ON THE REGULARITY OF STATIONARY MEASURES

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Abstract. Extending a construction of Bourgain for SL(2,R), we construct on any semisimple real Lie group a finitely supported and Zariski dense probability measure whose stationary measure on the Furstenberg boundary has a smooth density.

1. Introduction

1.1. **Notations.** Let G be a connected real semisimple Lie group and let $P \subset G$ be a parabolic subgroup. We recall that a parabolic subgroup is a subgroup P that contains a minimal parabolic subgroup $P_{\min}$ and that a minimal parabolic subgroup is a subgroup that is equal to the normalizer of a maximal unipotent subgroup of G. The homogeneous space $X := G/P$ is called a partial flag variety. The homogeneous space $G/P_{\min}$ is called the full flag variety or the Furstenberg boundary.

Let μ be a (Borel) probability measure on G. In this paper, the probability measure μ will often be a finite average of Dirac masses $\mu = |F|^{-1} \sum_{f\in F} \delta_f$ where F is a finite subset of G.

1.2. **Example.** The main example is the following : the group G is the special linear group $G = \mathrm{SL}(d, \mathbb{R})$, the parabolic subgroup P is the stabilizer in G of a line of $\mathbb{R}^d$ and X is the real projective space $X = \mathbb{P}(\mathbb{R}^d)$.

1.3. **Main result.** A probability measure ν on X is said to be μ-stationary if $\nu = \mu * \nu$ where $\mu * \nu = \int_G g_*\nu \, \mathrm{d}\mu(g)$.

The following fact which is the starting point of this note is due to Furstenberg in [13] and to Goldsheid and Margulis in [18]. We denote by Γ_μ the subgroup of G spanned by the support of μ. We will assume that Γ_μ is Zariski dense in G. Here, this means that no finite index subgroup of Γ_μ is included in a proper connected closed subgroup of G.

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Fact 1.1. *When Γ_μ is Zariski dense in G, there exists a unique μ-stationary probability measure ν on X.*

We will call this measure ν the Furstenberg measure. The importance of this measure relies on the fact that it controls the behavior of the random walk on G obtained by multiplying random elements of G chosen independently with law μ. See the articles [11], [15], [21], or the surveys [5], [6], [9], [23]. The question we address in this short note is : What is the regularity of ν? Our main result is the construction of examples where ν has regularity C^k.

Theorem 1.2. *Let G be a connected semisimple real Lie group, P be a parabolic subgroup of G and let $k \geq 1$. Then, there exists a finitely supported symmetric probability measure μ on G with Γ_μ dense in G whose stationary measure ν on the flag variety $X := G/P$ of G has a C^k-smooth density.*

With no loss of generality, we can assume that G has finite center and we denote by $K \subset G$ a maximal compact subgroup. For instance when $G = \mathrm{SL}(d, \mathbb{R})$, the maximal compact subgroup K is the special orthogonal group $K = \mathrm{SO}(d, \mathbb{R})$.

The conclusion of Theorem 1.2 means that one can write $\nu = \psi \, \mathrm{d}x$ where $\psi \in C^k(X)$ is a k-times continuously differentiable function on X and where $\mathrm{d}x$ is the K-invariant probability measure on X.

When $G = \mathrm{SL}(2, \mathbb{R})$, this existence theorem is due to B. Barany, M. Pollicott and K. Simon in [4, Section 9], if we do not insist on μ to be symmetric. If we insist on μ to be symmetric, the first example of such a measure μ when $G = \mathrm{SL}(2, \mathbb{R})$ is due to J. Bourgain in [7]. Moreover the example of Bourgain is given by an explicit construction. Our proof below will also give an explicit construction of such a measure μ.

1.4. **Related results.** We survey now a few regularity results for the Furstenberg measure which help to put our theorem in perspective. We fix a K-invariant Riemannian metric on X.

(*i*) *When μ has a C^1 density, then ν has a C^∞ density.* Just because the convolution by μ is then a regularizing operator : it sends measures with C^k density to measures with C^{k+1} density.

(*ii*) *If Γ_μ is Zariski dense in G and μ has a finite exponential moment, then ν is Hölder regular.* This means that there exists $\alpha > 0$ and $C > 0$ such that $\nu(B(x, r)) \leq C r^\alpha$ for all ball $B(x, r)$ in X of radius r. This fact is due to Guivarch in [19]. See also the survey [5, Chap. 13] (*iii*) *For any lattice* Γ *in* G*, one can find* μ *such that* $\Gamma_\mu = \Gamma$ *and* $\nu = \mathrm{d}x$*.* This fact is due to Furstenberg in [12] and to Lyons and Sullivan in [25]. See also [27]. Ballmann and Ledrappier have proved in [3] that one can choose μ to be symmetric. When Γ is cocompact, the construction of Lyons and Sullivan gives a probability measure μ with a finite exponential moment.

(*iv*) *If* $G = \mathrm{SL}(2, \mathbb{R})$*, if* Γ_μ *is a non-cocompact lattice in* G *and if* μ *has a finite first moment, then* ν *is singular with respect to* $\mathrm{d}x$*.* This fact is due to Guivarch and Le Jan in [20]. See also [8] and [16].

(*v*) *If* $G = \mathrm{SL}(d, \mathbb{R})$*, there exists a finitely supported symmetric probability measure* μ *on* G *such that* Γ_μ *is dense in* G *and* ν *is singular with respect to* $\mathrm{d}x$*.* This fact is due to Kaimanovich and Le Prince in [22] and the construction allows to obtain a Furstenberg measure ν whose Hausdorff dimension is arbitrarily small. The authors of [22] conjectured there that the Furstenberg measure ν of a finitely supported probability measure μ might always be singular. As we have already seen, the first counterexamples for $G = \mathrm{PSL}(2, \mathbb{R})$ are due to Barany, Pollicott and Simon in [4] and to Bourgain in [7] with a symmetric measure μ. The main theorem of this note is a counterexample for each semisimple Lie group G.

(*vi*) It is not known whether there exists a finitely supported probability measure μ on G with Γ_μ discrete and Zariski dense and whose Furstenberg measure is absolutely continuous with respect to $\mathrm{d}x$.

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2. Construction of the law

We begin now the proof of Theorem 1.2.

2.1. **First reductions.**

We notice that if Theorem 1.2 is true for two semisimple Lie groups, then it will be true for their product. Since moreover Bourgain has proved Theorem 1.2 for $G = \mathrm{PSL}(2, \mathbb{R})$, we can assume with no loss of generality that

$$G \text{ is a non-compact simple Lie group, } G \neq \mathrm{PSL}(2, \mathbb{R}).$$

We will first construct in Section 2.6 probability measures μ for which the Furstenberg measure ν has an L^2 density. We will explain then in Section 2.7 that the same method allows to construct probability measures μ for which the Furstenberg measure ν has a C^k density.

2.2. **Transfer operators.** We introduce some notation and a few remarks that will relate Theorem 1.2 to a spectral property of the transfer operators that we will prove later. We will use the Hilbert space

$$L^2(X) := \{\varphi : X \to \mathbb{C} \mid \|\varphi\|_{L^2}^2 := \int_X |\varphi(x)|^2 \, \mathrm{d}x < \infty\}.$$

The main tool will be the two transfer operators

$$P_\mu : L^2(X) \to L^2(X) \ \text{ and } \ P_\mu^* : L^2(X) \to L^2(X)$$

defined for compactly supported measures μ on G by, for all φ, ψ in $L^2(X)$,

$$\begin{array}{rcl} P_\mu\varphi\,(x) & = & \int_G \varphi(gx)\,\mathrm{d}\mu(g) \quad \text{and} \\ P_\mu^*\psi\,(x) & = & \int_G \psi(g^{-1}x)\mathrm{Jac}(g^{-1},x)\,\mathrm{d}\mu(g), \end{array}$$

where $\mathrm{Jac}(g^{-1}, x)$ is the Jacobian determinant of the map $x \mapsto g^{-1}x$ with respect to the volume form $\mathrm{d}x$.

Remark 2.1. (i) These operators P_μ and P_μ^* are bounded operators which are adjoint of one another, i.e. for all φ, ψ in $L^2(X)$, one has

$$\int_X P_\mu\varphi\,\psi\,\mathrm{d}x = \int_X \varphi\, P_\mu^*\psi\,\mathrm{d}x\,.$$

(ii) Their norms as operators of $L^2(X)$ are equal $\|P_\mu\|_{L^2} = \|P_\mu^*\|_{L^2}$.
- When μ is a symmetric probability measure $\mu = \sigma$ supported on K, one has the equalities

$$P_\sigma^* = P_\sigma \ \text{ and } \ \|P_\sigma\|_{L^2} = 1,$$

because the measure $\mathrm{d}x$ is K-invariant and because $P_\sigma\mathbf{1} = \mathbf{1}$.

(iii) For all compactly supported measures μ_1, μ_2 on G, one has

$$P_{\mu_1 * \mu_2} = P_{\mu_2}P_{\mu_1}.$$

(iv) Whenever the equation

$$P_\mu^*\psi = \psi$$

has a solution ψ in $L^2(X)$, the measure $\psi\,\mathrm{d}x$ is μ-stationary. In particular, if Γ_μ is Zariski dense, by uniqueness, the stationary measure ν must be proportional to $\psi\,\mathrm{d}x$, hence ν has an L^2 density. Moreover, whenever this solution ψ can be found in $C^k(X)$, the stationary measure ν has a C^k density.

(v) The equation $P_\mu\varphi = \varphi$ always has a solution in $L^2(X)$: the constant function $\varphi = \mathbf{1}$. Hence we will just have to use the following general Fact 2.3 which allows us sometimes to deduce that 1 is an eigenvalue of P_μ^* from the input that 1 is an eigenvalue of P_μ.

2.3. **Essential spectral radius.** Let E be a Banach space and let $T \in \mathcal{L}(E)$ be a bounded operator. We denote by E^* the dual Banach space and $T^* \in \mathcal{L}(E^*)$ the adjoint operator. We recall that the *spectral radius* of T is

$$\rho(T) = \lim_{n\to\infty} \|T^n\|_E^{1/n}$$

and that the *essential spectral radius* is

$$\rho_e(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n},$$

where $\gamma(T)$ is the infimum of the radii R such that the image $T(B(0,1))$ of the ball of radius 1 is included in a finite union of translates of the ball $B(0,R)$. The operator T is said to be *quasicompact* if one has $\rho_e(T) < \rho(T)$.

The following two related facts will be useful.

Fact 2.2. *One has $\rho_e(T) < 1$ if and only if, some positive power T^d of T can be written as a sum $T^d = T_0 + T_1$ of two operators with T_0 compact and $\|T_1\| < 1$.*

Fact 2.3. *Let λ be a complex number such that $|\lambda| > \rho_e(T)$. Then the following dimensions are finite and are equal :*

$$\dim \operatorname{Ker}(T^* - \lambda) = \dim \operatorname{Ker}(T - \lambda).$$

For a proof of these classical facts, see for instance [5, Prop. B.13]. For more on the essential spectral radius see [28] and [29, Section 2.4].

2.4. **Spectral gap.** We recall that G is now a non-compact simple Lie group of dimension $d > 3$ and with finite center.

Fact 2.4. *G contains a simple 3-dimensional compact subgroup S.*

This subgroup S is locally isomorphic to the orthogonal group $\mathrm{SO}(3,\mathbb{R})$. We will say that a probability measure σ on S has a *spectral gap* if there exists $\varepsilon > 0$ such that, for every unitary representation $(\mathcal{H}, \pi)$ of S with no S-invariant non-zero vectors, one has $\|\pi(\sigma)\| \leq 1 - \varepsilon$ where $\pi(\sigma)$ is the bounded operator of $\mathcal{H}$ given by $\pi(\sigma) := \int_G \pi(s)\, \mathrm{d}\sigma(s)$. The following fact is due to Drinfeld in [10] (see also [26]).

Fact 2.5. *There exists a finitely supported symmetric probability measure σ on S which has a spectral gap.*

Here are two comments on this well-known fact.

- An explicit example of such a probability measure σ on $\mathrm{SO}(3,\mathbb{R})$ has been given by Lubotzky, Phillips and Sarnak in [24] (see also [30, Section 2.5]). One can choose σ to be $\sigma = \frac{1}{6}\sum_{i\leq 3} \delta_{R_i} + \delta_{R_i^{-1}}$ where the R_i's are the rotations of angle $\arccos(-3/5)$ with respect to the i^{th}

coordinate axis. One has then $\|\pi(\sigma)\| = \sqrt{3}/5$.
- When a probability measure σ on S has a spectral gap, the subgroup spanned by the support of σ is dense in S. Conversely, it is conjectured that any probability measure σ on S whose support spans a dense subgroup has a spectral gap.

2.5. **Construction of μ.** We choose now a finitely supported symmetric probability measure σ on S with a spectral gap. We choose also a finitely supported symmetric probability measure μ_0 on G of the form $\mu_0 = |F_0|^{-1} \sum_{f\in F_0} \delta_f$ where
(i) F_0 is a symmetric finite subset of G with $|F_0| = 4d$,
(ii) F_0 is included in a ball $B(e,r)$ of center e and small radius r so that, for g in $B(e,r)$ one has $\|P_{\delta_g}\| \leq (1+\varepsilon_0)^{1/d}$ with $\varepsilon_0 = |F_0|^{-d}/2$
(iii) F_0 contains elliptic elements $g_i = e^{X_i}$ of infinite order where the elements X_i spans the Lie algebra $\mathfrak{g}$ of G.
(iv) One can find a finite sequence $f_1, \ldots, f_d$ in F_0, such that the Lie algebra $\mathfrak{s}$ of S together with the images $\mathrm{Ad}(f_1 \cdots f_i)(\mathfrak{s})$ with $1 \leq i \leq d$ span $\mathfrak{g}$ as a vector space.

It is elementary to construct such a finite set F_0.
- The equality $|F_0| = 2d$ is not important: it can be relaxed easily.
- The condition (iii) ensures that the subgroup Γ_{μ_0} is dense in G.
- The condition (iv) will be used to ensure that the set $Sf_1S \cdots f_dS$ has non empty interior.

We will choose the probability measure μ to be

$$\mu = \mu_n := \sigma^{*n} * \mu_0 * \sigma^{*n} \,,$$

for n large enough. The subgroup Γ_{μ_n} is also dense in G.

2.6. **Stationary measure with L^2 density.**

Proposition 2.6. *For n large enough, the essential spectral radius of P_{μ_n} in $L^2(X)$ is strictly smaller than 1 :*

$$\rho_e(P_{\mu_n}) < 1.$$

Hence the μ_n-stationary measure ν_n on X has an L^2 density.

Since 1 is an eigenvalue of P_{μ_n}, this Proposition 2.6 tells us also that the operator P_{μ_n} is quasicompact in $L^2(X)$.

Proof of Proposition 2.6. Let σ_∞ be the S-invariant probability measure on S and let $\mu_\infty := \sigma_\infty * \mu_0 * \sigma_\infty$. Since the probability measure σ has a spectral gap, and since the operator P_{σ_∞} is the orthogonal projection on the S-invariant vectors in $L^2(X)$, one has the convergences in $\mathcal{L}(L^2(X))$ for the norm topology,

$$P_{\sigma^{*n}} \xrightarrow[n\to\infty]{} P_{\sigma_\infty} \text{ and hence } P_{\mu_n} \xrightarrow[n\to\infty]{} P_{\mu_\infty}.$$

Since the essential spectral radius varies continuously in the norm topology, by Lemma 2.7 below, one has $\rho_e(P_{\mu_n}) < 1$ for n large enough.

Since 1 is always an eigenvalue of P_{μ_n} and since $\rho_e(P_{\mu_n}) < 1$, according to Fact 2.3, 1 is also an eigenvalue of $P^*_{\mu_n}$. Let $\psi_n \in L^2(X)$ be the corresponding eigenvector. According to Remark 2.1.*iv*, the μ_n-stationary probability measure ν_n on X is proportional to $\psi_n \, \mathrm{d}x$. In particular ν_n has an L^2 density. □

Lemma 2.7. *The essential spectral radius of P_{μ_∞} in $L^2(X)$ is strictly smaller than 1 : $\rho_e(P_{\mu_\infty}) < 1$.*

Proof of Lemma 2.7. Recall that $d = \dim G$ and $\varepsilon_0 := |F_0|^{-d}$.

We first claim that we can write,

$$\mu_\infty^{*d} = \varepsilon_0 \alpha_0 + (1-\varepsilon_0)\alpha_1, \tag{2.1}$$

with α_0, α_1 positive measures on G such that α_0 has a C^∞ density and

$$\|P_{\alpha_1}\| \leq 1{+}\varepsilon_0. \tag{2.2}$$

Indeed, by construction μ_∞^{*d} is the average of $|F_0|^d$ probability measures of the form

$$\sigma_\infty * \delta_{f_1} * \sigma_\infty * \cdots * \delta_{f_d} * \sigma_\infty,$$

with the f_i's in F_0. If one chooses $(f_1, \ldots, f_d)$ in F_0^d to be the d-tuple given by condition (iv), the map $\pi : S^{d+1} \to G$ given by

$$\pi(s_0, \ldots, s_d) = s_0 f_1 s_1 \cdots f_d s_d$$

is submersive near the point $(e, \ldots, e)$. Since this map π is algebraic, it is submersive on a non-empty Zariski open subset $U \subset S^{d+1}$. This open subset U has full $\sigma_\infty^{\otimes d+1}$-measure. Hence there exists a compactly supported function $\varphi \in C_c^\infty(U)$ with $0 \leq \varphi \leq 1$ on U such that $\int_U \varphi \, \mathrm{d}\sigma_\infty^{\otimes d+1} = 1/2$. The measure

$$\alpha_0 := \pi_*(2\varphi\sigma_\infty^{\otimes d+1})$$

is a probability measure on G with C^∞ density. By construction, one can write $\mu_\infty^{*d} = \varepsilon_0\alpha_0 + (1-\varepsilon_0)\alpha_1$ where α_1 is another probability measure on G. It remains only to check (2.2).

Notice that, by construction, the operator P_{α_1} is an average of operators of the form P_{δ_g} where $g = s_0 f_1 s_1 \cdots f_d s_d$ with the s_i's varying in S and the f_i's varying in F_0. The condition (ii) tells us that these operators P_{δ_g} have norm at most $1{+}\varepsilon_0$, hence one also has $\|P_{\alpha_1}\| \leq 1{+}\varepsilon_0$ as required.

Now, the operator $T := P_{\mu_\infty}^d$ of $L^2(X)$ is equal to the sum $T = T_0 + T_1$ where $T_0 := \varepsilon_0 P_{\alpha_0}$ and $T_1 := (1-\varepsilon_0)P_{\alpha_1}$. The measure α_0 has a C^∞ density, hence the convolution operator by α_0 is a continuous operator from $L^2(X)$ to $C^\infty(X)$. Because of Ascoli Theorem, the embedding $C^\infty(X) \hookrightarrow L^2(X)$ is compact, hence the first operator T_0 is a compact operator of $L^2(X)$. The norm of the second operator T_1 is bounded by

$$\|T_1\| \leq (1-\varepsilon_0)\|P_{\alpha_1}\| \leq 1-\varepsilon_0^2 < 1.$$

This proves that $\rho_e(P_{\mu_\infty}) < 1$ in $L^2(X)$. □

2.7. **Stationary measure with C^k-density.** We explain now how to modify the previous arguments to show that for n large enough the μ_n-stationary measure ν_n has a C^k-density.

The main modification is to replace the Hilbert space $L^2(X)$ by the Sobolev space $E = H^{-s}(X)$ and by its dual $E^* = H^s(X)$. We first recall the definition of Sobolev spaces. For more details, one can consult [1] for Sobolev spaces over $\mathbb{R}^n$ and [2, Chap. 2] for Sobolev spaces over Riemannian manifolds. We denote by $C^\infty(X)$ the Frechet space of C^∞-functions on X, and by $\mathcal{D}'(X)$ the Frechet space of generalized functions (or distributions) on X. By definition, $\mathcal{D}'(X)$ is the topological dual of $C^\infty(X)$. The duality on $C^\infty(X)$ given by, for all φ, ψ in $C^\infty(X)$,

$$(\varphi, \psi) := \int_X \varphi(x)\psi(x)\, \mathrm{d}x \tag{2.3}$$

identifies the space $C^\infty(X)$ with a dense subspace of $\mathcal{D}'(X)$.

We denote by Δ the Laplacian of the K-invariant Riemannian metric on X. It is a symmetric operator on $C^\infty(X)$ that has a unique continuous extension, also denoted by Δ, as an operator of $\mathcal{D}'(X)$. The operator $1 - \Delta$ is invertible both in $C^\infty(X)$ and in $\mathcal{D}'(X)$. For s in $\mathbb{R}$, the Sobolev spaces are given by

$$H^s(X) := \{\psi \in \mathcal{D}'(X) \mid (1-\Delta)^{s/2}\psi \in L^2(X)\}.$$

Note that we will only need this definition when s is an even integer. The Sobolev space $H^s(X)$ is a Hilbert space for the norm

$$\|\psi\|_{H^s} := \|(1-\Delta)^{s/2}\psi\|_{L^2}.$$

This Hilbert norm is K-invariant. When a probability measure μ on G has compact support, the operators P_μ and P_μ^* introduced in Section 2.2 have a unique continuous extension, also denoted by P_μ and P_μ^*, as operators of $\mathcal{D}'(X)$. These operators P_μ and P_μ^* preserve the Sobolev spaces. In what follows, we will assume $s > k + \frac{1}{2}\dim X$ so that, by the Sobolev embedding theorem, one has $H^s(X) \subset C^k(X)$. We will consider P_μ as a bounded operator of $H^{-s}(X)$ and P_μ^* as a bounded operator of $H^s(X)$:

$$P_\mu : H^{-s}(X) \to H^{-s}(X) \text{ and } P_\mu^* : H^s(X) \to H^s(X).$$

We recall that the duality (2.3) on $C^\infty(X)$ extends as a duality also denoted $(\cdot,\cdot)$ between $H^{-s}(X)$ and $H^s(X)$. This duality identifies $H^s(X)$ with the dual of $H^{-s}(X)$. The operators P_μ and P_μ^* are still adjoint to each other for this duality, i.e. one has, for all φ in $H^{-s}(X)$ and ψ in $H^s(X)$,

$$(P_\mu\varphi, \psi) = (\varphi, P_\mu^*\psi).$$

Proof of Theorem 1.2. We use the same probability measures σ and σ_∞ on K, μ_n and μ_∞ on G as in Sections 2.5 and 2.6, maybe with a smaller value of r and a larger value of n. Our claim follows from the previous discussion and the following Proposition 2.8. □

Proposition 2.8. *Let $s \geq 0$. For n large enough, the essential spectral radius of P_{μ_n} in $H^{-s}(X)$ is strictly smaller than 1 : $\rho_e(P_{\mu_n}) < 1$. Hence the μ_n-stationary measure ν_n on X has a H^s density.*

Since 1 is an eigenvalue of P_{μ_n}, this Proposition 2.8 tells us also that the operator P_{μ_n} is quasicompact in $H^{-s}(X)$.

Proof of Proposition 2.8. The proof is the same as for Proposition 2.6. We just replace Lemma 2.7 by Lemma 2.9 below. □

Lemma 2.9. *Let $s \geq 0$. For r small enough, the essential spectral radius of P_{μ_∞} in $H^{-s}(X)$ is strictly smaller than 1 : $\rho_e(P_{\mu_\infty}) < 1$.*

Proof of Lemma 2.9. The proof is the same as for Lemma 2.7. □

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