CONDITIONED LIMIT THEOREMS FOR HYPERBOLIC DYNAMICAL SYSTEMS

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ABSTRACT. Let $(\mathbb{X}, T)$ be a subshift of finite type equipped with the Gibbs measure ν and let f be a real-valued Hölder continuous function on $\mathbb{X}$ such that $\nu(f) = 0$. Consider the Birkhoff sums $S_n f = \sum_{k=0}^{n-1} f \circ T^k$, $n \geqslant 1$. For any $t \in \mathbb{R}$, denote by τ_t^f the first time when the sum $t + S_n f$ leaves the positive half-line for some $n \geqslant 1$. By analogy with the case of random walks with independent identically distributed increments, we study the asymptotic as $n \to \infty$ of the probabilities $\nu(x \in \mathbb{X} : \tau_t^f(x) > n)$ and $\nu(x \in \mathbb{X} : \tau_t^f(x) = n)$. We also establish integral and local type limit theorems for the sum $t + S_n f(x)$ conditioned on the set $\{x \in \mathbb{X} : \tau_t^f(x) > n\}$.

CONTENTS

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1. Statement of the results and motivation

1.1. **Main results.** Consider a subshift of finite type $(\mathbb{X}, T)$ endowed with a Gibbs measure ν and let f be a real-valued Hölder continuous function on $\mathbb{X}$ (the precise definitions are given in Section 2). Define the Birkhoff sums

$$S_n f = f + f \circ T + \ldots + f \circ T^{n-1}, \quad n \geqslant 1.$$

A fundamental result of the theory of dynamical systems is the celebrated Birkhoff ergodic theorem which asserts that ν-almost surely,

$$\lim_{n \to \infty} \frac{S_n f}{n} = \int_{\mathbb{X}} f(x) \nu(dx) =: \nu(f).$$

Much effort was made to establish another important property – the central limit theorem for $S_n f$. To formulate the corresponding statement, we first note that the following limit exists:

$$\sigma_f^2 = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{X}} (S_n f - n\nu(f))^2 d\nu.$$

It is known that $\sigma_f^2 = 0$ if and only if f is a coboundary w.r.t. to T, which means that there exists a Hölder continuous function g on $\mathbb{X}$ such that $f(x) = g(Tx) - g(x)$ for any $x \in \mathbb{X}$. In the case when $\sigma_f > 0$ (or equivalently when f is not a coboundary) the following central limit theorem holds: for any bounded continuous function $F : \mathbb{R} \mapsto \mathbb{R}$,

$$\lim_{n \to \infty} \int_{\mathbb{X}} F\left(\frac{S_n f(x) - n\nu(f)}{\sigma_f \sqrt{n}} \right) \nu(dx) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(t) e^{-\frac{t^2}{2}} dt. \tag{1.1}$$

All these statements, which can be found in the excellent book of Parry and Pollicott [24], are consequences of many successive works such as Sinai [27, 28], Ratner [25], Ruelle [26] and Denker-Phillip [9], to cite only a few. The goal of this paper is to complement the central limit theorem (1.1) by proving limit theorems for the Birkhoff sum $t + S_n f$ under the condition that the trajectory $(t + S_k f)_{1 \leqslant k \leqslant n}$ stays positive, where $t \in \mathbb{R}$ is a starting point.

There is a vast literature on the properties of conditioned random walks and their applications based on independent observations: a brief historical foray into the subject of conditioned limit theorems and our motivation are presented in Section 1.2. At this point let us note that finding the corresponding asymptotics for Birkhoff sums encounters major problems. One of them is related to the fact that Wiener-Hopf factorization techniques do not apply in these settings. The other, and this is one of the main findings of the paper, is that the asymptotic analysis requires the introduction of the new object – the harmonic measure – which makes an important difference with the case of simple random walks. Regarding potential applications, we note that counting for trajectories conditioned to stay in some conic domains of $\mathbb{R}^d$ (like for instance the Weyl chamber) is of interest in statistical physics, see Fisher [14]. Our study which deals with the one dimensional observable f is the first step in considering such problems, and open ways to cope also with observables taking values in $\mathbb{R}^d$.

To state our results assume that $\nu(f) = 0$ and that f is not a coboundary. For any $t \in \mathbb{R}$, the following exit time is finite for ν-almost every $x \in \mathbb{X}$:

$$\tau_t^f(x) := \inf\{k \geqslant 1 : t + S_k f(x) < 0\}.$$

Thus, by definition, $\{x \in \mathbb{X} : \tau_t^f(x) > n\}$ is the set where the trajectory $(t + S_k f)_{1 \leqslant k \leqslant n}$ stays non-negative, i.e. $t + S_k f \geqslant 0$ for $1 \leqslant k \leqslant n$.

Our first theorem states the existence of a special Radon measure which will play a central role in the paper and will be used in the formulations of the subsequent results.

Theorem 1.1. *Let f be a Hölder continuous function on $\mathbb{X}$ such that $\nu(f) = 0$ and f is not a coboundary. Then, there exists a unique Radon measure μ^f on $\mathbb{X} \times \mathbb{R}$ such that for any continuous compactly supported function φ on $\mathbb{X} \times \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t) S_n f(x) \mathbb{1}_{\left\{\tau_t^f(x) > n\right\}} \nu(dx) dt = \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t) \mu^f(dx, dt). \tag{1.2}$$

Moreover, the Radon measure μ^f satisfies the following quasi-invariance property: for every continuous compactly supported function φ on $\mathbb{X} \times \mathbb{R}$,

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t) \mu^f(dx, dt) = \int_{\mathbb{X}\times\mathbb{R}} \varphi\left(T^{-1}x, t - f(T^{-1}x)\right) \mathbb{1}_{\{t \geqslant 0\}} \mu^f(dx, dt). \tag{1.3}$$

The limit (1.2) takes a simpler form when the function φ does not depend on the first argument. Indeed, we will show in Section 3.3 that the marginal of μ^f on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure. Its density function is a non-decreasing function on $\mathbb{R}$ that will be denoted by V^f. In particular, by standard arguments, the asymptotic (1.2) is valid for functions φ of the form $\varphi(x,t) = \mathbb{1}_{[a,b]}(t)$ for $x \in \mathbb{X}$ and $t \in \mathbb{R}$. This leads to the following:

Corollary 1.2. *Let f be a Hölder continuous function on $\mathbb{X}$ such that $\nu(f) = 0$ and f is not a coboundary. Then, for any real numbers $-\infty < a < b < \infty$, we have*

$$\lim_{n\to\infty} \int_a^b \int_{\mathbb{X}} S_n f(x) \mathbb{1}_{\left\{\tau_t^f(x) > n\right\}} \nu(dx) dt = \mu^f(\mathbb{X} \times [a,b]) = \int_a^b V^f(t) dt. \tag{1.4}$$

Note that (1.2) and (1.4) are stated in integral forms with respect to t. It is an open question whether it is possible to give an asymptotic of the integral $\int_{\mathbb{X}} S_n f(x) \mathbb{1}_{\{\tau_t^f(x) > n\}} \nu(dx)$ for a fixed value of t.

The Radon measure μ^f appearing in Theorem 1.1 will be called the *harmonic measure* associated to the dynamical system $(\mathbb{X}, T, \nu)$ with the observable f. The reason for this is that the measure μ^f is related to the harmonicity property that appears in the study of killed random walks on the half line. We refer to Section 3.3 for precise statements.

The following results describe the limit behavior of the Birkhoff sum $t + S_n f$ under the condition that the trajectory $(t + S_k f)_{1 \leqslant k \leqslant n}$ stays non-negative. We start by giving the equivalent of the probability that the trajectory $(t + S_k f)_{1 \leqslant k \leqslant n}$ stays non-negative. Denote by $\check{\mu}^f$ the harmonic measure related to the reversed dynamical system $(\mathbb{X}, T^{-1}, \nu)$ with the observable $f \circ T^{-1}$.

Theorem 1.3. *Let f be a Hölder continuous function on $\mathbb{X}$ such that f is not a coboundary and $\nu(f) = 0$. Then, for any continuous compactly supported function φ on $\mathbb{X} \times \mathbb{R}$, we have*

$$\lim_{n\to\infty} \frac{\sigma_f \sqrt{2\pi n}}{2} \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t) \mathbb{1}_{\{\tau_t^f(x) > n\}} \nu(dx)dt = \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\mu^f(dx,dt) \tag{1.5}$$

and

$$\lim_{n\to\infty} \frac{\sigma_f \sqrt{2\pi n}}{2} \int_{\mathbb{X}\times\mathbb{R}} \varphi(T^n x, t + S_n f(x)) \mathbb{1}_{\{\tau_t^f(x) > n\}} \nu(dx)dt = \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^{(-f)}(dx,dt). \tag{1.6}$$

As the measure μ^f has absolutely continuous marginal on $\mathbb{R}$, Theorem 1.3 also applies to the function $\varphi(x,t) = \mathbb{1}_{[a,b]}(t)$ for $x \in \mathbb{X}$ and $t \in \mathbb{R}$. In particular, this gives the following corollary.

Corollary 1.4. *Let f be a Hölder continuous function on $\mathbb{X}$ such that f is not a coboundary and $\nu(f) = 0$. Then, for any real numbers $-\infty < a < b < \infty$, we have*

$$\lim_{n\to\infty} \frac{\sigma_f \sqrt{2\pi n}}{2} \int_a^b \nu\left(x \in \mathbb{X} : \tau_t^f(x) > n\right) dt = \int_a^b V^f(t)dt.$$

Now we give a conditioned central limit theorem for the Birkhoff sum $S_n f$, which states that the law of $S_n f$ conditioned to stay positive converges weakly to the Rayleigh law. In the sequel we denote by ϕ^+ and Φ^+ the Rayleigh density and cumulative distribution functions respectively:

$$\phi^+(u) = ue^{-u^2/2}\mathbb{1}_{\{u\geqslant 0\}}, \quad \Phi^+(u) = \left(1 - e^{-u^2/2}\right)\mathbb{1}_{\{u \geqslant 0\}}, \quad u \in \mathbb{R}. \tag{1.7}$$

Theorem 1.5. *Let f be a Hölder continuous function on $\mathbb{X}$ such that f is not a coboundary and $\nu(f) = 0$. Then, for any continuous compactly supported function F on $\mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}$, we have*

$$\begin{aligned}\lim_{n\to\infty} \frac{\sigma_f \sqrt{2\pi n}}{2} \int_{\mathbb{X}\times\mathbb{R}} F\left(x, T^n x, t, \frac{S_n f(x)}{\sigma_f\sqrt{n}}\right) \mathbb{1}_{\{\tau_t^f(x) > n\}} \nu(dx)dt \\ = \int_{\mathbb{X}\times\mathbb{R}} \int_{\mathbb{X}\times\mathbb{R}} F(x,x',t,t')\phi^+(t')\nu(dx')dt'\mu^f(dx,dt).\end{aligned}$$

As above, we can actually apply Theorem 1.5 to the function $F(x,x',t,t') = \mathbb{1}_{[a,b]}(t)\mathbb{1}_{[a',b']}(t')$ for $x,x' \in \mathbb{X}$ and $t,t' \in \mathbb{R}$. Therefore, this implies the following corollary:

Corollary 1.6. *Let f be a Hölder continuous function on $\mathbb{X}$ such that f is not a coboundary and $\nu(f) = 0$. Then, for any real numbers $-\infty < a < b < \infty$ and $-\infty < a' < b' < \infty$, we have*

$$\begin{aligned}\lim_{n\to\infty} \frac{\sigma_f \sqrt{2\pi n}}{2} \int_a^b \nu\left(x \in \mathbb{X} : \frac{S_n f(x)}{\sigma_f\sqrt{n}} \in [a',b'], \tau_t^f(x) > n\right) dt \\ = \int_a^b V^f(t)dt\, \left(\Phi^+(b') - \Phi^+(a')\right).\end{aligned}$$

Next we formulate a conditioned local limit theorem for $S_n f$, which is a refinement of the previous result.

Theorem 1.7. *Let f be a Hölder continuous function on $\mathbb{X}$ such that $\nu(f) = 0$. Assume that for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pf + q$ is not cohomologous to a function with* *values in $\mathbb{Z}$. Then, for any continuous compactly supported function F on $\mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}$, we have*

$$\lim_{n\to\infty}\frac{\sqrt{2\pi}\sigma_f^3 n^{3/2}}{2}\int_{\mathbb{X}\times\mathbb{R}} F(x,T^n x,t,t+S_n f(x))\mathbb{1}_{\left\{\tau_t^f(x)>n-1\right\}}\nu(dx)dt$$
$$=\int_{\mathbb{X}\times\mathbb{R}}\int_{\mathbb{X}\times\mathbb{R}} F(x,x',t,t')\mu^f(dx,dt)\check{\mu}^{(-f)}(dx',dt'). \qquad (1.8)$$

In Theorem 1.7, we assumed that the function f satisfies a non-arithmeticity condition. When this is not the case but f is still not cohomologous to 0, we could still get an analogue of this result by the same method.

In the particular case when the function F has the form $F(x,x',t,t')=\mathbb{1}_{[a,b]}(t)\mathbb{1}_{[a',b']}(t')$ for $x,x'\in\mathbb{X}$ and $t,t'\in\mathbb{R}$, from the previous theorem we obtain the following:

Corollary 1.8. *Let f be a Hölder continuous function on $\mathbb{X}$ such that $\nu(f)=0$. Assume that for any $p\neq 0$ and $q\in\mathbb{R}$, the function $pf+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Then, for any real numbers $-\infty<a<b<\infty$ and $-\infty<a'<b'<\infty$, we have*

$$\lim_{n\to\infty}\frac{\sqrt{2\pi}\sigma_f^3 n^{3/2}}{2}\int_a^b \nu\left(x\in\mathbb{X}: t+S_n f(x)\in[a',b'],\tau_t^f(x)>n-1\right)dt$$
$$=\int_a^b V^f(t)dt\int_{a'}^{b'}\check{V}^{(-f)}(t')dt'.$$

In this corollary we have denoted by $\check{V}^{(-f)}$ the density function with respect to the Lebesgue measure of the marginal on $\mathbb{R}$ of the Radon measure $\check{\mu}^{(-f)}$.

From Corollary 1.8 we get a local limit theorem for $\tau_t^f(x)$. Indeed, by taking the interval $[a',b')$ to be $[-c,0)$ for $c>0$ large enough, as f is bounded, we deduce the following:

Corollary 1.9. *Let f be a Hölder continuous function on $\mathbb{X}$ such that $\nu(f)=0$. Assume that for any $p\neq 0$ and $q\in\mathbb{R}$, the function $pf+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Then, for any real numbers $-\infty<a<b<\infty$, we have*

$$\lim_{n\to\infty}\frac{\sqrt{2\pi}\sigma_f^3 n^{3/2}}{2}\int_a^b \nu\left(x\in\mathbb{X}:\tau_t^f(x)=n\right)dt=\int_a^b V^f(t)dt\int_{-\infty}^0\check{V}^{(-f)}(t')dt'.$$

Our Corollary 1.9 could be extended without difficulties to the case when one only assumes that f is not cohomologous to 0. This assertion could be deduced from a version of Theorem 1.7 for functions f that are cohomologous to functions with values in a set of the form $\alpha\mathbb{Z}+\beta$ for some $\alpha,\beta\in\mathbb{R}$.

Similarly to the comment after Corollary 1.2, Theorems 1.3, 1.5 and 1.7 are stated in integral forms with respect to t. It is an open problem to obtain asymptotics for a fixed value of $t\in\mathbb{R}$ of the following probabilities:

$$\nu\left(x\in\mathbb{X}:\tau_t^f(x)>n\right),\quad \nu\left(x\in\mathbb{X}:\frac{S_n f(x)}{\sigma_f\sqrt{n}}\in[a',b'],\tau_t^f(x)>n\right),$$
$$\nu\left(x\in\mathbb{X}:S_n f(x)\in[a',b'],\tau_t^f(x)>n\right).$$

Remark 1.10. In the previous theorems we have considered the two-sided subshift. However, all the above results apply as well to the case of one-sided subshift. The latter is a particular case of the two-sided one with a function f depending only on the future (or on the past). Indeed, let $(\mathbb{X}^+,T)$ be the one-sided shift associated with $\mathbb{X}$, $\psi:\mathbb{X}^+\to\mathbb{R}$

be the potential of the Gibbs measures ν and the function f only depends on the future coordinates in $\mathbb{X}$. Then, for instance, the conclusions (1.2) and (1.3) of Theorem 1.1 may be rewritten as follows: for any continuous compactly supported function φ on $\mathbb{X}^+ \times \mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}^+\times\mathbb{R}} \varphi(x,t) S_n f(x) \mathbb{1}_{\left\{\tau_t^f(x)>n\right\}} \nu^+(dx)dt = \int_{\mathbb{X}^+\times\mathbb{R}} \varphi(x,t)\mu_+^f(dx,dt) \tag{1.9}$$

and

$$\int_{\mathbb{X}^+\times\mathbb{R}} \varphi(x,t)\mu_+^f(dx,dt) = \int_{\mathbb{X}^+\times\mathbb{R}} \sum_{Ty=x} e^{-\psi(y)} \varphi\left(y, t - f(y)\right) \mathbb{1}_{\{t\geqslant 0\}} \mu_+^f(dx,dt), \tag{1.10}$$

where ν^+ is the marginal of ν and μ_+^f is the marginal of μ^f on $\mathbb{X}^+ \times \mathbb{R}$. In Theorem 1.3, in the case of an one-sided shift, the limit in the right-hand side of (1.6) exists. Nevertheless, even if the function f depends only on future coordinates, in order to construct the marginal of the measure $\check{\mu}^{(-f)}$, we need to work in the full-shift $\mathbb{X}$ and to apply Theorem 1.1 to the inverse map T^{-1}. In the same way, in Theorem 1.7, the left-hand side of (1.8) makes sense in an one-sided shift, but we need to use the two-sided shift in order to make sense of the right-hand side. We refer to Subsections 2.1 and 2.2 for more details about the relation between one-sided and two-sided subshifts.

Due to the theory of Markov partitions (see Appendix III of [24] and Chapter 18.7 of [20]), Theorems 1.1, 1.3, 1.5 and 1.7 can be applied without any changes to hyperbolic dynamical systems. Finally, using the approach of this paper, one can obtain analogous results for hyperbolic flows. The latter is out of the scope of this article and will be done in another work.

1.2. **Previous work and motivation.** The first examples of conditioned limit theorems for sums of independent random variables are due to the pioneering work of Spitzer [29] and Feller [13]. Since then integral and local limit theorems for random walks conditioned to stay positive attracted a lot of attention. Very many authors contributed to this subject, among them Borovkov [3, 4, 5], Bolthausen [2], Iglehart [22], Eppel [11], Bertoin and Doney [1], Caravenna [6], Vatutin and Wachtel [32], Doney [10], Kersting and Vatutin [23]. Most of this work is based on the Wiener-Hopf factorization and various factorization identities. Varopoulos [30, 31], Eichelsbacher and König [12], Denisov and Wachtel [7, 8] have studied the setting of random walks in cones and have developed a new approach for obtaining exact asymptotics based on the construction of a harmonic function for a certain operator. This construction therefore avoids the use of the Wiener-Hopf factorization. Following this method, in the case of dependent random variables recent progress was made in [17, 15], where conditioned integral limit theorems for products of random matrices and for Markov chains satisfying spectral gap properties have been obtained. In [16] a conditioned local limit theorem for a Markov chain with finite state space was considered.

As far as we know, conditioned integral and local limit theorems for Birkhoff sums have not yet been considered in the literature. In establishing these results we encountered two main difficulties.

The first one is actually related with the statement of the conditioned limit theorems themselves. In the case of Markov chain, the statement of results requires the use of the corresponding harmonic function. In some cases the subshift comes with an auxiliary Markov chain and the statement of the conditioned central limit theorem can be deduced from the Markov case. But in general, to state the result for our dynamical system, we need a replacement for the harmonic function. Indeed, one of the major findings of the paper is that, in the case for the subshift of finite type $(\mathbb{X}, T, \nu)$ with a general Hölder continuous observable f, a more general object – the harmonic measure μ^f – has to be considered. The conditioned central limit theorem for the Birkhoff sum $t + S_n f$ is stated in terms of the harmonic measure μ^f, whose use cannot be avoided and which constitutes an essential characteristic of the model. The construction of μ^f is performed first for the sum $t + S_n f$ with an observable f depending only on the past coordinates, which in the reversed setting corresponds to studying a Markov chain. Then it is extended gradually to a function f depending on the whole set of coordinates using smoothing techniques and a vague convergence argument, see Section 3.

Once this construction is achieved we are able to adapt several statements from the Markov chain case, such as the conditioned central limit theorem, to the dynamical system setting. We were motivated by the previous developments in [15, 16] for the Markov chains. To put it in a nutshell, we shall first establish the corresponding theorems for the Birkhoff sum $t+S_n f$ with an observable f depending only on the future coordinates, which corresponds to dealing with some Markov chain. Then we extend them to the general case of subshifts of finite type, using the technique similar to that developed for the proof of the existence of the harmonic measure.

The second difficulty is related to the proof of the corresponding conditioned local limit theorem. For proving the conditioned local limit theorem in the case of finite Markov chains [16] it is necessary to consider the reversed walk, which in this particular case is again a Markov chain. For the subshift of finite type the situation is trickier, but can be handled using the reversed subshift. Once the harmonic measure μ^f is constructed for any Hölder continuous observable f, this construction can be applied to the reversed subshift $(\mathbb{X}, T^{-1}, \nu)$ with observable $-f \circ T^{-1}$ yielding the reverse harmonic measure $\check{\mu}^{-f}$, which is necessary to state the conditioned local limit theorem. To prove the conditioned local limit theorem we are able to patch up the two conditioned integral limit theorems for the direct and reversed walks to establish a conditioned local limit theorem, where both measures μ^f and $\check{\mu}^{-f}$ will show up. We use the techniques from [7] and [18] which deal with random walks with independent increments.

In a perspective, it is possible to apply the developed approach for studying conditioned local limit theorems for products of random matrices and more generally for Markov chains with values in general state spaces, in contrast to [16] where a chain with finite state spaces has been considered. This will be the subject of a forthcoming paper.

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2. Background and auxiliary statements

2.1. **Subshift of finite type and Gibbs measure.** We start by precisely introducing the subshift of finite type. Let $k \geqslant 2$ be an integer and $A = \{1, 2, \dots, k\}$. Let M be a transition matrix on A, that is to say, $M = (M(i,j))_{i,j \in A}$ is a matrix with coefficients in $\{0, 1\}$. We assume that the transition matrix M is aperiodic in the sense that there exists an integer $p \geqslant 1$ such that all the coefficients of the matrix M^p are strictly positive. Consider the associated subshift of finite type

$$\mathbb{X} = \left\{ x = (x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}} : M(x_n, x_{n+1}) = 1, n \in \mathbb{Z} \right\} \subset A^{\mathbb{Z}},$$

equipped with the shift map T defined by $(Tx)_n = x_{n+1}$ for $x \in \mathbb{X}$ and $n \in \mathbb{Z}$. The set $\{1, 2, \dots, k\}$ is equipped with the discrete topology, so the space $A^{\mathbb{Z}}$ is compact with the corresponding Tychonov product topology. We equip $\mathbb{X}$ with the induced topology, which is also compact. For any $x = (x_n)_{n\in\mathbb{Z}} \in \mathbb{X}$ and $y = (y_n)_{n\in\mathbb{Z}} \in \mathbb{X}$, define

$$\omega(x, y) = \min\{k \geqslant 0 : x_k \neq y_k \text{ or } x_{-k} \neq y_{-k}\}.$$

Note that for any constant $\alpha \in (0, 1)$, the function $(x, y) \mapsto \alpha^{\omega(x,y)}$ is a distance on $\mathbb{X}$ which induces the natural product topology.

=The space of real-valued continuous functions $f : \mathbb{X} \to \mathbb{R}$ is denoted by $\mathcal{C}(\mathbb{X})$. For any function $f \in \mathcal{C}(\mathbb{X})$, we say that f is Hölder continuous on $\mathbb{X}$ if there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that for all $x, y \in \mathbb{X}$,

$$|f(x) - f(y)| \leqslant C\alpha^{\omega(x,y)}. \tag{2.1}$$

For a fixed $\alpha \in (0, 1)$, denote by $\mathscr{B}_\alpha$ the space of all real-valued functions on $\mathbb{X}$ satisfying (2.1) for some constant C, equipped with the following norm

$$\|f\|_{\mathscr{B}_\alpha} := \sup_{x\in\mathbb{X}} |f(x)| + \sup_{x,y\in\mathbb{X}: x\neq y} \frac{|f(x) - f(y)|}{\alpha^{\omega(x,y)}}. \tag{2.2}$$

The function $d_\alpha : (x, y) \mapsto \alpha^{\omega(x,y)}$ is a distance on $\mathbb{X}$, and $\mathscr{B}_\alpha$ is the space of Lipschitz continuous functions with respect to the distance d_α. Note that the notion of Lipschitz continuity depends on the index α, but the notion of Hölder continuity does not. It is clear that the set of all real-valued Hölder continuous functions on $\mathbb{X}$ can be written as $\mathscr{B} = \cup_{0<\alpha<1} \mathscr{B}_\alpha$.

For any $f \in \mathscr{B}$, we consider the Birkhoff sum process $(S_n f)_{n\geqslant 0}$ by setting $S_0 f = 0$ and

$$S_n f = f + \ldots + f \circ T^{n-1}, \quad n \geqslant 1.$$

Let us denote by $\mathbb{X}^+ \subset A^{\mathbb{N}}$ the set

$$\mathbb{X}^+ = \left\{ x = (x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}} : M(x_n, x_{n+1}) = 1, n \in \mathbb{N} \right\} \subset A^{\mathbb{N}}.$$

The set of continuous functions on $\mathbb{X}^+$ is denoted by $\mathcal{C}(\mathbb{X}^+)$. By abuse of notation, the one-sided shift map $\mathbb{X}^+ \mapsto \mathbb{X}^+$ will still be denoted by T.

The Ruelle operator $\mathcal{L}_f : \mathcal{C}(\mathbb{X}^+) \to \mathcal{C}(\mathbb{X}^+)$ related to $f \in \mathcal{C}(\mathbb{X}^+)$ is defined as follows: for any $g \in \mathcal{C}(\mathbb{X}^+)$,

$$\mathcal{L}_f g(x) = \sum_{y:\, Ty=x} e^{-f(y)} g(y), \quad x \in \mathbb{X}^+. \tag{2.3}$$

One can easily see that $\mathcal{L}_f$ is a bounded linear operator on $\mathcal{C}(\mathbb{X}^+)$. From (2.3), by iteration, it follows that for any $n \geqslant 1$,

$$\mathcal{L}_f^n g(x) = \sum_{y:\, T^n y=x} e^{-S_n f(y)} g(y), \quad x \in \mathbb{X}^+.$$

Also, if $h \in \mathcal{C}(\mathbb{X}^+)$, we have the conjugacy relation

$$\mathcal{L}_{f+h\circ T-h} g = e^{-h} \mathcal{L}_f \left(e^h g\right), \tag{2.4}$$

which tells us that the spectral properties of the transfer operator $\mathcal{L}_f$ only depend on the cohomology class of f. We say that a real-valued and Hölder continuous function ψ on $\mathbb{X}^+$ is normalized if $\mathcal{L}_\psi \mathbb{1} = \mathbb{1}$. By [24, Chapter 2, Theorem 2.2], there exist a real-valued Hölder continuous function h and a real number λ such that $\mathcal{L}_\psi e^h = e^{\lambda+h}$. From the conjugacy relation (2.4), this tells us that the function $\psi - h\circ T + h + \lambda$ is also normalized. Therefore, all over the paper, we will assume that ψ is normalized. In this case, it is well known (e.g. [24]) that the adjoint operator $\mathcal{L}_\psi^*$ admits a unique invariant probability measure ν^+. The measure ν^+ is called the Gibbs measure related to the potential ψ. Since ψ is normalized, the measure ν^+ is T-invariant, that is, for any $f \in \mathcal{C}(\mathbb{X}^+)$,

$$\nu^+(f\circ T) = \nu^+(f), \tag{2.5}$$

see [24, Chapter 2].

Note that ν^+ is also T-ergodic, meaning that any T-invariant Borel subset B of $\mathbb{X}^+$ has ν^+ measure 0 or 1:

$$T^{-1}B = B \quad \Rightarrow \quad \nu^+(B) \in \{0,1\}.$$

Thanks to the following lemma, the measure ν^+ allows to define a T-invariant measure on $\mathbb{X}$.

Lemma 2.1. *Let ν^+ be a Borel probability measure on $\mathbb{X}^+$ which is T-invariant. Then there exists a unique T-invariant Borel probability measure ν on $\mathbb{X}$ such that the image of ν under the natural projection map $\mathbb{X} \to \mathbb{X}^+$ is equal to ν^+.*

The proof of this lemma is just a consequence of Kolmogorov's extension theorem. We will actually give an explicit formula for the measure ν below.

2.2. **Conditional measures on the past.** For any $z \in \mathbb{X}^+$, we shall construct a measure ν_z^-, which is the conditional measure of ν with respect to the map $x \in \mathbb{X} \mapsto x_+ \in \mathbb{X}^+$. To this end, for any $a \in A$, let

$$\mathbb{X}_a^- = \left\{ y \in A^{-\mathbb{N}^*} : M(y_{-1}, a) = 1, M(y_{-n-1}, y_{-n}) = 1, \forall n \geqslant 1 \right\},$$

where M is the transition matrix on the set A which was used to define the finite type subshift $\mathbb{X} \subset A^{\mathbb{Z}}$. For any $z \in \mathbb{X}^+$, we set $\mathbb{X}_z^- = \mathbb{X}_{z_0}^-$, where z_0 is the first coordinate of $z \in \mathbb{X}^+$. We have the decomposition

$$\mathbb{X} = \bigcup_{z\in\mathbb{X}^+} \mathbb{X}_z^- \times \{z\}.$$

The point z may be thought of as the future of the trajectory whereas the elements of $\mathbb{X}_z^-$ describe the pasts which are compatible with this future. Let us introduce some notation related to this decomposition. For any $z \in \mathbb{X}^+$ and $y \in \mathbb{X}_z^-$, we denote $y \cdot z = (y,z) \in \mathbb{X}$. For $z \in \mathbb{X}^+$ and $k \geqslant 1$, we set

$$A_z^k = \Big\{ (y_{-k}, \dots, y_{-1}) \in A^{\{-k,\dots,-1\}} : \\ M(y_{-1}, z_0) = 1, M(y_{-n-1}, y_{-n}) = 1, \forall 1 \leqslant n \leqslant k-1 \Big\}.$$

For $(y_{-k}, \dots, y_{-1}) \in A_z^k$, we set $y_{-k} \dots y_{-1} \cdot z$ to be the element $w \in \mathbb{X}^+$ defined by

$$w_n = \begin{cases} y_{n-k} & \text{if } 0 \leqslant n \leqslant k-1 \\ z_{n-k} & \text{if } n \geqslant k. \end{cases}$$

For $a \in A_z^k$, let

$$\mathbb{C}_{a,z} = \{y \in \mathbb{X}_z^- : y_{-k} = a_{-k}, \dots, y_{-1} = a_{-1}\} \tag{2.6}$$

be the associated cylinder of length k in $\mathbb{X}_z^-$.

Recall that the two-sided shift map $T : \mathbb{X} \to \mathbb{X}$ and its inverse T^{-1} are defined by $(Tx)_n = x_{n+1}$ and $(T^{-1}x)_n = x_{n-1}$ for any $x \in \mathbb{X}$ and $n \in \mathbb{Z}$. By abuse of notation, the one-sided forward shift map will be denoted by $T : \mathbb{X}^+ \mapsto \mathbb{X}^+$ and is defined by $T(x) = (x_1, x_2, x_3, \dots)$, for any $x = (x_0, x_1, x_2, \dots) \in \mathbb{X}^+$. Let us define conditional measures on the past of trajectories. For $k \geqslant 0$, define ν_z^k as a function on cylinders of length k in $\mathbb{X}_z^-$ by the formula

$$\nu_z^k(\mathbb{C}_{a,z}) = \exp(-S_k\psi(a \cdot z)), \tag{2.7}$$

for $a \in A_z^k$. Since $\mathcal{L}_\psi \mathbb{1} = \mathbb{1}$, we have that for any $a \in A_z^k$,

$$\nu_z^k(\mathbb{C}_{a,z}) = \sum_{\substack{b \in A \\ M(b,a_{-k})=1}} \nu_z^{k+1}(\mathbb{C}_{b\cdot a,z}). \tag{2.8}$$

By Kolmogorov's extension theorem, from equation (2.8) it follows that there exists a unique Borel probability measure ν_z^- on $\mathbb{X}_z^-$ such that for any $k \geqslant 0$, ν_z^k is the restriction of ν_z^- to cylinders of length k.

We can now give an explicit formula for the measure ν in terms of the measures ν^+ and ν_z^-.

Lemma 2.2. *Let $\varphi \in \mathcal{C}(\mathbb{X})$. Then we have*

$$\nu(\varphi) = \int_{\mathbb{X}^+} \int_{\mathbb{X}_z^-} \varphi(y \cdot z) \nu_z^-(dy) \nu^+(dz).$$

Proof. By Lemma 2.1, it suffices to prove that the measure ν on $\mathbb{X}$ defined by the above equation is T-invariant. This property is a direct consequence of the definition of the measures ν_z^-, $z \in \mathbb{X}^+$, and of the fact that ν^+ is $\mathcal{L}_\psi$-invariant. □

We shall use the fact that the measures ν_z^- and $\nu_{z'}^-$ are equivalent.

Lemma 2.3. *There exists a real-valued continuous function θ on the set*

$$\mathbb{X}_3 := \left\{(y, z, z') \in A^{-\mathbb{N}^*} \times \mathbb{X}^+ \times \mathbb{X}^+ : z_0 = z_0',\, y \in \mathbb{X}_z^- = \mathbb{X}_{z'}^-\right\}$$

such that for any $z, z' \in \mathbb{X}^+$ and any continuous function φ on $\mathbb{X}_z^-$, one has

$$\int_{\mathbb{X}_z^-} \varphi(y) \nu_{z'}^-(dy) = \int_{\mathbb{X}_z^-} \varphi(y) e^{\theta(y,z,z')} \nu_z^-(dy).$$

Besides, there exists a constant $c > 0$ such that for any $(y, z, z') \in \mathbb{X}_3$,

$$|\theta(y, z, z')| \leqslant c\alpha^{\omega(z,z')}.$$

Proof. Indeed, it suffices to set

$$\theta(y, z, z') = \sum_{k=1}^{\infty} \left(\psi(T^{-k}(y \cdot z)) - \psi(T^{-k}(y \cdot z'))\right).$$

□

2.3. **General properties of exit times.** From the following lemma it follows that the function $x \mapsto \tau_t^f(x)$ is finite ν-almost surely.

Lemma 2.4. *Let $f \in \mathscr{B}$ with $\nu(f) = 0$. Assume that f is not a coboundary. Then for ν-almost every $x \in \mathbb{X}$, one has*

$$\inf_{n \geqslant 1} S_n f(x) = -\infty.$$

Proof. Consider the Borel set

$$A = \left\{ x \in \mathbb{X} : \inf_{n \geqslant 1} S_n f(x) > -\infty \right\}.$$

It is clear that the set A is T-invariant. Therefore $\nu(A) = 0$ or $\nu(A) = 1$. Assume that $\nu(A) = 1$, then let us show that f is a coboundary. Indeed, for any $x \in A$, we have that $h(x) := \liminf_{n \to \infty} S_n f(x) > -\infty$. Since $\nu(f) = 0$, it is well known that $S_n f(x)$ admits finite limit points for ν-almost every $x \in \mathbb{X}$, so that $h(x) < \infty$. Now by definition, we have $h(Tx) = h(x) - f(x)$, hence f is a coboundary as a measurable function on $\mathbb{X}$. Therefore, by [24, Proposition 6.2], we get that f is a coboundary as a Hölder continuous function on $\mathbb{X}$. □

For notational reasons, it is more convenient to study objects defined by the reverse shift T^{-1}. Note that the two studies are equivalent.

Indeed, let us define the flip map $\iota : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by the following relation: for any $x = (\dots, x_{-1}, x_0, x_1, \dots) \in A^{\mathbb{Z}}$ it holds $\iota(x) = (\dots, x_1, x_0, x_{-1}, \dots) \in A^{\mathbb{Z}}$, that is $(\iota x)_n = x_{-n}$ for $n \in \mathbb{Z}$. The following lemma is classical (see [24, Chapter 2]).

Lemma 2.5. *The set $\iota\mathbb{X}$ is a subshift of finite type and the measure $\iota_*\nu$ is a Gibbs measure on $\iota\mathbb{X}$.*

For $f \in \mathscr{B}$, consider the reversed Birkhoff sum process $(\check{S}_n f)_{n \geqslant 1}$ which is defined as follows: for any $x \in \mathbb{X}$,

$$\check{S}_n f(x) = f(T^{-1}x) + f(T^{-2}x) + \ldots + f(T^{-n}x) = S_n f(T^{-n}x), \quad n \geqslant 1.$$

In the same way, denote by $\check{\tau}_t^f(x)$ the first time when $t + \check{S}_n f(x)$ becomes negative: for any $x \in \mathbb{X}$,

$$\check{\tau}_t^f(x) := \inf \left\{ k \geqslant 1, \, t + \check{S}_k f(x) < 0 \right\}. \tag{2.9}$$

Then the relation between the exit times τ_t^f and $\check{\tau}_t^{f \circ \iota}$ is given by

$$\tau_t^f(Tx) = \check{\tau}_t^{f \circ \iota}(\iota x), \quad x \in \mathbb{X}.$$

In the present paper we deal with the measure

$$\nu\left(x \in \mathbb{X} : \tau_t^f(x) > n \right) \tag{2.10}$$

which, by the discussion above, is equivalent to studying the measure

$$\nu\left(x \in \mathbb{X} : \check{\tau}_t^f(x) > n \right). \tag{2.11}$$

In turn, Lemma 2.2 shows that in order to study (2.11), it suffices to investigate

$$\nu_z^-\left(y \in \mathbb{X}_z^- : \check{\tau}_t^f(y \cdot z) > n \right), \tag{2.12}$$

for $z \in \mathbb{X}^+$. We will do it by using tools from the theory of Markov chains [15]. In particular, we will make use of the martingale approximation for the process $(\check{S}_n f)_{n \geqslant 1}$.

2.4. **Martingale approximation.** Recall that $\mathscr{B} = \cup_{0<\alpha<1}\mathscr{B}_\alpha$, where $\mathscr{B}_\alpha$ is the space of real-valued α-Hölder continuous functions on $\mathbb{X}$ endowed with the norm (2.2). In the same way, we denote by $\mathscr{B}_\alpha^+$ the space of real-valued α-Hölder continuous functions on $\mathbb{X}^+$ endowed with the norm

$$\|f\|_{\mathscr{B}_\alpha^+} := \sup_{x\in\mathbb{X}^+} |f(x)| + \sup_{x,y\in\mathbb{X}^+ : x\neq y} \frac{|f(x)-f(y)|}{\alpha^{\omega(x,y)}}.$$

Let $\mathscr{B}^+ = \cup_{0<\alpha<1}\mathscr{B}_\alpha^+$. Note that every Hölder continuous function f on $\mathbb{X}^+$ can be extended to a Hölder continuous function on $\mathbb{X}$ defined by

$$x = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathbb{X} \mapsto f(x_0, x_1, \dots),$$

so we can identify $\mathscr{B}^+$ with a subspace of $\mathscr{B}$.

Let $f \in \mathscr{B}$. Define the cohomology class of f as the following set of Hölder continuous functions:

$$\mathscr{C}(f) = \{f_0 \in \mathscr{B} \mid f_0 = f - h\circ T + h,\ h \in \mathscr{B}\}.$$

The following proposition tells us that there exists a natural choice in $\mathscr{C}(f)$.

Proposition 2.6. *Let $f \in \mathscr{B}$ be such that $\nu(f) = 0$. Then there exists a unique function $f_0 \in \mathscr{B}^+$ such that $\mathcal{L}_\psi f_0 = 0$ and its extension on $\mathbb{X}$ belongs to $\mathscr{C}(f)$.*

Proof. First we prove the existence of f_0. By Proposition 1.2 in [24], there exists a Hölder continuous function g on $\mathbb{X}^+$, whose extension to $\mathbb{X}$ is cohomologous to f. As $\nu(f) = 0$, we have $\nu^+(g) = 0$. Now we choose $\alpha \in (0,1)$ close enough to 1 so that $\mathcal{L}_\psi$ is bounded on $\mathscr{B}_\alpha$ and $g \in \mathscr{B}_\alpha$. By the spectral gap property for the operator $\mathcal{L}_\psi$ (see Theorem 2.2 of [24]), there exists a Hölder continuous function $h \in \mathscr{B}_\alpha$ such that

$$h - \mathcal{L}_\psi h = \mathcal{L}_\psi g. \tag{2.13}$$

Since $h = h\mathcal{L}_\psi \mathbb{1} = \mathcal{L}_\psi(h\circ T)$, it follows that

$$\mathcal{L}_\psi(g - h\circ T + h) = 0.$$

Hence there exists a function $f_0 := g - h\circ T + h \in \mathscr{C}(f)$ satisfying $\mathcal{L}_\psi f_0 = 0$.

Now we prove the uniqueness of f_0. Suppose that there exist $f_0, f_0' \in \mathscr{C}(f)$ such that $\mathcal{L}_\psi f_0 = \mathcal{L}_\psi f_0' = 0$. Then $f_0 - f_0'$ is a coboundary, namely, there exists $h_1 \in \mathscr{B}$ such that $f_0 - f_0' = h_1\circ T - h_1$. As f_0 and f_0' depend only on the future coordinates, it is well known that h_1 depends only on the future coordinates. It follows that $\mathcal{L}_\psi(h_1\circ T - h_1) = 0$ and hence $\mathcal{L}_\psi h_1 = h_1$. This implies that h_1 is a constant and therefore $f_0' = f_0$. □

For any $z \in \mathbb{X}^+$, we have defined a probability measure ν_z^- on the set $\mathbb{X}_z^- \subset A^{-\mathbb{N}^*}$ of past sequences which are compatible with z. For $n \geqslant 1$, we let $\mathscr{F}_n$ denote the σ-algebra of subsets of $A^{-\mathbb{N}^*}$ generated by the coordinate maps $y \mapsto (y_{-1}, \dots, y_{-n})$. By convention, we also define $\mathscr{F}_0$ as the trivial σ-algebra. And we let $\mathscr{F}_n^z$ be the σ-algebra induced on $\mathbb{X}_z^-$. The following proposition is a classical result from [24]:

Proposition 2.7. *Let $f_0 \in \mathcal{C}(\mathbb{X}^+)$. Then $\mathcal{L}_\psi f_0 = 0$ if and only if for any $z \in \mathbb{X}^+$, the sequence of random variables*

$$y \in \mathbb{X}_z^- \mapsto \check{S}_n f_0(y\cdot z), \quad n \geqslant 0$$

is a martingale on $\mathbb{X}_z^-$ equipped with the probability measure ν_z^- w.r.t. the filtration $(\mathscr{F}_n^z)_{n\geqslant 0}$.

Proof. Denote by $g_n^z : \mathbb{X}_z^- \to \mathbb{R}$ the function $y \mapsto \check{S}_n f_0(y \cdot z)$. Then for $y \in \mathbb{X}_z^-$ and $n \geqslant 1$, we have by the definition of the measure ν_z^-,

$$\nu_z^- \left(g_n^z \,\middle|\, \mathscr{F}_{n-1}^z \right)(y) = g_{n-1}^z(y) + \mathcal{L}_\psi f_0(T^{-n}(y \cdot z)).$$

From this identity, the assertion follows. □

The following result shows that the difference $\check{S}_n f - \check{S}_n g$ is bounded, for f and g in the same cohomology class.

Lemma 2.8. *Let $f \in \mathscr{B}$ and $g \in \mathscr{C}(f)$. Let $h \in \mathscr{B}$ be such that $f - g = h \circ T - h$. Then, for any $x \in \mathbb{X}$ and any $n \geqslant 1$, we have*

$$\left| \check{S}_n f(x) - \check{S}_n g(x) \right| \leqslant c = 2\|h\|_\infty.$$

Proof. Indeed, we have $S_n f - S_n g = h \circ T^n - h$. Since $\check{S}_n f = (S_n f) \circ T^{-n}$, we get $\check{S}_n f - \check{S}_n g = h - h \circ T^{-n}$, which proves the assertion. □

2.5. **The Hölder continuity and approximation.** We establish several technical results which will be used in the proofs of the main results of the paper. In particular, they will allow us to prove that several convergences hold uniformly in $z \in \mathbb{X}^+$.

Lemma 2.9. *For any $g \in \mathscr{B}$, there exist constants $\alpha \in (0, 1)$ and $c_0 > 0$ such that for any $n \geqslant 1$, $z, z' \in \mathbb{X}^+$ with $z_0 = z_0'$ and $y \in \mathbb{X}_z^- (= \mathbb{X}_{z'}^-)$, one has*

$$\left| \check{S}_n g(y \cdot z) - \check{S}_n g(y \cdot z') \right| \leqslant c_0 \alpha^{\omega(z,z')}. \tag{2.14}$$

In particular, for any $g \in \mathscr{B}$, there exists a constant $c_0 > 0$ such that for any $n \geqslant 1$, $z, z' \in \mathbb{X}^+$ with $z_0 = z_0'$ and $y \in \mathbb{X}_z^- (= \mathbb{X}_{z'}^-)$, it holds

$$\check{S}_n g(y \cdot z) \leqslant \check{S}_n g(y \cdot z') + c_0.$$

Proof. Since $g \in \mathscr{B}$, there exists a constant L_g such that for any $x, x' \in \mathbb{X}$,

$$|g(x) - g(x')| \leqslant L_g \alpha^{\omega(x,x')},$$

where $0 < \alpha < 1$. Hence for any $z, z' \in \mathbb{X}^+$ with $z_0 = z_0'$ and $y \in \mathbb{X}_z^-$, and $n \geqslant 1$, one has

$$\begin{aligned}
\left| \check{S}_n g(y \cdot z) - \check{S}_n g(y \cdot z') \right| &\leqslant \sum_{k=0}^{n-1} L_g \alpha^{n-k+\omega(z,z')} \\
&\leqslant L_g \frac{\alpha^{1+\omega(z,z')}}{1-\alpha} =: c_0 \alpha^{w(z,z')}.
\end{aligned}$$

The desired result follows. □

Corollary 2.10. *For any $g \in \mathscr{B}$, there exist constants $\alpha \in (0, 1)$ and $c_0 > 0$ such that for any $n \geqslant 1$, $z, z' \in \mathbb{X}^+$ with $z_0 = z_0'$ and $y \in \mathbb{X}_z^- (= \mathbb{X}_{z'}^-)$, we have*

$$\left| \min_{1 \leqslant j \leqslant n} \check{S}_j g(y \cdot z) - \min_{1 \leqslant j \leqslant n} \check{S}_j g(y \cdot z') \right| \leqslant c_0 \alpha^{w(z,z')}. \tag{2.15}$$

Proof. By Lemma 2.9, there exist constants $c_0 > 0$ and $\alpha \in (0, 1)$ such that for any $n \geqslant j \geqslant 1$,

$$\min_{1 \leqslant j \leqslant n} \check{S}_j g(y \cdot z) \leqslant \check{S}_j g(y \cdot z) \leqslant \check{S}_j g(y \cdot z') + c_0 \alpha^{w(z,z')}.$$

Taking the minimum over $1 \leqslant j \leqslant n$ on the right-hand side, we get

$$\min_{1\leqslant j\leqslant n} \check{S}_j g(y\cdot z) \leqslant \min_{1\leqslant j\leqslant n} \check{S}_j g(y\cdot z') + c_0\alpha^{w(z,z')}. \tag{2.16}$$

In the same way, again by Lemma 2.9, there exist constants $c_0 > 0$ and $\alpha \in (0,1)$ such that for any $n \geqslant j \geqslant 1$,

$$\check{S}_j g(y\cdot z) \geqslant \check{S}_j g(y\cdot z') - c_0\alpha^{w(z,z')} \geqslant \min_{1\leqslant j\leqslant n} \check{S}_j g(y\cdot z') - c_0\alpha^{w(z,z')}.$$

Taking the minimum over $1 \leqslant j \leqslant n$ on the left-hand side, we get

$$\min_{1\leqslant j\leqslant n} \check{S}_j g(y\cdot z) \geqslant \min_{1\leqslant j\leqslant n} \check{S}_j g(y\cdot z') - c_0\alpha^{w(z,z')}. \tag{2.17}$$

Combining (2.16) and (2.17), we conclude the proof of (2.15). □

We will also need the following technical lemma that allows us to approximate the function $g \in \mathscr{B}$ by a function $x \mapsto g_m(x)$ on $\mathbb{X}$ which only depends on the coordinates $\{x_k\}_{k\geqslant -m}$.

Lemma 2.11. *Let $g \in \mathscr{B}$. Then there exist constants $\alpha \in (0,1)$, $c_1 > 0$ and a sequence of Hölder continuous functions $(g_m)_{m\geqslant 0}$ on $\mathbb{X}$ which only depend on the coordinates $\{x_k\}_{k\geqslant -m}$ such that $\mathcal{L}_\psi g_0 = 0$ and for any $m \geqslant 0$,*

$$\sup_{n\geqslant 1} \left\| \check{S}_n g_m - \check{S}_n g \right\|_\infty \leqslant c_1\alpha^m. \tag{2.18}$$

Proof. By Proposition 2.6, there exist $g_0 \in \mathscr{B}^+$ and $h \in \mathscr{B}$ with $\mathcal{L}_\psi g_0 = 0$ and

$$g_0 = g - h\circ T + h. \tag{2.19}$$

Since $h \in \mathscr{B}$, there is $\alpha \in (0,1)$ such that $h \in \mathscr{B}_\alpha$. Then, for any $m \geqslant 0$, there exists a Hölder continuous function h_m on $\mathbb{X}$ which only depends on the coordinates $\{x_k\}_{k\geqslant -m}$ such that

$$\|h - h_m\|_\infty \leqslant c_1\alpha^m, \tag{2.20}$$

where $c_1 > 0$ is a fixed constant not depending on m, and by convention $h_0 = 0$. We define for any $m \geqslant 0$,

$$g_m = g_0 + h_m\circ T - h_m. \tag{2.21}$$

From (2.19), (2.20) and (2.21), we get (2.18). □

2.6. **Duality.** The next duality property will be crucial in the proof of the main results.

Lemma 2.12. *Let $g \in \mathscr{B}$. For any $n \geqslant 1$ and any non-negative measurable function $F : \mathbb{X}\times\mathbb{R}\times\mathbb{X}\times\mathbb{R} \to \mathbb{R}$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{X}} F\left(x,t,T^{-n}x, t+\check{S}_n g(x)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(x)>n-1\right\}}\nu(dx)dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{X}} F\Big(T^n x, u - S_n g(x), x, u\Big)\mathbb{1}_{\left\{\tau_u^{-g}(x)>n-1\right\}}\nu(dx)du.
\end{aligned}$$

Proof. By a change of variable $t = u - \check{S}_n g(x)$, it follows that

$$\begin{aligned}
I :&= \int_{\mathbb{R}}\int_{\mathbb{X}} F\left(x,t,T^{-n}x, t+\check{S}_n g(x)\right) \mathbb{1}_{\left\{t+\check{S}_{n-1}g(x)\geqslant 0,\dots,t+\check{S}_1 g(x)\geqslant 0\right\}}\nu(dx)dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{X}} F\left(x, u-\check{S}_n g(x), T^{-n}x, u\right) \\
&\qquad \times \mathbb{1}_{\{u-g(T^{-n}x)\geqslant 0,\dots,u-g(T^{-n}x)-\dots-g(T^{-2}x)\geqslant 0\}}\nu(dx)du.
\end{aligned}$$

Since the measure ν is T^{-1}-invariant, we obtain

$$I = \int_{\mathbb{R}} \int_{\mathbb{X}} F\Big(T^n x, u - S_n g(x), x, u\Big) \mathbb{1}_{\{u - S_1 g(x) \geqslant 0, \dots, u - S_{n-1} g(x) \geqslant 0\}} \nu(dx) du,$$

which ends the proof of the lemma. □

3. Harmonicity for dynamical system

3.1. **Existence of the harmonic function.** The aim of this section is to prove the existence of a function V^f on the state space $\mathbb{R}$ which we call the harmonic function of f by analogy with the theory developed for Markov chains in [15]. Our main result is the following theorem:

Theorem 3.1. *Let f be a Hölder continuous function on $\mathbb{X}$ such that f is not a coboundary and $\nu(f) = 0$. Then there exists a unique non-decreasing and right continuous function $V^f : \mathbb{R} \to \mathbb{R}_+$ such that for any continuous compactly supported function φ on $\mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}} S_n f(x) \mathbb{1}_{\left\{\tau_t^f(x) > n\right\}} \nu(dx) dt = \int_{\mathbb{R}} \varphi(t) V^f(t) dt. \tag{3.1}$$

Besides, there exists a constant $c > 0$ such that for any $t \in \mathbb{R}$,

$$\max\{t - c, 0\} \leqslant V^f(t) \leqslant \max\{t, 0\} + c. \tag{3.2}$$

Note that, the bound (3.2) implies that $V^f(t)/t \to 1$ as $t \to \infty$.

The proof of Theorem 3.1 will be given at the end of this section. At this point we start by giving an explicit formula for the harmonic function in the case where the observable only depends on future coordinates. Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Let g_0 be the unique element of $\mathscr{B}^+$ such that $\mathcal{L}_\psi g_0 = 0$ and g_0 is cohomologous to g, as in Proposition 2.6. For $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$, we set

$$\check{V}^g(z,t) = -\int_{\mathbb{X}_z^-} \check{S}_{\check{\tau}_t^g(y\cdot z)} g_0(y \cdot z) \nu_z^-(dy). \tag{3.3}$$

This integral makes sense. Indeed, first, by Lemma 3.3, the stopping time $y \mapsto \check{\tau}_t^g(y \cdot z)$ is finite ν_z^--almost everywhere. Second, the Birkhoff sum $t + \check{S}_{\check{\tau}_t^g(y\cdot z)} g(y \cdot z)$ takes values in the interval $[-\|g\|_\infty, 0]$ when t is non-negative, and in the interval $[t - \|g\|_\infty, 0]$ when t is negative. Third, by Lemma 2.8, the difference of the Birkhoff sums for g and g_0 is uniformly bounded.

The function $\check{V}^g(z, \cdot)$ plays a crucial role in proving conditioned limit theorems for products of random matrices and more generally for Markov chains, see [17] and [15]. From the results of [15] it follows that $\check{V}^g(z, \cdot)$ has the following harmonicity property.

Lemma 3.2. *Let g be in $\mathscr{B}^+$ such that $\nu^+(g) = 0$ and g is not a coboundary. Then for any $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$, we have*

$$\check{V}^g(z,t) = \sum_{z' \in \mathbb{X}^+ : T(z') = z} e^{-\psi(z')} \mathbb{1}_{\{t + g(z') \geqslant 0\}} \check{V}^g(z', t + g(z')). \tag{3.4}$$

The proof of the existence of the harmonic function $\check{V}^g$ given in [15] is rather difficult. In the case of the subshift of finite type (since the jumps are bounded) it is possible to give a much shorter direct proof, which is not included because of the space limitations.

We shall extend the definition of $\check{V}^g(z, \cdot)$ to the case of any function $g \in \mathscr{B}$, that is, the case of a function g that depends on both the past and the future coordinates. We will use the following technical assertion:

Lemma 3.3. *Let $g \in \mathscr{B}$ such that $\nu(g) = 0$ and g is not a coboundary with respect to T. Then, for any $t \in \mathbb{R}$, it holds uniformly in $z \in \mathbb{X}^+$ that*

$$\lim_{n\to\infty} \nu_z^- \Big(y \in \mathbb{X}_z^- : \check{\tau}_t^g(y\cdot z) > n \Big) = 0.$$

Proof. Let $c_0 > 0$ be as in Lemma 2.9. By Lemma 2.4 and Fubini's theorem, for any $a \in A$, we can find $z' \in \mathbb{X}^+$ such that $z'_0 = a$ and the function $y \mapsto \check{\tau}^g_{t+c_0}(y\cdot z')$ on $\mathbb{X}^-_{z'}$ is finite $\nu^-_{z'}$-almost everywhere. Then for any $z \in \mathbb{X}^+$ with $z_0 = a$, we have

$$\Big\{ y \in \mathbb{X}_z^- : \check{\tau}_t^g(y\cdot z) > n \Big\} \subseteq \Big\{ y \in \mathbb{X}_{z'}^- : \check{\tau}_{t+c_0}^g(y\cdot z') > n \Big\}.$$

From Lemma 2.3, we get that for some constant $c > 0$,

$$\nu_z^- \Big(y \in \mathbb{X}_z^- : \check{\tau}_t^g(y\cdot z) > n \Big) \leqslant c\nu_{z'}^- \Big(y \in \mathbb{X}_{z'}^- : \check{\tau}_{t+c_0}^g(y\cdot z') \Big).$$

Thus, the lemma follows from the fact that $\nu_{z'}^- \Big(y \in \mathbb{X}_{z'}^- : \check{\tau}_{t+c_0}^g(y\cdot z') > n \Big)$ converges to 0 as $n \to \infty$. $\square$

Now we give an alternative definition of the function $\check{V}^g(z,\cdot)$ for $g \in \mathscr{B}^+$, where the key point is that in this case, the function $y \mapsto \check{\tau}^g_t(y\cdot z)$ is a stopping time with respect to the filtration $\{\mathscr{F}_k^z\}_{k\geqslant 0}$.

Lemma 3.4. *Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Let g_0 be the unique element of $\mathscr{B}^+$ such that $\mathcal{L}_\psi(g_0) = 0$ and g_0 is cohomologous to g. Then, for any $t \in \mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$,*

$$\begin{aligned}\check{V}^g(z,t) &= \lim_{n\to\infty} \int_{\mathbb{X}_z^-} \Big(t + \check{S}_n g(y\cdot z) \Big) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) \\ &= \lim_{n\to\infty} \int_{\mathbb{X}_z^-} \check{S}_n g_0(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy). \end{aligned} \tag{3.5}$$

In addition, there is a constant $c > 0$ such that, for any $z \in \mathbb{X}^+$, $t \in \mathbb{R}$ and $n \geqslant 1$,

$$\int_{\mathbb{X}_z^-} \check{S}_n g(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) \leqslant \max\{t, 0\} + c, \tag{3.6}$$

for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}_+$,

$$t - c \leqslant \check{V}^g(z,t) \leqslant t + c, \tag{3.7}$$

and for any $z \in \mathbb{X}^+$ and $t < -c$, it holds that $\check{V}^{g_0}(z,t) = 0$.

Moreover, for any $z \in \mathbb{X}^+$, the function $\check{V}^g(z,\cdot)$ is non-decreasing on $\mathbb{R}$.

Proof. Since g is cohomologous to g_0, by Lemma 2.8, all Birkhoff sums $\check{S}_n g(y\cdot z)$ stay at bounded distance from the Birkhoff sums $\check{S}_n g_0(y\cdot z)$. Therefore one can deal with $\check{S}_n g_0(y\cdot z)$ instead of $\check{S}_n g(y\cdot z)$. By the optional stopping theorem, for any $n \geqslant 1$,

$$\begin{aligned} &\int_{\mathbb{X}_z^-} \check{S}_n g_0(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) \\ &= \int_{\mathbb{X}_z^-} \check{S}_n g_0(y\cdot z) \nu_z^-(dy) - \int_{\mathbb{X}_z^-} \check{S}_n g_0(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) \leqslant n\right\}} \nu_z^-(dy) \\ &= -\int_{\mathbb{X}_z^-} \check{S}_{\check{\tau}_t^g(y\cdot z)} g_0(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) \leqslant n\right\}} \nu_z^-(dy). \end{aligned}$$

The bound (3.6) follows since $t + \check{S}_{\check{\tau}_t^g(y\cdot z)} g_0(y\cdot z)$ is bounded from below, because $t + \check{S}_{\check{\tau}_t^g(y\cdot z)-1} g(y\cdot z) \geqslant 0$ and g is bounded.

Note that, as $t+\check S_{\check\tau_t^g(y\cdot z)}g(y\cdot z)<0$, the quantity $t+\check S_{\check\tau_t^g(y\cdot z)}g_0(y\cdot z)$ is also bounded from above. Therefore, by Lemma 3.3, we obtain (3.5) uniformly in $z\in\mathbb{X}^+$.

Still because the function $y\mapsto t+\check S_{\check\tau_t^g(y\cdot z)}g_0(y\cdot z)$ is uniformly bounded, we get $\check V^g(z,t)\in[t-c,t+c]$, for some constant $c>0$. Besides, if $t<-\|g\|_\infty$, we get $\check\tau_t^g(y\cdot z)=1$ everywhere for all $z\in\mathbb{X}^+$, thus by (2.7) and (2.3) we have

$$\check V^g(z,t)=\int_{A_z^1}(g_0(y_{-1}\cdot z))\,\nu_z^1(dy_{-1})=\mathcal{L}_\psi(g_0)(z)=0.$$

It remains to prove the monotonicity of $t\mapsto\check V^g(z,t)$. As $\check\tau_{t_1}^g\leqslant\check\tau_{t_2}^g$ for any $t_1\leqslant t_2$, and $t_2+\check S_n g\geqslant 0$ on the set $\{\check\tau_{t_2}^g>n\}$, it follows that

$$\begin{aligned}&\int_{\mathbb{X}_z^-}\left(t_1+\check S_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check\tau_{t_1}^g(y\cdot z)>n\right\}}\nu_z^-(dy)\\ &\leqslant\int_{\mathbb{X}_z^-}\left(t_2+\check S_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check\tau_{t_2}^g(y\cdot z)>n\right\}}\nu_z^-(dy).\end{aligned}$$

Letting $n\to\infty$ yields that the function $\check V^g(z,\cdot)$ is non-decreasing on $\mathbb{R}$. □

By using Lemma 3.4, we can now give a definition of $\check V^g$ for a function g only depending on finitely many negative coordinates.

Lemma 3.5. *Let $g\in\mathscr{B}$ such that $\nu(g)=0$ and g is not a coboundary. Assume that g only depends on m negative coordinates for some $m\geqslant 0$. In other words, the function $h=g\circ T^m\in\mathscr{B}^+$. Then, for any $t\in\mathbb{R}$, we have uniformly in $z\in\mathbb{X}^+$,*

$$\lim_{n\to\infty}\int_{\mathbb{X}_z^-}\check S_n g(y\cdot z)\mathbb{1}_{\left\{\check\tau_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)=\mathcal{L}_\psi^m\left(\check V^h(\cdot,t)\right)(z).$$

Let g and h be as in Lemma 3.5. We set for $z\in\mathbb{X}^+$ and $t\in\mathbb{R}$,

$$\check V^g(z,t)=\mathcal{L}_\psi^m\left(\check V^h(\cdot,t)\right)(z).$$

Lemma 3.5 implies that this notation is coherent with that introduced in (3.3).

Proof of Lemma 3.5. By conditioning over the m first coordinates of y, we get for $n\geqslant 0$,

$$\begin{aligned}&\int_{\mathbb{X}_z^-}\check S_n g(y\cdot z)\mathbb{1}_{\left\{\check\tau_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)\\ &=\sum_{a\in A_z^m}\exp(-S_m\psi(a\cdot z))\int_{\mathbb{X}_{a\cdot z}^-}\check S_n g((y\cdot a)\cdot z)\mathbb{1}_{\left\{\check\tau_t^g((y\cdot a)\cdot z)>n\right\}}\nu_{a\cdot z}^-(dy)\\ &=\sum_{a\in A_z^m}\exp(-S_m\psi(a\cdot z))\int_{\mathbb{X}_{a\cdot z}^-}\check S_n h(T^{-m}(y\cdot a)\cdot z)\mathbb{1}_{\left\{\check\tau_t^{h\circ T^{-m}}((y\cdot a)\cdot z)>n\right\}}\nu_{a\cdot z}^-(dy)\\ &=\sum_{a\in A_z^m}\exp(-S_m\psi(a\cdot z))\int_{\mathbb{X}_{a\cdot z}^-}\check S_n h(y\cdot(a\cdot z))\mathbb{1}_{\left\{\check\tau_t^h(y\cdot(a\cdot z))>n\right\}}\nu_{a\cdot z}^-(dy),\end{aligned}$$

where we have used the relations $(y\cdot a)\cdot z=T^m(y\cdot(a\cdot z))$ and $\check\tau_t^{h\circ T^{-m}}=\check\tau_t^h\circ T^{-m}$. The conclusion now follows from Lemma 3.4 and the definition of the transfer operator $\mathcal{L}_\psi^m$. □

We will prove that the convergence in Lemma 3.5 holds in a weak sense for every function $g\in\mathscr{B}$. The key step to prove Theorem 3.1 is the following technical lemma which shows that the convergence of Lemma 3.5 holds for all functions $g\in\mathscr{B}$ in a weak sense.

Lemma 3.6. *Assume that $g \in \mathscr{B}$ is not a coboundary w.r.t. to T and $\nu(g) = 0$. Then, for any continuous compactly supported function φ on $\mathbb{R}$, uniformly in $z \in \mathbb{X}^+$, the following limit exists and is finite:*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}_z^-} \check{S}_n g(y \cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g (y\cdot z)>n\right\}} \nu_z^-(dy) dt.$$

Proof. Assume that $g \in \mathscr{B}$. Let $(g_m)_{m\geqslant 0}$, $c_1 > 0$ and $\alpha \in (0,1)$ be as in Lemma 2.11. Set

$$W_n(z,t) = \int_{\mathbb{X}_z^-} \left(t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g (y\cdot z)>n\right\}} \nu_z^-(dy)$$

and

$$W_{n,m}(z,t) = \int_{\mathbb{X}_z^-} \left(t + \check{S}_n g_m(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^{g_m} (y\cdot z)>n\right\}} \nu_z^-(dy).$$

By (2.18), we have the inclusions

$$\left\{\check{\tau}_{t-2c_1\alpha^m}^{g_m} > n\right\} \subseteq \{\check{\tau}_t^g > n\} \subseteq \left\{\check{\tau}_{t+2c_1\alpha^m}^{g_m} > n\right\},$$

which imply that

$$W_{n,m}(z, t - 2c_1\alpha^m) \leqslant W_n(z,t) \leqslant W_{n,m}(z, t + 2c_1\alpha^m). \tag{3.8}$$

In the same way, we have

$$W_{n,m}(z,t) \leqslant W_{n,0}(z, t+2c_1) \leqslant \max\{t, 0\} + c_2, \tag{3.9}$$

where the last bound follows from (3.6).

By lemma 3.5, for fixed $m \geqslant 0$, as $n \to \infty$, the function $W_{n,m}(z,t)$ converges to $\check{V}^{g_m}(z,t)$, uniformly in $z \in \mathbb{X}^+$. From (3.8) we get

$$\check{V}^{g_m}(z, t - 2c_1\alpha^m) \leqslant \liminf_{n\to\infty} W_n(z,t) \leqslant \limsup_{n\to\infty} W_n(z,t) \leqslant \check{V}^{g_m}(z, t + 2c_1\alpha^m).$$

Now we have

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(t) \left[\check{V}^{g_m}(z, t+2c_1\alpha^m) - \check{V}^{g_m}(z, t-2c_1\alpha^m)\right] dt \\
&= \int_{\mathbb{R}} \left[\varphi(t - 2c_1\alpha^m) - \varphi(t+2c_1\alpha^m)\right] \check{V}^{g_m}(z,t) dt.
\end{aligned}$$

Using (3.9) and Lemma 3.5, we have that $\check{V}^{g_m}(z,t) \leqslant c_2 + \max\{t, 0\}$. As φ is continuous on $\mathbb{R}$ with compact support, by the Lebesgue dominated convergence theorem, we get that uniformly in $z \in \mathbb{X}^+$,

$$\lim_{m\to\infty} \int_{\mathbb{R}} \varphi(t) \left[\check{V}^{g_m}(z, t+2c_1\alpha^m) - \check{V}^{g_m}(z, t-2c_1\alpha^m)\right] dt = 0.$$

This tells us that $\int_{\mathbb{R}} \varphi(t) W_n(z,t) dt$ has a uniform limit as $n \to \infty$. $\square$

We will use the previous lemma to build a function $\check{V}^g(z,t)$. The existence of this function will be deduced from the following elementary fact from the theory of distributions.

Lemma 3.7. *Let $(V_n)_{n\geqslant 1}$ be a sequence of non-decreasing functions on $\mathbb{R}$. Assume that for every continuous compactly supported function φ on $\mathbb{R}$, the sequence $\int_{\mathbb{R}} V_n(t)\varphi(t)dt$ admits a finite limit. Then there exists a unique right continuous and non-decreasing function V on $\mathbb{R}$ such that for any continuous compactly supported function φ, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} V_n(t)\varphi(t) dt = \int_{\mathbb{R}} V(t)\varphi(t) dt.$$

Now we construct the function $\check{V}^g(z,t)$ for any $g \in \mathscr{B}$.

Lemma 3.8. *Assume that $g \in \mathscr{B}$ is not a coboundary w.r.t. to T and $\nu(g) = 0$. Then, for any $z \in \mathbb{X}^+$, there exists a unique non-decreasing and right continuous function $\check{V}^g(z,\cdot)$ on $\mathbb{R}$ such that*
1. For any continuous compactly supported function φ on $\mathbb{R}$, uniformly in $z \in \mathbb{X}^+$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}_z^-} \check{S}_n g(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)dt = \int_{\mathbb{R}} \varphi(t)\check{V}^g(z,t)dt. \tag{3.10}$$

2. For any continuous compactly supported function φ on $\mathbb{R}$, the mapping $z \mapsto \int_{\mathbb{R}} \varphi(t)\check{V}^g(z,t)dt$ is continuous on $\mathbb{X}^+$.
3. There exists a constant $c > 0$ such that for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}_+$,

$$t - c \leqslant \check{V}^g(z,\cdot) \leqslant t + c. \tag{3.11}$$

In addition, for any $z \in \mathbb{X}^+$ and $t \leqslant -c$, we have $\check{V}^g(z,t) = 0$.

By Lemma 3.4, in the case $g \in \mathscr{B}^+$, the notation $\check{V}^g(z,\cdot)$ is coherent with the one in (3.3).

Proof of Lemma 3.8. Fix $z \in \mathbb{X}^+$. By Lemmas 3.3 and 3.6, the following limit exists: for any continuous compactly supported function φ on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}_z^-} \left(t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)dt. \tag{3.12}$$

For $t \in \mathbb{R}$, set

$$\check{V}_n^g(z,t) = \int_{\mathbb{X}_z^-} \left(t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy). \tag{3.13}$$

Then the function $\check{V}_n^g(z,\cdot)$ is non-decreasing on $\mathbb{R}$. By Lemma 3.7, there exists a unique non-decreasing and right continuous function $\check{V}^g(z,\cdot)$ on $\mathbb{R}$ such that for any continuous function φ on $\mathbb{R}$ with compact support,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \check{V}_n^g(z,t)\varphi(t)dt = \int_{\mathbb{R}} \check{V}^g(z,t)\varphi(t)dt.$$

Note that for $t < -\|g\|_\infty$, we have $\check{\tau}_t^g = 1$ everywhere. Hence $\check{V}^g(z,t) = 0$ for $t \leqslant -c$.

We now prove (3.11). By Proposition 2.6, there exists $g_0 \in \mathscr{B}^+$ such that $\mathcal{L}_\psi(g_0) = 0$ and g is cohomologous to g_0. By Lemma 2.11, we can choose a constant $c > 0$ large enough such that for any $n \geqslant 1$, it holds that $\|\check{S}_n g - \check{S}_n g_0\|_\infty \leqslant c$. By Lemmas 3.3 and 3.6, we have, for any continuous non-negative function φ on $\mathbb{R}$ with compact support,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}_z^-} \left(t + c + \check{S}_n g_0(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)dt = \int_{\mathbb{R}} \check{V}^g(z,t)\varphi(t)dt.$$

Note that from Lemma 3.3, we have $\nu_z^-(y \in \mathbb{X}_z^- : \check{\tau}_t^g(y\cdot z) > n) \to 0$ as $n \to \infty$. As we have the following inclusion: for any $t \in \mathbb{R}$,

$$\left\{\check{\tau}_{t-c}^{g_0} > n\right\} \subset \left\{\check{\tau}_t^g > n\right\} \subset \left\{\check{\tau}_{t+c}^{g_0} > n\right\},$$

and as $t + c + \check{S}_n g_0 \geqslant 0$ on the set $\{\check{\tau}_{t+c}^{g_0} > n\}$, we get

$$\int_{\mathbb{R}} \check{V}^{g_0}(z,t-c)\varphi(t)dt \leqslant \int_{\mathbb{R}} \check{V}^g(z,t)\varphi(t)dt \leqslant \int_{\mathbb{R}} \check{V}^{g_0}(z,t+c)\varphi(t)dt.$$

Since this holds for any continuous non-negative test function φ on $\mathbb{R}$, we obtain

$$\check{V}^{g_0}(z,t-c) \leqslant \check{V}^{g}(z,t) \leqslant \check{V}^{g_0}(z,t+c). \tag{3.14}$$

This, together with Lemma 3.7, concludes the proof of (3.11).

We now want to prove the continuity in $z \in \mathbb{X}^+$ of the function $z \mapsto \int_{\mathbb{R}} \varphi(t)\check{V}^g(z,t)dt$. To this aim, we establish a uniform bound for the quantity $\check{V}^g_n(z,t)$ defined in (3.13). Indeed, as usual, we have $\check{V}^g_n(z,t) \leqslant \check{V}^{g_0}_n(z,t+c)$. Now the optional stopping theorem gives

$$\begin{aligned}\check{V}^{g_0}_n(z,t) &= t\nu^-_z\left(y \in \mathbb{X}^-_z : \check{\tau}^{g_0}_t(y\cdot z) > n\right) \\ &\quad - \int_{\mathbb{X}^-_z} \check{S}_{\check{\tau}^{g_0}_t(y\cdot z)} g_0(y\cdot z)\mathbb{1}_{\left\{\check{\tau}^{g_0}_t(y\cdot z)\leqslant n\right\}}\nu^-_z(dy) \\ &\leqslant |t| + |t| + \|g_0\|_\infty = 2|t| + \|g_0\|_\infty.\end{aligned}$$

From (3.14) we get

$$\check{V}^g_n(z,t) \leqslant 2|t| + 2c + \|g_0\|_\infty. \tag{3.15}$$

It remains to prove that for any continuous compactly supported function φ on $\mathbb{R}$, the mapping $z \mapsto \int_{\mathbb{R}} \varphi(t)\check{V}^g(z,t)dt$ is continuous on $\mathbb{X}^+$. It suffices to prove that for any $n \geqslant 1$, the mapping $z \mapsto \int_{\mathbb{R}} \varphi(t)\check{V}^g_n(z,t)dt$ is continuous on $\mathbb{X}^+$.

A priori, for fixed $t \in \mathbb{R}$, the function $z \mapsto \check{V}^g_n(z,t)$ is not continuous. Nevertheless, we claim that it satisfies the following weak continuity property: for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for any $z, z' \in \mathbb{X}^+$ with $w(z,z') \geqslant k$ we have

$$\check{V}^g_n(z,t-\varepsilon) \leqslant \check{V}^g_n(z',t) \leqslant \check{V}^g_n(z,t+\varepsilon).$$

Indeed, this follows from the inequality (2.14) in Lemma 2.9. This, together with the bound (3.15) and the uniform continuity of the function φ, implies that the mapping $z \mapsto \int_{\mathbb{R}} \varphi(t)\check{V}^g_n(z,t)dt$ is continuous on $\mathbb{X}^+$. □

The previous statements can be summarized as follows:

Theorem 3.9. *Let g be a Hölder continuous function on $\mathbb{X}$ such that $\nu(g) = 0$ and g is not a coboundary. Then there exists a unique non-decreasing and right continuous function $\check{V}^g : \mathbb{R} \to \mathbb{R}_+$ with the following properties:*

1. For any continuous compactly supported function φ on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}} \check{S}_n g(x)\mathbb{1}_{\left\{\check{\tau}^g_t(x)>n\right\}}\nu(dx)dt = \int_{\mathbb{R}} \varphi(t)\check{V}^g(t)dt. \tag{3.16}$$

2. There exists a constant $c > 0$ such that for any $t \in \mathbb{R}$ it holds

$$\max\{t-c, 0\} \leqslant \check{V}^g(t) \leqslant \max\{t,0\} + c. \tag{3.17}$$

Proof. Let $g \in \mathscr{B}$. For $t \in \mathbb{R}$, we set

$$\check{V}^g(t) = \int_{\mathbb{X}^+} \check{V}^g(z,t)\nu^+(dz).$$

Then the points 1 and 2 of Theorem 3.9 follow from (3.10) and (3.11) in Lemma 3.8, respectively. □

Proof of Theorem 3.1. It is easy to see that Theorem 3.1 is equivalent to Theorem 3.9 for the reversed dynamics, i.e. by replacing f with $g = f \circ T^{-1} \circ \iota = f \circ \iota \circ T$, and ν with $\iota_*\nu$. □

3.2. **Properties of the harmonic function.** The goal of this section is to give some additional properties of the harmonic function $\check{V}^g$ which will be necessary for the proof of Theorem 1.3. We start with a continuity result on the cohomology class of the function g.

Lemma 3.10. *Let $g \in \mathscr{B}$ with $\nu(g) = 0$. Assume that g is not a coboundary. Let $\alpha \in (0, 1)$ and $(h_n)_{n \geqslant 0}$ be a sequence of element of $\mathscr{B}_\alpha$ that converges to 0 with respect to the Hölder norm $\|\cdot\|_\alpha$. For $n \geqslant 0$, set $g_n = g + h_n \circ T - h_n$. Then, there exists a constant $c > 0$ such that for any $n \geqslant 0$, $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$, one has*

$$\check{V}^{g_n}(z, t) \leqslant \max\{t, 0\} + c. \tag{3.18}$$

Moreover, for any continuous compactly supported function φ on $\mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t)\check{V}^{g_n}(z, t)dt = \int_{\mathbb{R}} \varphi(t)\check{V}^{g}(z, t)dt. \tag{3.19}$$

Proof. The bound (3.18) follows from (3.11) and the relation $g_n = g + h_n \circ T - h_n$. The construction of the function $\check{V}^g$ in (3.19) can be performed in the same way as in Lemmas 3.6 and 3.8. □

We can also describe how the function $\check{V}^g$ behaves when the function g is shifted by the dynamics.

Lemma 3.11. *Let $g \in \mathscr{B}$ with $\nu(g) = 0$. Assume that g is not a coboundary. Then, for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$, we have*

$$\check{V}^{g\circ T^{-1}}(z, t) = \mathcal{L}_\psi \left(V^g(\cdot, t)\right)(z).$$

Proof. By Lemma 3.8, for any continuous compactly supported function φ on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} \varphi(t)\check{V}^{g\circ T^{-1}}(z, t)dt = \lim_{n\to\infty} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{X}_z^-} \check{S}_n g(T^{-1}(y \cdot z))\mathbb{1}_{\{\check{\tau}_t^{g\circ T^{-1}}(y\cdot z)>n\}}\nu_z^-(dy)dt.$$

By conditioning on the coordinate y_{-1}, we get

$$\begin{aligned}
&\int_{\mathbb{X}_z^-} \check{S}_n g(T^{-1}(y \cdot z))\mathbb{1}_{\left\{\check{\tau}_t^{g\circ T^{-1}}(y\cdot z)>n\right\}}\nu_z^-(dy) \\
&= \int_{A_z^1} \int_{\mathbb{X}_z^-} \check{S}_n g(w \cdot (y_{-1} \cdot z))\mathbb{1}_{\left\{\check{\tau}_t^{g}(w\cdot(y_{-1}\cdot z))>n\right\}}\nu_z^-(dy)\nu_z^1(dy_{-1}).
\end{aligned}$$

Again by Lemma 3.8, we obtain

$$\check{V}^{g\circ T^{-1}}(z, t) = \int_{A_z^1} \check{V}^g(y_{-1} \cdot z, t)\nu_z^1(dy_{-1}) = \mathcal{L}_\psi \left(V^g(\cdot, t)\right)(z),$$

as desired. □

3.3. **The harmonic measure and the proof of Theorem 1.1.** In the case when g depends only on the future ($g \in \mathscr{B}^+$), the function $\check{V}^g$ satisfies the harmonicity equation (3.4). In general, when g depends also on the past, this property may not hold. It turns out that equation (3.4) can be reinterpreted as a kind of invariance property of a certain Radon measure, which we will introduce at the end of this section. Indeed, we have:

Lemma 3.12. *Let g be in $\mathscr{B}^+$ and let V be a locally integrable non-negative function on $\mathbb{X}^+ \times \mathbb{R}$. Then the following are equivalent:*

(1) For $\nu^+ \otimes dt$ almost every (z,t) in $\mathbb{X}^+ \times \mathbb{R}$, we have

$$V(z,t) = \sum_{z' \in \mathbb{X}^+ : T(z') = z} e^{-\psi(z')} \mathbb{1}_{\{t+g(z') \geqslant 0\}} V(z', t+g(z')).$$

(2) For any continuous compactly supported function φ on $\mathbb{X}^+ \times \mathbb{R}$, we have

$$\int_{\mathbb{X}^+ \times \mathbb{R}} \varphi(z,t) V(z,t) \nu^+(dz) dt = \int_{\mathbb{X}^+} \int_0^{\infty} \varphi(Tz, t - g(z)) V(z,t) \nu^+(dz) dt. \tag{3.20}$$

Proof. The proof is a direct computation. Indeed, for any continuous compactly supported function φ on $\mathbb{X}^+ \times \mathbb{R}$, by a change of variable, the right-hand side of (3.20) can be written as

$$\int_{\mathbb{R}} \int_{\mathbb{X}^+} \varphi(Tz, t) \mathbb{1}_{\{t+g(z) \geqslant 0\}} V(z, t+g(z)) \nu^+(dz) dt.$$

As ν^+ is $\mathcal{L}_\psi$ invariant, by using (2.5), we get for $t \in \mathbb{R}$,

$$\begin{aligned} &\int_{\mathbb{X}^+} \varphi(Tz, t) \mathbb{1}_{\{t+g(z) \geqslant 0\}} V(z, t+g(z)) \nu^+(dz) \\ &= \int_{\mathbb{X}^+} \varphi(z,t) \left(\sum_{Tz'=z} e^{-\psi(z')} \mathbb{1}_{\{t+g(z') \geqslant 0\}} V(z', t+g(z')) \right) \nu^+(dz). \end{aligned}$$

This proves the lemma. $\square$

We will now show that the functions $\check{V}^g$ and V^g can be seen as the densities with respect to the Lebesgue measure on $\mathbb{R}$ of the projections on $\mathbb{R}$ of certain natural Radon measures $\check{\mu}^g$ and μ^g on $\mathbb{X} \times \mathbb{R}$, which satisfy an invariance property similar to (3.20). Those measures will play a key role in the statement of the conditioned local limit theorem. The purpose of this subsection is to build them. This construction will follow the same lines as the one of the harmonic functions. We will first use Markov chain arguments to define these objects when $g \in \mathscr{B}^+$ and then use approximation arguments to extend the definition to the general case.

We first assume that g is in $\mathscr{B}^+$. In that case, for $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$ with $\check{V}^g(z,t) > 0$, let us introduce a Borel probability measure $\check{\mu}^{g,-}_{z,t}$ on $\mathbb{X}^-_z$. To do this, for $n \geqslant 1$, let A^n_z be as in the definition (2.7). For $a \in A^n_z$, let us write $a \cdot z$ for the element $\mathbb{X}^+$ whose n first coordinates are $a_{-n}, \dots, a_{-1}$ and whose k-th coordinate is z_{k-n} for $k \geqslant n$.

Lemma 3.13. *Let g be in $\mathscr{B}^+$ such that $\nu^+(g) = 0$ and g is not cohomologous to 0. Let (z,t) be in $\mathbb{X}^+ \times \mathbb{R}$ with $\check{V}^g(z,t) > 0$. Then, there exists a unique Borel probability measure $\check{\mu}^{g,-}_{z,t}$ on $\mathbb{X}^-_z$ such that for any $n \geqslant 0$ and any $a \in A^n_z$ we have*

$$\begin{aligned} &\check{\mu}^{g,-}_{z,t}(\{y \in \mathbb{X}^-_z : y_{-n} = a_{-n}, \dots, y_{-1} = a_{-1}\}) \\ &= \frac{1}{\check{V}^g(z,t)} \exp(-S_n \psi(a \cdot z)) \check{V}^g(a \cdot z, t + S_n g(a \cdot z)), \end{aligned} \tag{3.21}$$

as soon as $t + S_k g(T^k(a \cdot z)) \geqslant 0$ for all $1 \leqslant k \leqslant n$.

Proof. The proof is a translation of the general construction of the Markov measures on the set of trajectories of a Markov chain.

Recall that, for $a \in A_z^n$, we denoted by $\mathbb{C}_{a,z}$ (see (2.6)) the associated cylinder of length n in $\mathbb{X}_z^-$. For $n \geqslant 0$, define $\check{\mu}_{z,t}^{g,n}$ as a function on cylinders of length n in $\mathbb{X}_z^-$ by the formula

$$\check{\mu}_{z,t}^{g,n}(\mathbb{C}_{a,z}) = \frac{1}{\check{V}^g(z,t)} \exp(-S_n\psi(a\cdot z))\check{V}^g(a\cdot z, t+S_ng(a\cdot z)),$$

if $t + S_kg(T^k(a\cdot z)) \geqslant 0$ for all $1 \leqslant k \leqslant n$; if not, we set $\check{\mu}_{z,t}^{g,n}(\mathbb{C}_{a,z}) = 0$, (compare with (2.7)). We claim that for any $a \in A_z^n$, we have

$$\check{\mu}_{z,t}^{g,n}(\mathbb{C}_{a,z}) = \sum_{\substack{b\in A \\ M(b,a_{-n})=1}} \check{\mu}_{z,t}^{g,n+1}(\mathbb{C}_{b\cdot a,z}), \tag{3.22}$$

(compare with (2.8)). Indeed, this follows from the harmonicity property of the function $\check{V}^g$ established in Lemma 3.2. By Kolmogorov's extension theorem, equation (3.22) implies that there exists a unique Borel probability measure $\check{\mu}_{z,t}^{g,-}$ on $\mathbb{X}_z^-$ such that for any $n \geqslant 0$, $\check{\mu}_{z,t}^{g,n}$ is the restriction of $\check{\mu}_{z,t}^{g,-}$ to cylinders of length n. The lemma follows. □

In the same way as for the function $\check{V}^g$, we can give an alternative definition of the measures $\check{\mu}_{z,t}^{g,-}$, which relies on a convergence property.

Lemma 3.14. *Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Let (z,t) be in $\mathbb{X}^+ \times \mathbb{R}$ and φ be a continuous function on $\mathbb{X}_z^-$. Then, we have*

$$\check{\mu}_{z,t}^{g,-}(\varphi)\check{V}^g(z,t) = \lim_{n\to\infty} \int_{\mathbb{X}_z^-} \check{S}_ng(y\cdot z)\varphi(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy). \tag{3.23}$$

Proof. By Lemma 3.3, the limit in equation (3.23) is the same as the limit of

$$\int_{\mathbb{X}_z^-} \left(t + \check{S}_ng(y\cdot z)\right)\varphi(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy).$$

The latter quantity is non-negative whenever φ is non-negative. Besides, if $\varphi = \mathbb{1}$, the convergence follows from Lemma 3.4. Therefore, it suffices to check the convergence when φ is the indicator function of a cylinder set. Thus, let $m \geqslant 0$ be an integer. Pick $a \in A_z^m$ and let $\mathbb{C}_{a,z}$ be the associated cylinder in $\mathbb{X}_z^-$. If $S_kg(T^k(a\cdot z)) < 0$ for some $1 \leqslant k \leqslant m$, we have for $n \geqslant m$,

$$\int_{\mathbb{X}_z^-} \left(t + \check{S}_ng(y\cdot z)\right)\mathbb{1}_{\mathbb{C}_{a,z}}(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy) = 0.$$

If not, we have for $n \geqslant m$,

$$\begin{aligned}\int_{\mathbb{X}_z^-} \left(t + \check{S}_ng(y\cdot z)\right)\mathbb{1}_{\mathbb{C}_{a,z}}(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy) &= \exp(-S_m\psi(a\cdot z)) \\ \times \int_{\mathbb{X}_{a\cdot z}^-} \left(t + \check{S}_{n-m}g(y\cdot a\cdot z) + S_mg(a\cdot z)\right)&\mathbb{1}_{\left\{\check{\tau}_{t+S_mg(a\cdot z)}^g(y\cdot a\cdot z)>n-m\right\}}\nu_{a\cdot z}^-(dy).\end{aligned}$$

By Lemma 3.4, as $n \to \infty$, this converges to

$$\exp(-S_m\psi(a\cdot z))\check{V}^g(a\cdot z, t+S_mg(a\cdot z)),$$

which, by the definition of $\check{\mu}_{z,t}^{g,-}$ in Lemma 3.13, is equal to $\check{\mu}_{z,t}^{g,-}(\mathbb{C}_{a,z})\check{V}^g(z,t)$. □

Using Lemma 3.14, we can now give a definition of $\check{\mu}_{z,t}^{g,-}$ for a function g only depending on finitely many negative coordinates.

Lemma 3.15. *Let $g \in \mathscr{B}$ such that $\nu(g) = 0$ and g is not a coboundary. Assume that g only depends on m negative coordinates for some $m \geqslant 0$. In other words, the function $h = g \circ T^m \in \mathscr{B}^+$. Let (z, t) be in $\mathbb{X}^+ \times \mathbb{R}$ and φ be a continuous function on $\mathbb{X}_z^-$. For $a \in A_z^m$, set φ_a to be the function $y \mapsto \varphi(y \cdot a)$ on $\mathbb{X}_{a \cdot z}^-$. Then, we have*

$$\begin{aligned}&\lim_{n\to\infty} \int_{\mathbb{X}_z^-} \check{S}_n g(y \cdot z)\varphi(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)\\ &= \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z))\check{V}^h(a\cdot z, t)\check{\mu}_{a\cdot z,t}^{h,-}(\varphi_a).\end{aligned}$$

Before proving this lemma, we recall some useful facts. Let g and h be as in Lemma 3.15. For $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\check{V}^g(z,t) = \mathcal{L}_\psi^m \left(\check{V}^h(\cdot, t)\right)(z) = \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z))\check{V}^h(a\cdot z, t).$$

If $\check{V}^g(z, t) > 0$ and φ is a continuous function on $\mathbb{X}_z^-$, we set

$$\check{\mu}_{z,t}^{g,-}(\varphi) = \frac{1}{\check{V}^g(z,t)} \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z))\check{V}^h(a\cdot z,t)\check{\mu}_{a\cdot z,t}^{h,-}(\varphi_a). \tag{3.24}$$

Lemma 3.15 implies that the notation (3.24) is coherent with that introduced in Lemma 3.13.

Proof of Lemma 3.15. As in the proof of Lemma 3.5, by conditioning over the m first coordinates of y, we get for $n \geqslant 0$,

$$\begin{aligned}&\int_{\mathbb{X}_z^-} \check{S}_n g(y \cdot z)\varphi(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)\\ &= \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z)) \int_{\mathbb{X}_{a\cdot z}^-} \check{S}_n g((y\cdot a)\cdot z)\varphi(y\cdot a)\mathbb{1}_{\left\{\check{\tau}_t^g((y\cdot a)\cdot z)>n\right\}}\nu_{a\cdot z}^-(dy)\\ &= \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z)) \int_{\mathbb{X}_{a\cdot z}^-} \check{S}_n h(T^{-m}(y\cdot a)\cdot z)\varphi_a(y)\mathbb{1}_{\left\{\check{\tau}_t^{h\circ T^{-m}}((y\cdot a)\cdot z)>n\right\}}\nu_{a\cdot z}^-(dy)\\ &= \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z)) \int_{\mathbb{X}_{a\cdot z}^-} \check{S}_n h(y\cdot (a\cdot z))\varphi_a(y)\mathbb{1}_{\left\{\check{\tau}_t^h(y\cdot(a\cdot z))>n\right\}}\nu_{a\cdot z}^-(dy),\end{aligned}$$

where we have used the relations $(y \cdot a) \cdot z = T^m(y \cdot (a \cdot z))$ and $\check{\tau}_t^{h\circ T^{-m}} = \check{\tau}_t^h \circ T^{-m}$. The conclusion now follows from Lemma 3.14. □

Now we prove that the convergence in Lemma 3.15 holds in a weak sense for every function $g \in \mathscr{B}$.

Lemma 3.16. *Assume that $g \in \mathscr{B}$ is not a coboundary w.r.t. to T and $\nu(g) = 0$. Then, for any $z \in \mathbb{X}^+$, for any continuous compactly supported function φ on $\mathbb{X}_z^- \times \mathbb{R}$, the following limit exists and is finite:*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \int_{\mathbb{X}_z^-} \check{S}_n g(y\cdot z)\varphi(y,t)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt.$$

Proof. First let us assume that φ is of the form $(y, t) \mapsto \varphi_1(y)\varphi_2(t)$, where φ_1 and φ_2 are non-negative continuous functions on $\mathbb{X}_z^-$ and $\mathbb{R}$, and φ_2 is compactly supported. In that case, let $(g_m)_{m\geqslant 0}$, $c_1>0$ and $\alpha\in(0,1)$ be as in Lemma 2.11. Set

$$W_n(z,t)=\int_{\mathbb{X}_z^-}\left(t+\check{S}_n g(y\cdot z)\right)\varphi_1(y)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)$$

and

$$W_{n,m}(z,t)=\int_{\mathbb{X}_z^-}\left(t+\check{S}_n g_m(y\cdot z)\right)\varphi_1(y)\mathbb{1}_{\left\{\check{\tau}_t^{g_m}(y\cdot z)>n\right\}}\nu_z^-(dy).$$

By (2.18), we have the inclusions

$$\left\{\check{\tau}^{g_m}_{t-2c_1\alpha^m}>n\right\}\subseteq\{\check{\tau}_t^g>n\}\subseteq\left\{\check{\tau}^{g_m}_{t+2c_1\alpha^m}>n\right\},$$

which imply that

$$W_{n,m}(z,t-2c_1\alpha^m)\leqslant W_n(z,t)\leqslant W_{n,m}(z,t+2c_1\alpha^m). \tag{3.25}$$

By lemma 3.15, for fixed $m\geqslant 0$, as $n\to\infty$, the function $W_{n,m}(z,t)$ converges to $\check{\mu}_{z,t}^{g_m,-}(\varphi_1)\check{V}^{g_m}(z,t)$. From (3.25) we get

$$\begin{aligned}\check{\mu}^{g_m,-}_{z,t-2c_1\alpha^m}(\varphi_1)\check{V}^{g_m}(z,t-2c_1\alpha^m)&\leqslant\liminf_{n\to\infty}W_n(z,t)\leqslant\limsup_{n\to\infty}W_n(z,t)\\&\leqslant\check{\mu}^{g_m,-}_{z,t+2c_1\alpha^m}(\varphi_1)\check{V}^{g_m}(z,t+2c_1\alpha^m).\end{aligned}$$

Now we have

$$\begin{aligned}&\int_{\mathbb{R}}\varphi_2(t)\left[\check{\mu}^{g_m,-}_{z,t+2c_1\alpha^m}(\varphi_1)\check{V}^{g_m}(z,t+2c_1\alpha^m)-\check{\mu}^{g_m,-}_{z,t-2c_1\alpha^m}(\varphi_1)\check{V}^{g_m}(z,t-2c_1\alpha^m)\right]dt\\&=\int_{\mathbb{R}}\left[\varphi_2(t-2c_1\alpha^m)-\varphi_2(t+2c_1\alpha^m)\right]\check{\mu}^{g_m,-}_{z,t}(\varphi_1)\check{V}^{g_m}(z,t)dt.\end{aligned}\tag{3.26}$$

Using (3.9) and Lemma 3.5, we have that $\check{V}^{g_m}(z,t)\leqslant c_2+\max\{t,0\}$. As φ_2 is continuous on $\mathbb{R}$ with compact support, by the Lebesgue dominated convergence theorem, we get that the left-hand side of (3.26) converges to 0 as $m\to\infty$. This tells us that $\int_{\mathbb{R}}\varphi_2(t)W_n(z,t)dt$ has a limit as $n\to\infty$. In other words, the lemma holds for the function $\varphi(y,t)=\varphi_1(y)\varphi_2(t)$. This is also true when φ_1 and φ_2 are not necessarily non-negative.

The general case follows from a standard but tedious approximation argument. Indeed, we can find a continuous compactly supported function θ on $\mathbb{R}$ with support K such that for any $\varepsilon>0$, there exist an integer $p\geqslant 0$ and continuous functions $\varphi_{i,1}$ on $\mathbb{X}_z^-$ and continuous compactly supported functions $\varphi_{i,2}$ on $\mathbb{R}$ with support included in K, $1\leqslant i\leqslant p$, with

$$\sup_{y\in\mathbb{X}_z^-}|\varphi(y,t)-\varphi_\varepsilon(y,t)|\leqslant\varepsilon\theta(t),\quad t\in\mathbb{R},\tag{3.27}$$

where $\varphi_\varepsilon(y,t)=\sum_{i=1}^p\varphi_{i,1}(y)\varphi_{i,2}(t)$. We set $t_0=\sup_{t\in K}|t|$. By Lemma 3.3, we need to show that

$$U_n=\int_{\mathbb{R}}\int_{\mathbb{X}_z^-}\left(t_0+\check{S}_n g(y\cdot z)\right)\varphi(y,t)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt$$

has a limit as $n\to\infty$. By the first case, we know that

$$U_{n,\varepsilon}=\int_{\mathbb{R}}\int_{\mathbb{X}_z^-}\left(t_0+\check{S}_n g(y\cdot z)\right)\varphi_\varepsilon(y,t)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt$$

has a limit U_ε as $n\to\infty$. Besides, by Lemma 3.6, we get that

$$\int_{\mathbb{R}}\int_{\mathbb{X}_z^-}\left(t_0+\check{S}_n g(y\cdot z)\right)\theta(t)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt$$

converges to $\int_{\mathbb{R}} \check{V}^g(z,t)\theta(t)dt$. By (3.27), we have

$$U_\varepsilon - \varepsilon \int_{\mathbb{R}} \check{V}^g(z,t)\theta(t)dt \leqslant \liminf_{n\to\infty} U_n \leqslant \limsup_{n\to\infty} U_n \leqslant U_\varepsilon + \varepsilon \int_{\mathbb{R}} \check{V}^g(z,t)\theta(t)dt,$$

which gives

$$\limsup_{n\to\infty} U_n - \liminf_{n\to\infty} U_n \leqslant 2\varepsilon \int_{\mathbb{R}} \check{V}^g(z,t)\theta(t)dt.$$

Hence the proof of Lemma 3.16 is complete. □

Now we use the previous lemma to build a Radon measure $\check{\mu}_z^{g,-}$ on $\mathbb{X}_z^- \times \mathbb{R}$ for any $g \in \mathscr{B}$.

Lemma 3.17. *Assume that $g \in \mathscr{B}$ is not a coboundary w.r.t. to T and $\nu(g) = 0$. Then, for any $z \in \mathbb{X}^+$, there exists a unique Radon measure $\check{\mu}_z^{g,-}$ on $\mathbb{X}_z^- \times \mathbb{R}$ such that for any continuous compactly supported function φ on $\mathbb{X}_z^- \times \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \int_{\mathbb{X}_z^-} \varphi(y,t) \check{S}_n g(y\cdot z) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)dt = \int_{\mathbb{R}} \int_{\mathbb{X}_z^-} \varphi(y,t)\check{\mu}_z^{g,-}(dy,dt). \tag{3.28}$$

Besides, the marginal measure of $\check{\mu}_z^{g,-}$ on $\mathbb{R}$ under the natural projection map is the absolutely continuous measure $\check{V}^g(z,t)dt$.

Proof. By Lemma 3.16, the limit on the left-hand side of (3.28) exists. By Lemma 3.3, the limit is the same as the one of

$$\lim_{n\to\infty} \int_{\mathbb{R}} \int_{\mathbb{X}_z^-} \varphi(y,t) \left(t_0 + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)dt,$$

where $t_0 > 0$ is arbitrarily large. In particular, this limit is non-negative. By Riesz representation theorem, it may be written as $\check{\mu}_z^{g,-}(\varphi)$, where $\check{\mu}_z^{g,-}$ is a Radon measure on $\mathbb{X}_z^- \times \mathbb{R}$. By Lemma 3.8, the marginal measure of $\check{\mu}_z^{g,-}$ on $\mathbb{R}$ under the natural projection map is the absolutely continuous measure $\check{V}^g(z,t)dt$. □

We define the Radon measure $\check{\mu}^g$ on $\mathbb{X} \times \mathbb{R}$ by setting, for any continuous compactly supported function φ on $\mathbb{X} \times \mathbb{R}$,

$$\check{\mu}^g(\varphi) = \int_{\mathbb{X}^+} \int_{\mathbb{R}} \int_{\mathbb{X}_z^-} \varphi(y\cdot z,t)\check{\mu}_z^{g,-}(dy,dt)\nu^+(dz).$$

The main result of this section is stated as follows.

Theorem 3.18. *Let g be a Hölder continuous function on $\mathbb{X}$ such that $\nu(g) = 0$ and g is not a coboundary. Then, for any continuous compactly supported function φ on $\mathbb{X} \times \mathbb{R}$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{X}} \int_{\mathbb{R}} \varphi(x,t) \check{S}_n g(x) \mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}} \nu(dx)dt = \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^g(dx,dt). \tag{3.29}$$

Moreover, the following harmonicity property holds:

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^g(dx,dt) = \int_{\mathbb{X}} \int_0^\infty \varphi(Tx, t-g(x))\check{\mu}^g(dx,dt). \tag{3.30}$$

Proof. We can assume that φ is non-negative. By Lemma 3.17, for every $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}&\lim_{n\to\infty} \int_{\mathbb{X}_z^-} \int_{\mathbb{R}} \varphi(y\cdot z,t) \left(t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)dt \\ &= \int_{\mathbb{X}_z^-} \int_{\mathbb{R}} \varphi(y\cdot z,t)\check{\mu}_z^{g,-}(dy,dt).\end{aligned}$$

Thanks to the dominated convergence theorem, this will imply (3.29). Indeed, for $t \in \mathbb{R}$, set $\theta(t) = \sup_{x\in\mathbb{X}} \varphi(x,t)$, so that θ is a continuous compactly supported function on $\mathbb{R}$. Note that

$$\begin{aligned}&\int_{\mathbb{X}_z^-}\int_{\mathbb{R}} \varphi(y\cdot z,t)\left(t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt\\ &\leqslant \int_{\mathbb{X}_z^-}\int_{\mathbb{R}} \theta(t)\left(t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt.\end{aligned}$$

By Lemma 3.8, we have, uniformly in $z\in\mathbb{X}^+$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} \theta(t)\left(t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy)dt = \int_{\mathbb{R}}\theta(t)\check{V}^g(z,t)dt.$$

By the dominated convergence theorem, we get (3.29).

Now we prove (3.30). By (3.29),

$$\int_{\mathbb{X}\times\mathbb{R}}\varphi(x,t)\check{\mu}^g(dx,dt) = \lim_{n\to\infty}\int_{\mathbb{X}}\int_{\mathbb{R}}\varphi(x,t)\check{S}_n g(x)\mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}}\nu(dx)dt.$$

As ν is T-invariant, we have

$$\begin{aligned}&\int_{\mathbb{X}}\int_{\mathbb{R}}\varphi(x,t)\check{S}_n g(x)\mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}}\nu(dx)dt\\ &=\int_{\mathbb{X}}\int_{\mathbb{R}}\varphi(Tx,t)\left(\check{S}_{n-1}g(x)+g(x)\right)\mathbb{1}_{\{\check{\tau}^g_{t+g(x)}(x)>n-1\}}\mathbb{1}_{\{t+g(x)\geqslant 0\}}\nu(dx)dt\\ &=\int_{\mathbb{X}}\int_{\mathbb{R}}\varphi(Tx,t-g(x))\left(\check{S}_{n-1}g(x)+g(x)\right)\mathbb{1}_{\{\check{\tau}^g_{t}(x)>n-1\}}\mathbb{1}_{\{t\geqslant 0\}}\nu(dx)dt.\end{aligned}$$

By Lemma 3.3, the latter has the same limit, as $n\to\infty$, as

$$\int_{\mathbb{X}}\int_{\mathbb{R}}\varphi(Tx,t-g(x))\check{S}_{n-1}g(x)\mathbb{1}_{\{\check{\tau}^g_{t}(x)>n-1\}}\mathbb{1}_{\{t\geqslant 0\}}\nu(dx)dt.$$

We will prove below that we can apply (3.29) to the function $(x,t)\mapsto\varphi(Tx,t-g(x))\mathbb{1}_{\{t\geqslant 0\}}$ to get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{X}}\int_{\mathbb{R}}\varphi(Tx,t-g(x))\check{S}_{n-1}g(x)\mathbb{1}_{\{\check{\tau}^g_{t}(x)>n-1\}}\mathbb{1}_{\{t\geqslant 0\}}\nu(dx)dt\\ &=\int_{\mathbb{X}}\int_0^\infty\varphi(Tx,t-g(x))\check{\mu}^g(dx,dt),\end{aligned}$$

which proves (3.30).

To finish the proof, we need to show that (3.29) implies that for any continuous compact supported function φ on $\mathbb{X}\times\mathbb{R}$, as $n\to\infty$, the quantity

$$I_n = \int_{\mathbb{X}}\int_0^\infty\varphi(x,t)\check{S}_n g(x)\mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}}\nu(dx)dt$$

converges to

$$\int_{\mathbb{X}}\int_0^\infty\varphi(x,t)\check{\mu}^g(dx,dt). \tag{3.31}$$

This is a standard argument by an approximation. Indeed, for $\varepsilon>0$ and $t\in\mathbb{R}$, set $\chi_\varepsilon^-(t) = 0$ if $t<0$; $\chi_\varepsilon^-(t)=\frac{t}{\varepsilon}$ if $0\leqslant t\leqslant\varepsilon$ and $\chi_\varepsilon^-(t)=1$ if $t>\varepsilon$. Define also

$\chi_\varepsilon^+(t) = \chi_\varepsilon^-(t+\varepsilon)$. Then, for any $n \geqslant 0$, we have

$$\int_{\mathbb{X}\times\mathbb{R}} \chi_\varepsilon^-(t)\varphi(x,t)\left(t+\check{S}_n g(x)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}}\nu(dx)dt$$
$$\leqslant I_n \leqslant \int_{\mathbb{X}\times\mathbb{R}} \chi_\varepsilon^+(t)\varphi(x,t)\left(t+\check{S}_n g(x)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}}\nu(dx)dt.$$

By (3.30) and Lemma 3.3, we get

$$\int_{\mathbb{X}\times\mathbb{R}} \chi_\varepsilon^-(t)\varphi(x,t)\check{\mu}^g(dx,dt)$$
$$\leqslant \liminf_{n\to\infty} I_n \leqslant \limsup_{n\to\infty} I_n \leqslant \int_{\mathbb{X}\times\mathbb{R}} \chi_\varepsilon^+(t)\varphi(x,t)\check{\mu}^g(dx,dt).$$

We claim that the left and right-hand sides of the latter inequality converge to the integral in (3.31). Indeed, for $(x,t) \in \mathbb{X}\times\mathbb{R}$, we have that $|\chi_\varepsilon^+(t)\varphi(x,t)|$ and $|\chi_\varepsilon^-(t)\varphi(x,t)|$ are dominated by $|\varphi(x,t)|$. The conclusion now follows from the dominated convergence theorem. □

As for the function $\check{V}^g$, the measure $\check{\mu}^g$ enjoys the following continuity property on cohomology classes.

Lemma 3.19. *Let $g \in \mathscr{B}$ with $\nu(g) = 0$. Assume that g is not a coboundary. Let $\alpha \in (0,1)$ and $(h_m)_{m\geqslant 0}$ be a sequence of element of $\mathscr{B}_\alpha$ that converges to 0 with respect to the Hölder norm $\|\cdot\|_\alpha$. For $m \geqslant 0$, set $g_m = g + h_m \circ T - h_m$. Then, for any continuous compactly supported function φ on $\mathbb{X}\times\mathbb{R}$, we have,*

$$\lim_{m\to\infty} \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^{g_m}(dx,dt) = \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^g(dx,dt). \tag{3.32}$$

Proof. We can assume that φ is non-negative. By Theorem 3.18, for $m \geqslant 0$, we have

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\mu^{g_m}(dx,dt) = \lim_{n\to\infty} \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\left(t+\check{S}_n g_m(x)\right)\mathbb{1}_{\left\{\check{\tau}_t^{g_m}(x)>n\right\}}\nu(dx)dt.$$

For any $n \geqslant 0$, we have $S_n g_m \leqslant S_n g + 2\|h_m\|_\infty$. Hence, for $t \in \mathbb{R}$, we have $\check{\tau}_t^{g_m} \leqslant \check{\tau}^g_{t+2\|h_m\|_\infty}$. We get

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\left(t+\check{S}_n g_m(x)\right)\mathbb{1}_{\left\{\check{\tau}_t^{g_m}(x)>n\right\}}\nu(dx)dt$$
$$\leqslant \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\left(t+\check{S}_n g_m(x)\right)\mathbb{1}_{\left\{\check{\tau}^g_{t+2\|h_m\|_\infty}>n\right\}}\nu(dx)dt.$$

Again by Theorem 3.18, as $n\to\infty$, the latter quantity converges to

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t-2\|h_m\|_\infty)\check{\mu}^g(dx,dt).$$

Thus we have

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^{g_m}(dx,dt) \leqslant \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t-2\|h_m\|_\infty)\check{\mu}^g(dx,dt).$$

In the same way, one also has

$$\int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t)\check{\mu}^{g_m}(dx,dt) \geqslant \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t+2\|h_m\|_\infty)\check{\mu}^g(dx,dt).$$

As φ is continuous, the conclusion follows from the dominated convergence theorem. □

Proof of Theorem 1.1. So far we have proved Theorem 3.18 which is an analogue of Theorem 1.1 for the reversed dynamical system $(\mathbb{X}, T^{-1}, \nu)$. By Lemma 2.5, this dynamical system is isomorphic to a subshift of finite type equipped with a Gibbs measure. Therefore, Theorem 1.1 is actually equivalent to Theorem 3.18. Formally, the former can be obtained from the latter by replacing f with $g = f \circ T^{-1} \circ \iota = f \circ \iota \circ T$, and ν with $\iota_* \nu$. $\square$

The reader may notice that (3.20) is a particular case of (1.3), which is the reason to call the Radon measure μ^f harmonic.

4. Conditioned limit theorems

In this section we prove Theorems 1.3 and 1.5.

4.1. Proof of Theorem 1.3.

As in the construction of the harmonic function $\check{V}^g$ and the harmonic measure $\check{\mu}^g$, we will prove Theorem 1.3 in several steps. The first step is to deal with the case of functions g depending only on the future. The following result follows from the general result for Markov chains established in [15, Theorem 2.3]. The assumptions of this statement can be checked to hold thanks to the spectral gap properties of the Ruelle operator formulated in Section 5.1.

Lemma 4.1. *Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Then, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n}\, \nu_z^- \left(y \in \mathbb{X}_z^- : \check{\tau}_t^g (y \cdot z) > n \right) = 2\check{V}^g(z,t).$$

We have to strengthen Lemma 4.1 by proving the following integral form:

Lemma 4.2. *Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Then, for any continuous compactly supported function φ on $\mathbb{X}_z^-$, we have, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} \varphi(y) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) = 2\check{V}^g(z,t) \int_{\mathbb{X}_z^-} \varphi(y) \check{\mu}_{z,t}^{g,-}(dy).$$

Proof. It suffices to prove this result when φ is the indicator function of a cylinder set in $\mathbb{X}_z^-$, since the general case follows by a standard approximation argument. Thus, let $m \geqslant 0$ and $a \in A_z^m$ and, as before, denote by $\mathbb{C}_{a,z}$ the associated cylinder in $\mathbb{X}_z^-$ (see (2.6)).

If $t + S_k g(T^{m-k}(a \cdot z)) \geqslant 0$ for every $1 \leqslant k \leqslant m$, we have

$$\begin{aligned} &\sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} \mathbb{1}_{\mathbb{C}_{a,z}}(y) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) \\ &= \sigma_g \sqrt{2\pi n} \exp(-S_m \psi(a\cdot z)) \int_{\mathbb{X}_{a\cdot z}^-} \mathbb{1}_{\left\{\check{\tau}^g_{t+S_m g(a\cdot z)}(y\cdot(a\cdot z)) > n-m\right\}} \nu_{a\cdot z}^-(dy). \end{aligned}$$

By Lemma 4.1, as $n \to \infty$, the latter quantity converges, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$, to

$$2\check{V}^g(a\cdot z, t + S_m g(a\cdot z)) \exp(-S_m \psi(a \cdot z)),$$

which, by definition, is equal to $2\check{V}^g(z,t)\check{\mu}_{z,t}^{g,-}(\mathbb{C}_{a,z})$.

If there exists $1 \leqslant k \leqslant m$ with $t + S_k g(T^{m-k}(a\cdot z)) < 0$, we have $\check{\mu}_{z,t}^{g,-}(\mathbb{C}_{a,z}) = 0$ and

$$\int_{\mathbb{X}_z^-} \mathbb{1}_{\mathbb{C}_{a,z}}(y) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) = 0,$$

for $n > k$. The conclusion follows. $\square$

From Lemmas 4.1 and 4.2, we deduce the analogous result for functions which depend only on finitely many negative coordinates.

Lemma 4.3. *Let $g \in \mathscr{B}$ be such that $\nu(g) = 0$ and there exists $m \geqslant 0$ with $g \circ T^m \in \mathscr{B}^+$. Assume that g is not a coboundary. Then, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n}\, \nu_z^- \left(y \in \mathbb{X}_z^- : \check{\tau}_t^g(y \cdot z) > n \right) = 2\check{V}^g(z,t).$$

Moreover, for any continuous compactly supported function φ on $\mathbb{X}_z^-$, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} \varphi(y) \mathbb{1}_{\left\{ \check{\tau}_t^g(y\cdot z) > n \right\}} \nu_z^-(dy) = 2\check{V}^g(z,t) \int_{\mathbb{X}_z^-} \varphi(y) \check{\mu}_{z,t}^{g,-}(dy).$$

Proof. As in Lemma 3.15, for $a \in A_z^m$, let φ_a be the continuous function $y \mapsto \varphi(a \cdot y)$ on $\mathbb{X}_{a\cdot z}^-$. We have, by setting $h = g \circ T^m$,

$$\begin{aligned}
&\int_{\mathbb{X}_z^-} \varphi(y) \mathbb{1}_{\left\{ \check{\tau}_t^g(y\cdot z) > n \right\}} \nu_z^-(dy) \\
&= \sum_{a \in A_z^m} \exp(-S_m \psi(a\cdot z)) \int_{\mathbb{X}_{a\cdot z}^-} \varphi_a(y) \mathbb{1}_{\left\{ \check{\tau}_t^h(y\cdot (a\cdot z)) > n \right\}} \nu_{a\cdot z}^-(dy).
\end{aligned}$$

The conclusion now follows from Lemmas 4.1 and 4.2 and (3.24). □

Now we use the same approximation argument as before to deduce from Lemma 4.3 a slightly weaker statement that works for every function g in $\mathscr{B}$. This is the main result of this section.

Theorem 4.4. *Let $g \in \mathscr{B}$ be such that $\nu(g) = 0$. Assume that g is not a coboundary. Then, for any continuous compactly supported function φ on $\mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{R}} \varphi(t) \nu_z^- \left(y \in \mathbb{X}_z^- : \check{\tau}_t^g(y \cdot z) > n \right) dt = 2 \int_{\mathbb{R}} \varphi(t) \check{V}^g(z,t) dt.$$

Moreover, for any continuous compactly supported function φ on $\mathbb{X} \times \mathbb{R}$, we have

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t) \mathbb{1}_{\left\{ \check{\tau}_t^g(x) > n \right\}} \nu(dx) dt = 2 \int_{\mathbb{X}\times\mathbb{R}} \varphi(x,t) \check{\mu}^g(dx,dt).$$

Proof. For $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$, denote

$$\check{V}_n^g(z,t) = \frac{1}{2} \sigma_g \sqrt{2\pi n}\, \nu_z^- \left(y \in \mathbb{X}_z^- : \check{\tau}_t^g(y\cdot z) > n \right).$$

Let $(g_m)_{m\geqslant 0}$ be the sequence of Hölder continuous functions as in Lemma 2.11. For $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$, we have

$$\check{V}_n^{g_m}(z, t - 2c_1\alpha^m) \leqslant \check{V}_n^g(z,t) \leqslant \check{V}_n^{g_m}(z, t + 2c_1\alpha^m).$$

By taking the limit as $n \to \infty$, we get by Lemma 4.3,

$$\check{V}^{g_m}(z, t - 2c_1\alpha^m) \leqslant \liminf_{n\to\infty} \check{V}_n^g(z,t) \leqslant \limsup_{n\to\infty} \check{V}_n^g(z,t) \leqslant \check{V}^{g_m}(z, t + 2c_1\alpha^m).$$

The first part of the lemma now follows from Lemma 3.10.

Let now φ be a non-negative continuous compactly supported function on $\mathbb{X}\times\mathbb{R}$. For $m,n\geqslant 0$, we have

$$\begin{aligned}\int_{\mathbb{X}\times\mathbb{R}}\varphi(x,t)\mathbb{1}_{\left\{\check{\tau}^{g_m}_{t-2c_1\alpha^m}(x)>n\right\}}\nu(dx)dt &\leqslant \int_{\mathbb{X}\times\mathbb{R}}\varphi(x,t)\mathbb{1}_{\left\{\check{\tau}^{g}_{t}(x)>n\right\}}\nu(dx)dt\\ &\leqslant \int_{\mathbb{X}\times\mathbb{R}}\varphi(x,t)\mathbb{1}_{\left\{\check{\tau}^{g_m}_{t+2c_1\alpha^m}(x)>n\right\}}\nu(dx)dt.\end{aligned}$$

The conclusion follows from Lemmas 3.19 and 4.3. □

Now we prove Theorem 1.3.

Proof of Theorem 1.3. The first assertion of Theorem 1.3 follows from the second assertion of Theorem 4.4 by replacing f with $g=f\circ T^{-1}\circ\iota=f\circ\iota\circ T$, and ν with $\iota_*\nu$. The second assertion is also obtained from Theorem 4.4 by using Lemma 2.12. □

From Theorem 1.3, we get the following coarse domination which will be used in the proof of the conditioned local limit theorem (Theorem 1.7).

Corollary 4.5. *Let g be in $\mathscr{B}^+$ with $\nu(g)=0$. Assume that g is not cohomologous to 0. Let G be a continuous compactly supported function on $\mathbb{X}^+\times\mathbb{R}$. Then there exists a constant $c>0$ such that for any $n\geqslant 1$,*

$$\int_{\mathbb{R}}\sup_{z\in\mathbb{X}^+}\int_{\mathbb{X}^-_z}G(T^{-n}(y\cdot z)_+,t+\check{S}_ng(y\cdot z))\mathbb{1}_{\left\{\check{\tau}^g_t(y\cdot z)>n\right\}}\nu^-_z(dy)dt\leqslant\frac{c}{\sqrt{n}}.$$

Proof. By replacing G with the function $\sup_{z\in\mathbb{X}^+}|G(z,t)|$, we can assume that G does not depend on the first coordinate. Let c_0 be as in Lemma 2.9. For $t\in\mathbb{R}$, set $G_1(t)=\sup_{|t'-t|\leqslant c_0}|G(t')|$. Then for any $t\in\mathbb{R}$ and $z,z'\in\mathbb{X}^+$ with $z_0=z'_0$, we have

$$\begin{aligned}&\int_{\mathbb{X}^-_z}G(t+\check{S}_ng(y\cdot z))\mathbb{1}_{\left\{\check{\tau}^g_t(y\cdot z)>n\right\}}\nu^-_z(dy)\\ &\leqslant\int_{\mathbb{X}^-_z}G_1(t+\check{S}_ng(y\cdot z'))\mathbb{1}_{\left\{\check{\tau}^g_{t+c_0}(y\cdot z')>n\right\}}\nu^-_z(dy)\\ &\leqslant c\int_{\mathbb{X}^-_{z'}}G_1(t+\check{S}_ng(y\cdot z'))\mathbb{1}_{\left\{\check{\tau}^g_{t+c_0}(y\cdot z')>n\right\}}\nu^-_{z'}(dy),\end{aligned}$$

for some constant $c>0$ coming from Lemma 2.3. By integrating over $z'\in\mathbb{X}^+$, we get

$$\begin{aligned}&\sup_{z\in\mathbb{X}^+}\int_{\mathbb{X}^-_z}G(t+\check{S}_ng(y\cdot z))\mathbb{1}_{\left\{\check{\tau}^g_t(y\cdot z)>n\right\}}\nu^-_z(dy)\\ &\leqslant\frac{c}{c_1}\int_{\mathbb{X}}G_1(t-c_0+\check{S}_ng(x))\mathbb{1}_{\left\{\check{\tau}^g_t(x)>n\right\}}\nu(dx),\end{aligned}$$

where $c_1=\inf_{a\in A}\nu^+\{z'\in\mathbb{X}^+:z'_0=a\}$. Integrating over $t\in\mathbb{R}$, we get the result by Theorem 1.3. □

4.2. **Proof of Theorem 1.5.** Again we start with the case of Markov chains. As in the previous section, using the argument of [15, Theorem 2.5], we get the following result.

Lemma 4.6. *Let $g\in\mathscr{B}^+$ with $\nu(g)=0$ and assume that g is not a coboundary. Then, for any continuous compactly supported function F on $\mathbb{X}^+\times\mathbb{R}$, we have, uniformly in $z\in\mathbb{X}^+$*

and t in compact subsets of $\mathbb{R}$,

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} F\left(\left(T^{-n}(y\cdot z)\right)_+, \frac{\check{S}_n g(y\cdot z)}{\sigma_g \sqrt{n}} \right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= 2\check{V}^g(z,t) \int_{\mathbb{X}^+\times\mathbb{R}} F(z',u)\phi^+(u)du\nu^+(dz').$$

We shall extend the previous lemma to allow functions F depending on the past coordinates in $\mathbb{X}$.

Lemma 4.7. *Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Then, for any continuous compactly supported function F on $\mathbb{X}\times\mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} F\left(T^{-n}(y\cdot z), \frac{\check{S}_n g(y\cdot z)}{\sigma_g \sqrt{n}} \right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= 2\check{V}^g(z,t) \int_{\mathbb{X}\times\mathbb{R}} F(x,u)\phi^+(u)du\nu(dx).$$

Proof. For $a \in A^{\{-m,\cdots,-1\}}$, set $\mathbb{C}_a = \{x \in \mathbb{X} : x_{-m} = a_{-m}, \dots, x_{-1} = a_{-1}\}$. By a standard approximation argument, it suffices to prove the result for the set of functions F of the form $(x,t) \mapsto \mathbb{1}_{\mathbb{C}_a}(x)F_1(x_+,t)$, where F_1 is a continuous compactly supported function on $\mathbb{X}^+\times\mathbb{R}$, and $a \in A^{\{-m,\cdots,-1\}}$ with $M(a_{i-1},a_i) = 1$ for $-m+1 \leqslant i \leqslant -1$. We want to determine the limit as $n\to\infty$,

$$I_n := \int_{\mathbb{X}_z^-} F\left(T^{-n}(y\cdot z), \frac{\check{S}_n g(y\cdot z)}{\sigma_g \sqrt{n}} \right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy).$$

Note that in this integral, all the terms only depend on the coordinates $y_{-n}, y_{-n+1}, \dots, y_{-1}$ except $T^{-n}(y\cdot z)$. By integrating first over the deep past coordinates $\dots, y_{-n-2}, y_{-n-1}$, we get by using Lemma 2.2,

$$I_n = \int_{\mathbb{X}_z^-} F_2\left(\left(T^{-n}(y\cdot z)\right)_+, \frac{\check{S}_n g(y\cdot z)}{\sigma_g \sqrt{n}} \right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy),$$

where, for $(z',t) \in \mathbb{X}^+\times\mathbb{R}$,

$$F_2(z',t) = \exp(-S_m\psi(a_{-m}\dots a_{-1}\cdot z'))F_1(z',t).$$

Lemma 4.6 gives uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} F_2\left(\left(T^{-n}(y\cdot z)\right)_+, \frac{\check{S}_n g(y\cdot z)}{\sigma_g \sqrt{n}} \right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= 2\check{V}^g(z,t) \int_{\mathbb{X}^+\times\mathbb{R}} F_2(z',u)\phi^+(u)du\nu^+(dz').$$

By construction of the measure ν in Lemma 2.2, we have

$$\int_{\mathbb{X}^+\times\mathbb{R}} F_2(z',u)\phi^+(u)du\nu^+(dz') = \int_{\mathbb{X}\times\mathbb{R}} F(x,u)\phi^+(u)du\nu(dx),$$

which ends the proof of the lemma. □

As for Theorem 1.3, we get the following version of Lemma 4.7, where we add a source target function.

Lemma 4.8. *Let $g \in \mathscr{B}^+$ with $\nu(g) = 0$ and assume that g is not a coboundary. Then, for any continuous compactly supported function F on $\mathbb{X}_z^- \times \mathbb{X} \times \mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} F\left(y, T^{-n}(y\cdot z), \frac{\check{S}_n g(y\cdot z)}{\sigma_g\sqrt{n}}\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= 2\check{V}^g(z,t) \int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F(y',x,u)\phi^+(u)\check{\mu}_{z,t}^{g,-}(dy')\nu(dx)du.$$

The proof of Lemma 4.8 can be carried out in the same way as that of Lemma 4.2 and therefore is left to the reader. By using again conditioning and Lemma 3.15, we extend the previous lemma to functions g depending on finitely many coordinates of the past.

Lemma 4.9. *Let $g \in \mathscr{B}$ be such that $\nu(g) = 0$ and there exists $m \geqslant 0$ with $g \circ T^m \in \mathscr{B}^+$. Assume that g is not a coboundary. Then, for any continuous compactly supported function F on $\mathbb{X} \times \mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} F\left(T^{-n}(y\cdot z), \frac{\check{S}_n g(y\cdot z)}{\sigma_g\sqrt{n}}\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= 2\check{V}^g(z,t) \int_{\mathbb{X}\times\mathbb{R}} F(x,u)\phi^+(u)\nu(dx)du.$$

Moreover, for any continuous compactly supported function F on $\mathbb{X}_z^- \times \mathbb{X} \times \mathbb{R}$, we have, uniformly in $z \in \mathbb{X}^+$ and t in compact subsets of $\mathbb{R}$,

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}_z^-} F\left(y, T^{-n}(y\cdot z), \frac{\check{S}_n g(y\cdot z)}{\sigma_g\sqrt{n}}\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= 2\check{V}^g(z,t) \int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F(y',x,u)\phi^+(u)\check{\mu}_{z,t}^{g,-}(dy')\nu(dx)du.$$

Proof. We prove only the second assertion, since the first one is a particular case of the second. As in Lemma 3.15, for $a \in A_z^m$, set F_a to be the function on $\mathbb{X}_{a\cdot z}^- \times \mathbb{X} \times \mathbb{R}$ defined by $F_a(y,x,t) = F(y\cdot a, T^m x, t)$. We have, by setting $h = g \circ T^m$,

$$\int_{\mathbb{X}_z^-} F\left(y, T^{-n}(y\cdot z), \frac{\check{S}_n g(y\cdot z)}{\sigma_g\sqrt{n}}\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}} \nu_z^-(dy)$$
$$= \sum_{a\in A_z^m} \exp(-S_m\psi(a\cdot z)) \int_{\mathbb{X}_{a\cdot z}^-} F_a\left(y, T^{-n}(y\cdot(a\cdot z)), \frac{\check{S}_n h(y\cdot(a\cdot z))}{\sigma_g\sqrt{n}}\right)$$
$$\times \mathbb{1}_{\left\{\check{\tau}_t^h(y\cdot(a\cdot z))>n\right\}} \nu_{a\cdot z}^-(dy).$$

The conclusion now follows from Lemma 4.8 and (3.24). □

The same technique as in Lemma 4.4 gives

Lemma 4.10. *Let $g \in \mathscr{B}$ with $\nu(g) = 0$ and assume that g is not a coboundary. Then, for any continuous compactly supported function F on $\mathbb{X} \times \mathbb{X} \times \mathbb{R} \times \mathbb{R}$, we have*

$$\lim_{n\to\infty} \sigma_g \sqrt{2\pi n} \int_{\mathbb{X}\times\mathbb{R}} F\left(x, T^{-n}x, t, \frac{\check{S}_n g(y\cdot z)}{\sigma_g\sqrt{n}}\right) \mathbb{1}_{\left\{\check{\tau}_t^g(x)>n\right\}} \nu(dx)dt$$
$$= 2\int_{\mathbb{X}\times\mathbb{R}\times\mathbb{X}\times\mathbb{R}} F(x,x',t,t')\phi^+(t')\nu(dx')dt'\check{\mu}^g(dx,dt).$$

Theorem 1.5 easily follows from Lemma 4.10.

5. Effective local limit theorems

So far we have adapted some results from the theory of Markov chains to the case of hyperbolic dynamical systems by constructing the analogs of the harmonic functions V^g and $\check{V}^g$ and building the harmonic measures μ^g and $\check{\mu}^g$. In the remaining part of the paper we shall use these objects to establish conditioned limit theorems, by adapting the strategy from the case of sums of independent random variables [18]. We start with formulating an effective version of the ordinary local limit theorem which is adapted to our needs.

5.1. **Spectral gap theory.** Fix $\alpha \in (0,1)$ such that $\psi \in \mathscr{B}_\alpha^+$, where ψ is the potential function used for the construction of the Gibbs measure ν (see Section 2.1). Denote by $\mathscr{L}(\mathscr{B}_\alpha^+, \mathscr{B}_\alpha^+)$ the set of all bounded linear operators from $\mathscr{B}_\alpha^+$ to $\mathscr{B}_\alpha^+$ equipped with the standard operator norm $\|\cdot\|_{\mathscr{B}_\alpha^+ \to \mathscr{B}_\alpha^+}$. From the general construction of the Ruelle operator, every $f \in \mathscr{B}_\alpha^+$ gives rise to a family of perturbed operators $(\mathcal{L}_{\psi+\mathbf{i}tf})$ defined as follows: for any $\varphi \in \mathscr{B}_\alpha^+$,

$$\mathcal{L}_{\psi+\mathbf{i}tf}\varphi(z) = \sum_{z' :\, Tz'=z} e^{-\psi(z') - \mathbf{i}tf(z')}\varphi(z'), \quad z \in \mathbb{X}^+,\ t \in \mathbb{R}. \tag{5.1}$$

By iteration, it follows that for any $\psi, f \in \mathscr{B}_\alpha$ and $t \in \mathbb{R}$,

$$\mathcal{L}^n_{\psi+\mathbf{i}tf}\varphi(z) = \sum_{z' :\, T^n z'=z} e^{-S_n(\psi+\mathbf{i}tf)(z')}\varphi(z'), \quad z \in \mathbb{X}^+.$$

The following result (see [24]) provides the spectral gap properties for the perturbed operator $\mathcal{L}_{\psi+\mathbf{i}tf}$. For similar statements in the case of Markov chains we refer to [21].

Lemma 5.1. *Assume that $f \in \mathscr{B}_\alpha^+$ is not a coboundary and that $\nu(f) = 0$. Then, there exists a constant $\delta > 0$ such that for any $t \in (-\delta, \delta)$,*

$$\mathcal{L}^n_{\psi+\mathbf{i}tf} = \lambda_t^n \Pi_t + N_t^n, \quad n \geqslant 1, \tag{5.2}$$

where the mappings $t \mapsto \Pi_t : (-\delta,\delta) \to \mathscr{L}(\mathscr{B}_\alpha^+, \mathscr{B}_\alpha^+)$ and $z \mapsto N_t : (-\delta,\delta) \to \mathscr{L}(\mathscr{B}_\alpha^+, \mathscr{B}_\alpha^+)$ are analytic in the operator norm topology, Π_t is a rank-one projection with $\Pi_0(\varphi)(z) = \nu^+(\varphi)$ for any $\varphi \in \mathscr{B}_\alpha^+$ and $z \in \mathbb{X}^+$, $\Pi_t N_t = N_t \Pi_t = 0$. Moreover, there exist $n_0 \geqslant 1$ and $q \in (0,1)$ such that for any $t \in (-\delta,\delta)$ the $\|N_t^{n_0}\|_{\mathscr{B}_\alpha^+ \to \mathscr{B}_\alpha^+} \leqslant q$.

The eigenvalue λ_t has the asymptotic expansion: as $t \to 0$,

$$\lambda_t = 1 - \frac{\sigma_f^2}{2}t^2 + O(|t|^3). \tag{5.3}$$

Note that since f is not a coboundary w.r.t. to T, the asymptotic variance σ_f^2 appearing in (5.3) is strictly positive.

Lemma 5.2. *Let $f \in \mathscr{B}_\alpha^+$ and $t \neq 0$. Assume that for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pf + q$ is not cohomologous to a function with values in $\mathbb{Z}$. Then, for any $t \neq 0$, the operator $\mathcal{L}_{\psi+\mathbf{i}tf}$ has spectral radius strictly less than 1 in $\mathscr{B}_\alpha^+$. More precisely, for any compact set $K \subset \mathbb{R} \setminus \{0\}$, there exist constants $c_K, c'_K > 0$ such that for any $\varphi \in \mathscr{B}_\alpha^+$ and $n \geqslant 1$,*

$$\sup_{t \in K} \|\mathcal{L}^n_{\psi+\mathbf{i}tf}\varphi\|_{\mathscr{B}_\alpha^+} \leqslant c'_K e^{-c_K n}\|\varphi\|_{\mathscr{B}_\alpha^+}. \tag{5.4}$$

Proof. The proof of the first assertion can be found in [24, Theorem 4.5]. Now we prove (5.4). For every $t \in K$, there exist $n_0(t) \geqslant 1$ and $\alpha(t) \in (0,1)$ such that $\|\mathcal{L}^{n_0(t)}_{\psi+\mathbf{i}tf}\|_{\mathscr{B}^+_\alpha \to \mathscr{B}^+_\alpha} < \alpha(t)$. As the operator $\mathcal{L}_{\psi+\mathbf{i}tf}$ depends continuously on t for the operator norm topology, there exists $\delta = \delta(t)$ such that for any $s \in (t-\delta(t), t+\delta(t))$, we still have $\|\mathcal{L}^{n_0(t)}_{\psi+\mathbf{i}sf}\|_{\mathscr{B}^+_\alpha \to \mathscr{B}^+_\alpha} < 1$. In particular, for every $n \geqslant 0$ we have $\|\mathcal{L}^{n}_{\psi+\mathbf{i}sf}\|_{\mathscr{B}^+_\alpha \to \mathscr{B}^+_\alpha} \leqslant c(t)\alpha(t)^{n/n_0(t)}$, for some $c(t) > 0$. By compactness, we can find $t_1, \dots, t_r \in K$ such that $K \subset \bigcup_{i=1}^r (t_i - \delta(t_i), t_i + \delta(t_i))$. In particular, by setting $c = \max_{1 \leqslant i \leqslant r} c(t_i)$, $\alpha = \max_{1 \leqslant i \leqslant r} \alpha(t_i)$ and $n_0 = \max_{1 \leqslant i \leqslant r} n_0(t_i)$, we get for any $s \in K$ and $n \geqslant 0$, $\|\mathcal{L}^{n}_{\psi+\mathbf{i}sf}\|_{\mathscr{B}^+_\alpha \to \mathscr{B}^+_\alpha} \leqslant c\alpha^{n/n_0}$. □

5.2. **Local limit theorem for smooth target functions.** In the following we establish a local limit theorem for Markov chains with a precise estimation of the remainder term. Let F be a measurable non-negative bounded target function on $\mathbb{X} \times \mathbb{R}$. The probability we are interested in can be written as follows: for any $z \in \mathbb{X}^+$,

$$\int_{\mathbb{X}^-_z} F\left((T^{-n}y \cdot z)_+, \check{S}_n g(y \cdot z)\right) \nu^-_z(dy).$$

The main difficulty is to give a local limit theorem with the explicit dependence of the remainder terms on F.

We first describe the kind of target functions that we will use.

Lemma 5.3. *Let X be a compact metric space and $\alpha > 0$. Let F be a real-valued function on $X \times \mathbb{R}$ such that*

1. *For any $t \in \mathbb{R}$, the function $z \mapsto F(z,t)$ is α-Hölder continuous on X.*
2. *For any $z \in X$, the function $t \mapsto F(z,t)$ is measurable on $\mathbb{R}$.*

Then, the function $(z,t) \mapsto F(z,t)$ is measurable on $X \times \mathbb{R}$ and the function $t \mapsto \|F(\cdot,t)\|_\alpha$ is measurable on $\mathbb{R}$, where the norm $\|\cdot\|_\alpha$ is the usual norm on the space of α-Hölder continuous functions on X. Moreover, if the integral $\int_{\mathbb{R}} \|F(\cdot,t)\|_\alpha dt$ is finite, we define the partial Fourier transform $\widehat{F}$ of F by setting for any $z \in X$ and $u \in \mathbb{R}$,

$$\widehat{F}(z,u) = \int_{\mathbb{R}} e^{itu} F(z,t) dt.$$

This is a continuous function on $X \times \mathbb{R}$. In addition, for every $u \in \mathbb{R}$, the function $z \mapsto \widehat{F}(z,u)$ is α-Hölder continuous and $\|\widehat{F}(\cdot,u)\|_\alpha \leqslant \int_{\mathbb{R}} \|F(\cdot,t)\|_\alpha dt$.

Proof. Since the space X is separable and the function $z \mapsto F(z,t)$ is continuous on X for any $t \in \mathbb{R}$, the supremum $\sup_{z \in X} |F(z,t)|$ can be taken over a countable dense subset, so that $t \mapsto \sup_{z \in X} |F(z,t)|$ is measurable. In the same way, since the function $z \mapsto F(z,t)$ is α-Hölder continuous on X for any $t \in \mathbb{R}$, one can also verify that $\sup_{z,z' \in X} \frac{|F(z,t) - F(z',t)|}{\alpha^{\omega(z,z')}}$ is a measurable function in t.

In case the integral $\int_{\mathbb{R}} \|F(\cdot,t)\|_\alpha dt$ is finite, the partial Fourier transform $\widehat{F}$ is well defined and continuous by the dominated convergence theorem. The norm domination is obvious. □

We denote by $\mathscr{H}^+_\alpha$ the set of real-valued functions on $\mathbb{X}^+ \times \mathbb{R}$ such that conditions (1) and (2) of Lemma 5.3 hold and the integral $\int_{\mathbb{R}} \|F(\cdot,t)\|_{\mathscr{B}^+_\alpha} dt$ is finite. For any compact set $K \subset \mathbb{R}$, denote by $\mathscr{H}^+_{\alpha,K}$ the set of functions $F \in \mathscr{H}^+_\alpha$ such that the Fourier transform

$\widehat{F}(z,\cdot)$ has a support contained in K for any $z \in \mathbb{X}^+$. Let ϕ be the standard normal density:

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \quad u \in \mathbb{R}.$$

Theorem 5.4. *Let $\alpha \in (0,1)$. Assume that $g \in \mathscr{B}_\alpha^+$ such that $\nu^+(g) = 0$ and for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pg + q$ is not cohomologous to a function with values in $\mathbb{Z}$. Let $K \subset \mathbb{R}$ be a compact set. Then there exists a constant $c_K > 0$ such that for any $F \in \mathscr{H}_{\alpha,K}^+$, $n \geqslant 1$ and $z \in \mathbb{X}^+$,*

$$\Bigg| \sqrt{n} \int_{\mathbb{X}_z^-} F\left((T^{-n} y \cdot z)_+, \check{S}_n g(y \cdot z) \right) \nu_z^-(dy) \\ - \int_{\mathbb{X}^+ \times \mathbb{R}} \frac{1}{\sigma_g} \phi\left(\frac{u}{\sigma_g \sqrt{n}} \right) F\left(z', u\right) du \nu^+(dz') \Bigg| \leqslant \frac{c_K}{\sqrt{n}} \int_{\mathbb{R}} \|F(\cdot,t)\|_{\mathscr{B}_\alpha^+} dt. \tag{5.5}$$

Proof. Without loss of generality, we assume that $\sigma_g = 1$. By the Fourier inversion formula, the Fubini theorem and a change of variable t to $\frac{t}{\sqrt{n}}$, we get

$$\begin{aligned} &\sqrt{n} \int_{\mathbb{X}_z^-} F\left((T^{-n} y \cdot z)_+, \check{S}_n g(y \cdot z) \right) \nu_z^-(dy) \\ &= \frac{\sqrt{n}}{2\pi} \int_{\mathbb{X}_z^- \times \mathbb{R}} e^{-\mathbf{i} t \check{S}_n g(y\cdot z)} \widehat{F}((T^{-n} y \cdot z)_+, t) \nu_z^-(dy) dt \\ &= \frac{1}{2\pi} \int_{\mathbb{X}_z^- \times \mathbb{R}} e^{-\frac{\mathbf{i}t}{\sqrt{n}} \check{S}_n g(y\cdot z)} \widehat{F}\left((T^{-n} y \cdot z)_+, \frac{t}{\sqrt{n}} \right) \nu_z^-(dy) dt =: I. \end{aligned}$$

Note that the Fubini theorem can be applied since the integral on $\mathbb{X}_z^-$ is in fact a finite sum. Denote

$$J(t) = \int_{\mathbb{X}_z^-} e^{-\frac{\mathbf{i}t}{\sqrt{n}} \check{S}_n g(y\cdot z)} \widehat{F}\left((T^{-n} y \cdot z)_+, \frac{t}{\sqrt{n}} \right) \nu_z^-(dy) \\ - e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z', \frac{t}{\sqrt{n}} \right) \nu^+(dz').$$

We decompose the integral I into three parts: $I = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{|t| \leqslant \delta\sqrt{n}} J(t) dt, \\ I_2 &= \frac{1}{2\pi} \int_{\delta\sqrt{n} < |t|} \left[\int_{\mathbb{X}_z^-} e^{-\frac{\mathbf{i}t}{\sqrt{n}} \check{S}_n g(y\cdot z)} \widehat{F}\left((T^{-n} y \cdot z)_+, \frac{t}{\sqrt{n}} \right) \nu_z^-(dy) \right] dt, \\ I_3 &= \frac{1}{2\pi} \int_{|t| \leqslant \delta\sqrt{n}} \left[e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z', \frac{t}{\sqrt{n}} \right) \nu^+(dz') \right] dt. \end{aligned}$$

Estimate of I_1. Since $\int_{\mathbb{R}} \|F(\cdot,u)\|_{\mathscr{B}_\alpha^+} du < \infty$, the function $z \mapsto \widehat{F}(z,t)$ is Hölder continuous on $\mathbb{X}^+$ with Hölder norm at most $\int_{\mathbb{R}} \|F(\cdot,u)\|_{\mathscr{B}_\alpha^+} du$, for any fixed $t \in \mathbb{R}$. Applying

(5.2), we get

$$\begin{aligned} J(t) &= \mathcal{L}^n_{\psi+\frac{\mathbf{i}t}{\sqrt{n}}g} \widehat{F}\left(\cdot, \frac{t}{\sqrt{n}}\right)(z) - e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z', \frac{t}{\sqrt{n}}\right) \nu^+(dz') \\ &= \left(\lambda^n_{\frac{t}{\sqrt{n}}} - e^{-\frac{t^2}{2}}\right) \Pi_{\frac{t}{\sqrt{n}}} \widehat{F}\left(\cdot, \frac{t}{\sqrt{n}}\right)(z) \\ &\quad + e^{-\frac{t^2}{2}} \left(\Pi_{\frac{t}{\sqrt{n}}} - \Pi_0\right) \widehat{F}\left(\cdot, \frac{t}{\sqrt{n}}\right)(z) + N^n_{\frac{t}{\sqrt{n}}} \widehat{F}\left(\cdot, \frac{t}{\sqrt{n}}\right)(z) \\ &=: J_1(t) + J_2(t) + J_3(t). \end{aligned}$$

For the first term, by (5.3) and simple calculations, we get

$$|J_1(t)| \leqslant c \left| \lambda^n_{\frac{t}{\sqrt{n}}} - e^{-\frac{t^2}{2}} \right| \sup_{|t'| \leqslant \delta} \left\| \widehat{F}(\cdot, t') \right\|_{\mathscr{B}^+_\alpha} \leqslant \frac{C}{\sqrt{n}} e^{-\frac{t^2}{4}} \int_{\mathbb{R}} \|F(\cdot, u)\|_{\mathscr{B}^+_\alpha} du.$$

For the second and third terms, using again Lemma 5.1, we obtain

$$|J_2(t)| + |J_3(t)| \leqslant C \left(\frac{|t|}{\sqrt{n}} e^{-\frac{t^2}{2}} + e^{-cn} \right) \int_{\mathbb{R}} \|F(\cdot, u)\|_{\mathscr{B}^+_\alpha} du.$$

Therefore, we obtain the following upper bound for I_1:

$$|I_1| \leqslant \left(\frac{C}{\sqrt{n}} + C e^{-cn} \right) \int_{\mathbb{R}} \|F(\cdot, u)\|_{\mathscr{B}^+_\alpha} du \leqslant \frac{C}{\sqrt{n}} \int_{\mathbb{R}} \|F(\cdot, u)\|_{\mathscr{B}^+_\alpha} du. \tag{5.6}$$

Estimate of I_2. Since the function $\widehat{F}(z, \cdot)$ is compactly supported on $K \subset [-C_1, C_1]$, where $C_1 > 0$ is a constant not depending on $z \in \mathbb{X}^+$, we have

$$\begin{aligned} I_2 &= \frac{1}{2\pi} \int_{\mathbb{X}^-_z} \left[\int_{\delta\sqrt{n} < |t| \leqslant C_1 \sqrt{n}} e^{-\frac{\mathbf{i}t}{\sqrt{n}} \check{S}_n g(y \cdot z)} \widehat{F}\left((T^{-n} y \cdot z)_+, \frac{t}{\sqrt{n}} \right) dt \right] \nu^-_z(dy) \\ &= \frac{\sqrt{n}}{2\pi} \int_{\delta < |t| \leqslant C_1} \left[\mathcal{L}^n_{\psi + \mathbf{i}tf} \widehat{F}(\cdot, t)(z) \right] dt. \end{aligned}$$

Note that, for any t satisfying $\delta < |t| \leqslant C_1$,

$$\sup_{z \in \mathbb{X}^+} \left| \mathcal{L}^n_{\psi+\mathbf{i}tf} \widehat{F}(\cdot, t)(z) \right| \leqslant \left\| \mathcal{L}^n_{\psi+\mathbf{i}tf} \widehat{F}(\cdot, t) \right\|_{\mathscr{B}^+_\alpha} \leqslant \left\| \mathcal{L}^n_{\psi+\mathbf{i}tf} \right\|_{\mathcal{L}(\mathscr{B}^+_\alpha, \mathscr{B}^+_\alpha)} \|\widehat{F}(\cdot, t)\|_{\mathscr{B}^+_\alpha}.$$

Then, by Lemma 5.2, it follows that

$$\begin{aligned} |I_2| &= \frac{1}{2\pi} \int_{\delta < |t| \leqslant C_1} \sqrt{n} \left\| \mathcal{L}^n_{\psi+\mathbf{i}tf} \right\|_{\mathcal{L}(\mathscr{B}^+_\alpha, \mathscr{B}^+_\alpha)} dt \sup_{|t'| \in [\delta, C_1]} \|\widehat{F}(\cdot, t')\|_{\mathscr{B}^+_\alpha} \\ &\leqslant c'_K \sqrt{n} e^{-c_K n} \sup_{|t'| \in [\delta, C_1]} \|\widehat{F}(\cdot, t')\|_{\mathscr{B}^+_\alpha} \leqslant c'_K e^{-c_K n} \int_{\mathbb{R}} \|F(\cdot, t)\|_{\mathscr{B}^+_\alpha} dt. \end{aligned} \tag{5.7}$$

Estimate of I_3. Notice that

$$\begin{aligned} I_3 &= \frac{1}{2\pi} \int_{\mathbb{R}} \left[e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z, \frac{t}{\sqrt{n}}\right) \nu^+(dz) \right] dt \\ &\quad - \frac{1}{2\pi} \int_{|t| > \delta\sqrt{n}} \left[e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z, \frac{t}{\sqrt{n}}\right) \nu^+(dz) \right] dt. \end{aligned}$$

For the first term, by the Fourier inversion formula,

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z, \frac{t}{\sqrt{n}}\right) \nu^+(dz)dt = \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{X}^+}\int_{\mathbb{R}} e^{-\frac{t^2}{2n}} F\left(z,t\right) dt \nu^+(dz). \tag{5.8}$$

For the second term, using the fact that $\widehat{F}\left(z, \frac{t}{\sqrt{n}}\right) \leqslant \int_{\mathbb{R}} |F(z,u)| du$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|t|>\delta\sqrt{n}} \left[e^{-\frac{t^2}{2}} \int_{\mathbb{X}^+} \widehat{F}\left(z, \frac{t}{\sqrt{n}}\right) \nu^+(dz)\right] dt \\
&\leqslant \frac{1}{2\pi}\int_{|t|>\delta\sqrt{n}} e^{-\frac{t^2}{2}} dt \int_{\mathbb{X}^+\times\mathbb{R}} |F(z,u)| du \nu^+(dz) \leqslant c e^{-\frac{\delta^2}{4}n} \int_{\mathbb{R}} \|F(\cdot,u)\|_{\mathscr{B}_\alpha^+} du. \qquad (5.9)
\end{aligned}$$

Combining (5.6), (5.7), (5.8) and (5.9), and taking into account that δ is a fixed constant, we conclude the proof of (5.5). □

5.3. **Local limit theorem for ε-dominated target functions.** Let $\varepsilon > 0$. Let f, g be functions on $\mathbb{R}$. We say that the function g ε-dominates the function f (or f ε-minorates g) if for any $t \in \mathbb{R}$, it holds that

$$f(t) \leqslant g(t+v), \quad \forall\ |v| \leqslant \varepsilon.$$

In this case we write $f \leqslant_\varepsilon g$ or $g \geqslant_\varepsilon f$. For any functions F and G on $\mathbb{X}^+ \times \mathbb{R}$, we say that $F \leqslant_\varepsilon G$ if $F(z,\cdot) \leqslant_\varepsilon G(z,\cdot)$ for any $z \in \mathbb{X}^+$.

In the proofs we make use of the following assertion. Denote by ρ the non-negative density function on $\mathbb{R}$, which is the Fourier transform of the function $(1-|t|)\mathbb{1}_{|t|\leqslant 1}$ for $t \in \mathbb{R}$. Set $\rho_\varepsilon(u) = \frac{1}{\varepsilon}\rho(\frac{u}{\varepsilon})$ for $u \in \mathbb{R}$ and $\varepsilon > 0$.

Lemma 5.5. *Let $\varepsilon \in (0, 1/4)$. Let $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R} \to \mathbb{R}_+$ be integrable functions satisfying $f \leqslant_\varepsilon g$. Then for any $u \in \mathbb{R}$,*

$$f(u) \leqslant \frac{1}{1-2\varepsilon} g * \rho_{\varepsilon^2}(u), \quad g(u) \geqslant f * \rho_{\varepsilon^2}(u) - \int_{|v|>\varepsilon} f\left(u-v\right) \rho_{\varepsilon^2}(v) dv.$$

Remark 5.6. The domination property $\leqslant_\varepsilon$ implies, in particular, that if $f \leqslant_\varepsilon g$ and the function g is integrable, then f is bounded and $\lim_{u\to\infty} f(u) = 0$, $\lim_{u\to-\infty} f(u) = 0$. Indeed, since $f \leqslant_\varepsilon g$ and g is an integrable function, by Lemma 5.5 we have $f \leqslant \frac{1}{1-2\varepsilon} g * \rho_{\varepsilon^2}$. Since the Fourier transform of $g * \rho_{\varepsilon^2}$ is compactly supported on $[-\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^2}]$, by the Fourier inversion formula,

$$|g * \rho_{\varepsilon^2}(x)| = \left| \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} \widehat{g}(t) \widehat{\rho}_{\varepsilon^2}(t) dt \right| \leqslant c.$$

Therefore, $g * \rho_{\varepsilon^2}$ is bounded on $\mathbb{R}$, so that f is bounded on $\mathbb{R}$.

Below, for any function $F \in \mathscr{H}_\alpha^+$, we use the notation

$$F * \rho_{\varepsilon^2}(z,t) = \int_{\mathbb{R}} F(z, t-v) \rho_{\varepsilon^2}(v) dv, \quad z \in \mathbb{X}^+,\ t \in \mathbb{R},$$

and

$$\|F\|_{\mathscr{H}_\alpha^+} = \int_{\mathbb{R}} \|F(\cdot,u)\|_{\mathscr{B}_\alpha^+} du, \qquad \|F\|_{\nu^+\otimes \mathrm{Leb}} = \int_{\mathbb{R}}\int_{\mathbb{X}^+} |F(z,u)| \nu^+(dz) du.$$

The following properties will be useful in the proofs:

Lemma 5.7. *Let $F \in \mathscr{H}_\alpha^+$ and $\rho \in L^1(\mathbb{R})$. Then $F * \rho \in \mathscr{H}_\alpha^+$ and $\|F * \rho\|_{\mathscr{H}_\alpha^+} \leqslant \|F\|_{\mathscr{H}_\alpha^+} \|\rho\|_{L^1(\mathbb{R})}$.*

Theorem 5.8. *Let $\alpha \in (0,1)$ and $g \in \mathscr{B}_\alpha^+$ be such that $\nu^+(g) = 0$. Assume that for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pg + q$ is not cohomologous to a function with values in $\mathbb{Z}$. There exists $c > 0$ with the following property: for any $\varepsilon \in (0, \frac{1}{8})$, there exists a constant $c_\varepsilon > 0$ such that for any non-negative function F and any function $G \in \mathscr{H}_\alpha^+$ satisfying $F \leqslant_\varepsilon G$, $n \geqslant 1$ and $z \in \mathbb{X}^+$,*

$$\begin{aligned}\int_{\mathbb{X}_z^-} & F\left((T^{-n}y \cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\leqslant \frac{1}{\sqrt{n}} \int_{\mathbb{X}^+}\int_{\mathbb{R}} \frac{1}{\sigma_g}\phi\left(\frac{u}{\sigma_g\sqrt{n}}\right) G\left(z', u\right) du \nu^+(dz') \\ &\quad + \frac{c\varepsilon}{\sqrt{n}}\|G\|_{\nu^+\otimes \mathrm{Leb}} + \frac{c_\varepsilon}{n}\|G\|_{\mathscr{H}_\alpha^+}, \end{aligned} \tag{5.10}$$

and for any non-negative function F and non-negative functions $G, H \in \mathscr{H}_\alpha^+$ satisfying $H \leqslant_\varepsilon F \leqslant_\varepsilon G$, $n \geqslant 1$ and $z \in \mathbb{X}^+$,

$$\begin{aligned}\int_{\mathbb{X}_z^-} & F\left((T^{-n}y \cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\geqslant \frac{1}{\sqrt{n}} \int_{\mathbb{X}^+}\int_{\mathbb{R}} \frac{1}{\sigma_g}\phi\left(\frac{u}{\sigma_g\sqrt{n}}\right) H\left(z', u\right) du \nu^+(dz') \\ &\quad - \frac{c\varepsilon}{\sqrt{n}}\|G\|_{\nu^+\otimes \mathrm{Leb}} - \frac{c_\varepsilon}{n}\left(\|G\|_{\mathscr{H}_\alpha^+} + \|H\|_{\mathscr{H}_\alpha^+}\right). \end{aligned} \tag{5.11}$$

Proof. Without loss of generality, we assume that $\sigma_g = 1$. We first prove the upper bound (5.10). By Lemma 5.5, we have $F \leqslant (1+4\varepsilon) G * \rho_{\varepsilon^2}$, and hence

$$\begin{aligned}\int_{\mathbb{X}_z^-} & F\left((T^{-n}y \cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\leqslant (1+4\varepsilon)\int_{\mathbb{X}_z^-} G * \rho_{\varepsilon^2}\left((T^{-n}y \cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy). \end{aligned} \tag{5.12}$$

By Lemma 5.7, $\widehat{G * \rho_{\varepsilon^2}} \in \mathscr{H}_\alpha^+$, and the support of the function $\widehat{G * \rho_{\varepsilon^2}}(z,\cdot) = \widehat{G}(z,\cdot)\widehat{\rho}_{\varepsilon^2}(\cdot)$. is included in $[-\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^2}]$, for all $z \in \mathbb{X}^+$. Using Theorem 5.4, for any $\varepsilon \in (0, \frac{1}{4})$, there exists $c_\varepsilon > 0$ such that for all $n \geqslant 1$ and $z \in \mathbb{X}^+$,

$$\begin{aligned}\int_{\mathbb{X}_z^-} & G * \rho_{\varepsilon^2}\left((T^{-n}y \cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\leqslant \frac{1}{\sqrt{n}} \int_{\mathbb{X}^+}\int_{\mathbb{R}} \phi\left(\frac{u}{\sqrt{n}}\right) G * \rho_{\varepsilon^2}\left(z, u\right) du \nu^+(dz) + \frac{c_\varepsilon}{n}\|G\|_{\mathscr{H}_\alpha^+}. \end{aligned} \tag{5.13}$$

By a change of variable and Fubini's theorem, we have for any $z \in \mathbb{X}^+$,

$$\int_{\mathbb{R}} \phi\left(\frac{u}{\sqrt{n}}\right) G * \rho_{\varepsilon^2}\left(z, u\right) du = \sqrt{n}\int_{\mathbb{R}} \phi_{\sqrt{n}} * \rho_{\varepsilon^2}\left(t\right) G(z,t)dt, \tag{5.14}$$

where $\phi_{\sqrt{n}}(t) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{t^2}{2n}}$, $t \in \mathbb{R}$. For brevity, denote $\psi(t) = \sup_{|v|\leqslant \varepsilon} \phi_{\sqrt{n}}(t+v)$, $t \in \mathbb{R}$. Using the second inequality in Lemma 5.5, we have

$$\begin{aligned}\int_{\mathbb{R}} & \phi_{\sqrt{n}} * \rho_{\varepsilon^2}\left(t\right) G(z,t)dt \\ &\leqslant \int_{\mathbb{R}} \psi\left(t\right) G(z,t)dt + \int_{\mathbb{R}}\int_{|v|\geqslant\varepsilon} \phi_{\sqrt{n}}\left(t-v\right)\rho_{\varepsilon^2}(v)dv G(z,t)dt =: J_1 + J_2. \end{aligned}$$

For J_1, by Taylor's expansion and the fact that the function ϕ' is bounded on $\mathbb{R}$, we derive that

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{n}} \left[\int_{-\infty}^{-\varepsilon} \phi\left(\frac{t+\varepsilon}{\sqrt{n}}\right) G(z,t)dt + \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} G(z,t)dt + \int_{\varepsilon}^{\infty} \phi\left(\frac{t-\varepsilon}{\sqrt{n}}\right) G(z,t)dt \right] \\ &\leqslant \frac{1}{\sqrt{n}} \int_{\mathbb{R}} \phi\left(\frac{t}{\sqrt{n}}\right) G(z,t)dt + \frac{c\varepsilon}{\sqrt{n}} \int_{\mathbb{R}} G(z,t)dt. \end{aligned} \tag{5.15}$$

For J_2, since $\phi_{\sqrt{n}} \leqslant \frac{1}{\sqrt{n}}$ and $\int_{|v|\geqslant \varepsilon} \rho_{\varepsilon^2}(v)dv \leqslant 2\varepsilon$, we get

$$J_2 \leqslant \frac{1}{\sqrt{n}} \int_{\mathbb{R}} \left(\int_{|v|\geqslant\varepsilon} \rho_{\varepsilon^2}(v)dv \right) G(z,t)dt \leqslant \frac{2\varepsilon}{\sqrt{n}} \int_{\mathbb{R}} G(z,t)dt. \tag{5.16}$$

From (5.15) and (5.16), together with (5.12) and (5.13), we get (5.10).

Now we prove the lower bound (5.11). Since $F \geqslant_{\varepsilon} H$, using the second inequality in Lemma 5.5, we get

$$\begin{aligned} &\int_{\mathbb{X}_z^-} F\left((T^{-n}y\cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\geqslant \int_{\mathbb{X}_z^-} H * \rho_{\varepsilon^2}\left((T^{-n}y\cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\quad - \int_{\mathbb{X}_z^-} \int_{|v|\geqslant\varepsilon} H\left((T^{-n}y\cdot z)_+, \check{S}_n g(y\cdot z) - v\right) \rho_{\varepsilon^2}(v) dv \nu_z^-(dy). \end{aligned} \tag{5.17}$$

For the first term, by Theorem 5.4, for any $\varepsilon > 0$, there exists $c > 0$ such that for all $n \geqslant 1$ and $z \in \mathbb{X}^+$,

$$\begin{aligned} &\int_{\mathbb{X}_z^-} H * \rho_{\varepsilon^2}\left((T^{-n}y\cdot z)_+, \check{S}_n g(y\cdot z)\right) \nu_z^-(dy) \\ &\geqslant \frac{1}{\sqrt{n}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} \phi\left(\frac{u}{\sqrt{n}}\right) H * \rho_{\varepsilon^2}(z,u)\, du \nu^+(dz) - \frac{c_\varepsilon}{n} \int_{\mathbb{R}} \|H(\cdot,u)\|_{\mathscr{B}_\alpha^+} du. \end{aligned} \tag{5.18}$$

In the same way as in (5.14), we have

$$\int_{\mathbb{R}} \phi\left(\frac{u}{\sqrt{n}}\right) H * \rho_{\varepsilon^2}(z,u)\, du = \sqrt{n} \int_{\mathbb{R}} \phi_{\sqrt{n}} * \rho_{\varepsilon^2}(t)\, H(z,t)dt. \tag{5.19}$$

Using the first inequality in Lemma 5.5, we have $\phi_{\sqrt{n}} * \rho_{\varepsilon^2}(t) \geqslant (1-2\varepsilon)\psi(t)$, for $t \in \mathbb{R}$, where $\psi(t) = \inf_{|v|\leqslant\varepsilon} \phi_{\sqrt{n}}(t+v)$. Proceeding in the same way as in (5.15) and (5.16), we obtain that

$$\begin{aligned} &\int_{\mathbb{X}^+} \int_{\mathbb{R}} \phi\left(\frac{u}{\sqrt{n}}\right) H * \rho_{\varepsilon^2}(z,u)\, du \nu^+(dz) \\ &\geqslant \int_{\mathbb{X}^+} \int_{\mathbb{R}} \phi\left(\frac{u}{\sqrt{n}}\right) H(z,u)\, du \nu^+(dz) - c\varepsilon \int_{\mathbb{X}^+} \int_{\mathbb{R}} H(z,u) du \nu^+(dz). \end{aligned} \tag{5.20}$$

For the second term on the right-hand side of (5.17), using (5.10) and the fact that $H \leqslant_{2\varepsilon} G$ and $\phi \leqslant 1$, we get that there exist constants $c, c_\varepsilon > 0$ such that for any $v \in \mathbb{R}$ and $n \geqslant 1$,

$$\begin{aligned} &\int_{\mathbb{X}_z^-} H\left((T^{-n}y\cdot z)_+, \check{S}_n g(y\cdot z) - v\right) \nu_z^-(dy) \\ &\leqslant \frac{c}{\sqrt{n}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} G(z,u)\, du \nu^+(dz) + \frac{c_{2\varepsilon}}{n} \int_{\mathbb{R}} \|G(\cdot,u)\|_{\mathscr{B}_\alpha^+} du. \end{aligned}$$

This, together with the fact that $\int_{|v|\geqslant\varepsilon}\rho_{\varepsilon^2}(v)dv \leqslant 2\varepsilon$, implies

$$\begin{aligned}
&\int_{\mathbb{X}_z^-}\int_{|v|\geqslant\varepsilon} H\left((T^{-n}y\cdot z)_+, \check{S}_n g(y\cdot z)-v\right)\rho_{\varepsilon^2}(v)dv\nu_z^-(dy) \\
&\leqslant \frac{2c\varepsilon}{\sqrt{n}}\int_{\mathbb{X}^+}\int_{\mathbb{R}} G(z,u)\,du\nu^+(dz) + \frac{c_{2\varepsilon}}{n}\int_{\mathbb{R}}\|G(\cdot,u)\|_{\mathscr{B}_\alpha^+}du. \qquad (5.21)
\end{aligned}$$

From (5.17), (5.18), (5.20) and (5.21), we obtain the lower bound (5.11). □

6. Effective conditioned local limit theorems

6.1. **Formulation of the result.** We will prove the following conditioned local limit theorem for Markov chains which provides a rate of order n^{-1}. This result will serve as an intermediate step between the conditioned central limit Theorem 1.5 and the conditioned local limit Theorem 1.7. The interest of this result lies in the fact that it is uniform in the function F. In particular, the theorem is effective when the support of the function F moves to infinity with the rate $\sqrt{n}$. This strategy is inspired by [7] for random walks in cones of $\mathbb{R}^d$, see also [16] for finite Markov chains and [18] for random walks on $\mathbb{R}$. For a different approach based on the Wiener-Hopf factorisation we refer to [6, 10, 32].

Theorem 6.1. *Let $\alpha \in (0,1)$ and $g \in \mathscr{B}_\alpha^+$ be such that $\nu^+(g)=0$. Assume that for any $p\neq 0$ and $q\in\mathbb{R}$, the function $pg+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Let $t_0\in\mathbb{R}_+$. Then, there exist a constant $c>0$ and a sequence (r_n) of positive numbers satisfying $\lim_{n\to\infty} r_n = 0$ with the following properties.*

1. *For any $\varepsilon\in(0,\frac{1}{8})$, there exists a constant $c_\varepsilon>0$ such that for any $n\geqslant 1$, $z\in\mathbb{X}^+$, $t\leqslant t_0$, any functions $F,G:\mathbb{X}^+\times\mathbb{R}\to\mathbb{R}_+$ satisfying $F\leqslant_\varepsilon G$, $G\in\mathscr{H}_\alpha^+$,*

$$\begin{aligned}
&n\int_{\mathbb{X}_z^-} F\left((T^{-n}y\cdot z)_+, t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy) \\
&\leqslant \frac{2\check{V}^g(z,t)}{\sigma_g^2\sqrt{2\pi}}\int_{\mathbb{X}^+}\int_{\mathbb{R}} G(z',u')\,\phi^+\left(\frac{u'}{\sigma_g\sqrt{n}}\right)du'\nu^+(dz') \\
&+ c\left(\varepsilon^{1/4}+\frac{r_n}{\varepsilon^{1/4}}\right)\|G\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{\sqrt{n}}\|G\|_{\mathscr{H}_\alpha^+}. \qquad (6.1)
\end{aligned}$$

2. *For any $\varepsilon\in(0,\frac{1}{8})$, there exists a constant $c_\varepsilon>0$ such that for any $n\geqslant 1$, $z\in\mathbb{X}^+$, $t\leqslant t_0$, any functions $F,G,H:\mathbb{X}^+\times\mathbb{R}\to\mathbb{R}_+$ satisfying $H\leqslant_\varepsilon F\leqslant_\varepsilon G$, $G,H\in\mathscr{H}_\alpha^+$,*

$$\begin{aligned}
&n\int_{\mathbb{X}_z^-} F\left((T^{-n}y\cdot z)_+, t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy) \\
&\geqslant \frac{2\check{V}^g(z,t)}{\sigma_g^2\sqrt{2\pi}}\int_{\mathbb{X}^+}\int_{\mathbb{R}} H(z',u')\phi^+\left(\frac{u'}{\sigma_g\sqrt{n}}\right)du'\nu^+(dz') \\
&- c\left(\varepsilon^{1/12}+\frac{r_n}{\varepsilon^{1/4}}\right)\|G\|_{\nu^+\otimes\mathrm{Leb}} - \frac{c_\varepsilon}{\sqrt{n}}\left(\|G\|_{\mathscr{H}_\alpha^+}+\|H\|_{\mathscr{H}_\alpha^+}\right). \qquad (6.2)
\end{aligned}$$

6.2. **Preparatory statements.** The normal density of variance $v>0$ is denoted by

$$\phi_v(x) = \frac{1}{\sqrt{2\pi v}}e^{-\frac{x^2}{2v}}, \quad x\in\mathbb{R},$$

and the Rayleigh density with scale parameter $\sqrt{v}$ is denoted by

$$\phi_v^+(x) = \frac{x}{v} e^{-\frac{x^2}{2v}} \mathbb{1}_{\mathbb{R}_+}(x), \quad x \in \mathbb{R}.$$

The standard normal density is denoted by $\phi(x) = \phi_1(x)$, $x \in \mathbb{R}$. The following lemma from [18] shows that when v is small the convolution $\phi_v * \phi_{1-v}^+$ behaves like the Rayleigh density.

Lemma 6.2. *For any $v \in (0, 1/2]$ and $x \in \mathbb{R}$, it holds*

$$-|x| e^{-\frac{x^2}{2}} \mathbb{1}_{\{x<0\}} \leqslant \phi_v * \phi_{1-v}^+(x) - \sqrt{1-v}\phi^+(x) \leqslant \sqrt{v} e^{-\frac{x^2}{2v}} + |x| e^{-\frac{x^2}{2}} \mathbb{1}_{\{x<0\}}.$$

We need the following inequality of Haeusler [19, Lemma 1], which is a generalisation of Fuk's inequality for martingales.

Lemma 6.3. *Let $\xi_1, \ldots, \xi_n$ be a martingale difference sequence with respect to the non-decreasing σ-fields $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_n$. Then, for all $u, v, w > 0$,*

$$\mathbb{P}\left(\max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^k \xi_i \right| \geqslant u \right) \leqslant 2 \exp\left\{ \frac{u}{v} \left(1 - \log \frac{uv}{w} \right) \right\} + \sum_{i=1}^n \mathbb{P}(|\xi_i| > v) + 2\mathbb{P}\left(\sum_{i=1}^n \mathbb{E}\left(\xi_i^2 | \mathscr{F}_{i-1} \right) > w \right). \tag{6.3}$$

Using this lemma we establish the following Fuk-type inequality involving a target function on the Markov chain $((T^{-n} y \cdot z)_+)_{n \geqslant 0}$.

Lemma 6.4. *Let $\alpha \in (0,1)$, $g \in \mathscr{B}_\alpha^+$ such that $\nu^+(g) = 0$ and $\varphi \in \mathscr{B}_\alpha^+$ be nonnegative. There exist constants $c, c', c_0 > 0$ such that for any $M > c_0$ and $n \geqslant 1$,*

$$\begin{aligned} I &:= \int_{\mathbb{X}_z^-} \left[\varphi\left((T^{-n} y \cdot z)_+ \right) \mathbb{1}_{\left\{ \max_{1 \leqslant j \leqslant n} |\check{S}_j g(y \cdot z)| \geqslant M\sqrt{n} \right\}} \right] \nu_z^-(dy) \\ &\leqslant 2\nu^+(\varphi) \exp(-cM) + c' e^{-cn^{1/3}} \|\varphi\|_{\mathscr{B}_\alpha^+}. \end{aligned}$$

Proof. By Propositions 2.6, 2.7 and Lemma 2.8 , there exists a Hölder continuous function g_0 on $\mathbb{X}^+$ satisfying $\mathcal{L}_\psi g_0 = 0$ such that $\{y \mapsto \check{S}_k g_0((y \cdot z)_+)\}_{k \geqslant 0}$ is a martingale on $\mathbb{X}_z^-$ and $\sup_{k \geqslant 0} \|\check{S}_k g_0 - \check{S}_k g\|_\infty \leqslant c$ for some constant $c > 0$. Let $c_0 = 1 + \|g_0\|_\infty < \infty$. In addition, with $p = n - [n^{1/3}]$, we have $\max_{1 \leqslant j \leqslant n} |\check{S}_j g_0(x')| \leqslant \max_{1 \leqslant j \leqslant p} |\check{S}_j g_0(x')| + c_0 n^{1/3}$. With these properties, it holds that, for n large enough,

$$I \leqslant J_n(z) := \int_{\mathbb{X}} \left[\varphi\left((T^{-n} y \cdot z)_+ \right) \mathbb{1}_{\left\{ \max_{1 \leqslant j \leqslant p} |\check{S}_j g_0(y \cdot z)| \geqslant \frac{1}{2} M\sqrt{n} \right\}} \right] \nu_z^-(dy). \tag{6.4}$$

Note that $\mathcal{L}_\psi^k \varphi(z) = \int_{\mathbb{X}_z^-} \varphi\left((T^{-k} y \cdot z)_+ \right) \nu_z^-(dy)$ with $k \geqslant 1$, where $\mathcal{L}_\psi^k$ is defined by (2.3). Moreover, by Lemma 5.1 with $t = 0$, for $k \geqslant 1$,

$$\sup_{z \in \mathbb{X}^+} \left| \mathcal{L}_\psi^k \varphi(z) - \nu^+(\varphi) \right| \leqslant c' e^{-ck} \|\varphi\|_{\mathscr{B}_\alpha^+}.$$

By the Markov property, we have that for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\begin{aligned} J_n(z) &= \int_{\mathbb{X}_z^-} \mathcal{L}_\psi^{[n^{1/3}]} \varphi\left((T^{-p} y \cdot z)_+ \right) \mathbb{1}_{\left\{ \max_{1 \leqslant j \leqslant p} |\check{S}_j g_0(y \cdot z)| \geqslant \frac{1}{2} M\sqrt{n} \right\}} \nu_z^-(dy) \\ &\leqslant \nu^+(\varphi) \nu_z^-\left(y \in \mathbb{X}_z^- : \max_{1 \leqslant j \leqslant p} |\check{S}_j g_0(y \cdot z)| \geqslant \frac{1}{2} M\sqrt{n} \right) + c' e^{-cn^{1/3}} \|\varphi\|_{\mathscr{B}_\alpha^+}. \end{aligned} \tag{6.5}$$

We apply Fuk's inequality for martingales (Lemma 6.3) with $u = \frac{1}{2}M\sqrt{n}$, $v = c_0^2\sqrt{n}$ and $w = \frac{c_0^2}{8}Mn$, so that the second and the third terms in the right-hand side of (6.3) vanish. This gives

$$\nu_z^- \left(y \in \mathbb{X}_z^- : \max_{1\leqslant j\leqslant p} |\check{S}_j g_0(y\cdot z)| \geqslant \frac{1}{2}M\sqrt{n} \right) \leqslant \nu_z^- \left(y \in \mathbb{X}_z^- : \max_{1\leqslant j\leqslant n} |\check{S}_j g_0(y\cdot z)| \geqslant \frac{1}{2}M\sqrt{n} \right)$$
$$\leqslant 2\exp\left(\frac{u}{v}\left(1 - \log\frac{uv}{w}\right)\right) = \exp\left(-\frac{M}{2c_0^2}\log\frac{4}{e}\right).$$

This ends the proof of the lemma. □

In order to control certain natural quantities appearing in the proof, we shall need the following definitions. For $\varepsilon > 0$,

$$\chi_\varepsilon(u) = 0 \text{ for } u \leqslant -\varepsilon,\ \chi_\varepsilon(u) = \frac{u+\varepsilon}{\varepsilon} \text{ for } u \in (-\varepsilon, 0),\ \chi_\varepsilon(u) = 1 \text{ for } u \geqslant 0. \tag{6.6}$$

Denote $\overline{\chi}_\varepsilon(u) = 1 - \chi_\varepsilon(u)$ and note that

$$\chi_\varepsilon(t-\varepsilon) \leqslant \mathbb{1}_{(0,\infty)}(t) \leqslant \chi_\varepsilon(t), \quad \overline{\chi}_\varepsilon(t) \leqslant \mathbb{1}_{(-\infty,0]}(t) \leqslant \overline{\chi}_\varepsilon(t-\varepsilon). \tag{6.7}$$

Lemma 6.5. *Let $\alpha \in (0,1)$ and $g \in \mathscr{B}_\alpha^+$ be such that $\nu^+(g) = 0$. Assume that g is not a coboundary. Let κ be a smooth compactly supported function on $\mathbb{R}$ and $\varepsilon > 0$. Then there exists a constant $c > 0$ such that for any $G \in \mathscr{H}_\alpha^+$ and any $m \geqslant 1$, the function A_m defined on $\mathbb{X}^+ \times \mathbb{R}$ by*

$$A_m(z,t) := \int_{\mathbb{X}_z^-} G * \kappa\left((T^{-m}y\cdot z)_+, t + \check{S}_m g(y\cdot z)\right)$$
$$\times\, \overline{\chi}_\varepsilon\left(t - \varepsilon + \min_{1\leqslant j\leqslant m} \check{S}_j g(y\cdot z)\right) \nu_z^-(dy),$$

belongs to $\mathscr{H}_\alpha^+$ and satisfies

$$\|A_m\|_{\nu^+\otimes\mathrm{Leb}} \leqslant \int_{\mathbb{R}} |\kappa(t)|dt \|G\|_{\nu^+\otimes\mathrm{Leb}}, \qquad \|A_m\|_{\mathscr{H}_\alpha^+} \leqslant \frac{c}{\varepsilon}\|G\|_{\mathscr{H}_\alpha^+}.$$

Proof. For the first inequality, we write

$$|A_m(z,t)| \leqslant \int_{\mathbb{X}_z^-} |G*\kappa|\left((T^{-m}y\cdot z)_+, t + \check{S}_m g(y\cdot z)\right) \nu_z^-(dy),$$

which gives

$$\|A_m\|_{\nu^+\otimes\mathrm{Leb}} \leqslant \int_{\mathbb{X}\times\mathbb{R}} |G*\kappa|\left((T^{-m}x)_+, t + \check{S}_m g(x)\right) \nu(dx)dt$$
$$= \int_{\mathbb{X}\times\mathbb{R}} |G*\kappa|\,(x_+, t)\,\nu(dx)dt \leqslant \int_{\mathbb{R}} |\kappa(t)|dt\|G\|_{\nu^+\otimes\mathrm{Leb}}.$$

This finishes the proof of the first inequality.

For the second inequality, recall that

$$\|A_m\|_{\mathscr{H}_\alpha^+} = \int_{\mathbb{R}} \sup_{z\in\mathbb{X}^+} |A_m(z,t)|dt + \int_{\mathbb{R}} \sup_{z,z'\in\mathbb{X}^+} \frac{|A_m(z,t) - A_m(z',t)|}{\alpha^{\omega(z,z')}}dt.$$

We pick $c_0 > 0$ as in Lemma 2.9 and for $t \in \mathbb{R}$ we set $\kappa_1(t) = \sup_{|s|\leqslant c_0} |\kappa(t+s)|$ and $H(t) = \sup_{z\in\mathbb{X}^+} |G(z,t)|$. We get for $z, z' \in \mathbb{X}^+$ with $z_0 = z_0'$ and $t \in \mathbb{R}$,

$$|A_m(z,t)| \leqslant \int_{\mathbb{X}_z^-} H * \kappa_1\left(t + \check{S}_m g(y\cdot z')\right) \nu_z^-(dy).$$

By Lemma 2.3, we get

$$|A_m(z,t)| \leqslant c \int_{\mathbb{X}^-_{z'}} H * \kappa_1 \left(t + \check{S}_m g(y \cdot z')\right) \nu^-_{z'}(dy),$$

for some constant c. By integrating over z', we get

$$|A_m(z,t)| \leqslant c' \int_{\mathbb{X}} H * \kappa_1 \left(t + \check{S}_m g(x)\right) \nu(dx).$$

By integrating over t, it follows that

$$\int_{\mathbb{R}} \sup_{z \in \mathbb{X}^+} |A_m(z,t)| dt \leqslant c' \int_{\mathbb{R}} H * \kappa_1 (t)\, dt = c' \int_{\mathbb{R}} |\kappa_1(t)| dt \int_{\mathbb{R}} H(t) dt.$$

Now we dominate the second term in the norm $\|A_m\|_{\mathscr{H}^+_\alpha}$. For $t \in \mathbb{R}$, set $\kappa_2(t) = \sup_{|s| \leqslant c_0} |\kappa'(t+s)|$, where c_0 is the constant from Lemma 2.9. We get for $|t - t'| \leqslant c_0$ and $z \in \mathbb{X}^+$,

$$|G * \kappa(z,t) - G * \kappa(z,t')| \leqslant |t - t'| H * \kappa_2(t).$$

Hence for $z, z', z'' \in \mathbb{X}^+$ with $z_0 = z'_0 = z''_0$ and $t \in \mathbb{R}$,

$$\begin{aligned} I_1(z,z',t) := \Bigg| A_m(z,t) - \int_{\mathbb{X}^-_z} & G * \kappa \left((T^{-m} y \cdot z)_+, t + \check{S}_m g(y \cdot z')\right) \\ & \times \overline{\chi}_\varepsilon \left(t - \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) \nu^-_z(dy) \Bigg| \\ \leqslant c\alpha^{\omega(z,z')} & \int_{\mathbb{X}^-_z} H * \kappa_2 \left(t + \check{S}_m g(y \cdot z)\right) \nu^-_z(dy) \\ \leqslant c\alpha^{\omega(z,z')} & \int_{\mathbb{X}^-_z} H * \kappa_3 \left(t + \check{S}_m g(y \cdot z'')\right) \nu^-_z(dy) \\ \leqslant c_1\alpha^{\omega(z,z')} & \int_{\mathbb{X}^-_{z''}} H * \kappa_3 \left(t + \check{S}_m g(y \cdot z'')\right) \nu^-_{z''}(dy), \end{aligned}$$

where $\kappa_3(t) = \sup_{|s| \leqslant c_0} |\kappa_2(t+s)|$ with c_0 from Lemma 2.9; for the second inequality we have applied Lemma 2.9 and for the last inequality we have used Lemma 2.3. Again by integrating over z'', we get that

$$I_1(z,z',t) \leqslant c_2 \alpha^{\omega(z,z')} \int_{\mathbb{X}} H * \kappa_3 \left(t + \check{S}_m g(x)\right) \nu(dx). \tag{6.8}$$

Besides, as G is in $\mathscr{H}^+_\alpha$, the function $L(t) = \sup_{z,z' \in \mathbb{X}^+} \alpha^{-\omega(z.z')} |G(z,t) - G(z',t)|$ is integrable on $\mathbb{R}$ and for $z, z' \in \mathbb{X}^+$ with $z_0 = z'_0$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} I_2(z,z',t) := \Bigg| \int_{\mathbb{X}^-_z} \Big[& G * \kappa \left((T^{-m} y \cdot z)_+, t + \check{S}_m g(y \cdot z')\right) \\ & - G * \kappa \left((T^{-m} y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right) \Big] \\ & \times \overline{\chi}_\varepsilon \left(t - \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) \nu^-_z(dy) \Bigg| \\ \leqslant \alpha^{\omega(z.z')} & \int_{\mathbb{X}^-_z} L * \kappa \left(t + \check{S}_m g(y \cdot z')\right) \nu^-_z(dy) \\ \leqslant c\alpha^{\omega(z.z')} & \int_{\mathbb{X}} L * \kappa_1 \left(t + \check{S}_m g(x)\right) \nu(dx), \end{aligned} \tag{6.9}$$

where we have again used Lemmas 2.3 and 2.9.

As $\overline{\chi}_\varepsilon$ is $1/\varepsilon$-Lipschitz continuous on $\mathbb{R}$, by reasoning in the same way and using Corollary 2.10, we get

$$\begin{aligned} I_3(z,z',t) &:= \Bigg| \int_{\mathbb{X}_z^-} G * \kappa\left((T^{-m}y\cdot z')_+, t+\check{S}_m g(y\cdot z')\right) \\ &\times \left[\overline{\chi}_\varepsilon\left(t-\varepsilon+\min_{1\leqslant j\leqslant m}\check{S}_j g(y\cdot z)\right) - \overline{\chi}_\varepsilon\left(t-\varepsilon+\min_{1\leqslant j\leqslant m}\check{S}_j g(y\cdot z')\right)\right]\nu_z^-(dy)\Bigg| \\ &\leqslant \frac{c}{\varepsilon}\alpha^{\omega(z.z')}\int_{\mathbb{X}} H*\kappa_1\left(t+\check{S}_m g(x)\right)\nu(dx). \end{aligned} \tag{6.10}$$

By Lemma 2.3, we have

$$\begin{aligned} I_4(z,z',t) := \Bigg| \int_{\mathbb{X}_z^-} & G * \kappa\left((T^{-m}y\cdot z')_+, t+\check{S}_m g(y\cdot z')\right) \\ & \times \overline{\chi}_\varepsilon\left(t-\varepsilon+\min_{1\leqslant j\leqslant m}\check{S}_j g(y\cdot z')\right)\nu_z^-(dy) - A_m(z',t)\Bigg| \\ &\leqslant c\alpha^{\omega(z.z')}\int_{\mathbb{X}} H*\kappa_1\left(t+\check{S}_m g(x)\right)\nu(dx). \end{aligned} \tag{6.11}$$

Putting (6.8), (6.9), (6.10) and (6.11) together, and integrating over $t\in\mathbb{R}$, yields the required domination. □

6.3. **Proof of the upper bound.** We prove the inequality (6.1) in Theorem 6.1. It is enough to prove (6.1) only for sufficiently large $n > n_0(\varepsilon)$, where $n_0(\varepsilon)$ depends on ε, otherwise the bound becomes trivial.

Without loss of generality, we assume that $\sigma_g = 1$. Let $\varepsilon\in(0,\frac{1}{8})$. With $\delta=\sqrt{\varepsilon}$, set $m=[\delta n]$ and $k=n-m$. Note that $\frac{1}{2}\delta\leqslant\frac{m}{k}\leqslant\frac{\delta}{1-\delta}$ for $n\geqslant\frac{2}{\sqrt{\varepsilon}}$. Denote, for $z\in\mathbb{X}^+$ and $t\in\mathbb{R}$,

$$\Psi_n(z,t) = \int_{\mathbb{X}_z^-} F\left((T^{-n}y\cdot z)_+, t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n\right\}}\nu_z^-(dy).$$

By the Markov property, we have that for any $z\in\mathbb{X}^+$ and $t\in\mathbb{R}$,

$$\Psi_n(z,t) = \int_{\mathbb{X}_z^-} \Psi_m\left((T^{-k}y\cdot z)_+, t+\check{S}_k g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>k\right\}}\nu_z^-(dy). \tag{6.12}$$

By bounding the indicator function by $\mathbb{1}_{\left\{t+\check{S}_m g(y\cdot z)\geqslant 0\right\}}$ in the definition of Ψ_m, we get

$$\Psi_m(z,t) \leqslant \int_{\mathbb{X}_z^-} F\left((T^{-m}y\cdot z)_+, t+\check{S}_m g(y\cdot z)\right)\mathbb{1}_{\left\{t+\check{S}_m g(y\cdot z)\geqslant 0\right\}}\nu_z^-(dy) =: J_m(z,t). \tag{6.13}$$

Let $G_\varepsilon(z,u) = G(z,u)\chi_\varepsilon(u-\varepsilon)$ for $z\in\mathbb{X}^+$ and $u\in\mathbb{R}$, where $\varepsilon\in(0,1)$ and χ_ε is defined in (6.6). By the local limit theorem (cf. Theorem 5.8), there exist constants $c, c_\varepsilon > 0$ such that for any $m\geqslant 1$, $z\in\mathbb{X}^+$ and $t\in\mathbb{R}$,

$$J_m(z,t) \leqslant H_m(t) + \frac{c\varepsilon}{\sqrt{m}}\|G_\varepsilon\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{m}\|G_\varepsilon\|_{\mathscr{H}_\alpha^+}, \tag{6.14}$$

where for brevity we set

$$H_m(t) = \int_{\mathbb{X}^+}\int_{\mathbb{R}} G_\varepsilon(z,u)\frac{1}{\sqrt{m}}\phi\left(\frac{u-t}{\sqrt{m}}\right)du\,\nu^+(dz). \tag{6.15}$$

Using (6.12), (6.13) and (6.14), and Lemma 4.1, we get that uniformly in $z \in \mathbb{X}^+$ and $t \leqslant t_0$,

$$\Psi_n(z,t) \leqslant \int_{\mathbb{X}_z^-} H_m\left(t + \check{S}_k g(y \cdot z)\right) \mathbb{1}_{\{\check{\tau}_t^g(y\cdot z)>k\}} \nu_z^-(dy) + \frac{c\varepsilon}{\sqrt{mk}} \|G_\varepsilon\|_{\nu^+\otimes \mathrm{Leb}} + \frac{c_\varepsilon}{m\sqrt{k}} \|G_\varepsilon\|_{\mathscr{H}_\alpha^+}. \tag{6.16}$$

Now we deal with the first term on the right-hand side of (6.16). Denote $L_m(s) = H_m(\sqrt{k}s)$ for $s \in \mathbb{R}$. We have

$$L_m(s) = \int_{\mathbb{X}^+} \int_{\mathbb{R}} G_\varepsilon\left(z, \sqrt{k}u\right) \frac{1}{\sqrt{m/k}} \phi\left(\frac{s-u}{\sqrt{m/k}}\right) du \nu^+(dz). \tag{6.17}$$

Since the function $s \mapsto L_m(s)$ is differentiable on $\mathbb{R}$ and vanishes as $s \to +\infty$, using integration by parts, we have, for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\begin{aligned} H_{m,k}(z,t) :&= \int_{\mathbb{X}_z^-} H_m\left(t + \check{S}_k g(y \cdot z)\right) \mathbb{1}_{\{\check{\tau}_t^g(y\cdot z)>k\}} \nu_z^-(dy) \\ &= \int_{\mathbb{X}_z^-} L_m\left(\frac{t + \check{S}_k g(y \cdot z)}{\sqrt{k}}\right) \mathbb{1}_{\{\check{\tau}_t^g(y\cdot z)>k\}} \nu_z^-(dy) \\ &= \int_{\mathbb{R}_+} L_m'(s) \nu_z^-\left(\frac{t + \check{S}_k g(y \cdot z)}{\sqrt{k}} > s, \check{\tau}_t^g(y\cdot z) > k\right) ds. \end{aligned} \tag{6.18}$$

Applying the conditioned central limit theorem (see Lemma 4.6), we have

$$H_{m,k}(z,t) \leqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi k}} \int_{\mathbb{R}_+} L_m'(s)\left(1 - \Phi^+(s)\right) ds + \frac{r_k}{k^{1/2}} \int_{\mathbb{R}_+} |L_m'(s)| ds, \tag{6.19}$$

where $r_k \to 0$ as $k \to \infty$ and by Φ^+ we denoted the Rayleigh cumulative distribution function (1.7). By (6.17), we have

$$\begin{aligned} \int_{\mathbb{R}_+} |L_m'(s)| ds &= \int_{\mathbb{X}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\varepsilon\left(z, \sqrt{m}\frac{u}{\sqrt{m/k}}\right) \phi'\left(\frac{s-u}{\sqrt{m/k}}\right) \frac{du}{\sqrt{m/k}} \frac{ds}{\sqrt{m/k}} \nu^+(dz) \\ &= \int_{\mathbb{X}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\varepsilon\left(z, \sqrt{m}u\right) \phi'\left(s-u\right) ds du \nu^+(dz) \\ &\leqslant c \int_{\mathbb{X}^+} \int_{\mathbb{R}} G_\varepsilon\left(z, \sqrt{m}u\right) du \nu^+(dz) = \frac{c}{\sqrt{m}} \|G_\varepsilon\|_{\nu^+\otimes \mathrm{Leb}}. \end{aligned} \tag{6.20}$$

By integration by parts and a change of variable, we have

$$\int_{\mathbb{R}_+} L_m'(s)\left(1 - \Phi^+(s)\right) ds = \int_{\mathbb{R}_+} H_m(s) \phi^+\left(\frac{s}{\sqrt{k}}\right) \frac{ds}{\sqrt{k}}.$$

Hence, from (6.18), (6.19) and (6.20), we obtain

$$H_{m,k}(z,t) \leqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi k}} \int_{\mathbb{R}_+} H_m(s) \phi^+\left(\frac{s}{\sqrt{k}}\right) \frac{ds}{\sqrt{k}} + \frac{r_k}{\sqrt{km}} \|G_\varepsilon\|_{\nu^+\otimes \mathrm{Leb}}. \tag{6.21}$$

Implementing this bound into (6.16) and using the fact that $\|G_\varepsilon\|_{\nu^+\otimes\mathrm{Leb}} \leqslant \|G\|_{\nu^+\otimes\mathrm{Leb}}$ and $\|G_\varepsilon\|_{\mathscr{H}_\alpha^+} \leqslant \|G\|_{\mathscr{H}_\alpha^+}$, we get, uniformly in $z \in \mathbb{X}^+$ and $t \leqslant t_0$,

$$\begin{aligned}\Psi_n(z,t) &\leqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}} I_{m,k} + \frac{c\varepsilon + r_k}{\sqrt{km}}\|G_\varepsilon\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{m\sqrt{k}}\|G_\varepsilon\|_{\mathscr{H}_\alpha^+} \\ &\leqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}} I_{m,k} + \frac{c\varepsilon + r_k}{\sqrt{km}}\|G\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{m\sqrt{k}}\|G\|_{\mathscr{H}_\alpha^+},\end{aligned} \tag{6.22}$$

where

$$I_{m,k} = \frac{1}{\sqrt{k}}\int_{\mathbb{R}_+} H_m(s)\phi^+\left(\frac{s}{\sqrt{k}}\right)\frac{ds}{\sqrt{k}}.$$

By the definition of H_m (cf. (6.15)) and Fubini's theorem, it follows that

$$\begin{aligned}I_{m,k} &= \int_{\mathbb{R}_+}\int_{\mathbb{X}^+}\int_{\mathbb{R}} \phi_{\sqrt{m}}(u-s) G_\varepsilon(z,u)\, du\nu^+(dz)\phi^+\left(\frac{s}{\sqrt{k}}\right)\frac{ds}{k} \\ &= \int_{\mathbb{X}^+}\int_{\mathbb{R}} G_\varepsilon(z,u)\left[\int_{\mathbb{R}_+}\phi_{\sqrt{m}}(u-s)\phi^+\left(\frac{s}{\sqrt{k}}\right)\frac{ds}{k}\right] du\nu^+(dz).\end{aligned}$$

Denote $\delta_n = \frac{m}{n} = \frac{[\delta n]}{n}$. By a change of variable, we have

$$\begin{aligned}I_{m,k} &= \frac{1}{\sqrt{n}}\int_{\mathbb{X}^+}\int_{\mathbb{R}} G_\varepsilon\left(z,\sqrt{n}u\right)\left[\int_{\mathbb{R}_+}\phi_{\delta_n}(u-s)\phi^+_{1-\delta_n}(s)\, ds\right] du\nu^+(dz) \\ &= \frac{1}{\sqrt{n}}\int_{\mathbb{X}^+}\int_{\mathbb{R}} G_\varepsilon\left(z,\sqrt{n}u\right)\phi_{\delta_n} * \phi^+_{1-\delta_n}(u) du\nu^+(dz) \\ &= \frac{1}{n}\int_{\mathbb{X}^+}\int_{\mathbb{R}} G_\varepsilon(z,u)\,\phi_{\delta_n} * \phi^+_{1-\delta_n}\left(\frac{u}{\sqrt{n}}\right) du\nu^+(dz) \\ &= \frac{1}{n}\int_{\mathbb{X}^+}\int_{-\varepsilon}^{\infty} G_\varepsilon(z,u)\,\phi_{\delta_n} * \phi^+_{1-\delta_n}\left(\frac{u}{\sqrt{n}}\right) du\nu^+(dz),\end{aligned} \tag{6.23}$$

where in the last line we used the fact that $G_\varepsilon(z,u) = 0$ for any $z \in \mathbb{X}^+$ and $u \leqslant -\varepsilon$. We handle the convolution $\phi_{\delta_n} * \phi^+_{1-\delta_n}$ using Lemma 6.2 together with the fact that $\delta_n = \frac{m}{n}$, $1-\delta_n = \frac{k}{n}$ and $u \geqslant -\varepsilon$:

$$\begin{aligned}\phi_{\delta_n} * \phi^+_{1-\delta_n}\left(\frac{u}{\sqrt{n}}\right) &\leqslant \sqrt{1-\delta_n}\phi^+\left(\frac{u}{\sqrt{n}}\right) + \sqrt{\delta_n}e^{-\frac{u^2}{2n\delta_n}} + \left|\frac{u}{\sqrt{n}}\right| e^{-\frac{u^2}{2n}}\mathbb{1}_{\{u<0\}} \\ &= \sqrt{\frac{k}{n}}\phi^+\left(\frac{u}{\sqrt{n}}\right) + \sqrt{\frac{m}{n}}e^{-\frac{u^2}{2m}} + \left|\frac{u}{\sqrt{n}}\right| e^{-\frac{u^2}{2n}}\mathbb{1}_{\{u<0\}} \\ &\leqslant \sqrt{\frac{k}{n}}\phi^+\left(\frac{u}{\sqrt{n}}\right) + \sqrt{\frac{m}{n}} + \frac{\varepsilon}{\sqrt{n}}.\end{aligned}$$

Since $G_\varepsilon \leqslant G$, it follows that

$$\begin{aligned} I_{m,k} &\leqslant \frac{\sqrt{k}}{n^{3/2}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} G(z,u)\, \phi^+\left(\frac{u}{\sqrt{n}}\right) du \nu^+(dz) \\ &\quad + \frac{\sqrt{m}}{n^{3/2}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} G(z,u)\, du \nu^+(dz) + \frac{\varepsilon}{n^{3/2}} \int_{\mathbb{X}^+} \int_{-\infty}^{0} G(z,u)\, du \nu^+(dz) \\ &\leqslant \frac{\sqrt{k}}{n^{3/2}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} G(z,u)\, \phi^+\left(\frac{u}{\sqrt{n}}\right) du \nu^+(dz) + \frac{2\varepsilon^{1/4}}{n} \|G\|_{\nu^+ \otimes \mathrm{Leb}}. \end{aligned}$$

Substituting this into (6.22), and using the fact that $\check{V}^g(z,t) \leqslant t + c$ gives

$$\begin{aligned} \Psi_n(z,t) &\leqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}} \frac{\sqrt{k}}{n^{3/2}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} G(z,u)\, \phi^+\left(\frac{u}{\sqrt{n}}\right) du \nu^+(dz) \\ &\quad + c\left(\frac{c\varepsilon + r_k}{\sqrt{mk}} + \frac{\varepsilon^{1/4}}{n}\right) \|G\|_{\nu^+ \otimes \mathrm{Leb}} + \frac{c_\varepsilon}{m\sqrt{k}} \|G\|_{\mathscr{H}_\alpha^+}. \end{aligned}$$

Since $\varepsilon^{1/2} n \geqslant m \geqslant \frac{1}{2}\varepsilon^{1/2} n$ and $n > k \geqslant \frac{1}{2} n$, we obtain

$$\begin{aligned} \Psi_n(z,t) &\leqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi n}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} G(z,u)\, \phi^+\left(\frac{u}{\sqrt{n}}\right) du \nu^+(dz) \\ &\quad + \frac{c}{n}\left(\varepsilon^{1/4} + \frac{r_n}{\varepsilon^{1/4}}\right) \|G\|_{\nu^+ \otimes \mathrm{Leb}} + \frac{c_\varepsilon}{n^{3/2}} \|G\|_{\mathscr{H}_\alpha^+}, \end{aligned}$$

which finishes the proof of the upper bound (6.1).

6.4. **Proof of the lower bound.** We now proceed to prove the second assertion (6.2) of Theorem 6.1. We use the same notation as that in the proof of the upper bound. Recall that $\delta = \sqrt{\varepsilon}$, $m = [\delta n]$ and $k = n - m$. For $z \in \mathbb{X}^+$, $t \in \mathbb{R}$ and $n \geqslant 1$, denote

$$\Psi_n(z,t) := \int_{\mathbb{X}_z^-} F\left((T^{-n} y \cdot z)_+, t + \check{S}_n g(y \cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy).$$

Note that $\Psi_n(z,t) = 0$ for $t \leqslant -c_0 = -\|g\|_\infty$ because $\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} = 0$ for these values of t, and therefore in the sequel we can consider that $t \leqslant t_0$. By the Markov property, we have that for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\Psi_n(z,t) = \int_{\mathbb{X}_z^-} \Psi_m\left((T^{-k} y \cdot z)_+, t + \check{S}_k g(y \cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > k\right\}} \nu_z^-(dy). \tag{6.24}$$

We write Ψ_m as a sum of two terms: for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\Psi_m(z,t) = A_m(z,t) - \overline{A}_m(z,t), \tag{6.25}$$

where

$$A_m(z,t) = \int_{\mathbb{X}_z^-} F\left((T^{-m} y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) \nu_z^-(dy), \tag{6.26}$$

$$\overline{A}_m(z,t) = \int_{\mathbb{X}_z^-} F\left((T^{-m} y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) \leqslant m\right\}} \nu_z^-(dy). \tag{6.27}$$

This implies that for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\Psi_n(z,t) = J_n(z,t) - K_n(z,t), \tag{6.28}$$

where

$$J_n(z,t) := \int_{\mathbb{X}_z^-} A_m\left((T^{-k}y\cdot z)_+, t+\check{S}_k g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>k\right\}} \nu_z^-(dy), \tag{6.29}$$

$$K_n(z,t) := \int_{\mathbb{X}_z^-} \overline{A}_m\left((T^{-k}y\cdot z)_+, t+\check{S}_k g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>k\right\}} \nu_z^-(dy). \tag{6.30}$$

We proceed to give a lower bound for the term $J_n(z,t)$ in (6.28). It can be handled as the case of the upper bound, but here the situation is more complicated. By the local limit theorem (cf. Theorem 5.8), we get that there exist constants $c, c_\varepsilon > 0$ such that for any $m \geqslant 1$, $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$A_m(z,t) \geqslant H_m(t) - \frac{c\varepsilon}{\sqrt{m}}\|G\|_{\nu^+\otimes\mathrm{Leb}} - \frac{c_\varepsilon}{m}\left(\|G\|_{\mathscr{H}_\alpha^+} + \|H\|_{\mathscr{H}_\alpha^+}\right), \tag{6.31}$$

where for brevity we set

$$H_m(t) = \int_{\mathbb{X}^+}\int_{\mathbb{R}} \frac{1}{\sqrt{m}}\phi\left(\frac{u-t}{\sqrt{m}}\right) H\left(z,u\right) du \nu^+(dz). \tag{6.32}$$

Using (6.26), (6.29) and (6.31), and Lemma 4.1, we get that for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\begin{aligned} J_n(z,t) \geqslant &\int_{\mathbb{X}_z^-} H_m\left(t+\check{S}_k g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>k\right\}} \nu_z^-(dy) \\ &- \frac{c\varepsilon}{\sqrt{km}}\|G\|_{\nu^+\otimes\mathrm{Leb}} - \frac{c_\varepsilon}{\sqrt{km}}\left(\|G\|_{\mathscr{H}_\alpha^+} + \|H\|_{\mathscr{H}_\alpha^+}\right). \end{aligned} \tag{6.33}$$

For the first term on the right-hand side of (6.33), proceeding in the same way as that in the proof of (6.21), using the lower bound in the conditioned central limit theorem (see Lemma 4.6), one can verify that

$$\begin{aligned} &\int_{\mathbb{X}_z^-} H_m\left(t+\check{S}_k g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>k\right\}} \nu_z^-(dy) \\ &\geqslant \frac{2\check{V}^g(z,t)}{\sqrt{2\pi k}}\int_{\mathbb{R}_+} H_m(s)\phi^+\left(\frac{s}{\sqrt{k}}\right)\frac{ds}{\sqrt{k}} - \frac{r_k}{\sqrt{km}}\|H\|_{\nu^+\otimes\mathrm{Leb}}. \end{aligned} \tag{6.34}$$

Implementing this bound into (6.33), we get that for any $z \in \mathbb{X}^+$,

$$\begin{aligned} J_n(z,t) \geqslant &\frac{2\check{V}^g(z,t)}{\sqrt{2\pi}} I_{m,k} - \frac{r_k}{\sqrt{km}}\|H\|_{\nu^+\otimes\mathrm{Leb}} \\ &- \frac{c\varepsilon}{\sqrt{km}}\|G\|_{\nu^+\otimes\mathrm{Leb}} - \frac{c_\varepsilon}{\sqrt{km}}\left(\|G\|_{\mathscr{H}_\alpha^+} + \|H\|_{\mathscr{H}_\alpha^+}\right). \end{aligned} \tag{6.35}$$

where

$$I_{m,k} := \int_{\mathbb{R}_+} H_m(s)\phi^+\left(\frac{s}{\sqrt{k}}\right)\frac{ds}{k}.$$

In the same way as in the proof of (6.23), we have

$$\begin{aligned} I_{m,k} &= \frac{1}{n} \int_{\mathbb{X}^+} \int_{\mathbb{R}} H(z,u)\, \phi_{\delta_n} * \phi^+_{1-\delta_n} \left(\frac{u}{\sqrt{n}} \right) du \nu^+(dz) \\ &\geqslant \frac{1}{n} \int_{\mathbb{X}^+} \int_0^\infty H(z,u)\, \phi_{\delta_n} * \phi^+_{1-\delta_n} \left(\frac{u}{\sqrt{n}} \right) du \nu^+(dz) \\ &\geqslant \frac{\sqrt{k}}{n^{3/2}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} H(z,u)\, \phi^+ \left(\frac{u}{\sqrt{n}} \right) du \nu^+(dz), \end{aligned}$$

where in the last inequality we applied Lemma 6.2 and the fact that $\phi^+(u) = 0$ for $u < 0$. Substituting this into (6.35), and using the fact that $H \leqslant_\varepsilon G$ and $\check{V}^g(z,t) \leqslant t + c$, we get

$$\begin{aligned} J_n(z,t) \geqslant\ & \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}} \frac{\sqrt{k}}{n^{3/2}} \int_{\mathbb{X}^+} \int_{\mathbb{R}} H(z,u)\, \phi^+ \left(\frac{u}{\sqrt{n}} \right) du \nu^+(dz) \\ & - \frac{c\varepsilon + r_k}{\sqrt{km}} \|G\|_{\nu^+ \otimes \mathrm{Leb}} - \frac{c_\varepsilon}{\sqrt{km}} \left(\|G\|_{\mathscr{H}^+_\alpha} + \|H\|_{\mathscr{H}^+_\alpha} \right). \end{aligned}$$

Since $\sqrt{\frac{n}{k}} \leqslant 1 + c\varepsilon^{1/4}$, $m \geqslant \frac{1}{2}\varepsilon^{1/2} n$ and $k \geqslant \frac{1}{2} n$, using again $H \leqslant_\varepsilon G$ we deduce that for n sufficiently large,

$$\begin{aligned} J_n(z,t) \geqslant\ & \frac{2\check{V}^g(z,t)}{\sqrt{2\pi} n} \int_{\mathbb{X}^+} \int_{\mathbb{R}} H(z,u)\, \phi^+ \left(\frac{u}{\sqrt{n}} \right) du \nu^+(dz) \\ & - \frac{c}{n} \left(\varepsilon^{1/4} + \frac{r_n}{\varepsilon^{1/4}} \right) \|G\|_{\nu^+ \otimes \mathrm{Leb}} - \frac{c_\varepsilon}{n^{3/2}} \left(\|G\|_{\mathscr{H}^+_\alpha} + \|H\|_{\mathscr{H}^+_\alpha} \right). \end{aligned} \tag{6.36}$$

We now deal with $K_n(z,t)$ which is the second term in (6.28). Bounding $K_n(z,t)$ is one of the difficult points of the paper and needs to make use of the duality. We start by splitting $K_n(z,t)$ into two parts according to whether the values of $t + \check{S}_k g(y \cdot z)$ are less or larger than $\varepsilon\sqrt{n}$: for $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$K_n(z,t) = K_1 + K_2, \tag{6.37}$$

where

$$\begin{aligned} K_1 &= \int_{\mathbb{X}^-_z} \overline{A}_m \left((T^{-k} y \cdot z)_+, t + \check{S}_k g(y \cdot z) \right) \mathbb{1}_{\left\{ t + \check{S}_k g(y\cdot z) \leqslant \varepsilon^{1/6} \sqrt{n} \right\}} \mathbb{1}_{\left\{ \check{\tau}^g_t(y\cdot z) > k \right\}} \nu^-_z(dy), \\ K_2 &= \int_{\mathbb{X}^-_z} \overline{A}_m \left((T^{-k} y \cdot z)_+, t + \check{S}_k g(y \cdot z) \right) \mathbb{1}_{\left\{ t + \check{S}_k g(y\cdot z) > \varepsilon^{1/6} \sqrt{n} \right\}} \mathbb{1}_{\left\{ \check{\tau}^g_t(y\cdot z) > k \right\}} \nu^-_z(dy). \end{aligned}$$

For K_1, using the upper bound in the local limit theorem (cf. Theorem 5.8) and taking into account that $\phi \leqslant 1$, we get

$$\overline{A}_m(z,t) \leqslant \frac{L_m(\varepsilon)}{\sqrt{m}}, \quad \text{where } L_m(\varepsilon) = c\|G\|_{\nu^+ \otimes \mathrm{Leb}} + \frac{c_\varepsilon}{\sqrt{m}} \|G\|_{\mathscr{H}^+_\alpha}.$$

This and the fact that $\sqrt{\frac{n}{k}} \leqslant c$ imply

$$\begin{aligned} K_1 &\leqslant \frac{L_m(\varepsilon)}{\sqrt{m}} \int_{\mathbb{X}^-_z} \mathbb{1}_{\left\{ t + \check{S}_k g(y\cdot z) \leqslant \varepsilon^{1/6} \sqrt{n} \right\}} \mathbb{1}_{\left\{ \check{\tau}^g_t(y\cdot z) > k \right\}} \nu^-_z(dy) \\ &\leqslant \frac{L_m(\varepsilon)}{\sqrt{m}}\, \nu^-_z \left(\frac{t + \check{S}_k g(y \cdot z)}{\sqrt{k}} \leqslant c\varepsilon^{1/6},\ \check{\tau}^g_t(y \cdot z) > k \right). \end{aligned}$$

Using Lemma 4.6 and the fact that $m = [\varepsilon^{1/2} n]$, we get that uniformly in $z \in \mathbb{X}^+$,

$$\begin{aligned}
K_1 &\leqslant \frac{L_m(\varepsilon)}{\sqrt{m}} \left(\frac{2\check{V}^g(z,t)}{\sqrt{2\pi k}} \int_0^{c\varepsilon^{1/6}} \phi^+(t')\, dt' + \frac{o(1)}{k^{1/2}} \right) \\
&\leqslant \frac{L_m(\varepsilon)}{\sqrt{mk}} \left(\int_0^{c\varepsilon^{1/6}} \phi^+(t')\, dt' + o(1) \right) \\
&\leqslant c \frac{L_m(\varepsilon)}{\varepsilon^{\frac{1}{4}} n} \left(\varepsilon^{1/3} + o(1) \right) \\
&\leqslant c \frac{L_m(\varepsilon)}{n} \left(\varepsilon^{\frac{1}{12}} + \frac{o(1)}{\varepsilon^{\frac{1}{4}}} \right) \\
&= \frac{c}{n} \left(\|G\|_{\nu^+ \otimes \mathrm{Leb}} + \frac{c_\varepsilon}{\sqrt{m}} \|G\|_{\mathscr{H}_\alpha^+} \right) \left(c\varepsilon^{\frac{1}{12}} + \frac{o(1)}{\varepsilon^{\frac{1}{4}}} \right) \\
&\leqslant \frac{c}{n} \left(\varepsilon^{\frac{1}{12}} + \frac{o(1)}{\varepsilon^{\frac{1}{4}}} \right) \|G\|_{\nu^+ \otimes \mathrm{Leb}} + \frac{c_\varepsilon}{n^{3/2}} \|G\|_{\mathscr{H}_\alpha^+}. \qquad (6.38)
\end{aligned}$$

We proceed to give an upper bound for K_2, see (6.37). Recall that the function $(z,t) \mapsto \overline{A}_m(z,t)$, which is involved in the definition of K_2, is defined by (6.27) and does not in general belong to the space $\mathscr{H}_\alpha^+$. We start by smoothing the indicator function in (6.27). Let κ be a non-negative smooth compactly supported function in $[-1,1]$ such that $\int_{-1}^1 \kappa(u) du = 1$ and set $\kappa_\varepsilon(u) = \frac{1}{\varepsilon}\kappa(\frac{u}{\varepsilon})$ for $u \in \mathbb{R}$. Define

$$\begin{aligned}
\overline{A}_{m,\varepsilon}(z,t) := \int_{\mathbb{X}_z^-} & G * \kappa_{\varepsilon/2} \left((T^{-m} y \cdot z)_+, t + \check{S}_m g(y \cdot z) \right) \\
& \times \overline{\chi}_\varepsilon \left(t - \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \right) \nu_z^-(dy),
\end{aligned}$$

where χ_ε is the same as in (6.6) and $\overline{\chi}_\varepsilon = 1 - \chi_\varepsilon$. Note that the function F is $\varepsilon/2$-dominated by the function $G * \kappa_{\varepsilon/2}$. By the identity

$$\mathbb{1}_{\left\{\check{\tau}_t^g(y \cdot z) > m\right\}} = \mathbb{1}_{[0,\infty)} \left(t + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \right), \qquad (6.39)$$

using the bounds (6.7) and $F(z,\cdot) \leqslant G * \kappa_{\varepsilon/2}(z,\cdot)$, we get that the function $\overline{A}_m$ is $\varepsilon/2$-dominated by the function $\overline{A}_{m,\varepsilon}$. Moreover, by Lemma 6.5, there exists a constant c_ε such that for any $m \geqslant 1$, the function $\overline{A}_{m,\varepsilon}$ belongs to $\mathscr{H}_\alpha^+$ and satisfies

$$\|\overline{A}_{m,\varepsilon}\|_{\mathscr{H}_\alpha^+} \leqslant c_\varepsilon \|G\|_{\mathscr{H}_\alpha^+}, \qquad \|\overline{A}_{m,\varepsilon}\|_{\nu^+ \otimes \mathrm{Leb}} \leqslant \|G\|_{\nu^+ \otimes \mathrm{Leb}}.$$

Denote

$$W_{m,\varepsilon}(z,t) = \overline{A}_{m,\varepsilon}(z,t)\, \mathbb{1}_{\{t \geqslant \varepsilon^{1/6} \sqrt{n}\}}. \qquad (6.40)$$

Using the upper bound (6.1) and the fact that $\phi^+ \leqslant 1$, we obtain

$$\begin{aligned}
K_2 &\leqslant \int_{\mathbb{X}_z^-} W_{m,\varepsilon} \left((T^{-k} y \cdot z)_+, t + \check{S}_k g(y \cdot z) \right) \mathbb{1}_{\left\{\check{\tau}_t^g(y \cdot z) > k\right\}} \nu_z^-(dy) \\
&\leqslant \left(\frac{2\check{V}^g(z,t)}{\sqrt{2\pi k}} + \frac{c}{k} \left(\varepsilon^{1/4} + \frac{r_n}{\varepsilon^{1/4}} \right) \right) \|W_{m,\varepsilon}\|_{\nu^+ \otimes \mathrm{Leb}} + \frac{c_\varepsilon}{\sqrt{n}k} \|W_{m,\varepsilon}\|_{\mathscr{H}_\alpha^+}. \qquad (6.41)
\end{aligned}$$

For the first term on the right-hand side of (6.41), by the definition of $W_{m,\varepsilon}$ and Fubini's theorem, we have

$$\begin{aligned}\|W_{m,\varepsilon}\|_{\nu^+\otimes\mathrm{Leb}} &= \int_{\mathbb{X}^+}\int_{\mathbb{R}} W_{m,\varepsilon}\left(z',u\right) du\nu^+(dz') \\ &\leqslant \int_{\mathbb{X}}\int_{\mathbb{R}} \Big[G*\kappa_{\varepsilon/2}\left((T^{-m}x)_+, u+\check{S}_m g(x)\right) \\ &\qquad \times \mathbb{1}_{\left\{u+\min_{1\leqslant j\leqslant m}\check{S}_j g(x)\leqslant 0\right\}} \mathbb{1}_{\{u\geqslant \varepsilon^{1/6}\sqrt{n}\}}\Big] \nu(dx)du =: U. \end{aligned} \tag{6.42}$$

Using the duality (Lemma 2.12) yields that

$$U = \int_{\mathbb{X}}\int_{\mathbb{R}} \left[G*\kappa_{\varepsilon/2}\left(x'_+, u'\right) \mathbb{1}_{\left\{u'-\min_{1\leqslant j\leqslant m} S_j g(x')\leqslant 0\right\}} \mathbb{1}_{\left\{u'-S_m g(x')\geqslant \varepsilon^{1/6}\sqrt{n}\right\}} \right] \nu(dx')du'.$$

Since the measure ν is T-invariant, it follows that

$$\begin{aligned} U &= \int_{\mathbb{X}}\int_{\mathbb{R}} \Big[G*\kappa_{\varepsilon/2}\left((T^{-m}x')_+, u'\right) \mathbb{1}_{\left\{u'-\min_{1\leqslant j\leqslant m} S_j g(T^{-m}x')\leqslant 0\right\}} \\ &\qquad \times \mathbb{1}_{\left\{u'-S_m g(T^{-m}x')\geqslant \varepsilon^{1/6}\sqrt{n}\right\}} \Big] \nu(dx')du' \\ &= \int_{\mathbb{X}}\int_{\mathbb{R}} \Big[G*\kappa_{\varepsilon/2}\left((T^{-m}x')_+, u'\right) \mathbb{1}_{\left\{u'-\check{S}_m g(x')+\max_{1\leqslant j\leqslant m}\check{S}_{m-j} g(x')\leqslant 0\right\}} \\ &\qquad \times \mathbb{1}_{\left\{u'-\check{S}_m g(x')\geqslant \varepsilon^{1/6}\sqrt{n}\right\}} \Big] \nu(dx')du' \\ &\leqslant \int_{\mathbb{X}}\int_{\mathbb{R}} \left[G*\kappa_{\varepsilon/2}\left((T^{-m}x')_+, u'\right) \mathbb{1}_{\left\{\max_{1\leqslant j\leqslant m}\check{S}_j g(x')\leqslant -\varepsilon^{1/6}\sqrt{n}\right\}} \right] \nu(dx')du' \\ &= \int_{\mathbb{R}}\int_{\mathbb{X}^+}\int_{\mathbb{X}_z^-} \Big[G*\kappa_{\varepsilon/2}\left((T^{-m}y\cdot z)_+, u'\right) \mathbb{1}_{\left\{\max_{1\leqslant j\leqslant m}|\check{S}_j g(y\cdot z)|\geqslant \varepsilon^{1/6}\sqrt{n}\right\}} \Big] \\ &\qquad \times \nu_z^-(dy)\nu^+(dz)du', \end{aligned} \tag{6.43}$$

where for the last line we made use of Lemma 2.2. By the Fuk inequality of Lemma 6.4 with $M=\varepsilon^{-1/12}$ and ε small enough, it follows that

$$\begin{aligned} &\int_{\mathbb{X}_z^-} \left[G*\kappa_{\varepsilon/2}\left((T^{-m}y\cdot z)_+, u'\right) \mathbb{1}_{\left\{\max_{1\leqslant j\leqslant m}|\check{S}_j g(y\cdot z)|\geqslant \varepsilon^{1/6}\sqrt{n}\right\}} \right] \nu_z^-(dy) \\ &\leqslant \int_{\mathbb{X}_z^-} \left[G*\kappa_{\varepsilon/2}\left((T^{-m}y\cdot z)_+, u'\right) \mathbb{1}_{\left\{\max_{1\leqslant j\leqslant m}|\check{S}_j g(y\cdot z)|\geqslant M\sqrt{m}\right\}} \right] \nu_z^-(dy) \\ &\leqslant 2e^{-c\varepsilon^{-1/12}} \int_{\mathbb{X}^+} G*\kappa_{\varepsilon/2}\left(z,u'\right)\nu^+(dz) + c' e^{-c\varepsilon^{1/6}n^{1/3}} \|G*\kappa_{\varepsilon/2}(\cdot,u')\|_{\mathscr{B}_\alpha^+}. \end{aligned}$$

Implementing this into (6.43), by (6.42), we have

$$\begin{aligned}\|W_{m,\varepsilon}\|_{\nu^+\otimes\mathrm{Leb}} &\leqslant 2e^{-c\varepsilon^{-1/12}}\int_{\mathbb{R}}\int_{\mathbb{X}^+} G*\kappa_{\varepsilon/2}\,(z,u')\,\nu^+(dz)du' \\ &\qquad + c'e^{-c\varepsilon^{1/6}n^{1/3}}\int_{\mathbb{R}}\|G*\kappa_{\varepsilon/2}(\cdot,u')\|_{\mathscr{B}^+_\alpha}du' \\ &\leqslant c\int_{\mathbb{R}}\kappa_{\varepsilon/2}(u')du'e^{-c\varepsilon^{-1/12}}\|G\|_{\nu^+\otimes Leb} + c'\varepsilon^{-1}e^{-c\varepsilon^{1/6}n^{1/3}}\|G\|_{\mathscr{H}^+_\alpha} \\ &\leqslant ce^{-c\varepsilon^{-1/12}}\|G\|_{\nu^+\otimes\mathrm{Leb}} + c'\varepsilon^{-1}e^{-c\varepsilon^{1/6}n^{1/3}}\|G\|_{\mathscr{H}^+_\alpha},\end{aligned} \tag{6.44}$$

where for the last line we made use of bounds similar to those in Lemma 6.5.

The norm $\|W_{m,\varepsilon}\|_{\mathscr{H}^+_\alpha}$ in the second term on the right-hand side of (6.41) is bounded using Lemma 6.5. Taking into account (6.40), we get

$$\|W_{m,\varepsilon}\|_{\mathscr{H}^+_\alpha} \leqslant \|\overline{A}_{m,\varepsilon}\|_{\mathscr{H}^+_\alpha} \leqslant c_\varepsilon\|G\|_{\mathscr{H}^+_\alpha}. \tag{6.45}$$

Therefore, from (6.41), (6.44) and (6.45), we derive the upper bound for K_2: uniformly in $z\in\mathbb{X}^+$ and $t\leqslant t_0$,

$$\begin{aligned}K_2 &\leqslant 2\left(\frac{2\check{V}^g(z,t)}{\sqrt{2\pi}n} + \frac{c}{n}\left(\varepsilon^{1/4}+\frac{r_n}{\varepsilon^{1/4}}\right)\right)\exp\left(-c\varepsilon^{-1/12}\right)\|G\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{n^{3/2}}\|G\|_{\mathscr{H}^+_\alpha} \\ &\leqslant \frac{c\varepsilon^{1/4}}{n}\|G\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{n^{3/2}}\|G\|_{\mathscr{H}^+_\alpha}.\end{aligned} \tag{6.46}$$

Combining (6.28), (6.36), (6.37), (6.38) and (6.46), the lower bound (6.2) follows.

7. Proof of Theorem 1.7

As for Theorems 1.3 and 1.5, we first establish the result when g is in $\mathscr{B}^+$. The general case of a function g in $\mathscr{B}$ will follow using the same method as in Section 4.

Theorem 7.1. *Let $g\in\mathscr{B}^+$ be such that $\nu^+(g)=0$. Assume that for any $p\neq 0$ and $q\in\mathbb{R}$, the function $pg+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Let F be a continuous compactly supported function on $\mathbb{X}^+\times\mathbb{R}$. Then, we have, uniformly in $z\in\mathbb{X}^+$ and t in a compact subset of $\mathbb{R}$,*

$$\begin{aligned}\lim_{n\to\infty} n^{3/2}\int_{\mathbb{X}^-_z} F\left((T^{-n}y\cdot z)_+, t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}^g_t(y\cdot z)>n-1\right\}}\nu^-_z(dy) \\ = \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3}\int_{\mathbb{X}\times\mathbb{R}} F\left(x'_+,t'\right)\mu^{(-g)}(dx',dt').\end{aligned}$$

In the proof of this theorem, we will make use of several technical lemmas which are stated below. We say that a function G on $\mathbb{X}^+\times\mathbb{R}$ is α-regular if there is a constant c such that for any (z,t) and (z',t') in $\mathbb{X}^+\times\mathbb{R}$, we have $|G(z,t)-G(z',t')|\leqslant c(|t-t'|+\alpha^{\omega(z,z')})$. In other words, a function is α-regular if and only if it is Lipschitz continuous on $\mathbb{X}^+\times\mathbb{R}$ when $\mathbb{R}$ is equipped with the standard distance and $\mathbb{X}^+$ is equipped with the distance $(z,z')\mapsto\alpha^{\omega(z,z')}$. The following result is similar to Lemma 6.5. It will allow us to smooth certain functions appearing in the proof of Theorem 7.1 in order to be able to apply Theorem 6.1. Recall that for $\varepsilon\in(0,1)$, $\chi_\varepsilon(u)=0$ for $u\leqslant-\varepsilon$, $\chi_\varepsilon(u)=\frac{u+\varepsilon}{\varepsilon}$ for $u\in(-\varepsilon,0)$, and $\chi_\varepsilon(u)=1$ for $u\geqslant 0$.

Lemma 7.2. *Let $\alpha \in (0,1)$ and $g \in \mathscr{B}_\alpha^+$ be such that $\nu^+(g) = 0$. Assume that g is not a coboundary. Let G be an α-regular function with compact support on $\mathbb{X}^+ \times \mathbb{R}$. For $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$, $m \geqslant 1$ and $\varepsilon > 0$, define*

$$\overline{\Psi}_{m,\varepsilon}(z,t) := \int_{\mathbb{X}_z^-} G\left((T^{-m}y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) \\ \times \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) \nu_z^-(dy).$$

Then $\overline{\Psi}_{m,\varepsilon} \in \mathscr{H}_\alpha^+$ and $\|\overline{\Psi}_{m,\varepsilon}\|_{\mathscr{H}_\alpha^+} \leqslant \frac{c}{\varepsilon\sqrt{m}}$.

Proof. It is enough to prove the lemma for a non-negative function G. Recall that

$$\|\overline{\Psi}_{m,\varepsilon}\|_{\mathscr{H}_\alpha^+} = \int_{\mathbb{R}} \sup_{z \in \mathbb{X}^+} |\overline{\Psi}_{m,\varepsilon}(z,t)| dt + \int_{\mathbb{R}} \sup_{z,z' \in \mathbb{X}^+} \frac{|\overline{\Psi}_{m,\varepsilon}(z,t) - \overline{\Psi}_{m,\varepsilon}(z',t)|}{\alpha^{\omega(z,z')}} dt.$$

By Corollary 4.5, the first term is dominated by $c/\sqrt{m}$ for some constant $c > 0$.

For the second term, we start by noticing that by Lemma 2.9, there exists a constant $c_0 > 0$ such that for any $z, z' \in \mathbb{X}^+$ with $z_0 = z_0'$, $t \in \mathbb{R}$ and $y \in \mathbb{X}_z^-$,

$$\begin{aligned} &\chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) - \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \\ &= \left(\chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) - \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right)\right) \\ &\quad \times \mathbb{1}_{\{t + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \geqslant -c_0\}} \\ &\leqslant \frac{1}{\varepsilon}\left|\min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) - \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right| \mathbb{1}_{\{t + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \geqslant -c_0\}} \\ &\leqslant \frac{c_1}{\varepsilon} \alpha^{\omega(z,z')} \mathbb{1}_{\{t + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \geqslant -c_0\}}, \end{aligned}$$

where in the last inequality we used Corollary 2.10. It follows that

$$\begin{aligned} &\int_{\mathbb{X}_z^-} G\left((T^{-m}y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) \\ &\quad \times \left|\chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) - \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right)\right| \nu_z^-(dy) \\ &\leqslant \frac{c_1}{\varepsilon} \alpha^{\omega(z,z')} \int_{\mathbb{X}_z^-} G\left((T^{-m}y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) \mathbb{1}_{\{t + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \geqslant -c_0\}} \nu_z^-(dy). \end{aligned}$$

By using again Corollary 4.5, we get

$$\begin{aligned} &\int_{\mathbb{R}} \sup_{z,z' \in \mathbb{X}^+ : z_0 = z_0'} \alpha^{-\omega(z,z')} \int_{\mathbb{X}_z^-} G\left((T^{-m}y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) \\ &\quad \times \left|\chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z)\right) - \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right)\right| \nu_z^-(dy) \\ &\leqslant \frac{c_2}{\varepsilon\sqrt{m}}. \end{aligned} \tag{7.1}$$

Besides, as G is α-regular and has compact support, we have for any $z, z' \in \mathbb{X}^+$ with $z_0 = z'_0$, and $t \in \mathbb{R}$, by Lemma 2.9,

$$\begin{aligned}&\int_{\mathbb{X}_z^-} \left| G\left((T^{-m}y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) - G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right)\right| \\ &\qquad \times \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \nu_z^-(dy) \\ &\leqslant c_3 \alpha^{\omega(z,z')} H(t) \nu_z^-\left(y \in \mathbb{X}_z^- : t + c' + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z) \geqslant 0\right),\end{aligned}$$

for some compactly supported continuous function H on $\mathbb{R}$ and some $c' > 0$. Again by Corollary 4.5, we get

$$\begin{aligned}&\int_{\mathbb{R}} \sup_{z,z' \in \mathbb{X}^+ : z_0 = z'_0} \alpha^{-\omega(z,z')} \\ &\times \int_{\mathbb{X}_z^-} \left| G\left((T^{-m}y \cdot z)_+, t + \check{S}_m g(y \cdot z)\right) - G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right)\right| \\ &\times \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \nu_z^-(dy) dt \leqslant \frac{c_4}{\sqrt{m}}.\end{aligned} \tag{7.2}$$

Finally, for any $z, z' \in \mathbb{X}^+$ with $z_0 = z'_0$, $t \in \mathbb{R}$, we have

$$\begin{aligned}&\int_{\mathbb{X}_z^-} G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right) \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \nu_z^-(dy) \\ &= \int_{\mathbb{X}_{z'}^-} G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right) \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) e^{\theta(y,z',z)} \nu_{z'}^-(dy),\end{aligned}$$

where θ is as in Lemma 2.3. By the Hölder continuous domination of θ in Lemma 2.3, we derive that

$$\begin{aligned}&\int_{\mathbb{R}} \sup_{z,z' \in \mathbb{X}^+ : z_0 = z'_0} \alpha^{-\omega(z,z')} \\ &\Bigg| \int_{\mathbb{X}_z^-} G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right) \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \nu_z^-(dy) \\ &- \int_{\mathbb{X}_{z'}^-} G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right) \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \nu_{z'}^-(dy) \Bigg| dt \\ &\leqslant c_4 \int_{\mathbb{R}} \sup_{z' \in \mathbb{X}^+} \int_{\mathbb{X}_{z'}^-} G\left((T^{-m}y \cdot z')_+, t + \check{S}_m g(y \cdot z')\right) \chi_\varepsilon\left(t + \varepsilon + \min_{1 \leqslant j \leqslant m} \check{S}_j g(y \cdot z')\right) \nu_{z'}^-(dy) dt \\ &\leqslant \frac{c_5}{\sqrt{m}},\end{aligned} \tag{7.3}$$

where the last inequality follows from Corollary 4.5. Putting together (7.1), (7.2) and (7.3) gives

$$\int_{\mathbb{R}} \sup_{z,z' \in \mathbb{X}^+ : z_0 = z'_0} \frac{\left|\overline{\Psi}_{m,\varepsilon}(z,t) - \overline{\Psi}_{m,\varepsilon}(z',t)\right|}{\alpha^{\omega(z,z')}} dt \leqslant \frac{c_6}{\varepsilon \sqrt{m}}.$$

The lemma follows. $\square$

Now we write a technical version of Theorem 7.1.

Lemma 7.3. *Let $\alpha \in (0,1)$ and $g \in \mathscr{B}_\alpha^+$ be such that $\nu^+(g) = 0$. Assume that for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pg + q$ is not cohomologous to a function with values in $\mathbb{Z}$. Let $t \in \mathbb{R}$. Then, for any $\varepsilon \in (0, \frac{1}{8})$ and $z \in \mathbb{X}^+$, and for any non-negative function F and non-negative α-regular compactly supported functions G, H satisfying $H \leqslant_\varepsilon F \leqslant_\varepsilon G$, we have*

$$\limsup_{n\to\infty} n^{3/2} \int_{\mathbb{X}_z^-} F\left((T^{-n}y \cdot z)_+, t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n-1\right\}} \nu_z^-(dy) \leqslant \frac{2\check{V}^g(z,t)}{\sigma_g^3\sqrt{2\pi}} \int_{\mathbb{X}} \int_{\mathbb{R}} G(x_+, t) \mu^{(-g)}(dx, dt) \tag{7.4}$$

and

$$\liminf_{n\to\infty} n^{3/2} \int_{\mathbb{X}_z^-} F\left((T^{-n}y \cdot z)_+, t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n-1\right\}} \nu_z^-(dy) \geqslant \frac{2\check{V}^g(z,t)}{\sigma_g^3\sqrt{2\pi}} \int_{\mathbb{X}} \int_{\mathbb{R}} H(x_+, t) \mu^{(-g)}(dx, dt). \tag{7.5}$$

Proof. We first prove (7.4). As in (6.24), denote, for $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\Psi_n(z,t) = \int_{\mathbb{X}_z^-} F\left((T^{-n}y \cdot z)_+, t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n-1\right\}} \nu_z^-(dy).$$

Set $m = [n/2]$ and $k = n - m$. By the Markov property we have that for any $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$,

$$\Psi_n(z,t) = \int_{\mathbb{X}_z^-} \Psi_m\left((T^{-k}y \cdot z)_+, t + \check{S}_k g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > k\right\}} \nu_z^-(dy). \tag{7.6}$$

For any $z' \in \mathbb{X}^+$ and $t' \in \mathbb{R}$, we set

$$\overline{\Psi}_m(z',t') := \int_{\mathbb{X}_{z'}^-} G\left((T^{-m}y \cdot z')_+, t' + \check{S}_m g(y\cdot z')\right) \times \chi_\varepsilon\left(t' + \varepsilon + \min_{1\leqslant j\leqslant m} \check{S}_j g(y\cdot z')\right) \nu_{z'}^-(dy).$$

By using $F \leqslant_\varepsilon G$, we get that $\Psi_m \leqslant_\varepsilon \overline{\Psi}_m$. Note that by Lemma 7.2, the function $\overline{\Psi}_m$ belongs to the space $\mathscr{H}_\alpha^+$, so that we are exactly in the setting of Theorem 6.1. Therefore, using the bound (6.1) of Theorem 6.1, we get

$$\begin{aligned}\Psi_n(z,t) &\leqslant \frac{2\check{V}^g(z,t)}{\sigma_g^2\sqrt{2\pi}k} \int_{\mathbb{X}^+} \int_{\mathbb{R}_+} \overline{\Psi}_m(z',u')\, \phi^+\left(\frac{u'}{\sigma_g\sqrt{k}}\right) du' \nu^+(dz') \\ &\quad + \frac{c}{k}\left(\varepsilon^{1/4} + \frac{r_k}{\varepsilon^{1/4}}\right) \|\overline{\Psi}_m\|_{\nu^+\otimes\mathrm{Leb}} + \frac{c_\varepsilon}{k^{3/2}} \|\overline{\Psi}_m\|_{\mathscr{H}_\alpha^+} \\ &=: J_1 + J_2 + J_3.\end{aligned}$$

For J_1, applying the duality (Lemma 2.12), we deduce that

$$\begin{aligned}
&\int_{\mathbb{X}^+}\int_{\mathbb{R}_+} \overline{\Psi}_m(z,u)\,\phi^+\left(\frac{u}{\sigma_g\sqrt{k}}\right) du\nu^+(dz) \\
&\leqslant \int_{\mathbb{X}^+}\int_{\mathbb{R}_+}\int_{\mathbb{X}_z^-} G\left((T^{-m}y\cdot z)_+, u+\check{S}_m g(y\cdot z)\right) \\
&\qquad \times \mathbb{1}_{\left\{\check{\tau}^g_{u+2\varepsilon}(y\cdot z)>m-1\right\}}\nu_z^-(dy)\phi^+\left(\frac{u}{\sigma_g\sqrt{k}}\right) du\nu^+(dz) \\
&= \int_{\mathbb{X}}\int_{\mathbb{R}_+} G\left((T^{-m}x)_+, u+\check{S}_m g(x)\right)\phi^+\left(\frac{u}{\sigma_g\sqrt{k}}\right)\mathbb{1}_{\left\{\check{\tau}^g_{u+2\varepsilon}(x)>m-1\right\}} du\nu(dx) \\
&= \int_{\mathbb{X}}\int_{\mathbb{R}} G(x_+, t-2\varepsilon)\,\phi^+\left(\frac{t-S_m g(x)-2\varepsilon}{\sigma_g\sqrt{k}}\right)\mathbb{1}_{\left\{\tau_t^{-g}(x)>m-1\right\}} dt\nu(dx).
\end{aligned}$$

Using the conditioned central limit theorem (Theorem 1.5), we get

$$\begin{aligned}
&\lim_{n\to\infty}\sigma_g\sqrt{2\pi m}\int_{\mathbb{X}}\int_{\mathbb{R}} G(x_+, t-2\varepsilon)\,\phi^+\left(\frac{t-S_m g(x)-2\varepsilon}{\sigma_g\sqrt{k}}\right)\mathbb{1}_{\left\{\tau_t^{-g}(x)>m-1\right\}} dt\nu(dx) \\
&= 2\int_{\mathbb{X}}\int_{\mathbb{R}} G(x_+, t-2\varepsilon)\mu^{-g}(dx,dt)\int_{\mathbb{R}_+}\left(\phi^+(t')\right)^2 dt' \\
&= \frac{\sqrt{\pi}}{2}\int_{\mathbb{X}}\int_{\mathbb{R}} G(x_+, t-2\varepsilon)\mu^{(-g)}(dx,dt).
\end{aligned}$$

Therefore, we obtain

$$\lim_{n\to\infty} n^{3/2}J_1 = \frac{2\check{V}^g(z,t)}{\sigma_g^3\sqrt{2\pi}}\int_{\mathbb{X}}\int_{\mathbb{R}} G(x_+, t-2\varepsilon)\mu^{(-g)}(dx,dt).$$

For J_2, by Corollary 4.5, we have

$$\|\overline{\Psi}_m\|_{\nu^+\otimes\mathrm{Leb}} \leqslant \frac{c}{\sqrt{m}}.$$

Taking into account that $m=[n/2]$ and $k=n-m$, we get $\limsup_{n\to\infty} n^{3/2}J_2 \leqslant c\varepsilon^{1/4}$.

For J_3, by Lemma 7.2, we have $\lim_{n\to\infty} n^{3/2}J_3 = 0$. This finishes the proof of the upper bound. The proof of the lower bound can be carried out in the same way. □

From Lemma 7.3, we get Theorem 7.1 by a standard approximation procedure.

Lemma 7.4. *Fix $\alpha\in(0,1)$. Let F be a non-negative continuous compactly supported function on $\mathbb{X}^+\times\mathbb{R}$. Then, there exist a decreasing sequence $(G_k)_{k\geqslant 1}$ and an increasing sequence $(H_k)_{k\geqslant 1}$ of compactly supported α-regular functions, such that $H_k \leqslant_{1/k} F \leqslant_{1/k} G_k$ for any $k\geqslant 1$, and G_k and H_k converge uniformly to F as $k\to\infty$.*

Proof of Theorem 7.1. This follows directly from Lemmas 7.3 and 7.4. □

From Theorem 7.1 we deduce a new lemma in which the target function F may depend on the past coordinates.

Lemma 7.5. *Let $g\in\mathscr{B}^+$ be such that $\nu^+(g)=0$. Assume that for any $p\neq 0$ and $q\in\mathbb{R}$, the function $pg+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Let F be a continuous* *compactly supported function on* $\mathbb{X} \times \mathbb{R}$. *Then, uniformly in* $z \in \mathbb{X}_+$ *and* t *in a compact subset of* $\mathbb{R}$,

$$\lim_{n\to\infty} n^{3/2} \int_{\mathbb{X}_z^-} F\left(T^{-n}(y\cdot z), t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n-1\right\}} \nu_z^-(dy)$$
$$= \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}\times\mathbb{R}} F\left(x', t'\right) \mu^{(-g)}(dx', dt').$$

Proof. As in the proof of Lemma 4.7, it suffices to prove this result when F is of the form $(x,t) \mapsto \mathbb{1}_{\mathbb{C}_a} F_1(x_+, t)$, where $a \in A^{\{-m,\cdots,-1\}}$ satisfies $M(a_{i-1}, a_i) = 1$ for $-m+1 \leqslant i \leqslant -1$, and F_1 is a continuous compactly supported function on $\mathbb{X}^+ \times \mathbb{R}$. For such a function, we have

$$\int_{\mathbb{X}_z^-} F\left(T^{-n}(y\cdot z), \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n-1\right\}} \nu_z^-(dy)$$
$$= \int_{\mathbb{X}_z^-} F_2\left(\left(T^{-n}(y\cdot z)\right)_+, \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n-1\right\}} \nu_z^-(dy),$$

where, for $(z', t') \in \mathbb{X}^+ \times \mathbb{R}$,

$$F_2(z',t') = \exp(-S_m\psi(a_{-m}\dots a_{-1}\cdot z'))F_1(z',t') = \int_{\mathbb{X}_{z'}^-} F(y\cdot z', t')\nu_{z'}^-(dy).$$

Since $F_2(\cdot, t')$ depends only on the future, we can apply Theorem 7.1, which gives

$$\lim_{n\to\infty} n^{3/2} \int_{\mathbb{X}_z^-} F_2\left(\left(T^{-n}(y\cdot z)\right)_+, \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n-1\right\}} \nu_z^-(dy)$$
$$= \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}\times\mathbb{R}} F_2\left(x'_+, t'\right) \mu^{(-g)}(dx', dt').$$

To conclude, it remains to show that

$$\int_{\mathbb{X}\times\mathbb{R}} F_2\left(x_+, t\right) \mu^{(-g)}(dx, dt) = \int_{\mathbb{X}\times\mathbb{R}} F\left(x, t\right) \mu^{(-g)}(dx, dt).$$

Indeed, by the definition of the measure $\mu^{(-g)}$ (see Theorem 1.1) and by using Lemma 2.2, we get

$$\begin{aligned}
&\int_{\mathbb{X}\times\mathbb{R}} F\left(x, t\right) \mu^{(-g)}(dx, dt) \\
&= \lim_{n\to\infty} \int_{\mathbb{X}\times\mathbb{R}} F\left(x, t\right) (-S_n g(x)) \mathbb{1}_{\{\tau_t^{(-g)}(x)>n\}} \nu(dx)dt \\
&= \lim_{n\to\infty} \int_{\mathbb{X}_+} \int_{\mathbb{R}} (-S_n g(z)) \mathbb{1}_{\{\tau_t^{(-g)}(z)>n\}} \int_{\mathbb{X}_z^-} F(y\cdot z, t)\nu_z^-(dy)dt\nu^+(dz) \\
&= \lim_{n\to\infty} \int_{\mathbb{X}_+} \int_{\mathbb{R}} (-S_n g(z)) \mathbb{1}_{\{\tau_t^{(-g)}(z)>n\}} F_2(z,t)dt\nu^+(dz) \\
&= \int_{\mathbb{X}\times\mathbb{R}} F_2\left(x_+, t\right) \mu^{(-g)}(dx, dt),
\end{aligned}$$

which ends the proof of the lemma. $\square$

Now we will put a target on the starting point $y \in \mathbb{X}_z^-$.

Lemma 7.6. *Let $g \in \mathscr{B}^+$ be such that $\nu^+(g) = 0$. Assume that for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pg + q$ is not cohomologous to a function with values in $\mathbb{Z}$. Then, for any $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$ and any continuous compactly supported function F on $\mathbb{X}_z^- \times \mathbb{X} \times \mathbb{R}$, we have*

$$\begin{aligned}\lim_{n\to\infty} n^{3/2} \int_{\mathbb{X}_z^-} F\left(y, T^{-n}y \cdot z, t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n-1\right\}} \nu_z^-(dy) \\ = \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}_z^- \times \mathbb{X}\times\mathbb{R}} F\left(y',x',t'\right) \mu^{(-g)}(dx',dt') \check{\mu}_{z,t}^{g,-}(dy').\end{aligned}$$

Proof. As usual, it suffices to prove the lemma when F is of the form $(y,x,t') \mapsto \mathbb{1}_{\mathbb{C}_{a,z}}(y) G(x,t')$, where $a \in A_z^m$ and G is a continuous compactly supported function on $\mathbb{X}\times\mathbb{R}$.

If $t + S_k g(T^{m-k}(a\cdot z)) \geqslant 0$ for every $1 \leqslant k \leqslant m$, we have that for $n > m$,

$$\begin{aligned}&n^{3/2} \int_{\mathbb{X}_z^-} F\left(y, T^{-n}y\cdot z, t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n-1\right\}} \nu_z^-(dy) \\ &= n^{3/2} \exp(-S_m\psi(a\cdot z)) \\ &\quad \times \int_{\mathbb{X}_{a\cdot z}^-} G\left(T^{-(n-m)}y\cdot(a\cdot z), t + S_m g(a\cdot z) + \check{S}_{n-m} g(y\cdot(a\cdot z))\right) \\ &\quad \times \mathbb{1}_{\left\{\check{\tau}^g_{t+S_m g(a\cdot z)}(y\cdot(a\cdot z)) > n-m-1\right\}} \nu_{a\cdot z}^-(dy).\end{aligned}$$

By Lemma 7.5, as $n \to \infty$, the latter quantity converges to

$$\frac{2\check{V}^g(a\cdot z, t + S_m g(a\cdot z))}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}\times\mathbb{R}} G\left(x',t'\right) \mu^{(-g)}(dx',dt') \exp(-S_m\psi(a\cdot z)),$$

which, by the definition of measure $\check{\mu}_{z,t}^{g,-}$ (see (3.21)), is equal to

$$\begin{aligned}&\frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3} \check{\mu}_{z,t}^{g,-}(\mathbb{C}_{a,z}) \int_{\mathbb{X}\times\mathbb{R}} G\left(x',t'\right) \mu^{(-g)}(dx',dt') \\ &= \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F\left(y',x',t'\right) \mu^{(-g)}(dx',dt') \check{\mu}_{z,t}^{g,-}(dy').\end{aligned}$$

If there exists $1 \leqslant k \leqslant m$ with $t + S_k g(T^{m-k}(a\cdot z)) < 0$, we have $\check{\mu}_{z,t}^{g,-}(\mathbb{C}_{a,z}) = 0$ and

$$\int_{\mathbb{X}_z^-} F\left(y, T^{-n}y\cdot z, t + \check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n\right\}} \nu_z^-(dy) = 0$$

for $n > k$. The conclusion follows. $\square$

As usual, from Lemma 7.6, we want to deduce the analogous result for functions which depend only on finitely many negative coordinates. We shall use the following easy formula that relates the measures μ^g and $\mu^{g\circ T}$.

Lemma 7.7. *Let $g \in \mathscr{B}$ be such that $\nu(g) = 0$ and g is not a coboundary. Then, for any continuous compactly supported function F on $\mathbb{X}\times\mathbb{R}$, we have*

$$\int_{\mathbb{X}\times\mathbb{R}} F(x,t) \mu^{g\circ T}(dx,dt) = \int_{\mathbb{X}\times\mathbb{R}} F(T^{-1}x,t) \mu^g(dx,dt).$$

Proof. By using the relation $\tau_t^{g\circ T} = \tau_t^g \circ T$, we get

$$\begin{aligned}\int_{\mathbb{X}\times\mathbb{R}} F(x,t)\mu^{g\circ T}(dx,dt) &= \lim_{n\to\infty}\int_{\mathbb{X}\times\mathbb{R}} F(x,t)S_n(g\circ T)(x)\mathbb{1}_{\{\tau_t^{g\circ T}(x)>n\}}\nu(dx)dt \\ &= \lim_{n\to\infty}\int_{\mathbb{X}\times\mathbb{R}} F(T^{-1}x,t)S_n g(x)\mathbb{1}_{\{\tau_t^{g}(x)>n\}}\nu(dx)dt \\ &= \int_{\mathbb{X}\times\mathbb{R}} F(T^{-1}x,t)\mu^{g}(dx,dt),\end{aligned}$$

as desired. □

Lemma 7.8. *Let $g \in \mathscr{B}$ be such that $\nu(g) = 0$ and there exists $m \geqslant 0$ with $g\circ T^m \in \mathscr{B}^+$. Assume that for any $p \neq 0$ and $q \in \mathbb{R}$, the function $pg+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Then, for any $(z,t) \in \mathbb{X}^+\times\mathbb{R}$ and any continuous compactly supported function F on $\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}$, we have*

$$\begin{aligned}&\lim_{n\to\infty} n^{3/2}\int_{\mathbb{X}_z^-} F\left(y, T^{-n}y\cdot z, t+\check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n-1\right\}}\nu_z^-(dy) \\ &\qquad = \frac{2\check{V}^g(z,t)}{\sqrt{2\pi}\sigma_g^3}\int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F\left(y',x',t'\right)\mu^{(-g)}(dx',dt')\check{\mu}_{z,t}^{g,-}(dy').\end{aligned}$$

Proof. As in Lemma 3.15, for $a \in A_z^m$, set F_a to be the function on $\mathbb{X}_{a\cdot z}^-\times\mathbb{X}\times\mathbb{R}$ defined by $F_a(y,x,t) = F(y\cdot a, T^m x, t)$. Then we have, by setting $h = g\circ T^m$,

$$\begin{aligned}&n^{3/2}\int_{\mathbb{X}_z^-} F\left(y, T^{-n}(y\cdot z), \check{S}_n g(y\cdot z)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z)>n-1\right\}}\nu_z^-(dy) \\ &= n^{3/2}\sum_{a\in A_z^m}\exp(-S_m\psi(a\cdot z))\int_{\mathbb{X}_{a\cdot z}^-} F_a\left(y, T^{-n}(y\cdot(a\cdot z)), \check{S}_n h(y\cdot(a\cdot z))\right) \\ &\qquad \times \mathbb{1}_{\left\{\check{\tau}_t^h(y\cdot(a\cdot z))>n-1\right\}}\nu_{a\cdot z}^-(dy).\end{aligned}$$

By Lemma 7.6, as $n\to\infty$, this converges to

$$\begin{aligned}&\sum_{a\in A_z^m}\exp(-S_m\psi(a\cdot z))\frac{2\check{V}^h(a\cdot z,t)}{\sqrt{2\pi}\sigma_g^3} \\ &\qquad\times\int_{\mathbb{X}_{a\cdot z}^-\times\mathbb{X}\times\mathbb{R}} F_a\left(y',x',t'\right)\mu^{(-h)}(dx',dt')\check{\mu}_{a\cdot z,t}^{h,-}(dy').\end{aligned}$$

By (3.24), the latter quantity is equal to

$$\int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F\left(y',T^m x',t'\right)\mu^{(-h)}(dx',dt')\check{\mu}_{z,t}^{g,-}(dy').$$

As $h = g\circ T^m$, the conclusion now follows from Lemma 7.7. □

Now we can give a result for any function g in $\mathscr{B}$.

Lemma 7.9. *Let $g\in\mathscr{B}$ be such that $\nu(g) = 0$. Assume that for any $p\neq 0$ and $q\in\mathbb{R}$, the function $pg+q$ is not cohomologous to a function with values in $\mathbb{Z}$. Then, for any continuous compactly supported function F on $\mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}$, we have*

$$\begin{aligned}&\lim_{n\to\infty} n^{3/2}\int_{\mathbb{X}\times\mathbb{R}} F\left(x, T^{-n}x, t, t+\check{S}_n g(x)\right)\mathbb{1}_{\left\{\check{\tau}_t^g(x)>n-1\right\}}\nu(dx)dt \\ &\qquad = \frac{2}{\sqrt{2\pi}\sigma_g^3}\int_{\mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}} F\left(x,x',t,t'\right)\mu^{(-g)}(dx',dt')\check{\mu}^{g}(dx,dt).\end{aligned}$$

Proof. We can assume that the function F is non-negative. For $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$, denote

$$W_n(z,t) = n^{3/2} \int_{\mathbb{X}_z^-} F\left(y\cdot z, T^{-n}(y\cdot z), t, t+\check{S}_n g(y\cdot z)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(y\cdot z) > n-1\right\}} \nu_z^-(dy).$$

Let $(g_m)_{m\geqslant 0}$ be the sequence of Hölder continuous functions as in Lemma 2.11. For any $n, m \geqslant 0$, we set

$$F_m^+(x,x',t,t') = \sup_{|s'|\leqslant 2c_1\alpha^m} F(x,x',t-2c_1\alpha^m, t'+s'),$$
$$F_m^-(x,x',t,t') = \inf_{|s'|\leqslant 2c_1\alpha^m} F(x,x',t+2c_1\alpha^m, t'+s'),$$

and

$$W_{n,m}^+(z,t) = n^{3/2} \int_{\mathbb{X}_z^-} F_m^+\left(y\cdot z, T^{-n}(y\cdot z), t, t+\check{S}_n g_m(y\cdot z)\right)$$
$$\times \mathbb{1}_{\left\{\check{\tau}_t^{g_m}(y\cdot z) > n-1\right\}} \nu_z^-(dy),$$
$$W_{n,m}^-(z,t) = n^{3/2} \int_{\mathbb{X}_z^-} F_m^-\left(y\cdot z, T^{-n}(y\cdot z), t, t+\check{S}_n g_m(y\cdot z)\right)$$
$$\times \mathbb{1}_{\left\{\check{\tau}_t^{g_m}(y\cdot z) > n-1\right\}} \nu_z^-(dy).$$

For $z \in \mathbb{X}^+$ and $t \in \mathbb{R}$, it holds that

$$W_{n,m}^-(z,t-2c_1\alpha^m) \leqslant W_n(z,t) \leqslant W_{n,m}^+(z,t+2c_1\alpha^m).$$

By taking the limit as $n\to\infty$, we get by Lemma 7.8,

$$\begin{aligned}
&\frac{2\check{V}^{g_m}(z,t-2c_1\alpha^m)}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F_m^-\left(y\cdot z, x', t-2c_1\alpha^m, t'\right) \\
&\qquad\qquad \times \mu^{(-g_m)}(dx',dt') \check{\mu}_{z,t-2c_1\alpha^m}^{g_m,-}(dy) \\
&\leqslant \liminf_{n\to\infty} W_n(z,t) \leqslant \limsup_{n\to\infty} W_n(z,t) \\
&\leqslant \frac{2\check{V}^{g_m}(z,t+2c_1\alpha^m)}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}_z^-\times\mathbb{X}\times\mathbb{R}} F_m^+\left(y\cdot z, x', t+2c_1\alpha^m, t'\right) \\
&\qquad\qquad \times \mu^{(-g_m)}(dx',dt') \check{\mu}_{z,t+2c_1\alpha^m}^{g_m,-}(dy).
\end{aligned} \tag{7.7}$$

On the one hand, after integrating over $\mathbb{X}^+\times\mathbb{R}$ in (7.7) with respect to the product of ν^+ with the Lebesgue measure, by Fatou's lemma and Lemma 7.8, we get

$$\begin{aligned}
&\frac{2}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}} F_m^-\left(x,x',t,t'\right) \mu^{(-g_m)}(dx',dt') \check{\mu}^{g_m}(dx,dt) \\
&\leqslant \liminf_{n\to\infty} n^{3/2} \int_{\mathbb{X}\times\mathbb{R}} F\left(x, T^{-n}x, t, t+\check{S}_n g(x)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(x) > n-1\right\}} \nu(dx) dt.
\end{aligned} \tag{7.8}$$

To conclude, we need to show the reverse Fatou property holds. To this aim, we choose a non-negative continuous compactly supported function G on $\mathbb{R}$ such that for any $(x,x',t,t') \in \mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}$, one has $F_0^+(x,x',t,t') \leqslant G(t)G(t')$. Then, we get for $(z,t)\in\mathbb{X}^+\times\mathbb{R}$,

$$W_{n,0}^+(z,t) \leqslant U_n(z,t) := n^{3/2} G(t) \int_{\mathbb{X}_z^-} G(t+\check{S}_n g_0(y\cdot z)) \mathbb{1}_{\left\{\check{\tau}_t^{g_0}(y\cdot z) > n-1\right\}} \nu_z^-(dy).$$

By Lemma 7.1, $U_n(z,t)$ converges uniformly in $(z,t) \in \mathbb{X}^+ \times \mathbb{R}$. Therefore, by applying Fatou's lemma to the sequence $U_n(z,t) - W_n(z,t)$, we get by integrating over $\mathbb{X}^+ \times \mathbb{R}$ in (7.7) with respect to the product of ν^+ with the Lebesgue measure,

$$\begin{aligned} &\frac{2}{\sqrt{2\pi}\sigma_g^3} \int_{\mathbb{X}\times\mathbb{X}\times\mathbb{R}\times\mathbb{R}} F_m^+(x,x',t,t')\, \mu^{(-g_m)}(dx',dt')\check{\mu}^{g_m}(dx,dt) \\ &\geqslant \limsup_{n\to\infty} n^{3/2} \int_{\mathbb{X}\times\mathbb{R}} F\left(x, T^{-n}x, t, t+\check{S}_n g(x)\right) \mathbb{1}_{\left\{\check{\tau}_t^g(x)>n-1\right\}} \nu(dx)dt. \end{aligned} \tag{7.9}$$

By letting $m \to \infty$, the conclusion follows from (7.8), (7.9) and Lemma 3.19. □

Proof of Theorem 1.7. By the duality lemma (Lemma 2.12), we have

$$\begin{aligned} &\int_{\mathbb{X}\times\mathbb{R}} F(x, T^n x, t, t+S_n f(x)) \mathbb{1}_{\left\{\tau_t^f(x)>n-1\right\}} \nu(dx)dt \\ &\quad = \int_{\mathbb{X}\times\mathbb{R}} F(T^{-n}x, x, t-\check{S}_n f(x), t) \mathbb{1}_{\left\{\check{\tau}_t^{(-f)}(x)>n-1\right\}} \nu(dx)dt. \end{aligned}$$

Now the conclusion follows from Lemma 7.9. □

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