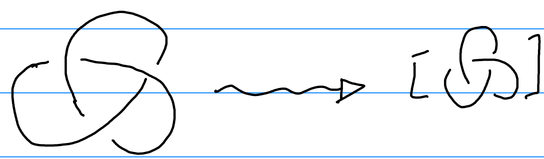
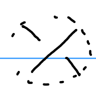
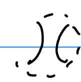
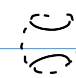
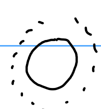
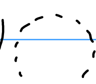
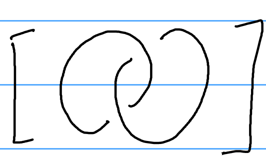
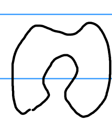
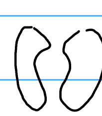
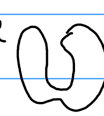
Positivity of surface skein algebras and categorification.

<u>Goal</u>: prove that surface skein algebras (unpunctured surfaces) admit positive bases using categorification.



 $= A$  $+ A^{-1}$ 

 $= (-A^2 - A^{-2})$ 

 $= A^2$  $+$  $+$  $+ A^{-2}$ 

$$= \underbrace{(A^6 + 2A^4 + 2 - A^{-2})}_{\text{Jones polynomial}} \cdot \emptyset$$

Aside:

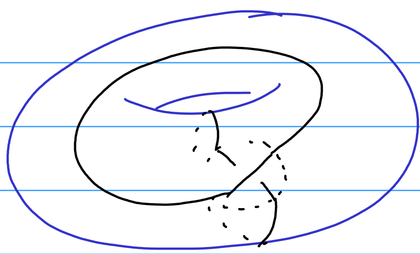
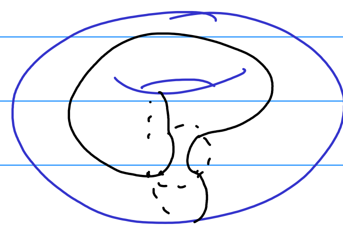
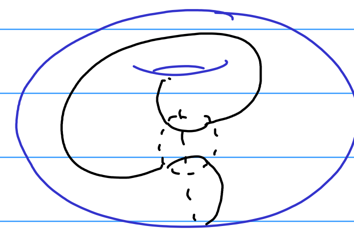
B^3 or S^3 $\cup$ link (Jones) $\xrightarrow{\text{root of } 1}$ M^3 $\cup$ link (WRT)

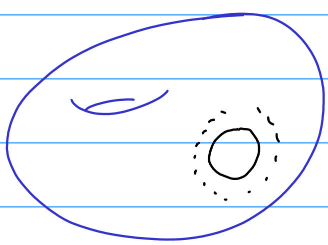
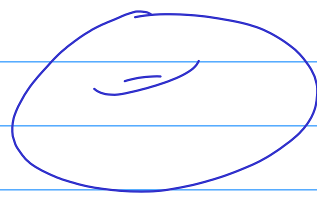
B^3 or S^3 $\cup$ link (Jones) $\longrightarrow$ M^3 $\cup$ link: $Sk(M^3)$ [Przytycki; Turaev] $\rightsquigarrow$ $M^3 = H_g \cup_{S_g} H_g$

$$\Downarrow$$

$$Sk(M^3) = Sk(H_g) \otimes_{Sk(S_g)} Sk(H_g)$$

algebra structure is central!

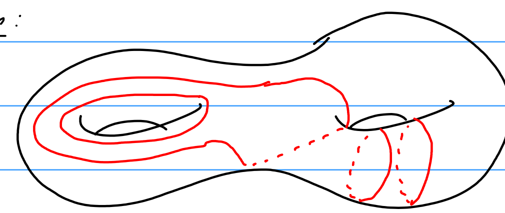
 $= A$  $+ A^{-1}$ 

 $= (-A^2 - A^{-2})$ 

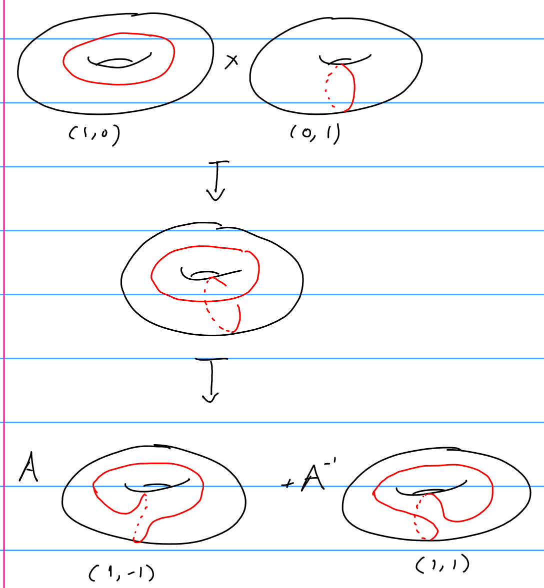
$Sk(\mathbb{R}^2) \simeq \mathbb{Z}[A, A^{-1}]$ with generator $\emptyset$.

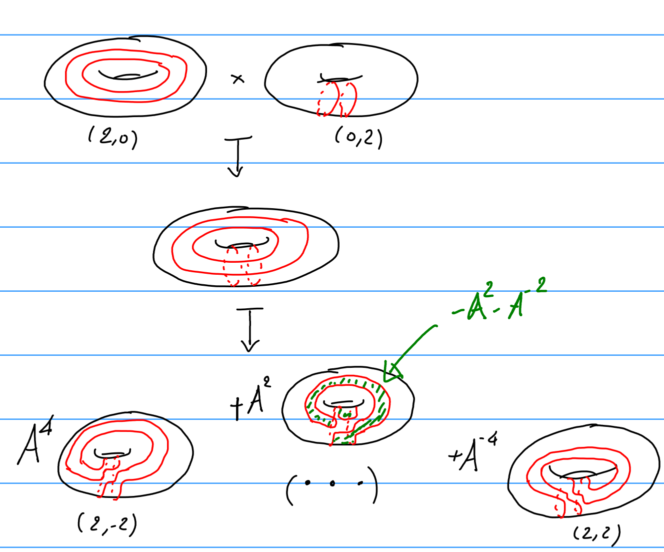
<u>Basis:</u> for S general, a basis is given by multicurves: embedded 1-manifolds, closed, with no inessential loop.

<u>Example:</u>

<u>Torus:</u>  $\longleftrightarrow$ $(2,2) = (1,1)^2$ <u>Higher genus:</u> 

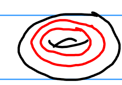
Algebra structure: stacking





Chebyshev polynomials:

1st kind	2nd kind
$T_0 = 2,\ T_1 = X$	$U_0 = 1,\ U_1 = X$
$T_n = X T_{n-1} - T_{n-2}$	$U_n = X U_{n-1} - U_{n-2}$

$T_2((1,0)) =$  $- 2$ 

Theorem [Frohman-Gelca 00]

For $S = \mathbb{T}^2$, the Chebyshev basis $\{\emptyset\} \sqcup \{T_k(m,n),\ m \wedge n = 1,\ k>0\}$ is positive, with explicit formulas:

$$(p,q)_T \times (r,s)_T = A^{\left|\begin{smallmatrix} p & r \\ q & s\end{smallmatrix}\right|}(p-r, q-s)_T + A^{-\left|\begin{smallmatrix} p & r \\ q & s\end{smallmatrix}\right|}(p+r, q+s)_T$$

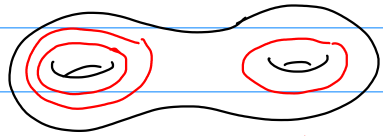
$$(p,q)_T = T_{p\wedge q}\left(\left(\frac{p}{p\wedge q}, \frac{q}{p\wedge q}\right)\right)$$

Chebyshev basis

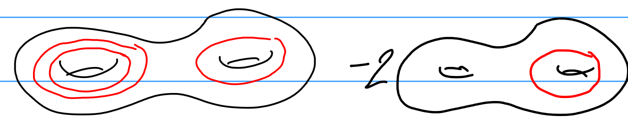
For general S:

multicurve basis $\longleftrightarrow$ Chebyshev basis

$\{\gamma\}$ $\qquad\qquad$ $\{\gamma_T\}$



$\gamma_1^2 \; \gamma_2$



$\gamma_1^2 \; \gamma_2 \; - \; 2 \; \gamma_2$

Conjecture: [Fock-Goncharov, Thurston, Lê]

$\{\gamma_T\}$ is positive.

[$S = T^2$ [FG], $\forall S$ $A=1$ [Thurston], $S = T^2 \setminus \{*\}$, $S^2 \setminus 4$pts [Bousseau], general [Mandel-Qin]]

Idea: categorification

Steps: ① $Sk(S)$ as a module

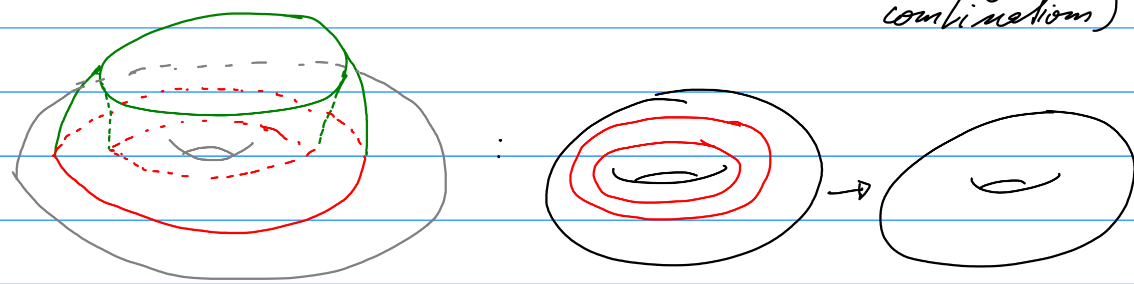
② algebra structure

③ lift up the positivity property

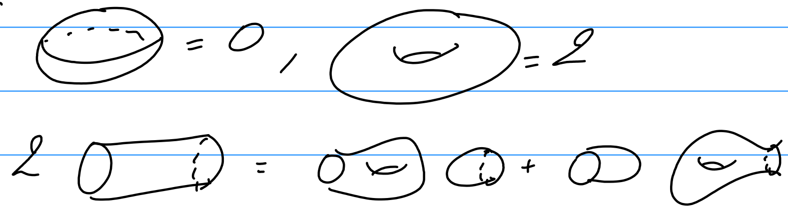
Categorification of $Sk(S)$: Bar-Natan's cobordisms

- objects: curves on S with no relations! (and $\oplus$, formal shift)
- morphisms: cobordisms in $S \times [0,1]$ mod relations (matrices of $\mathbb{Q}$-linear combinations)

Example:



Relations:

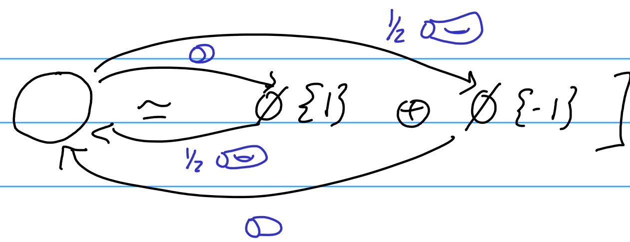


$S_g, g > 1 = 0.$

Grading: $-\chi(S)$

Theorem [Khovanov, Bar-Natan]: $K_0(BN(S)) \simeq Sk(S)$

$[\bigcirc] = -A^2 - A^{-2} \longleftrightarrow$



Trouble: this category is hard to handle...

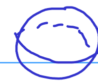
Theorem: [QW18] If $S \neq \mathbb{D}^2$ or $\mathbb{S}^2$, S orientable, then $BN(S)$ is non-negatively graded.

Important because:
- one can hope that $BN(S)^0$ is easier to handle
- $BN(S) \twoheadrightarrow BN(S)^0$ is a functor.

Defining the grading goes in two steps:

① any object $\simeq$ $\bigoplus$ essential multicurve $\{\ \}$ $\rightsquigarrow$ $BN(S)^{red}$

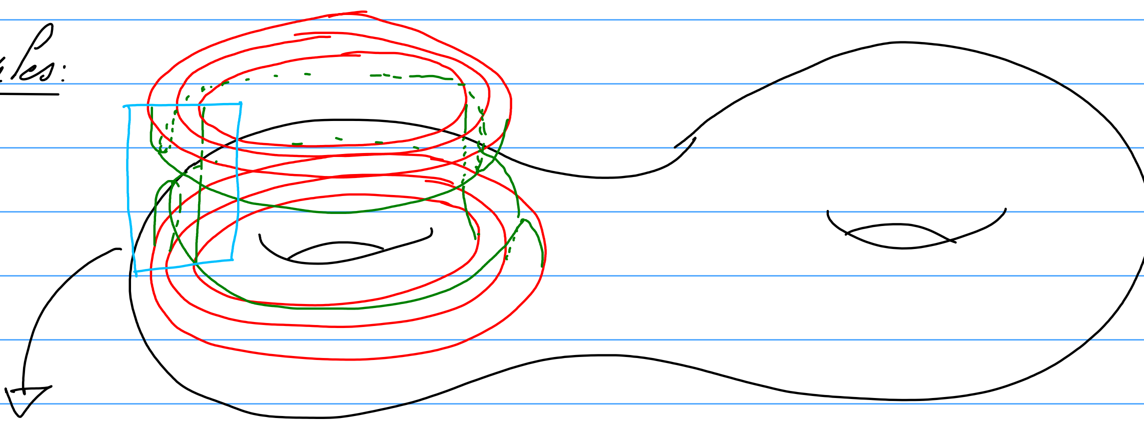
② **Lemma:** if $S \neq S^2$, then a connected surface properly embedded in $S \times [0,1]$ is:

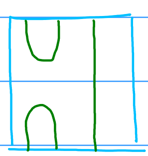
- a disk $\leftarrow$ not in $BN(S)^{red}$
- a sphere bounding a ball $\leftarrow$  $= 0$
- of non-positive χ

↳ induces a non-negative grading on $BN(S)^{red}$

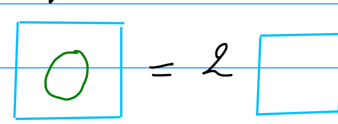
↳ that propagates in $BN(S)$

Simples:



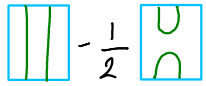


with relations:

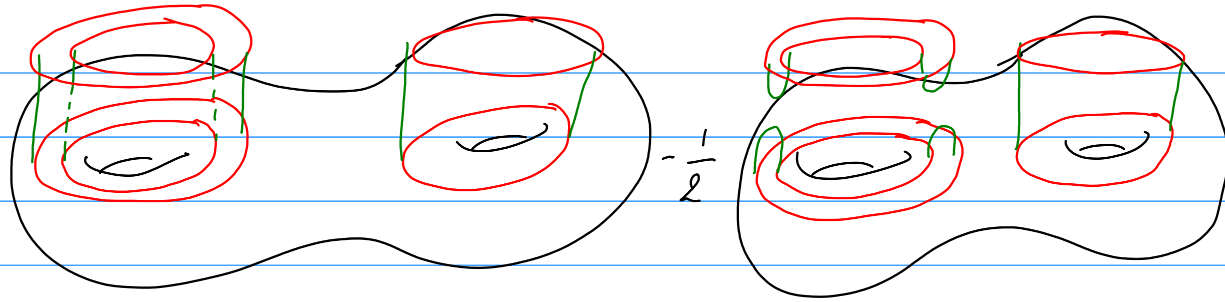
- isotopies
- 

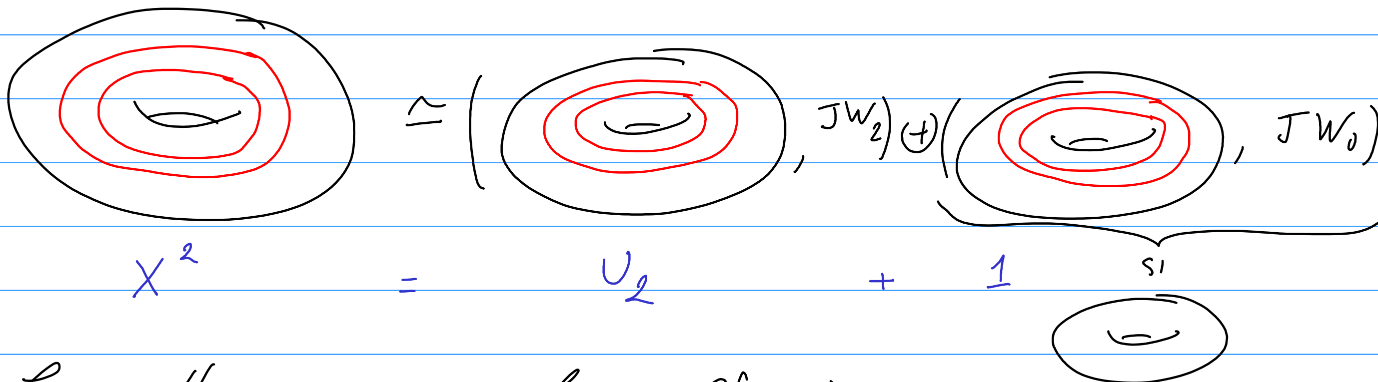
$\Big\}$ $TL(2)$

Theorem [QW18]: If $S \neq \mathbb{T}^2$ then $\mathrm{Kar}(BN(S)^0)$ is semi-simple with simples: unions of $JW \times S^1$ over non-equivalent curves.

$JW_2 =$ 

Example:





$$X^2 = U_2 + 1$$

Lemma: the corresponding basis in $Sk(S)$ is:
$\|$ $\{\gamma_{JW}\}$ made of Chebyshev polynomials of the 2^{nd} kind.

Conjecture: If $S \neq \mathbb{T}^2$, $\{\gamma_{JW}\}$ is positive.

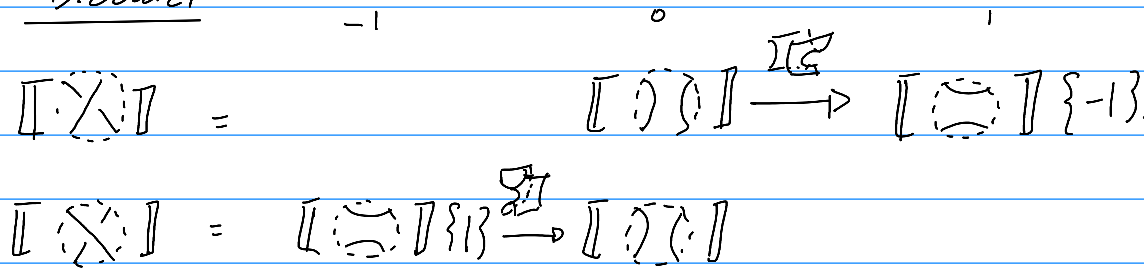
To prove this:

① need $\otimes$ $\longrightarrow$ takes functorial versions
$\downarrow$
go to $\mathfrak{gl}_2$ foams: full functoriality needed

② find a categorical analog of positivity $\leftrightsquigarrow$ heart of a t-structure

③ prove it's stable under $\otimes$

Product:



$\in K(BN(S))$

positive $\longleftrightarrow$ linear: $\overset{k}{\circ}\{-k\}$

Claim: $K_0(\text{linear in } Kh(Kar(BN^{red}(S)))) = \text{positive in } \{\gamma_{JW}\}$.

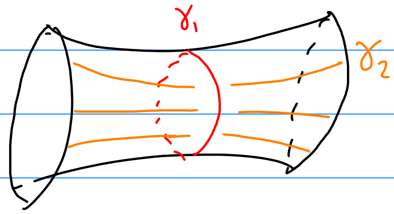
Product: $\underset{\substack{\cap \\ Kh(Kar(BN(S)))^{\otimes 2}}}{((\gamma_1, p_1), (\gamma_2, p_2))} \rightsquigarrow \underset{\substack{\cap \\ \text{Foam}}}{(\gamma_1 * \gamma_2, p_1 * p_2)} \xrightarrow{Kh} \underset{\substack{\cap \\ Kh(Kar(BN(S)))}}{-}$

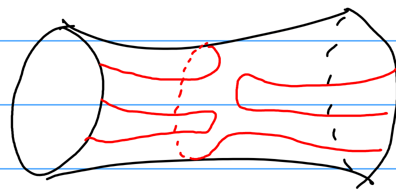
Main proof: we want: $\gamma_1^{(JW)} * \gamma_2^{(JW)}$ linear.

- $\gamma_1^{(JW)}$ direct summand in γ_1, so:
  $\gamma_1 * \gamma_2$ linear $\Longrightarrow$ $\gamma_1^{(JW)} * \gamma_2^{(JW)}$ linear

- Associativity: can assume $\underset{k}{\gamma_1}$ simple closed curve.

Then: $\gamma_1 * \gamma_2 \simeq \dots \oplus w\{-k-r\} \oplus \dots$





$\rightsquigarrow \quad \mathrm{HOM}(w, \gamma_1 * \gamma_2) \simeq \mathrm{HOM}(\underbrace{\gamma_1^{*} * w}_{\text{non-linear}}, \gamma_2)$

Theorem: [Q.22] If $S \neq \mathbb{T}^2$, $\{\gamma_{JW}\}$ is positive in $Sk(S)$