = \int_\Omega \left(\nabla\psi, \nabla\tilde{\psi}\right)\,\mathrm{d}\Omega = \int_\Omega f\,\tilde{\psi}\,\mathrm{d}\Omega$$

If we restrict ourselves to the conforming subspace spanned by the nodal functions

$$\psi^h = \sum_{i=0}^{N}\sum_{j=0}^{N} \psi_{i,j} h_i(\xi) h_j(\eta)$$

we obtain the discrete Laplace equation

$$\mathbb{E}^{1,0\,T}\mathbb{M}^{(1)}\mathbb{E}^{1,0}\vec{\psi} = \mathbb{M}^{(0)}\vec{f}$$

Note that the discrete Laplacian only consists of a mass matrix and incidence matrix.

Volumetric Laplace problem I

Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta\phi = f \qquad \text{in } \Omega$$

with $\partial\phi/\partial n = 0$

Volumetric Laplace problem I

Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1,1]^2$

$$-\Delta\phi = f \qquad \text{in } \Omega$$

with $\partial\phi/\partial n = 0$

If we ϕ^h in terms of edge functions, like

$$\phi^h = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{i,j} e_i(\xi) e_j(\eta)$$

Now the derivative of an edge function $e_i(\xi)$ does not exist. We therefore have to go to the mixed formulation

$$\left\{ \begin{array}{rcccc} \mathbf{u} & - & \nabla\phi & = & 0 \\ \nabla\cdot\mathbf{u} & & & & f \end{array} \right.$$

Volumetric Laplace problem I

Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1,1]^2$

$$-\Delta\phi = f \qquad \text{in } \Omega$$

with $\partial\phi/\partial n = 0$

If we ϕ^h in terms of edge functions, like

$$\phi^h = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{i,j} e_i(\xi) e_j(\eta)$$

Now the derivative of an edge function $e_i(\xi)$ does not exist. We therefore have to go to the mixed formulation

$$\left\{ \begin{array}{rcccc} \mathbf{u} & - & \nabla\phi & = & 0 \\ \nabla\cdot\mathbf{u} & & & & f \end{array} \right.$$

If we multiply the first equation by $\mathbf{v} \in H_0(\text{div};\Omega)$ and the second equation by $q \in L^2(\Omega)$, we obtain

$$\begin{array}{rcccc} (\mathbf{v},\mathbf{u}) & - & (\mathbf{v},\nabla\phi) & = & 0 \\ (q,\nabla\cdot\mathbf{u}) & & & & (q,f) \end{array}$$

Volumetric Laplace problem I

Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1,1]^2$

$$-\Delta\phi = f \qquad \text{in } \Omega$$

with $\partial\phi/\partial n = 0$

If we ϕ^h in terms of edge functions, like

$$\phi^h = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{i,j} e_i(\xi) e_j(\eta)$$

Now the derivative of an edge function $e_i(\xi)$ does not exist. We therefore have to go to the mixed formulation

$$\begin{cases} \mathbf{u} & - & \nabla\phi & = & 0 \\ \nabla\cdot\mathbf{u} & & & & f \end{cases}$$

If we multiply the first equation by $\mathbf{v} \in H_0(\text{div};\Omega)$ and the second equation by $q \in L^2(\Omega)$, we obtain

$$\begin{array}{rcccc} (\mathbf{v},\mathbf{u}) & + & (\nabla\cdot\mathbf{v},\phi) & = & 0 \\ (q,\nabla\cdot\mathbf{u}) & & & & (q,f) \end{array}$$

Volumetric Laplace problem I

Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1,1]^2$

$$-\Delta\phi = f \qquad \text{in } \Omega$$

with $\partial\phi/\partial n = 0$

If we ϕ^h in terms of edge functions, like

$$\phi^h = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{i,j} e_i(\xi) e_j(\eta)$$

Now the derivative of an edge function $e_i(\xi)$ does not exist. We therefore have to go to the mixed formulation

$$\left\{ \begin{array}{rcccl} \mathbf{u} & - & \nabla\phi & = & 0 \\ \nabla\cdot\mathbf{u} & & & & f \end{array} \right.$$

If we multiply the first equation by $\mathbf{v} \in H_0(\mathrm{div};\Omega)$ and the second equation by $q \in L^2(\Omega)$, we obtain

$$\begin{array}{rcccl} (\mathbf{v},\mathbf{u}) & + & (\nabla\cdot\mathbf{v},\phi) & = & 0 \\ (q,\nabla\cdot\mathbf{u}) & & & & (q,f) \end{array}$$

Inserting our expansions gives

$$\begin{array}{rcccl} \mathbb{M}^{(1)}\vec{u} & + & \mathbb{E}^{2,1\,T}\mathbb{M}^{(2)}\vec{\phi} & = & 0 \\ \mathbb{M}^{(2)}\mathbb{E}^{2,1}\vec{u} & & & & \mathbb{M}^{(2)}\vec{f} \end{array}$$

TUDelft

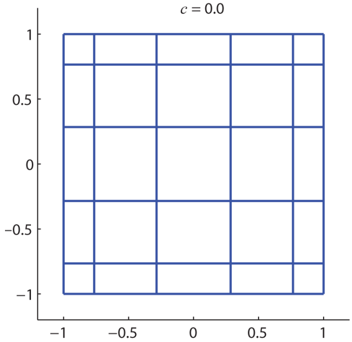
Volumetric Laplace problem II

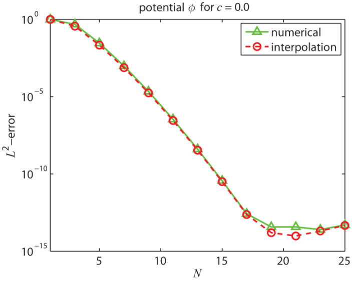
$$\begin{array}{rcccl} \mathbb{M}^{(1)}\vec{u} & + & {\mathbb{E}^{2,1}}^T\mathbb{M}^{(2)}\vec{\phi} & = & 0 \\ \mathbb{M}^{(2)}\mathbb{E}^{2,1}\vec{u} & & & & \mathbb{M}^{(2)}\vec{f} \end{array}$$

If we eliminate **u** again,we obtain

$$\mathbb{M}^{(2)}\mathbb{E}^{2,1}{\mathbb{M}^{(1)}}^{-1}{\mathbb{E}^{2,1}}^T\mathbb{M}^{(2)}\vec{\phi} = \mathbb{M}^{(2)}\vec{f}$$

Again, this system matrix only consists of mass matrices and an incidence matrix.





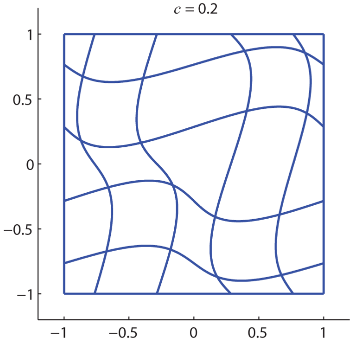
Volumetric Laplace problem II

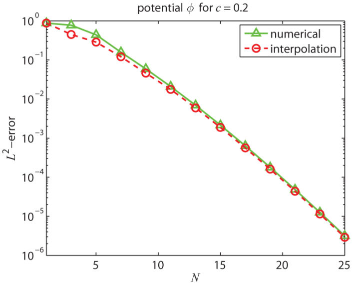
$$
\begin{array}{rcl}
\mathbb{M}^{(1)}\vec{u} & + \quad \mathbb{E}^{2,1\,T}\mathbb{M}^{(2)}\vec{\phi} & = \quad 0 \\
\mathbb{M}^{(2)}\mathbb{E}^{2,1}\vec{u} & & \quad \mathbb{M}^{(2)}\vec{f}
\end{array}
$$

If we eliminate **u** again,we obtain

$$
\mathbb{M}^{(2)}\mathbb{E}^{2,1}\mathbb{M}^{(1)\,-1}\mathbb{E}^{2,1\,T}\mathbb{M}^{(2)}\vec{\phi} = \mathbb{M}^{(2)}\vec{f}
$$

Again, this system matrix only consists of mass matrices and an incidence matrix.





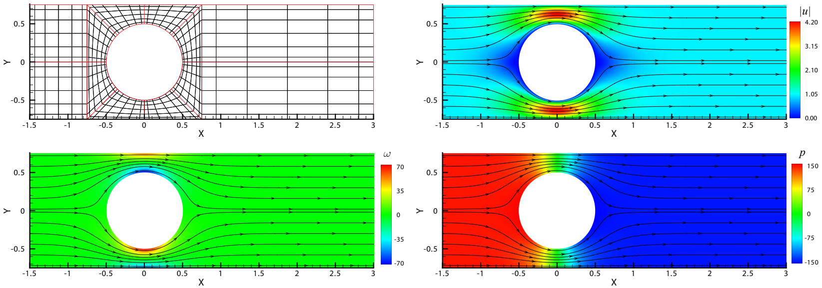
Stokes

$$\left\{ \begin{aligned} \nabla \times \mathbf{u} - \omega = 0 \qquad & \text{in } \Omega \\ \nabla \times \omega + \nabla p = \mathbf{f} \qquad & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 \qquad & \text{in } \Omega \end{aligned} \right. ,$$

Using the same approach, we can discretize this as

$$\begin{pmatrix} \mathbb{M}^{(0)} & {\mathbb{E}^{1,0}}^T \mathbb{M}^{(1)} & 0 \\ \mathbb{M}^{(1)}\mathbb{E}^{1,0} & 0 & {\mathbb{E}^{2,1}}^T \mathbb{M}^{(2)} \\ 0 & \mathbb{M}^{(2)}\mathbb{E}^{2,1} & 0 \end{pmatrix} \begin{pmatrix} \vec{\omega} \\ \vec{u} \\ \vec{p} \end{pmatrix} - \begin{pmatrix} 0 \\ \mathbb{M}^{(1)}\vec{f} \\ 0 \end{pmatrix}$$

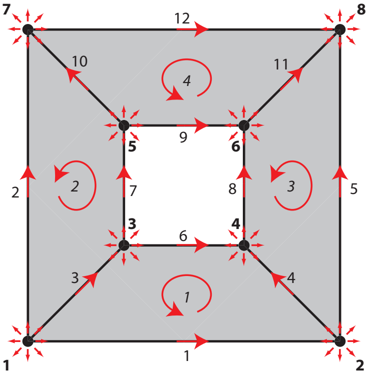
Stokes flow around cylinder



[Jasper Kreeft, JCP 2014]

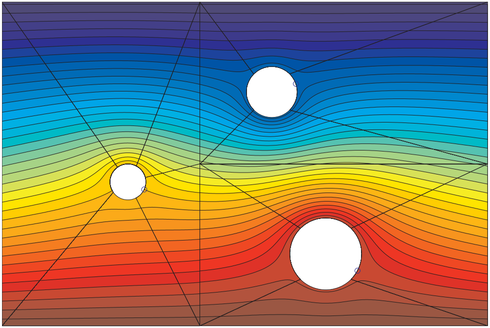
Domain with holes I

If we have holes in the domain, there exist vectors in in the null space of $\mathbb{E}^{2,1}$ which is not in the range of $\mathbb{E}^{1,0}$



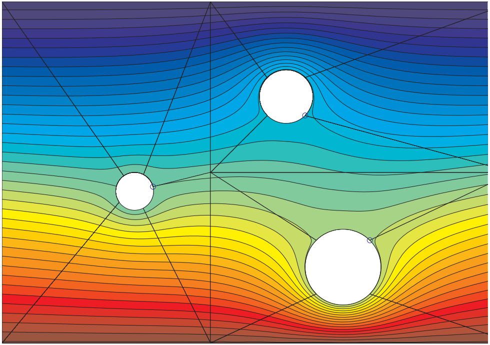
Domain with holes II

Application: Potential flow around 3 cylinders (Isogeometric)



Domain with holes II

Application: Potential flow around 3 cylinders (Isogeometric)



Final Remarks

- Further reading: Consult papers of this week's speakers!
- Do it yourself!
- See work by: Mikhail Shashkov, Enzo Tonti, Alain Bossavit, Matthieu Desbrun (DEC), Arnold, Falk & Winther (FEEC), Jerome Bonelle, Pierre Cantin & Alexandre Ern (CO schemes), Franco Brezzi, Stanly Steinberg, Mac Hyman & Pavel Bochev
- *The Geometry of Physics*, by Theodore Frankel.

Final Remarks

- Further reading: Consult papers of this week's speakers!
- Do it yourself!
- See work by: Mikhail Shashkov, Enzo Tonti, Alain Bossavit, Matthieu Desbrun (DEC), Arnold, Falk & Winther (FEEC), Jerome Bonelle, Pierre Cantin & Alexandre Ern (CO schemes), Franco Brezzi, Stanly Steinberg, Mac Hyman & Pavel Bochev
- *The Geometry of Physics*, by Theodore Frankel.

Thank you for your attention and enjoy the remainder of the summer school.