An introduction to the study of dynamical systems on homogeneous spaces

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Contents

Chapter 1

Lattices

1.1 Haar measure

In this section, we briefly recall the elementary properties of Haar measures that we will use. First, we have the following fundamental existence result:

Theorem 1.1.1. *Let G be a locally compact topological group. Then G admits a non zero Radon measure that is invariant under left translations. This measure is unique up to multiplication by a non zero real number.*

Such a measure is called a (left) Haar measure on G (and sometimes the Haar measure of G, since it is essentially unique).

Example 1.1.2. Haar measure of $\mathbb{R}^d$ is Lebesgue measure. Haar measure of discrete groups is the counting measure.

Example 1.1.3. One of the difficulties which one encounters in the study of Haar measure is that it is not in general right invariant. For example, if P denotes the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \quad a, b \in \mathbb{R}, \quad a \neq 0,$$

then one easily checks that the measure $\frac{1}{|a|}\mathrm{d}a\mathrm{d}b$ is left invariant, but that it is not right invariant.

Definition 1.1.4. Let G be a locally compact topological group with left Haar measure $\mathrm{d}g$. The modular function of G is the unique function Δ_G :

$G \to \mathbb{R}_+^*$ such that, for any continuous function φ with compact support on G, for any h in G, one has

$$\int_G \varphi(gh^{-1})\mathrm{d}g = \Delta_G(h) \int_G \varphi(g)\mathrm{d}g.$$

The existence of Δ_G follows from the uniqueness part in Theorem 1.1.1. Note that Δ_G is a continuous multiplicative morphism.

Example 1.1.5. If P is as above, one has $\Delta_P \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \left|\frac{1}{a^2}\right|$.

Definition 1.1.6. A locally compact topological group is said to be unimodular if its Haar measures are both left and right invariant.

Example 1.1.7. Discrete groups are unimodular. Abelian groups (and more generally, nilpotent groups) are unimodular. Compact groups are unimodular since the modular function is a morphism and $\mathbb{R}_+^*$ does not admit any compact subgroup. The groups $\mathrm{SL}_d(\mathbb{R})$ and $\mathrm{GL}_d(\mathbb{R})$, $d \geq 2$, (and more generally, any reductive Lie group) are unimodular.

Let us now study Haar measure on quotient spaces. Recall that, if G is a locally compact topological group and H is a closed subgroup of G, then the quotient topology on G/H is Hausdorff and locally compact.

Proposition 1.1.8. *Let G be a locally compact topological group and H be a closed subgroup of G. Then there exists a non zero G-invariant Radon measure on G/H if and only if the modular functions of G and H are equal on H. This measure is then unique up to multiplication by a scalar.*

In particular, if both G and H are unimodular, such a measure exists (and this is the main case of application of this result).

Remark 1.1.9. There is a formula which relates measure on quotient spaces and Haar measures on G and H. Let $\mathrm{d}g$ and $\mathrm{d}h$ denote Haar measures on G and H. For any continuous compactly supported function φ on G, set, for g in G, $\overline{\varphi}(gH) = \int_H \varphi(gh)\mathrm{d}h$. Then $\overline{\varphi}$ is a continuous compactly supported function on G/H and, if the assumptions of the proposition hold, one can chose the G-invariant measure μ on G/H in such a way that

$$\int_{G/H} \overline{\varphi}\mathrm{d}\mu = \int_G \varphi(g)\mathrm{d}g$$

(that is, to integrate on G, one first integrate on fibers of the projection $G \to G/H$ and then on the base space G/H).

When G is second countable (which is of course the interesting case for geoemtric purposes!), this formula can easily be extended to any Borel function φ which is integrable with respect to Haar measure.

Note that if H is discrete, one can chose the counting measure as Haar measure, so that, for g in G, $\overline{\varphi}(gH) = \sum_{h\in H} \varphi(gh)$.

Example 1.1.10. Since $G = \mathrm{SL}_2(\mathbb{R})$ is unimodular and its subgroup

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\} \simeq \mathbb{R}$$

is unimodular, the space G/N admits an invariant measure: indeed, as a homogeneous space, this space identifies with $\mathbb{R}^2 \smallsetminus \{0\}$ and this measure is nothing but the restriction of the Lebesgue measure of $\mathbb{R}^2$.

The projective line does not admit any Radon measure that is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$, since it is the quotient of $\mathrm{SL}_2(\mathbb{R})$ by the group P of Example 1.1.3 which is not unimodular. The fact that $\mathbb{P}^1_{\mathbb{R}}$ does not admit any $\mathrm{SL}_2(\mathbb{R})$-invariant measure can of course be proved directly!

1.2 Definition and first examples of lattices

Lattices are discrete subgroups that are very large:

Definition 1.2.1. Let G be a locally compact topological group and Γ be a discrete subgroup of G. We say that Γ is a lattice in G if the locally compact topological space G/Γ admits a finite Radon measure that is invariant under the natural left action of G.

Example 1.2.2. For any integer $d \geq 1$, the subgroup $\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$. Indeed, since they are abelian groups, the quotient is an abelian group and its Haar measure is invariant under translations. Since the quotient is compact, this measure is finite.

As we saw in Proposition 1.1.8, if H is a closed subgroup of G, there is no reason for the homogeneous space G/H to admit a G-invariant Radon measure. In fact, we have

Proposition 1.2.3. *Let G be a locally compact topological group which admits a lattice. Then G is unimodular.*

Proof. Let Γ be a lattice in G. Since G/Γ admits a G-invariant measure, by Proposition 1.1.8, the modular function Δ_G of G is trivial on Γ. Let H be the kernel of Δ_G, so that $H \supset \Gamma$. The image of the G-invariant measure on G/Γ is a finite G-invariant measure on G/H. This means that the quotientgroup G/H has finite Haar measure, that is, it is compact. Now, Δ_G factors as an injective morphism $G/H \to \mathbb{R}_+^*$. Since this latter group does not admit non trivial compact subgroups, we get $H = G$ as required. □

By using the elementary properties of Haar measure, one can show

Proposition 1.2.4. *Let G be a locally compact topological group and Γ be a discrete subgroup of G. If Γ is cocompact in G (that is, if the space G/Γ is compact), then Γ is a lattice.*

Example 1.2.5. Let H (resp. Γ) be the Heisenberg group (resp. the discrete Heisenberg group), that is, the group of matrices of the form

$$h_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with x, y, z in $\mathbb{R}$ (resp. $\mathbb{Z}$) and let $Z = \{h_{0,0,z} | z \in \mathbb{R}\}$. One easily proves that Z is the center of H, hence in particular, that Z is a normal subgroup of H. One has $H/Z \simeq \mathbb{R}^2$ and ΓZ is a closed subgroup of H and $\Gamma Z/Z$ is a cocompact lattice in H/Z. Therefore, since $\Gamma \cap Z$ is also a cocompact lattice in Z, the space H/Γ is compact and Γ is a lattice in H.

Example 1.2.6. Let G be the sol group, that is the semidirect product $U \ltimes V$, where $U = \mathbb{R}$ acts on $V = \mathbb{R}^2$ through the one-parameter group of automorphisms $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}_{t \in \mathbb{R}}$. Now, let A be a hyperbolic element of $\mathrm{SL}_2(\mathbb{Z})$ with positive eigenvalues (for example, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$). Then there exists g in $\mathrm{SL}_2(\mathbb{R})$ such that $B = gAg^{-1}$ is diagonal. By construction, B preserves the lattice $\Lambda = g\mathbb{Z}^2$ in V. We set $\Gamma = B^{\mathbb{Z}}\Lambda$. One easily checks that, since Λ is cocompact in V and $B^{\mathbb{Z}}$ is cocompact in U, the group Γ is cocompact in G and hence, that it is a lattice.

Example 1.2.7. Let S be a closed orientable surface of genius $g \geq 2$ with fundamental group Γ. Choose a uniformization of S, that is, equivalently, equip S with a complex manifold structure or with a Riemannian metric with constant curvature -1. Then the universal cover of S is identified with the real hyperbolic plane $\mathbb{H}^2$ and Γ acts on $\mathbb{H}^2$ by orientation preserving isometries, that is, one is given an injective morphism from Γ to $\mathrm{PSL}_2(\mathbb{R})$ (which has discrete image). Since, as a $\mathrm{PSL}_2(\mathbb{R})$-homogeneous space, one has $\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$ and since $S = \Gamma\backslash\mathbb{H}^2$ is compact, the image of Γ in $\mathrm{PSL}_2(\mathbb{R})$ is a discrete cocompact subgroup, that is, it is a lattice.

In the sequel, we will prove that, for $d \geq 2$, $\mathrm{SL}_d(\mathbb{Z})$ is a non cocompact lattice in $\mathrm{SL}_d(\mathbb{Z})$. Nevertheless, there are groups all of whose lattices are cocompact. Indeed, one has the

Proposition 1.2.8. *Let G be a locally compact topological group. Assume G is nilpotent. Then every lattice in G is cocompact.*

Remark 1.2.9. This does not mean that every such group admits a lattice! Indeed, by dimension arguments, one can easily prove that there are a lot of 14-dimensional nilpotent Lie groups which do not admit lattices.

For solvable groups, the situation is more complicated. On one hand, one has the

Theorem 1.2.10 (Mostow)**.** *Le G be a solvable Lie group. Every lattice in G is cocompact.*

On the other hand, there exists an example by Bader-Caprace-Gelander-Mozes of a locally compact solvable group G with a non cocompact lattice. In this example, the group G is not compactly generated. To my knowledge, is is not known wether this can be done with a compactly generated group.

Let us now describe compact subsets of spaces of the form G/Γ, where Γ is a lattice in G.

Proposition 1.2.11. *Let G be a locally compact topological group, Γ be a lattice in G and $A \subset G/\Gamma$. Then A is relatively compact if and only if there exists a neighborhood U of e in G such that, for any $x = g\Gamma$ in A, for any $\gamma \neq e$ in Γ, one has $g\gamma g^{-1} \notin U$.*

Remark 1.2.12. This propoition is a group-theoretic translation of the following fact: if a Riemannian manifold M has finite Riemannian volume but is not compact, then, close to infinity in M, the injectivity radius is small.

Proof. Assume A is contained in a compact subset L. For every $x = g\Gamma$ in L, the group $g\Gamma g^{-1}$ is discrete in G, hence there exists a neighborhood W_x of e in G such that $g\Gamma g^{-1} \cap W_x = \{e\}$. Let us prove that we can chose a neighborhood of e that does not depend on x in L. Indeed, there exists a symmetric neighborhood V_x of e such that $V_x V_x V_x \subset W_x$. By construction, for any h in V_x, we have

$$hg\Gamma g^{-1}h^{-1} \cap V_x = h(g\Gamma g^{-1} \cap h^{-1}V_x h)h^{-1} \subset h(g\Gamma g^{-1} \cap W_x)h^{-1} = \{e\}.$$

Now, we pick $x_1, \dots, x_r$ in L such that $L \subset V_{x_1}x_1 \cup \dots \cup V_{x_r}x_r$ and we set $U = V_{x_1} \cap \dots \cap V_{x_r}$. By construction, we have, for any $x = g\Gamma$ in L, $g\Gamma g^{-1} \cap U \subset \{e\}$.

Conversely, assume there exists such a compact neighborhood U. Let μ be the invariant measure on G/Γ, that we normalize in such a way that the formula from Remark 1.1.9 holds, when G is equipped with a given Haar measure and Γ is equipped with the counting measure. Choose a non zero continuous compactly supported function $\varphi \geq 0$ on G, with support contained in U. Set $\varepsilon = \int_G \varphi(g)\mathrm{d}(g)$, so that, for any x in A, we have $\mu(Ux) \geq \varepsilon$. If $F \subset A$ is a finite subset such that for any $x \neq y$ in F, $Ux \cap Uy = \emptyset$, one has $\sharp F \leq \varepsilon^{-1}$, hence one can choose such a F to be maximal. Then, for any $z \in A \backslash F$, one has $Uz \cap UF \neq \emptyset$ and hence $A \subset U^{-1}UF$, so that A is relatively compact. $\square$

Corollary 1.2.13. *Let G be a locally compact topological group, H be a closed subgroup of G and Γ be a discrete subgroup of G such that $\Gamma \cap H$ is a lattice in H. Then the natural map $H/(\Gamma \cap H) \to G/\Gamma$ is proper. In particular, the set $H/(\Gamma \cap H)$ is a closed subset of G/Γ.*

Remark 1.2.14. If $\Gamma \cap H$ is cocompact in H, there is nothing to prove.

Example 1.2.15. This is far from being true if one does not assume $\Gamma \cap H$ to be a lattice in H: for example, if $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and $H = \mathbb{Z}\alpha$, where α belongs to $\mathbb{R} \smallsetminus \mathbb{Q}$, one has $H \cap \Gamma = \{e\}$ and H has dense image in G/Γ!

Proof. Let L be a compact subset of G/Γ. Note that the first part of the proof of Proposition 1.2.11 only uses the fact that Γ is a discrete subgroup of G, so that, reasoning in the same way, we get that there exists a neighborhood U of e in G such that, for any $x = g\Gamma$ in G/Γ, one has $g\Gamma g^{-1} \cap U = \{e\}$. If x also belongs to $H/(\Gamma \cap H)$, we get $g(\Gamma \cap U)g^{-1} \cap (U \cap H) = \{e\}$. By Proposition 1.2.11, we get that $L \cap (H/(\Gamma \cap H))$ is a compact subset of $H/(\Gamma \cap H)$, what should be proved. $\square$

Finally, let us give an elementary criterion for a group to be a lattice:

Proposition 1.2.16. *Let G be a unimodular second countable locally compact group and Γ be a discrete subgroup of G. Then Γ is a lattice if and only if there exists a Borel subset S of G with finite Haar measure such that $G = S\Gamma$. If this is the case, one can choose S in such a way that the natural map $G \to G/\Gamma$ is injective on S.*

Proof. Assume first that such a set S exist and apply the formula from Remark 1.1.9 to the indicator function φ of the set S. Since $G = S\Gamma$, we have $\overline{\varphi} \geq 1$ on G/Γ, hence, since S has finite Haar measure, G/Γ has finite G-invariant measure.

Conversely, let us first construct a Borel subset S of G such that $G = S\Gamma$ and the natural map $G \to G/\Gamma$ is injective on S. Since Γ is discrete in G, for any g in G, there exists an open neighborhood U_g of g in G such that the map $h \mapsto h\Gamma, G \to G/\Gamma$ is injective on U_g. Since G/Γ is second countable, there exits a sequence $(g_k)_{k\geq 0}$ of elements of G such that the union $\bigcup_{k\geq 0} U_{g_k}$ covers G. We set

$$S = \bigcup_{k\geq 0} \left(U_{g_k} \smallsetminus \bigcup_{\ell<k} U_\ell \Gamma \right)$$

and we are done. In particular, if φ is the indicator function of S, we have $\overline{\varphi} = 1$ on G/Γ, so that, again by Remark 1.1.9, if G/Γ has finite G-invariant measure, S has finite Haar measure. $\square$

1.3 Lattices in $\mathbb{R}^d$ and the group $\mathrm{SL}_d(\mathbb{Z})$

We will now prove that $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$. To this aim, we first study lattices in $\mathbb{R}^d$. We equip $\mathbb{R}$ with the usual scalar product and Lebesgue measure.

We have the following elementary

Proposition 1.3.1. *Let Λ be a discrete subgroup of $\mathbb{R}^d$. Then there exists $k \leq d$ and a linearily independent family of vectors $v_1, \ldots, v_k$ in $\mathbb{R}^d$ such that Λ is the subgroup spanned by $v_1, \ldots, v_k$. In particular, Λ is a free abelian group and Λ is a lattice if and only if it has rank d.*

If Λ is a lattice in $\mathbb{R}^d$, we equip $\mathbb{R}^d/\Lambda$ with the unique Radon measure that is invariant under translations associated to the choice of Lebesgue measure on $\mathbb{R}^d$ and the counting measure on Λ (see Remark 1.1.9). The total measure of this quotient space is called the covolume of Λ; in other words, the covolume of Λ is the absolute value of the determinant of an algebraic base of Λ. We say Λ is unimodular if it has covolume 1 (this has nothing to do with the notion of a unimodular group).

Proposition 1.3.1 implies that every lattice in $\mathbb{R}^d$ is the image of the standard lattice $\mathbb{Z}^d$ by an element of $\mathrm{GL}_d(\mathbb{R})$. Since the stabilizer of $\mathbb{Z}^d$ in $\mathrm{GL}_d(\mathbb{R})$ is $\mathrm{GL}_d(\mathbb{Z})$, the space of lattices of $\mathbb{R}^d$ identifies with $\mathrm{GL}_d(\mathbb{R})/\mathrm{GL}_d(\mathbb{Z})$. In the same way, the space of unimodular lattices of $\mathbb{R}^d$ identifies with $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. Since we know that $\mathrm{SL}_d(\mathbb{R})$ is unimodular, by Proposition 1.2.16, to prove that $\mathrm{SL}_d(\mathbb{Z})$ is a lattice, we only have to exhibit a Borel subset S of $\mathrm{SL}_d(\mathbb{R})$ with finite Haar measure such that $\mathrm{SL}_d(\mathbb{R}) = S\mathrm{SL}_d(\mathbb{Z})$. The starting point of the construction of S is the following elementary lemma which tells us that lattices are not too far away from 0:

Lemma 1.3.2. *Let Λ be a unimodular lattice in $\mathbb{R}^2$. Then Λ contains a non zero vector with euclidean norm $\leq \left(\frac{4}{3}\right)^{\frac{1}{4}}$.*

Proof. Let v be a non zero vector with minimal norm in Λ. If $\|v\| \leq 1$, we are done. Else, let D be the open disk with center 0 and radius $\|v\|$. By assumption, we have $D \cap \Lambda = \{e\}$. Now, since v is a primitive vector of the free abelian group Λ, there exists a vector w in Λ such that (v, w) is a basis of Λ, and hence the determinant of (v, w) is ± 1. Thus, Λ contains an element whose distance to the line $\mathbb{R}v$ is $\frac{1}{\|v\|}$. Now, the set E of such vectors is a union of two affine lines whose intersection with D are intervals of length $2\sqrt{\|v^2\| - \frac{1}{\|v\|^2}}$. Since $E \cap \Lambda$ is stable under the translations by v and does not encounter D, we have

$$2\sqrt{\|v^2\| - \frac{1}{\|v\|^2}} \leq \|v\|,$$

that is,

$$\|v\|^4 \leq \frac{4}{3},$$

which should be proved. □

Now, to construct the set S, we need some structure results about $\mathrm{SL}_d(\mathbb{R})$. We let K denote $\mathrm{SO}(d)$, A denote the group of diagonal matrices with positive entries and N denote the group of upper-triangular matrices all of whose eigenvalues are equal to 1.

Proposition 1.3.3 (Iwasawa decomposition for $\mathrm{SL}_d(\mathbb{R})$)**.** *One has* $\mathrm{SL}_d(\mathbb{R}) = KAN$*. More precisely, the product map* $K \times A \times N \to \mathrm{SL}_d(\mathbb{R})$ *is a homeomorphism.*

Proof. This is a translation of Gram-Schmidt orthonormalization process. $\square$

Remark 1.3.4. Here is an interpretation of Iwasawa decompostion: if e_1 denotes the vector $(1, 0, \dots, 0)$ and if g is an element of $\mathrm{SL}_d(\mathbb{R})$ with Iwasawa decomposition kan, then $a_{1,1}$ is the norm of the vector ge_1.

The set S that we will build will be given as a product set in the Iwasawa decomposition. In order to compute its Haar measure, we need the

Lemma 1.3.5. *Equip the groups* K*,* A *and* N *with Haar measures. Then the image by the product map* $K \times A \times N \to \mathrm{SL}_d(\mathbb{R})$ *of the measure*

$$\prod_{1 \leq i < j \leq d} \frac{a_{i,i}}{a_{j,j}} \mathrm{d}k \mathrm{d}a \mathrm{d}n$$

is a Haar measure on $\mathrm{SL}_d(\mathbb{R})$*.*

Proof. Consider the product map $K \times AN \to \mathrm{SL}_d(\mathbb{R}), (k, p) \mapsto k^{-1}p$. Since $\mathrm{SL}_d(\mathbb{R})$ is unimodular, the inverse image of a Haar measure by this map is a measure on $K \times AN$ that is right $K \times AN$ invariant, so that it is the product of right Haar measures of K and AN. Since K is compact, it is unimodular and this right Haar measure is also a left Haar measure, so that it only remains to prove that the measure $\prod_{1 \leq i < j \leq d} \frac{a_{i,i}}{a_{j,j}} \mathrm{d}a \mathrm{d}n$ is a right Haar measure on AN.

An elementary matrix computation shows that one can choose $\mathrm{d}n$ to be the product measure $\prod_{i<j} \mathrm{d}n_{i,j}$. Now, N is a normal subgroup of AN and the adjoint action of an element a of A multiplies this measure by $\prod_{1 \leq i < j \leq d} \frac{a_{j,j}}{a_{i,i}}$. The result follows by easy computations. $\square$

For $t, u > 0$, we set

$$\begin{aligned} A_t &= \{a \in A | \forall 1 \leq i \leq d-1 \quad a_{i,i} \leq t a_{i+1,i+1}\} \\ N_u &= \{n \in N | \forall 1 \leq i < j \leq d \quad |n_{i,j}| \leq u\} \\ \text{and } \mathfrak{S}_{t,u} &= K A_t N_u. \end{aligned}$$

The set $\mathfrak{S}_{t,u}$ is called a Siegel domain. It is our candidate for being the set S in Proposition 1.2.16.

First, we check that Siegel domains have finite Haar measure:

Lemma 1.3.6. *For any $t, u > 0$, the set $\mathfrak{S}_{t,u}$ has finite Haar measure in* $\mathrm{SL}_d(\mathbb{R})$.

Proof. Since K and N_u are compact, by Lemma 1.3.5, it suffices to prove that one has

$$\int_{A_t} \prod_{1\leq i<j\leq d} \frac{a_{i,i}}{a_{j,j}} \mathrm{d}a < \infty,$$

where $\mathrm{d}a$ is a Haar measure on A.

Now, one easily checks that the map

$$A \to \mathbb{R}^{d-1}, a \mapsto \left(\log \frac{a_{2,2}}{a_{1,1}}, \dots, \log \frac{a_{d,d}}{a_{d-1,d-1}}\right)$$

is a topological group isomorphism. Therefore, we can choose $\mathrm{d}a$ in such a way that

$$\begin{aligned}\int_{A_t} \prod_{1\leq i<j\leq d} \frac{a_{i,i}}{a_{j,j}} \mathrm{d}a &= \int_{\mathbb{R}^{d-1}} \prod_{1\leq i<j\leq d} e^{-s_i-\dots-s_{j-1}} \mathbf{1}_{s_1,\dots,s_{d-1}\geq -\log t} \mathrm{d}s_1 \cdots \mathrm{d}s_{d-1} \\ &= \prod_{k=1}^{d-1} \int_{-\log t}^{\infty} e^{-k(d-k)s_k} \mathrm{d}s_k.\end{aligned}$$

The lemma follows, since the latter integral is finite. □

Example 1.3.7. This proof might seem mysterious at a first glance. Let us check what it means for $d = 2$. In this case, the space $\mathrm{SL}_2(\mathbb{R})/N$ is $\mathbb{R}^2 \smallsetminus \{0\}$ and the $\mathrm{SL}_2(\mathbb{R})$-invariant measure is the restriction of the Lebesgue measure of $\mathbb{R}^2$. Under this identification, the right N-invariant set KA_tN may be seen as the intersection of $\mathbb{R}^2 \smallsetminus \{0\}$ with the euclidean disk of radius $\sqrt{t}$. This set is not compact in $\mathbb{R}^2 \smallsetminus \{0\}$, but it has finite $\mathrm{SL}_2(\mathbb{R})$-invariant measure.

We now have all the tools in hand to prove the

Theorem 1.3.8 (Hermite)**.** *For any $d \geq 2$, the subgroup $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$. More precisely, one has $\mathrm{SL}_d(\mathbb{R}) = \mathfrak{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}} \mathrm{SL}_d(\mathbb{Z})$.*

Remark 1.3.9. Note that this lattice is not cocompact. Indeed, if one sets

$$\gamma = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & 1_{d-2} & \\ 0 & 0 & & & \end{pmatrix} \text{ and } g = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & 1_{d-2} & \\ 0 & 0 & & & \end{pmatrix}$$

(where by 1_{d-2} we mean a $(d-2)\times(d-2)$ identity matrix), then γ is an element of $\mathrm{SL}_d(\mathbb{Z})$ and $g^{-n}\gamma g^n \xrightarrow[n\to\infty]{} e$, so that, by Proposition 1.2.11, the sequence $(g^{-n}\mathrm{SL}_d(\mathbb{Z}))$ is not relatively compact in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$.

Proof. Note that the second part of the Theorem implies the first one, thanks to Proposition 1.2.16 and Lemma 1.3.6. Thus, let us prove that one has $\mathrm{SL}_d(\mathbb{R}) = \mathfrak{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}\mathrm{SL}_d(\mathbb{Z})$.

First, we prove by induction on $d \geq 1$ that one has

$$\mathrm{SL}_d(\mathbb{R}) = KA_{\frac{2}{\sqrt{3}}}N\mathrm{SL}_d(\mathbb{Z}).$$

Indeed, for $d=1$, there is nothing to prove. Assume $d\geq 2$ and the result is true for $d-1$. Pick g in $\mathrm{SL}_d(\mathbb{R})$, set $\Lambda = g\mathbb{Z}^d$ and let v_1 be a non zero element with minimal euclidean norm in Λ. Since v_1 is a principal vector of the free abelian group Λ, there exists $v_2,\dots,v_d$ in Λ such that $v_1,\dots,v_d$ is a basis of Λ. Now, let γ be the element of $\mathrm{GL}_d(\mathbb{Z})$ that sends $e_1,\dots,e_d$ to $g^{-1}v_1,\dots,g^{-1}v_d$ (where $e_1,\dots,e_d$ is the canonical basis of $\mathbb{R}^d$). After maybe replacing v_1 by $-v_1$, we can assume γ belongs to $\mathrm{SL}_d(\mathbb{Z})$. By construction, the matrix $g\gamma$ sends e_1 to v_1. Let k be an element of K that sends v_1 to $a_{1,1}e_1$, where $a_{1,1} = \|v_1\|$. Then the matrix $kg\gamma$ has the form

$$\begin{pmatrix} a_{1,1} & * \\ 0 & h \end{pmatrix},$$

where h is an element of $\mathrm{GL}_{d-1}(\mathbb{R})$ with determinant $a_{1,1}^{-1}$. By applying the induction assupmption to the matrix $a_{1,1}^{\frac{1}{d-1}}h$, we can find ℓ in K, δ in $\mathrm{SL}_d(\mathbb{Z})$ and $a_{2,2},\dots,a_{d,d} > 0$, with $a_{i,i} \leq \frac{2}{\sqrt{3}}a_{i+1,i+1}$, $2\leq i\leq d-1$, such that $\ell g\delta$ belongs to the set aN, where a is the diagonal matrix with diagonal entries $a_{1,1},\dots,a_{d,d}$. To conclude, we only have to prove that one has $a_{1,1}\leq \frac{2}{\sqrt{3}}a_{2,2}$. To this aim, we will use Lemma 1.3.2. Indeed, recall that $a_{1,1}$ is the minimal norm of a non zero vector in $\Lambda = g\mathbb{Z}^d$. Now, consider the discrete group

$$\Delta = g\delta(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) \subset \Lambda.$$

By construction, $(a_{1,1}a_{2,2})^{-\frac{1}{2}}\Delta$ is a unimodular lattice in the euclidean plane $g\delta(\mathbb{R}e_1\oplus\mathbb{R}e_2)$, all of whose vectors have norm $\geq \left(\frac{a_{1,1}}{a_{2,2}}\right)^{\frac{1}{2}}$. By Lemma 1.3.2, we get $a_{1,1}\leq\frac{2}{\sqrt{3}}a_{2,2}$ as required.

To finish the proof, it suffices to prove that one has $N = N_{\frac{1}{2}}(N \cap \mathrm{SL}_d(\mathbb{Z}))$. Again, we will prove this by induction on $d \geq 1$. For $d = 1$, there is nothing to prove. Assume $d \geq 2$ and the result is true for $d-1$. Pick g in N and let $k_2, \ldots, k_d$ be relative integers such that $|g_{1,j} + k_j| \leq \frac{1}{2}$, $2 \leq j \leq d$, and let γ be the matrix in N with entries $\gamma_{1,j} = k_j$, $2 \leq j \leq d$, and $\gamma_{i,j} = 0$, $2 \leq i < j \leq d$. Set $g' = g\gamma$. Then all the coefficients $g'_{1,j}$, $2 \leq j \leq d$ have absolute value $\leq \frac{1}{2}$. The result follows by applying the induction assumption to the lower right $(d-1) \times (d-1)$-block in g'. $\square$

Now that we know that $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$, Proposition 1.2.11 gives a characterization of compact subsets of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ in eterms of the adjoint action of $\mathrm{SL}_d(\mathbb{R})$. We can also give a characterization of them in terms of the identification of this space with the space of unimodular lattices in $\mathbb{R}^d$. This will use the

Definition 1.3.10. Let Λ be a lattice in $\mathbb{R}^d$. Then the systole $s(\Lambda)$ of Λ is the smallest norm of a non zero vector of Λ.

Corollary 1.3.11. *Let E be a subset of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. Then E is relatively compact if and only if there exists $\varepsilon > 0$ such that, for any $x = g\mathrm{SL}_d(\mathbb{Z})$ in E, one has $s(g\Lambda) \geq \varepsilon$.*

Proof. Assume E is relatively compact and let L be a compact subset of $\mathrm{SL}_d(\mathbb{R})$ such that $E \subset L\mathrm{SL}_d(\mathbb{Z})$. Set $C = \sup_{g \in L} \|g^{-1}\|$. Then, for any $v \neq 0$ in $\mathbb{Z}^d$ and g in L, one has $\|gv\| \geq \frac{1}{C}$.

Conversely, assume there exists $\varepsilon > 0$ as in the statement of the corollary. For $t > 0$, set

$$\begin{aligned} A_t^\varepsilon &= \{a \in A | a_{1,1} \geq \varepsilon \text{ and } \forall 1 \leq i \leq d-1 \quad a_{i,i} \leq t a_{i+1,i+1}\} \\ &= \{a \in A_t | a_{1,1} \geq \varepsilon\}. \end{aligned}$$

We claim first that $E \subset K A^\varepsilon_{\frac{2}{\sqrt{3}}} N_{\frac{1}{2}} \mathrm{SL}_d(\mathbb{Z})$. Indeed, by Theorem 1.3.8, for g in $\mathrm{SL}_d(\mathbb{R})$, one can write g as $kan\gamma$, where k is in K, a is in $A_{\frac{2}{\sqrt{3}}}$, n is in $N_{\frac{1}{2}}$ and γ is in $\mathrm{SL}_d(\mathbb{Z})$. In particular, one has $\|g\gamma^{-1}e_1\| = a_{1,1}$, so that, if $g\mathrm{SL}_d(\mathbb{Z})$ belongs to E, a belongs to $A^\varepsilon_{\frac{2}{\sqrt{3}}}$. Now, we claim that for any $t > 0$, A_t^ε is a compact subset of A. Indeed, for any a in A_t^ε, one has, for $1 \leq i \leq d$, $a_{i,i} \geq t^{1-i} a_{1,1} \geq t^{1-i}\varepsilon$, whereas since $a_{d,d} = \frac{1}{a_{1,1} \cdots a_{d-1,d-1}}$,

$$a_{i,i} \leq t^{d-i} a_{d,d} = \frac{t^{d-i}}{a_{1,1} \cdots a_{d-1,d-1}} \leq t^{\frac{d(d+1)}{2} - i} \varepsilon^{1-d},$$

that is, the coordinate functions are bounded on A_t^ε, so that it is compact. Both statements imply that E is relatively compact. $\square$

Chapter 2

Howe-Moore ergodicity Theorem

In this chapter, we present a general theorem which shows how group-theoretic tools can be used in order to prove dynamical results.

2.1 Group actions and representations

In this section, we make the link between group actions on probability spaces and unitary representations in Hilbert spaces.

To be precise, we need to introduce a natural assumption on the probability spaces we shall encounter. Let $(X, \mathcal{A}, \mu)$ be a probability space, that is X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a probability measure on $(X, \mathcal{A})$. Recall that the space $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$ may be viewed as the dual space to the space $\mathrm{L}^1(X, \mathcal{A}, \mu)$ of classes of integrable functions on X, so that it admits a weak-$*$ topology. In general, this space is not separable.

Definition 2.1.1. We say $(X, \mathcal{A}, \mu)$ is a Lebesgue probability space if the space $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$ contains a separable weak-$*$ dense subalgebra A which separates points, that is, there exists a measurable subset E of X with $\mu(E) = 1$ such that, for any $x \neq y$ in E, there exists f in A with $f(x) \neq f(y)$.

Every natural probability space that appears in geometry is a Lebesgue space. More precisely, we have

Proposition 2.1.2. *Let X be a locally compact second countable space, $\mathcal{B}$*

be its Borel σ-algebra and μ be a Borel probability measure on X. Then $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space.

Proof. Take A to be the image in $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$ of the algebra of bounded continuous functions on X. $\square$

Up to atoms (and to isomorphism!), there exists only one Lebesgue probability space:

Proposition 2.1.3. *Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space. Then, there exists t in $[0, 1]$ such that, as a probability space, $(X, \mathcal{A}, \mu)$ is isomorphic to the disjoint union of $[0, t]$, equipped with the restriction of Lebesgue measure of $\mathbb{R}$, and of $\mathbb{N}$, equipped with the full σ-algebra and a measure of mass $1 - t$.*

Now, we will study group actions on these spaces.

Definition 2.1.4. Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space ang G be a locally compact group. A measure preserving action of G on X is an action of G on X such that the action map $G \times X \to X$ is measurable and that, for any g in G, the map $x \mapsto gx, X \to X$ preserves the measure μ.

For example, if X is a locally compact second countable space and G acts continuously on X, if μ is a Borel probability measure on X that is preserved by the elements of G, we get a measure preserving action of G. Of course, this is the main example of such an action. But, even in geometric situation, it may happen that certain operations (such as taking quotients) break the topological structure.

Given a measure preserving action of G on $(X, \mathcal{A}, \mu)$, we get actions by isometries of G on the spaces $\mathrm{L}^p(X, \mathcal{A}, \mu)$, $1 \leq p \leq \infty$, defined by $gf = f \circ g^{-1}$, $g \in G$, $f \in \mathrm{L}^p(X, \mathcal{A}, \mu)$. The understanding of these representations plays a great role in the study of the dynamical properties of the action of G. We first define a notion of continuity for actions on Banach spaces.

Definition 2.1.5. Let G be a locally compact group and E be a Banach space and let G act on E by isometries. We say that the action is strongly continuous if the action map $G \times E \to E$ is continuous.

One easily checks that this amounts to say that, for any v in E, the orbit map $g \mapsto gv, G \to E$ is continuous at e.

Proposition 2.1.6. *Let G be a locally compact group and $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space, equipped with a measure preserving action of G. Then the associated actions of G on $\mathrm{L}^p(X, \mathcal{A}, \mu)$, $1 \leq p < \infty$, are strongly continuous.*

Example 2.1.7. In general, the action of G on $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$ is not strongly continuous. For example, if $G = X$ is the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, acting on itself by translations, the orbit map associated to the characteristic function of an proper interval is not continuous.

The proof uses the following elementary property of Haar measure:

Lemma 2.1.8. *Let G be a locally compact group and B be a measurable subset of G with finite positive Haar measure. Then, the set $B^{-1}B$ is a neighborhood of e in G.*

Proof. Let λ be a right Haar measure on G. Since the Haar measure is a Radon measure, B contains a compact subset with positive measure K. Again since the Haar measure is a Radon measure, there exists an open subset V of G containing K with $\lambda(V) < 2\lambda(K)$. Finally, as K is compact, there exists a neighborhood U of e in G with $KU \subset V$. We claim that $U \subset K^{-1}K \subset B^{-1}B$. Indeed, for any g in U, we have $Kg \cup K \subset V$ and $\lambda(Kg) + \lambda(K) = 2\lambda(K) > \lambda(V)$, hence $Kg \cap K \neq \emptyset$, which should be proved. $\square$

Proof of Proposition 2.1.6. Let f be in $\mathrm{L}^p(X, \mathcal{A}, \mu)$ and let $(h_k)_{k \geq 0}$ be a dense sequence in $\mathrm{L}^p(X, \mathcal{A}, \mu)$ (which exists since the probability space is Lebesgue). Let K be a compact subset of G with positive Haar measure. Fix $\varepsilon > 0$. For any h in $\mathrm{L}^p(X, \mathcal{A}, \mu)$, there exists k in $\mathbb{N}$ with $\|h - h_k\|_1 \leq \varepsilon$. For k in $\mathbb{N}$, the map

$$\begin{aligned} K \times X &\to \mathbb{R} \\ (t, x) &\mapsto \left|f(g^{-1}x) - h_k(x)\right| \end{aligned}$$

is measurable. Therefore, by Fubini Theorem, the map

$$\begin{aligned} K &\to \mathbb{R} \\ t &\mapsto \|gf - h_k\|_p \end{aligned}$$

is measurable and the set

$$B_k = \{g \in K | \, \|gf - h_k\|_p \leq \varepsilon\}$$

is a measurable subset of G. As we have $K = \bigcup_{k\in\mathbb{N}} B_k$, we can find a k such that B_k has positive Haar measure. By Lemma 2.1.8, there exists a neighborhood U of e in G such that $U \subset B_k^{-1}B_k$. Then, if g belongs to U, there exists r and s in B_k with $g = r^{-1}s$ and we have

$$\|gf - f\|_p = \left\|r^{-1}sf - f\right\|_p = \|sf - rf\|_p \leq \|rf - h_k\|_p + \|sf - h_k\| \leq 2\varepsilon,$$

since r and s belong to B_k. Hence the map $g \mapsto gf$ is continuous at e, which should be proved. $\square$

2.2 Howe-Moore Theorem for $\mathrm{SL}_2(\mathbb{R})$

We will now focus on the study of unitary representations of groups in Hilbert spaces. We first start by giving a property of the representations of $\mathrm{SL}_2(\mathbb{R})$, which will later be extended to all connected semisimple Lie groups. The following theorem says that, in unitary representations of $\mathrm{SL}_2(\mathbb{R})$ with no invariant vectors, the coefficient functions decay.

Theorem 2.2.1 (Howe-Moore). *Let H be a Hilbert space equipped with a strongly continuous unitary action of $\mathrm{SL}_2(\mathbb{R})$. Assume H does not admit any non zero $\mathrm{SL}_2(\mathbb{R})$-invariant vector. Then, for any v, w in H, one has $\langle gv, w\rangle \xrightarrow[g\to\infty]{} 0$.*

In this statement, $\langle gv, w\rangle \xrightarrow[g\to\infty]{} 0$ means that, for any $\varepsilon > 0$, there exists a compact subset K of $\mathrm{SL}_2(\mathbb{R})$ such that, for any g in $\mathrm{SL}_2(\mathbb{R}) \smallsetminus K$, one has $|\langle gv, w\rangle| \leq \varepsilon$.

Remark 2.2.2. Assume G is a locally compact abelian group. Then Howe-Moore theorem does not apply to G. Indeed, by Pontryagin theory, the characters of G, that is the continuous morphisms from G to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, separate points. Let χ be a non trivial character. Define a unitary representation of G in the Hilbert space $H = \mathbb{C}$ by letting an element g of G act through the multiplication by $e^{2i\pi\chi(g)}$. Then this unitary representation does not admit non zero G-invariant vectors, but for any g in G, we have $\langle g1, 1\rangle = e^{2i\pi\chi(g)}$ which has constant modulus 1.

In other words, Howe-Moore theorem is a way of saying that $\mathrm{SL}_2(\mathbb{R})$ is far away from being abelian.

To prove the theorem, we need a new decomposition in linear groups. We let A^+ denote the set of diagonal matrices a with positive entries in $\mathrm{SL}_d(\mathbb{R})$ such that $a_{1,1} \geq \cdots \geq a_{d,d}$.

Proposition 2.2.3 (Cartan decomposition for $\mathrm{SL}_d(\mathbb{R})$)**.** *One has* $\mathrm{SL}_d(\mathbb{R}) = KA^+K$. *More precisely, for any* g *in* $\mathrm{SL}_d(\mathbb{R})$, *there exists a unique* a *in* $A^+ \cap KgK$.

Proof. Let g be in $\mathrm{SL}_d(\mathbb{R})$ then the matrix $g^t g$ is symmetric and positive, so that it admits a positive symmetric square root s. By construction, one has $g = ks$ with k in K. The existence of the decomposition follows by diagonalizing s. Uniqueness comes from the uniqueness of the square root. □

Proof of Theorem 2.2.1. Let v be in H. We have to prove that one has $gv \xrightarrow[g\to\infty]{} 0$ in H for the weak topology. Since, by Banach-Alaoglu theorem, the closed balls of H are weakly compact, it suffices to prove that all the weak cluster points of gv as $g \to \infty$ are 0. Since $\mathrm{SL}_2(\mathbb{R})$ is second countable, after having replaced H by the closure of the subspace spanned by $\mathrm{SL}_2(\mathbb{R})v$, it suffices to prove that, for any sequence (g_p) going to infinity in G, if $g_p v \xrightarrow[p\to\infty]{} u$ weakly, for some u in H, then $u = 0$.

Now, let u, v and (g_p) be as above. For any p, write a Cartan decomposition of g_p as $k_p a_p \ell_p$, so that a_p goes to infinity in A^+. After having extracted a subsequence, we can assume $k_p \xrightarrow[p\to\infty]{} k$ and $\ell_p \xrightarrow[p\to\infty]{} \ell$, for some k, ℓ in K. We claim that $a_p \ell v \xrightarrow[p\to\infty]{} k^{-1}u$. Indeed, for any p, one has

$$\begin{aligned}&\left|\langle a_p\ell_p v, k_p^{-1}w\rangle - \langle a_p\ell v, k^{-1}w\rangle\right|\\ &\qquad\leq \left|\langle a_p\ell_p v, k_p^{-1}w\rangle - \langle a_p\ell_p v, k^{-1}w\rangle\right| + \left|\langle a_p\ell_p v, k^{-1}w\rangle - \langle a_p\ell v, k^{-1}w\rangle\right|\end{aligned}$$

and, on one hand,

$$\begin{aligned}\left|\langle a_p\ell_p v, k_p^{-1}w\rangle - \langle a_p\ell_p v, k^{-1}w\rangle\right| &= \left|\langle a_p\ell_p v, k_p^{-1}w - k^{-1}w\rangle\right|\\ &\leq \|v\| \left\|k_p^{-1}w - k^{-1}w\right\|,\end{aligned}$$

whereas, on the other hand,

$$\left|\langle a_p\ell_p v, k^{-1}w\rangle - \langle a_p\ell v, k^{-1}w\rangle\right| = \left|\langle \ell_p v - \ell v, a_p^{-1}k^{-1}w\rangle\right| \leq \|\ell_p v - \ell v\| \, \|w\|,$$

so that
$$\left|\langle a_p\ell_p v, k_p^{-1}w\rangle - \langle a_p\ell v, k^{-1}w\rangle\right| \xrightarrow[p\to\infty]{} 0.$$
Since $g_p v \xrightarrow[p\to\infty]{} u$ weakly, we get
$$\langle a_p\ell_p v, k_p^{-1}w\rangle \xrightarrow[p\to\infty]{} \langle u, w\rangle = \langle k^{-1}u, k^{-1}w\rangle$$
and we are done.

We set $v' = \ell v$ and $u' = k^{-1}u$. We claim that u' is N-invariant. Fix n in N. First, let us note that, for any n in N, since a_p goes to infinity in A^+, one has $a_p^{-1}na_p \xrightarrow[p\to\infty]{} e$. Now, we write, for any w in H, for any p,
$$\langle na_pv', w\rangle = \langle a_p^{-1}na_pv', a_p^{-1}w\rangle$$
and
$$\begin{aligned}\left|\langle a_p^{-1}na_pv', a_p^{-1}w\rangle - \langle v', a_p^{-1}w\rangle\right| &= \left|\langle a_p^{-1}na_pv' - v', a_p^{-1}w\rangle\right| \\ &\leq \left\|a_p^{-1}na_pv' - v'\right\| \|w\| \xrightarrow[p\to\infty]{} 0.\end{aligned}$$
Thus, we get
$$\langle na_pv', w\rangle - \langle a_pv', w\rangle \xrightarrow[p\to\infty]{} 0,$$
hence $\langle nu', w\rangle = \langle u', w\rangle$, that is, $nu' = u'$.

We will now prove that this implies that u' is actually $\mathrm{SL}_2(\mathbb{R})$-invariant, which finishes the proof, since then, by assumption $u' = 0$, hence $u = ku' = 0$.

Indeed, for g in G, set $\varphi(g) = \langle gu', u'\rangle$. The function φ is continuous and left and right N-invariant. Consider φ as a function on $G/N \simeq \mathbb{R}^2 \smallsetminus \{0\}$: it is constant on N-orbits in $\mathbb{R}^2 \smallsetminus \{0\}$. Now, for every $y \neq 0$ in $\mathbb{R}$, the N-orbit of $(0, y)$ in $\mathbb{R}^2$ is the line $\mathbb{R} \times \{y\}$, so that φ is constant on each of these lines. Since φ is continuous, it is also constant on $\mathbb{R}^* \times \{0\}$. In other words, for any p in P, we have $\langle pu', u'\rangle = \|u'\|^2$, hence, by the equality case in Cauchy-Schwarz inequality, $pu' = u'$, that is, u' is P-invariant and φ is left and right P-invariant. Now, consider φ as a function on $G/P \simeq \mathbb{P}^1_{\mathbb{R}}$. Again, it is constant on P-orbits in $\mathbb{P}^1_{\mathbb{R}}$. Since the P-orbit of $\mathbb{R}(0, 1)$ is equal to $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \mathbb{R}(1, 0)$, φ is constant, hence u' is $\mathrm{SL}_2(\mathbb{R})$-invariant, which should be proved. □

Remark 2.2.4. This proof may be seen as a translation in group theoretic language of Hopf's proof of mixing for geodesic flows.

2.3 Howe-Moore Theorem for $\mathrm{SL}_d(\mathbb{R})$

We will now extend the proof of Howe-Moore theorem for any $d \geq 2$.

Theorem 2.3.1 (Howe-Moore)**.** *Let H be a Hilbert space equipped with a strongly continuous unitary action of $\mathrm{SL}_d(\mathbb{R})$. Assume H does not admit any non zero $\mathrm{SL}_d(\mathbb{R})$-invariant vector. Then, for any v, w in H, one has $\langle gv, w\rangle \xrightarrow[g\to\infty]{} 0$.*

Proof. Fix v in H. Again, it suffices to prove that, if (g_p) is a sequence that goes to infinity in $\mathrm{SL}_d(\mathbb{R})$ such that $g_p v \xrightarrow[p\to\infty]{} u$ for some u in H, then u is $\mathrm{SL}_d(\mathbb{R})$-invariant. By using Cartan decomposition as in the proof of the case where $d = 2$, one can assume $(g_p) = (a_p)$ takes values in A^+. In particular, one has $\frac{(a_p)_{1,1}}{(a_p)_{d,d}} \xrightarrow[p\to\infty]{} \infty$.

For any $1 \leq i < j \leq d$, let $N_{i,j}$ be the group of those n in N such that $n_{k,\ell} = 0$ for any $1 \leq k < \ell \leq d$ with $(i,j) \neq (k,\ell)$. For n in $N_{1,d}$, we have $a_p^{-1} n a_p \xrightarrow[p\to\infty]{} e$ in $\mathrm{SL}_d(\mathbb{R})$, so that, reasoning again as in the case where $d = 2$, we get that the vector u is $N_{1,d}$-invariant. Now, for $1 \leq i < j \leq d$, let $S_{i,j}$ be the group of matrices of $\mathrm{SL}_d(\mathbb{R})$ of the form

$$\begin{pmatrix} 1_{i-1} & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 \\ 0 & 0 & 1_{j-i-1} & 0 & 0 \\ 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1_{d-j} \end{pmatrix}$$

(where, for any k, 1_k means an identity square block of size k). In other words, the group $S_{i,j}$ is the subgroup spanned by $N_{i,j}$ and by its image by transposition of matrices. The group $S_{i,j}$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$, so that, by the case where $d = 2$, we get that the vector u is $S_{1,d}$-invariant. In particular, u is invariant under the matrix

$$a_d = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1_{d-2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Fix $2 \leq j \leq d$. Again, since, for any n in $N_{1,j}$, we have $a_d^{-k} n a_d^k \xrightarrow[k\to\infty]{} e$, we get that u is $N_{1,j}$-invariant. Hence u is $S_{1,j}$-invariant. Set

$$a_j = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1_{j-2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1_{d-j} \end{pmatrix}.$$

For any $1 \leq i < j \leq d$, for any n in $N_{i,j}$, we have $a_j^{-k} n a_j^k \xrightarrow[k\to\infty]{} e$, so that u is $N_{i,j}$-invariant, hence $S_{i,j}$-invariant. Since $\mathrm{SL}_d(\mathbb{R})$ is spanned by these subgroups, we are done. □

The combinatorial game that appears in this proof may be extended to any connected simple Lie group by using the general structure theory of these groups (which we are trying to avoid to do in these notes). One then gets the full Howe-Moore Theorem:

Theorem 2.3.2 (Howe-Moore)**.** *Let G be a connected simple Lie group and H be a Hilbert space equipped with a strongly continuous unitary action of G. Assume H does not admit any non zero G-invariant vector. Then, for any v, w in H, one has $\langle gv, w\rangle \xrightarrow[g\to\infty]{} 0$.*

2.4 Dynamical consequences of Howe-Moore theorem

Let us now give an interpretation of these results in the language of ergodic theory. Recall that, if $(X, \mathcal{A}, \mu)$ is a probability space, if T is measure preserving ransformation of X, then T is said to be ergodic if, for any measurable subset A of X such that $T^{-1}A = A$ almost everywhere (that is, $\mu(T^{-1}A\Delta A) = 0$), one has $\mu(A) = 0$ or $\mu(A) = 1$. In other words, the transformation T is ergodic if and only if the isometry $f \mapsto f \circ T$ of $\mathrm{L}^2(X, \mathcal{A}, \mu)$ does not admit any invariant vector besides the constant functions.

Remark 2.4.1. Note that, if A is a subset of X such that $T^{-1}A = A$ almost everywhere, there exists a measurable subset A' such that $T^{-1}A' = A'$ and $\mu(A\Delta A') = 0$, so that in the definition of ergodicity one can forget the words "almost everywhere".

Now, the transformation T is said to be (strongly) mixing if for any measurable subsets A, B of X, one has

$$\mu(T^{-n}A \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B).$$

Strong mixing implies ergodicity, since, if $T^{-1}A = A$, one gets $\mu(A) = \mu(T^{-n}A \cap A) = \mu(A)^2$ hence $\mu(A) \in \{0, 1\}$. One easily checks that T is (strongly) mixing if and only if, for any φ, ψ in $\mathrm{L}^2(X, \mathcal{A}, \mu)$, one has

$$\int_X \varphi \circ T^n \psi \mathrm{d}\mu \xrightarrow[n\to\infty]{} \int_X \varphi \mathrm{d}\mu \int_X \psi \mathrm{d}\mu.$$

Example 2.4.2. Equip $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure and the transformation $x \mapsto 2x$. Then this dynamical system is strongly mixing. Indeed, for any k in $\mathbb{Z}$ and x in $\mathbb{T}$, set $e_k(x) = e^{2i\pi kx}$. Then, for any k, ℓ in $\mathbb{Z}$, $k \neq 0$, one has

$$\int_{\mathbb{T}} e_k(2^n x) e_\ell(x) \mathrm{d}x = \int_{\mathbb{T}} e_{2^n k}(x) e_\ell(x) \mathrm{d}x = 0$$

as soon as $|2^n k| > |\ell|$, whence the result by density of the trigonometric polynomials in the space of square-integrable functions on the torus.

Now, consider the transformation $x \mapsto x + \alpha$, where α belongs to $\mathbb{R} \smallsetminus \mathbb{Q}$. This transformation is ergodic, but it is not mixing: indeed, for anu n, one has

$$\int_{\mathbb{T}} e_1(x + n\alpha) e_{-1}(x) \mathrm{d}x = e_1(n\alpha),$$

which has modulus 1! This is the dynamical version of the phenomenon that is described in Remark 2.2.2.

Let us now draw the conclusions of Howe-Moore Theorem for the action of semisimple groups on finite volume homogeneous spaces. We begin with a special case:

Corollary 2.4.3. *Let g be an element of $\mathrm{SL}_d(\mathbb{R})$ that is not contained in a compact subgroup. Then the map $x \mapsto gx$ of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ is mixing for the $\mathrm{SL}_d(\mathbb{R})$-invariant measure.*

This will follow from Howe-Moore that we will apply to the natural action of $\mathrm{SL}_d(\mathbb{R})$ on $\mathrm{L}^2(\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}))$. To prove that the only invariant functions are the constant ones, we will need the following extension of Remark 2.4.1:

Lemma 2.4.4. *Let G be a locally compact second countable toplogical group and $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space, equipped with a measure preserving G action. Let φ be a measurable function on X such that, for any g in G, one has $\varphi \circ g = \varphi$ almost everywhere. Then, there exists a measurable function φ_1 on X such that $\varphi = \varphi_1$ almost everywhere and $\varphi_1 \circ g = \varphi_1$ everywhere, for any g in G.*

Proof. Let λ be a right Haar measure on G and let E be the set of those (g, h, x) in G such that $\varphi(gx) = \varphi(hx)$. Then, by Fubini Theorem, E has full measure for $\lambda \otimes \lambda \otimes \mu$. Therefore, again by Fubini Theorem, the set A of those x in X such that, for $\lambda \otimes \lambda$-almost every (g, h) in G^2, $\varphi(gx) = \varphi(hx)$ is a measurable subset of full measure for μ. Since λ is right invariant, this set is G-invariant. For any x in A, the map $g \mapsto \varphi(gx)$ is λ-almost everywhere constant on G. We let $\varphi_1(x)$ denote its almost constant value. Again, since λ is right invariant, φ_1 is a G-invariant function on A. We extand φ_1 to all of X by setting $\varphi_1 = 0$ on $X \smallsetminus A$. This is a G-invariant function. Now, again by Fubini Theorem, for μ-almost any x in X, for λ-almost any g in G, we have $\varphi(gx) = \varphi(x)$, so that $\varphi_1(x) = \varphi(x)$ and the result follows. □

Now, we need to prove that, if an element g of $\mathrm{SL}_d(\mathbb{R})$ does not belong to a compact subgroup, its powers go to infinity. This we could do directly by studying the Jordan decomposition of g. Instead, we will prove that this is a general fact in locally compact groups by establishing a general dynamical property.

Lemma 2.4.5. *Let X be a locally compact topological space and $T : X \to X$ be a continuous map (resp. $(\varphi_t)_{t \geq 0}$ be a continuous semiflow). Assume, for every x in X, the semiorbit $\{T^n x | n \geq 0\}$ (resp. $\{\varphi_t(x) | t \geq 0\}$) is dense in X. Then X is compact.*

By a consinuous semiflow, we mean a family $(\varphi_t)_{t \geq 0}$ of continuous maps such that $\varphi_{t+s} = \varphi_t \circ \varphi_s$, $s, t \geq 0$, and the map $\mathbb{R}_+ \times X, (t, x) \mapsto \varphi_t(x)$ is continuous.

Proof. We prove the result in the case of one map, the proof for semiflows being analoguous. Let Y be a compact subset of X that contains a non empty open subset U. For any x in X, there exists $n \geq 1$ such that $T^n x \in U$. For x in X, we set $\tau(x) = \min\{n \geq 1 | T^n x \in U\}$. Since U is open, τ is an upper semicontinuous function. In particular, Y being compact, τ is bounded on Y

by some constant $m \geq 1$. Thus, one has $\bigcup_{n\geq 0} T^n Y = \bigcup_{0\leq n\leq m} T^n Y$ and this set is compact. Since it is stable by T, it is equal to X and X is compact. $\square$

Corollary 2.4.6. *Let G be a locally compact group and g be an element of G. Assume $g^{\mathbb{Z}}$ is dense in G. Then either $G = g^{\mathbb{Z}}$ or G is compact.*

Proof. Assume G is different from $g^{\mathbb{Z}}$, that is, G is not discrete. Then, for any neighborhood V of e, for any integer $n \geq 0$, there exists an integer $p \geq n$ such that $g^p \in V$. Indeed, one can assume V to be symmetric. Now, since G is not discrete, the set $V \smallsetminus \{g^k | -n < k < n\}$ has non empty interior, hence, it contains some g^p, with $|p| \geq n$. Since V is symmetric, it contains $g^{|p|}$ and we are done.

Now, let us prove that $g^{\mathbb{N}}$ is dense in G. Indeed, for any open subset U of G, there exists k in $\mathbb{Z}$ such that g^k belongs to U. Let V be a neighborhood of e such that $Vg^k \subset U$ and let $p \geq |k|$ be such that g^p belongs to V. We get $g^{p+k} \in U$ and $p + k \geq 0$, and the result follows.

Equip the group G with the map $x \mapsto gx$. Then every semiorbit $g^{\mathbb{N}}x$ is dense in G. By Lemma 2.4.5, G is compact. $\square$

Proof of Corollary 2.4.3. Consider the natural action of $\mathrm{SL}_d(\mathbb{R})$ on the space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. Let φ be an element of $\mathrm{L}^2(\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}))$ that is $\mathrm{SL}_d(\mathbb{R})$-invariant. By Lemma 2.4.4, φ admits a representative that is an $\mathrm{SL}_d(\mathbb{R})$-invariant function. As the action of $\mathrm{SL}_d(\mathbb{R})$ on $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ is transitive, φ is constant. Therefore, the space of functions with zero integral in $\mathrm{L}^2(\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}))$ does not admit any non zero $\mathrm{SL}_d(\mathbb{R})$-invariant vector. The mixing property now follos from Howe-Moore Theorem, since, by Corollary 2.4.6, for any g in $\mathrm{SL}_d(\mathbb{R})$ that does not belong to a compact subgroup, one has $g^n \xrightarrow[n\to\infty]{} \infty$ in $\mathrm{SL}_d(\mathbb{R})$. $\square$

Let us give an extension of this result for actions on quotients of semisimple groups by lattices. Note that, in case one is working with semisimple but non simple groups, the result may be untrue in general. Indeed, for example, in the group $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, the subgroup $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is a lattice, but the action of an element of the form (g, e) on G/Γ is never ergodic! Thus, we need to introduce a new notion:

Definition 2.4.7. Let G be a connected semisimple Lie group and Γ be a lattice in G. We say Γ is irreducible if, for any non discrete proper normal closed subgroup H of G, Γ has dense image in G/H.

Example 2.4.8. By using the same technique as in the proof that $\mathrm{SL}_2(\mathbb{Z})$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, one can show that the image of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by the map $g \mapsto (g, \sigma(g))$ (where σ is the non-trivial automorphism of the field $\mathbb{Q}[\sqrt{2}]$) is a lattice. It is clearly irreducible.

The study of the quotients of semisimple groups by lattices essentially reduces to the study of quotients by irreducible ones:

Proposition 2.4.9. *Let G be a connected semisimple Lie group and Γ be a lattice in G. There exist normal closed connected subgroups $G_1, \dots, G_r$ of G such that*

(i) $G = G_1 \cdots G_r$.

(ii) for any $1 \leq i < j \leq r$, $G_i \cap G_j$ *is discrete.*

(iii) for any $1 \leq i \leq r$, $\Gamma_i = \Gamma \cap G_i$ *is a lattice in* G_i.

(iv) the group $\Gamma_1 \cdots \Gamma_r$ *has finite index in* Γ.

Now we can state a very general mixing property:

Corollary 2.4.10. *Let G be a conneceted semisimple Lie group, Γ be an irreducible lattice in G and g be an element of G that is not contained in a compact subgroup. Then the map $x \mapsto gx$ of G/Γ is mixing for the* $\mathrm{SL}_d(\mathbb{R})$*-invariant measure.*

Chapter 3

Recurrence of unipotent flows

In this chapter, we will establish a fundamental result by Dani and Margulis about the trajectories of certain one-parameter subgroups on quotients of semisimple groups by lattices. It has applications in itself, in particular in Diophantine approximation, and plays also a key-role in the proof of Ratner's Theorem.

Let us state this result. We will say that a one-parameter subgroup $(u_t)_{t\in\mathbb{R}}$ of $\mathrm{SL}_d(\mathbb{R})$ is unipotent if its derivative is nilpotent, that is, if there exists a nilpotent $d \times d$ matrix X such that, for any t, $u_t = \exp(tX)$. We will establish the following

Theorem 3.0.11 (Dani-Margulis)**.** *Let (u_t) be a unipotent one-parameter subgroup of $\mathrm{SL}_d(\mathbb{R})$ and x be an element in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. Then, for any $\varepsilon > 0$, there exists a compact subset K of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ such that, for any $T > 0$, one has*

$$|\{0 \leq t \leq T | u_t x \in K\}| \geq (1 - \varepsilon)T.$$

By $|.|$, we mean Lebesgue measure of $\mathbb{R}$.

Remark 3.0.12. Let μ be the $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure of the space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ and (K_n) be a sequence of compact subsets inside $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ such that $\mu(K_n) \xrightarrow[n\to\infty]{} 1$. We know that the action of (u_t) on $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ is mixing, hence ergodic with respect to the $\mathrm{SL}_d(\mathbb{R})$-invariant measure μ. Therefore, we know from Birkhoff's ergodic theorem that, for μ-almost any x in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, for any n, one has

$$\frac{1}{T} |\{0 \leq t \leq T | u_t x \in K_n\}| \xrightarrow[n\to\infty]{} \mu(K_n),$$

so that the theorem clearly holds for μ-almost any x in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. But it is much more difficult to prove it for any x!

Example 3.0.13. The assumption that the one-parameter subgroup is unipotent is crucial for the theorem to hold. Indeed, assume $d = 2$ and consider the point $x = \mathrm{SL}_2(\mathbb{Z})$ and the one-parameter subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}_{t\in\mathbb{R}}$. Then, in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, by Corollary 1.3.11, one has $a_t x \xrightarrow[t\to\infty]{} \infty$, since $a_t\mathbb{Z}^2$ contains a non-zero vector with norm e^{-t}.

3.1 Recurrence in case $d = 2$

We start by giving the proof of Dani-Margulis Theorem in case $d = 2$, which is simpler than the general case.

Proof of Theorem 3.0.11 in case $d = 2$. All the unipotent subgroups of the group $\mathrm{SL}_2(\mathbb{R})$ are conjugate, so that we can assume, for any t, one has

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Now, set $\Lambda = g\mathbb{Z}^2$, where g is in $\mathrm{SL}_2(\mathbb{R})$ and $x = g\mathrm{SL}_2(\mathbb{Z})$. The set Λ is a unimodular lattice in $\mathbb{R}^2$ and we distinguish between two cases.

First, assume Λ contains a non zero vector in the coordinate axis $\mathbb{R}e_1$. Choose v to be a generator of $\Lambda \cap \mathbb{R}e_1$ and w in $\mathbb{R}^2$ such that (v, w) is a basis of Λ, so that, in particular, $w_2 \neq 0$. Then, for any t, we have $u_t(w) = (w_1 + tw_2, w_2)$, hence, if $t_0 = \frac{v_1}{w_2}$, $u_{t_0}(w) = w + v$ and we have $u_{t_0}(\Lambda) = \Lambda$, that is $u_{t_0}x = x$. Therefore, the orbit of x under (u_t) is compact and the result naturally follows.

Now, assume $\Lambda \cap \mathbb{R}e_1 = \{0\}$. We will use Corollary 1.3.11 to prove that the trajectory $(u_t x)$ spends most of its time in compact subsets, that is, we will prove that, most of the time, the systole of the lattice $u_t\Lambda$ is not too small. Set

$$E = \left\{ t \in \mathbb{R} \,\middle|\, s(u_t\Lambda) < \frac{1}{2}\min(s(\Lambda), 1) \right\},$$

where s denotes systole. For any t in $\mathbb{R}$, let $F_t \subset \Lambda$ denote the set of vectors v in Λ such that $\|u_t v\| = s(u_t\Lambda)$ and, for any v in $\Lambda \smallsetminus \{0\}$, let E_v denote the set of t in E such that $v \in F_t$. We claim that E_v is both open and closed in E and that, for any t in E_v, $F_t = \{\pm v\}$. On one hand, since

$$E_v = \{t \in E | \forall w \in \Lambda \smallsetminus \{0\} \quad \|u_t w\| \geq \|u_t v\|\},$$

E_v is clearly closed in E. On the other hand, let t be in E_v and I be a neighborhood of t in $\mathbb{R}$ such that, for any s in I, one has $\|u_s v\| < \frac{1}{2}$. We claim that, for s in I, $F_s = \{\pm v\}$. Indeed, for any v' in F_s, we have $\|u_s v'\| \leq \|u_s v\| < \frac{1}{2}$, hence

$$\det(u_t v, u_t v') \leq \|u_t v\| \, \|u_t v'\| < \frac{1}{4},$$

whereas, Λ being unimodular, we have $\det(v, v') = \det(u_t v, u_t v') \in \mathbb{Z}$, hence $\det(u_t v, u_t v') = 0$. This gives $v' = \pm v$, since $\|u_t v\| \leq \|u_t v'\|$ and $\|u_s v'\| \leq \|u_s v\|$, and we are done.

Let $\mathcal{I}$ be the set of connected components of E, which is at most countable. Fix I in $\mathcal{I}$. By the property above, there exists a vector v^I in Λ such that, for any t in I, one has $s(u_t \Lambda) = \left\| u_t v^I \right\|$. By assumption, we have $v^I \notin \mathbb{R} e_1$, so that $\|u_t v_I\| \xrightarrow[|t| \to \infty]{} \infty$ and I is a bounded interval (a, b). Note that, by construction, we have $\left\| u_a v^I \right\| = \frac{1}{2} \min(s(\Lambda), 1)$. Fix $0 < \varepsilon \leq \frac{1}{4} \min(s(\Lambda, 1)$ and let us study the set of t in I such that $s(u_t \Lambda) \leq \varepsilon$. Set $w = u_a v^I$, so that if $t = a + r$ is in I and $s(u_t \Lambda) \leq \varepsilon$, we have

$$(w_1 + r w_2)^2 + w_2^2 = \|u_r w\|^2 \leq \varepsilon^2.$$

If $w_2 > \varepsilon$, this inequation has no solution. If not, we have

$$w_1^2 = \frac{1}{4} \min(s(\Lambda), 1)^2 - w_2^2 \geq \frac{1}{4} \min(s(\Lambda), 1)^2 - \varepsilon^2 \geq \frac{3}{16} \min(s(\Lambda), 1)^2$$

hence, for $T > a$, the set

$$\{a \leq t \leq T | \, \|u_t v\| \leq \varepsilon\}$$

is empty if $T \leq a + \frac{1}{4 w_2} \min(s(\Lambda), 1)$. Else, we have

$$|\{a \leq t \leq T | \, \|u_t v\| \leq \varepsilon\}| \leq \frac{2\varepsilon}{|w_2|} \leq 8\varepsilon \min(s(\Lambda), 1)(T - a).$$

In any case, we get, for any $T > 0$,

$$|\{a \leq t \leq \min(T, b) | s(u_t \Lambda) \leq \varepsilon\}| \leq 8\varepsilon \min(s(\Lambda), 1) \, |[0, T] \cap (a, b)| \, .$$

Hence, we have

$$\begin{aligned}
|\{0 \le t \le T | s(u_t\Lambda) \le \varepsilon\}| &= \sum_{I\in\mathcal{I}} |\{0 \le t \le T | s(u_t\Lambda) \le \varepsilon\}| \\
&\le \sum_{I\in\mathcal{I}} |\{t \in [0,T] \cap I | s(u_t\Lambda) \le \varepsilon\}| \\
&\le 8\varepsilon \min(s(\Lambda),1) \sum_{I\in\mathcal{I}} |[0,T]\cap I| \\
&\le 8\varepsilon \min(s(\Lambda),1) T
\end{aligned}$$

and the result now follows from Corollary 1.3.11. □

3.2 Preliminaries from multilinear algebra

We will now give the full proof of Dani-Margulis Theorem. As in case $d = 2$, we need to control the amount of time when polynomial functions take small values, but now this polynomial functions have larger degrees. This is not a great difficulty, and we will prove below an elementary lemma on polynomial functions with bounded degrees (whose conceptual statement will turn out to be easier to handle that the very explicit computations we made in case $d = 2$). The main problem when the dimension d is ≥ 3 is that the vector of a lattice in $\mathbb{R}^d$ that achieves the minimum of the norm has no reason to be unique. Indeed, for example, for any $t > 0$, there are (up to sign change) two vectors of the lattice $\Lambda_t = e^{-t}\mathbb{Z}e_1 \oplus e^{-t}\mathbb{Z}e_2 \oplus e^{2t}\mathbb{Z}e_3$. In this example, this phenomenon is related to the fact that the dual lattice Λ_t^* has systole e^{-2t}, that is, much smaller than the systole of Λ_t (where, by the dual lattice of a lattice Λ in $\mathbb{R}^d$, we mean the set of linear forms φ in the dual space of $\mathbb{R}^d$ such that $\varphi(\Lambda) \subset \mathbb{Z}$). For this reason, to control efficiently the way a family of lattices goes to infinity in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, we need to introduce new functions on this space, which are other versions of the systole. These functions will be defined by using notions from multilinear algebra. When $d = 3$, there is only one new function to consider, which is precisely the systole of the dual lattice of a given lattice in $\mathbb{R}^3$.

Let us proceed to this definition. As we said, we need to recall elements of multilinear algebra. Given a finite-dimensional real vector space V with dimension d and an integer $k \ge 0$, we let $\wedge^k V$ denote its k-th exterior power (with the convention that $\wedge^0 V = \mathbb{R}$): this space may be seen as the space of alternate k-linear forms on the dual space V^* of V. In particular, it is 0 if $k > d$. If T is a linear endomorphism of V, we let $\wedge^k T$ denote the associated linear endomorphism of $\wedge^k V$, so that, for $v_1, \dots, v_k$ in V, one has

$$\wedge^k T(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k).$$

We let $\mathfrak{S}_k$ denote the group of permutations of $\{1, \dots, k\}$ and $\varepsilon : \mathfrak{S}_k \to \{\pm 1\}$ be the signature morphism.

Lemma 3.2.1. *Let V be a euclidean space and $k \geq 0$. There exists a unique scalar product on $\wedge^k V$ such that, for any $v_1, \dots, v_k$ in V, one has*

$$\|v_1 \wedge \cdots \wedge v_k\|^2 = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \prod_{i=1}^{k} \langle v_i, v_{\sigma(i)} \rangle.$$

Proof. For $v_1, \dots, v_k$ and $w_1, \dots, w_k$ in V, define

$$\varphi(v_1, \dots, v_k, w_1, \dots, w_k) = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \prod_{i=1}^{k} \langle v_i, w_{\sigma(i)} \rangle.$$

Then φ is $2k$-linear and it is alternate in the k-first variables and in the k-last variables. Therefore, φ defines a bilinear for on $\Lambda^k V$, which we still denote by φ. Now, let $e_1, \dots, e_d$ be an orthnormal basis of V, so that $(e_{i_1} \wedge \cdots \wedge e_{i_k})_{1 \leq i_1 < \cdots < i_k \leq d}$ is a basis of $\wedge^k V$. One easily checks that this basis is orthonormal for φ, hence that φ is a scalar product. Existence follows. Uniqueness is evident. □

In the sequel, we shall always equip the exterior powers of a euclidean space with this scalar product. Here is a geometric interpretation of this construction:

Lemma 3.2.2. *Let V be a euclidean space and $v_1, \dots, v_k$ be a free set of vectors in V. Then the euclidean volume of the set*

$$C(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} t_i v_i \,\middle|\, \forall 1 \leq i \leq k \quad t_i \in [0, 1] \right\}$$

is equal to the norm of $v_1 \wedge \cdots \wedge v_k$.

Proof. It suffices to prove this result when $k = d$. By Gram-Schmidt theorem, there exists an orthogonal basis $w_1, \dots, w_d$ of V such that the matrix of the basis $w_1, \dots, w_d$ with respect to $v_1, \dots, v_d$ is unipotent. In particular, the unique linear map T that sends v_i to w_i, $1 \le i \le d$, has determinant 1, hence T preserves the euclidean volume and $\wedge^d T$ is the identity map in $\wedge^d V$. Since $TC(v_1, \dots, v_d) = C(w_1, \dots, w_d)$, it suffices to prove the result when $v_1, \dots, v_d$ is orthogonal. In this case, one clearly has

$$|C(v_1, \dots, v_d)| = \|v_1\| \cdots \|v_d\| = \|v_1 \wedge \cdots \wedge v_d\| .$$

□

When k varies, these norms are related by a nice inequality.

Lemma 3.2.3. *Let V be a euclidean space, $h, k, \ell \ge 0$ be integers and $u = u_1 \wedge \ldots \wedge u_h \in \wedge^h V$, $v = v_1 \wedge \ldots \wedge v_k \in \wedge^k V$ and $w = w_1 \wedge \ldots \wedge w_\ell \in \wedge^\ell V$ be pure tensors. One has*

$$\|u\| \, \|u \wedge v \wedge w\| \le \|u \wedge v\| \, \|u \wedge w\| .$$

Proof. We can assume u, v and w are $\neq 0$. Since $u \wedge v$, $u \wedge w$ and $u \wedge v \wedge w$ are not changed by adding linear combinations of $u_1, \dots, u_h$ to $v_1, \dots, v_k$ and $w_1, \dots, w_\ell$, we can assume that $v_1, \dots, v_k$ and $w_1, \dots, w_\ell$ are orthogonal to $u_1, \dots, u_h$. Now, the left member of the inequality we have to prove is equal to $\|u\|^2 \|v \wedge w\|$ whereas the right member is equal to $\|u\|^2 \|v\| \, \|w\|$. So that we are reduced to proving the simpler inequality

$$\|v \wedge w\| \le \|v\| \, \|w\| .$$

Let W be the linear span of $w_1, \dots, w_\ell$, $W^\perp$ be its orthogonal subspace and, for $1 \le i \le k$, let v'_i be the orthogonal projection of v_i on $W^\perp$, so that $v_i \in v'_i + W$. We have $v \wedge w = v' \wedge w$, hence

$$\|v \wedge w\| = \|v' \wedge w\| = \|v'\| \, \|w\| .$$

Besides, by using the definition of the scalar product in $\wedge^k V$, one easily checks that the tensors v' and $v - v'$ are orthogonal, so that $\|v'\| \le \|v\|$ and we are done. □

Let us now introduce unipotent flows and make explicit their link with polynomial functions. Let (u_t) be a one-parameter subgroup in $\mathrm{GL}(V)$. Recall (u_t) is said to be unipotent if its generator is nilpotent, that is, if there exists a nilpotent endomorphism N of V such that, for $t \geq 0$, $u_t = \exp(tN) = \sum_{p=0}^{d-1} \frac{1}{p!} N^p$. In particular, we at once get

Lemma 3.2.4. *Let V be a a euclidean space and (u_t) be a unipotent one-parameter subgroup in $\mathrm{GL}(V)$. Then for any integer $k \geq 0$ and any v in $\wedge^k V$, the function $t \mapsto \left\|\wedge^k u_t v\right\|^2$ is polynomial wth degree $\leq k(d-1)$.*

We will now use these objects to define new systoles in lattices.

Definition 3.2.5. Let Λ be a lattice in $\mathbb{R}^d$ and k be an integer in $[1, d]$. The k-systole of Λ is the positive real number $s_k(\Lambda)$ defined by

$$s_k(\Lambda) = \min\{\|v_1 \wedge \cdots \wedge v_k\| \,|v_1, \ldots, v_k \in \Lambda \quad v_1 \wedge \cdots \wedge v_k \neq 0\}.$$

In other words, by Lemma 3.2.2, $s_k(\Lambda)$ is the smallest covolume of the intersection of Λ with a k-dimensional subspace of $\mathbb{R}^d$. In particular, $s_d(\Lambda)$ is the covolume of Λ.

Assume Λ is unimodular. We can identify the space $\wedge^{d-1}\mathbb{R}^d$ with the dual space of $\mathbb{R}^d$ through the map that sends a pure tensor $w_1 \wedge \cdots \wedge w_{d-1}$ to the linear form $v \mapsto \det(w_1, \ldots, w_{d-1}, v)$. Under this identification, the scalar product on $\wedge^{d-1}\mathbb{R}^d$ is the one one obtains on the dual space of $\mathbb{R}^d$, through the natural identification of a euclidean space with its dual space. Then, the lattice $\wedge^{d-1}\Lambda$ identifies with the dual lattice Λ^* of Λ and the number $s_{d-1}(\Lambda)$ is the systole of Λ^*. More generally, for any $1 \leq k \leq d-1$, one has $s_k(\Lambda) = s_{d-k}(\Lambda^*)$.

The systoles are comparable to each other.

Lemma 3.2.6. *Fix $d \geq 1$. There exists a constant $C > 1$ such that, for any unimodular lattice Λ in $\mathbb{R}^d$, for any $1 \leq k \leq \ell \leq d$, one has*

$$\frac{1}{C} s_\ell(\Lambda)^{\frac{d-k}{d-l}} \leq s_k(\Lambda) \leq C s_\ell(\Lambda)^{\frac{k}{\ell}}.$$

Proof. Let us sketch this briefly, since we will note use it later. First, note that there exists $C_d > 1$, such that, if $a_1, \ldots, a_d$ are positive numbers with $a_1 \cdots a_d = 1$ and $a_i \leq \frac{2}{\sqrt{3}} a_{i+1}$, $1 \leq i \leq d-1$, one has $a_1 \cdots a_k \leq C_d$. By Theorem 1.3.8, this tells us that any unimodular lattice Λ in $\mathbb{R}^d$ satisfies $s_k(\Lambda) \leq C_d$.

Let Λ be a unimodular lattice in $\mathbb{R}^d$, $1 \leq k \leq \ell \leq d$ and V be an ℓ-dimensional subspace of $\mathbb{R}^d$ such that $\Lambda \cap V$ is a lattice with covolume $s_\ell(\Lambda)$ in V. By the remark above, there exists $v_1, \dots, v_k$ in $\Lambda \cap V$ with

$$\|v_1 \wedge \cdots \wedge v_k\| \leq C_\ell s_\ell(\Lambda)^{\frac{k}{\ell}},$$

hence

$$s_k(\Lambda) \leq C_\ell s_\ell(\Lambda)^{\frac{k}{\ell}}.$$

Now, recall that, since Λ is unimodular, we have $s_k(\Lambda) = s_{d-k}(\Lambda^*)$ and $s_\ell(\Lambda) = s_{d-\ell}(\Lambda^*)$. By applying the same inequality, we get

$$s_\ell(\Lambda) \leq C_{d-k} s_k(\Lambda)^{\frac{d-\ell}{d-k}}.$$

□

Given a unipotent one-parameter subgroup (u_t) of $\mathrm{SL}_d(\mathbb{R})$, we will study the behavior of the functions $t \mapsto s_k(u_t\Lambda)^2$. As these are piecewise polynomial function with bounded degrees, we will need an elementary lemma about the small values of polynomial functions.

Lemma 3.2.7. *Let $m \geq 0$ be an integer. For any $\varepsilon > 0$, there exists $\alpha > 0$ such that, for any polynomial function φ with degree $\leq m$ and $\max_{[0,1]} |\varphi| \geq 1$, one has*

$$|\{0 \leq t \leq 1 | \, |\varphi(t)| \leq \alpha\}| \leq \varepsilon.$$

Proof. It suffices to prove the result for polynomial functions φ with

$$\max_{[0,1]} |\varphi| = 1.$$

Equip the space of polynomial functions with degree $\leq m$ with the norm $\varphi \mapsto \max_{[0,1]} |\varphi|$, so that, in particular, the unit sphere S is compact.

For $\alpha > 0$ and $\varphi \in S$, set $F_\alpha(\varphi) = |\{0 \leq t \leq 1 | \, |\varphi(t)| \leq \alpha\}|$. We claim that the function F_α is upper semicontinuous: indeed, for any φ in S, one has

$$F_\alpha(\varphi) = \inf_{\substack{\theta \in \mathcal{C}^0([0,1]) \\ \theta \geq 1_{[0,\alpha]}}} \int_0^1 \theta(\varphi(t)) \mathrm{d}t,$$

so that F_α is the infimum of a family of continuous functions.

Now, for any φ in S, we have $F_\alpha(\varphi) \xrightarrow[\alpha \to 0]{} 0$, since φ has but a finite numbers of 0 in $[0,1]$. In particular, we have $\bigcap_{\alpha > 0} F_\alpha^{-1}([\varepsilon, \infty[) = \emptyset$. Since S is compact, there exists $\alpha > 0$ such that $F_\alpha^{-1}([\varepsilon, \infty[) = \emptyset$, which should be proved. □

3.3 Besicovitch covering Theorem

In the course of the proof of Theorem 3.0.11, one splits the interval $[0, T]$ into smaller intervals where the systole functions are polynomial. One then needs to build these intervals together and to control their overlapings. This will be done by using the one-dimensional case of Besicovitch covering Theorem:

Lemma 3.3.1 (Besicovitch covering Theorem in dimension 1)**.** *Let A be a bounded subset of $\mathbb{R}$ and $T : A \to \mathbb{R}_+^*$ be a bounded function. For x in A, set $I(x) = [x - T(x), x + T(x)]$. There exists $B \subset A$ such that $A \subset \bigcup_{x \in B} I(x)$ and that, for any x, y, z in B that are two by two distinct, one has $I(x) \cap I(y) \cap I(z) = \emptyset$.*

In other words, the covering $\bigcup_{x \in B} I(x)$ has multiplicity 2. This result holds in any dimension, but with a bound on the multiplicity of the covering that grows with the dimension.

Proof. First, we construct a sequence of elements of A by induction. We start by chosing x_0 in A with $T(x_0) \geq \frac{1}{2} \sup_A T$. Then, if $n \geq 1$ and $x_0, \ldots, x_{n-1}$ are constructed, we pick x_n in $A \setminus \bigcup_{k=0}^{n-1} I(x_k)$ such that

$$T(x_n) \geq \frac{1}{2} \sup_{A \setminus \bigcup_{k=0}^{n-1} I(x_k)} T.$$

For any n, set $T_n = T(x_n)$ and $I_n = I(x_n)$. We will construct B as a subset of the set $\{x_n | n \geq 0\}$. First, let us draw some direct consequences of the construction of the sequence.

We claim that $T_n \xrightarrow[n \to \infty]{} 0$. Indeed, for any $n < p$, one has $x_p \notin I_n$, hence $|x_n - x_p| \geq T_n$. As A is bounded, (x_n) admits a converging subsequence, hence (T_n) admits a subsequence converging to 0. Since, for any $n < p$, one has $T_p \leq 2T_n$, (T_n) goes to 0. In particular, this implies that $A \subset \bigcup_{n \geq 0} I_n$. Indeed, for every x in A, there exists n such that, for $p \geq n$, one has one has

$$T(x) > 2T_n \geq \sup_{A \setminus \bigcup_{k=0}^{n-1} I(x_k)} T,$$

hence $x \in \bigcup_{k=0}^{n-1} I(x_k)$.

For x in A, we let $N(x)$ be the set of n in $\mathbb{N}$ such that I_n contains x and we claim that $N(x)$ is finite. Write $N(x) = N_+(x) \cup N_-(x)$ where $N_+(x)$

(resp. $N_-(x)$) is the set of n in $N(x)$ such that $x_n \geq x$ (resp. $x_n \leq x$). Assume $N_+(x)$ is not empty, and let us prove it is finite. Fix n in $N_+(x)$. Then, there exists $p \geq n+1$ such that, for any $q \geq p$, one has $T_q \leq T_n$. For such a q, as x_q does not belong to I_n, one has $q \notin N_+(x)$. In the same way, $N_-(x)$ is finite.

We are now ready to construct the set B. This will again need an induction procedure, that relies on the following observation. Assume n, p are integers. We claim that, if $x_n \leq x_p$, one has $x_n - T_n \leq x_p - T_p$. Indeed, if $n > p$, one has $x_n < x_p - T_p$ and, if $n < p$, one has $x_p > x_n + T_n$ and $T_p \leq 2T_n$. In the same way, if $x_n \geq x_p$, one has $x_n + T_n \geq x_p + T_p$. Now, assume n, p, q are integers with $x_n < x_p < x_q$ and $I_n \cap I_q \neq \emptyset$. Then, on one hand, $I_n \cup I_q = [x_n - T_n, x_q + T_q]$ is an interval and, on the other hand, $x_p - T_p \geq x_n - T_n$ and $x_p + T_p \leq x_q + T_q$. We get $I_p \subset I_n \cup I_q$. In other words, as soon as such a configuration of three points arises, we can erase the one in the middle. Let us do this precisely.

We define inductively a decreasing sequence of subsets of $\mathbb{N}$. We set $E_0 = \mathbb{N}$. Assume the construction was achieved for $0, 1, \ldots, n$. If there exists p, q in E_n such that $x_p < x_n < x_q$ and $I_p \cap I_q \neq \emptyset$, we set $E_{n+1} = E_n \smallsetminus \{n\}$. If not, we set $E_{n+1} = E_n$. In any case, we have $\bigcup_{p \in E_{n+1}} I_p = \bigcup_{p \in E_{n+1}} I_p \supset A$. Finally, we set $E = \bigcap_{n \geq 0} B_n$, $B = \{x_n | n \in E\}$ and we claim that B has the required properties.

First, let n, p, q be two by two distincts elements of E and assume, for example $x_n < x_p < x_q$. Then, since n and q belong to E_p and p belongs to E_{p+1}, we have $I_n \cap I_p \cap I_q = \emptyset$.

Now, recall that, for any x in A, the set $N(x)$ of n in $\mathbb{N}$ such that $x \in I_n$ is finite. Hence the family $(N(x) \cap E_n)_{n \geq 0}$ is a decreasing sequence of finite subsets. Since it is never empty, it has non-empty intersection, hence, there exists n in E such that x belongs to I_n, which should be proved. $\square$

To give an idea of how powerful this covering theorem is, we give an application that is a strong extension of Lebesgue derivation Theorem.

Corollary 3.3.2. *Let μ and ν be Radon measures on $\mathbb{R}$. Then, for ν-almost any x in $\mathbb{R}$, one has*

$$\frac{\mu([x-\varepsilon, x+\varepsilon])}{\nu([x-\varepsilon, x+\varepsilon])} \xrightarrow[\varepsilon \to 0]{} \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x).$$

Proof. We can of course assume μ and ν are concentrated in a bounded subset of $\mathbb{R}$.

First, let us assume that μ is absolutely continuous with respect to ν, that is, $\mu = \varphi\nu$, for some φ in $\mathrm{L}^1(\mathbb{R},\nu)$. In this case, we will prove that the result follows from the fact that it holds when φ is continuous and from the denseness of continuous functions in $\mathrm{L}^1(\mathbb{R},\nu)$. To make this possible, we must prove that the quantities we want to converge do not change very much when one changes the function. This is where Besicovitch Theorem comes into play.

For any ψ in $\mathrm{L}^1(\mathbb{R},\nu)$, we introduce the associated maximal function ψ^* defined for ν-almost every x by

$$\psi^*(x) = \sup_{0<\varepsilon\leq 1} \frac{\left|\int_{[x-\varepsilon,x+\varepsilon]} \psi \mathrm{d}\nu\right|}{\nu([x-\varepsilon,x+\varepsilon])}.$$

We claim that, for any $\lambda > 0$, this function satisfies the maximal inequality

$$\nu(\{x\in\mathbb{R}|\psi^*(x)\geq\lambda\}) \leq \frac{2}{\lambda}\left\|\psi\right\|_1.$$

Indeed, let I be a bouded interval of $\mathbb{R}$ which contains the support of ν and set $A_\lambda = \{x \in I|\psi^*(x) \geq \lambda\}$. For any x in A_λ, we choose some $\varepsilon_x > 0$ such that

$$\left|\int_{[x-\varepsilon_x,x+\varepsilon_x]} \psi \mathrm{d}\nu\right| \geq \lambda\nu([x-\varepsilon_x,x+\varepsilon_x]).$$

By Lemma 3.3.1, there exists a set $B_\lambda \subset A_\lambda$ such that $A_\lambda \subset \bigcup_{x\in B_\lambda}[x-\varepsilon_x,x+\varepsilon_x]$ and the covering $\bigcup_{x\in B_\lambda}[x-\varepsilon_x,x+\varepsilon_x]$ has multiplicity ≤ 2. We have

$$\begin{aligned}\nu(A_\lambda) \leq \sum_{x\in B_\lambda}\nu([x-\varepsilon_x,x+\varepsilon_x]) &\leq \frac{1}{\lambda}\sum_{x\in B_\lambda}\left|\int_{[x-\varepsilon_x,x+\varepsilon_x]}\psi\mathrm{d}\nu\right| \\ &\leq \frac{1}{\lambda}\sum_{x\in B_\lambda}\int_{[x-\varepsilon_x,x+\varepsilon_x]}|\psi|\,\mathrm{d}\nu \leq \frac{2}{\lambda}\left\|\psi\right\|_1,\end{aligned}$$

where the last inequality follows from the fact that the covering has multiplicity 2.

Let us deduce from this inequality the almost sure convergence of the ratios $\frac{\int_{[x-\varepsilon,x+\varepsilon]}\varphi\mathrm{d}\nu}{\nu([x-\varepsilon,x+\varepsilon])}$ towards φ. More precisely, we will prove that, for any $\alpha > 0$, one has

$$\limsup_{\varepsilon\to 0}\left|\frac{\int_{[x-\varepsilon,x+\varepsilon]}\varphi\mathrm{d}\nu}{\nu([x-\varepsilon,x+\varepsilon])} - \varphi(x)\right| \leq 2\alpha$$

on a set whose complement has measure $\leq 3\alpha$, which implies the result. Indeed, pick a continuous compactly supported function ψ on $\mathbb{R}$ such that $\|\varphi-\psi\|_1 \leq \alpha^2$. On one hand, one has

$$\nu(\{x \in \mathbb{R} | \, |\varphi(x)-\psi(x)| \geq \alpha\}) \leq \alpha$$

and, on the other hand, form the maximal inequality,

$$\mu(\{x \in \mathbb{R} | (\varphi-\psi)^*(x) \geq \alpha\}) \leq 2\alpha,$$

which implies the claim, hence the lemma in the case where μ is absolutely continuous with respect to ν.

Now, to get the general case, we just have to deal with the case where μ is orthogonal to ν, that is, there exists a Borel set A which has measure 0 for ν and full measure for μ. In this sitation, since both μ and ν are absolutely continuous with respect to $\mu+\nu$, we get from the previous case that, for ν-almost any x in A, one has

$$\frac{\mu([x-\varepsilon, x+\varepsilon])}{\mu([x-\varepsilon, x+\varepsilon]) + \nu([x-\varepsilon, x+\varepsilon])} \xrightarrow[\varepsilon\to 0]{} 0,$$

which amounts to saying

$$\frac{\mu([x-\varepsilon, x+\varepsilon])}{\nu([x-\varepsilon, x+\varepsilon])} \xrightarrow[\varepsilon\to 0]{} 0,$$

which should be proved. □

3.4 Dani-Margulis induction

We now come back to the proof of the recurrence theorem.

Let us introduce some terminology about subgroups of lattices. We let $\mathcal{P}(\Lambda)$ denote the set of proper non trivial subgroups of Λ which are direct factors in Λ. For any Δ in $\mathcal{P}(\Lambda)$, we have $\Delta = \Lambda \cap V$, where V is the subspace of $\mathbb{R}^d$ that is spanned by Δ. We let $d(\Delta)$ be the covolume of Δ in V, that is $d(\Delta) = \|v_1 \wedge \cdots \wedge v_k\|$, where $v_1, \ldots, v_k$ is a basis of Δ. Finally, we define a primitive flag of Λ as a (finite) subset $\mathcal{F}$ of $\mathcal{P}(\Lambda)$ that is totally ordered by inclusion. For such a flag $\mathcal{F}$, we let $\mathcal{C}(\mathcal{F})$ denote the set of primitve subgroups in $\mathcal{P}(\Lambda) \smallsetminus \mathcal{F}$ which are comparable with all the elements of $\mathcal{F}$.

The core of the proof of Dani-Margulis theorem is the following lemma, which relies on successive applications of Lemma 3.2.7:

Lemma 3.4.1. *Let Λ be a unimodular lattice in $\mathbb{R}^d$ and (u_t) be a unipotent one-parameter subgroup in $\mathrm{SL}_d(\mathbb{R})$. Fix $\varepsilon > 0$. Then, there exists $0 < \sigma \leq \rho$ with the following property: for any $T > 0$, there exists a Borel subset $E_T \subset [0, T]$ with $|E_T| \geq (1-\varepsilon)T$, such that, for any t in E_T, there exists a primitive flag $\mathcal{F}$ of Δ with*

$$\begin{aligned} d(u_t\Delta) \geq \sigma \quad & \text{for any } \Delta \text{ in } \mathcal{F} \cup \mathcal{C}(\mathcal{F}) \\ \text{and } d(u_t\Delta) \leq \rho \quad & \text{for any } \Delta \text{ in } \mathcal{F}. \end{aligned}$$

This conclusion might seem a bit strange, but let us show that it permits to conclude:

Proof of Theorem 3.0.11 in the general case. Keep the notations of Lemma 3.4.1 and let us prove that, for any t in E_T, one has $s_1(u_t\Delta) \geq \frac{\sigma}{\min(\rho,1)}$, which, by Corollary 1.3.11, implies the result.

Indeed, for such a t, let $\mathcal{F}$ be as in the statement and v be a primitive vector of Λ. If v belongs to all the elements of $\mathcal{F}$, then $\mathbb{Z}v$ belongs to $\mathcal{F} \cup \mathcal{C}(\mathcal{F})$, hence $\|v\| \geq \sigma$. Else, let Δ be the largest element of $\mathcal{F}$ that does not contain v. Then $\mathbb{Z}v \oplus \Delta$ belongs to $\mathcal{F} \cup \mathcal{C}(\mathcal{F})$, hence

$$\sigma \leq d(u_t(\mathbb{Z}v \oplus \Delta)) \leq \|u_t v\| \, d(u_t\Delta) \leq \|u_t v\| \, \rho$$

(where the second inequality follows from Lemma 3.2.3) and we are done. □

It remains to prove Lemma 3.4.1. To build the flags $\mathcal{F}$, we will proceed by induction. More precisely, we will assume that a small flag $\mathcal{G}$ has already been constructed at most points of E_T and that it is constant on some intervals. Then, we will prove that, at most points, we can select a nice element of $\mathcal{C}(\mathcal{G})$. Once written precisely, the induction statement is the following:

Lemma 3.4.2. *Fix $\rho, \varepsilon > 0$. Then, there exists $0 < \sigma \leq \rho$ with the following property: for any unimodular lattice Λ in $\mathbb{R}^d$ and any primitive flag $\mathcal{G}$ in Λ and any compact interval $I \subset \mathbb{R}$ such that*

$$\min_{\Delta \in \mathcal{C}(\mathcal{G})} \max_{t \in I} d(u_t\Delta) \geq \rho,$$

there exists a Borel subset $E_I \subset I$ with $|E_I| \geq (1-\varepsilon)\,|I|$, such that, for any t in E_I, there exists a primitive flag $\mathcal{F}$ of Δ with $\mathcal{F} \subset \mathcal{G} \cup \mathcal{C}(\mathcal{G})$ and

$$\begin{aligned} d(u_t\Delta) \geq \sigma \quad & \text{for any } \Delta \text{ in } \mathcal{F} \cup \mathcal{C}(\mathcal{F}) \smallsetminus \mathcal{G} \\ \text{and } d(u_t\Delta) \leq \rho \quad & \text{for any } \Delta \text{ in } \mathcal{F} \smallsetminus \mathcal{G}. \end{aligned}$$

In this statement, one has to think to $\mathcal{G}$ as the part of the flag $\mathcal{F}$ that has already been constructed.

This allows to recover Lemma 3.4.1:

Proof of Lemma 3.4.1. This is the particular case where $\mathcal{G}$ is the empty flag, ρ is the minimum of the $d(\Delta)$, where Δ belongs to $\mathcal{P}(\Lambda)$, and $I = [0, T]$. $\square$

It only remains to give the

Proof of Lemma 3.4.2. We will prove this statement when one fixes the cardinality r of the flags $\mathcal{G}$. This, we will do by reverse induction on r, that is an element in $0, \dots, d-1$. If $r = d-1$, we can take $E_I = I$ and $\mathcal{F} = \mathcal{G}$, since $\mathcal{C}(\mathcal{G}) = \emptyset$.

Hence, assume r belongs to $0, \dots, d-2$ and the statement holds for $r+1$. Let σ' the constant that is given by this induction assumption for flags of cardinality $r+1$ and when ε is replaced with $\frac{\varepsilon}{12}$. From Lemmas 3.2.4 and 3.2.7, we know that there exists $\alpha > 0$ such that, for any compact interval I of $\mathbb{R}$, for any Δ in $\mathcal{P}(\Lambda)$, if $\max_{t \in I} d(u_t \Delta) = \rho$, then $|\{t \in I | d(u_t \Delta) \leq \alpha\}| \leq \frac{\varepsilon}{12} |I|$. We set $\sigma = \min(\sigma', \alpha)$ and we will prove that it satisfies the conclusions of the Lemma.

Let $\mathcal{G}$ be a primitive flag in Λ with cardinality r and I be a compact interval in $\mathbb{R}$ such that

$$\min_{\Delta \in \mathcal{C}(\mathcal{G})} \max_{t \in I} d(u_t \Delta) \geq \rho.$$

We will split I into smaller intervals such that, on each of these, there is a good choice of an element of $\mathcal{C}(G)$. Then, we will apply the induction statement to the small intervals and to the flag one obtains by adding the marked element of $\mathcal{C}(\mathcal{G})$ to $\mathcal{G}$.

Let us be more precise. We let $A \subset I$ be the set of those t in I such that there exists Δ in $\mathcal{C}(\mathcal{G})$ with $d(u_t \Delta) < \rho$. For t in $I \smallsetminus A$, we note that $\mathcal{F} = \mathcal{G}$ satisfies the conclusions of the lemma.

Now, for any t in A, there exists only finitely many Δ in $\mathcal{C}(\mathcal{G})$ with $d(u_t \Delta) < \rho$: indeed, these correspond to vectors with bounded norm in the lattices $\wedge^k \Lambda$, $1 \leq k \leq d-1$. We let $T_t > 0$ be the smallest real number such that

$$\min_{\Delta \in \mathcal{C}(\mathcal{G})} \max_{s \in [t - T_t, t + T_t]} d(u_s \Delta) \geq \rho$$

and we pick some $\Delta_t \in \mathcal{C}(\mathcal{G})$ such that, for any s in $(t - T_t, t + T_t)$, one has $d(u_s \Delta_t) < \rho$. Note that, by assumption, we have $T_t \leq |I|$. We apply the induction assumption to the interval $[t - T_t, t + T_t]$ and the flag $\mathcal{G} \cup \{\Delta_t\}$. This gives us a Borel subset $E_t \subset [t - T_t, t + T_t]$ with $|E_t| \geq (1 - \frac{\varepsilon}{12})2T_t$ such that, for any s in E_t, there exists a primitive flag $\mathcal{F}_s$ of Λ with $\mathcal{F}_s \subset \mathcal{G} \cup \{\Delta_t\} \cup \mathcal{C}(\mathcal{G} \cup \{\Delta_t\})$ and

$$\begin{aligned} d(u_s\Delta) \geq \sigma' \quad & \text{for any } \Delta \text{ in } \mathcal{F}_s \cup \mathcal{C}(\mathcal{F}_s) \smallsetminus (\mathcal{G} \cup \Delta_t) \\ \text{and } d(u_s\Delta) \leq \rho \quad & \text{for any } \Delta \text{ in } \mathcal{F}_s \smallsetminus (\mathcal{G} \cup \{\Delta_t\}). \end{aligned}$$

To conclude, we need to take out the points s where $d(u_s\Delta_t)$ is too small. We set

$$F_t = \{s \in [t - T_t, t + T_t] | d(u_s\Delta_t) \geq \alpha\}.$$

By definition of α, we have $|F_t| \geq (1 - \frac{\varepsilon}{12})2T_t$. For s in $F_t \cap E_t$, the flag $\mathcal{F}_s$ associated to s satisfies the conclusions of the lemma.

Now, we need to go back to the full interval I. To do this we apply Lemma 3.3.1 to the set A and the intervals $[t - T_t, t + T_t]$, $t \in A$. This gives us a subset B of A such that $A \subset \bigcup_{t \in B} [t - T_t, t + T_t]$ and, for any two by two distinct t, t', t'' in B, the three associated intervals do not intersect. We set

$$E_I = (I \smallsetminus A) \cup \bigcup_{t \in K} E_t \cap F_t.$$

To conclude, we only need to prove that the Lebesgue measure of the complement of this set is small. Indeed, we have

$$|I \smallsetminus E_I| \leq \sum_{t \in B} |[t - T_t, t + T_t] \smallsetminus (E_t \cap F_t)| \leq \sum_{t \in B} \frac{\varepsilon}{3} T_t \leq \frac{\varepsilon}{3} \left| \bigcup_{t \in B} [t - T_t, t + T_t] \right|$$

(where the latter inequality follows from the multiplicity bound on the covering). Now, since, for any t in B, we have $T_t \leq |I|$, we get

$$\left| \bigcup_{t \in B} [t - T_t, t + T_t] \right| \leq 3 |I|$$

and we are done. $\square$