Moduli problems for operadic algebras

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Abstract

A theorem of Pridham and Lurie provides an equivalence between formal moduli problems and Lie algebras in characteristic zero. We prove a generalization of this correspondence, relating formal moduli problems parametrized by algebras over a Koszul operad to algebras over its Koszul dual operad. In particular, when the Lie algebra associated to a deformation problem is induced from a pre-Lie structure it corresponds to a permutative formal moduli problem. As another example we obtain a correspondence between operadic formal moduli problems and augmented operads.

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1 Introduction

A classical heuristic in deformation theory asserts that the infinitesimal deformations of an algebro-geometric object over a field $k$ of characteristic zero are controlled by a differential graded Lie algebra. A first instance of this can already be found in the work of Kodaira–Spencer on deformations of complex manifolds $[KS58]$; its recognition as a key principle in deformation theory traces back to ideas of Deligne and Drinfeld. These ideas have been further developed in the work of various authors $[GM88, Hin01, Kon, Man04]$, leading to a precise mathematical formulation of the above heuristic as an equivalence of categories between deformation problems and dg-Lie algebras $[Pri10, Lur11]$.

More precisely, following work of Schlessinger $[Sch68]$, one can describe the infinitesimal deformations of an algebro-geometric object $X$ over $k$ by a functor

$$def_X : Art_k \longrightarrow Set$$

from the category of (commutative) Artin local $k$-algebras with residue field $k$. This functor sends each Artin local $k$-algebra $A$ to the set of deformations of $X$ over $A$. The aforementioned works have led to two modifications of this idea.

First, the deformations of $X$ typically have automorphisms and homotopies between them, leading to the study of deformation functors with values in spaces or simplicial sets. Second, it has been observed that the deformation theory of an object $X$ usually comes with an additional obstruction theory, which is not encoded by the functor $def_X$. A key idea, tracing back to Drinfeld, is to incorporate such an obstruction theory by extending $def_X$ to the category of dg-Artin local $k$-algebras. One is therefore led to contemplate deformation functors

$$Def_X : CAlg_{\text{dg}}^{\text{conn}}_k \longrightarrow S$$

from the $\infty$-category of (connective) dg-Artin local $k$-algebras to the $\infty$-category of spaces. Such deformation problems satisfy a variant of the Schlessinger conditions: their value on $k$ is contractible and they preserve fiber products along maps inducing a surjection on $H^0$ (see Section 4.1 for more details). Following Lurie $[Lur11]$, we will refer to such functors as formal moduli problems. The work of Lurie $[Lur11]$ and Pridham $[Pri10]$ now provides an equivalence of $\infty$-categories

$$\text{FMP}_k \sim \text{Lie}_k$$

between formal moduli problems and differential graded Lie algebras over $k$. 
The equivalence between Lie algebras and formal moduli problems indexed by commutative algebras can be viewed as a manifestation of the Koszul duality between the commutative operad and the Lie operad. In fact, there is a similar equivalence between associative algebras and formal moduli problems indexed by associative algebras [Lur11], which can be thought of as an incarnation of the Koszul self-duality of the associative operad. These two equivalences are related in a natural way: if a Lie algebra arises from an associative algebra by taking the commutator bracket, then the corresponding commutative formal moduli problem is the restriction of an associative formal moduli problem.

Statement of results

The purpose of this paper is to generalize the above results to more general pairs of Koszul dual operads over a field of characteristic zero. More precisely, for any augmented operad \( P \) one can define an \( \infty \)-category \( \text{Alg}_{\text{sm}}^P \) of small, or ‘Artin’, \( P \)-algebras. A \( P \)-algebraic formal moduli problem is then given by a functor

\[
F : \text{Alg}_{\text{sm}}^P \longrightarrow S
\]

satisfying a natural analogue of the Schlessinger conditions (see Section 4.1 for more details). We denote the \( \infty \)-category of such functors by \( \text{FMP}_P \). When \( P \) is a Koszul binary quadratic operad, we prove that such \( P \)-algebraic formal moduli problems can be classified by algebras over its Koszul dual operad:

**Theorem 1.1.** Let \( k \) be a field of characteristic zero and consider:

- a Koszul binary quadratic operad \( P \) in nonpositive cohomological degrees.
- its Koszul dual operad \( P' \).

Then there is an equivalence of \( \infty \)-categories

\[
\text{FMP}_P \longrightarrow \text{Alg}_P; \ F \mapsto T(F)[−1].
\]

Here \( T(F) \) denotes the tangent complex of the formal moduli problem, as defined by Lurie [Lur11] (see also Definition 4.13).

This recovers the aforementioned results of Lurie and Pridham, taking \( P \) to be the commutative operad, whose Koszul dual is the Lie operad, or the associative operad. It also applies to many other Koszul dual pairs of algebraic operads (see Section 3). For example, taking \( P \) to be the permutative operad, whose Koszul dual is the pre-Lie operad [CL01], we obtain a classification of permutative formal moduli problems in terms of pre-Lie algebras. Such pre-Lie algebras indeed appear naturally in the deformation theory of operadic algebras, see the work of Dotsenko, Shadrin and Vallette [DSV15] (in fact, this was the original motivation for the present paper). From the point of view of deformation theory, a Lie algebra underlies a pre-Lie algebra structure whenever the corresponding commutative formal moduli problem is the restriction of a permutative formal moduli problem.

Our proof of Theorem 1.1 will make little use of the Koszul property of \( P \): the Koszul property mainly serves to guarantee that the operad \( P \) admits a resolution with good properties. More precisely, we will deduce Theorem 1.1 from a statement about algebras over the dual of the bar construction of an augmented dg-operad. In fact, it will be convenient to work in a slightly more general setting:

(a) We work with coloured dg-operads.
Instead of taking dg-operads over the base field, we will consider operads which are augmented over a connective dg-algebra or, in the coloured case, over a connective dg-category \( k \) (here connective means that the cohomology groups are concentrated in nonpositive degrees). More precisely, we will consider coloured dg-operads \( \mathcal{P} \) which fits into a retract diagram of operads
\[
\begin{array}{ccc}
k & \longrightarrow & \mathcal{P} \\
& \longrightarrow & \\
\mathcal{P} & \longrightarrow & k
\end{array}
\]

Given a dg-category \( k \), we will refer to such objects as (augmented) \( k \)-operads.

The usual operadic homological algebra (as in [LV], for example) has an analogue for augmented \( k \)-operads; Section 9 provides all the results and definitions that we will need. In particular, every (augmented) \( k \)-operad \( \mathcal{P} \) has a dual \( k^{op} \)-operad, given somewhat informally as
\[
\mathcal{D}(\mathcal{P}) = \mathcal{D}_k(\mathcal{P}) = \text{Ext}_k(k,k).
\]

More precisely, we can make the following definition:

**Definition 1.2.** Let \( k \) be a dg-category and let \( \mathcal{P} \) be a (augmented) \( k \)-operad, which we assume to be cofibrant as a left \( k \)-module throughout this introduction. We define the dual operad to be the \( k \)-linear dual of the bar construction of \( \mathcal{P} \) over \( k \)
\[
\mathcal{D}_k(\mathcal{P}) := B_k(\mathcal{P})^\vee.
\]

The \( k^{op} \)-operad structure arises from the \( k \)-cooperad structure on the bar construction. See Section 9.2 for more details.

With these definitions, our main result is then the following:

**Theorem 1.3.** Let \( k \) be a dg-category over \( \mathbb{Q} \) and let \( \mathcal{P} \) be an augmented \( k \)-operad. Suppose that the following conditions are satisfied:

1. \( k \) and \( \mathcal{P} \) are both connective, i.e. their cohomology is concentrated in nonpositive degrees.
2. \( k \) is cohomologically bounded, i.e. there exists an \( N \in \mathbb{N} \) such that all \( H^*(k)(c,d) \) are concentrated in degrees \([−N,0]\).
3. The derived relative composition product
\[
\mathcal{P}(1) \circ_{\mathcal{P}(2)}^h \mathcal{P}(1)
\]

is concentrated in increasingly negative cohomological degrees as the arity increases (cf. Definition 3.17).

Then there is an equivalence of \( \infty \)-categories
\[
\begin{array}{ccc}
\text{FMP}_\mathcal{P} & \longrightarrow & \text{Alg}_{\mathcal{D}_k(\mathcal{P})} \\
F & \longrightarrow & T(F)
\end{array}
\]
(1.4)
where \( T(F) \) denotes the tangent complex (Definition 4.13). Furthermore, this equivalence is natural in \( \mathcal{P} \).

We will denote the inverse equivalence (1.4) by
\[
\begin{array}{ccc}
\text{MC}: \text{Alg}_{\mathcal{D}_k(\mathcal{P})} & \longrightarrow & \text{FMP}_\mathcal{P}
\end{array}
\]
and think of it as sending a \( \mathcal{D}(\mathcal{P}) \)-algebra to the ‘formal \( \mathcal{P} \)-algebraic stack of solutions to the Maurer–Cartan equation’. Some justification for this terminology is provided in Section
where we show that for various operads $P$, this inverse functor does indeed admit a description in terms of Maurer–Cartan elements of dg-Lie algebras. This is a by-product of our proof of Theorem 1.3, which relies on a careful analysis of the adjoint pair

$$
\mathcal{D} : \text{Alg}_P \longrightarrow \text{Alg}_{\mathcal{D}(P)}' : \mathcal{D}'.
$$

Here $\mathcal{D}$ sends a $P$-algebra to the $k$-linear dual of its operadic bar construction. Our argument differs slightly from the arguments of Lurie and Pridham: when $P$ is the commutative operad, we study the behaviour of the functor $\mathcal{D}$ (the Harrison complex) instead of the functor $\mathcal{D}'$ (the Chevalley–Eilenberg complex).

The conditions of Theorem 1.3 hold for Koszul binary quadratic operads, leading to the following proof of Theorem 1.1:

**Proof of Theorem 1.1 (from Theorem 1.3).** The Koszul property of $P$ asserts that there are weak equivalences of operads

$$
\Omega P \sim \longrightarrow P \quad \quad \text{and} \quad \quad \mathcal{D}(P) \sim \longrightarrow P'\{-1\}.
$$

Since $P$ is generated by binary operations in degrees $\leq 0$, the quadratic dual cooperad $P^!$ is generated by binary operations in degrees $\leq -1$. It follows that the generators of $\Omega P$ are concentrated in increasingly negative degrees as the arity increases. In light of Corollary 9.25, the operad $P$ then satisfies the conditions of Theorem 1.3, and the sequence of equivalences

$$
\text{FMP}_P \xrightarrow{F \mapsto T(F)} \text{Alg}_{\mathcal{D}(P)} \xleftarrow{\sim} \text{Alg}_{P^!(-1)} \xrightarrow{V \mapsto V[-1]} \text{Alg}_{P^!}
$$

provides the desired result.

Suppose that $P$ is an augmented $k$-operad arising as the bar dual of a (sufficiently nice) $k^\text{op}$-operad. Theorem 1.3 then gives an interpretation of the $\infty$-category $\text{Alg}_P$ in terms of formal moduli problems. One may wonder if there is a similar interpretation of the $\infty$-category of algebras over an arbitrary augmented $k$-operad $P$. Somewhat informally, one expects a $P$-algebra to correspond to some homotopy-theoretic, or geometric, analogue of a ‘conilpotent coalgebra over a conilpotent cooperad’. Theorem 1.3 precisely provides us with a geometric way to think about this conilpotent cooperad, as a formal moduli problem

$$
\text{MC}_P : \text{Op}^{\text{fm}}_k \longrightarrow S
$$

on the category of small (i.e. nilpotent, finite-dimensional) operads. This is discussed in more detail in Section 8 and relies on the fact that the operad for nonunital symmetric operads is Koszul self-dual, relative to the dg-category of finite sets and bijections ($k$-linearizing all sets of maps). One may informally think of the functor $\text{MC}_P$ as encoding a family of finite-dimensional nilpotent operads, corresponding to the family of linear duals of the finite-dimensional conilpotent sub-cooperads of the conilpotent cooperad $B_P$.

Given a formal moduli problem $X : \text{Op}^{\text{fm}}_k \longrightarrow S$, there is a natural notion of formal moduli problem over $X$. Indeed, in a similar way as one usually defines quasi-coherent sheaves on moduli functors in algebraic geometry, one can define a formal moduli problem over $X$ to consist of the following data:

1. a $Q$-algebraic formal moduli problem $F_x \in \text{FMP}_Q$ for every $x \in X(Q)$.
2. for every map $f : Q \longrightarrow Q'$ and every $x \in X(Q)$, an equivalence

$$
F_{f_*(x)} \sim \longrightarrow f_*F_x
$$

5
together with coherence data between them (see Definition 8.16). Here \( f_*F_x \) denotes the restriction of \( F_x \) along the forgetful functor \( \text{Alg}^\text{sm}_Q \rightarrow \text{Alg}^\text{sm}_Q \).

Informally, these formal moduli problems can be thought of as geometric analogues of conilpotent coalgebras over conilpotent cooperads. Indeed, a formal moduli problem over \( X \) describes a coherent collection of finite-dimensional nilpotent algebras over finite-dimensional nilpotent operads. This roughly corresponds to the collection of linear duals of the finite-dimensional conilpotent sub-coalgebras of a conilpotent coalgebra.

We then have the following result:

**Theorem 1.6.** Let \( \mathcal{P} \) be a 1-coloured augmented operad. Then there is an equivalence of \( \infty \)-categories

\[
\text{FMP}_{MC_{\mathcal{P}}} \simeq \text{Alg}_{\mathcal{P}}; \quad F \mapsto T(F).
\]

One may consider this as a geometric, or \( \infty \)-categorical, version of the relation between algebras over \( \mathcal{P} \) and conilpotent coalgebras over the conilpotent cooperad \( B\mathcal{P} \) [Val14].

**Outline of the paper**

Let us briefly describe the structure of the rest of the paper. In Section 3, we will discuss various (non-)examples and special cases of our main theorem; these include many of the well-known operads. The reader may want to simultaneously consult Section 4, which recalls the notion of a formal moduli problem (Section 4.1) and describes the main ingredients that go into our proof of Theorem 1.3, such as the adjoint pair (1.5) (see Theorem 4.26).

Section 5 is the technical heart of the proof (Theorem 5.1); in this section, we verify the technical hypotheses that allow us to apply the axiomatic argument (Theorem 4.34) described in Section 4.3. In Section 6, we discuss the naturality of the equivalence (1.4) in the operad \( \mathcal{P} \). Under certain conditions on the operad \( \mathcal{P} \), we describe this equivalence more concretely in terms of Maurer–Cartan elements in Section 7 (see Theorem 7.18).

The special case of the (\( \mathbb{N} \)-coloured) operad for operads is treated in Section 8. We prove that the operad for operads is, in a relative sense, Koszul self-dual. We therefore obtain an equivalence between (augmented) operads and operadic formal moduli problems. The proof of Theorem 1.6 is given in Section 8.3.

Finally, Section 9 provides an extensive treatment of the theory of augmented operads over dg-categories. This includes a discussion of the bar-cobar construction for such augmented operads and algebras over them.

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**2 Vocabulary and conventions**

Throughout, we work over a field \( k \) of characteristic zero and all objects involved are differential graded (with differentials of degree +1), even if this is not said explicitly. All
operads (resp. cooperads) are assumed to be unital and augmented (resp. counital and coaugmented) and have no other constraints in arities 0 and 1, unless otherwise explicitly written. Given \( k \in \mathbb{Z} \) and a graded vector space \( V \) we denote by \( V[k] \) its degree shift satisfying \((V[k])_d = V_{k+d}\).

**Model categories and \( \infty \)-categories.** Since certain functors are only defined at the level of \( \infty \)-categories, we will need to distinguish between model categories or relative categories, and the \( \infty \)-categories obtained from them by inverting the weak equivalences. We will employ the following basic convention: we will denote by \( \mathcal{C} \) a certain category of dg-objects, e.g. operads or algebras over them, and by \( \mathcal{C}^\infty \) the underlying \( \infty \)-category. For example:

\[
\text{Alg}_{dg}^\mathcal{P} = \{ \text{dg-algebras over } \mathcal{P} \} \quad \text{Alg}_P = \text{Alg}_{dg}^\mathcal{P}[\text{quasi-iso}^{-1}].
\]

We will typically refer to the objects of each of these two categories as \( \mathcal{P} \)-algebras, leaving the differential graded structure implicit (except for dg-categories, in order not to confuse these with ordinary categories or \( \infty \)-categories).

**Linear algebra.** Let \( \mathcal{A} \) be a dg-category (over our base field of characteristic zero). Recall that a left \( \mathcal{A} \)-module is a dg-functor \( \mathcal{A} \to \text{Ch}_k \) to the category of cochain complexes and a right \( \mathcal{A} \)-module is a functor \( \mathcal{A}^{op} \to \text{Ch}_k \), i.e. a left \( \mathcal{A}^{op} \)-module. By default, modules are left modules. For an object \( c \in \mathcal{A} \), the free \( \mathcal{A} \)-module at \( c \) is the corepresentable functor

\[
\mathcal{A}_c : \mathcal{A} \to \text{Ch}_k; \quad d \mapsto \mathcal{A}(c,d).
\]

An \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule is a \( \mathcal{B}^{op} \otimes \mathcal{A} \)-module. We will denote the canonical \( \mathcal{A} \)-bimodule by

\[
\mathcal{A} : \mathcal{A}^{op} \otimes \mathcal{A} \to \text{Ch}_k; \quad (c,d) \mapsto \mathcal{A}(c,d).
\]

If \( E \) is a \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule and \( M \) is a \( \mathcal{B} \)-module, then we can form the \( \mathcal{A} \)-module \( E \otimes_{\mathcal{B}} M \). It actually defines a functor

\[
E \otimes_{\mathcal{B}} - : \text{LMod}^{dg}_{\mathcal{B}} \to \text{LMod}^{dg}_{\mathcal{A}}.
\]

This satisfies the obvious identities, e.g. \( \mathcal{A} \otimes_{\mathcal{A}} M = M, \ E \otimes_{\mathcal{B}} \mathcal{B}_c = E_c = E(c,-) \). Similarly, we have a functor

\[
- \otimes_{\mathcal{A}} E : \text{RMod}^{dg}_{\mathcal{A}} \to \text{RMod}^{dg}_{\mathcal{B}}.
\]

As usual, composing two such functors coincides with tensoring bimodules: if \( E \) is a \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule, and \( F \) a \( \mathcal{B} \)-\( \mathcal{C} \)-bimodule, then

\[
(E \otimes_{\mathcal{B}} -) \circ (F \otimes_{\mathcal{C}} -) \simeq (E \otimes_{\mathcal{B}} F) \otimes_{\mathcal{C}} -.
\]

The functor \( E \otimes_{\mathcal{B}} - \) has a right adjoint, given by \( \text{Hom}_{\mathcal{A}}(E,-) \), where the left \( \mathcal{B} \)-module structure on \( \text{Hom}_{\mathcal{A}}(E,N) \) comes from the right \( \mathcal{B} \)-action on \( E \). Similarly, the right adjoint to \( - \otimes_{\mathcal{A}} E \) is \( \text{Hom}_{\mathcal{B}}(E,-) \).

Finally, for \( M \) a left \( \mathcal{A} \)-module, we will denote

\[
M' := \text{Hom}_{\mathcal{A}}(M,\mathcal{A}).
\]

This is a right \( \mathcal{A} \)-module via the canonical right action of \( \mathcal{A} \) on itself.

**Remark 2.1.** Let \( \mathcal{A} \) be a small dg-category with finitely many objects and consider its (ordinary) category of left modules. This category has a single compact projective generator, given by the direct sum of all free modules

\[
P = \bigoplus_{c \in \mathcal{A}} A_c.
\]
Likewise, the category of right $A$-modules has a single compact generator

$$Q = \bigoplus_{c \in A} cA$$

It follows from Morita theory that the category of left $A$-modules is equivalent to the category of modules over the dg-algebra $B = \text{End}_A(P)^{op} \cong \text{End}_{A^{op}}(Q)$. Unraveling the definition, this algebra is given by the cochain complex

$$B \cong \prod_{c,d} A(c, d)$$

and the product of two morphisms is their composition whenever they are composable, and 0 otherwise. The equivalence from left $A$-modules to $B$-modules simply sends a left $A$-module $M$ to $Q \otimes A M \cong \bigoplus_c M(c)$, with the obvious action maps arising from $A(c, d) \otimes M(e) \to M(d)$.

**Operads.** Throughout we will employ the theory of coloured operads relative to a base dg-category, which we will denote by $k$. More precisely, suppose that $k$ has a set of objects $S$. By a $k$-operad $P$ we will mean an $S$-coloured (symmetric, differential graded) operad together with maps of $S$-coloured operads $k \to P \to k$ that compose to the identity (cf. Proposition 9.2 for a slightly different perspective). In particular, we will always assume that $P$ is *augmented* over $k$, unless explicitly stated otherwise.

An algebra over a $k$-operad $P$ is simply an algebra over the underlying coloured operad. In particular, each $P$-algebra has an underlying left $k$-module and the usual constructions with $P$-algebras, such as the free $P$-algebra or the bar construction, can be performed at the level of $k$-modules as well. We refer to Section 9 for an extensive discussion of the usual operadic homological algebra relative to a dg-category $k$.

For any operad $P$ we denote by $P[k]$ its degree shift by $k$, such that $V$ is a $P[k]$ algebra if and only if $V[k]$ is a $P$ algebra. In particular, if $P$ is concentrated in degree 0, $P[1](n)$ is concentrated in degree $-n + 1$.

**Definition 2.2.** We will say that a $k$-operad $P$ is *n-reduced* if the map $k \to P \to k$ is an isomorphism in arities $\leq n$ (in particular, it is trivial in arities $\neq 1$ and $\leq n$).

**Definition 2.3.** Let $Q$ be a coloured symmetric sequence of chain complexes (e.g. a $k$-operad). We will say that $Q$ is *connective* if for all tuples of colours,

$$H^*(Q(c_1, \ldots, c_p; c_0)) = 0 \quad \text{for all } * > 0.$$  

Furthermore, $Q$ is *eventually highly connective* if for every $n \in \mathbb{Z}$, there exists an $p(n) \in \mathbb{N}$ such that $H^*(Q)$ vanishes in degrees $* \geq n$ in arities $\geq p(n)$.

**Assumption 2.4** (Cofibrancy assumptions). Since we are not working over a field, various point-set level constructions involving tensor products and $k$-linear duals are only well-behaved when applied to left $k$-modules that are cofibrant. For this reason, we will typically (tacitly) assume throughout the text that our $k$-operads are cofibrant as left $k$-modules and that $k$-cooperads are filtered-cofibrant left $k$-modules (Definition 9.12).

Notice that this assumption is automatically satisfied if $k$ is of the form $k[G]$, where $G$ is any locally finite groupoid, by Maschke’s theorem. Furthermore, since our main results are formulated in homotopy-invariant terms, one can always replace $P$ by a $k$-operad for which this assumption holds.

**Remark 2.5.** Everything we do also works over an arbitrary ring, instead of a field of characteristic zero, if one works with *nonsymmetric* operads.
3 Examples and applications

In this section we discuss various examples and applications of our main result. We will start by discussing some applications of Theorem 1.1 to the deformation theory of operads: here one encounters deformation problems parametrized by permutative algebras, or by operads themselves, which are Koszul dually controlled by pre-Lie algebra structures on deformation complexes, or by convolution operads.

In Section 3.3, we will give some examples of operads satisfying the conditions of Theorem 1.3; of these conditions, the most important one is condition (3), which we discuss in some detail.

3.1 Permutative deformation theory

In this section we consider the classical deformation problems whose associated Lie algebra arises from a pre-Lie algebra.

The permutative operad

Definition 3.1 ([Cha01]). A Perm-algebra, or permutative algebra, is an associative monoid \((X, \cdot)\) such that for every \(x, y, z \in X\),

\[ x \cdot (y \cdot z) = x \cdot (z \cdot y). \]

One easily sees that permutative algebras are algebras over an operad in sets, denoted \(\text{Perm}\), which is binary quadratic. Abusing notation, we will also write \(\text{Perm}\) for its \(k\)-linearization.

The \(k\)-linear operad \(\text{Perm}\) has the operad \(\text{preLie}\) as its quadratic dual. Recall that a pre-Lie algebra is a vector space \(V\) equipped with bilinear operation \(\{-, -\}\) such that for every \(x, y, z \in V\),

\[ \{\{x, y\}, z\} - \{x, \{y, z\}\} = \{\{x, z\}, y\} - \{x, \{z, y\}\}. \]

The operads \(\text{Perm}\) and \(\text{preLie}\) are Koszul [CL01]. Therefore, Theorem 1.1 tells us that there is an equivalence of \(\infty\)-categories between pre-Lie algebras and permutative formal moduli problems

\[ F : \mathcal{A}_{\text{Alg}_{\text{Perm}}} \longrightarrow S. \]

The domain of these formal moduli problems is the \(\infty\)-category of small permutative algebras: these are permutative algebras \(A\) such that \(H^*(A)\) is finite-dimensional and concentrated in degrees \(\leq 0\), and such that \(H^0(A)\) is nilpotent (see Lemma 4.7).

Furthermore, let us observe that the operad \(\text{Perm}\) fits into a sequence of Koszul binary quadratic operads

\[ \text{As} \longrightarrow \text{Perm} \longrightarrow \text{Com}, \]

whose quadratic dual sequence is

\[ \text{As} \leftarrow \text{preLie} \leftarrow \text{Lie}. \]

This dual sequence sends the Lie bracket to the commutator of the pre-Lie structure (respectively, of the associative product). As a consequence of Proposition 6.10, we then
obtain a commuting diagram of ∞-categories

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{A}s} & \sim & \text{FMP}_{\mathcal{A}s} \\
\downarrow & & \downarrow \\
\text{Alg}_{\text{preLie}} & \sim & \text{FMP}_{\text{Perm}} \\
\downarrow & & \downarrow \\
\text{Alg}_{\text{Lie}} & \sim & \text{FMP}_{\text{Com}}.
\end{array}
\]

This tells us in particular that a commutative formal moduli problem lifts to a permutative one (respectively, to an associative one) if and only if the Lie bracket on its tangent complex arises from a pre-Lie structure (respectively, an associative structure).

**Remark 3.3.** More precisely, and more generally, let \( \mathcal{P} \to \Omega \to \mathcal{R} \) be a sequence of Koszul binary quadratic operads. Then we have the following sequence of morphisms between Koszul twisting morphisms

\[
\begin{array}{ccc}
\mathcal{P} & \to & \Omega & \to & \mathcal{R} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}' & \to & \Omega' & \to & \mathcal{R}'.
\end{array}
\]

We also have a sequence of equivalences between operads

\[
\begin{array}{ccc}
\mathcal{D}(\mathcal{P}) & \sim & \mathcal{D}(\Omega) & \sim & \mathcal{D}(\mathcal{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}'{-1} & \sim & \Omega'{-1} & \sim & \mathcal{R}'{-1}.
\end{array}
\]

By Proposition 6.10, we then get the following commuting diagram of ∞-categories, where all horizontal functors are equivalences:

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{P}'} & \sim & \text{Alg}_{\mathcal{D}(\mathcal{P})} & \sim & \text{FMP}_{\mathcal{P}} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Alg}_{\mathcal{Q}'} & \sim & \text{Alg}_{\mathcal{D}(\mathcal{Q})} & \sim & \text{FMP}_{\mathcal{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Alg}_{\mathcal{R}'} & \sim & \text{Alg}_{\mathcal{D}(\mathcal{R})} & \sim & \text{FMP}_{\mathcal{R}}.
\end{array}
\]

Before providing several examples, recall that it follows from Theorem 7.18 that the permutative formal moduli problem classified by a pre-Lie algebra \( g \) sends a finite-dimensional nilpotent permutative algebra \( A \) to the simplicial set of Maurer–Cartan elements

\[
MC_g(A) = MC(g \otimes A \otimes \Omega_{\bullet}).
\]

Here the Lie bracket on \( g \otimes B \), for \( B \) any permutative algebra (e.g. \( A \otimes \Omega_n \)), is simply given by the commutator of the pre-Lie structure

\[
[x \otimes b, y \otimes c] := (-1)^{|b||y|} \{x, y\} \otimes b \cdot c - (-1)^{|b|+|x|} \{y, x\} \otimes b \cdot c.
\]
Example: two-parameter permutative deformations. Let $A$ be the quotient of the free permutative algebra on two generators $a, b$ by the relations $ab = b^2$. As a vector space, it admits the following basis: $\{a^n, ba^m | n \geq 1, m \geq 0\}$.

**Lemma 3.4.** For every pre-Lie algebra $\mathfrak{g}$, the Maurer–Cartan set of $\mathfrak{g} \otimes A$ consists of pairs $(X, Y)$ of degree one elements in $\mathfrak{g}[a]$ such that

$$X(0) = 0, \quad dX + \{X, X\} = 0 \quad \text{and} \quad \nabla_X(Y) = 0,$$

where $\nabla_X = d - (-1)^{|-|}\{-, X\}$.

**Proof.** A degree one element $\gamma$ in $\mathfrak{g} \otimes A$ is a (finite) linear combination

$$\gamma = \sum_{n \geq 1} X_n \otimes a^n + \sum_{m \geq 0} Y_m \otimes ba^m,$$

where the $X_n$ and $Y_m$ have degree one. The Maurer–Cartan equation $d\gamma + \{\gamma, \gamma\} = 0$ then translates into two infinite families of equations, that we describe now.

First, for every $n \geq 1$, the coefficient of $a^n$ gives

$$dX_n + \sum_{k+l=n} \{X_k, X_l\} = 0.$$

This family of equations is equivalent to the single equation $dX + \{X, X\} = 0$ for $X := \sum_{n \geq 1} X_n a^n \in a \cdot \mathfrak{g}[a]$.

Second, for every $m \geq 0$, the coefficient of $ba^m$ gives

$$dY_m + \sum_{k+l=m} \{Y_k, X_l\} = 0.$$

This family of equations is equivalent to the single equation $dY + \{Y, X\} = 0$ for $Y := \sum_{m \geq 0} Y_m a^n \in \mathfrak{g}[a]$, and $X$ as above.

**Remark 3.5.** Observe that this is different from what we would get by looking at the Maurer–Cartan set of $\mathfrak{g} \otimes C$, for the commutative algebra $C = k^+[a, b]/(b^2)$: the equation $\nabla_X(Y) = 0$ would have to be replaced by $d_X(Y) = 0$, where $d_X = d + [X, -]$.

The permutative algebra $A$ introduced above is not small, but each finite-dimensional quotient $A_n = A/(a^n)$ is. Using Lemma 3.4, one sees for instance that the space of Maurer–Cartan elements in $\mathfrak{g} \otimes A_2$ is the space of pairs of 1-cocycles $(Y, X)$ in $\mathfrak{g}$ together with a null-homotopy of $\{Y, X\}$:

$$\text{MC}_\mathfrak{g}(A_2) \simeq \text{hofib} \left( \tau_{\leq 1} \mathfrak{g} \xrightarrow{\cdot 1} \tau_{\leq 2} \mathfrak{g} \right).$$

**Deforming trivial morphisms of operads**

A standard source of pre-Lie algebras is given by convolution pre-Lie algebras [LV, Section 6.4]. Indeed, let $\mathfrak{C}$ be a $k$-cooperad, $\mathfrak{C}$ the kernel of its coaugmentation and let $\mathcal{P}$ be a $k$-linear operad (not necessarily augmented). Then we have a $\mathbb{Z}_{\geq 0}$-graded pre-Lie algebra

$$\mathfrak{g}^{gr} = \bigoplus_{p \geq 0} \left( \text{Hom}(\mathfrak{C}(p), \mathcal{P}(p))_{\Sigma_p} \right) = \bigoplus_{p \geq 0} \left( \text{Hom}(\mathfrak{C}(p), \mathcal{P}(p))^{\Sigma_p} \right).$$
with pre-Lie bracket having weight $-1$ with respect to the $\mathbb{Z}_{\geq 0}$-grading. The permutative FMP corresponding to the so-called convolution pre-Lie algebra
\[
\mathfrak{g} := \prod_{p \geq 0} \mathfrak{g}^{gr}(p).
\]

sends a finite-dimensional, nilpotent permutative algebra $A$ to
\[
\text{MC}_g(A) = \text{MC}(\mathfrak{g} \otimes A \otimes \Omega_\bullet).
\]

This permutative FMP can also be interpreted as follows. Observe that for every permutative algebra $B$, there is a nonunital operad $P \otimes B$, such that:

- the underlying symmetric sequence is given by $(P \otimes B)(n) := P(n) \otimes B$;
- the composition operation reads as
  \[
  (\psi_0 \otimes b_0) \circ (\psi_1 \otimes b_1, \ldots, \psi_p \otimes b_p) := \pm (\psi_0 \circ (\psi_1, \ldots, \psi_p)) \otimes b_0 \cdot b_1 \cdots b_p.
  \]

Associativity of the composition follows from the associativity of permutative algebras and the permutative axiom (3.2), while the equivariance/commutativity directly follows from (3.2).

Now let $\Omega C$ denote the augmentation ideal of the cobar construction of $C$ [LV, Section 6.5] (or see Section 9.2) and consider the functor
\[
F: \text{Alg}_{\text{perm}}^{\text{fin}} \longrightarrow S; \quad A \mapsto \text{Map}_{\text{Op}^{nu}}(\Omega C, P \otimes A)
\]

sending every small permutative algebra to the space of nonunital operad maps $\Omega C \longrightarrow P \otimes A$. This space of maps can be described by the simplicial set whose $n$-simplices are maps $\Omega^{n} C \longrightarrow P \otimes A \otimes \Omega_\bullet$. Using that $P \otimes (-)$ sends homotopy pullbacks of permutative algebras to homotopy pullbacks of nonunital operads, one sees that $F$ is a permutative FMP. One can think of $F$ as the FMP describing permutative deformations of the trivial map of operads $\Omega C \longrightarrow k$.

Proposition 3.6. Let $A$ be a finite-dimensional nilpotent permutative algebra. Then there is a natural equivalence of spaces
\[
\text{MC}_g(A) \overset{\sim}{\longrightarrow} F(A) = \text{Map}_{\text{Op}^{nu}}(\Omega C, P \otimes A).
\]

In other words, the convolution pre-Lie algebra $\mathfrak{g}$ classifies deformations of the trivial operad map $\Omega C \longrightarrow P$.

Proof. We compute both sides of (3.7) using simplicial resolutions. For the codomain, fix a simplicial resolution $\mathcal{P}_\bullet$ of the operad $\mathcal{P}$. The mapping space can be presented by the simplicial set of nonunital operad maps $\Omega C \longrightarrow \mathcal{P}_\bullet \otimes A$.

By Remark 7.21, the space $\text{MC}_g(A)$ can be presented by the simplicial set $\text{MC}(A \otimes \mathfrak{g}_\bullet)$ for any simplicial resolution $\mathfrak{g}_\bullet$ of the pre-Lie algebra $\mathfrak{g}$. One such resolution is given by $\mathfrak{g} \otimes \Omega_\bullet$ (as used above), but for convolution pre-Lie algebras one can also use
\[
\mathfrak{g}_\bullet = \prod_{p \geq 0} \text{Hom}(\mathcal{C}(p), \mathcal{P}_\bullet(p)) \Sigma_p.
\]
Now recall that nonunital operad maps \( \Omega C \to P \otimes A \) correspond bijectively to twisting morphisms \( C \to P \otimes A \) [LV, Theorem 6.5.10]. Since \( A \) is finite-dimensional, these twisting morphisms correspond to Maurer–Cartan elements of the convolution pre-Lie algebra \( g \otimes A \) (cf. Remark 9.18), so that we obtain an isomorphism of simplicial sets

\[
MC(g \otimes A) \cong \text{Hom}_{\text{Op}^{nu}}(\Omega C, P \otimes A).
\]

This isomorphism of simplicial sets presents the equivalence (3.7). \( \square \)

### 3.2 Operadic deformation theory

In the previous section, we have seen how the convolution pre-Lie algebra

\[
g = \prod_{p \geq 0} \text{Hom}(C(p), P(p))_{\Sigma_p}
\]

classifies permutative deformations of the trivial map of operads \( \Omega C \to k \to P \). In fact, the pre-Lie algebra \( g \) arises from an even richer algebraic structure: the sequence of mapping complexes \( \text{Hom}(C(p), P(p)) \) forms a nonunital operad, the convolution operad of \( C \) and \( P \).

In this section, we will explain how this additional algebraic structure can be understood from a deformation theoretic point of view. More precisely, we will describe a commuting square of \( \infty \)-categories (see Corollary 3.16)

\[
\begin{array}{ccc}
\text{Op}^{\text{ns}} & \xrightarrow{\sim} & \text{FM}_{\text{Op}^{\text{sym}}} \\
\downarrow & & \downarrow \\
\text{preLie} & \xrightarrow{\sim} & \text{FM}_{\text{Perm}}
\end{array}
\]

where the left vertical functor sends a nonunital operad \( P \) to \( \prod_{p \geq 0} P(p)_{\Sigma_p} \) and the right vertical functor restricts operadic formal moduli problems to premutative formal moduli problems. The convolution operad then classifies deformations of \( \Omega C \to P \) not just along permutative algebras, but also along nonunital operads.

**Operads for operads**

To see how deformation problems parametrized by operads fit into the framework of Theorem 1.3, let us recall how (unicoloured) operads themselves are algebras over a coloured operad.

**The operad of nonsymmetric operads.** Let us consider the linear category \( k^{\text{ns}} \) having nonnegative integers as objects and morphisms

\[
k^{\text{ns}}(m, n) := \begin{cases} 
0 & \text{if } q \neq p \\
1 & \text{else.}
\end{cases}
\]

Note that \( k^{\text{ns}} \)-operads are just (augmented) \( \mathbb{Z}_{\geq 0} \)-coloured operads. In particular, there is a quadratic \( k^{\text{ns}} \)-operad \( O^{\text{ns}} \) such that \( O^{\text{ns}} \)-algebras are nonunital nonsymmetric operads (see Definition 8.1(1)). The \( k^{\text{ns}} \)-operad \( O^{\text{ns}} \) is Koszul self-dual [VdL03, Theorem 4.3].

There is a close relation between the operad \( O^{\text{ns}} \) and the operad \( \text{Perm} \). To see this, let us consider the \( k^{\text{ns}} \)-symmetric sequence \( E \) defined by

\[
E(n_1, \ldots, n_p; n_0) := k.
\]
This sequence gives rise to a tensor-hom adjunction between $k$-linear symmetric sequences and $k^{ns}$-symmetric sequences, given explicitly by

$$(E \otimes M)(n_1, \ldots, n_p; n_0) = M(p)$$

and

$$\text{Hom}_k(E, N)(p) = \prod_{n_0, n_1, \ldots, n_p} N(n_1, \ldots, n_p; n_0).$$

One easily sees that $E \otimes -$ is strong monoidal for the composition product, so that it sends $k$-operads to $k^{ns}$-operads, and similarly for algebras.

As we have already seen in Section 3.1, every permutative algebra defines a nonsymmetric operad which is constant as a $\mathbb{Z}_{>0}$-sequence. This fact is reflected in the existence of a map of $k^{ns}$-operads

$$O^{ns} \to E \otimes \text{Perm}.$$

In concrete terms, it sends every generator $\circ_i$ of $O^{ns}$ to the generating operation of $\text{Perm}$; one easily sees that the defining relation (8.2) is sent to the associativity relation, while the defining relation (3.2) is sent to the permutative relation (3.2).

The operad of (symmetric) operads. Let us introduce another linear category $k[\Sigma]$, having objects the nonnegative integers, and

$$k[\Sigma](p, q) := \begin{cases} 0 & \text{if } q \neq p \\ k[\Sigma_p] & \text{else} \end{cases}$$

There is a quadratic $k[\Sigma]$-operad $O^{sym}$ such that $O^{sym}$-algebras are nonunital symmetric operads. As a symmetric $k[\Sigma]$-bimodule, there is an isomorphism (see Definition 8.1(2) and [DV15, Proposition 1])

$$O^{sym} \cong k[\Sigma] \circ O^{ns}.$$

The operad $O^{sym}$ is Koszul self-dual as a $k[\Sigma]$-operad (Corollary 8.12). Furthermore, it can be related to the operad $O^{ns}$ for nonsymmetric operads using the following construction:

Construction 3.10. Let us denote by $\text{triv} \dashv \text{inv}$ the pair of adjoint functors between symmetric $k^{ns}$-bimodules and symmetric $k[\Sigma]$-bimodules given as follows:

$$\text{triv}(M)(n_1, \ldots, n_k; n_0) = M(n_1, \ldots, n_k; n_0)$$

equipped with the trivial $\Sigma_{n_0} \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-module structure, while

$$\text{inv}(N)(n_1, \ldots, n_k; n_0) = N(n_1, \ldots, n_k; n_0) \Sigma_{n_0} \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$$

is the $\Sigma_{n_0} \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-invariant submodule. The left adjoint $\text{triv}$ is strong monoidal, so that the adjunction can be promoted to an adjunction between $k^{ns}$-operads and $k[\Sigma]$-operads.

It follows from the $k[\Sigma]$-distributive law defining $O^{sym}$ that there is a morphism

$$O^{sym} \to \text{triv}(O^{ns}),$$

reflecting the fact that every nonsymmetric operad defines a symmetric operad where the symmetric groups act trivially. Every morphism of $k^{ns}$-operads $O^{ns} \to P$ therefore gives rise to a morphism of $k[\Sigma]$-operads $O^{sym} \to \text{triv}(P)$. In particular, we have a morphism

$$O^{sym} \to \text{triv}(E \otimes_k \text{Perm}),$$

which reflects the fact that every permutative algebra leads to a symmetric operad, which is constant as a symmetric sequence and with trivial actions of the symmetric groups.
Connecting permutative and operadic deformation theories

Because of the Koszul self-duality of $O_{\text{ns}}$ and $O_{\text{sym}}$, Theorem 1.3 provides equivalences

\[ \text{Op}^{\text{nu}, \text{ns}} \simeq \text{FMP}_{O_{\text{ns}}} \quad \text{Op}^{\text{nu}} \simeq \text{FMP}_{O_{\text{sym}}} \]

between (nonsymmetric) nonunital operads and (nonsymmetric) operadic formal moduli problems. The latter are FMPs indexed by nonunital (nonsymmetric) operads whose cohomology forms a nilpotent operad of total finite dimension. In this paragraph we would like to understand how these equivalences are related to the permutative deformation theory discussed in Section 3.1.

Let us concentrate on the symmetric case and first introduce a piece of notation:

\[ L := \text{triv}(E \otimes -) \quad \text{and} \quad R = \text{Hom}_k(E, \text{inv}(-)) . \]

Formulas (3.8) and (3.11) show that tensoring with $E$ and applying $\text{triv}$ both preserve quadratic duality. The sequence of $k[\Sigma]$-operads

\[ O_{\text{sym}} \rightarrow \text{triv}(O_{\text{ns}}) \rightarrow L(\text{Perm}) \]

therefore induces a Koszul dual sequence

\[ L(\text{preLie}) \rightarrow \text{triv}(O_{\text{ns}}) \rightarrow O_{\text{sym}} \]

telling us that every symmetric operad is naturally a $\mathbb{Z}_{\geq 0}$-graded pre-Lie algebra (endowed with trivial $\Sigma_p$-actions on its graded pieces).

More precisely, let $\mathcal{P}$ be a symmetric operad, that is to say $k[\Sigma]$-symmetric sequence concentrated in arity 0 together with the structure of a left $O_{\text{sym}}$-module $O_{\text{sym}} \circ_k \mathcal{P} \rightarrow \mathcal{P}$. By restriction, we get a graded pre-Lie algebra structure $L(\text{preLie}) \circ_k \mathcal{P} \rightarrow \mathcal{P}$ on $\mathcal{P}$. Using the unit of the adjunction $L \dashv R$ and lax monoidality of $R$ (which follows from the fact that $\text{triv}$ and tensoring with $E$ are both strong monoidal), we have a map

\[ \text{preLie} \circ_k R(\mathcal{P}) \rightarrow RL(\text{preLie}) \circ_k R(\mathcal{P}) \rightarrow R(L(\text{preLie}) \circ_k \mathcal{P}) \rightarrow R(\mathcal{P}). \]

This defines a pre-Lie algebra structure on $R(\mathcal{P}) = \prod_n \mathcal{P}(n)^{\Sigma_p}$.

**Remark 3.12.** In the nonsymmetric case, the map $O_{\text{ns}} \rightarrow E \otimes \text{Perm}$ determines a quadratic dual map $E \otimes \text{preLie} \rightarrow O_{\text{ns}}$, reflecting that every nonsymmetric operad $\mathcal{P}$ forms a $\mathbb{Z}_{\geq 0}$-graded pre-Lie algebra. Using lax monoidality of the hom functor (3.9), one finds that $\text{Hom}_{O_{\text{ns}}}(E, \mathcal{P}) = \prod_{p \geq 0} \mathcal{P}(p)$ has a natural pre-Lie structure as well.

We will now describe how the functor $R : \text{Op} \rightarrow \text{preLie}$ arises from the point of view of deformation theory. We will do this in two steps, resulting in Corollary 3.16:

(1) We relate permutative FMPs (equivalently: pre-Lie algebras) to FMPs over $L(\text{Perm})$-algebras (equivalently: $L(\text{preLie})$-algebras).

(2) We relate $L(\text{Perm})$-algebraic FMPs to operadic FMPs.

Since (2) is an immediate application of the functoriality results from Section 6, we will continue with the first step, which is a little more subtle (as we do not have a general enough functoriality result for changing the base dg-category).
Working slightly more generally, let $Q$ be a $k$-linear Koszul operad, which will play the role of $\text{Perm}$. When restricted to symmetric sequences concentrated in arity 0, the adjunction $L \dashv R$ lifts, for every $k$-operad $Q$, to a Quillen adjunction $L: \text{Alg}_{Q}^{d_{g}} \rightarrow \text{Alg}_{L(Q)}^{d_{g}} : R$.

Note that the induced right adjoint functor of $\infty$-categories preserves small algebras, since it preserves pullbacks and trivial algebras.

On the other hand, the left adjoint simply sends a $Q$-algebra $A$ to the symmetric sequence $L(A)(p) = A$ with the trivial $\Sigma_{p}$-action. Taking such constant symmetric sequences preserves tensor products and linear duals, and hence commutes with the functor taking the dual of the (operadic) bar construction. In other words, we obtain a commuting diagram of $\infty$-categories

$$
\begin{array}{ccc}
\text{Alg}_{Q}^{op} & \xrightarrow{B(-)^{\vee}} & \text{Alg}_{Q}^{!} \\
L & \downarrow & \downarrow L
\end{array}
\rightleftharpoons
\begin{array}{ccc}
\text{Alg}_{L(Q)}^{op} & \xrightarrow{B(-)^{\vee}} & \text{Alg}_{L(Q)}^{!} \\
\hat{L} & \downarrow \hat{L}
\end{array}
\xrightarrow{\sim}
\begin{array}{ccc}
\text{FMP}_{Q} & \xrightarrow{\sim} & \text{FMP}_{L(Q)}.
\end{array}
$$

The composite horizontal functors have a very simple description: they send a $Q$-algebra $A$ to the formal moduli problem $\text{Spf}(A) = \text{Map}_{Q}(A, -)$ of Example 4.12. Consequently, the right vertical functor $\hat{L}$ sends the formal moduli problem corepresented by a small $Q$-algebra $A$ to the formal moduli problem $\text{Spf}(\hat{L}(A)) = \text{Map}(\text{Spf}(\hat{L}(A)), F) \simeq \lim_{p \rightarrow \infty} F(L(A)_{\leq p})$.

Passing to right adjoints, we now obtain a commuting square

$$
\begin{array}{ccc}
\text{Alg}_{Q} & \xrightarrow{\sim} & \text{FMP}_{Q} \\
R & \uparrow & \uparrow R^{*}
\end{array}
\rightleftharpoons
\begin{array}{ccc}
\text{Alg}_{L(Q)}^{!} & \xrightarrow{\sim} & \text{FMP}_{L(Q)}
\end{array}
$$

where the right vertical functor sends a formal moduli problem $F$ over $L(Q)$ to the formal moduli problem

$$
\hat{L}^{*}F(A) = \text{Map}(\text{Spf}(L(A)), F) \simeq \lim_{p \rightarrow \infty} F(L(A)_{\leq p}).
$$

In other words, $\hat{L}^{*}F$ parametrizes deformations along (pro-small) $L(Q)$-algebras of the form $L(A)$, with $A$ a small $Q$-algebra.

**Remark 3.15.** Note that for any small $Q$-algebra $A$, there is an equivalence of $L(Q)$-algebraic formal moduli problems

$$
\text{Map}_{L(Q)}(L(A), -) \simeq \text{Map}_{Q}(A, R(-)).
$$

---

1Technically, the $L(Q)$-algebras $L(A)_{\leq p}$ are not small: their underlying $k[\Sigma]$-module is not quasi-free, since all symmetric groups act trivially. However, they are retracts of small algebras and one can equivalently define formal moduli problems to take such retracts as input (cf. Remark 8.15).
Since the $\infty$-category of formal moduli problems is generated by the corepresentable ones, it follows formally from this that $\hat{L} = R^*$ is simply given by restriction along

$$R: \text{Alg}_{L(Q)}^{\text{sm}} \longrightarrow \text{Alg}_Q^{\text{sm}}.$$  

**Corollary 3.16.** There is a commuting square of $\infty$-categories

$$\begin{array}{ccc} \text{Op}_{\text{sym}}^{\text{nu}} & \longrightarrow & \text{FMP}_{\text{sym}} \\ \downarrow R & & \downarrow \hat{L}^* \\ \text{preLie} & \longrightarrow & \text{FMP}_{\text{perm}}. \end{array}$$

Here the left vertical functor sends $\mathcal{P} \mapsto \prod_{p \geq 0} \mathcal{P}(p)^{\Sigma_p}$ and the right vertical functor sends an operadic formal moduli problem $F$ to the permutative formal moduli problem (3.14).

In other words, a pre-Lie algebra $\mathfrak{g}$ arises as $\mathfrak{g} = \prod_{p \geq 0} \mathfrak{g}(p)^{\Sigma_p}$ if the corresponding permutative deformation problem lifts to an operadic deformation problem. In this case, the deformation problem classified by $\mathfrak{g}$ is obtained by restricting to the operads $L(A)(p) = A$ described in Section 3.1, with $A$ a small permutative algebra.

**Proof.** Compose the square (3.13), with $Q = \text{Perm}$ and $Q^! = \text{preLie}$, with the square

$$\begin{array}{ccc} \text{Op}_{\text{sym}}^{\text{nu}} & \longrightarrow & \text{FMP}_{\text{sym}} \\ \downarrow R & & \downarrow \hat{L}^* \\ \text{Alg}_{L(\text{preLie})}^{\text{sym}} & \longrightarrow & \text{FMP}_{L(\text{Perm})} \end{array}$$

obtained by applying Proposition 6.10 to the map of $k[\Sigma]$-operads $\mathcal{O}_{\text{sym}} \longrightarrow L(\text{Perm})$.

**Deforming trivial morphisms of operads (continued)**

The convolution pre-Lie algebras from the previous subsection typically arise as images of a (convolution) $k$-operad through the functor $R$. Let us recall the setup of Section 3.1.

We were given a $k$-cooperad $\mathcal{C}$ and a $k$-linear operad $\mathcal{P}$, and considered the convolution pre-Lie algebra

$$\mathfrak{g} := \prod_{p \geq 2} \text{Hom}_k(\mathcal{C}(p), \mathcal{P}(p))^{\Sigma_p} = R(\text{Conv}(\mathcal{C}, \mathcal{P})),$$

where $\text{Conv}(\mathcal{C}, \mathcal{P})(p) := \text{Hom}_k(\mathcal{C}(p), \mathcal{P}(p))$. As we have seen in Proposition 3.6, the deformation functor associated with $\mathfrak{g}$ sends a very small permutative algebra $A$ to the space of twisting morphisms $\mathcal{C} \longrightarrow \mathcal{P} \otimes A$. Equivalently, this is the space of nonunital operad maps $\mathcal{C} \longrightarrow \mathcal{P} \otimes A$, or when $\mathcal{C} = B(\mathcal{Q})$, the space of nonunital $\infty$-morphisms $\mathcal{Q} \sim \mathcal{P} \otimes A$.

Observe that $\mathcal{P} \otimes A$ is nothing but the Hadamard tensor product $\mathcal{P} \otimes_H L(A)$, where $L(A)$ is an operad via the morphism $\mathcal{O}_{\text{sym}} \longrightarrow L(\text{Perm})$. More generally, one can associate to any small nonunital operad $\mathcal{R}$ the space of of twisting morphisms $\mathcal{C} \longrightarrow \mathcal{P} \otimes_H \mathcal{R}$.

Again, this is equivalent to the space of nonunital operad maps $\mathcal{O}(\mathcal{C}) \longrightarrow \mathcal{P} \otimes_H \mathcal{R}$ or, whenever $\mathcal{C} = B(\mathcal{Q})$ is the reduced bar construction of a $k$-operad, to the space of $\infty$-morphisms of operads $\mathcal{Q} \sim \mathcal{P} \otimes_H \mathcal{R}$. This defines an operadic deformation problem, classified by the convolution operad $\text{Conv}(\mathcal{C}, \mathcal{P})$, which extends the permutative deformation problem classified by the underlying convolution pre-Lie algebra of $\text{Conv}(\mathcal{C}, \mathcal{P})$. 

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3.3 Splendid operads

The main technical condition of Theorem 1.3 is Condition (3), which asserts that the operad is splendid in the following sense:

**Definition 3.17.** Let \( P \) be a \( k \)-operad. We will say that \( P \) is splendid if its 0-reduced part \( P^\geq 1 \) satisfies the following condition: the derived relative composition product

\[
P(1) \circ_{P^\geq 1}^h P(1)
\]

is eventually highly connective (Definition 2.3).

**Remark 3.18.** At least for connective \( P \), this definition should be considered as a homotopy-invariant reformulation of the following condition: \( P^\geq 1 \) admits a free resolution whose generators are in increasingly negative degrees (as the arity increases). See Section 9.4.

An immediate natural question to ask is therefore whether a given operad is splendid. Let us start by making some general observations about the property of being splendid. First of all, let us observe that more Koszul operads are splendid than just the binary ones considered in the Introduction, so that Theorem 1.1 applies to these as well:

**Observation 3.19.** A (non-necessarily binary) Koszul quadratic operad \( T(E)/(R) \) living in nonpositive degrees generated by a symmetric sequence \( E \) with generators in bounded arity (i.e. \( E(n) = 0 \) for \( n \gg 0 \)) is splendid. Indeed, its Koszul resolution has generators sitting in increasingly negative degrees by the same argument as in the proof of Theorem 1.1.

**Example 3.20 (and non-example).** A quadratic operad that does not fit the constraints of the previous observation is the gravity operad \( \text{Grav} \) [Get94, Theorem 4.5]. The operad \( \text{Grav} \) is generated by a sequence \( E \) such that \( E(n) \) is 1-dimensional and concentrated in degree \(-1\). Clearly such an operad cannot be splendid, as the generators of a resolution need to cover all generators of \( \text{Grav} \).

In fact, there is some ambiguity in the literature regarding the degrees of these operads. We denote by \( \text{Grav} \) what we will also call the gravity operad, which has the same quadratic presentation but with generators \( V(n) \) a 1-dimensional space concentrated in degree \( 2 - n \). In other words, \( \text{Grav} \) is obtained from \( \overline{\text{Grav}} \) by reversing the degrees and operadically shifting down by 1.

The operad \( \text{Grav} \) is Koszul and its Koszul dual is the operad \( \text{HyperCom} \) of hypercommutative algebras [Get95], generated by one operation in arity \( n \) in degree \( 2(n-2) \) for all \( n \geq 2 \). It follows that \( \text{Grav} \) is splendid and from Theorem 1.3 we deduce that the \( \infty \)-category \( FMP_{\text{Grav}} \) is equivalent to the \( \infty \)-category of hypercommutative algebras.

Note that one cannot exchange the roles of \( \text{Grav} \) and \( \text{HyperCom} \) in this statement: \( \text{HyperCom} \) is not splendid and Theorem 1.3 does not hold for hypercommutative formal moduli problems.

**Remark 3.21.** Suppose that \( P \) is a 1-coloured augmented operad which is 1-reduced:

\[
P(0) = 0 \quad \text{and} \quad P(1) = k \cdot 1.
\]

If \( P \) is connective, then the shifted operad \( P\{1\} \) satisfies the conditions of Theorem 1.3.

Next, note that an operad typically satisfies the conditions of Theorem 1.3 as soon as its cohomology does:

**Lemma 3.22.** Let \( P \) be a (coloured) connective operad over \( \mathbb{Q} \). If \( H^*(P) \) is splendid, then \( P \) is splendid as well.
Proof. We can assume that $\mathcal{P}$ is 0-reduced and consider the simplicial resolution

$$\cdots \longrightarrow (\mathcal{P}(1) \circ \mathcal{P}(1)) \longrightarrow (\mathcal{P}(1) \circ \mathcal{P}(1)) \longrightarrow (\mathcal{P}(1) \circ \mathcal{P}(1)).$$

Taking (at each tuple of colours) the corresponding normalized cochains, we obtain a (cohomologically) $\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\leq 0}$-graded bicomplex, with an associated convergent spectral sequence

$$E^{r,s}_2 = H^r\left( H^s(\mathcal{P}(1) \circ_{H^*\mathcal{P}}^b H^s(\mathcal{P}(1)) \right) \longrightarrow H^{r+s}(\mathcal{P}(1) \circ_{\mathcal{P}}^b \mathcal{P}(1)).$$

Here $H^s(\mathcal{P}(1) \circ_{H^*\mathcal{P}}^b H^s(\mathcal{P}(1))$ is computed in the category of (nonpositively) graded symmetric sequences of complexes. If $H^*\mathcal{P}$ is splendid, then $p$-ary part of the $E_1$-page is concentrated in degrees $s \leq 0$ and $r \leq f(p) \leq 0$, with $f(p) \xrightarrow{p \to \infty} -\infty$. Then the $p$-ary part of $\mathcal{P}(1) \circ_{\mathcal{P}}^b \mathcal{P}(1)$ is also concentrated in cohomological degrees $\leq f(p)$, and we conclude that $\mathcal{P}$ is splendid. \qed

Example 3.23 (Variants of the little discs operads). Using this lemma one can show that the little $n$-discs operad $E_n$ is splendid without having to use that it is quasi-isomorphic to its homology $e_n$. Indeed, $e_n$ is a binary quadratic Koszul operad [LV, Section 13.3.16] and is therefore splendid. In the next section we will show that the homology of the framed little $n$-discs operad is also splendid.

Another application of Lemma 3.22 involves the result of Hoefel and Livernet [HL13] that the homology of the Swiss–Cheese operad ($\mathcal{S}^{\text{cofree}}$ in loc. cit.) is a quadratic binary Koszul colored operad. This shows the Swiss–Cheese operad is splendid, even though we do not know a simple model for its dual.

In the following subsections we will look at a few examples in a bit more detail.

The homology of the framed little discs operad

The homology of the (0-reduced) framed little $n$-discs operad $H_n(\mathcal{E}_n^{fr}) = e_n = e_n \rtimes H_*(\text{SO}_n)$ can be expressed as the semi-direct product of the ordinary little $n$-discs operad $e_n$ with the Hopf algebra $H_*(\text{SO}_n)$ [SW03].

As we will see, the framed little discs operads are in the conditions of the main Theorem, essentially due to the following lemma expressed in terms of a distributive law (in the sense of [LV, Section 8.6]).

Lemma 3.24. Let $\mathcal{P}$ be a connective splendid operad and let $H$ be a Hopf algebra given by an universal enveloping algebra $U(\mathfrak{g})$ of an abelian Lie algebra $\mathfrak{g}$ concentrated in nonpositive degrees. Then, seeing $H$ as an operad concentrated in arity 1, the map induced by its coproduct $\Delta: H \circ \mathcal{P} \to \mathcal{P} \circ H$ is a distributive law and the corresponding operad $\mathcal{P} \circ_{\Delta} H$ is a splendid operad.

Proof. Property (I) from [LV, 8.6.1] is verified due to the coassociativity and cocommutativity of the coproduct in $H$, while property (II) follows from the compatibility of the product and the coproduct in $H$. It follows that $\Delta$ is indeed a distributive law.

Let $\Omega(\mathcal{C}) \to \mathcal{P}$ be a cofibrant resolution with $\mathcal{C}$ in increasingly negative degrees. There are natural quasi-isomorphisms of associative algebras $U_{\mathfrak{g}} \cong U\mathcal{L}\mathcal{C}_{\mathfrak{g}} = \Omega\mathcal{C}_{\mathfrak{g}}$, where $\mathcal{L}$ and $\mathcal{C}$ denote the unshifted cobar-bar adjunction with respect to the Lie-cocommutative structures:

$$\mathcal{L}: \{\text{cocommutative coalgebras}\} \xrightarrow{\cong} \{\text{Lie algebras}\} : \mathcal{C}.$$  

We consider their unshifted versions in order to be compatible with the operadic bar-cobar adjunctions.\footnote{In the sense that the convention is that the bar construction of an operad is a cooperad and not a shifted cooperad.} Concretely, since $\mathfrak{g}$ is abelian, $\mathcal{C}_{\mathfrak{g}} = S^0(\mathfrak{g}[1])$ has a natural Hopf algebra
structure. We denote by $m: C \circ \mathfrak{g} \to \mathfrak{g} \circ C$ the map induced by the product on $\mathfrak{g}$, which is a codistributive law, by a similar argument as before.

Notice that since $\mathfrak{g}$ is concentrated in non-positive degrees, $\Omega(C \circ_m \mathfrak{g})$ is splendid. We claim that the twisting morphisms $\phi: C \to \mathcal{P}$ and $\psi: \mathfrak{g} \circ C \to H$ induce a quasi-isomorphism $\Omega(C \circ_m \mathfrak{g}) \to \mathcal{P} \circ_\Delta H$.

First we have to show that $\phi \circ \psi: C \circ_m \mathfrak{g} \to \mathcal{P} \circ_\Delta H$ is a twisting morphism. We notice that $\phi \circ \psi$ can be expressed as a sum of two maps $\bar{\phi} + \bar{\psi}$ where

\[
\bar{\phi}: C \circ_m \mathfrak{g} \to \mathcal{P} \circ C \mathfrak{g} \to \mathcal{P} \circ H \to \mathcal{P} \circ_\Delta H \\
\bar{\psi}: C \circ_m \mathfrak{g} \to C \circ (\mathfrak{g} \circ \mathfrak{g}) \to \mathcal{P} \circ (H \otimes H) \to \mathcal{P} \circ_\Delta H.
\]

In particular, $\bar{\phi}$ and $\bar{\psi}$ commute. It follows that

\[
\partial(\bar{\phi} + \bar{\psi}) + \frac{1}{2}[\bar{\phi} + \bar{\psi}, \bar{\phi} + \bar{\psi}] = \left( \partial \bar{\phi} + \frac{1}{2}[\bar{\phi}, \bar{\phi}] \right) + \frac{1}{2}[\bar{\psi}, \bar{\psi}] + [\bar{\phi}, \bar{\psi}] = 0 + 0 + 0.
\]

To see that $\Omega(C \circ_m \mathfrak{g}) \to \mathcal{P} \circ_\Delta H$ is a quasi-isomorphism, we consider the Koszul complex $\mathcal{P} \circ_\Delta H$. By filtering with respect to the degree in $\mathfrak{g}$, using the Koszulity of $\phi$ we recover on the first page of the spectral sequence the Koszul complex of $\psi$, which vanishes on the second page of the spectral sequence, thus concluding the proof. \hfill $\square$

### The even dimensional framed little discs operad

In the case $n = 2$ we recover the Batalin–Vilkovisky operad $\mathcal{B}V$. More generally, in the even case $n = 2k$, $H_*(SO_{2k})$ is the symmetric algebra on the Pontryagin classes $p_3, \ldots, p_{4i-1}, \ldots, p_{4k-5}$ and the Euler class $\chi_{2k-1}$ living in degrees symmetric to their indices. Homologically, the Pontryagin classes act trivially while the Euler class acts in the same way as the BV operator in the case $n = 2$. In particular we have that the operad $e_{2k}^f$ can be expressed in terms of a distributive law $e_{2k}^f = \mathcal{B}V_{2k} \circ_\Delta S^c(p_3, \ldots, p_{4k-5})$.

Here, the map $\Delta: S^c(p_3, \ldots, p_{4k-5}) \circ \mathcal{B}V_{2k} \to \mathcal{B}V_{2k} \circ S^c(p_3, \ldots, p_{4k-5})$ is induced by the cocommutative coproduct and $\mathcal{B}V_{2k}$ is the operad with the following quadratic-linear presentation:

- There are two binary generators $m$ of degree 0 and $l$ of degree $-2k + 1$ subject to the Pois$_{2k}$ relations.
- There is a unary generator $\Delta$ subject to $\Delta \circ \Delta = 0$.
- There is a quadratic-linear relation $l = \Delta \circ m - m \circ_1 \Delta - m \circ_2 \Delta$.

This quadratic-linear presentation is Koszul as one can see by adapting the proof done in [GCTV12] for $k = 1$. The Koszul resolution has generators in increasingly negative degrees and therefore $\mathcal{B}V_{2k}$ is splendid. It follows from Lemma 3.24 that $e_{2k}^f$ is a splendid operad.

### The odd dimensional framed little discs operad

For odd $n$ the action of $SO_n$ on $\mathbb{E}_n$ is homologically trivial. As a Hopf algebra, $H_*(SO_{2k})$ is the symmetric algebra on the Pontryagin classes $p_3, \ldots, p_{4i-1}, \ldots, p_{2n-3}$ and therefore we have a distributive law $e_{n}^f = e_n \circ_\Delta S(p_3, \ldots, p_{2n-3})$, where $\Delta$ denotes the Hopf cocommutative coproduct as before.

Since $e_n$ is a splendid operad, we are again in the conditions of Lemma 3.24 so we conclude that $e_{n}^f$ is splendid.
Remark 3.25. For odd $n$, the action of $\text{SO}_n$ on $E_n$ is not homotopically trivial. In [KW17], Khoroshkin and Willwacher show that for odd $n$ the framed little discs operad is not formal over $\mathbb{R}$. They also show that for even $n$ there is formality over $\mathbb{R}$ of $E_n$. Note that Lemma 3.22 and the above discussion imply that Theorem 1.3 also holds for $E_n$, even if it is not formal.

The BD-operad

For each $n \geq 0$, there is a $k[\hbar]$-linear operad $BD_n$ which agrees with the (0-reduced) $E_n$-operad away from $\hbar = 0$ and with the (0-reduced) shifted Poisson operad at $\hbar = 0$ [CPT+17]:

$$BD_n \otimes_{k[\hbar]} k[\hbar^\pm] \simeq E_n[k^\pm], \quad BD_n \otimes_{k[\hbar]} k[\hbar]/\hbar \simeq \text{Pois}_n.$$ 

For example, the $BD_0$-operad from [CG16, CG18b] (see also [BD04]) is the $k[\hbar]$-operad generated by a commutative product and a Lie bracket of degree 1 satisfying the Leibniz rule, and equipped with the differential $d(-,-) = \hbar[-,-]$.

Similarly, the operad $BD_1$ is obtained as the Rees construction of the associative operad, equipped with the PBW-filtration [CG18b, CPT+17]; explicitly, a $BD_1$-algebra is a $k[\hbar]$-module equipped with a (nonunital) associative product $*$ and a Lie bracket $[-,-]$ satisfying

$$[a, b * c] = [a, b] * c + b * [a, c] \quad a * b - b * a = \hbar[a, b].$$

Proposition 3.26. The $k[\hbar]$-operads $BD_0$ and $BD_1$ are Koszul self-dual

$$\mathcal{O}(BD_0) \simeq BD_0 \quad \mathcal{O}(BD_1) \simeq BD_1 \{-1\}$$

(relative to $k[\hbar]$) and satisfy the conditions of Theorem 1.3, so that there are equivalences

$$\text{FMP}_{BD_n} \xrightarrow{\sim} \text{Alg}_{BD_n} ; \quad X \longrightarrow TX[-n].$$

Proof. Case $n = 0$: note that $BD_0 = \text{Free}(E)/R$ is a binary quadratic operad on two generators $\mu = (-,-)$ and $\lambda = [-,-]$, with differential $d\mu = \hbar \cdot \lambda$. Since the relations are the ones of the usual Poisson operad, its quadratic dual $BD_0' = \text{Free}(E')/R$ is isomorphic to $BD_0\{1\}$. To see that it is Koszul, it suffices to see that

$$BD_0' = \text{coFree}(E[1], R[2]) \longrightarrow B(BD_0)$$

is a quasi-isomorphism of $k[\hbar]$-modules. To see this, it suffices to verify that the maps

$$BD_0' \otimes_{k[\hbar]} k[\hbar^\pm] \longrightarrow B(BD_0) \otimes_{k[\hbar]} k[\hbar^\pm]$$

$$BD_0' \otimes_{k[\hbar]} k[\hbar]/\hbar \longrightarrow B(BD_0) \otimes_{k[\hbar]} k[\hbar]/\hbar$$

are both quasi-isomorphisms (by derived Nakayama, the second condition implies that the localizations at $\hbar = 0$ are quasi-isomorphic). Because extension of scalars is symmetric monoidal and all $BD_0(p)$ and $BD_0'(p)$ are finite complexes of free $k[\hbar]$-modules (so that we do not have to derive the tensor product), the above two maps agree with the maps

$$\left( BD_0 \otimes_{k[\hbar]} k[\hbar^\pm] \right)^i \longrightarrow B\left( BD_0 \otimes_{k[\hbar]} k[\hbar^\pm] \right)$$

$$\left( BD_0 \otimes_{k[\hbar]} k[\hbar]/\hbar \right)^i \longrightarrow B\left( BD_0 \otimes_{k[\hbar]} k[\hbar]/\hbar \right)$$
The first map is a quasi-isomorphism between two cooperads which are both quasi-isomorphic to the trivial cooperad $k[h]$, while the second map is a quasi-isomorphism because $BD_0 \otimes_k [h] k[h]/h \cong P_0$ is a quadratic Koszul operad.

**Case** $n = 1$: note that $BD_1 = \text{Free}(E)/R$ is a quadratic operad on two binary generators $\mu = - \cdot -$, $\lambda = [\cdot, \cdot]$, on which $\Sigma_2$ acts trivially, resp. by the sign representation. The module of relations $R$ is generated by:

(J) Jacobi relation $[a, [b, c]] + [b, [c, a]] + [c, [a, b]]$.

(L) Leibniz rule $[a, b \cdot c] - c \cdot [a, b] + b \cdot [a, c]$.

(A) associativity for $a \ast b = a \cdot b + h[a, b]$, or explicitly:

$$
(a \cdot (b \cdot c) - c \cdot (a \cdot b)) + h\left(a \cdot [b, c] - c \cdot [a, b] + [a, b \cdot c] + [c, a \cdot b]\right) + h^2\left([a, [b, c]] + [c, [a, b]]\right).
$$

Note that $\text{Free}(E)(p)$ and $BD_1(p)$ are finitely generated projective (equivalently, torsion free) $k[h]$-modules for all $p$; for $BD_1(p)$ this follows from the fact that it arises as the Rees construction of a vector space with an increasing filtration. It follows that $R$ is also finitely generated and projective. Note that $R$ has rank 6, since its fiber at $h = 0$ is the vector space of relations for the Poisson operad $P_1$, which has dimension 6.

Now consider the inner product on $E$ of signature $(1, 1)$, determined by $\langle \mu, \lambda \rangle = 1$. This induces an inner product on $\text{Free}(E)(3)$ of signature $(6, 6)$, and an explicit computation shows that $R \subseteq \text{Free}(E)(3)$ is isotropic, hence Lagrangian. For example, one has

$$
\langle (A); (J) \rangle = \langle a \cdot (b \cdot c); [a, [b, c]] \rangle = \langle c \cdot (a \cdot b); [c \cdot [a, b]] \rangle = 1 - 1 = 0
$$

$$
\langle (A); (L) \rangle = \langle h a \cdot [b, c]; [a, b \cdot c] \rangle - \langle h [c, a \cdot b]; c \cdot [a, b] \rangle = h - h = 0
$$

and $\langle (A); (A) \rangle$ is given by $2h^2$ times

$$
\langle a \cdot (b \cdot c); [a, [b, c]] \rangle - \langle c \cdot (a \cdot b); [c, [a, b]] \rangle + \langle a \cdot [b, c]; [a \cdot (b \cdot c)] \rangle - \langle c \cdot [a, b]; c, (a \cdot b) \rangle = 0.
$$

Now consider the quadratic dual $BD_1^1 = \text{Free}(E')/R^\perp$. Identifying $\mu' \leftrightarrow \mu$ and $\lambda' \leftrightarrow \mu$ using the inner product described above and using that the inner product identifies the Lagrangian $R$ with $R^\perp$, we obtain an isomorphism $BD_1^1 \cong BD_1$.

It remains to verify that $BD_1$ is Koszul. This follows as in the case of $BD_0$: we have to show that the map $BD_1 \rightarrow R(BD_1)$ is a quasi-isomorphism, which can be checked at $h = 0$ and after inverting $h$. Since each $BD_1(p)$ is a finitely generated projective $k[h]$-module and extension of scalars is symmetric monoidal, one then reduces to checking that $BD_1 \otimes_k k[h]^{\pm}$ and $BD_1 \otimes_k k$ are Koszul operads. But these are just the associative and $P_1$-operads. □

**Remark 3.27.** Proposition 3.26 also applies to the operads $BD_n$ with $n \geq 2$, which are defined as the Rees construction of the $E_n$-operads, endowed with their Postnikov filtration. Indeed, by the (rational) formality of the $E_n$-operad [Wil18], this filtration splits and one can identify $BD_n \cong e_n[h]$. Probably one can also deduce Proposition 3.26 directly from the self-duality of the $E_n$-operad [Fre11].

**Operads with only nullary operations**

The following baby-example might also be useful to illustrate what happens for operads with nullary operations, when the category $k$ has nontrivial (endo)morphisms. Let $k$ be a connective dg-algebra and let $V$ be a connective left $k$-module. There is a $k$-operad $\mathcal{P}$ whose algebras are left $k$-modules $W$ with a $k$-linear map $V \rightarrow W$: $\mathcal{P}(0) = V$, $\mathcal{P}(1) = k$, and $\mathcal{P}(n) =$
0 for every \( n \geq 2 \). The dual operad (relative to \( k \)) is the \( k^{\text{op}} \)-operad \( \mathfrak{D}_k(P) \) whose algebras are left \( k^{\text{op}} \)-modules \( W \) endowed with an \( k^{\text{op}} \)-linear map \( V^\vee[-1] = \text{Hom}_k(V[1], k) \to W \).

In this case, the Koszul duality functor \( \mathfrak{D} : \text{Alg}_P \to \text{Alg}_D^{\text{op}}(P) \) can be identified with the functor

\[
\begin{array}{ccc}
V/L\text{Mod}_k & \to & \left(V^\vee[-1]/R\text{Mod}_k\right)^{\text{op}} \\
(V \to W) & \mapsto & \text{fib}(W^\vee \to V^\vee).
\end{array}
\]

(3.28)

The category of small \( P \)-algebras can be identified with the category of \( V \to W \), where \( W \) is a finitely presented \( k \)-module, with generators in nonpositive degrees. Using this, one sees that

\[
\text{FMP}_P \simeq \text{Ind}(\mathcal{C}_1^{\text{op}}) \quad \mathcal{C}_1 = \{ V \to W : \text{perfect } W \} \subseteq V/L\text{Mod}_k.
\]

Similarly, the category of right \( k \)-modules under \( V^\vee \) is compactly generated, so that there is an equivalence

\[
\text{Alg}_D^{\text{op}}(P) \simeq \text{Ind}(\mathcal{C}_2) \quad \mathcal{C}_2 = \{ V^\vee \to W : \text{perfect cofiber} \} \subseteq V^\vee/R\text{Mod}_k.
\]

Theorem 1.3 then reduces to the assertion that the functor (3.28) establishes a contravariant equivalence between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \).

## 4 Operadic formal moduli problems

In this section we introduce the notion of a formal moduli problem for algebras over a (augmented) \( k \)-operad \( P \) (Section 4.1), as well as the main ingredients that will be used to relate such formal moduli problems to algebras over the dual operad \( \mathfrak{D}(P) \). In particular, we describe an adjoint pair of \( \infty \)-categories

\[
\mathfrak{D} : \text{Alg}_P \to \text{Alg}_D^{\text{op}}(P) ; \mathfrak{D}'
\]

sending an algebra to its (bar) dual algebra (see Section 4.2). This adjunction is an example of a weak Koszul duality context in the sense of [CG18a] and will be the main actor in the proof of our main theorem (Theorem 1.3). Indeed, the axiomatic framework developed in [Lur11, CG18a] provides explicit conditions under which this adjoint pair induces an equivalence between \( \mathfrak{D}(P) \)-algebras and formal moduli problems over \( P \). We will recall these conditions in Section 4.3 (see Theorem 4.34).

Throughout this section, we will follow Assumption 2.4: all \( k \)-(co)operads are assumed to be (co)augmented and (filtered) cofibrant as left \( k \)-modules.

### 4.1 Moduli problems for algebras over operads

Recall that a classical (commutative) formal deformation functor is a functor

\[
\text{Art}_k \to \text{Set}
\]

from the category of Artin local commutative \( k \)-algebras satisfying (some version of) the Schlessinger conditions. To describe the notion of a formal moduli problem for algebras over a \( k \)-operad \( P \), we will replace the category of Artin local rings by the following category of small \( P \)-algebras.
Definition 4.1. Let $\mathcal{P}$ be a dg-category and let $\mathcal{P}$ be a $k$-operad. A **trivial algebra** is a $\mathcal{P}$-algebra obtained from a $k$-module by restriction along the augmentation map $\mathcal{P} \to k$. We will denote by

$$k_c[n] := k(c,-)[n]$$

the trivial algebra whose underlying $k$-module is free on a generator at the object $c \in k$, of cohomological degree $-n$. We denote its cone by $k_c[n,n+1]$.

Definition 4.2 (cf. [Lur11, Definition 1.1.8]). The $\infty$-category $\text{Alg}^{\text{sm}}_\mathcal{P}$ of small $\mathcal{P}$-algebras is the smallest full subcategory of the $\infty$-category of $\mathcal{P}$-algebras such that:

1. the trivial algebra $k_c[n]$ is small for every object $c \in k$ and every $n \geq 0$.
2. for any small $\mathcal{P}$-algebra $A$ and any map $A \to k_c[n]$ with $n \geq 1$, the homotopy pullback $A \times^h_{k_c[n]} 0$ is also small.

By definition, being small is a homotopy-invariant condition: any algebra quasi-isomorphic to a small algebra is itself small. As we see in Example 4.9, in the case $\mathcal{P}$ is the commutative operad, we recover the usual notion of an Artin local algebra.

Example 4.3. Suppose that $A$ is a small $\mathcal{P}$-algebra and that

$$k_c[n] \to B \to A$$

is a square zero extension of $A$ by the trivial $A$-module $k_c[n]$, for $n \geq 0$. Then $B$ is small as well. Indeed, pulling back to a quasi-free resolution of $A$ if necessary, we may assume that $A$ is a quasi-free $\mathcal{P}$-algebra. In this case, we can write $B = A \oplus k_c[n]$ as a split square zero extension, with differential of the form

$$d(a, v) = (da, dv + \chi(a)) \quad a \in A \quad v \in k_c[n].$$

The map $\chi$ defines a $\mathcal{P}$-algebra map $\chi: A \to k_c[n+1]$, and one can easily verify that $A'$ fits into a pullback square

$$\begin{array}{ccc}
B = A \oplus k_c[n] & \to & k_c[n,n+1] \\
\downarrow & & \downarrow \\
A & \to & k_c[n+1].
\end{array}$$

Since $k_c[n,n+1]$ is contractible, we find that $B \simeq A \times^h_{k_c[n+1]} 0$ is small.

Remark 4.4. In fact, the argument from Example 4.3 can be used to give the following chain-level description of the small $\mathcal{P}$-algebras: they form the smallest class of $\mathcal{P}$-algebras that is closed under quasi-isomorphisms and square zero extensions by the trivial modules $k_c[n]$ with $n \geq 0$.

Definition 4.5. A $\mathcal{P}$-algebra $A$ is **very small** if it admits a filtration

$$A = A^{(n)} \to \ldots \to A^{(0)} = 0$$

with the property that each $A^{(i)} \to A^{(i-1)}$ is a square zero extension with kernel $k_{c_i}[p_i]$, for some $c_i \in k$ and $p_i \geq 0$.

An iterated application of Example 4.3 shows that a very small $\mathcal{P}$-algebra is small. Conversely, if $\mathcal{P}$ is a cofibrant $k$-operad, then every small $\mathcal{P}$-algebra is quasi-isomorphic to a very small $\mathcal{P}$-algebra (see Lemma 5.12).
Remark 4.6. The $k$-module underlying a very small $\mathcal{P}$-algebra is cofibrant, and quasi-freely generated (i.e. disregarding differentials) by finitely many generators of degree $\leq 0$. In particular, it is a perfect left $k$-module.

Let us remark that in favourable cases, being small reduces to a condition at the level of the cohomology groups of a $\mathcal{P}$-algebra:

Lemma 4.7. Suppose that $k = k$ is a field and that $\mathcal{P}$ is a connective operad. Then a $\mathcal{P}$-algebra $A$ is small if and only if it satisfies the following conditions:

• $H^i(A) = 0$ for $i > 0$ and $i \ll 0$.

• each $H^i(A)$ is a finite-dimensional vector space.

• each $H^i(A)$ is a nilpotent module over the $H^0(\mathcal{P})$-algebra $H^0(A)$, in the following sense: consider the action maps

$$
\mu(a_1, \ldots, a_{q-1}, -) : H^i(A) \longrightarrow H^i(A)
$$

for $\mu \in H^0(\mathcal{P})(q)$ and $a_i \in H^0(A)$. Then there exists an $n$ such that any $n$-fold composition of such (possibly different) action maps is zero.

Proof. Consider a homotopy pullback of $\mathcal{P}$-algebras of the form $B \simeq A \times_{k[n]}^h 0$. Then the map $H^*(B) \longrightarrow H^*(A)$ on cohomology is a square zero extension of $H^0(\mathcal{P})$-algebras with kernel $k[n - 1]$. Using this inductively, one verifies the above conditions for every small $\mathcal{P}$-algebra $A$.

For the converse, we may assume $\mathcal{P}$ is a cofibrant operad and by homotopy transfer [LV, Section 10.3] that $A$ is minimal, so that $H^i(A) = A_i$. Let $i \leq 0$ be the minimal number such that $A_i \neq 0$. We claim that there exists a nonzero $v \in A_i$ such that $\mu(a_1, \ldots, a_p, v) = 0$ for any operation $\mu$. Assuming this, we find that $\langle v \rangle \longrightarrow A \longrightarrow A/\langle v \rangle$ is a square zero extension by $k[i]$. Example 4.3 then shows that $A$ is small if $A/\langle v \rangle$ is small, and the result follows by induction.

Since we assumed $\mathcal{P}$ to be connective, degree reasons dictate that the claim is equivalent to the following: there exists a $v \in A_i$ on which the $H^0(\mathcal{P})$-algebra $A_0$ acts trivially. Let $n$ be the minimal number such that any $n$-fold composition of action maps (4.8) is zero. If $n = 0$ then $A_0$ acts trivially on $A_i$ and we are done. For $n \geq 1$, there exists by assumption an $(n - 1)$-fold composite of action maps which is nonzero. Any nontrivial element $v$ in its image is then annihilated by all of $A_0$.

Example 4.9. Let $\mathcal{P} = \text{Com}$ be the 1-reduced commutative operad. A small algebra is exactly a nonunital cdga $\mathfrak{m}$ with finite dimensional cohomology groups which are zero in degrees $> 0$ and $\ll 0$, and with $H^0(\mathfrak{m})$ nilpotent. These are exactly the augmentation ideals of the unital Artin dg-$k$-algebras from [Lur11, Proposition 1.1.11].

With the small $\mathcal{P}$-algebras playing the role of local Artin dg-algebras, we now define a "$\mathcal{P}$-algebraic formal moduli problem" to be a functor $\text{Alg}_{\mathcal{P}}^{\text{sm}} \longrightarrow S$ satisfying the Schlessinger conditions.

Definition 4.10. Let $\mathcal{P}$ be a $k$-operad. A formal moduli problem over $\mathcal{P}$ is a functor

$$
F : \text{Alg}_{\mathcal{P}}^{\text{sm}} \longrightarrow S
$$

to the $\infty$-category of spaces, satisfying the following two conditions:

1. $F(0) \simeq \ast$, where $0$ is the zero algebra.
(2) $F$ sends a pullback diagram in $\text{Alg}_{\mathcal{P}}^{sm}$ of the form

\[
\begin{array}{c}
A' \\
\downarrow \\
A \\
\end{array} \rightarrow k_c[n]
\]

(4.11)
to a pullback square of spaces, for every colour $c \in k$ and $n \geq 1$.

We will denote the $\infty$-category of formal moduli problems over $\mathcal{P}$ by $\text{FMP}_{\mathcal{P}}$.

**Example 4.12.** To every $\mathcal{P}$-algebra $B$ we can associate its formal spectrum, a formal moduli problem $\text{Spf}(B) : \text{Alg}_{\mathcal{P}}^{sm} \rightarrow \mathcal{S}$, given by $A \mapsto \text{Map}_{\mathcal{P}}(B, A)$.

If one thinks of the functor $F$ as assigning to a $\mathcal{P}$-algebra $A$ the space of deformations of a certain object $X$, then the above conditions encode the usual obstruction theory for deformations along square zero extensions. Indeed, note that the pullback square (4.11) exhibits $A'$ as a square zero extension of $A$ by the trivial $A$-module $k_c[n - 1]$ (cf. Example 4.3). For every deformation $X_A \in F(A)$, one obtains an ‘obstruction class’

\[\text{ob}(X_A) \in \pi_0 F(k_c[n])\]

by applying the functor $F$ to the map $A \rightarrow k_c[n]$. This obstruction class is zero if and only if $X_A$ lifts to a deformation over the square zero extension $A'$.

Let us recall that there is a more cohomological way of interpreting these kinds of obstruction classes, as follows. Applying condition (2) in the case where $A = 0$, one obtains a natural sequence of equivalences

\[F(k_c) \cong \Omega F(k_c[1]) \cong \Omega^2 F(k_c[2]) \cong \cdots\]

In other words, the sequence of spaces $F(k_c[n])_{n \geq 0}$ forms an $\Omega$-spectrum $T(F)_c$.

**Definition 4.13.** We refer to $T(F)_c$ as the tangent complex of $F$ at $c$, and to the spectra $T(F)_{c \in k}$ collectively as the tangent complex of $F$.

In fact, the tangent complex admits a canonical $k^{op}$-module structure, as we will see in the next lemma. We denote the category of spectra by $\text{Sp}$. Recall that there is a functor $\text{Mod}_Z \rightarrow \text{Sp}$ sending $X$ to the iterated truncations $X_{\leq 0} \leftarrow X_{\leq 1} \leftarrow \cdots$.

**Lemma 4.14.** For any formal moduli problem $F$ over $\mathcal{P}$, the tangent complex has a unique inverse image under the forgetful functor

\[\text{Mod}_{k^{op}} \rightarrow \prod_{c \in k^{op}} \text{Mod}_Z \rightarrow \prod_{c \in k^{op}} \text{Sp}\]

(4.15)

with the following property: for all free $k^{op}$-modules generated by $c \in k^{op}$ in degree $n \geq 0$, there is a natural equivalence

\[\text{Map}_{\text{Mod}_{k^{op}}}(k^{op}_{-n}, T(F)) \simeq F(k_c[n]).\]

In other words, the obstructions to lifting deformations along square zero extensions are given by classes in the cohomology of the $k^{op}$-module $T(F)$.

**Remark 4.16.** The first functor in (4.15) takes a $k^{op}$-module $V$ to the collection of chain complexes $V(c)$. Equivalently, one can consider these as $HZ$-module spectra (via the Dold-Kan correspondence [SS03]). The second functor then forgets the $HZ$-module structure. The composite functor preserves both limits and colimits, since its left adjoint preserves compact generators: it sends $(0, \ldots, 0, \mathbb{S}^{n}, 0, \ldots, 0)$, with a sphere at place $c$, to the free $k^{op}$-module $k^{op}_{c[n]}$. 26
Proof. Uniqueness follows from the fact that the free modules $k^{\text{op}}[-n]$ with $n \geq 0$ generate the $\infty$-category $\text{Mod}_{k^{\text{op}}}$ under colimits. Existence follows either from Theorem 4.34, or from the following argument. Let $C^\leq n \subseteq \text{Mod}_{k^{\text{op}}}$ denote the subcategory generated by the free $k^{\text{op}}$-modules $k^{\text{op}}[n]$ under finite limits and let $C = \text{colim}_n C^\leq n$ be their union. Consider the functors

$$X_n: (C^\leq n)^{\text{op}} \rightarrow \mathcal{S}; \quad V \mapsto \Omega^n F(V[-n]).$$

These functors are well-defined because the trivial $\mathcal{P}$-algebra $V[-n]^\vee$ is small for all $V \in C^\leq n$. Because $F$ is a formal moduli problem, there are natural equivalences $X_n \cong X_{n+1}|_{C^\leq n}$, so that one obtains a functor

$$X: C^{\text{op}} \rightarrow \mathcal{S}; \quad V \mapsto \Omega^n F(V[-n]^\vee).$$

This functor sends finite colimits in $C$ to limits of spaces, since $F$ is a formal moduli problem. But $C \subseteq \text{Mod}_{k^{\text{op}}}$ contains all free $k^{\text{op}}$-modules and is closed under finite colimits, so it follows that $X$ is representable by a $k^{\text{op}}$-module $T(F)$.

Unravelling the definitions, this is exactly the desired $k^{\text{op}}$-module $T(F)$.

4.2 Duality for algebras over operads

Let $k$ be a dg-category and let $\phi: C \rightarrow \mathcal{P}$ be a twisting morphism from a $k$-cooperad to a $k$-operad (see Construction 9.16). Recall our convention that $C$ (resp. $\mathcal{P}$) is always assumed to be filtered-cofibrant (resp. cofibrant) as a left $k$-module, see Assumption 2.4. By Proposition 9.31, the twisting morphism $\phi$ gives rise to an adjoint pair

$$\Omega_\phi: \text{CoAlg}_{\mathcal{P}} \rightarrow \text{Alg}_{k^{\text{op}}} \quad \text{B}_\phi.$$

Taking the linear dual of the bar construction, we obtain a functor

$$\text{Alg}_{d\mathcal{P}} \xrightarrow{\text{B}_\phi} \text{CoAlg}_{d\mathcal{P}} \rightarrow \text{Alg}_{d\mathcal{P}}$$

with values in algebras over the dual $k^{\text{op}}$-operad $C^\vee$ (cf. Proposition 9.14). By Lemma 9.32, this functor preserves quasi-isomorphisms between algebras which are cofibrant as left $k$-modules. Consequently, it induces a functor of $\infty$-categories

$$\mathcal{D}_\phi: \text{Alg}_{\mathcal{P}} \rightarrow \text{Alg}_{d\mathcal{P}}^{\text{op}}.$$

Lemma 4.17. Suppose that $\phi: C \rightarrow \mathcal{P}$ is weakly Koszul (Definition 9.26). Then the following assertions hold:

1. For any $\mathcal{P}$-algebra $A$, there is a natural equivalence of $k^{\text{op}}$-modules

$$\mathcal{D}_\phi(A) \simeq \text{RDer}_{\mathcal{P}}(A, k)$$

2. The functor $\mathcal{D}_\phi$ preserves all colimits, so it is the left adjoint in an adjoint pair

$$\mathcal{D}_\phi: \text{Alg}_{\mathcal{P}} \rightarrow \text{Alg}_{d\mathcal{P}}^{\text{op}}; \mathcal{D}'_\phi. \quad (4.18)$$

In the terminology of [CG18a], the adjoint pair (4.18) is an example of a weak Koszul duality context (this is essentially the assertion of Corollary 4.23).
Proof. The first assertion implies the second: indeed, the functor $\mathcal{D}_φ$ preserves colimits (and hence admits a right adjoint by the adjoint functor theorem [Lur09, Corollary 5.5.2.9]) if and only if the composite

$$\text{Alg}^{dg}_{\mathcal{P}} \xrightarrow{B_φ} \text{CoAlg}^{dg}_{\mathcal{C}} \xrightarrow{(-)^\vee} \text{Alg}^{dg, op}_{\mathcal{C}^\vee} \xrightarrow{\text{forget}} \text{Mod}^{dg, op}_{\mathcal{C}}$$

preserves colimits. The functor $\mathbb{R} \text{Der}(-, k)$ taking derived modules of derivations clearly has this property.

Since $\mathcal{C} \rightarrow \mathcal{B}\mathcal{P}$ is a quasi-isomorphism between cofibrant left $k$-modules, the functor (4.19) is naturally equivalent to the functor associated to the universal twisting morphism $ϕ^{uni}: \mathcal{B}\mathcal{P} \rightarrow \mathcal{P}$. It will therefore suffice to prove assertion (1) for $ϕ = ϕ^{uni}$. In this case, consider the Quillen pair

$I: \text{Alg}^{dg}_{\mathcal{P}} \xleftarrow{\text{triv}} \text{Mod}^{dg}_{k} \xrightarrow{\text{forget}}$ where the right adjoint takes the trivial $\mathcal{P}$-algebra (using the augmentation $\mathcal{P} \rightarrow k$) and $I$ sends a $\mathcal{P}$-algebra to its module of indecomposables. Unraveling the definitions, one sees that there is an isomorphism of chain complexes

$$B_ϕ(A)^\vee \cong I\left(Ω_φ B_ϕ(A)\right)^\vee \cong \text{Der}_{\mathcal{P}}(Ω_φ B_ϕ(A), k)$$

where we have used that $I(B)^\vee = \text{Der}_{\mathcal{P}}(B, k)$. By Lemma 9.33, the map $Ω_φ B_ϕ(A) \rightarrow A$ is a quasi-isomorphism whenever $A$ is cofibrant as a $k$-module, i.e. it provides a functorial cofibrant replacement of $A$. The composite

$$\text{Alg}^{dg}_{\mathcal{P}} \xrightarrow{\mathcal{D}_{\phi}} \text{Alg}^{dg, op}_{\mathcal{C}^\vee} \xrightarrow{\text{triv}} \text{Mod}^{dg, op}_{k}$$

therefore agrees with the derived functor of the Quillen functor $A \mapsto \text{Der}(A, k)$, and hence preserves colimits.

Let us note that the adjoint pair (4.18) depends naturally on $ϕ$, in the following sense (we will come back to this in Section 6):

**Lemma 4.20.** Consider a commuting square

$$\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\phi} & \mathcal{P} \\
g & \downarrow & f \\
\mathcal{D} & \xrightarrow{\psi} & \mathcal{Q} 
\end{array}$$

(4.21)

where $g$ is a map of $k$-cooperads, $f$ is a map of $k$-operads and $ϕ$ and $ψ$ are weakly Koszul twisting morphisms. Then there is a natural transformation of $\mathcal{D}^\vee$-algebras

$$μ: B_φ(f_!A)^\vee \xrightarrow{g^*} B_φ(A)^\vee.$$ 

When $A$ is a cofibrant $\mathcal{P}$-algebra, this map is a weak equivalence.

In other words, a diagram like (4.21) induces a square of $\infty$-categories

$$\begin{array}{ccc}
\text{Alg}_{\mathcal{P}} & \xrightarrow{\mathcal{D}_{\phi}} & \text{Alg}^{op}_{\mathcal{C}^\vee} \\
\text{Alg}_{\mathcal{Q}} & \xrightarrow{\mathcal{D}_{\psi}} & \text{Alg}^{op}_{\mathcal{D}^\vee} \\
\text{f}_! & \downarrow & g^* \\
\sim & & \sim 
\end{array}$$

(4.22)
commuting up to a natural equivalence $\mu$. In particular, $D_\phi$ is a homotopy invariant of the map $\phi$, in the following sense: if $f$ (and, by Lemma 9.20, also $g$) is a quasi-isomorphism, then the vertical functors in (4.22) are equivalences by Corollary 9.6 which intertwine $D_\phi$ and $D_\psi$.

Proof. We define $\mu$ to be the dual of a natural map of $D$-coalgebras

$$g^*B_\phi(A) \longrightarrow B_\psi(f!A).$$

Without differentials, this map is given by the map $C(A) \longrightarrow D(f!A)$, defined on cogenerators by $C(A) \longrightarrow A \longrightarrow f!A$. This map of $D$-coalgebras indeed preserves the bar differential.

To see that it is a weak equivalence when $A$ is cofibrant, we can work at the level of the underlying chain complexes. In that case, we have a weak equivalence

$$\text{Der}(A, k) \sim \rightarrow \text{Der}(\Omega_\phi B_\phi(A), k) \cong B_\phi(A)^\vee$$

from the complex of $P$-algebra derivations of $A$ (see Lemma 4.17). We obtain a commuting square of chain complexes

$$\begin{array}{ccc}
\text{Der}_\Omega(f!A, k) & \longrightarrow & \text{Der}_P(A, k) \\
\sim & \sim \\
B_\psi(f!A)^\vee & \underset{\mu}{\longrightarrow} & B_\phi(A)^\vee
\end{array}$$

The top horizontal map is an isomorphism, so the result follows. \qed

**Corollary 4.23.** For any weakly Koszul twisting morphism $\phi: C \longrightarrow P$ and any $k$-module $V$, there is a natural equivalence of $C^\vee$-algebras

$$D_\phi(P(V)) \longrightarrow \text{triv}(V^\vee).$$

Consequently, for any algebra $g$ over the $k^\text{op}$-operad $D(P)$, the underlying $k$-module of $D_\phi'(g)$ is given by the derived functor of derivations

$$D_\phi'(g) \cong \mathbb{R}\text{Der}_{D(D(P))}(g, k^\text{op}).$$

Proof. The first assertion is a special case of Lemma 4.20 and the second assertion follows by passing to right adjoints. \qed

**Definition 4.24.** Let $P$ be a $k$-operad and let $D(P) := (B^D(P))^\vee$ be its dual operad. We will denote by

$$D: \text{Alg}_P \longrightarrow \text{Alg}_{D(D(P))}: D'$$

the adjunction associated to the universal twisting morphism $\pi: B^P \longrightarrow P$. By the discussion after Lemma 4.20, we are allowed to model $D(P)$ and $P$ using any quasi-isomorphic twisting morphism $\phi: C \longrightarrow P'$.

We conclude by giving an explicit description of the right adjoint $D'$.

**Theorem 4.26.** For any $k$-operad $P$, there exists a natural map

$$\eta: P \longrightarrow D(D(P))$$

in the $\infty$-category of $k$-operads, and the right adjoint functor $D_\phi': \text{Alg}_{D(D(P))} \longrightarrow \text{Alg}_P$ is naturally equivalent to the functor

$$\text{Alg}_{D(D(P))} \longrightarrow \text{Alg}_{D(D(P))} \longrightarrow \text{Alg}_P.$$
To define the map \( \eta \), which will arise from a zig-zag of maps at the chain level, let us make the following observation:

**Construction 4.27.** Let \( \mathcal{C} \) be a \( k \)-cooperad and \( \mathcal{D} \) a \( k^{\text{op}} \)-cooperad. Then there is an isomorphism of convolution Lie algebras (cf. Remark 9.18)

\[
\text{Hom}_{\text{BiMod}_{\Sigma k^{\text{op}}}}(\mathcal{D}, \mathcal{C}^\vee) \cong \text{Hom}_{\text{BiMod}_{\Sigma k}}(\mathcal{C}, \mathcal{D}^\vee)
\]

sending a linear map \( \psi: \mathcal{D} \to \mathcal{C}^\vee \) to its adjoint \( \psi^\top: \mathcal{C} \to \mathcal{D}^\vee \). In particular, this restricts to a bijection between twisting morphisms. For a twisting morphism \( \psi \) and a \( \mathcal{D} \)-coalgebra \( X \), there is a natural map of \( \mathcal{C}^\vee \)-algebras

\[
\Omega_\psi(X) \longrightarrow (B_\phi^\top (X^\vee))^\vee
\]

given on generators by the obvious inclusion \( X \longrightarrow X^\vee \longrightarrow (\mathcal{D} \circ X^\vee)^\vee \).

Let us now fix a \( k \)-cooperad \( \mathcal{C} \) which is filtered-cofibrant as a left \( k \)-module and let \( \epsilon: Q \longrightarrow \mathcal{C}^\vee \) denote a replacement of \( \mathcal{C}^\vee \) by a \( k^{\text{op}} \)-operad which is cofibrant as a left \( k^{\text{op}} \)-module. Consider the canonical twisting morphisms

\[
\phi: \mathcal{C} \longrightarrow \Omega \mathcal{C} \quad \quad \phi^\top: \mathcal{B} Q \longrightarrow \Omega \mathcal{C}
\]

Applying Construction 4.27 to the case where \( \mathcal{D} = \mathcal{B} Q \), the twisting morphism \( \epsilon \circ \phi^\top: \mathcal{B} Q \longrightarrow \mathcal{C}^\vee \) has an adjoint twisting morphism \( (\epsilon \circ \phi^\top)^\top: \mathcal{C} \longrightarrow \mathcal{B}(Q)^\vee \). We can write \( (\epsilon \circ \phi^\top)^\top = \eta \circ \phi \), where

\[
\eta: \Omega \mathcal{C} \longrightarrow \mathcal{B}(Q)^\vee
\]

is the corresponding map of \( k \)-operads out of the cobar construction. In this setting, we have the following identification of the right adjoint \( \mathcal{D}^\prime \phi \):

**Proposition 4.29.** In the above situation, the right adjoint \( \mathcal{D}^\prime_{\phi}: \text{Alg}_{\mathcal{C}^\vee}^{\text{op}} \longrightarrow \text{Alg}_{\mathcal{C}^\vee}^{\Omega \mathcal{C}} \) is naturally equivalent to the functor

\[
\text{Alg}_{\mathcal{C}^\vee}^{\text{op}} \longrightarrow \text{Alg}_{\mathcal{C}^\vee}^{\Omega \mathcal{C}} \longrightarrow \text{Alg}_{\mathcal{C}^\vee}^{\mathcal{B}(Q)^\vee}
\]

**Remark 4.30.** When \( \mathcal{C} \) is finite dimensional the map \( \eta: \Omega(\mathcal{C}) \longrightarrow \mathcal{B}(\mathcal{C}^\vee)^\vee \) is an isomorphism and we can simply identify \( \mathcal{D}^\prime_{\phi} \) with \( \mathcal{D}_{\phi^\top} \).

**Proof.** Our first goal will be to define a natural map of \( \Omega \mathcal{C} \)-algebras

\[
\eta^* \mathcal{D}_{\phi^\top}(\epsilon^* g) \longrightarrow \mathcal{D}_{\phi}(g)
\]

for every \( \mathcal{C}^\vee \)-algebra \( g \). By adjunction, it suffices to provide a natural map

\[
g \longrightarrow \mathcal{D}_{\phi}(\eta^* \mathcal{D}_{\phi^\top}(\epsilon^* g))
\]

in the \( \infty \)-category of \( \mathcal{C}^\vee \)-algebras. To do this, note that \( B_{\phi}(\eta^* -) \cong B_{\phi^\top}(-) \) and \( B_{\phi^\top}(\epsilon^* -) = B_{\phi^\top}(-) \) both preserve objects that are cofibrant as left modules. Consequently, we can compute

\[
\mathcal{D}_{\phi}(\eta^* \mathcal{D}_{\phi^\top}(g)) \cong \left( B_{\eta \phi}(B_{\phi^\top} g)^\vee \right)^\vee
\]
whenever \( g \) is cofibrant as a left \( k \)-module. Now apply Construction 4.27 to the case where \( \mathcal{D} = B(\Omega) \) and to the twisting morphisms

\[
\psi = \epsilon \phi^\dagger: \mathcal{D} = B(\Omega) \longrightarrow \mathcal{C}^\vee \quad \quad \psi^\top = \eta \phi: \mathcal{C} \longrightarrow \mathcal{D}^\vee.
\]

For the \( \mathcal{D} \)-coalgebra \( X = B_{\epsilon \phi} g \), the map (4.28) then gives a natural map of \( \mathcal{C}^\vee \)-algebras

\[
\Omega_{\epsilon \phi} (B_{\epsilon \phi} g) \longrightarrow \left( B_{\eta \phi} (B_{\epsilon \phi} g) \right)^{\vee}.
\]

The domain is the usual bar-cobar construction of \( g \), which comes with a natural quasi-isomorphism \( \Omega_{\epsilon \phi} (B_{\epsilon \phi} g) \longrightarrow g \) when \( g \) is cofibrant as a \( k^{\text{op}} \)-module (Lemma 9.33). At the level of \( \infty \)-categories, we therefore obtain the desired map (4.32) and the adjoint comparison map (4.31).

We now have to check that the comparison map (4.31) is an equivalence, for which it suffices to see that the underlying map of \( k \)-modules is an equivalence. Note that Lemma 4.17 and Corollary 4.23 produce natural equivalences of \( k \)-modules

\[
\mathcal{D}_{\epsilon \phi} (g) \simeq \mathcal{R}\text{Der}_{\mathcal{C}^\vee} (g, k^{\text{op}}) \quad \quad \mathcal{D}^\vee (g) \simeq \mathcal{R}\text{Der}_{\mathcal{C}^\vee} (g, k^{\text{op}}).
\]

Under these equivalences, the comparison map (4.31) corresponds to a natural endomorphism of

\[
\mathcal{R}\text{Der}_{\mathcal{C}^\vee} (g, k^{\text{op}}) \simeq \mathcal{R}\text{Der}_{\mathcal{C}^\vee} (\Omega_{\epsilon \phi} B_{\epsilon \phi} g, k^{\text{op}}) \simeq B_{\epsilon \phi} g^{\vee}.
\]

Unravelling the definitions, this endomorphism can be described as follows: an element \( \alpha \in B_{\epsilon \phi} g^{\vee} \) is sent to the \( k^{\text{op}} \)-linear map

\[
f_\alpha: B_{\epsilon \phi} g \longrightarrow \Omega_{\epsilon \phi} B_{\epsilon \phi} g \xrightarrow{\Omega_{\epsilon \phi} B_{\epsilon \phi} g^{\vee}} B_{\eta \phi} (B_{\epsilon \phi} g)^{\vee} \longrightarrow B_{\epsilon \phi} g^{\vee} \longrightarrow k^{\text{op}}.
\]

Here the first map is the inclusion of the generators and the third map is the projection, dual to the inclusion of \( B_{\epsilon \phi} g^{\vee} \) into its bar construction (as the primitive elements). The last map evaluates an element of the bidual at \( \alpha \). One easily sees that the assignment \( \alpha \mapsto f_\alpha \) is an isomorphism, so that (4.31) is indeed an equivalence.

**Proof (of Theorem 4.26).** Suppose that \( \mathcal{P} \) is cofibrant as a left \( k \)-module and consider the situation of Proposition 4.29 in the case where \( \mathcal{C} = B \mathcal{P} \). Then

\[
\mathcal{C}^\vee \simeq \Omega \simeq \mathcal{D} (\mathcal{P}) \quad \text{and} \quad B(\Omega)^{\vee} \simeq \mathcal{D} (\mathcal{D} (\mathcal{P}))
\]

so that the natural zig-zag \( \mathcal{P} \xleftarrow{\Omega B \mathcal{P}} \mathcal{P} \xrightarrow{\mathcal{D}} B(\Omega)^{\vee} \) defines a natural map in the \( \infty \)-category of \( k \)-operads \( \eta: \mathcal{P} \longrightarrow \mathcal{D} (\mathcal{D} (\mathcal{P})) \). Theorem 4.26 then follows from Proposition 4.29. \( \square \)

### 4.3 Axiomatic argument

We will now describe the strategy of the proof of our main result, Theorem 1.3. Our strategy follows the axiomatic frameworks developed in [Lur11, CG18a]. More precisely, let us consider the adjunction

\[
\mathcal{D}: \text{Alg}_{\mathcal{P}} \underrightarrows{\text{Alg}_{\mathcal{D}(\mathcal{P})}}: \mathcal{D}'.
\]

This adjunction is essentially never an equivalence, because it involves taking duals: both \( \mathcal{D} \) and \( \mathcal{D}' \) send an algebra to its module of derivations with coefficients in \( k \) (Lemma 4.17
and Corollary 4.23). Instead, one can try to refine the above adjunction to an equivalence between $D(P)$-algebras and formal moduli problems over $P$, using the following construction: every $D(P)$-algebra $g$ defines a functor

$$\text{Alg}_{D(P)}^{\text{sm}} \longrightarrow S; \quad A \longmapsto \text{Map}_{D(P)}(D(A), g).$$

Under suitable conditions on the functor $D$, this functor will satisfy the axioms of a formal moduli problem (Definition 4.10). Furthermore, the results of [Lur11, CG18a] provide general conditions on $D$ under which this construction becomes an equivalence. In the current situation, we can summarize these results as follows:

**Theorem 4.34.** Let $P$ be a $k$-operad. Then there is an equivalence of $\infty$-categories

$$\text{MC}: \text{Alg}_{D(P)}^{\text{sm}} \sim \text{FMP}_P; \quad g \longmapsto \text{Map}_{D(P)}(D(-), g).$$

if the following conditions are satisfied:

1. **(A)** For every small $P$-algebra $A$, the unit map $A \longrightarrow D'(D(A)$ is an equivalence.
2. **(B)** For every trivial algebra $k_c[n]$ generated by a single element of degree $n \geq 0$, the $D(P)$-algebra $D(k_c[n])$ is freely generated by $k_c[n]$.\(\right)
3. **(C)** The functor $D$ sends every pullback square of small algebras

$$\begin{array}{ccc}
A' & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
A & \longrightarrow & k_c[n]
\end{array}$$

with $n \geq 1$ to a pushout square of $D(P)$-algebras.

In this case, the inverse of the functor $\text{MC}$ sends a formal moduli problem $F$ to its tangent complex $T(F)$, endowed with some $D(P)$-algebra structure.

**Remark 4.35.** The notation $\text{MC}$ is supposed to be suggestive: when $P$ satisfies suitable finite-dimensionality conditions, the formal moduli problem $\text{MC}_g$ can indeed be described concretely in terms of Maurer–Cartan simplicial sets of Lie algebras. We will discuss this in more detail in Section 7.

The technical part of the proof of our main result (Theorem 1.3) will consist of verifying the above conditions for a suitable class of operads. This will be done in Section 5. In the remainder of this section, we will describe how Theorem 4.34 follows from the results of [Lur11, CG18a].

**Proof.** Condition (C) guarantees that for every $D(P)$-algebra $g$, the functor

$$\text{MC}_g: \text{Alg}_{D(P)}^{\text{sm}} \longrightarrow S; \quad A \longmapsto \text{Map}_{D(P)}(D(A), g)$$

does indeed define a formal moduli problem. Consequently, we obtain a well-defined functor $\text{MC}: \text{Alg}_{D(P)} \longrightarrow \text{FMP}_P$. By condition (B), we have that

$$\text{MC}_g(k_c[n]) = \text{Map}_{D(P)}(D(k_c[n]), g) \simeq \text{Map}_{D(P)}(\text{Free}(k_c[n]), g) \simeq \text{Map}_{k^{op}}(\mathbb{k}_c^n[-n], g).$$
It then follows from Lemma 4.14 that the tangent complex of the formal moduli problem $MC_g$ is given by

\[ T(MC_g) \simeq g. \] (4.36)

In particular, if $MC$ admits an inverse, then this inverse will necessarily send a formal moduli $F$ to $T(F)$, endowed with a $\mathcal{D}(\mathcal{P})$-algebra structure. To see that $MC$ indeed does admit an inverse, let us recall the following terminology [CG18a, Definition 2.15]. The class of good $\mathcal{D}(\mathcal{P})$-algebras is the smallest class of algebras such that:

1. It contains the free algebras $\text{Free}(k_{\mathcal{P}}^{op}[n])$ for $n \leq 0$.
2. For any pushout square

\[
\begin{array}{ccc}
\text{Free}(k_{\mathcal{P}}^{op}[n]) & \longrightarrow & g \\
\downarrow & & \downarrow \\
0 & \longrightarrow & h
\end{array}
\] (4.37)

where $g$ is good and $n \leq -1$, $h$ is good as well.

By condition (A), the functor $\mathcal{D}$ restricts to a fully faithful embedding of the $\infty$-category of small $\mathcal{P}$-algebras into $\text{Alg}_{\mathcal{D}(\mathcal{P})}^{op}$. By conditions (B) and (C), the essential image of this embedding is (the opposite of) a full subcategory of $\text{Alg}_{\mathcal{D}(\mathcal{P})}$ which satisfies conditions (1) and (2). In particular, it contains the good $\mathcal{D}(\mathcal{P})$-algebras. But then the image of the good $\mathcal{D}(\mathcal{P})$-algebras under $\mathcal{D}'$ is a full subcategory of the small $\mathcal{P}$-algebras that satisfies the conditions of Definition 4.2. Since the small algebras were the smallest subcategory with these properties, we conclude that $\mathcal{D}$ and $\mathcal{D}'$ induce an equivalence

\[ \mathcal{D}: \text{Alg}_{\mathcal{P}}^{\text{sm}} \rightleftarrows (\text{Alg}_{\mathcal{D}(\mathcal{P})}^{\text{good}})^{op}: \mathcal{D}' \] (4.38)

It will now follow from [Lur11, Theorem 1.3.12] that the functor $MC$ is an equivalence. Indeed, the conditions of [Lur11, Definition 1.3.1] hold precisely because $\mathcal{D}$ restricts to the equivalence (4.38). The remaining condition [Lur11, Definition 1.3.5] asserts that the functor

\[ \text{Alg}_{\mathcal{D}(\mathcal{P})} \xrightarrow{MC} \text{FMP}_{\mathcal{P}} \xrightarrow{T} \prod_c \text{Sp} \]

preserves sifted colimits. But it follows from (4.36) that this functor is naturally equivalent to the composite

\[ \text{Alg}_{\mathcal{D}(\mathcal{P})} \xrightarrow{\text{forget}} \text{Mod}_{k_{\mathcal{P}}} \xrightarrow{\prod_c \text{Sp}} \]

Forgetting the structure of an algebra over an operad always preserves sifted homotopy colimits [HNP19, Appendix A] and the second functor preserves all colimits (see Remark 4.16).

5 Cohomology of small algebras

Let $\mathcal{P}$ be a $k$-operad and consider the adjoint pair whose left adjoint sends a $\mathcal{P}$-algebra to its dual $\mathcal{D}(\mathcal{P})$-algebra

\[ \mathcal{D}: \text{Alg}_{\mathcal{P}} \rightleftarrows \text{Alg}_{\mathcal{D}(\mathcal{P})}^{op}: \mathcal{D}'. \]

The purpose of this section is to show that under certain conditions on the operad $\mathcal{P}$, the functor $\mathcal{D}$ is well-behaved when restricted to the class of small $\mathcal{P}$-algebras. In particular, it

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satisfies the assumptions of Theorem 4.34, so that the above adjunction can be refined to an equivalence between $D(P)$-algebras and formal moduli problems over $P$. More precisely, will prove the following:

**Theorem 5.1 (Theorem 1.3).** Let $k$ be a dg-category and $P$ an (augmented) $k$-operad. Assume that the following conditions hold:

1. $k$ and $P$ are both connective.
2. $k$ is cohomologically bounded, i.e. there exists an $n \in \mathbb{N}$ such that all $H^*(k)(c,d)$ are concentrated in degrees $[-n,0]$.
3. $P$ is splendid.

Then the following assertions hold:

1. For any small $P$-algebra (Definition 4.2), the unit map $A \to D^!(D(A))$ is an equivalence.
2. $D(k_c[n])$ is freely generated by $k_c[n]^\vee$, for all $c$ and $n \geq 0$.
3. The functor $D$ sends every pullback square of small algebras

$$
\begin{array}{ccc}
A' & \to & 0 \\
\downarrow & & \downarrow \\
A & \to & k_c[n]
\end{array}
$$

with $n \geq 1$ to a pushout square of $D(P)$-algebras.

In particular, Theorem 4.34 applies and there is an equivalence

$$
FMP_P \simeq \text{Alg}_D(P).
$$

**Assumption 5.3.** We will assume, as usual, that $P$ is cofibrant as a left $k$-module. Because $k$ is assumed to be connective and cohomologically bounded, we will furthermore make the following chain-level assumption throughout this section: we will assume that $k$ is a dg-category such that every $k(c,d)$ is concentrated in degrees $[-N,0]$, for some fixed $N$.

### 5.1 Polynomial subalgebras

Let us start with the following general observation. Let $\phi: C \to P$ be a weakly Koszul twisting morphism from an $k$-cooperad to a $k$-operad. Then $D_\phi(A) = B_\phi(A)^\vee$ is given by

$$
D_\phi(A) = \left( \bigoplus_{p \geq 0} (C(p) \otimes_{\Sigma_p, k^\otimes p} A^\otimes p) \right)^\vee
\cong \prod_{p \geq 0} \left( (C(p) \otimes_{\Sigma_p, k^\otimes p} A^\otimes p) \right)^\vee.
$$

Consider the graded $C^\vee$-subalgebra

$$
D_\phi^{poly}(A) := \bigoplus_{p \geq 0} \left( (C(p) \otimes_{\Sigma_p, k^\otimes p} A^\otimes p) \right)^\vee \subseteq D_\phi(A).
$$

Note that this is not necessarily closed under the differential, but it will be if $A$ satisfies the following condition:
Definition 5.5. A $\mathcal{P}$-algebra $A$ is nilpotent if $A$ is annihilated by all operations of arity $\geq p$, for some $p$.

Remark 5.6. When $\mathcal{P}$ is concentrated in arity $\geq 2$, then a nilpotent algebra is annihilated by any composition of $\geq p$ operations in $\mathcal{P}$, for some $p$. Conversely, if $\mathcal{P}$ is generated by operations in finitely many arities and $A$ is annihilated by any composition of $\geq p$ operations, then $A$ is nilpotent.

Remark 5.7. The algebra $\mathcal{D}^{\text{poly}}_\phi(A)$ is not homotopy invariant: it depends on the point-set choices for $A$ and the twisting morphism $\phi$. Note that an algebra that is quasi-isomorphic to a nilpotent algebra need not be nilpotent itself.

To prove Theorem 5.1, it will be much more convenient to work with $\mathcal{D}^{\text{poly}}(A)$ instead of $\mathcal{D}(A)$. Indeed, the following result shows that $\mathcal{D}^{\text{poly}}(A)$ is typically much better behaved than $\mathcal{D}(A)$:

Lemma 5.8. Let $\phi: \mathcal{C} \longrightarrow \mathcal{P}$ be a Koszul twisting morphism. If $A$ is a very small, nilpotent $\mathcal{P}$-algebra, then $\mathcal{D}^{\text{poly}}_\phi(A)$ is a cofibrant $\mathcal{C}^\vee$-algebra.

Proof. Let us start with the following general observation: if $A \longrightarrow B$ is a square zero extension of $\mathcal{P}$-algebras by $k_c[n]$, with $n \geq 0$, then their bar constructions fit into a pullback square of $\mathcal{C}$-coalgebras

$$
\begin{array}{ccc}
B_\phi(A) & \longrightarrow & \mathcal{C}(k_c[n, n+1]) \\
\downarrow & & \downarrow \\
B_\phi(B) & \longrightarrow & \mathcal{C}(k_c[n+1]).
\end{array}
$$

(5.9)

Indeed, this follows from writing $A \cong B \oplus k_c[n]$ as $k$-modules (without differential), so that $B_\phi(A)$ is obtained from $B_\phi(B)$ by adding cogenerators from $k_c[n]$. Assuming that $A$ and (hence) $B$ are nilpotent, we can take duals and restrict to ‘polynomial’ subalgebras to obtain a square

$$
\begin{array}{ccc}
\mathcal{C}^\vee(k_c^\text{op}[-n-1]) & \longrightarrow & \mathcal{D}^{\text{poly}}_\phi(B) \\
\downarrow & & \downarrow \\
\mathcal{C}^\vee(k_c^\text{op}[-n, -n-1]) & \longrightarrow & \mathcal{D}^{\text{poly}}_\phi(A).
\end{array}
$$

(5.10)

Without differentials, this square is a pushout square of $\mathcal{C}^\vee$-algebras, so the same is true with differentials. Since the left vertical map is a (generating) cofibration of $\mathcal{C}^\vee$-algebras, it follows that $\mathcal{D}^{\text{poly}}_\phi(A)$ is cofibrant as soon as $\mathcal{D}^{\text{poly}}_\phi(B)$ is.

Now suppose that $A$ is very small and nilpotent. By definition, $A$ fits into a sequence $A = A^{(n)} \longrightarrow \ldots \longrightarrow A^{(0)} = 0$ of square zero extensions by various $k_c[p_i]$. Proceeding by induction, it follows that $\mathcal{D}^{\text{poly}}_\phi(A)$ is cofibrant.

To reduce statements about $\mathcal{D}(A)$ to statements about the more tractable algebra $\mathcal{D}^{\text{poly}}(A)$, we will use the following result:

Proposition 5.11. Suppose that $k$ is as in Assumption 5.3 and that $\mathcal{P}$ is a splendid $k$-operad, concentrated in nonpositive degrees. Let $\pi: \mathcal{B}^{\text{op}} \longrightarrow \mathcal{P}$ be the universal Koszul twisting morphism and let $A$ be a $\mathcal{P}$-algebra which is very small and nilpotent. Then the map of $\mathcal{B}^{\text{op}}$-algebras

$$
\mathcal{D}^{\text{poly}}_\pi(A) \longrightarrow \mathcal{D}_\pi(A)
$$

is a quasi-isomorphism.
This proposition forms the technical heart of our proof of Theorem 5.1 (and hence Theorem 1.3). In particular, its proof is somewhat involved and is proven in increasing levels of generality, using some of the results of Section 9. We will therefore postpone the proof to Section 5.3 and instead discuss how it can be used to prove Theorem 5.1.

As a first application of Proposition 5.11, we find that every small $\mathcal{P}$-algebra can be modelled by a nilpotent algebra. More precisely, we have the following:

**Lemma 5.12.** Consider a retract diagram of $k$-operads $\mathcal{P} \xrightarrow{\sim} \Omega \mathcal{C} \xrightarrow{\sim} \mathcal{P}$, where $\mathcal{C}$ is filtered-cofibrant as a left $k$-module. Then the following assertions hold:

1. Every small $\mathcal{P}$-algebra is quasi-isomorphic to a very small $\mathcal{P}$-algebra (Definition 4.5).
2. Suppose that the (Koszul) twisting morphism $\phi: \mathcal{C} \to \mathcal{P}$ associated to $\Omega \mathcal{C} \to \mathcal{P}$ has the following property: for every $A$ which is very small and nilpotent, the map $D^{\text{poly}}_\phi(A) \to D_{\phi}(A)$ is a quasi-isomorphism. Then every small $\mathcal{P}$-algebra is quasi-isomorphic to a very small $\mathcal{P}$-algebra which is furthermore nilpotent.

Every cofibrant $k$-operad $\mathcal{P}$ fits into a retract diagram $\mathcal{P} \xrightarrow{\sim} \Omega \mathcal{B} \mathcal{P} \xrightarrow{\sim} \mathcal{P}$. Consequently, part (1) asserts that small algebras over cofibrant operads are quasi-isomorphic to very small algebras.

**Proof.** The small $\mathcal{P}$-algebras form the smallest class of $\mathcal{P}$-algebras that is closed under homotopy pullbacks along the maps of trivial algebras $0 \to k_c[n] + 1$ (with $n \geq 0$). It therefore suffices to show the following: let $A$ be a very small $\mathcal{P}$-algebra, let $\Omega_\phi B_\phi(A) \to A$ be its bar-cobar resolution (cofibrant by Lemma 9.33) and consider any (homotopy) pullback diagram of the form

$$
\begin{array}{ccc}
Y & \xrightarrow{\chi} & k_c[n, n + 1] \\
\downarrow & & \downarrow \\
\Omega_\phi B_\phi(A) & \xrightarrow{\chi} & k_c[n + 1].
\end{array}
$$

(5.13)

Then the map $Y \to \Omega_\phi B_\phi(A)$ is naturally quasi-isomorphic to a square zero extension $B \to A$ with kernel $k_c[n]$. For part (2), we must furthermore show that $B$ can be taken nilpotent, assuming $A$ is nilpotent.

We will only prove this assertion for part (2); the argument for part (1) is similar but easier. Let us denote by $i: \mathcal{P} \to \Omega \mathcal{C}$ and $r: \Omega \mathcal{C} \to \mathcal{P}$ the inclusion and retraction, and let $\psi: \mathcal{C} \to \Omega \mathcal{C}$ denote the universal twisting morphism. There are natural maps

$$
B_\phi(A) \xrightarrow{\sim} B_\phi(r^* A) \quad i^* \Omega_\phi(C) \xrightarrow{\sim} \Omega_\phi(C)
$$

for a $\mathcal{P}$-algebra $A$ and a $\mathcal{C}$-coalgebra $C$. The first map is an isomorphism and the second map is obtained by applying $i^*$ to the natural map $\Omega_\phi(C) \to r^* \Omega_\phi(C)$. Now observe that there are bijections

$$
\chi: \Omega_\phi B_\phi(A) \to k_c[n + 1] \quad \iff \quad \chi: B_\phi(A) \to \mathcal{C}(k_c[n + 1])
$$

where $\mathcal{C}(k_c[n + 1])$ is the cofree $\mathcal{C}$-coalgebra on a single generator of degree $n + 1$ at place $c$. A map $\chi: \Omega_\phi B_\phi(A) \to k_c[n + 1]$ therefore corresponds to a degree $-(n + 1)$ cycle $\chi \in D_{\phi}(A)(c)$. Homologous cycles correspond to homotopic maps, and hence give rise to weakly equivalent homotopy pullbacks $Y$. We may therefore change $\chi$ by a coboundary and assume that it is contained in the image of the quasi-isomorphism

$$
D^{\text{poly}}_\phi(A) \xrightarrow{\sim} D_{\phi}(A).
$$

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Now consider the pullback square of $\mathcal{C}$-coalgebras

\[
\begin{array}{ccc}
C' & \longrightarrow & \mathcal{C}(k_c[n, n+1]) \\
\downarrow & & \downarrow \\
B_\phi(A) & \xrightarrow{\chi} & \mathcal{C}(k_c[n+1]).
\end{array}
\] (5.14)

Unravelling the definitions, one sees that the map $C' \longrightarrow B_\phi(A)$ is isomorphic to a map of the form $B_\phi(A') \longrightarrow B_{\psi}(r^*A)$, where $A' \longrightarrow r^*A$ is a square zero extension of $\Omega\mathcal{C}$-algebras with kernel $k_c[n]$. In particular, $A'$ is a very small $\Omega\mathcal{C}$-algebra.

To see that $A'$ is a nilpotent $\Omega\mathcal{C}$-algebra, we use that $A' \cong A \oplus k_c[n]$ is a square zero extension of $r^*A$ by a trivial $r^*A$-module. Each generator $\mu \in \mathcal{C}[-1] \subseteq \Omega\mathcal{C}$ acts on $A'$ by

$$
\mu_{A'}: (A \oplus k_c[n])^{\otimes p} \longrightarrow A^{\otimes p} (\mu, \chi(\mu, -)) A \oplus k_c[n].$$

Here $\chi(\mu, -)$ denotes the composite

$$
A^{\otimes p} \longrightarrow \mathcal{C}(p)[-1] \otimes A^{\otimes p} \subseteq B_\phi(A) \xrightarrow{\chi[-1]} k_c[n].$$

By our assumption that $\chi$ lies in the image of $\mathcal{D}_\text{poly}(A)$, the generating operations $\chi(\mu, -)$ vanish when the arity of $\mu$ is high enough. Furthermore, the composition of at least two such generating operations maps $A$ to $A$ and vanishes on $k_c[n]$. Because $A$ was assumed to be a nilpotent $\mathcal{P}$-algebra, it follows that such composite operations also vanish if their arity is high enough. We conclude that $A'$ is a nilpotent $\Omega\mathcal{C}$-algebra.

Now, applying functor $i^*\Omega_{\psi}$ to (5.14) and using that there is a natural map $i^*\Omega_{\psi} \longrightarrow \Omega_{\phi}$, we obtain a diagram of $\mathcal{P}$-algebras

\[
\begin{array}{ccc}
i^*A' & \xleftarrow{\sim} & i^*\Omega_{\psi}B_\psi(A') \longrightarrow \Omega_{\phi}B_\phi(k_c[n+1, n]) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\sim} & \Omega_{\phi}B_\phi(A) \longrightarrow \Omega_{\phi}B_\phi(k_c[n+1]) \xrightarrow{\sim} k_c[n+1].
\end{array}
\] (5.15)

Taking $B = i^*A'$, we obtain a nilpotent square zero extension of $A$. The above diagram shows that it is related to the pullback $Y$ of (5.13) by a zig-zag

$$
B = i^*A' \xleftarrow{\sim} i^*\Omega_{\psi}B_\psi(A') \longrightarrow Y.
$$

It remains to verify that the right map is a quasi-isomorphism, for which we can work at the level of the underlying complexes. But forgetting $\mathcal{P}$-algebra structures, there are natural sections $i^*A' \longrightarrow i^*\Omega_{\psi}B_\psi(A')$ and $A \longrightarrow \Omega_{\phi}B_\phi(A)$ that make the composition of the three squares a (homotopy) pullback square of chain complexes. Consequently, we find maps of complexes

$$
B = A \times^{h}_{k_c[n+1]} 0 \longrightarrow \Omega_{\psi}B_\psi(A') \longrightarrow \Omega_{\phi}B_\phi(A) \times^{h}_{k_c[n+1]} 0 = Y.
$$

The first map and the composite map are quasi-isomorphisms, so that $i^*\Omega_{\psi}B_\psi(A') \longrightarrow Y$ is a quasi-isomorphism, as desired.

**Corollary 5.16.** Suppose that $k$ is as in Assumption 5.3 and that $\mathcal{P}$ is a splendid cofibrant $k$-operad, concentrated in nonpositive cohomological degrees. Then every small $\mathcal{P}$-algebra is quasi-isomorphic to a very small $\mathcal{P}$-algebra which is nilpotent.

**Proof.** Apply part (2) of Lemma 5.12 to the retract diagram $\mathcal{P} \longrightarrow \Omega B\mathcal{P} \longrightarrow \mathcal{P}$, where first map exists since $\mathcal{P}$ is assumed cofibrant. \[\square\]
5.2 Proof of Theorem 5.1

In this section, we will prove Theorem 5.1, and hence Theorem 1.3, using Proposition 5.11 (whose proof will be taken up in Section 5.3). Since the statement of Theorem 5.1 only depends on the quasi-isomorphism classes of $k$ and $\mathcal{P}$, we are allowed to make the following assumptions throughout this section: we will assume that $k$ is bounded, as in Assumption 5.3, and that $\mathcal{P}$ is a cofibrant $k$-operad which is concentrated in nonpositive cohomological degrees. We denote by

$$\pi: B\mathcal{P} \longrightarrow \mathcal{P}$$

the universal Koszul twisting morphism and will model $\mathcal{D}: \text{Alg}_{\mathcal{P}} \longrightarrow \text{Alg}_{\mathcal{D}(\mathcal{P})}^\text{op}$ by $\mathcal{D}_\pi$.

**Proof of Theorem 5.1 (A).** Suppose that $A$ is a small $\mathcal{P}$-algebra. By Corollary 5.16, we can assume that $A$ is very small and nilpotent. To verify that the unit map

$$A \longrightarrow \mathcal{D}_\pi(A)$$

is an equivalence, it suffices to work at the level of the underlying $k$-modules. By Corollary 4.23, the functor $\mathcal{D}_\pi$ is given at the level of $k$-modules by the derived functor of $B \mapsto \text{Der}(B,k)$. By Lemma 5.8 and Proposition 5.11, a cofibrant resolution of $\mathcal{D}_\pi(A)$ is given by the polynomial subalgebra $\mathcal{D}_\pi^\text{poly}(A)$. It therefore suffices to verify that the natural map

$$A \longrightarrow \text{Der}(\mathcal{D}_\pi^\text{poly}(A), k)$$

is a quasi-isomorphism. Since $\mathcal{D}_\pi^\text{poly}(A)$ is the free graded algebra on $A^\vee$, one can identify the underlying map of graded $k$-modules with the canonical map

$$A \longrightarrow A^\vee.$$

This map is an isomorphism since $A$ is a finitely generated quasi-free $k$-module (Remark 4.6). \qed

For part (B) of Theorem 5.1, let us make the following more general observation:

**Proposition 5.17.** Let $k$ be a bounded connective dg-category and $f: \mathcal{P} \longrightarrow \mathcal{Q}$ a map of augmented $k$-operads which are connective and splendid. Let $\mathcal{D}(f): \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{P})$ be the induced map on bar dual operads. For every $\mathcal{Q}$-algebra $A$, there is a natural map of $\mathcal{D}(\mathcal{P})$-algebras

$$(\mathcal{D}(f)_*)\mathcal{D}(A) \longrightarrow \mathcal{D}(f^*(A)).$$

This map is an equivalence whenever $A$ is a small $\mathcal{Q}$-algebra.

**Proof.** We can assume that $\mathcal{P}$ and $\mathcal{Q}$ are cofibrant $k$-operads and consider the map between twisting morphisms

$$\begin{array}{ccc}
B\mathcal{P} & \xrightarrow{\phi} & \mathcal{P} \\
\downarrow Bf & & \downarrow f \\
B\mathcal{Q} & \xrightarrow{\psi} & \mathcal{Q}
\end{array}$$

Let $B(f)^*$ denote the forgetful functor from $B\mathcal{P}$-coalgebras to $B\mathcal{Q}$-coalgebras. Then there is a natural map of $B\mathcal{Q}$-coalgebras for every $\mathcal{Q}$-algebra $A$

$$B(f)^*B\phi(f^*(A)) \longrightarrow B\psi(A).$$
Without differentials, this is given by the obvious map $B(P(A) \to BQ(A)$ into the cofree $BQ$-coalgebra on $A$. Taking duals gives a map of $D\Omega$-algebras
\[
D\phi(A) \longrightarrow D(f)^* D\phi(f^*(A)).
\]
The desired natural map of algebras over $D(P)$ is then obtained by adjunction, i.e. by (derived) inducing up along $D\phi \to D\phi'$. 

Now suppose that $A$ is a small $Q$-algebra. By Corollary 5.16, we may assume that $A$ is very small and nilpotent. By Proposition 5.11 and Lemma 5.8, there are cofibrant resolutions
\[
D\text{poly} \phi(A) \sim \to D\phi(\bar{f}^*(A)) \to D\phi(f^*(A)).
\]
In particular, we obtain a commuting square
\[
\begin{array}{ccc}
D(f)_! & 
\longrightarrow & D\text{poly}(A) \\
\downarrow & & \sim \\
D(f)_! D\phi(A) & 
\longrightarrow & D\phi(f^*(A))
\end{array}
\]
where in the second row, $D(f)_!$ (implicitly) denotes the derived functor. Unravelling the definitions, the top horizontal map is given without differentials by the natural map
\[
D(P) \circ D(Q) \circ \kappa[n,n+1] \to D(P) \circ \kappa[n+1].
\]
This map is an isomorphism, so the result follows.

**Proof of Theorem 5.1 (B).** This is the special case of Proposition 5.17 where the map $P \to P' = \kappa$ is the augmentation map.

**Proof of Theorem 5.1 (C).** Let $A$ be a small $P$-algebra and consider a pullback square (5.2) in the $\infty$-category of $P$-algebras. Inspecting the proof of Lemma 5.12 (cf. Diagram (5.15)), one can present such a square in the $\infty$-category of $P$-algebras by a strict diagram of $P$-algebras of the form
\[
\begin{array}{ccc}
B & \sim & \tilde{B} \\
p & & \downarrow \\
A & \sim & \bar{A}
\end{array}
\]
where $p: B \to A$ is a square zero extension of very small, nilpotent $P$-algebras. By a standard model categorical argument, one can in fact assume that the surjective map $\tilde{B} \to \bar{A}$ is given by $\Omega B(p): \Omega_B B\phi(B) \to \Omega_B B\phi(A)$, and that the left two quasi-isomorphisms are the canonical maps from the bar-cobar resolution.

Now apply the bar construction $B\phi$ to the above diagram. Then the left two weak equivalences admit canonical sections. Using these canonical sections, one obtains a composite square of $B\phi$-coalgebras of the form (5.9), which is cartesian. After dualizing, one obtains a square of the form
\[
\begin{array}{ccc}
D\phi(k_c[n+1]) & \longrightarrow & D\phi(A) \\
\downarrow & & \downarrow \\
D\phi(k_c[n,n+1]) & \longrightarrow & D\phi(B)
\end{array}
\]

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We have to show that this square is a homotopy pushout square of $\mathcal{D}(\mathcal{P})$-algebras. Since all $\mathcal{P}$-algebras involved in this square are very small and conilpotent, Proposition 5.11 implies the above square is naturally equivalent to the square (5.10) of polynomial subalgebras. But then the proof of Lemma 5.8 shows that this square is a (homotopy) pushout square of $\mathcal{D}(\mathcal{P})$-algebras (cf. Diagram (5.10)).

We conclude that the functor $\mathcal{D} : \text{Alg}_{\mathcal{P}} \to \text{Alg}_{\mathcal{P}}^{\text{op}}$ satisfies the conditions of Theorem 4.34. In particular, this says that the functor

$$\text{MC} : \text{Alg}_{\mathcal{D}(\mathcal{P})} \xrightarrow{\sim} \text{FMP}_{\mathcal{P}} : \mathfrak{g} \mapsto \text{Map}_{\mathcal{D}(\mathcal{P})}(\mathcal{D}(-), \mathfrak{g})$$

is an equivalence of $\infty$-categories, with inverse sending a formal moduli problem $F$ to $T(F)$. This proves Theorem 1.3.

**Variant 5.18.** Let $\mathcal{C}$ be a $k$-cooperad which is filtered-cofibrant as a $k$-module and let $\iota : \mathcal{C} \to \Omega \mathcal{C} = \mathcal{P}$ be the universal twisting morphism. Inspecting the above proof, one sees that the conclusions of Theorem 5.1 remain valid as long as $\mathcal{D}^{\text{poly}}(\iota(A)) \to \mathcal{D}_{\iota}(A)$ is a quasi-isomorphism for every $A$ that is very small and nilpotent. Consequently, Theorem 1.3 then holds for the operad $\mathcal{P} = \Omega \mathcal{C}$.

As an important example of this situation, let us record the following. Suppose that $k$ is a dg-category such that all $k(c,d)$ are concentrated in some fixed interval $[a,b]$, and suppose $\mathcal{C}$ is a $k$-cooperad with the following property: $\mathcal{C}(p)$ is concentrated in degrees $\leq f(p)$, with

$$f(p) \xrightarrow{p \to \infty} -\infty.$$ 

Note that when $A$ is very small, there is an $n$ such that any $n$-fold composition of generating operations acts trivially on $A$. Since $A$ is concentrated in finitely many degrees (Remark 4.6), this means that such $A$ is automatically nilpotent (Definition 5.5). Furthermore, the map $\mathcal{D}^{\text{poly}}(\iota(A)) \to \mathcal{D}_{\iota}(A)$ is then an isomorphism for degree reasons (cf. the proof of Lemma 5.19). The above proof and Theorem 4.34 then imply that there is equivalence of $\infty$-categories

$$\text{Alg}_{\mathcal{C}^\vee} = \text{Alg}_{\mathcal{D}(\mathcal{P})} \xrightarrow{\sim} \text{FMP}_{\mathcal{P}}.$$ 

Note that $\mathcal{C}$ may have contributions from positive degrees, as long as it is eventually concentrated in sufficiently negative degrees. In particular, this hold when $\mathcal{C}$ concentrated in finitely many arities.

### 5.3 Proof of Proposition 5.11

This section is devoted to the proof of Proposition 5.11. Throughout, we assume that $k$ is as in Assumption 5.3, i.e. concentrated in cohomological degrees $[-N,0]$, and that $\mathcal{P}$ is a $k$-operad in nonpositive degrees. We will prove Proposition 5.11 in increasing levels of generality, starting with the following special case:

**Lemma 5.19.** Suppose that $\mathcal{P} = \mathcal{P}^\leq_{\leq p}$ is nonpositively graded and concentrated in arities $\leq p$, and let $\pi : B^\mathcal{P} \to \mathcal{P}$ be the universal twisting morphism. If $A$ is a nonpositively graded $\mathcal{P}$-algebra, then the map of $B\mathcal{P}^{\vee}$-algebras

$$\mathcal{D}^{\text{poly}}(\pi(A)) \to \mathcal{D}_{\pi}(A)$$

is an isomorphism.
Proof. Since \( \mathcal{P} \) is concentrated in arities \( \leq p \), its bar construction is generated by operations in arities \( \leq p \) and degrees \( \leq -1 \). This means that the arity \( q \) part of \( B\mathcal{P} \) is concentrated in degrees \( \leq -q/p \). Consequently, each term

\[
B\mathcal{P}(q) \otimes_{\Sigma_k \times k^\otimes q} A^\otimes q
\]

is concentrated in degrees \( \leq -q/p \). Since \( k \) is concentrated in degrees \([-N, 0]\), the \( k \)-linear dual is concentrated in degrees \( \geq q/p - N \) in arity \( q \). Consequently, in each degree there are only finitely many arities that contribute to \( D(A) \), i.e. the map

\[
\bigoplus_{q \geq 0} \left( B\mathcal{P}(q) \otimes_{\Sigma_k \times k^\otimes q} A^\otimes q \right)^\vee \longrightarrow \prod_{q \geq 0} \left( B\mathcal{P}(q) \otimes_{\Sigma_k \times k^\otimes q} A^\otimes q \right)^\vee
\]

is an isomorphism in each individual degree.

Let us next consider the case of a \( 0 \)-reduced \( k \)-operad \( \mathcal{P} \), i.e. \( \mathcal{P}(0) = 0 \). Then the tower of quotients

\[
\mathcal{P} \longrightarrow \ldots \longrightarrow \mathcal{P}^{\leq p} \longrightarrow \mathcal{P}^{\leq p-1} \longrightarrow \ldots
\]

is a tower of operads. By definition, every nilpotent \( \mathcal{P} \)-algebra \( A \) can be considered as a \( \mathcal{P}^{\leq p_0} \)-algebra, for some \( p_0 \).

Lemma 5.20. Let \( \mathcal{P} \) be a \( 0 \)-reduced \( k \)-operad, concentrated in nonpositive degrees, and let

\[
\pi : B\mathcal{P} \longrightarrow \mathcal{P} \quad \text{and} \quad \pi^{\leq p} : B(\mathcal{P}^{\leq p}) \longrightarrow \mathcal{P}^{\leq p}
\]

denote the universal twisting morphisms. For each \( \mathcal{P}^{\leq p} \)-algebra \( A \) in nonpositive degrees there is a natural square of chain complexes

\[
\begin{array}{ccc}
\text{colim}_{p \geq p_0} D_{\pi \leq p}^\text{poly}(A) & \longrightarrow & D_{\pi}^\text{poly}(A) \\
\cong \downarrow & & \downarrow \\
\text{colim}_{p \geq p_0} D_{\pi \leq p}(A) & \longrightarrow & D_{\pi}(A)
\end{array}
\]

in which the two marked arrows are isomorphisms.

Proof. Recall that for every map of twisting morphisms \( \phi \longrightarrow \phi' \), there is a natural map of chain complexes \( D_{\phi'}(A) \longrightarrow D_{\phi}(A) \). When \( A \) is nilpotent, this restricts to polynomial subalgebras. This gives the desired square. The vertical arrow is an isomorphism by Lemma 5.19 and the horizontal arrow is given without differentials by the map

\[
\bigoplus_{q \geq 0} \text{colim}_{p \geq p_0} \left( B(\mathcal{P}^{\leq p})(q) \otimes_{\Sigma_k \times k^\otimes q} A^\otimes q \right)^\vee \longrightarrow \bigoplus_{q \geq 0} \left( B\mathcal{P}(q) \otimes_{\Sigma_k \times k^\otimes q} A^\otimes q \right)^\vee
\]

This map is an isomorphism. Indeed, the tower

\[
B\mathcal{P}(q) \longrightarrow \ldots \longrightarrow B(\mathcal{P}^{\leq p+1})(q) \longrightarrow B(\mathcal{P}^{\leq p})(q) \longrightarrow \ldots
\]

becomes stationary as soon as \( p \geq q \), so that the sequence obtained by tensoring with \( A^\otimes q \) and taking \( k \)-linear duals becomes stationary for \( p \geq q \) as well. \( \square \)
Corollary 5.21. Let $\mathcal{P}$ be a 0-reduced $\mathbb{k}$-operad in nonnegative degrees and let $A$ be a $\mathcal{P}^{\leq p_0}$-algebra in nonnegative degrees, for some $p_0$. Let $\pi: B\mathcal{P} \to \mathcal{P}$ be the universal twisting morphism. Then the map of complexes

$$D^\text{poly}_\pi(A) \to D_\pi(A)$$

can be identified with the natural map

$$\text{hocolim}_{p \geq p_0} D^\leq p(A) \to D(A)$$

where $D^\leq p: \text{Alg}_{\mathcal{P}^{\leq p}} \to \text{Alg}_{D(\mathcal{P}^{\leq p})}$ and $D: \text{Alg}_\mathcal{P} \to \text{Alg}_{D(\mathcal{P})}$.

In particular, Corollary 5.21 furnishes a homotopy-invariant characterization of the map $D^\text{poly}_\pi(A) \to D_\pi(A)$, as long as we take all our operads and algebras to be nonpositively graded: it no longer depends on the specific point-set models for the twisting morphism $\pi$ or $A$ (as long as these models are nonpositively graded).

Proposition 5.22. Let $\mathbb{k}$ be a dg-category in degrees $[-N, 0]$ and let $\mathcal{P}$ be a splendid, 0-reduced $\mathbb{k}$-operad, concentrated in degrees $\leq 0$. Let $A$ be a $\mathcal{P}^{\leq p_0}$-algebra which is freely generated as a $\mathbb{k}$-module by generators of degrees $\leq 0$, with finitely many generators of degree 0. Then the map

$$\text{hocolim}_{p \geq p_0} D^\leq p(A) \to D(A)$$

is an equivalence.

Proof. We can work at the level of chain complexes. Since $D$ and $D^\leq p$ are homotopy invariant, we may resolve the tower $\mathcal{P} \to \ldots \to \mathcal{P}^{\leq p} \to \ldots$ by a tower of cofibrant $\mathbb{k}$-operads

$$\Omega \to \ldots \to \Omega^{(p)} \to \Omega^{(p-1)} \to \ldots$$

with the properties described in Proposition 9.42. In particular, each $\Omega^{(p)}$ is a quasi-free $\mathbb{k}$-operad generated by a nonnegatively graded, cofibrant $\mathbb{k}$-symmetric sequence $V^{(p)}$.

For each the quasi-free $\mathbb{k}$-operad $Q = \text{Free}(V)$, the right $Q$-module $\mathbb{k}$ admits a cofibrant resolution of the form

$$K = \text{Cone}(V \circ_k \Omega \to \Omega).$$

By Remark 9.30, the underlying complex of $D(A)$ can be identified with

$$\left(K \circ Q A\right)^\vee \cong \left((1 \oplus V[1]) \circ_k A\right)^\vee$$

with some differential. Let us now decompose $A = A_0 \oplus \overline{A}$, where $A_0$ is the $\mathbb{k}$-module generated by the (finitely many) degree 0 generators and $\overline{A}$ is generated by elements of degree $< 0$. By Proposition 9.42, $V^{(p)}$ is concentrated in increasingly negative degrees as its arity increases. Using that $A_0$ is free on finitely many generators, one then sees that

$$\left(K \circ Q^{(p)} A\right)^\vee \cong \prod_{q, r \geq 0} M(q, r)$$

with some differential, where

$$M(q, r) := \left((1 \oplus V[1])(q + r) \otimes \Sigma_{k \otimes k \otimes r} \overline{A}^{\otimes q} \otimes \Sigma_{r \otimes k \otimes r} (A_0^{(q)})^{\otimes r}\right)^\vee.$$
Since $\overline{A}$ is concentrated in degrees $\leq -1$ and $k$ is concentrated in degrees $[-N,0]$, $M(q,r)$ is concentrated in degrees $\geq q-N$. Consequently, in each fixed cohomological degree there are only contributions of the $M(q,r)$ for finitely many $q$.

Similarly, $V$ is concentrated in increasingly negative degrees as the arity increases. Consequently, in each fixed cohomological degree there are only contributions of the $M(q,r)$ for finitely many $r$. It follows that the above product over $q$ and $r$ is isomorphic to a direct sum, so that

$$\mathcal{D}(A) \simeq \left( \mathcal{K} \circ \mathcal{Q}(p) \ A \right)^\vee \cong \bigoplus_{q,r \geq 0} M(q,r). \quad (5.23)$$

The same analysis applies to each of the graded-free operads $\mathcal{P}(p) = \text{Free}(V(p))$ with $p \geq p_0$. Consequently, one finds that the sequence of chain complexes \ldots $\rightarrow \mathcal{D}^{\leq p}(A) \rightarrow \mathcal{D}^{\leq p+1}(A) \rightarrow \ldots \rightarrow \mathcal{D}(A)$ is quasi-isomorphic to the sequence of sums

$$\ldots \rightarrow \bigoplus_{q,r \geq 0} M^{(p)}(q,r) \rightarrow \bigoplus_{q,r \geq 0} M^{(p+1)}(q,r) \rightarrow \ldots \rightarrow \bigoplus_{q,r \geq 0} M(q,r) \quad (5.24)$$

endowed with some differential. We claim that this sequence is a colimit sequence of complexes. This means that it is also a homotopy colimit, which proves the proposition.

To see that (5.24) is a colimit sequence, it suffices to prove that for every fixed $r$, the sequence of graded vector spaces

$$\bigoplus_{q \geq 0} \left( (1 \oplus V^{(p)}[1])(q+r) \otimes \Sigma_{k \geq 0} \overline{A}^{\otimes q} \right)^\vee$$

is a colimit sequence. We claim that this sequence becomes stationary in every fixed cohomological degree. Indeed, since the $q$-th summand is concentrated in degrees $\geq q$ (since $\overline{A}$ is generated by elements of degree $\leq -1$), only finitely many summands contribute to each individual degree. It therefore suffices to verify that for each $q$ and $r$, the sequence of graded vector spaces

$$\left( (1 \oplus V^{(p)}[1])(q+r) \otimes \Sigma_{k \geq 0} \overline{A}^{\otimes q} \right)^\vee$$

becomes stationary as $p \rightarrow \infty$. But this follows from the construction of Proposition 9.42, which guaranteed that $V^{(p)}(q+r)$ is constant for $p \geq q+r$. \hfill $\square$

To deduce Proposition 5.11 from Proposition 5.22, we now only have deal with the extra operations in arity zero that obstruct the use of the tower of quotients $\mathcal{P} \rightarrow \mathcal{P}^{\leq p}$. This is done by a filtration argument:

**Construction 5.25.** Let $\mathcal{P}$ be a $k$-operad which is cofibrant as a left $k$-module and let $\pi: B\mathcal{P} \rightarrow \mathcal{P}$ be the universal twisting morphism. For any $\mathcal{P}$-algebra $A$ which is cofibrant as a $k$-module, we can filter the bar construction $B_{\pi}(A) = B\mathcal{P} \circ_k A$ by word length in the nullary operations of $\mathcal{P}$. This is an increasing filtration by left $k$-modules which preserves the bar differential.

The associated graded can be described as follows: let $\mathcal{P}^{\geq 1}$ denote part of $\mathcal{P}$ in nonzero arity and let $\pi^{\geq 1}: B(\mathcal{P}^{\geq 1}) \rightarrow \mathcal{P}^{\geq 1}$ be the universal twisting morphism. Then we can identify

$$\text{gr}(B_{\pi}(A)) = B_{\pi^{\geq 1}}(A \oplus \mathcal{P}(0)[1])$$

where $A \oplus \mathcal{P}(0)[1]$ is the product of $A$, considered as a $\mathcal{P}^{\geq 1}$-algebra by restriction, and the trivial algebra $\mathcal{P}(0)[1]$. Since all pieces are cofibrant as $k$-modules, dualizing yields a complete Hausdorff filtration on $\mathcal{D}_{\pi}(A)$ whose associated graded is

$$\text{gr}(\mathcal{D}_{\pi}(A)) = \mathcal{D}_{\pi^{\geq 1}}(A \oplus \mathcal{P}(0)[1])$$

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Lemma 5.26. Suppose that $\mathcal{P}$ is a $k$-operad in nonpositive degrees and let $A$ be a $\mathcal{P}$-algebra which is very small and nilpotent. Then the complete Hausdorff filtration on $D_\pi(A)$ from Construction 5.25 restricts to a complete Hausdorff filtration on $D_\pi^{\text{poly}}(A)$. Furthermore, the map $D_\pi^{\text{poly}}(A) \to D_{\pi^+}(A)$ induces the obvious map

$$D_\pi^{\text{poly}}(A \oplus \mathcal{P}(0)[1]) \to D_{\pi^+}(A \oplus \mathcal{P}(0)[1])$$

at the level of the associated graded.

Proof. Let us first check that the induced filtration on $D_\pi^{\text{poly}}(A)$ is complete Hausdorff. By Construction 5.25, we can write

$$B_\pi(A) \cong \bigoplus_{q,r \geq 0} \left( B(\mathcal{P}^{\geq 1})(q + r) \otimes _{\Sigma_q \times k^{\otimes q}} \mathcal{P}(0)^{\otimes q} \right) [q] \otimes _{\Sigma_r \times k^{\otimes r}} A^{\otimes r}$$

as left $k$-modules, with some differential. The filtration is indexed by $q$. Since $A$ is finitely generated quasi-free over $k$, the $k$-linear dual of each of summand is given by

$$N(q,r) := \left( B(\mathcal{P}^{\geq 1})(q + r) \otimes _{\Sigma_q \times k^{\otimes q}} \mathcal{P}(0)^{\otimes q} \right)^{\vee} [-q] \otimes _{\Sigma_r \times (k^{\otimes r})^{\otimes r}} (A^{\vee})^{\otimes r}$$

and we have that

$$D_\pi(A) \cong \prod_{q,r \geq 0} N(q,r) \quad \text{and} \quad D_\pi^{\text{poly}}(A) = \bigoplus_{r \geq 0} \prod_{q \geq 0} N(q,r).$$

Since $k$ is concentrated in degrees $[-N,0]$ and both $A$ and $\mathcal{P}$ are concentrated in nonpositive degrees, we have that $N(q,r)$ is concentrated in degrees $q \geq N$, for all values of $r$. Consequently, the natural map

$$D_\pi^{\text{poly}}(A) = \bigoplus_r \prod_{q \geq 0} N(q,r) \to \prod_{q \geq 0} \bigoplus_r N(q,r)$$

is an isomorphism in each cohomological degree. Now note that $D_\pi^{\text{poly}}(A) \cong \prod_{q \geq 0} \bigoplus_r N(q,r)$ is manifestly complete Hausdorff with respect to the filtration by $q$. Furthermore, we see that, without differential, there is an inclusion

$$\text{gr}(D_\pi^{\text{poly}}) \subseteq \text{gr}(D_\pi)$$

given in degree $q$ by the obvious inclusion $\bigoplus_r N(q,r) \to \prod_r N(q,r)$. Since $\text{gr}(D_\pi(A)) \cong D_{\pi^+}(A \oplus \mathcal{P}(0)[1])$, the second part of the lemma then follows by unravelling the definitions. \qed

Proof (of Proposition 5.11). Suppose that $k$ is concentrated in degrees $[-N,0]$, that $\mathcal{P}$ is concentrated in nonpositive degrees and that $A$ is nilpotent and very small. In particular, $A$ is quasi-free and finitely generated over $k$ (Remark 4.6). To see that $D_\pi^{\text{poly}}(A) \to D_\pi(A)$ is a quasi-isomorphism, we can work at the level of complexes.

Endow both $D_\pi^{\text{poly}}(A)$ and $D_\pi(A)$ with the filtration by the number of nullary operations from $\mathcal{P}$, as in Lemma 5.26. Since these filtrations are complete and Hausdorff, it suffices to show that the induced map on the associated graded

$$D_\pi^{\text{poly}}(A \oplus \mathcal{P}(0)[1]) \to D_{\pi^+}(A \oplus \mathcal{P}(0)[1])$$

is a quasi-isomorphism, where $\pi^+ : B(\mathcal{P}^{\geq 1}) \to \mathcal{P}^{\geq 1}$ is the universal twisting morphism. Note that the $\mathcal{P}^{\geq 1}$-algebra $A \oplus \mathcal{P}(0)[1]$ satisfies the conditions of Proposition 5.22: it is quasi-free over $k$ and it has finitely many generators in degree 0, all coming from $A$. The result then follows from Corollary 5.21 and Proposition 5.22. \qed
6 Change of operads

In this section we describe the functoriality of the equivalence

\[ \text{MC}: \text{Alg}_{\mathcal{D}(P)} \longrightarrow \text{FMP}_P \]

in the operad \( P \). We will start by considering the functoriality of the adjoint pair \((\mathcal{D}_\phi, \mathcal{D}'_\phi)\) in the twisting morphism \( \phi \).

6.1 Naturality of weak Koszul duality

To study the dependence of the adjoint pair \((\mathcal{D}_\phi, \mathcal{D}'_\phi)\) on the twisting morphism \( \phi: \mathcal{C} \longrightarrow \mathcal{P} \), let us consider the following category of twisting morphisms:

**Definition 6.1.** Let Koszul denote the category whose
- objects are weakly Koszul twisting morphisms \( \phi: \mathcal{C} \longrightarrow \mathcal{P} \) from a \( k \)-cooperad \( \mathcal{C} \) to a \( k \)-operad \( \mathcal{P} \). When considered as a left \( k \)-module, \( \mathcal{C} \) is filtered-cofibrant and \( \mathcal{P} \) is cofibrant by assumption.
- morphisms consist of a \( k \)-operad map \( f: \mathcal{P} \longrightarrow \Omega \) and a \( k \)-cooperad map \( g: \mathcal{C} \longrightarrow \mathcal{D} \), fitting into a commuting square

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\phi} & \mathcal{P} \\
\downarrow{g} & & \downarrow{f} \\
\mathcal{D} & \xrightarrow{\psi} & \Omega.
\end{array}
\]

A map between such twisting morphisms is a weak equivalence if \( f \) (and hence also \( g \)) is a quasi-isomorphism.

**Remark 6.3.** There is an obvious projection map \( \pi: \text{Koszul} \longrightarrow \text{Op}_{k}^{dg} \) sending a twisting morphism to its codomain. This projection admits a section sending \( \mathcal{P} \) to the universal twisting morphism \( B\mathcal{P} \longrightarrow \mathcal{P} \). Every weakly Koszul twisting morphism \( \phi: \mathcal{C} \longrightarrow \mathcal{P} \) admits a natural weak equivalence to \( B\mathcal{P} \longrightarrow \mathcal{P} \), so that \( \pi \) induces an equivalence of \( \infty \)-categories after inverting the weak equivalences.

Consider the following functors with values in the \( \infty \)-category of \( \infty \)-categories and left adjoint functors between them

\[ \text{Alg}: \text{Koszul} \longrightarrow \text{Cat}_{\infty}^{L} \quad \text{Alg}^{\text{dual}}: \text{Koszul} \longrightarrow \text{Cat}_{\infty}^{L}. \]

These two functors send a map (6.2) to the left adjoint functors

\[ f: \text{Alg}_{\mathcal{P}} \longrightarrow \text{Alg}_{\Omega} \quad (g')^*: \text{Alg}_{\mathcal{C}}^{\text{op}} \longrightarrow \text{Alg}_{\mathcal{D}}^{\text{op}}. \]

We then have the following homotopy coherent upgrade of Lemma 4.20:

**Proposition 6.5.** There is a natural transformation of functors

\[
\begin{array}{ccc}
\text{Alg} & \xrightarrow{f} & \text{Cat}_{\infty}^{L} \\
\text{Alg}_{\mathcal{D}} & \xleftarrow{\text{Alg}^{\text{dual}}} & \text{Cat}_{\infty}^{L}
\end{array}
\]

whose value at a weakly Koszul twisting morphism \( \phi: \mathcal{C} \longrightarrow \mathcal{P} \) is given by

\[ \text{Alg}_{\mathcal{P}} \xrightarrow{D_{\phi}} \text{Alg}_{\mathcal{C}}^{\text{op}}. \]
Note that the functors $\text{Alg}$ and $\text{Alg}^{\text{dual}}$ send weak equivalences between twisting morphisms to equivalences of $\infty$-categories (Corollary 9.6), and hence descend to functors on the $\infty$-categorical localizations (cf. the discussion preceding Definition 4.24). By Remark 6.3, we therefore obtain the following:

**Corollary 6.7.** Let $\text{Op}_k$ be the $\infty$-category of (augmented) $k$-operads. Then there is a natural transformation of functors

$$\text{Op}^{\text{aug}}_k \xrightarrow{\text{Alg}} \text{Alg}^{\text{dual}} \rightarrow \text{Cat}^L_\infty$$

given on objects by

$$\text{Alg}(\mathcal{P}) := \text{Alg}_\mathcal{P} \xrightarrow{\mathcal{D}} \text{Alg}^{\text{op}}_{\mathcal{D}(\mathcal{P})} := \text{Alg}^{\text{dual}}(\mathcal{P}).$$

Recall from Lemma 4.20 that a single map of twisting morphisms induces a square of $\infty$-categories commuting up to natural equivalence. For this reason, it will be more convenient to establish Proposition 6.5 in terms of fibrations.

**Construction 6.8.** Let $\text{Alg}^{\text{dg}}$ denote the category whose

- objects are tuples $(\phi: \mathcal{C} \rightarrow \mathcal{P}, A)$ consisting of a weakly Koszul twisting morphism, together with a cofibrant $\mathcal{P}$-algebra $A$.
- morphisms $(\phi: \mathcal{C} \rightarrow \mathcal{P}, A) \rightarrow (\psi: \mathcal{D} \rightarrow \Omega, B)$ consist of a map (6.2) and a map of $\mathcal{P}$-algebras $A \rightarrow f^*B$.

Similarly, let $\text{Alg}^{\text{dual,dg}}$ denote the category whose

- objects are tuples $(\phi: \mathcal{C} \rightarrow \mathcal{P}, \mathfrak{g})$ consisting of a weakly Koszul twisting morphism, together with a $\mathcal{C}^\vee$-algebra $\mathfrak{g}$.
- morphisms $(\phi: \mathcal{C} \rightarrow \mathcal{P}, \mathfrak{g}) \rightarrow (\psi: \mathcal{D} \rightarrow \Omega, \mathfrak{h})$ consist of a map (6.2) and a map of $\mathcal{P}$-algebras $(\mathfrak{g}^\vee)^*(\mathfrak{g}) \rightarrow \mathfrak{h}$.

There are obvious projections

$$\begin{align*}
\text{Alg}^{\text{dg}} & \longrightarrow \text{Koszul} \\
\text{Alg}^{\text{dual,dg}} & \longrightarrow \text{Koszul},
\end{align*}$$

whose fibers over $\phi: \mathcal{C} \rightarrow \mathcal{P}$ are given by the categories of cofibrant $\mathcal{P}$-algebras and of $\mathcal{C}^\vee$-algebras, respectively. We will say that a map in $\text{Alg}^{\text{dg}}$ is a fiberwise weak equivalence if it is a quasi-isomorphism of algebras that covers the identity in the base category Koszul.

Note that both projections are cocartesian fibrations: given a map (6.2) in the base category Koszul, the induced functors between the fibers are given by $f_!$ and $(g^\vee)^*$. By construction, these change-of-fiber functors preserve fiberwise weak equivalences, so that inverting the fiberwise weak equivalences yields cocartesian fibrations [Hin16, Proposition 2.1.4]

$$\begin{align*}
\text{Alg} & \longrightarrow \text{Koszul} \\
\text{Alg}^{\text{dual}} & \longrightarrow \text{Koszul}.
\end{align*}$$

These are exactly the cocartesian fibrations classified by the functors $\text{Alg}$ and $\text{Alg}^{\text{dual}}$ from (6.4). Since these functors take values in $\infty$-categories and left adjoint functors between them, the projections are cartesian fibrations as well [Lur09, Corollary 5.2.2.5].
Proof (of Proposition 6.5). Consider the commuting triangle

\[
\begin{array}{ccc}
\text{Alg}^{dg} & \xrightarrow{B^\vee} & \text{Alg}^{dual,dg} \\
\downarrow & & \downarrow \\
\text{Koszul} & & \text{Koszul}
\end{array}
\]

where the vertical functors are the projections and the top horizontal functor is given by

\[
(\phi: \mathcal{C} \to \mathcal{P}, A) \mapsto (\phi: \mathcal{C} \to \mathcal{P}, B_\phi(A)^\vee).
\]

This functor sends fiberwise weak equivalences in \(\text{Alg}^{dg}\) to fiberwise weak equivalences in \(\text{Alg}^{dual,dg}\) by Lemma 9.32. Consequently, it descends to a functor between the \(\infty\)-categorical localizations, which we will denote by

\[
\begin{array}{ccc}
\text{Alg} & \xrightarrow{\mathcal{D}} & \text{Alg}^{dual} \\
\downarrow & & \downarrow \\
\text{Koszul} & & \text{Koszul}
\end{array}
\] (6.9)

When restricted to the fiber over a weakly Koszul twisting morphism \(\phi\), this functor is given by \(\mathcal{D}_\phi\) and admits a right adjoint \(\mathcal{D}_\phi^\vee\) by Lemma 4.17. Furthermore, the functor \(\mathcal{D}: \text{Alg} \to \text{Alg}^{dual}\) preserves cocartesian edges. Indeed, unraveling the definitions, this is exactly the assertion of Lemma 4.20. It follows from [Lur17, Proposition 7.3.2.6] (and its dual) that the functor \(\mathcal{D}\) has a right adjoint which commutes with the projections, preserves cartesian edges and is given fiberwise by \(\mathcal{D}_\phi^\vee\). Under straightening, this means precisely that \(\mathcal{D}\) determines a natural transformation of the form (6.6). \(\square\)

6.2 Naturality of the main theorem

We will now use Corollary 6.7 to show that the equivalence between formal moduli problems and algebras of Theorem 1.3 depends functorially on the operad:

Proposition 6.10. Let \(k\) be a bounded connective dg-category and let \(\text{Op}_k^+\) denote the \(\infty\)-category of splendid connective \(k\)-operads. There is a natural equivalence of functors

\[
\begin{array}{ccc}
\text{Op}_k^+ & \xrightarrow{\text{Alg}_\mathcal{D}} & \text{Pr}^R \\
\downarrow & \xrightarrow{\text{FMP}} & \downarrow \\
\text{FMP} & \xrightarrow{\text{FMP}_\mathcal{D}} & \text{FMP}_\mathcal{D}
\end{array}
\]

with values in the \(\infty\)-category of locally presentable \(\infty\)-categories and right adjoint functors. The value of this natural equivalence at a map \(f: \mathcal{P} \to \mathcal{Q}\) is given by the commuting square of right adjoints

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{D}(\mathcal{P})} & \xrightarrow{\text{MC}} & \text{FMP}_{\mathcal{P}} \\
\downarrow & \xrightarrow{(f^*)^*} & \downarrow \\
\text{Alg}_{\mathcal{D}(\mathcal{Q})} & \xrightarrow{\text{MC}} & \text{FMP}_{\mathcal{D}}
\end{array}
\] (6.11)

where the right vertical functor restricts a formal moduli problem along the forgetful functor \(f^*: \text{Alg}^{sm}_{\mathcal{Q}} \to \text{Alg}^{sm}_{\mathcal{P}}\).
Lemma 6.12. Let $\mathbb{k}$ be a bounded connective dg-category and let $\text{Op}_k^+$ denote the $\infty$-category of splendid connective $\mathbb{k}$-operads. Then there is a natural transformation of functors

$$
\begin{array}{c}
\text{Op}_k^+ \to \text{Cat}_\infty
\end{array}
$$

whose value on a map $f: \mathcal{P} \to \mathcal{Q}$ is given by

$$
\begin{array}{c}
\text{Alg}_{\mathcal{Q}}^+ \to \text{Alg}_{\mathcal{D}(\mathcal{Q})}^+
\end{array}
$$

Proof. Recall the commuting triangle $(6.9)$, where the vertical projections are cartesian and cocartesian fibrations and the top horizontal functor $\mathcal{D}$ sends $(\phi: \mathcal{C} \to \mathcal{P}, A)$ to $((\phi: \mathcal{C} \to \mathcal{P}, \mathcal{D}_\phi(A))$. Let us consider the following subcategories of the $\infty$-categories appearing in that triangle:

- Let $\text{Koszul}^+ \subseteq \text{Koszul}$ denote the subcategory of universal twisting morphisms $B\mathcal{P} \to \mathcal{P}$ where $\mathcal{P}$ is a splendid connective $\mathbb{k}$-operad, with maps between those given by tuples $f: \mathcal{P} \to \mathcal{Q}$ and $B(f): B\mathcal{P} \to B\mathcal{Q}$. Inverting the weak equivalences in $\text{Koszul}^+$ yields the $\infty$-category $\text{Op}_k^+$.

- Let $\text{Alg}^{+, \text{dual}} = \text{Alg} \times_{\text{Koszul}} \text{Koszul}^+$ be the restriction of $\text{Alg}^{\text{dual}}$ to the category $\text{Koszul}^+$.

- Let $\text{Alg}^{+, \text{sm}} \subseteq \text{Alg} \times_{\text{Koszul}} \text{Koszul}^+$ be the full subcategory of tuples $(\phi: B\mathcal{P} \to \mathcal{P}, A)$ with $A$ a small $\mathcal{P}$-algebra.

The functors appearing in $(6.9)$ then restrict to

$$
\begin{array}{c}
\text{Alg}^{+, \text{sm}} \to \text{Koszul}^+
\end{array}
$$

Note that the projection $\text{Alg}^{+, \text{dual}} \to \text{Koszul}^+$ is (the restriction of) a cocartesian and cartesian fibration. Since the restriction of a small algebra along a map of operads $\mathcal{P} \to \mathcal{Q}$ is again small, the projection $\text{Alg}^{+, \text{sm}} \to \text{Koszul}^+$ is a cartesian fibration as well. Recall that $\mathcal{D}$ sends a tuple $(\phi: \mathcal{C} \to \mathcal{P}, A)$ to $(\phi: \mathcal{C} \to \mathcal{P}, \mathcal{D}_\phi(A))$. In particular, Proposition 5.17 shows that it preserves cartesian edges, so that we obtain a natural transformation

$$
\begin{array}{c}
\text{Koszul}^{+, \text{op}} \to \text{Cat}_\infty
\end{array}
$$

The domain of this natural transformation is given by $\mathcal{P} \to \text{Alg}_{\mathcal{P}}^+$ and the codomain is given by $\mathcal{P} \to \text{Alg}_{\mathcal{D}(\mathcal{P})}^+$. Since these functors send quasi-isomorphisms to equivalences of $\infty$-categories, the natural transformation descends to the localization of $\text{Koszul}^+$ at the quasi-isomorphisms, yielding the desired natural transformation $(6.13)$.
Proof. Consider the natural transformation $(6.13)$ from Lemma 6.12. Taking the opposites of its values, one obtains a functor with values in $\infty$-categories sending a map $f: \mathcal{P} \rightarrow \mathcal{Q}$ to

$$\begin{align*}
\text{Alg}_{\mathcal{Q}}^{\text{sm}, \text{op}} & \xrightarrow{\mathcal{D}} \text{Alg}_{\mathcal{Q}}(\mathcal{D}(f)) \\
\text{Alg}_{\mathcal{P}}^{\text{sm}, \text{op}} & \xrightarrow{\mathcal{D}} \text{Alg}_{\mathcal{Q}}(\mathcal{D}(f)).
\end{align*}$$

By the universal property of presheaf categories [Lur09, Theorem 5.1.5.6], one obtains a natural transformation of functors $\text{Op}^{+} \rightarrow \text{Pr}^{R}$ with values in presentable $\infty$-categories and right adjoint functors, whose value on $f$ is given by

$$\begin{align*}
\text{Alg}_{\mathcal{Q}}(\mathcal{D}(P)) & \xrightarrow{\mathcal{D}^*} \text{Fun}(\text{Alg}_{\mathcal{P}}^{\text{sm}}, \mathcal{S}) \\
\text{Alg}_{\mathcal{Q}}(\mathcal{D}(Q)) & \xrightarrow{\mathcal{D}^*} \text{Fun}(\text{Alg}_{\mathcal{P}}^{\text{sm}}, \mathcal{S}).
\end{align*}$$

Here the functor $(f^*)^*$ restricts a (co)presheaf along $f^*$ and $\mathcal{D}^*$ sends a $\mathcal{D}(\mathcal{P})$-algebra $g$ to the functor $A \mapsto \text{Map}_{\mathcal{D}(\mathcal{P})}(\mathcal{D}(A), g)$. By part (C) of Theorem 5.1, the natural transformation $\mathcal{D}^*$ takes values in diagram of full subcategories $\mathcal{P} \rightarrow \text{FMP}_{\mathcal{P}}$. The result then follows from the fact that $\text{MC}: \text{Alg}_{\mathcal{D}(\mathcal{P})} \rightarrow \text{FMP}_{\mathcal{P}} \xhookrightarrow{\text{MC}} \text{Fun}(\text{Alg}_{\mathcal{P}}^{\text{sm}}, \mathcal{S})$ agrees with $\mathcal{D}^*$ by definition. \qed

7 Maurer–Cartan equation

In the previous sections we have discussed how – for a suitable augmented $k$-operad $\mathcal{P}$ – every algebra $g$ over the dual operad $\mathcal{D}(\mathcal{P})$ determines a formal moduli problem

$$\text{MC}_g: \text{Alg}_{\mathcal{P}}^{\text{sm}}/k \rightarrow \mathcal{S}.$$ 

The formal moduli problem has been defined in $\infty$-categorical terms by the formula

$$\text{MC}_g(A) = \text{Map}_{\mathcal{D}(\mathcal{P})}(\mathcal{D}(A), g).$$

The purpose of this section is to give a more explicit chain-level description of this functor in terms of Maurer–Cartan elements of (nilpotent) $L_\infty$-algebras (see Theorem 7.18). In particular, in the classical case where $\mathcal{P} = \text{Com}$ and $g$ is a Lie algebra, we recover the usual formula (see e.g. [Hin01])

$$\text{MC}_g(A) = \text{MC}(A \otimes g \otimes \Omega_\cdot)$$

describing the formal moduli problem classified by $g$ in terms of simplicial sets of Maurer–Cartan elements (see Example 7.20).

We will start by recalling some models for $\infty$-categories of algebras by simplicially enriched categories. Under certain finiteness conditions on the $k$-operad $\mathcal{P}$ (see Assumption 7.6), we can then present the $\infty$-functor $\mathcal{D}$ on small $\mathcal{P}$-algebras by a simplicially enriched functor that sends a very small $\mathcal{P}$-algebra to the cobar construction of its linear dual. The results of Section 9.3 then allow us to describe $\text{MC}_g$ in terms of Maurer–Cartan elements.
7.1 Simplicial categories of algebras

Recall that for any $k$-operad $\mathcal{P}$, the $\infty$-category of $\mathcal{P}$-algebras is defined to be the $\infty$-category obtained from the model category of $\mathcal{P}$-algebras by localizing at the quasi-isomorphisms. Such localizations can be modelled by simplicially enriched categories, using the simplicial localization of Dwyer and Kan, and often its mapping spaces can be computed using fibrant resolutions [DK80]:

**Definition 7.1.** Given a $k$-operad $\mathcal{P}$, the naive simplicial category of $\mathcal{P}$-algebras $\text{Alg}_{\mathcal{P}}$ is the following simplicially enriched category:

- objects are $\mathcal{P}$-algebras.
- for two $\mathcal{P}$-algebras $A$ and $B$, the simplicial set $\text{Map}_{\mathcal{P}}(A, B)$ of maps between them has $n$-simplices given by maps of $\mathcal{P}$-algebras $A \to B \otimes \Omega_n$ where $\Omega_n = \Omega[\Delta^n]$ denotes the cdga of differential forms on the $n$-simplex. Equivalently, these are maps of $\mathcal{P} \otimes \Omega_n$-algebras $A \otimes \Omega_n \to B \otimes \Omega_n$.

Furthermore, let $\text{Alg}^\circ \subseteq \text{Alg}_\mathcal{P}$ denote the full simplicial subcategory on the cofibrant $\mathcal{P}$-algebras.

Let $\text{Alg}^d_\mathcal{P}$ denote the (ordinary) category of $\mathcal{P}$-algebras and let $\text{Alg}^d_\mathcal{P} \subseteq \text{Alg}_\mathcal{P}$ denote the subcategory of cofibrant $\mathcal{P}$-algebras. We then have a commuting square of simplicial categories

$$
\begin{array}{ccc}
\text{Alg}^d_\mathcal{P} & \xrightarrow{i} & \text{Alg}^d_\mathcal{P} \\
\downarrow & & \downarrow \\
\text{Alg}^\circ_\mathcal{P} & \xrightarrow{i} & \text{Alg}_\mathcal{P}
\end{array}
$$

where the vertical functors simply include the vertices of the mapping spaces.

After taking simplicial localizations at the quasi-isomorphisms, each of the above functors yields a weak equivalence of simplicial categories. Indeed, taking cofibrant replacements produces a functor $Q: \text{Alg}^d_\mathcal{P} \to \text{Alg}^d_\mathcal{P}$ such that $Q \circ i$ and $i \circ Q$ are naturally quasi-isomorphic to the identity. It follows that $i$ induces a weak equivalence after simplicial localization at the quasi-isomorphisms, and similarly for the inclusion $\text{Alg}^\circ_\mathcal{P} \to \text{Alg}_\mathcal{P}$. The right vertical functor induces an equivalence after localizations because for every $\mathcal{P}$-algebra $A$, the simplicial presheaf

$$
\text{Map}_{\mathcal{P}}(\cdot, A): \text{Alg}^d_\mathcal{P} \to \text{sSet}
$$

is representable by the simplicial diagram of $\mathcal{P}$-algebras $A \otimes \Omega[\Delta^n]$, all of which are quasi-isomorphic (see e.g. [DK87] or [Nui16, Corollary 2.9]). Now note that every quasi-isomorphism in $\text{Alg}^\circ_\mathcal{P}$ is already a homotopy equivalence [Hin97, Lemma 4.8.4]. Consequently, $\text{Alg}^\circ_\mathcal{P}$ is weakly equivalent to its simplicial localization and we obtain the following:

**Lemma 7.2.** If $\mathcal{P}$ is a $k$-operad, then the $\infty$-category of $\mathcal{P}$-algebras can be modelled by the simplicial category $\text{Alg}^\circ_\mathcal{P}$.

Given a Koszul twisting morphism $\phi: \mathcal{C} \to \mathcal{P}$, one can also describe the localization using $\infty$-morphisms:
Definition 7.3. Given a Koszul twisting morphism $\phi: \mathcal{C} \rightarrow \mathcal{P}$ (Definition 9.26), we define $\text{Alg}_{\mathcal{P}}^\infty$ to be the following simplicially enriched category:

- the objects of $\text{Alg}_{\mathcal{P}}^\infty$ are $\mathcal{P}$-algebras which are cofibrant as $k$-modules.
- for two such $\mathcal{P}$-algebras $A$ and $B$, the simplicial set $\text{Map}^\infty_{\mathcal{P}}(A,B)$ of maps between them has $n$-simplices given by $\infty$-morphisms $A \sim \rightarrow B \otimes \Omega_n$.

Equivalently, an $n$-simplex is a map of $C \otimes \Omega_n$-coalgebras

$$B_\phi(A) \otimes \Omega_n \longrightarrow B_\phi(B) \otimes \Omega_n.$$  \hfill (7.4)

Lemma 7.5. If $\phi: \mathcal{C} \rightarrow \mathcal{P}$ is a Koszul morphism over $k$, then the $\infty$-category of $\mathcal{P}$-algebras can be modelled by the simplicial category $\text{Alg}_{\mathcal{P}}^\infty$.

Proof. Including the strict morphisms into the $\infty$-morphisms and sending an $\infty$-morphism $A \sim \rightarrow B$ to the strict morphism $\Omega_\phi B_\phi(A) \rightarrow \Omega_\phi B_\phi(B)$ induces simplicially enriched functors

$$j: \text{Alg}_\mathcal{P}^\circ \longrightarrow \text{Alg}_{\mathcal{P}}^\infty \quad \Omega_\phi B_\phi: \text{Alg}_{\mathcal{P}}^\infty \longrightarrow \text{Alg}_\mathcal{P}^\circ.$$  

The natural homotopy equivalences $\Omega_\phi B_\phi(A) \rightarrow A$ (Lemma 9.33) and $A \sim \rightarrow \Omega_\phi B_\phi(A)$ show that $j$ and $\Omega$ define a homotopy equivalence of simplicial categories. \hfill $\Box$

7.2 Simplicial categories of small algebras

We will now specialize to the case where $\mathcal{P} = \Omega \mathcal{C}$ arises as the cobar construction of a $k$-cooperad satisfying suitable finiteness hypotheses:

Assumption 7.6. For the remainder of this section, let us fix a dg-category $k$ and a $k$-cooperad $\mathcal{C}$ which is filtered-cofibrant as a left $k$-module, and let $\mathcal{P} = \Omega \mathcal{C}$ denote its cobar construction. We will assume that $k$ is concentrated in degrees $[0,N]$, for some $N$, and that $\mathcal{C}$ satisfies the following conditions:

1. for all colours $c_1, \ldots, c_p \in S$, the left $k$-module $\mathcal{C}(c_1, \ldots, c_p; -)$ is quasi-free and finitely generated.
2. each $\mathcal{C}(p)$ is concentrated in degrees $\leq f(p)$, where $f(p)$ tends to $-\infty$ as the arity $p$ tends to $\infty$.

Example 7.7. Let $\mathcal{C} = \text{coFree}(E,R)$ be a quadratic cooperad over a field $k$, where $E$ is finite dimensional and in cohomological degrees $\leq 1$. Then $\mathcal{C}$ satisfies the conditions of Assumption 7.6. In particular, this applies to the quadratic dual cooperads of classical quadratic operads such as $\text{Com}, \text{As}, \text{Lie}$ and $\text{Perm}$.

Let us record the following immediate consequences of these assumptions:

Remark 7.8. The $k$-operad $\mathcal{P} = \Omega \mathcal{C}$ satisfies the conditions of Variant 5.18. In particular, for degree reasons every small $\mathcal{P}$-algebra $A$ is automatically nilpotent and the inclusion $\mathcal{D}_\phi^\text{poly}(A) \subseteq \mathcal{D}_\phi(A)$ is the identity. Theorem 4.34 then provides an equivalence of $\infty$-categories between formal moduli problems over $\mathcal{P}$ and $\mathcal{C}^\vee$-algebras.
Remark 7.9. There is a canonical map of $k$-operads $P = \Omega C \to B(\mathcal{C})^\vee$. Because each $\mathcal{C}(p)$ is finitely generated over $k$, this map identifies $B(\mathcal{C})^\vee$ with the completion $P^\wedge$ of $P$ at its augmentation ideal.

Note that a very small (hence nilpotent) $P$-algebra $A$ has a canonical $P^\wedge$-algebra structure. Since such $A$ is perfect over $k$, its linear dual $A^\vee$ has the canonical structure of a $B(\mathcal{C})^\vee$-coalgebra. Unravelling the definitions, one then obtains a natural isomorphism of $\mathcal{C}$-algebras

$$\Omega_{\phi^!}(A^\vee) \xrightarrow{\cong} \mathcal{D}^{\text{poly}}_{\phi}(A) = \mathcal{D}_\phi(A)$$

where $\phi^!: B(\mathcal{C}) \to \mathcal{C}$ is the canonical twisting morphism.

Definition 7.10 (Simplicial category of small $P$-algebras). For $P = \Omega C$ as in Assumption 7.6, let us define

$$\text{Alg}_{^\text{sm},\infty}^P \subseteq \text{Alg}_{\infty}^P$$

to be the full simplicial subcategory on the $P$-algebras that are very small (Definition 4.5).

Lemma 7.11. In the situation of Assumption 7.6, the $\infty$-category of small $P$-algebras can be presented by the simplicial category $\text{Alg}_{^\text{sm},\infty}^P$.

Proof. The simplicial category $\text{Alg}_{^\text{sm},\infty}^P$ presents a full subcategory of the $\infty$-category of $P$-algebras by Lemma 7.5. It presents the subcategory of small $P$-algebras by Lemma 5.12, which applies by Variant 5.18. \hfill \square

Assumption 7.6 now allows us to give a very simple description of the functor $\mathcal{D}_\phi$ on the $\infty$-category of small $P$-algebras:

Lemma 7.12. In the situation of Assumption 7.6, there is a (strictly) fully faithful functor of simplicial categories

$$\mathcal{D}_\phi: \text{Alg}_{^\text{sm},\infty}^P \longrightarrow \text{Alg}_{\text{poly}}^P; \quad A \mapsto \mathcal{D}_\phi(A) := \Omega_{\phi^!}(A^\vee). \quad (7.13)$$

This simplicial functor presents the functor of $\infty$-categories $\mathcal{D}_\phi: \text{Alg}_{^\text{sm}}^P \longrightarrow \text{Alg}_{\text{poly}}^P$ from Section 4.2.

Proof. Let us start by defining the functor (7.13) more precisely. By Remark 7.9, we have that $\mathcal{D}_\phi(A) = \mathcal{D}_{\phi^!}^\text{poly}(A) \cong \Omega_{\phi^!}(A^\vee)$ is cofibrant whenever $A$ is very small (and hence nilpotent, cf. Remark 7.8). Let us now define the functor $\mathcal{D}_\phi$ on simplicial sets of morphisms

$$\mathcal{D}_\phi: \text{Map}_P^\infty(A, B) \longrightarrow \text{Map}_{\text{poly}}^\infty(\mathcal{D}_\phi(B), \mathcal{D}_\phi(A)).$$

To this end, recall that an $n$-simplex in $\text{Alg}_{^\text{sm},\infty}^P$ is given by a map of $\mathcal{C} \otimes \Omega_n$-coalgebras

$$B_\phi(A) \otimes \Omega_n \longrightarrow B_\phi(B) \otimes \Omega_n. \quad (7.14)$$

Because $\mathcal{C}$ satisfies the conditions (1) and (2) of Assumption 7.6 and because $A$ is perfect over $k$, we have that $B_\phi(A) \otimes \Omega_n = \mathcal{C}(A) \otimes \Omega_n$ is quasi-projective and finitely generated as a left $k \otimes \Omega_n$-module. The $k \otimes \Omega_n$-linear dual of $B_\phi(A) \otimes \Omega_n$ is then given by

$$\text{Hom}_{k \otimes \Omega_n}(B_\phi(A), k \otimes \Omega_n) \cong \mathcal{D}_\phi(A) \otimes \Omega_n.$$

On (higher) morphisms, we can therefore simply define $\mathcal{D}_\phi$ to send (7.14) to its $k \otimes \Omega_n$-linear dual. The resulting map of simplicial sets is an isomorphism, with inverse taking the
\( \mathcal{C}^{op} \otimes \Omega_n \)-linear dual of a map of \( \mathcal{C}^{\vee} \otimes \Omega_n \)-algebras. We therefore obtain the desired fully faithful functor \((7.13)\).

To see that this enriched functor indeed presents the \( \infty \)-functor \( \mathcal{D}_\phi \) defined in Section 4.2, consider the following commuting diagram:

\[
\begin{array}{c}
\text{Alg}_{c^{\infty}}^{sm} \longrightarrow \text{Alg}_{c^{\infty}} \leftarrow \text{Alg}_{d^{c^{\infty}}}^{dg} \\
\mathcal{D}_\phi \downarrow \quad \quad \quad \quad \downarrow \mathcal{D}_\phi \\
\text{Alg}_{c^{\infty}}^{op} \quad \sim \quad \text{Alg}_{c^{op}}^{op} \quad \sim \quad \text{Alg}_{d^{c^{\infty}}}^{dg}.
\end{array}
\]

By Lemma 7.11, \( \text{Alg}_{c^{\infty}}^{sm} \longrightarrow \text{Alg}_{c^{\infty}} \) models the inclusion of the small \( \mathcal{P} \)-algebras in the \( \infty \)-category of all \( \mathcal{P} \)-algebras. The assertion then follows by noting that the marked arrows become equivalences after localizing at the quasi-isomorphisms. \( \square \)

### 7.3 Formal moduli problems from the Maurer–Cartan equation

We will now describe the equivalence provided by Theorem 4.34

\[
\text{MC}: \text{Alg}_{c^{\infty}} \longrightarrow \text{FMP}_\mathcal{P}
\]

more concretely in terms of simplicial sets of Maurer–Cartan elements, at least for 1-reduced cooperads \( \mathcal{C} \) satisfying the finiteness hypotheses of Assumption 7.6. Let us start with the following observation:

**Lemma 7.15.** Fix a dg-category \( k \) and a 1-reduced \( k \)-cooperad \( \mathcal{C} \) as in Assumption 7.6. For any twisting morphism \( \phi: \mathcal{C} \longrightarrow \mathcal{P} \), there exists a functor

\[
\text{Alg}_{c^{\infty}}^{dg} \otimes \text{Alg}_{c^{\infty}}^{dg} \longrightarrow \text{Alg}_{L_{\infty}^{-1}}^{dg}; \quad (g, A) \longrightarrow g \otimes_k A
\]

to the category of shifted \( L_{\infty} \)-algebras. When \( A \) or \( g \) is nilpotent, \( g \otimes_k A \) is a nilpotent shifted \( L_{\infty} \)-algebra.

**Construction 7.16** (Hadamard tensor product of \( k \)-operads). Let \( k \) be a dg-category with a set of objects \( S \). Given a \( k^{op} \)-operad \( \mathcal{P} \) and a \( k \)-operad \( \mathcal{Q} \), we can construct a (single-coloured) operad \( \mathcal{P} \otimes H \mathcal{Q} \) over the base field \( k \), their (internal) Hadamard tensor product\(^3\), as follows. For two tuples of objects \( c = (c_1, \ldots, c_p) \) and \( d = (d_1, \ldots, d_p) \) in \( k \), consider the tensor product

\[
\mathcal{P}(c) \otimes_k \mathcal{Q}(d) := \mathcal{P}(c; c_0) \otimes \mathcal{Q}(d; c_0) = \left( \bigoplus_{c_0} \mathcal{P}(c; c_0) \otimes \mathcal{Q}(d; c_0) \right) / \sim.
\]

Explicitly, the tensor product over \( k \) (see Section 2) is computed as the quotient by relations

\[
(c \xrightarrow{\phi} c_0 \xrightarrow{\lambda} d_0) \otimes (d \xrightarrow{\psi} d_0) \sim (c \xrightarrow{\phi} c_0) \otimes (d \xrightarrow{\psi} d_0 \xrightarrow{\lambda} c_0)
\]

where \( \phi \) and \( \psi \) are operations in \( \mathcal{P} \) and \( \mathcal{Q} \), \( \lambda \) is an arrow in \( k \) and \( \lambda \) denotes the corresponding arrow in \( k^{op} \). We then define

\[
(\mathcal{P} \otimes \mathcal{Q})(c) := \text{Hom}_{k^{op} \otimes (k^{op})^{op}}(k^{\otimes p}, \mathcal{P} \otimes_k \mathcal{Q}) \subseteq \prod_{c = (c_1, \ldots, c_p)} \mathcal{P}(c) \otimes_k \mathcal{Q}(c).
\]

\(^3\)This construction differs from the (simpler) exterior Hadamard tensor product \((9.7)\).
Explicitly, its elements are $S^\times p$-tuples of the form
\[ \left( \sum_{c_0} \xi \xmapsto{\phi} c_0 \otimes \xi \xmapsto{\psi} c_0 \right)_{\xi \in S^\times p} \] (7.17)

such that for every tuple of maps $\lambda_i : d_i \rightarrow c_i$ in $k$
\[ \sum_{c_0} \left( \xi \xmapsto{\phi} c_0 \otimes \xi \xmapsto{\psi} c_0 \right) \circ_i \sum_{d_0} \left( d \xmapsto{\varphi} d_0 \otimes d \xmapsto{\psi} d_0 \right) = \sum_{c_0} \sum_{d_0=c_i} \phi_{i,0} \otimes \psi_{i,0} . \]
in the complex $P(\zeta) \otimes_k Q(d)$. These equations guarantee that $P \otimes_H Q$ carries a well-defined operad structure determined by
\[ \sum_{c_0} \left( \xi \xmapsto{\phi} c_0 \otimes \xi \xmapsto{\psi} c_0 \right) \circ_i \sum_{d_0} \left( d \xmapsto{\varphi} d_0 \otimes d \xmapsto{\psi} d_0 \right) = \sum \sum \phi_{i,0} \otimes \psi_{i,0} . \]
The operad $P \otimes_H Q$ is constructed in order for the following to hold: if $A$ is a $P$-algebra and $B$ is a $Q$-algebra, then the chain complex $A \otimes_k B$ is a $P \otimes_H Q$-algebra. Indeed, given $p$ elements in $A \otimes B$ of the form $\sum c_i \otimes b_i$, with $a_i \in A(c_i)$ and $b \in B(c_i)$, the operation (7.17) sends it to
\[ \sum c_0 \sum \phi_{i} (a_{c_1}, \ldots, a_{c_p}) \otimes \psi_{i} (b_{c_1}, \ldots, b_{c_p}) . \]

**Proof (of Lemma 7.15).** Note that for any $k$-cooperad $C$ and any $k$-operad $P$, there is a natural inclusion of operads over the ground field $k$
\[ C^\vee \otimes_H P \rightarrow \text{Conv}(C, P) . \]

Here $C^\vee \otimes_H P$ is the Hadamard tensor product (Construction 7.16) and $\text{Conv}(C, P)$ is the convolution operad (Remark 9.18). Given an element in $C^\vee \otimes_H P$ of the form (7.17), with $\phi \in C^\vee$ and $\psi \in P$, the corresponding map $C(p) \rightarrow P(p)$ is given by
\[ \xi \xmapsto{\phi} c \longrightarrow \sum_{c_0} \left( \xi \xmapsto{\phi} c_0 \xmapsto{\phi_{i,0}} c \right) . \]
The arrow $\langle \phi_{i,0} \rangle$ is the natural value of $\phi_{i,0} \in C^\vee$ on $\alpha \in C$, cf. Equation (9.10).

When $C$ is 1-reduced, the twisting morphism $\phi : C \rightarrow P$ determines a map of operads $L_{\infty} \{ -1 \} \rightarrow \text{Conv}(C, P)$ (Remark 9.18). When $C$ satisfies the finiteness conditions of Assumption 7.6, the universal twisting morphism defines a map of operads which factors (uniquely) as
\[ L_{\infty} \{ -1 \} \rightarrow C^\vee \otimes_H P \rightarrow \text{Conv}(C, P) . \]
Indeed, for every $\xi = (c_1, \ldots, c_p)$, Assumption 7.6 allows us to pick a finite base $e_{\xi,0} \in C(\xi,0)$ for the left $k$-module $C(\xi, -)$. Unravelling the definitions, one then sees that the generating $p$-ary operation of $L_{\infty} \{ -1 \}$ will be sent to the element
\[ \left( \sum_{c_0} c_0, (e_{\xi,0}) \right)_{\xi = (c_1, \ldots, c_p)} \in C^\vee \otimes_H P . \]

By Construction 7.16, $g \otimes_k A$ is a $C^\vee \otimes_H P$-algebra and hence an $L_{\infty} \{ -1 \}$-algebra by restriction. Furthermore, if $g$ is nilpotent, then there are only finitely many composites of $e_{\xi,0}$ that act nontrivially on $g$. Consequently, only finitely many composites of the generating operations in $L_{\infty} \{ -1 \}$ act nontrivially on $g \otimes_k A$. It follows that $g \otimes_k A$ is a nilpotent $L_{\infty} \{ -1 \}$-algebra, and similarly if $A$ is nilpotent.

\[ \square \]
Using the shifted $L_\infty$-structure from Lemma 7.15, we can now describe the formal moduli problem associated to a $\mathcal{C}^\vee$-algebra more precisely as follows:

**Theorem 7.18.** Consider a dg-category $\mathbf{k}$ and a 1-reduced $\mathbf{k}$-cooperad $\mathcal{C}$ satisfying the conditions from Assumption 7.6, and denote $\mathcal{P} = \Omega\mathcal{C}$. For every $\mathcal{C}^\vee$-algebra $g$, there is a simplicially enriched functor

$$\MC_g : \Alg_{\mathcal{P}}^{sm, \infty} \longrightarrow \sSet; \quad A \longmapsto \MC(g \otimes_\mathbf{k} A \otimes \Omega_{\bullet})$$

where $g \otimes_\mathbf{k} A$ carries the $L_\infty\{-1\}$-algebra structure from Lemma 7.15. This determines a simplicially enriched functor

$$\MC : \Alg_{\mathcal{C}^\vee}^{\Delta} \longrightarrow \Fun(\Alg_{\mathcal{P}}^{sm, \infty}, \sSet)$$

sending quasi-isomorphisms to pointwise homotopy equivalences. The associated functor between $\infty$-categories presents the fully faithful functor of Theorem 4.34

$$\MC : \Alg_{\mathcal{P}_G}^{\Delta} \longrightarrow \Fun(\Alg_{\mathcal{P}}^{sm}, S) , \quad (7.19)$$

whose essential image is the $\infty$-category of formal moduli problems over $\mathcal{P}$.

**Example 7.20.** Let $\mathcal{C} = \text{Lie}^\vee\{1\}$ be the shifted coLie cooperad, so that $\Omega\mathcal{C} = C_\infty$ is a resolution of the commutative operad. Suppose that $g$ is a Lie algebra and that $A$ is a strict unital Artin dg-algebra, i.e. an augmented unital cdga whose augmentation ideal $m_A$ is (strictly) finite dimensional and nilpotent. Theorem 7.18 then shows that

$$\MC_g(A) = \MC(m_A \otimes g \otimes \Omega_{\bullet}).$$

In other words, the value on $A$ of the formal moduli problem associated to $g$ by the equivalence of Lurie [Lur11, Theorem 2.0.2] coincides with the value of the deformation functor considered e.g. by Hinich [Hin01]. In addition, Theorem 7.18 shows that the full FMP associated to $g$ can be described similarly, by allowing $A$ to be an ’Artin’ (i.e. very small) $C_\infty$-algebra, in which case $m_A \otimes g$ is an $L_\infty$-algebra.

**Proof.** Consider the simplicially enriched functor $\Alg_{\mathcal{C}^\vee} \longrightarrow \Fun(\Alg^{\infty, sm}_\mathcal{P}, \sSet)$ sending $g$ to the enriched functor

$$A \longmapsto \Map_{\mathcal{C}^\vee}^\Delta(D_g(A), g).$$

By Lemma 7.12, this enriched functor presents the functor of $\infty$-categories (7.19) after inverting the weak equivalences. It therefore suffices to identify $\Map_{\mathcal{C}^\vee}^\Delta(D_g(A), g)$ with a simplicial set of Maurer–Cartan elements. But now recall that

$$D_g(A) = \Omega_{\mathbf{g}^\vee}(A^\vee)$$

is the cobar construction of the linear dual of $A$, which is a $B(\mathcal{C}^\vee)$-coalgebra. By the universal property of the cobar construction (Proposition 9.19), we then have that

$$\Map_{\mathcal{C}^\vee}^\Delta(D_g(A), g) \cong \MC(\Hom_{k^\vee}(A^\vee, g \otimes \Omega_{\bullet}))$$

is given by the simplicial set of Maurer–Cartan elements in the convolution $L_\infty\{-1\}$-algebra $\Hom_{k^\vee}(A^\vee, g \otimes \Omega_{\bullet})$ (Remark 9.28). The result now follows from the fact that the maps

$$A \otimes_\mathbf{k} g \longrightarrow A^\vee \otimes_\mathbf{k} g \longrightarrow \Hom_{k^\vee}(A^\vee, g)$$

are isomorphisms of $L_\infty\{-1\}$-algebras, since $A$ is quasi-free, finitely generated over $k$ (Remark 4.6). \qed
Remark 7.21. Suppose we are in the setting of Theorem 7.18 and fix a very small \( \mathcal{P} \)-algebra \( A \). The above proof shows that the space

\[
\text{MC}_g(A) \simeq \text{Map}_{\mathcal{C}^\vee}(\mathfrak{D}(A), g) \simeq \text{Map}_{\mathcal{P}^\vee}(\Omega_{\mathfrak{P}^\vee}(A^\vee), g)
\]
can be presented by the simplicial set

\[
\text{MC}(\text{Hom}_{k^{\text{op}}}(A^\vee, g_\bullet)) \cong \text{MC}(A \otimes_k g_\bullet)
\]
for any choice of fibrant simplicial resolution \( g_\bullet \) of the \( \mathcal{C}^\vee \)-algebra \( g \). The additional feature of the particular choice \( g_\bullet = g \otimes \Omega_\bullet \) is that one obtains a simplicially enriched functor in \( A \).

Remark 7.22. Let \( \mathcal{C} \) be a \( k \)-cooperad as in Assumption 7.6 with contributions in arity 0 or 1. Let us denote by \( \Omega_{\mathcal{C}}^{\text{nc}} \) the cobar construction of non-coaugmented cooperads, i.e. \( \Omega_{\mathcal{C}}^{\text{nc}}(\mathcal{C}) \) is by definition \( \Omega(k \oplus \mathcal{C}) \). A twisting morphism \( \phi: \mathcal{C} \rightarrow \mathcal{P} \) then determines an operad map

\[
\Omega_{\mathcal{C}}^{\text{nc}}(\text{coCom}_{\mathcal{U}}) \rightarrow \mathcal{C}^\vee \otimes \mathcal{H}_{\mathcal{P}}.
\]

from the non-coaugmented cobar construction on the counital cocommutative cooperad. This operad is freely generated by symmetric operations \( l_p \) of degree 1, with \( p \geq 0 \); the operation \( l_1 \) differs from the differential.

If \( A \) is a very small \( \Omega_{\mathcal{C}} \)-algebra, then maps of \( \mathcal{C}^\vee \)-algebras \( \Omega(A^\vee) \rightarrow g \) correspond bijectively to Maurer–Cartan elements of the nilpotent \( \Omega_{\mathcal{C}}^{\text{nc}}(\text{coCom}_{\mathcal{U}}) \)-algebra \( A \otimes_k g \), i.e. degree 0 elements satisfying

\[
dx + \sum_{p \geq 0} \frac{1}{p!} l_p(x, \ldots, x) = 0.
\]

One can then repeat the proof of Theorem 7.18 to show the following: the formal moduli problem associated to a \( \mathcal{C}^\vee \)-algebra \( g \) is represented by the simplicially enriched functor \( A \mapsto \text{MC}(A \otimes_k g \otimes \Omega_\bullet) \) on very small \( \Omega_{\mathcal{C}} \)-algebras.

8 Relative Koszul duality

We study the results of the previous sections in the context of the coloured operad whose algebras are (one coloured) operads.

Throughout this section we fix as set of colours \( S = \mathbb{Z}_{\geq 0} \) and we work with the linearized version of the symmetric groupoid, the dg-category \( k[\Sigma] \) whose Hom spaces are

\[
k[\Sigma](a, b) = k \cdot \text{Bij}(a, b),
\]

the \( k \)-vector space spanned by the bijections from \( a \) to \( b \). Notice that a \( k[\Sigma]^{op} \)-algebra is precisely a (one-coloured) symmetric sequence.

Definition 8.1 (See [DV15]). We define the following \( \mathbb{Z}_{\geq 0} \)-coloured \( k \)-operads:

1. Let \( \mathfrak{o}^\text{na} \) be the operad generated by \( \circ_i: (a, b) \rightarrow a + b - 1 \) for \( i = 1, \ldots, a \), subject to the relations

   \[
   (a, b, c) \xrightarrow{(a, \circ_j)} (a, b + c - 1) \quad (a_i, c) \xrightarrow{a_i} (a + b - 1, c) \xrightarrow{a_{i+j-1}} a + b + c - 2
   \]

   (8.2)
for $1 \leq i \leq a$ and $1 \leq j \leq b$, and

$$
\begin{array}{c} 
(a, b, c) \rightarrow (a, c, b) \rightarrow (a + c - 1, b) \\
\downarrow \quad (\circ_i, b) \\
(a + b - 1, c) \rightarrow a + b + c - 2
\end{array}
\tag{8.3}
$$

for $1 \leq i < j \leq a$.

(2) Let $O^{\text{sym}}$ be the unital non-augmented operad generated by $\circ_i$: $(a, b) \rightarrow a + b - 1$ and $\sigma: a \rightarrow a$ for $\sigma \in \Sigma_a$, subject to the equations (8.2) and (8.3), together with the group structure equations

$$
\sigma \cdot \tau = (\sigma \tau): a \rightarrow a \quad \sigma, \tau \in \Sigma_a,
$$

and the equations

$$
\begin{array}{c} 
(a, b) \rightarrow (a, b) \rightarrow (a + b - 1, b) \\
\downarrow \quad (a, \tau) \\
(a + b - 1, \tau / i) \rightarrow a + b - 1
\end{array}
\quad \begin{array}{c} 
(a, b) \rightarrow (a, b) \rightarrow (a + b - 1, b) \\
\downarrow \quad (a, \sigma) \\
(a + b - 1, i / \sigma) \rightarrow a + b - 1
\end{array}
\tag{8.4}
$$

where for $\tau / i$ and $i / \sigma$ are some permutations of $a + b - 1$ determined from $\tau, \sigma$ and the number $i$.

Finally, we impose the relation that for the identity of the group $1_a \in \Sigma_a$ is identified with the operadic unit $1_a \in O^{\text{sym}}(a; a)$.

The notation reflects that a $O^{\text{nss}}$-algebra is a nonsymmetric operad and a $O^{\text{sym}}$-algebra is a symmetric operad.

We note that unlike the usual conventions in this manuscript, we are forced to consider $O^{\text{sym}}$ as a unital non-augmented operad. The reason for this is that there is no way to define an augmentation since the relations $\sigma \cdot \sigma^{-1}$ produce the unit. There are other problems with this presentation (namely it is not quadratic) that heuristically come from seeing the symmetric groups as additional structure. In the next sections we will see how these problems can be circumvented by seeing $O^{\text{sym}}$ as a $k[\Sigma]$-operad.

### 8.1 Distributive laws

Let $k$ be a dg-category with set of objects $S$ and let $V$ be an $S$-coloured symmetric sequence. We say that a $k$-law on $V$ is a map of $S$-coloured symmetric sequences

$$
\Lambda: V \circ k \rightarrow k \circ V
$$

such that the following diagrams commute

$$
\begin{array}{ccc} 
V \circ k \circ k \rightarrow k \circ V \circ k & \lambda \circ \lambda & k \circ k \circ V \\
\downarrow \circ \mu \quad \downarrow \mu \circ \nu \quad \downarrow \nu \circ \lambda \\
V \circ k \rightarrow k \circ V
\end{array}
$$

and

$$
\begin{array}{ccc} 
V \circ k \rightarrow k \circ V & \Rightarrow & k \circ V \rightarrow k \circ V
\end{array}
$$

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Let $\Lambda: V \circ k \to k \circ V$ be a $k$-law on a symmetric sequence. Then $k \circ V$ has the natural structure of a symmetric $k$-bimodule via the maps

$$k \circ (k \circ V) \xrightarrow{\mu_0} k \circ V \quad (k \circ V) \circ k \xrightarrow{1 \circ \Lambda} k \circ k \circ V \xrightarrow{\mu_1} V.$$ 

If $V = P$ is a $k$-operad (augmented, as always), then $k \circ P$ inherits the structure of a $k$-operad via

$$(k \circ P) \circ_k (k \circ P) \cong k \circ (P \circ P) \to k \circ P$$

as long as the $k$-law is a *distributive law* in the sense of [LV, Section 8.6], i.e. it is also *right distributive*:

$$\begin{array}{cccc}
P \circ P \circ k & \xrightarrow{\Lambda \mu_1} & P \circ k \circ P & \xrightarrow{1 \circ \Lambda} k \circ P \circ P & k \circ k \end{array}$$

Similarly, if $V = C$ is a cooperad, then $k \circ C$ has the natural structure of a $k$-cooperad via

$$k \circ C \xrightarrow{\mu_0} k \circ C \circ C \cong (k \circ C) \circ_k (k \circ C)$$

as long as the $k$-law is *codistributive*:

$$\begin{array}{cccc}
C \circ k & \xrightarrow{\Lambda} & k \circ C & C \circ C \circ k & \xrightarrow{1 \circ \Lambda} k \circ C \circ C \end{array}$$

**Example 8.5.** Let $\Lambda: V \circ k \to k \circ V$ be a $k$-law. This induces a distributive law on the free operad $\text{Free}_{Op_k}(V)$ and a codistributive law on the cofree cooperad $\text{Cofree}_{Coop_k}(V)$.

**Definition 8.6.** Let $\phi: C \to P$ be a twisting morphism. We will say that a $k$-law on $\phi$ is the data of:

- a codistributive law $\Lambda_C$ on $C$.
- a distributive law $\Lambda_P$ on $P$.

such that $\phi$ intertwines $\Lambda_C$ and $\Lambda_P$.

**Lemma 8.7.** If $\phi: C \to P$ is a twisting morphism and $\Lambda$ is a distributive law on $\phi$, then $k \circ \phi: k \circ C \to k \circ P$ is a $k$-twisting morphism (see Construction 9.16).

*Proof.* This follows from checking that given maps $f, g: \overline{C} \to \overline{P}$ such that $f$ intertwines $\Lambda_C$ and $\Lambda_P$, the equation $(k \circ f) \star (k \circ g) = k \circ (f \star g)$ is satisfied.

**Example 8.8.** Let $\Lambda_P$ be a distributive law on an augmented operad $P$. This extends to a canonical $k$-law on the universal twisting morphism $\pi: B(P) \to P$ such that $k \circ \pi$ is the universal twisting morphism of the $k$-operad $k \circ P$. 
Koszul duality and distributive laws

Recall from standard Koszul theory \[ LV, \text{Section 7} \] that a graded (but non-dg) \( k \)-operad \( \mathcal{P} \) is said to be quadratic if it admits a presentation of the form \( \mathcal{P} = \text{Free}_{\text{Op}}(V)/(R) \), with \( R \subseteq V \circ(1) V = \text{Free}^{(2)}_{\text{Op}}(V) \). In this case, we write \( \mathcal{P} = Q(V, R) \).

Its Koszul dual cooperad denoted \( \mathcal{P}^! = Q_{\text{co}}(V[1], R[2]) \), is the conilpotent quadratic cooperad cogenerated by \( V[1] \) with corelations \( R[2] \). There is a canonical twisting morphism \( \mathcal{P}^! \rightarrow \mathcal{P} \) induced by the identity on (co)generators \( V \rightarrow V[1] \).

We denote by \( \mathcal{P}^! : = (\mathcal{P} \{-1\})^\vee \) the Koszul dual operad of \( \mathcal{P} \). If the induced map \( \Omega_{\mathcal{P}^!} \rightarrow \mathcal{P} \) is a quasi-isomorphism, we say that \( \mathcal{P} \) is a Koszul operad. In that case, the operad \( \mathcal{P}^! \) is quasi-isomorphic to \( D(\mathcal{P}) \{1\} \).

The following proposition shows that under good conditions, distributive laws are compatible with Koszul duality.

**Proposition 8.9.** Let \( \mathcal{P} = Q(V, R) \) be a quadratic operad and consider a \( k \)-law \( \Lambda : V \circ k \rightarrow k \circ V \) such that the induced distributive law on \( \text{Free}_{\text{Op}}(V) \) preserves the quadratic relations \( R \). In other words, \( \Lambda \) induces a distributive law on \( \mathcal{P} \). Then:

1. \( \Lambda[1] : V[1] \circ k \rightarrow k \circ V[1] \) induces a codistributive law on the quadratic dual \( \mathcal{P}^! \).
2. \( \Lambda \) and \( \Lambda[1] \) together determine a \( k \)-law on the twisting morphism \( \mathcal{P}^! \rightarrow \mathcal{P} \).
3. If \( \mathcal{P}^! \rightarrow \mathcal{P} \) is weakly Koszul, then the induced twisting morphism \( k \circ \mathcal{P}^! \rightarrow k \circ \mathcal{P} \) is weakly Koszul over \( k \).

*Proof.* (1) One can check that \( \Lambda[1] \) preserves the quadratic relations.

(2) The maps
\[
Q^{\text{co}}(V[1], R[2]) \rightarrow V[1] \rightarrow V \rightarrow Q(V, R)
\]
are all compatible with the distributive law by construction.

(3) The map of coalgebras \( \phi' : Q^{\text{co}}(V[1], R[2]) \rightarrow B(\mathcal{P}) \) is compatible with the \( \Lambda \), and consequently, there is an induced map of \( k \)-coalgebras
\[
k \circ \phi' : k \circ Q^{\text{co}}(V[1], R[2]) \rightarrow k \circ B(\mathcal{P}).
\]

Since the composition product (over the base field \( k \)) preserves quasi-isomorphisms, the result follows.

\[ \square \]

### 8.2 FMPs from operads

Notice that the presentation of the \( Z_{\geq 0} \)-colored operad of nonsymmetric operads \( \mathcal{O}^{\text{ns}} \) given in Definition 8.1 is quadratic and therefore fits the framework of Koszul duality. We recall the result of Van der Laan showing that \( \mathcal{O}^{\text{ns}} \) is both Koszul and Koszul self-dual.

**Proposition 8.10** ([VdL03], Theorem 4.3). The quadratic \( Z_{\geq 0} \)-coloured operad \( \mathcal{O}^{\text{ns}} \) is Koszul, and it is isomorphic to its Koszul dual operad \( \mathcal{O}^{\text{ns}}^! = \mathcal{O}^{\text{ns}} \).

A priori a similar result is not expectable for the operad of symmetric operads since the relations for the symmetric group are not quadratic. Dehling and Vallette [DV15] used curved Koszul duality theory to construct an appropriate cofibrant replacement functor over any ring. Crucially, they observed that the symmetric group data could be obtained as a distributive law.

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Proposition 8.11 ([DV15], Proposition 1). There is a $k[\Sigma]$-law on the quadratic data generating $O^{\text{ns}}$, such that

$$k[\Sigma] \circ O^{\text{ns}} \simeq O^{\text{sym}}.$$ 

Following the previous section we can therefore interpret $O^{\text{sym}}$ as a $k[\Sigma]$-operad which is quadratic. Note that there is a canonical equivalence $\text{inv}: \Sigma \rightarrow \Sigma^\text{op}$, given by sending a permutation to its inverse. We can use this equivalence to identify $\Sigma$-operads with $\Sigma^\text{op}$-operads.

As a consequence we conclude that as a $k[\Sigma]$-operad, $O^{\text{sym}}$ is Koszul and Koszul self-dual.

Corollary 8.12 (The operad of symmetric operads is Koszul). The quadratic $k[\Sigma]$-operad $O^{\text{sym}}$ is self-dual in the sense that $D_k[\Sigma](O^{\text{sym}}) \{1\} \simeq O^{\text{sym}}$.

Proof. Combining the above two propositions with Proposition 8.9, we have that:

$$D_k[\Sigma](O^{\text{sym}}) \{1\} \simeq \text{Hom}_k[\Sigma](B_k[\Sigma](k[\Sigma] \circ O^{\text{ns}}), k[\Sigma]) \quad \text{(Proposition 8.11)}$$

$$\simeq \text{Hom}_k(k[\Sigma] \circ B(O^{\text{ns}}), k[\Sigma]) \quad \text{(Example 8.8)}$$

$$\simeq k[\Sigma] \circ (O^{\text{ns}})^i \{1\} \quad \text{(Proposition 8.10)}$$

$$\simeq k[\Sigma] \circ O^{\text{ns}} \{1\} \quad \text{(Proposition 8.11)}.$$ 

\[\square\]

Proposition 8.13. Consider the twisting morphism $k[\Sigma] \circ (O^{\text{ns}})^i \rightarrow O^{\text{sym}}$ relative to $k[\Sigma]$. The induced bar-cobar adjunction can be identified with the usual bar-cobar adjunction $B: \text{Op}^{\text{sym}} \rightarrow \text{CoOp}^{\text{sym}}: \Omega$ between nonunital one-coloured $k$-operads and $k$-cooperads.

Proof (sketch). Note that coalgebras for the $k[\Sigma]$-cooperad $k[\Sigma] \circ (O^{\text{ns}})^i$ are symmetric sequences with a (conilpotent) non-counital cooperad structure. The bar construction then takes the cofree $k[\Sigma] \circ (O^{\text{ns}})^i$-coalgebra (in symmetric sequences) with some differential. Unraveling the definitions, this is exactly the cofree cooperad with the bar differential. \[\square\]

We can now apply our main Theorem 1.3 to the $k[\Sigma]$-operad $O^{\text{sym}}$ to get the following result.

Theorem 8.14. There is an equivalence of $\infty$-categories

$$\text{MC}: \text{Op}^{\text{sym}} \rightarrow \text{FMP}_{O^{\text{sym}}}$$

between the $\infty$-category of symmetric (single-coloured) $k$-operads and the $\infty$-category of formal moduli problems given by

$$X: (\text{Op}^{\text{sym}})^{\text{sm}} \rightarrow 8.$$

Proof. We apply the main Theorem 1.3. To apply this to the 1-reduced operad $O^{\text{sym}}$, we have to show that it is splendid. Since $O^{\text{sym}}(1) = k[\Sigma]$, this reduces to the assertion that

$$k[\Sigma] \circ O^{\text{sym}} = B_\Sigma(O^{\text{sym}}) \simeq (O^{\text{sym}} \{1\})^\vee$$

is eventually highly connective. This is the same argument as in the proof of Theorem 1.1. \[\square\]
Remark 8.15. Instead of indexing the operadic FMPs by small augmented symmetric operads, one may also take as the domain the category of retracts of small augmented symmetric operads. This will not change the theory, since an $S$-valued diagram on retracts of small operads is determined uniquely by its value on small operads.

The retracts of small operads have the following simple description, as in Lemma 4.7: they are those augmented symmetric operads $P$ such that $H^*(P)$ is concentrated in nonpositive degrees, of finite total dimensional (summing over degrees and arities) and $H^0(P)$ is a nilpotent augmented operad. The small symmetric operads are those for which the symmetric groups act freely on $H^*(P)$.

8.3 FMPs from algebras over operads

Since every small (augmented symmetric) $k$-operad $Q$ is in particular concentrated in finitely many arities, it is splendid. Proposition 6.10 and passing to opposite categories therefore shows that there is a triangle

commuting up to the natural equivalence $MC: \text{FMP}_{\mathcal{P}} \longrightarrow \text{Alg}_{\mathcal{D}(\mathcal{P})}$. Since $\text{Pr}^L$ admits all colimits, this induces a commuting diagram

where $\mathcal{D}$ and FMP are the unique colimit-preserving extensions of $\mathcal{D}$ and FMP.

Definition 8.16. Let $X: \text{Op}_{\text{sm}} \longrightarrow \mathcal{S}$ be a functor. We define $\text{FMP}_X$ to be the value of $\text{FMP}_!$ on $X$. One can identify $\text{FMP}_X$ with the limit of the diagram

sending

$$\left(\text{Op}_{\text{sm},\text{op}}/X\right)^{op} \longrightarrow \text{Cat}_\infty$$

Theorem 8.17 (Theorem 1.6). For any formal moduli problem $X: \text{Op}_{\text{sm}} \longrightarrow \mathcal{S}$, there is an equivalence of $\infty$-categories

$$\text{MC}: \text{Alg}_{\text{T}(X)} \longrightarrow \text{FMP}_X.$$ 

Lemma 8.18. The functor $\text{Alg}: \text{Op} \longrightarrow \text{Pr}^L; \mathcal{P} \mapsto \text{Alg}_{\mathcal{P}}$ preserves sifted colimits.
Proof. The functor $\text{Alg}$ is classified by a cartesian and cocartesian fibration $\text{Alg} \to \text{Op}$; in fact, this is just the functor obtained by localizing the functor $\text{Alg}_{dg} \to \text{Op}_{dg}$ of Construction 6.8. Note that $\text{Alg}$ is itself the category of algebras over a coloured operad (namely, the operad for operads with an algebra over them), and hence admits all limits and colimits.

We claim that the lemma follows from the following assertion: consider a cone diagram $F: \mathcal{K} \to \text{Alg}$ such that

- the full subcategory $\mathcal{K} \subseteq \mathcal{K}'$ is a sifted $\infty$-category.
- the composite $\mathcal{K}' \to \text{Alg} \to \text{Op}$ is a colimit diagram of operads.
- for each arrow in the subcategory $\mathcal{K} \subseteq \mathcal{K}'$, its image in $\text{Alg}$ is a cartesian arrow.

Then the diagram $F: \mathcal{K}' \to \text{Alg}$ is a colimit diagram if and only if for every arrow in $\mathcal{K}'$, its image in $\text{Alg}$ is cartesian.

Indeed, let $K' \to \text{Op}$ be a colimit diagram and consider the functor $\text{Map}_K((\mathcal{K}')^\otimes, \text{Alg}) \to \text{Map}_K((\mathcal{K})^\otimes, \text{Alg})$ which restricts a lift $\mathcal{K} \to \text{Alg}$ with values in cartesian edges to the full subcategory $\mathcal{K} \subseteq \mathcal{K}'$. By the assertion, this functor is an equivalence with inverse given by left Kan extension (cf. [Lur09, Proposition 4.3.2.15]) and the lemma then follows from [Lur09, Proposition 3.3.3.1].

To verify the assertion, note that the forgetful functor $\Phi = (\Phi_1, \Phi_2): \text{Alg} \to \text{Op} \times \text{Mod}_k; (P, A \in \text{Alg}_P) \mapsto (P, A)$ arises from restriction along a map of operads, and hence detects sifted colimits [Lur17, Corollary 3.2.3.2]. Furthermore, an arrow in $\text{Alg}$ is cartesian if and only if its image under $\Phi_2$ is essentially constant.

We now have a diagram $F: K' \to \text{Alg}$ such that $\Phi_1(F)$ is a colimit and diagram and $\Phi_2(F)[X]$ is essentially constant. Then $F$ is a colimit diagram iff $\Phi_2(F)$ is a colimit diagram, which is equivalent to $\Phi_2(F)$ being essentially constant, i.e. $F$ sends every arrow to a cartesian arrow.

Proof (of Theorem 8.17). By Lemma 8.18, we have sifted colimit-preserving functors that send a diagram $X: \text{Op}^\text{sm} \to \text{FMP}_X$ and $\text{Alg}_{\text{D}^h(X)}$. Since MC defines a natural equivalence between them on corepresentables, the same is true for all $X: \text{Op}^\text{sm} \to \text{S}$ that can be written as sifted colimits of corepresentables. In particular, this holds when $X$ is an FMP [Lur11, Proposition 1.5.8]. The result then follows from the fact that $T(X) = \text{D}^h(X)$ when $X$ is an FMP.

9 Operadic toolkit

In this section we introduce the operadic homotopical algebra required for our purposes throughout the text, notably in Section 8. The results from this section hold over a general dg-category $k$ but many of them are standard when the base $k$ is the point and similar statements can be found in [LV] or in [LH03].

We recall that we work over a fixed field $k$, and we denote by $S$ the set of objects of the dg-category $k$. When we use the term operad (resp. cooperad) we always mean unital augmented operad (resp. counital coaugmented cooperad) unless otherwise explicitly written.
9.1 Operads over a dg-category

A symmetric \( k \)-bimodule in \( S \)-coloured symmetric sequences is a family of chain complexes over \( k \)

\[
V(c_0) := V(c_1, \ldots, c_p; c_0), \quad c_i \in S
\]
together with maps

\[
k(c_0, d_0) \otimes V(c_0) \otimes \bigotimes_i k(d_{\phi^{-1}(i)}, c_i) \longrightarrow k(d_1, \ldots, d_p; d_0), \quad d_i \in S
\]

for every \( \phi : \{1, \ldots, p\} \longrightarrow \{1, \ldots, p\} \) which satisfy natural associativity conditions.

Definition 9.1. We denote by \( \text{BiMod}^{\Sigma, \text{dg}}_k \) the category of symmetric \( k \)-bimodules.

Note that a symmetric \( k \)-bimodule \( M \) has an arity \( p \) part \( M(p) \), which is a \( k \otimes k^p \)-bimodule with a \( \Sigma_p \)-action that is compatible with the right \( k^p \)-structure. In arity 0 this is simply a \( k \)-module in the usual sense, i.e. a functor \( k \otimes k^{op} \rightarrow \text{Ch}_k \), while a symmetric \( k \)-bimodule in arity 0 is a left \( k \)-module \( k \rightarrow \text{Ch}_k \).

The category \( \text{BiMod}^{\Sigma, \text{dg}}_k \) has a (nonsymmetric) monoidal structure given by the relative composition product \( \circ \). An element in \( M \circ N \) can be identified with a tree of height 2 with root vertex labeled by \( \phi \in M(c_1, \ldots, c_p; c_0) \) and all other vertices labeled by \( \psi_1, \ldots, \psi_p \in N \), with \( \psi_i \) having an output of color \( c_i \), subject to the relation that edges are \( k \)-equivariant. In other words, for all \( a \in k(c_i, c'_i) \),

\[
\text{labeling } (\phi \circ a), \psi_1, \ldots, \psi_p \sim \text{ labeling } \phi, \psi_1, \ldots, a\psi_i, \ldots, \psi_q.
\]

Proposition 9.2. The following categories carry model structures, in which the fibrations are the surjections and the weak equivalences are the quasi-isomorphisms:

(1) The category of \( k \)-operads, defined to be the category of augmented unital associative algebras in symmetric \( k \)-bimodules

\[
\text{Op}^\text{dg}_k := \text{Alg}^{\text{aug}}(\text{BiMod}^{\Sigma, \text{dg}}_k).
\]

(2) For any associative algebra \( P \) in symmetric \( k \)-bimodules, the categories of left and right \( P \)-modules

\[
\text{LMod}^\text{dg}_P := \text{LMod}_P(\text{BiMod}^{\Sigma, \text{dg}}_k) \quad \text{and} \quad \text{RMod}^\text{dg}_P := \text{RMod}_P(\text{BiMod}^{\Sigma, \text{dg}}_k).
\]

In particular, the category \( \text{BiMod}^{\Sigma, \text{dg}}_k \) itself.

(3) For any \( k \)-operad \( P \), the category of \( P \)-algebras, defined to be the category of left \( P \)-modules that are concentrated in arity 0

\[
\text{Alg}^\text{dg}_P := \text{LMod}_P(\text{LMod}^\text{dg}_k).
\]

Proof. The proposition follows essentially from the fact that over a field of characteristic zero, algebras over a coloured operad have a canonical model structure [Hin15]. For example, (1) the category of augmented unital \( k \)-operads can be identified with the category of nonnulal operads in symmetric \( k \)-bimodules; these are algebras over an operad with set of colours given by \( \prod_{n \geq 0} S^{n-1} \) (cf. Definition 8.1). Something similar holds for (2) left and right modules, and for (3) it suffices to observe that an algebra over a \( k \)-operad is simply an algebra over its underlying \( S \)-coloured operad. \( \square \)
Given a $k$-operad $\mathcal{P}$ together with a right module $M$ and a left module $N$, we denote by $M \circ_\mathcal{P} N$ the coequalizer of $M \circ_k \mathcal{P} \circ_k N \rightrightarrows M \circ_k N$.

**Example 9.3** (Free algebras). Let $\mathcal{P}$ be an operad and $V$ a left $k$-module. Then the free $\mathcal{P}$-algebra on $V$ is given by the usual formula

$$\mathcal{P}(V) := \mathcal{P} \circ_k V = \bigoplus_p \mathcal{P}(p) \otimes_{S^p \cdot k \circ \mathcal{P}} V^p.$$ 

**Lemma 9.4.** Let $\mathcal{P} \in \text{Op}_k$ and suppose that $M \in \text{RMod}_\mathcal{P}$ and $N \in \text{LMod}_\mathcal{P}$ are cofibrant. Then the two functors

$$M \circ_\mathcal{P} (-) : \text{RMod}^\mathrm{dg}_\mathcal{P} \longrightarrow \text{BiMod}^\Sigma_{\mathcal{P}}^\mathrm{dg}$$

\[ (-) \circ_\mathcal{P} N : \text{RMod}^\mathrm{dg}_\mathcal{P} \longrightarrow \text{BiMod}^\Sigma_{\mathcal{P}}^\mathrm{dg} \]

both preserve quasi-isomorphisms.

**Proof.** We will only deal with the first functor, the other is similar. Consider the simplicial resolution of $M$ as a right $\mathcal{P}$-module $M \circ_\mathcal{P} \mathcal{P} \Rightarrow M \circ_\mathcal{P} N$. Since $M$ is cofibrant, it is quasi-free as a right $\mathcal{P}$-module; in particular, without differentials this augmented simplicial object has extra degeneracies. Taking the relative composition product over $\mathcal{P}$ with a quasi-isomorphism $X \to Y$ yields a map of augmented simplicial objects

$$\begin{array}{ccc}
M \circ_\mathcal{P} X & \longleftarrow & M \circ_\mathcal{P} Y \\
\sim & & \sim \\
M \circ_\mathcal{P} \mathcal{P} & \longleftarrow & M \circ_\mathcal{P} \mathcal{P} \circ_\mathcal{P} X \longleftarrow \cdots
\end{array}$$

Since the composition product $\circ$ preserves quasi-isomorphisms, all marked vertical maps are quasi-isomorphisms. Without differentials, the rows are augmented simplicial objects with (natural) extra degeneracies, so that the above diagram provides a simplicial resolution of the map $M \circ_\mathcal{P} X \longrightarrow M \circ_\mathcal{P} Y$ and the result follows.

**Remark 9.5.** Lemma 9.4 implies that the composition product has a left derived functor, which we will denote by $M \circ_\mathcal{P}^h N$ and which can be computed by taking a cofibrant resolution of either $M$ or $N$. A quasi-isomorphism $\mathcal{P} \longrightarrow \mathcal{Q}$ induces a quasi-isomorphism $M \circ_\mathcal{P}^h N \longrightarrow M \circ_\mathcal{Q}^h N$ for any $M \in \text{RMod}^\mathrm{dg}_\mathcal{Q}$ and $N \in \text{LMod}^\mathrm{dg}_\mathcal{Q}$.

**Corollary 9.6.** Given a map $f : \mathcal{P} \longrightarrow \mathcal{Q}$ in $\text{Op}^\mathrm{dg}_k$, there are Quillen adjunctions

\[ f_* : \text{Alg}^\mathrm{dg}_\mathcal{P} \xleftarrow{\sim} \text{Alg}^\mathrm{dg}_\mathcal{Q} : f^* \]

\[ f_! : \text{LMod}^\mathrm{dg}_\mathcal{P} \xrightarrow{\sim} \text{LMod}^\mathrm{dg}_\mathcal{Q} : f^* \]

given by restriction and induction. When $f$ is a quasi-isomorphism, these are Quillen equivalences.

**Proof.** The restriction functor $f^*$ clearly preserves (and detects) fibrations and quasi-isomorphisms. When $f$ is a quasi-isomorphism, $(f_!, f^*)$ is a Quillen equivalence because the counit map $f_!f^*(M) \simeq \mathcal{Q} \circ_\mathcal{P} M \longrightarrow M$ is a quasi-isomorphism for all cofibrant $M$ by Lemma 9.4.
Dualizing

Given two dg-categories $k_1$ and $k_2$, one can take the exterior Hadamard tensor product

$$\text{BiMod}_{k_1}^{\Sigma,dg} \times \text{BiMod}_{k_2}^{\Sigma,dg} \longrightarrow \text{BiMod}_{k_1 \otimes k_2}^{\Sigma,dg}; \quad (M_1, M_2) \longrightarrow M_1 \otimes M_2 \quad (9.7)$$

where for any $c_i \in k_1$ and $d_j \in k_2$,

$$(M_1 \otimes M_2)((c_1, d_1), \ldots, (c_p, d_p); (c_0, d_0)) = M_1(c_1, \ldots, c_p; c_0) \otimes M_2(d_1, \ldots, d_p; d_0).$$

From the description of the composition product, one sees that it is compatible with the exterior Hadamard tensor product in the sense that there is a natural morphism

$$(M_1 \circ_{k_1} N_1) \otimes (M_2 \circ_{k_2} N_2) \longrightarrow (M_1 \otimes M_2) \circ_{k_1 \otimes k_2} (N_1 \otimes N_2). \quad (9.8)$$

An element in the domain can be represented by a tensor product of two trees of height two, with vertices labeled by $M_1$ and $N_1$, resp. by $M_2$ and $N_2$. Such a tensor product is sent to zero if the two trees are different and if the trees are the same, one labels its vertices by the corresponding elements in $M_1 \otimes M_2$ and $N_1 \otimes N_2$.

The exterior Hadamard tensor product preserves colimits in both of its variables. It follows that there are functors

$$\text{Hom}_{k_1}(-, -): \left(\text{BiMod}_{k_1}^{\Sigma,dg}\right)^{\text{op}} \times \text{BiMod}_{k_2}^{\Sigma,dg} \longrightarrow \text{BiMod}_{k_1 \otimes k_2}^{\Sigma,dg}$$

$$\text{Hom}_{k_2}(-, -): \left(\text{BiMod}_{k_2}^{\Sigma,dg}\right)^{\text{op}} \times \text{BiMod}_{k_1}^{\Sigma,dg} \longrightarrow \text{BiMod}_{k_1 \otimes k_2}^{\Sigma,dg}$$

such that for $M_1 \in \text{BiMod}_{k_1}^{\Sigma,dg}$, $M_2 \in \text{BiMod}_{k_2}^{\Sigma,dg}$ and $N \in \text{BiMod}_{k_1 \otimes k_2}^{\Sigma,dg}$, there are natural bijections

$$\text{Hom}(M_1, \text{Hom}_{k_1}(M_2, N)) \cong \text{Hom}(M_1 \otimes M_2, N) \cong \text{Hom}(M_2, \text{Hom}_{k_2}(M_1, N)).$$

We will be interested in applying this to the case where $k_1 = k$ and $k_2 = k^{\text{op}}$ is its opposite.

**Definition 9.9.** Let $\text{End}(k)$ denote the endomorphism operad of $k$, considered as a left $k \otimes k^{\text{op}}$-module. More precisely, $\text{End}(k)$ has set of colours $S \times S$ and $p$-ary morphisms $((c_1, d_1), \ldots, (c_p, d_p)) \longrightarrow (c_0, d_0)$ given by $k$-linear maps

$$k(c_1, d_1) \otimes \cdots \otimes k(c_p, d_p) \longrightarrow k(c_0, d_0).$$

This is a (non-augmented) $k \otimes k^{\text{op}}$-operad. We define the dual of a symmetric $k$-bimodule $M$ to be the symmetric $k^{\text{op}}$-bimodule

$$M^\vee := \text{Hom}_k(M, \text{End}(k)).$$

Unravelling the definition, one sees that $M^\vee$ is given in arity $p$ by the dual $M^\vee(p) = \text{Hom}_k(M(p), k)$ with respect to the left $k$-module structure on $M(p)$. The right $k$-action on $k$ and the right $k^{\text{op}}$-action on $M(p)$ endow $M^\vee$ with the structure of a symmetric $k^{\text{op}}$-module. Explicitly, we have

$$M^\vee(c_1, \ldots, c_p; c_0) = \text{Hom}_k(M(c_1, \ldots, c_p; -), k(c_0; -)). \quad (9.10)$$

To make sure that taking duals is homotopically well-behaved, the following condition will repeatedly show up:

**Definition 9.11.** A symmetric $k$-bimodule $M$ is cofibrant as a left $k$-module if for each tuple of objects $c_i \in S$, the left $k$-module $M(c_1, \ldots, c_p; -)$ is cofibrant. This holds in particular if $M$ is cofibrant in the model structure of Proposition 9.2. If $M$ and $N$ are cofibrant as left $k$-modules, then $M \circ_k N$ is as well.
Cooperads over a dg-category

**Definition 9.12.** A $\k$-cooperad $\mathcal{C}$ is a coaugmented counital coalgebra in the category $\text{BiMod}_\k^{\text{dg}}$. We will say that $\mathcal{C}$ is filtered-cofibrant as a left $\k$-module if it admits an exhaustive filtration

$$k = F_0\mathcal{C} \subseteq F_1\mathcal{C} \subseteq F_2\mathcal{C} \subseteq \ldots$$

such that $\Delta(F_i\mathcal{C}) \subseteq \bigoplus_{p+q=r} F_p\mathcal{C} \cdot F_q\mathcal{C}$ and each $F_i\mathcal{C}$ is cofibrant as a left $\k$-module. The first condition implies that $\mathcal{C}$ is conilpotent and the second is equivalent to the associated graded $\text{gr}(\mathcal{C})$ being cofibrant as a left $\k$-module.

**Remark 9.13.** Recall that one can always endow a $\k$-cooperad with its coradical filtration, where $F_0\mathcal{C} = \k \oplus \ker(\Delta')$. If $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = k$, then $\mathcal{C}$ is filtered-cofibrant as a left $\k$-module if and only if it is cofibrant as a left $\k$-module, using the filtration by arity.

**Proposition 9.14.** Let $\mathcal{C}$ be a $\k$-cooperad and let $\mathcal{C}'$ be its (left) $\k$-linear dual. Then $\mathcal{C}'$ has the natural structure of an operad. If $C$ is a $\mathcal{C}$-coalgebra, then the dual $C'$ has a natural $\mathcal{C}'$-algebra structure.

**Proof.** It suffices to verify that the functor $(-)^\vee$ is lax monoidal, in the sense that there is a natural map $M' \circ_{\k\text{-op}} N' \rightarrow (M \circ_k N)^\vee$. This map is the adjoint of

$$\left(M' \circ_{\k\text{-op}} N'\right) \otimes \left(M \circ_k N\right) \xrightarrow{(9.8)} \left(M' \otimes M\right) \circ_{\k\text{-op} \otimes k} \left(N' \otimes N\right) \xrightarrow{\text{End}(k) \circ_{\k\text{-op} \otimes k} \text{End}(k)} \text{End}(k),$$

where the second map arises from the evaluation map $\text{Hom}_k(M, \text{End}(k)) \otimes M \rightarrow \text{End}(k)$ and the last map uses that $\text{End}(k)$ is a (non-augmented) $\k^{\text{op}} \otimes \k$-operad. \qed

### 9.2 All we need about bar-cobar

From now on, all $\k$-objects (bimodules, operads) that we consider are assumed to be as in Assumption 2.4: they are cofibrant as left $\k$-modules, and filtered-cofibrant in the case of cooperads.

**Definition 9.15** (Bar-cobar constructions). Given a $\k$-operad $\mathcal{P}$, its bar construction $\text{BP}$ is the $\k$-cooperad constructed as the cofree conilpotent $\k$-cooperad on the augmentation ideal $\mathcal{T}_k[1]$, i.e.

$$\text{BP} = T_k(\mathcal{T}_k[1]) = \k \oplus \mathcal{T}_k[1] \oplus \mathcal{T}_k[1] \otimes_k \mathcal{T}_k[1] \oplus \ldots$$

with an additional bar differential given by contraction of trees along inner edges.

Similarly, given a conilpotent $\k$-cooperad $\mathcal{C}$, its cobar construction $\Omega \mathcal{C}$ is the free graded $\k$-operad on the coaugmentation coideal $\mathcal{C}[-1]$, i.e.

$$\Omega \mathcal{C} = T_k(\mathcal{C}[-1]) = \k \oplus \mathcal{C}[-1] \oplus \mathcal{C}[-1] \otimes_k \mathcal{C}[-1] \oplus \ldots$$

with an additional cobar differential given by decomposing trees along inner edges.

**Construction 9.16** ($\k$-twisting morphisms). Let $M$ and $N$ be symmetric $\k$-bimodules. Their infinitesimal composition product $M \circ_{(1)} N$ is the subobject of $M \circ_k N$ given by trees with 2 vertices, with root vertex labeled by $M$ and the other vertex labeled by $N$. There is a natural retraction

$$M \circ_{(1)} N \rightarrow M \circ_k N \rightarrow M \circ_{(1)} N$$

where the projection quotients out trees labeled by $M$ and $N$ with more than two vertices.
Let $P$ be a $k$-operad and $C$ a conilpotent $k$-cooperad. A twisting morphism $\phi: C \rightarrow P$ is a map of symmetric $k$-bimodules of cohomological degree 1, which vanishes both after composing with the augmentation and coaugmentation map, such that:

$$\partial \phi + \phi \star \phi = 0$$  \hspace{1cm} (9.17)

where $\phi \star \phi$ is the composite

$$C \rightarrow C \otimes_k C \rightarrow C \otimes_1 C \rightarrow P \circ_k P \rightarrow P,$$

and $\partial$ denotes the commutator of differentials in $\text{Hom}_{\text{BiMod}}^* (C, P)$. We denote by $\text{Tw}(C, P) \subset \text{Hom}_{\text{BiMod}}^* (C, P)$ the set of twisting morphisms.

**Remark 9.18.** Similar to [LV, Proposition 6.4.3], one checks that the sequence of complexes

$$\text{Conv}(C, P)(p) := \text{Hom}_k \otimes (k^{op}) \otimes p (C(p), P(p))$$

has the structure of an (ordinary) operad in chain complexes, called the convolution operad. As in [LV, Proposition 6.4.5], it follows that $(\text{Hom}_{\text{BiMod}}^* (C, P), \star, \partial)$ is a pre-Lie algebra and the twisting morphisms are its Maurer–Cartan elements. If $C$ or $P$ is 1-reduced, i.e. zero in arity 0 and $k$ in arity 1, then [Wie19, Section 7] shows that such Maurer–Cartan elements correspond bijectively to maps of operads $L_\infty^{-1} \rightarrow \text{Conv}(C, P)$ from the operadic suspension of the $L_\infty$-operad: the value on the generating $p$-ary operation $l_p$ of $L_\infty^{-1}$ is given by $\phi_p: C(p) \rightarrow P(p)$.

**Proposition 9.19.** Let $C$ be a conilpotent $k$-cooperad and $P$ a $k$-operad. Then there are natural bijections

$$\text{Hom}_{\text{CoOp}}^* (C, B P) \cong \text{Tw}(C, P) \cong \text{Hom}_{\text{Op}}^* (\Omega C, P).$$

*Proof.* Maps of bimodules $\varphi: C \rightarrow D$ which vanish both when composed with the augmentation and coaugmentation map are in one-to-one correspondence with maps of augmented operads from the free operad generated by $C$ to $D$. One can check that the compatibility with the differentials is given exactly by equation (9.17). A dual argument on the category of conilpotent cooperads shows that $\text{Hom}_{\text{CoOp}}^* (C, B P) \cong \text{Tw}(C, P)$, see [LV, Theorem 6.5.7] for the case $k = k$.

**Lemma 9.20.** Let $P \rightarrow \Omega$ be a quasi-isomorphism between two $k$-operads which are cofibrant as left $k$-modules. Then the map $B P \rightarrow B \Omega$ is a quasi-isomorphism of $k$-operads, which are filtered-cofibrant as left $k$-modules.

*Proof.* Endow both bar constructions with the (exhaustive) filtration by word length in $P$ and $\Omega$. The map on the associated graded is just the map $T^* (P[1]) \rightarrow T^* (\Omega[1])$. When $P$ and $\Omega$ are cofibrant as left $k$-modules, these associated gradeds are cofibrant as left $k$-modules, so that $B P$ and $B \Omega$ are filtered-cofibrant. Using Lemma 9.4, we conclude that the map at the level of the associated graded is a quasi-isomorphism.

**Proposition 9.21.** Let $P$ be a $k$-operad which is cofibrant as a left $k$-module. Then the counit of the bar-cobar adjunction $\Omega B P \rightarrow P$ is a quasi-isomorphism.
Proof. Ignoring degrees, elements of $\Omega B\mathcal{P}$ can be seen as trees whose vertices are themselves ("inner") trees whose vertices are labeled by $\mathcal{P}$. Filtering by the number of inner edges (bar word length) and using the cofibrancy of $\mathcal{P}$ as a left $k$-module we recover at the level of the associated graded only the piece of the differential corresponding to the one from $\mathcal{P}$ and a second one making an inner edges into an outer edge. One checks that the associated graded retracts into $\mathcal{P}$ by constructing a homotopy that makes an outer edge into an inner edge.  

Definition 9.22 (Twisted composition products). Given a twisting morphism $\phi : \mathcal{C} \rightarrow P$, the twisted composition product $\mathcal{C} \circ_\phi \mathcal{P}$ [LV, Section 6.4.11] is the symmetric $k$-bimodule $\mathcal{C} \circ_k \mathcal{P}$, but with differential twisted by the map

$$\mathcal{C} \circ_k \mathcal{P} \xrightarrow{\Delta(1) \circ 1} (\mathcal{C} \circ_1 \mathcal{C}) \circ_k \mathcal{P} \rightarrow \mathcal{C} \circ_k \mathcal{C} \circ_k \mathcal{P} \xrightarrow{1 \circ_1 1} \mathcal{C} \circ_k \mathcal{P} \circ_k \mathcal{P} \xrightarrow{1 \circ_1 \mu} \mathcal{C} \circ_k \mathcal{P}.$$ 

Similarly, the twisted composition product $\mathcal{P} \circ_\phi \mathcal{C}$ has differential twisted by

$$\mathcal{P} \circ_k \mathcal{C} \xrightarrow{1 \circ_1 \Delta} \mathcal{P} \circ_k \mathcal{C} \circ_k \mathcal{C} \xrightarrow{1 \circ_1 1} \mathcal{P} \circ_k \mathcal{C} \circ_k \mathcal{C} \xrightarrow{\mu(1) \circ 1} \mathcal{P} \circ_k \mathcal{C}.$$ 

Example 9.23. For the universal twisting morphism $\pi : B\mathcal{P} \rightarrow \mathcal{P}$, elements of $B\mathcal{P} \circ_\pi \mathcal{P}$ can be identified with trees whose vertices are labeled by elements in $\mathcal{P}[1]$, or by elements of $\overline{\mathcal{P}}$ for (some of the) leaf vertices. The differential then has three parts: (a) applying the differential of $\mathcal{P}$ to vertices, (b) contracting inner edges between $\mathcal{P}[1]$-labeled trees and (c) replacing an $\mathcal{P}[1]$-labeled vertex with only $\mathcal{P}$-labeled vertices above it by a $\mathcal{P}$-labeled vertex and contracting (at the same time) all inner edges above it.

Similarly, $\Omega \mathcal{C} \circ_1 \mathcal{C}$ consists of trees with vertices labeled by $\mathcal{C}[-1]$, or by $\overline{\mathcal{C}}$ for (some of the) leaf vertices, with differential having three terms: (a) applying the differential of $\mathcal{C}$, (b) partially decomposing along inner edges between $\mathcal{C}[-1]$-labeled vertices and (c) decomposing a $\mathcal{C}$-labeled leaf vertex into height 2 trees with root vertex labeled by $\mathcal{C}[-1]$.

Lemma 9.24. Let $\phi : \mathcal{C} \rightarrow P$ be a twisting morphism, where $\mathcal{C}$ and $\mathcal{P}$ are filtered-cofibrant, resp. cofibrant as left $k$-modules.

1. Let $M \rightarrow N$ be a quasi-isomorphism between left $\mathcal{P}$-modules that are cofibrant as left $k$-modules. Then $(\mathcal{C} \circ_\phi \mathcal{P}) \circ_\mathcal{P} M \rightarrow (\mathcal{C} \circ_\phi \mathcal{P}) \circ_\mathcal{P} N$ is a quasi-isomorphism between filtered-cofibrant left $k$-modules.

2. Let $M \rightarrow N$ be a quasi-isomorphism between right $\mathcal{P}$-modules. Then $M \circ_\mathcal{P} (\mathcal{P} \circ_\phi \mathcal{C}) \rightarrow N \circ_\mathcal{P} (\mathcal{P} \circ_\phi \mathcal{C})$ is a quasi-isomorphism.

3. The maps $B\mathcal{P} \circ_\pi \mathcal{P} \rightarrow k$ and $\Omega \mathcal{C} \circ_1 \mathcal{C} \rightarrow k$ are quasi-isomorphisms.

Proof. For (1), filter $\mathcal{C} \circ_\phi \mathcal{P}$ using the filtration on $\mathcal{C}$. The associated graded is $\text{gr}(\mathcal{C}) \circ_k \mathcal{P}$. The map $(\mathcal{C} \circ_\phi \mathcal{P}) \circ_\mathcal{P} M \rightarrow (\mathcal{C} \circ_\phi \mathcal{P}) \circ_\mathcal{P} N$ preserves the induced filtrations and is given on the associated graded by $\text{gr}(\mathcal{C}) \circ_k M \rightarrow \text{gr}(\mathcal{C}) \circ_k N$. This is a quasi-isomorphism by Lemma 9.4. The same argument applies to (2).

For (3), filter $B\mathcal{P} \circ_\pi \mathcal{P}$ by the number of inner edges. On the associated graded, one can then construct a contracting homotopy replacing a $\mathcal{P}$-labeled leaf vertex by a $\mathcal{P}[1]$-labeled leaf vertex.

Similarly, the filtration on $\mathcal{C}$ induces a total filtration on $\Omega \mathcal{C} \circ_1 \mathcal{C}$. The associated graded consists of trees with vertices labeled by the associated graded $\text{gr}(\overline{\mathcal{C}})[-1]$, or $\text{gr}(\overline{\mathcal{C}})$ for (some) leaf vertices. Since the cocomposition vanishes on $\text{gr}(\overline{\mathcal{C}})$, the differential has two remaining contributions: (a) the differential on $\text{gr}(\overline{\mathcal{C}})$ and its shift and (c) sending a $\text{gr}(\overline{\mathcal{C}})$-labeled leaf vertex to the corresponding $\text{gr}(\overline{\mathcal{C}})[-1]$-labeled vertex. This has a contracting homotopy by replacing $\text{gr}(\overline{\mathcal{C}})[-1]$-labeled leaf vertices by $\text{gr}(\overline{\mathcal{C}})$-labeled leaf vertices.  

\[\text{68}\]
Corollary 9.25. Let $P$ be a $k$-operad which is cofibrant as a left $k$-module. Then $BP \simeq k \circ_P k$.

9.3 The bar-cobar construction on algebras

Let $C$ be a $k$-cooperad and $P$ a $k$-operad, which are filtered-cofibrant, resp. cofibrant, as left $k$-modules.

Definition 9.26. A twisting morphism $\phi : C \rightarrow P$ is said to be Koszul if $\phi$ induces a quasi-isomorphism $\Omega C \rightarrow P$.

We will say that it is weakly Koszul if instead the map $C \rightarrow BP$ is a quasi-isomorphism. Since the bar construction preserves quasi-isomorphisms (Lemma 9.20), Koszul morphisms induce a quasi-isomorphism $B\Omega C \rightarrow BP$ and are therefore weakly Koszul.

Definition 9.27. Let $\phi : C \rightarrow P$ be a twisting morphism, $C$ a $C$-coalgebra (in left $k$-modules) and $A$ a $P$-algebra (in left $k$-modules). A twisting morphism $f : C \rightarrow A$ over $\phi$ is a left $k$-linear map of degree 0 satisfying

$$\partial f + \phi \circ f = 0$$

where $\phi \circ f : C \rightarrow A$ is given by

$$C \rightarrow C \circ_k C \xrightarrow{\phi \circ f} P \circ_k A \rightarrow A.$$

We denote by $\text{Tw}_\phi(C, A)$ the set of twisting morphisms over $\phi$.

Remark 9.28. If $C$ is a conilpotent $C$-coalgebra and $A$ is a $P$-algebra, then one can check that the complex $\text{Hom}_k(C, A)$ has the structure of an algebra over the convolution operad $\text{Conv}(C, P)$ of Remark 9.18 [Wie19, Proposition 7.1].

If $C$ or $P$ is 1-reduced, then a twisting morphism $\phi$ determines a map $L_\infty\{-1\} \rightarrow \text{Conv}(C, P)$, so that $\text{Hom}_k(C, A)$ has a shifted $L_\infty$-structure. As in loc. cit. the value $l_p(f_1, \ldots, f_p)$ of the generating $p$-ary operation $l_p$ in $L_\infty\{-1\}$ is given by

$$C \xrightarrow{\sigma} C(p) \otimes_k P \xrightarrow{\sum \phi(p) \otimes f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(p)}} P(p) \otimes_k A \otimes P \rightarrow A,$$

where the sum runs over $\sigma \in \Sigma_p$. The twisting morphisms $f : C \rightarrow A$ are exactly the degree 1 elements of this $L_\infty$-algebra satisfying the Maurer–Cartan equation $\sum_n \frac{1}{n!} l_n(f, \ldots, f) = 0$ [Wie19, Theorem 7.1]. Note that the infinite sum becomes finite when evaluated at some $c \in C$, because $C$ is a conilpotent $C$-coalgebra.

Definition 9.29 (Bar-cobar construction for algebras). Given a twisting morphism $\varphi : C \rightarrow P$ and $C$ a $C$-coalgebra, we define the bar construction $\Omega_\varphi C$, to be the free $P$-algebra on $C$, $P \circ_k C$, with differential given on generators by $d(c) = d_C(c) + \delta(c)$ with $\delta : C \rightarrow C \circ_k C \rightarrow P \circ_k C$.

Similarly, given $A$ a $P$-algebra, its bar construction $B_\varphi A$ is the cofree $C$-coalgebra on $A$, $C \circ_k A$, with differential given by $d_A + \delta$ with $\delta$ onto generators given by $C \circ_k A \rightarrow P \circ_k A \rightarrow A$.

Remark 9.30. One can also identify using twisted composition products (Definition 9.22) as $\Omega_\varphi C \cong (P \circ_k C) \circ_C C$ and $B_\varphi = (C \circ_k P) \circ_P A$. In particular, if $\pi : B_\varphi \rightarrow P$ is the universal twisting morphism, then Lemma 9.24 shows that for every $P$-algebra which is cofibrant as a left $k$-module,

$$B_\varphi A \cong (B \circ_k P \circ_P A) \cong k \circ_P A \simeq k \circ_P A.$$
Proposition 9.31. There are natural bijections
\[ \text{Hom}_{\text{Alg}_P}(\Omega \phi C, A) \cong \text{Tw}_\phi(C, A) \cong \text{Hom}_{\text{CoAlg}_e}(C, B \phi A). \]

Proof. The proof is similar to Proposition 9.19, see also [LV, Proposition 11.3.1].

Lemma 9.32. Let \( \phi : \mathcal{C} \to \mathcal{P} \) be a twisting morphism and \( A \) a \( \mathcal{P} \)-algebra. Then:

1. \( B \phi \) preserves quasi-isomorphisms between \( \mathcal{P} \)-algebras that are cofibrant as left \( \k \)-modules.
2. If \( A \) is cofibrant as a left \( \k \)-module and \( \mathcal{C} \) is filtered-cofibrant as a left \( \k \)-module, then \( B \phi(A) \) is filtered-cofibrant as a left \( \k \)-module.
3. In the setting of (2), \( \Omega \phi B \phi(A) \) is a cofibrant \( \mathcal{P} \)-algebra.

Proof. The first two points follow from Lemma 9.24 and Remark 9.30. In particular, the proof of Lemma 9.24 shows that \( B \phi(A) = (\mathcal{C} \circ_k A, d_A + d_B) \) carries a filtration induced from the filtration on \( \mathcal{C} \).

For the third point, note that \( \Omega \phi B \phi(A) \) inherits a filtration by subalgebras from the filtration on \( B \phi(A) \). Since \( \text{gr}(B \phi(A)) \) is a trivial coalgebra, \( \text{gr}(\Omega \phi B \phi(A)) \) is the free \( \mathcal{P} \)-algebra on \( \text{gr}(\mathcal{C}) \circ_k A \). Since \( \text{gr}(\mathcal{C}) \circ_k A \) is cofibrant as a graded left \( \k \)-module, an inductive argument shows that \( \Omega \phi B \phi(A) \) is cofibrant (see also [Val14, Proposition 2.8]).

Lemma 9.33.

1. Let \( \mathcal{C} \) be filtered-cofibrant as a left \( \k \)-module and let \( \iota : \mathcal{C} \to \Omega \mathcal{C} \) be the universal twisting morphism. Then the counit \( \Omega B \iota A \to A \) is a quasi-isomorphism for all \( A \in \text{Alg}_{\mathcal{C}} \) which are cofibrant as left \( \k \)-modules.
2. Let \( \phi : \mathcal{C} \to \mathcal{P} \) be a Koszul twisting morphism. Then \( \Omega \phi B \phi B \to B \) is a quasi-isomorphism for all \( B \in \text{Alg}_P \) which are cofibrant as left \( \k \)-modules.

Proof. For (1), note that \( \Omega B \iota A \) consists of trees with vertices labeled by \( \mathcal{C}[−1] \) or by \( \mathcal{C} \) for (some of the) leaf vertices, and with leaves labeled by \( A \). The differential has a contribution from the differential on \( \Omega \mathcal{C} \circ_k \mathcal{C} \) (Example 9.23) and a contribution by letting \( \mathcal{C} \)-labeled leaf vertices act on their leaves. Filtering \( \Omega B \iota A \) by the number of leaves, the associated graded is \( (\Omega \mathcal{C} \circ_k \mathcal{C}) \circ_k A \). The result then follows from Lemma 9.24.

For (2), let \( f : \Omega \mathcal{C} \to \mathcal{P} \) be the induced map and notice that \( \Omega \phi B \phi B = f_!(\Omega B(f^! B)) \). The result then follows from part (1), \( (f_!, f^!) \) being a Quillen equivalence (Corollary 9.6) and \( \Omega B(f^! B) \) being cofibrant (Lemma 9.32). 

9.4 Free resolutions of operads

The remainder of this section is devoted to a proof of the following result, relating the homotopy-invariant condition appearing in Theorem 1.3 to a more concrete condition in terms of quasi-free resolutions:

Proposition 9.34. Let \( \mathcal{P} \) be a connective 0-reduced \( \k \)-operad. Then the following are equivalent:

1. the symmetric sequence \( \mathcal{P}^{\leq 1} \phi^! \mathcal{P}^{\leq 1} \) is eventually highly connective.\(^4\)
2. \( \mathcal{P} \) is quasi-isomorphic to a quasi-free, non-positively graded \( \k \)-operad with higher arity generators in increasingly negative degrees. More precisely, for every \( n \in \mathbb{Z} \), there exists a \( p(n) \in \mathbb{N} \) such that all generators of arity \( \geq p(n) \) are in cohomological degrees \( < n \).

\(^4\)Here we use the natural left and right actions of \( \mathcal{P} \) on its quotient \( \mathcal{P}^{\leq 1} \).
Remark 9.35. Recall that every cofibrant $k$-operad is the retract of an operad which is quasi-freely generated by a (S-coloured) symmetric sequence of graded vector spaces (one can take for instance its cobar-bar construction). Conversely, if $\mathcal{P}$ is quasi-freely generated by a symmetric sequence of graded vector spaces in nonpositive degree, then $\mathcal{P}$ is cofibrant.\footnote{More generally, a triangulated quasi-free operad is cofibrant, see [LV, Proposition B.6.10].}

For the remaining of the section, all (co)operads are 0-reduced (trivial in arity zero). We will make use of the following Quillen adjunction between the categories of 0-reduced $k$-operads

$$\text{sk}_p : \text{Op}_k^{\text{nu, dg}} \xrightarrow{\text{Op}_k^{\text{nu, dg}} : (-)^{\leq p}}$$

The right adjoint is the “truncation to arity at most $p$” functor that quotients an operad $\mathcal{P}$ by the operadic ideal $\bigoplus_{k > p} \mathcal{P}(k)$. Its left adjoint is the “$p$-skeleton” functor that associates to $\mathcal{Q}$ the operad $\text{sk}_p(\mathcal{Q})$ which is given in arities $\leq p$ by $\mathcal{Q}$, and which is freely generated by this data.

Remark 9.36. A map $f : \mathcal{P} \to \mathcal{Q}$ between cofibrant operads induces a quasi-isomorphism $\text{sk}_p \mathcal{P} \to \text{sk}_p \mathcal{Q}$ as soon as it induces a quasi-isomorphism in arity $\leq p$. Indeed, factor $f$ as an acyclic cofibration $\mathcal{P} \to \mathcal{P}'$ followed by a fibration $f : \mathcal{P}' \to \mathcal{Q}$ and use Lemma 9.41 to resolve $f'$ by a map which is an isomorphism in arities $\leq p$. The result then follows from the fact that $\text{sk}_p$ is a left Quillen functor and which only depends on arity ($\leq p$)-parts.

Lemma 9.37. Let $\mathcal{P}$ be a cofibrant 0-reduced $k$-operad, let $p \geq 1$ and consider the cofiber sequences

$$\text{sk}_p(\mathcal{P}) \xrightarrow{} \mathcal{P} \xrightarrow{} X$$

$$\mathcal{P}^{\leq 1} \xrightarrow{c^h_{\text{sk}_p(\mathcal{P})}} \mathcal{P}^{\leq 1} \xrightarrow{} \mathcal{P}^{\leq 1} \xrightarrow{c^h_p} \mathcal{P} \xrightarrow{} Y.$$  

There is a natural map $X \to Y[-1]$, which is an equivalence in arity $p + 1$. Furthermore, the map $\mathcal{P}^{\leq 1} c^h_p \mathcal{P}^{\leq 1} \to Y$ is an equivalence in arity $p + 1$ as well.

Proof. Let $\mathcal{P}^{\geq 2}$ denote the kernel of the quotient $\mathcal{P} \to \mathcal{P}^{\leq 1}$, so that there is a cofiber sequence

$$\mathcal{P}^{\geq 2} \xrightarrow{c^h_p} \mathcal{P}^{\leq 1} \xrightarrow{} \mathcal{P}^{\leq 1} \xrightarrow{} \mathcal{P}^{\leq 1} \xrightarrow{c^h_p} \mathcal{P}^{\leq 1}.$$  

(9.38)

Using the same cofiber sequence for $\text{sk}_p(\mathcal{P})$ and unraveling the definitions, one sees that there is a natural cofiber sequence

$$\text{sk}_p(\mathcal{P})^{\geq 2} \xrightarrow{c^h_{\text{sk}_p(\mathcal{P})}} \mathcal{P}^{\leq 1} \xrightarrow{} \mathcal{P}^{\geq 2} \xrightarrow{c^h_p} \mathcal{P}^{\leq 1} \xrightarrow{} Y[-1].$$

There is a natural map $\mathcal{P} \to \mathcal{P}^{\geq 2} \to \mathcal{P}^{\geq 2} c^h_p \mathcal{P}^{\leq 1}$ (the first one quotients out the arity 1 part), and similarly for $\text{sk}_p(\mathcal{P})$. The desired map $X \to Y[-1]$ is the induced map on cofibers.

Now suppose that $\mathcal{P} = (\text{Free}_{\text{Op}_k}(V), d)$ is a cofibrant $k$-operad, quasi-freely generated by a symmetric $k$-bimodule $V$. Then $\mathcal{P}^{\geq 2}$ is a cofibrant right $\mathcal{P}$-module, given by $\mathcal{P}(1) \circ V^{\geq 2} \circ \mathcal{P}$ (with some differential), where $V^{\geq 2}$ is the arity $\geq 2$ piece of $V$. It follows that

$$\mathcal{P}^{\geq 2} c^h_p \mathcal{P}^{\leq 1} \simeq \mathcal{P}^{\leq 1} \circ V^{\geq 2} \circ \mathcal{P}^{\leq 1} = \mathcal{P}(1) \circ V^{\geq p + 1} \circ \mathcal{P}(1).$$  

(9.39)

with some differential. A similar equivalence holds for $\text{sk}_p(\mathcal{P})$, which is a cofibrant suboperad of $\mathcal{P}$ freely generated by $V^{\leq p}$, the arity $\leq p$ piece of $V$. One then deduces that

$$Y[-1] \simeq \mathcal{P}(1) \circ V^{\geq p + 1} \circ \mathcal{P}(1).$$
In particular, it agrees with $P^2 P_0^\bullet \circ \mathcal{P}(1)$ in arity $p+1$. Note that the above symmetric sequence consists exactly of the $(p+1)$-ary operations of $P$, modulo those that are compositions of $(\leq p)$-ary operations. This is exactly the $(p+1)$-ary part of the cofiber $X$.

**Corollary 9.40.** Let $f : \mathcal{P} \to \mathcal{Q}$ be a map of connective $\theta$-reduced $k$-operads such that the map of symmetric $k$-bimodules

$$\mathcal{P}(1) \circ \mathcal{P}(1) \to \mathcal{Q}(1) \circ \mathcal{Q}(1)$$

is a quasi-isomorphism. Then $f$ is a quasi-isomorphism.

**Proof.** We may assume that $\mathcal{P}$ and $\mathcal{Q}$ are cofibrant. In that case, the map $f$ induces a quasi-isomorphism in arity at most $p$ if and only if the induced maps on $p$-skeleta $sk_p \mathcal{P} \to sk_p \mathcal{Q}$ is a quasi-isomorphism (Remark 9.36). We check this for all $p$ by induction. For $p = 1$, note that $\mathcal{P}(1) \circ \mathcal{P}(1)$ is given in arity 1 by $\mathcal{P}(1)$; this follows from the cofiber sequence 9.38 and equation 9.39.

Next, notice that the arity $(p+1)$-part of $\mathcal{P}$ is quasi-isomorphic to the arity $(p+1)$-part of the cofiber $Y$ from Lemma 9.37. If $f$ induces a quasi-isomorphism on $p$-skeleta, then this cofiber is quasi-isomorphic to the corresponding cofiber for $\mathcal{Q}$. It then follows that $f$ also induces a quasi-isomorphism on $(p+1)$-skeleta.

**Proof of Proposition 9.34.** $(2) \Rightarrow (1)$: follows from the cofiber sequence (9.38) and the identification (9.39).

$(1) \Rightarrow (2)$: we can assume that $\mathcal{P}$ is cofibrant to begin with. It then suffices to show that $\mathcal{P}$ admits a free resolution with all generators in degrees at most 0 and with the following property at each arity $p \geq 2$:

If $H^*(\mathcal{P}(\leq 1) \circ \mathcal{P}(\leq 1))$ is concentrated in degrees $\leq n(p)$, then the generators of arity $p$ are concentrated in degrees $\leq n(p) + 1$.

We construct this resolution by induction on skeleta, using that

$$\mathcal{P}(\leq 1) \circ sk_\mathcal{P}(\mathcal{P}) \simeq \left(\mathcal{P}(\leq 1) \circ sk_\mathcal{P}(\mathcal{P}) \right)^\leq_p.$$

For the 1-skeleton $\mathcal{P}(\leq 1) = sk_1(\mathcal{P})$, there is no condition. Suppose we have found the desired presentation for $sk_{p-1}(\mathcal{P})$. It follows from Lemma 9.37 that in arity $p$, the cohomology of the cofiber $sk_p \mathcal{P} / sk_{p-1}(\mathcal{P})$ is concentrated in degrees $\leq n(p) + 1$ (and also in degrees $\leq 0$). This means that $sk_p \mathcal{P}$ can be obtained from $sk_{p-1}(\mathcal{P})$ by adding arity $p$ generators of degree $\leq n(p) + 1$ (as well as generators of higher arity).

In Section 5, we will need a slight refinement of Proposition 9.34 which provides a quasi-free resolution of the entire tower $\mathcal{P} \to \ldots \to \mathcal{P}(\leq n) \to \ldots$.

**Lemma 9.41.** Let $f : \mathcal{P} \to \mathcal{Q}$ be a fibration of connective operads such that $f$ induces a trivial fibration in arities $\leq p$. For any cofibrant resolution $\tilde{\mathcal{P}} \to \mathcal{Q}$, there exists a cofibrant resolution $\mathcal{P}$ of $\mathcal{P}$ which fits into a diagram

$$\begin{array}{ccc}
\tilde{\mathcal{P}} & \xrightarrow{\tilde{f}} & \tilde{\mathcal{Q}} \\
\sim & \downarrow & \sim \\
\mathcal{P} & \xrightarrow{f} & \mathcal{Q}
\end{array}$$

such that $\tilde{f}$ induces an isomorphism in arities $\leq p$. 

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Proof. Since \( \hat{\mathcal{Q}} \) is cofibrant, there exists a lift

\[
\begin{array}{c}
(\mathcal{P} \times_{\mathcal{Q}} \hat{\mathcal{Q}}) \leq p \\
\downarrow \sim \\
\hat{\mathcal{Q}} \longrightarrow (\hat{\mathcal{Q}}) \leq p
\end{array}
\]

and therefore, by adjunction we have a lift

\[
\begin{array}{c}
\mathcal{P} \times_{\mathcal{Q}} \hat{\mathcal{Q}} \\
\downarrow g \\
\sk_{p} \hat{\mathcal{Q}} \rightarrow \hat{\mathcal{Q}}.
\end{array}
\]

We can now factor the map \( g \) into a cofibration followed by a weak equivalence \( \sk_{p} \hat{\mathcal{Q}} \hookrightarrow \mathcal{P} \sim \mathcal{P} \times_{\mathcal{Q}} \hat{\mathcal{Q}} \).

Since all operads involved are connective, this can be done inductively by ‘adding cells to kill a cycle’. As \( g \) is already a weak equivalence in arity \( \leq p \), it suffices to add cells in arity \( \geq p + 1 \), which does not change the arity \( \leq p \) part. In particular, the composite map \( \mathcal{P} \rightarrow \hat{\mathcal{Q}} \) induces a weak equivalence in arities \( \leq p \). \( \Box \)

**Proposition 9.42.** Let \( \mathcal{P} \) be a connective \( k \)-operad and consider the tower of \( k \)-operads

\[
\mathcal{P} \longrightarrow \ldots \longrightarrow \mathcal{P} \leq p \longrightarrow \mathcal{P} \leq p - 1 \longrightarrow \ldots \longrightarrow \mathcal{P} \leq 1.
\]

Then there exists a resolution of this tower by a tower of quasi-free, non-positively graded \( k \)-operads \( \mathcal{Q} \longrightarrow \ldots \longrightarrow \mathcal{Q}^{(p)} \longrightarrow \ldots \) with the following properties:

(a) Each \( \mathcal{Q}^{(p)} \longrightarrow \mathcal{Q}^{(p-1)} \) induces an isomorphism in arity \( \leq p - 1 \).

(b) Each \( \mathcal{Q}^{(p)} \) has higher arity generators in increasingly negative degrees (in the sense of Proposition 9.34).

(c) \( \mathcal{Q} \) is the limit of the tower.

(d) For each \( p \geq 2 \), the generators of \( \mathcal{Q}^{(p)} \) in arity \( p \) are concentrated in degrees \( \leq n(p) + 1 \), where \( n(p) \) is such that

\[
H^{*}(\mathcal{P} \leq 1 \circ_{\mathcal{P}} \mathcal{P} \leq 1)(p) = 0 \quad * > n(p).
\]

In particular, \( \mathcal{Q} \) is a graded-free resolution of \( \mathcal{P} \) with higher arity generators in increasingly negative degrees.

Proof. We can assume from the start that \( \mathcal{P} \) is already cofibrant, and then construct such a tower of free resolutions inductively, as in Lemma 9.41. In each inductive step, it suffices to add generators of arity \( \geq p \) to \( \sk_{p-1}(\mathcal{Q}^{(p-1)}) \). In particular, we can always arrange for condition (a).

To see what kind of generators have to be added in arity \( p \), note that there is a quasi-isomorphism

\[
\sk_{p-1}(\mathcal{Q}^{(p-1)}) \longrightarrow \sk_{p-1}(\mathcal{P} \leq p) \cong \sk_{p-1}(\mathcal{P})
\]
since both are quasi-isomorphic in arities $\leq p - 1$ (Remark 9.36). One deduces that the cofiber of $sk_{p-1}(Q^{(p-1)}) \rightarrow P$ is given in arity $p$ by the arity $p$ part of $\mathcal{P}^{\leq 1} \circ_{\mathcal{P}^{\leq p}} \mathcal{P}^{\leq 1}$. Since

$$\mathcal{P}^{\leq 1} \circ_{\mathcal{P}^{\leq 1}} \mathcal{P}^{\leq 1} \rightarrow \mathcal{P}^{\leq 1} \circ_{\mathcal{P}^{\leq p}} \mathcal{P}^{\leq 1}$$

is an equivalence in arity $\leq p$, is follows that we only have to add arity $p$ generators in degrees $\leq n(p) + 1$ (together with generators of higher arity). This makes sure that we can arrange for condition (d).

For the remaining generators that we have to add, note that $\mathcal{P}^{\leq p}$ satisfies the equivalent conditions of Proposition 9.34, since it is zero in arities $\geq p + 1$. This implies that it suffices to add higher arity generators in increasingly negative degrees, so that we can arrange for (c).

Finally, define $Q$ to be the limit of the tower. Since the tower becomes stationary in every fixed arity, it follows that $Q$ is graded-free. Furthermore, one sees that the arity $p$ generators of $Q$ are concentrated in degrees $\leq n(p) + 1$, so they sit in increasingly negative degrees by the assumption on $\mathcal{P}$.

References


