CENTRAL LIMIT THEOREM ON HYPERBOLIC GROUPS

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Abstract. We prove a central limit theorem for random walks with finite variance on Gromov hyperbolic groups.

1. Introduction

1.1. **Central limit theorem.** Let (M, d) be a metric space and o be a point in M. Let $G := \mathrm{Isom}(M)$ be the group of isometries of M and set, for g in G, $\kappa(g) := d(go, o)$. Let μ be a Borel probability measure on G with a finite first moment : $\int_G \kappa(g)\,\mathrm{d}\mu(g) < \infty$. Let $g_1, \ldots, g_n, \ldots$ be independant random isometries of M chosen with law μ. We want to understand the behavior of the sequence of random variables $\kappa(g_n \ldots g_1)$. It is well-known that this sequence satisfy a law of large number : there exists a constant λ called the *escape rate* of μ such that, almost surely, $\lim\limits_{n\to\infty} \frac{1}{n}\kappa(g_n \ldots g_1) = \lambda$. In this paper we will prove that under suitable assumptions, this sequence satisfies a central limit theorem as soon as μ has a finite second moment : $\int_G \kappa(g)^2\,\mathrm{d}\mu(g) < \infty$. These suitable assumptions are : the metric space (M, d) is proper, quasiconvex, and Gromov hyperbolic (see Definition 2.1), and the law μ is non-elementary and non-arithmetic (see Definition 3.1).

Theorem 1.1. *Let (M, d) be a proper, quasiconvex, Gromov hyperbolic space, $o \in M$ and μ a non-elementary and non-arithmetic Borel probability measure on the group $G \subset \mathrm{Isom}(M)$ with finite second moment. Let λ be the escape rate of μ. Then the renormalized variables $\frac{1}{\sqrt{n}}(\kappa(g_n \cdots g_1) - n\lambda)$ converge in law to a non-degenerate gaussian law.*

Important examples of such Gromov hyperbolic spaces M are
(i) Metric trees,
(ii) Gromov hyperbolic groups Γ endowed with the left-invariant distance associated to a generating set S of Γ.
(iii) Universal covers of compact Riemannian manifolds with negative curvature.

If one replace the finite second moment assumption by a finite exponential moment assumption, i.e. $\int_G e^{\alpha_0\,\kappa(g)}\,\mathrm{d}\mu(g) < \infty$ for some $\alpha_0 > 0$, then the central limit theorem 1.1 is mainly due to Bjorklund in [5] (extending earlier central limit theorem for free groups due to Sawyer-Steger in [24] and Ledrappier in [22]).

Hence the key point of this paper is to get rid of this finite exponential moment assumption. For that we adapt the method we have introduced in [2] for linear groups. This method does not rely on spectral gap property of transfer operators, this is why it allows us to prove this central limit theorem without any further assumptions on the boundaries of M or on the support of μ. Even for the free group, our central limit theorem seems to be new when the law μ is only assumed to have finite second moment.

1.2. **Strategy.** We want to prove the central limit theorem for the random variables $\kappa(g_n \cdots g_1)$. Let X be the Busemann boundary of M (see Section 2.3). Since this function κ on G is very much related to the Busemann cocycle $\sigma : G \times X \to \mathbb{R}$ (see Sections 2.4 and 3.2), we are reduced to prove, for every x in X, a central limit theorem (Theorem 4.7) for the random variables $\sigma(g_n \cdots g_1, x)$. Adding a suitable coboundary, we will replace this cocycle σ by another cocycle σ_0 for which the "expected increase" is constant i.e. such that $\int_G \sigma_0(g,x)\,\mathrm{d}\mu(g) = \lambda$ for all x in X. This will allow us to use the classical central limit theorem for martingales due to Brown in [7]. This cocycle σ_0 will be given by $\sigma_0(g,x) = \sigma(g,x) - \psi(x) + \psi(gx)$ for a bounded function ψ on X (Proposition 4.6). As in [2], we give an explicit formula for this function ψ in terms of a $\check{\mu}$-stationary measure ν^* on X, where $\check{\mu}$ is the image of μ by $g \mapsto g^{-1}$. This formula is

$$\psi(x) = -2 \int_G (x|y)_o \,\mathrm{d}\nu^*(y), \tag{1.1}$$

where $(x|y)_o$ is the Gromov product on X (see Section 2.3). The main issue is to check that this integral is finite, i.e. that the stationary measure ν^* is log-regular, when the second moment of μ is finite (Proposition 4.2). As in [2], the key point is to prove the complete convergence of the sequence $\frac{1}{n}\kappa(g_n \cdots g_1)$ toward the escape rate λ (Proposition 4.1), generalizing Hsu-Robbins theorem.

Compared to the linear case in [2], the new technical difficulties which occur are due to the fact that the Busemann coboundary σ is a cocycle on the Busemann boundary X, while the behavior of the random walk is much easier to describe on the Gromov boundary ∂M (see Proposition 3.1). This forces us to change frequently our point of view, working sometimes with Buseman boundary and sometimes with Gromov boundary.

1.3. **Plan.** In Chapter 2, we recall without proof the basic definitions and properties for Gromov hyperbolic spaces and their boundaries.

In Chapter 3, we recall with short proofs basic results for random walks on Gromov hyperbolic groups.

In Chapter 4, we prove successively the complete convergence towards the escape rate, the log-regularity of the stationary measure on the Gromov boundary, the centerability of the Busemann cocycle on the Busemann boundary, and the central limit theorem 4.7.

In Chapter 5, we prove an optimal version (Proposition 5.1 and Example 5.4) of the log-regularity of the stationary measure on the Gromov boundary when G acts cocompactly on M. This optimal version is not needed for the proof of the central limit theorem but seems interesting in its own.

We thank M. Bjorklund for interesting discussions on this topic.

2. Hyperbolic spaces and their boundaries

In this chapter, we recall without proof the basic definitions and properties for Gromov hyperbolic spaces and their boundaries (see [14], [13], [9], [25] or [18] for more details).

2.1. **Gromov hyperbolic spaces.** Let (M, d) be a metric space, and o be a point of M. The *Gromov product* is defined, for o, m_1, m_2 in M, by

$$(m_1|m_2)_o := \tfrac{1}{2}(d(o, m_1) + d(o, m_2) - d(m_1, m_2)).$$

We assume that the metric space M is *Gromov hyperbolic* i.e. there exists a constant $\delta > 0$ such that, for every o, m_1, m_2, m_3 in M, one has

$$(m_1|m_3)_o \geq \min((m_1|m_2)_o, (m_2|m_3)_o) - \delta. \tag{2.1}$$

Definition 2.1. *The metric space M is said to be proper if the bounded closed subsets are compact. It is said to be quasi-convex, if there exists $C > 1$ such that every two points m, m' in M can be joined by a C-geodesic, i.e. there exists a sequence $m_1 = m, m_2, \ldots, m_n = m'$ of points in M, such that for $1 \leq i \leq j \leq n$, one has*

$$j - i - C \leq d(m_i, m_j) \leq j - i + C.$$

For instance, when M is *geodesic* i.e. when any two points of M are joined by a geodesic arc, then M is quasiconvex.

2.2. **Gromov boundary.** Let (M,d) be a proper quasiconvex Gromov hyperbolic space. The *Gromov boundary* ∂M is the set of equivalence classes $\xi = [m_n]$ of sequences (m_n) of points in M going to infinity such that $(m_n|m_p)_o \xrightarrow[n,p\to\infty]{} \infty$, where two sequences (m_n) and (m'_n) are said to be equivalent if $(m_n|m'_p)_o \xrightarrow[n,p\to\infty]{} \infty$. Since M is Gromov hyperbolic, this is an equivalence relation. Note that, since M is proper and quasiconvex, for every equivalence class ξ, we can choose a representative (m_n) which is a C-geodesic.

We extend the Gromov product to the Gromov boundary: for ξ, ξ' in ∂M and m in M, we set

$$(m|\xi)_o = (\xi|m)_o \;\; = \;\; \inf \liminf_{n\to\infty} (m|m_n)_o \;\text{ and} \tag{2.2}$$

$$(\xi|\xi')_o \;\; = \;\; \inf \liminf_{n\to\infty} (m_n|m'_n)_o \in [0,\infty], \tag{2.3}$$

where the infimum is over all the sequences m_n and m'_n such that $\xi = [m_n]$ and $\xi' = [m'_n]$.

We endow the union $M^* := M \cup \partial M$ with the following topology. A basis of neighborhoods of a point m in M is given by the balls

$$B(m,r) =: \{m' \in M \mid d(m,m') \le r\} \;\text{ for }\; r > 0.$$

A basis of neighborhoods of a point ξ in ∂M is given by the sets

$$\mathcal{V}(\xi,R) := \{\xi' \in M^* \mid (\xi|\xi')_o \ge R\} \;\text{ for }\; R > 0.$$

Since M is proper and quasiconvex, this topological space M^* is compact and metrizable. It is called the *Gromov compactification* of M.

For any ξ_1, ξ_2, ξ_3 in M^*, one still has

$$(\xi_1|\xi_3)_o \ge \min((\xi_1|\xi_2)_o, (\xi_2|\xi_3)_o) - \delta. \tag{2.4}$$

The Gromov product might not be continuous on M^* but we still have the following rough-continuity property: for any converging sequences $\xi_n \to \xi$ and $\xi'_n \to \xi'$ in M^*, one has the inequalities in $[0,\infty]$

$$(\xi|\xi')_o \;\; \le \;\; \liminf_{n\to\infty} (\xi_n|\xi'_n)_o \le \limsup_{n\to\infty} (\xi_n|\xi'_n)_o \le (\xi|\xi')_o + 2\delta. \tag{2.5}$$

2.3. **Busemann boundary.** Let (M,d) be a proper quasiconvex Gromov hyperbolic space. We recall that the *Busemann compactification* $\overline{M}$ of M is the set of equivalence classes x of sequences of points (m_n) in M such that, for every m in M, the limit

$$h_x(m) := \lim_{n\to\infty} d(m,m_n) - d(o,m_n) \tag{2.6}$$

exists, where two sequences (m_n) and (m'_n) are said to be equivalent if they define the same limit function on M. This function h_x is called the *Busemann function* at x. Identifying each element x in $\overline{M}$ with the corresponding Busemann function, we endow $\overline{M}$ with the topology of uniform convergence on compact sets. Since M is proper and since all these Busemann functions are 1-Lipschitzian and vanishes at the point o, this topological space $\overline{M}$ is compact and metrizable. Moreover, since M is quasiconvex, the space $\overline{M}$ contains M as an open subset. The space $X := \overline{M} \smallsetminus M$ is the *Busemann boundary* of M.

We also extend the Gromov product to the Busemann boundary: for x, x' in X and m in M, we set

$$(m|x)_o = (x|m)_o = \tfrac{1}{2}(d(m,o) - h_x(m)) \text{ and} \tag{2.7}$$

$$(x|x')_o := -\min_{m\in M} \tfrac{1}{2}(h_x(m) + h_{x'}(m)). \tag{2.8}$$

We denote by $\pi : \overline{M} \to M^*; x \mapsto \pi_x$ the natural projection between Busemann and Gromov compactifications. This is the unique continuous map such that $\pi_m = m$ for all m in M. This map is surjective.

The relationship between the Gromov product on the Busemann compactification $\overline{M}$ and on the Gromov compactification M^* is given by the following inequality : *There exists a constant* $C_0 > 0$*, such that for any* x*,* y *in* $\overline{M}$*,*

$$(\pi_x|\pi_y)_o - C_0 \quad\leq\quad (x|y)_o \leq (\pi_x|\pi_y)_o + C_0. \tag{2.9}$$

In particular, two points x and y in X have same image $\pi_x = \pi_y$ in ∂M if and only if $(x|y)_o = \infty$.

The Busemann functions allow us to control the distance function thanks to the following inequality: *for any* x*,* y *in* $\overline{M}$ *with* $\pi_x \neq \pi_y$*, there exists a constant* $C_{x,y} > 0$ *such that, for all* m *in* M*,*

$$\max(h_x(m), h_y(m) \geq d(o,m) - C_{x,y} \tag{2.10}$$

2.4. **Isometries of hyperbolic spaces.** Let $G := \mathrm{Isom}(M)$ be the group of isometries of M. We denote by $\sigma : G \times X \to \mathbb{R}$ the *Busemann cocycle*, it is the continuous cocycle given, for g in G and x in X, by

$$\sigma(g,x) := h_x(g^{-1}o). \tag{2.11}$$

For g in G, we define the length

$$\kappa(g) := d(go,o) \tag{2.12}$$

and define the *stable length* of g by

$$\ell(g) := \lim_{n\to\infty} \tfrac{1}{n}\, \kappa(g^n). \tag{2.13}$$

The group G acts continuously on both compactifications $\overline{M}$ and M^* and the projection $\pi : \overline{M} \to M^*$ is G-equivariant.

We collect a few useful properties of these functions: We first express that σ is a cocycle. We give then relations between the length $\kappa(g)$ with the cocycle and the Gromov product, these relations will be useful to prove the log-regularity of the stationary measure $\nu_{\partial M}$. The last equality is the key formula which will allow us to solve the cohomological equation 4.13.

Lemma 2.1. *Let (M,d) be a proper quasiconvex Gromov hyperbolic space. For any isometries g, g' of M and any x, y in the Busemann boundary X, one has the equalities*

$$\sigma(gg',x) = \sigma(g,g'x) + \sigma(g',x), \tag{2.14}$$

$$\sigma(g^{-1},x) = -\sigma(g,g^{-1}x), \tag{2.15}$$

$$\kappa(g) - \sigma(g,x) = 2\,(g^{-1}o|x)_o, \tag{2.16}$$

$$\kappa(g) + \sigma(g,x) = 2\,(go|gx)_o, \tag{2.17}$$

$$\sigma(g,x) = -2\,(x|g^{-1}y)_o + 2\,(gx|y)_o + \sigma(g^{-1},y) \tag{2.18}$$

Proof. Those equalities are straightforward applications of the definitions. □

The following lemma describes the dynamics of G on the Gromov boundary. It is a key ingredient in the proof of Proposition 3.1.

Lemma 2.2. *Let (M,d) be a proper quasiconvex Gromov hyperbolic space. Let ν be an atom-free Borel probability measure on the Gromov boundary ∂M, and g_n be a sequence of isometries of M such that the limit measure $\nu' := \lim_{n\to\infty} (g_n)_*\nu$ exists.*

If the sequence g_n is unbounded then ν' is a Dirac mass.

Conversely, if $\nu' = \delta_\xi$ then one has $\lim_{n\to\infty} g_n o = \xi$.

Proof. If the sequence g_n is unbounded, we may after extraction, assume that the following limits $\xi_+ = \lim_{n\to\infty} g_n o$ and $\xi_- = \lim_{n\to\infty} g_n^{-1} o$ exist in M^* and belong to ∂M. We first check that,

for any $\eta \neq \xi_-$ in ∂M, the sequence $g_n\eta$ converges to ξ_+.

Indeed, the sequence $(\eta|g_n^{-1}o)_o$ is bounded and hence the sequence $d(o,g_n^{-1}o) - (\eta|o)_{g_n^{-1}o}$ is also bounded, and one has $\lim_{n\to\infty} (g_n\eta|g_n o)_o = \infty$. This proves that the sequence $g_n\eta$ converges to ξ_+. But then, since $\nu(\{\xi_-\}) = 0$, the probability measure ν' has to be a Dirac mass at ξ_+.

Conversely, for the same reasons, if ν' is a Dirac mass at a point ξ, this point has to be equal to any cluster point of the sequence $g_n o$. □

2.5. **Hyperbolic group.** Let G be a locally compact group which is generated by a compact neighborhood V of e. We may assume that $V = V^{-1}$. We introduce the left-invariant distance d_V on G given, for g, h in G, by $d_V(g,h) := \inf\{n \geq 0 \mid g^{-1}h \in V^n\}$. The topology defined by this distance is the discrete topology.

A locally compact group G is said to be hyperbolic if it is generated by a compact neighborhood V of e and such that the distance d_V is hyperbolic. Note that this property does not depend on the choice of V.

According to [8, Cor 2.6] a locally compact group G is hyperbolic if and only if G has a continuous proper cocompact isometric action on a proper geodesic hyperbolic space (M,d).

3. Random walk on hyperbolic spaces

In this chapter we recall basic results for random walks on Gromov hyperbolic groups (see [20], [5], or [16] for more details; a comparison with the linear case as in [23],[12], [15], [6], [11], [3] might also be useful)

3.1. **Stationary measure on Gromov boundary.** Let (M,d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Let μ be a Borel probability measure on the group $G = \mathrm{Isom}(M)$ and G_μ be the smallest closed subgroup of G such that $\mu(G_\mu) = 1$. Let $g_1, \ldots, g_n, \ldots$ be independant random elements of G chosen with law μ. We want to understand the behavior of the random variables $g_n \ldots g_1 o$ and $g_1 \ldots g_n o$. We denote by μ^{*n} the $\mathrm{n^{th}}$-convolution power $\mu * \cdots * \mu$.

Definition 3.1. *We say that μ is non-elementary if the group G_μ is unbounded and if G_μ does not have any finite orbit in the Gromov boundary ∂M.*

We say that μ is non-arithmetic if there exists $n \geq 1$ and g, g' in the support of μ^{*n} such that $\ell(g) \neq \ell(g')$.

We denote by $(B, \mathcal{B}, \beta, S)$ the associated one-sided Bernoulli system i.e. $B = G^{\mathbb{N}^*}$ is the set of sequences $b = (b_1, b_2, \ldots)$ with b_n in G, $\mathcal{B}$ is the product σ-algebra, β is the product measure $\mu^{\otimes \mathbb{N}^*}$, and $S : B \to B$ is the shift given by $Sb = (b_2, b_3, \ldots)$. For $n \geq 1$, we denote by $\mathcal{B}_n$ the σ-algebra spanned by the first n coordinates $b_1, \ldots, b_n$.

A Borel probability measure ν on X or on ∂M is said to be μ-stationary if $\mu * \nu = \nu$.

Proposition 3.1. *Let (M,d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Let μ be a non-elementary Borel probability measure on the group $G = \mathrm{Isom}(M)$.*

a) For β-almost any b in B, one has $\lim\limits_{n\to\infty}\kappa(b_1..b_n)=\lim\limits_{n\to\infty}\kappa(b_n..b_1)=\infty$.
b) For β-almost any b in B, the limit $\xi_b := \lim\limits_{n\to\infty} b_1..b_n o$ *exists in ∂M, and one has the equality* $\xi_b = b_1\xi_{Sb}$.
c) There exists a unique μ-stationary Borel probability measure on the Gromov boundary ∂M which is $\nu_{\partial M} := \int_B \delta_{\xi_b}\, \mathrm{d}\beta(b)$.

Proof. Let ν be a μ-stationary probability measure on ∂M. Since G_μ does not have any finite orbit in ∂M, this measure ν is atom-free. By the martingale theorem, for β-almost all b in B, the limit probability measure $\nu_b := \lim\limits_{n\to\infty}(b_1..b_n)_*\nu$ exists. Since G_μ is unbounded, for β-almost all b the sequence $b_1..b_n$ is unbounded. Hence by Lemma 2.2, the measure ν_b is a Dirac mass at a point $\xi_b\in\partial M$ and one has $\lim\limits_{n\to\infty} b_1..b_n o = \xi_b$. Since $\nu = \int_B \delta_{\xi_b}\,\mathrm{d}\beta(b)$, the μ-stationary probability measure ν is unique. □

3.2. **Random walk on Busemann boundary.** The following proposition compares the behavior of the random variables $\kappa(b_n..b_1)$ and $\sigma(b_n..b_1, x)$.

Proposition 3.2. *Let (M, d) be a proper quasiconvex Gromov hyperbolic space and $o\in M$. Let μ be a non-elementary Borel probability measure on the group $G = \mathrm{Isom}(M)$.*
a) For all $\varepsilon > 0$ there exists $T > 0$ such that, for all x in X,

$$\beta(\{b\in B \mid \sup_{n\geq 1}(\kappa(b_n..b_1)-\sigma(b_n..b_1,x))\leq T\}) \;\;\geq\;\; 1-\varepsilon, \tag{3.1}$$

and hence, for all $n\geq 1$,

$$\mu^{*n}(\{g\in G \mid (\kappa(g)-\sigma(g,x))\leq T\}) \;\;\geq\;\; 1-\varepsilon. \tag{3.2}$$

b) For all x in X, for β-almost all b in B, one has $\lim\limits_{n\to\infty}\sigma(b_n..b_1,x)=\infty$.

Proof. a) According to Formulas (2.9) and (2.16), it is equivalent to prove the following assertion :

For all $\varepsilon > 0$ there exists $T > 0$ such that, for all ξ in ∂M,

$$\beta(\{b\in B \mid \sup_{n\geq 1}(b_1^{-1}..b_n^{-1}o|\xi)_o \leq T\}) \;\;\geq\;\; 1-\varepsilon, \tag{3.3}$$

According to Proposition 3.1, for β-almost any b in B, the following limit

$$\xi_{b^-} := \lim_{n\to\infty} b_1^{-1}..b_n^{-1}o \tag{3.4}$$

exists in ∂M.

On the one hand, since μ is non-elementary, the $\check{\mu}$-stationary probability measure $\nu^*_{\partial M} = \int_B \delta_{\xi_{b^-}}\,\mathrm{d}\beta(b)$ is atom free. Hence, for any $\varepsilon > 0$, there exists $R > 0$ such that, for all ξ in ∂M, one has

$$\beta(\{b \in B \mid (\xi_{b^-}|\xi)_o \leq R\}) \geq 1 - \tfrac{\varepsilon}{2}. \tag{3.5}$$

On the other hand, using (3.4), (2.4), and (2.5), for all $R > 0$, for β-almost any b in B, there exist $T_{R,b} > 0$ such that, for all ξ in ∂M with $(\xi_{b^-}|\xi)_o \leq R$, one has

$$\sup_{n\geq 1} (b_1^{-1}..b_n^{-1}o|\xi)_o \leq T_{R,b}. \tag{3.6}$$

We choose then a constant $T > 0$ such that

$$\beta(\{b \in B \mid T_{R,b} \leq T\}) \geq 1 - \tfrac{\varepsilon}{2}. \tag{3.7}$$

Then the wanted Equation (3.3) follows from (3.5), (3.6) and (3.7). □

3.3. **Escape rate of the random walk.** We assume in this section that the first moment of μ is finite $\int_G \kappa(g)\,\mathrm{d}\mu(g) < \infty$. In this case, by subadditivity, the limit

$$\lambda := \lim_{n\to\infty} \tfrac{1}{n} \int_G \kappa(g)\,\mathrm{d}\mu^{*n}(g)$$

exists and is called the *exponent* or the *escape rate* of μ.

There might be more than one stationary measure on the Busemann boundary X, but the following Proposition 3.3.c tells us that the Busemann cocycle σ on X has *unique average* λ.

Proposition 3.3. *Let (M, d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Let μ be a non-elementary Borel probability measure on the group $G = \mathrm{Isom}(M)$ such that $\int_G \kappa(g)\,\mathrm{d}\mu(g) < \infty$, and λ be the escape rate of μ.*
a) For β-almost all b in B, one has $\lambda = \lim_{n\to\infty} \frac{1}{n}\kappa(b_n \dots b_1)$.
b) For x in X and β-almost all b in B, one has $\lambda = \lim_{n\to\infty} \frac{1}{n}\sigma(b_n \dots b_1, x)$.
c) For all μ-stationary Borel probability measure ν on X, one has $\lambda = \int_{G\times X} \sigma(g, x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x)$.
d) The escape rate is positive $\lambda > 0$.

Proof. a) This follows from Kingman's subadditive ergodic theorem.
b) This follows from a) and Proposition 3.2.a.
c) This follows from b) and Birkhoff ergodic theorem.
d) This follows from b), Proposition 3.2.b and [6, Lemma II.2.2]. □

4. Central limit theorem

We begin now the proof of the central limit theorem 1.1. As in the linear case in [2], the key steps will be a regularity property for the μ-stationary measure on the Gromov boundary.

4.1. **Complete convergence toward the escape rate.** As in [2], we will need the following Proposition 4.1 which is an analog of Hsu-Robbins-Baum-Katz theorem in [19] and [1] for the convergence toward the escape rate in Proposition 3.3. When $p = 2$, it tells us that this convergence is complete.

Proposition 4.1. *Let $p > 1$. Let (M, d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Let μ be a non-elementary Borel probability measure on G such that $\int_G \kappa(g)^p \, \mathrm{d}\mu(g) < \infty$, and λ be the escape rate of μ. Then, for every $\varepsilon > 0$, there exist constants $C_n = C_n(p, \varepsilon, \mu)$ such that $\sum_{n \geq 1} n^{p-2} C_n < \infty$ and, for any x in X,*

$$\mu^{*n}(\{g \in G \text{ such that } |\sigma(g, x) - n\lambda| \geq \varepsilon n\}) \leq C_n, \text{ and} \tag{4.1}$$

$$\mu^{*n}(\{g \in G \text{ such that } |\kappa(g) - n\lambda| \geq \varepsilon n\}) \leq C_n. \tag{4.2}$$

Proof. *a*) Since the Busemann cocycle σ has unique average λ, this follows from [2, Prop. 3.2].

b) Formula (2.10) tells us that, for all x, y in X with $\pi_x \neq \pi_y$, there exists $C > 0$ such that, for all g in G,

$$\kappa(g) - C \leq \max(\sigma(g, x), \sigma(g, y)) \leq \kappa(g). \tag{4.3}$$

Hence, point *b*) with another constant C_n, follows from point *a*). □

4.2. **Log-regularity of the stationary measure.** The next proposition with $p = 2$ will be used to solve the cohomological equation (4.13).

Proposition 4.2. *Let $p > 1$. Let (M, d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Let μ be a non-elementary Borel probability measure on G such that $\int_G \kappa(g)^p \, \mathrm{d}\mu(g) < \infty$. Let ν be a μ-stationary Borel probability measure on the Busemann boundary X. Then, one has*

$$\sup_{y \in X} \int_X (x|y)_o^{p-1} \, \mathrm{d}\nu(x) < \infty. \tag{4.4}$$

Remark 4.3. Because of Formula (2.9), one can reformulate Equation (4.4) as,

$$\sup_{\eta\in\partial M}\int_{\partial M}(\xi|\eta)_o^{p-1}\,\mathrm{d}\nu_{\partial M}(\xi)<\infty, \tag{4.5}$$

where $\nu_{\partial M}$ is the unique μ-stationary Borel probability measure on he Gromov boundary ∂M.

Remark 4.4. When μ is assumed to have an exponential moment, one can prove that the stationary measure ν is much more regular: its Hausdorff dimension is finite, i.e. there exists $t>0$ such that

$$\sup_{y\in\mathbb{P}(V^*)}\int_X e^{-t(x|y)_o}\,\mathrm{d}\nu(x)<\infty. \tag{4.6}$$

Lemma 4.5. *Under the same assumptions as Proposition* 4.2*, there exist constants* $a>0$ *and* $C_n>0$ *with* $\sum_{n\geq 1}n^{p-2}C_n<\infty$*, and such that, for* $n\geq 1$*,* x*,* y *in* X*, one has*

$$\mu^{*n}(\{g\in G\mid (go|gx)_o\leq an\})\leq C_n, \tag{4.7}$$

$$\mu^{*n}(\{g\in G\mid (go|y)_o\geq an\})\leq C_n, \tag{4.8}$$

$$\mu^{*n}(\{g\in G\mid (gx|y)_o\geq an\})\leq C_n. \tag{4.9}$$

Proof. According to Proposition 3.3, the escape rate λ of μ is positive. We set $a=\frac{\lambda}{2}$. According to Proposition 4.1, there exist constants C_n such that $\sum_{n\geq 1}n^{p-2}C_n<\infty$ and such that, for $n\geq 1$, x, y in X, there exist subsets $G_{n,x,y}\subset G$ with $\mu^{*n}(G_{n,x,y})\geq 1-C_n$ such that, for g in $G_{n,x,y}$, the three quantities

$$|\kappa(g)-\lambda n|\,,\ |\sigma(g,x)-\lambda n|\,,\ |\sigma(g^{-1},y)-\lambda n|$$

are bounded by $\frac{\lambda n}{4}$. We will choose n_0 large enough, and prove the bounds (4.7), (4.8) and (4.9) only for $n\geq n_0$. We only have to check that for $n\geq n_0$ and g in $G_{n,x,y}$, one has

$$(go|gx)_o\geq an\ ,\ \ (go|y)_o\leq an\ \text{ and }\ (gx|y)_o\leq an.$$

We first notice that, according to Equation (2.17), one has

$$(go|gx)_o=\tfrac{1}{2}(\kappa(g)+\sigma(g,x))\geq\tfrac{3\lambda n}{4}.$$

This proves (4.7).

Using Equation (2.16), one has

$$(go|y)_o=\tfrac{1}{2}(\kappa(g)-\sigma(g^{-1},y))\leq\tfrac{\lambda n}{4}.$$

This proves (4.8).

Hence, combining these two equations with the bound

$$(go|y)_o\geq\min((go|gx)_o,(gx|y)_o)-\delta,$$

one gets, for $n \geq n_0 := \frac{4\delta}{\lambda}$,

$$(gx|y)_o \;\; \leq \;\; (go|y)_o + \delta \leq \tfrac{\lambda n}{2}.$$

This proves (4.9). □

Proof of Proposition 4.2. We choose a, C_n as in Lemma 4.5. We first check that, for $n \geq 1$ and y in X, one has

$$\nu(\{x \in X \mid (x|y)_o \geq an\}) \leq C_n. \tag{4.10}$$

Indeed, since $\nu = \mu^{*n} * \nu$, one computes using (4.9)

$$\begin{aligned} \nu(\{x \in X \mid (x|y)_o \geq an\}) &= \textstyle\int_X \mu^{*n}(\{g \in G \mid (gx|y)_o \geq an\}) \,\mathrm{d}\nu(x) \\ &\leq \textstyle\int_X C_n \,\mathrm{d}\nu(x) = C_n, \end{aligned}$$

Then cutting the integral (4.6) along the subsets $A_{n-1,y} \smallsetminus A_{n,y}$ where

$$A_{n,y} := \{x \in X \mid (x|y)_o \geq an\}$$

one gets the upperbound

$$\begin{aligned} \textstyle\int_X (x|y)_o^{p-1} \,\mathrm{d}\nu(x) &\leq \textstyle\sum_{n\geq 1} a^{p-1} n^{p-1} (\nu(A_{n-1,y}) - \nu(A_{n,y})) \\ &\leq a^{p-1} + a^{p-1} \textstyle\sum_{n\geq 1} ((n+1)^{p-1} - n^{p-1})\, C_n \\ &\leq a^{p-1} + (p-1)\, 2^p a^{p-1} \textstyle\sum_{n\geq 1} n^{p-2} C_n. \end{aligned}$$

which is finite. This proves (4.4). □

4.3. **Solving the cohomological equation.**

Proposition 4.6. *Let (M, d) be a proper, quasiconvex, Gromov hyperbolic space, o a point of M and μ a non-elementary Borel probability measure on G with a finite second moment. Then the Busemann cocycle σ on X is centerable i.e. there exists a bounded function ψ on X such that the cocycle σ_0 given, for (g, x) in $G \times X$, by*

$$\sigma_0(g, x) = \sigma(g, x) - \psi(x) + \psi(gx) \tag{4.11}$$

satisfies, for all x in X,

$$\textstyle\int_G \sigma_0(g, x) \,\mathrm{d}\mu(g) = \lambda. \tag{4.12}$$

Proof. We define the function ψ on X by the formula, for x on X,

$$\psi(x) = -2 \textstyle\int_X (x|y)_o \,\mathrm{d}\nu^*(y),$$

where ν^* is a $\check{\mu}$-stationary probability measure on X. According to Equality (2.18), for all g in G, x, y in X, one has

$$\sigma(g, x) = -2(x|g^{-1}y)_o + 2(gx|y)_o + \sigma(g^{-1}, y).$$

Integrating this equality on $G \times X$ for the measure $\mathrm{d}\mu(g)\,\mathrm{d}\nu^*(y)$ and using the $\check{\mu}$-stationarity of ν^*, one gets the equality, for all x in X,

$$\int_G \sigma(g,x)\,\mathrm{d}\mu(g) = \psi(x) - \int_G \psi(gx)\,\mathrm{d}\mu(g) + \lambda. \tag{4.13}$$

Hence the cocycle $\sigma_0 : (g,x) \mapsto \sigma(g,x) - \psi(x) + \psi(gx)$ satisfies Equation (4.12). $\square$

4.4. **Central limit theorem.** We restate more precisely our central limit theorem.

Theorem 4.7. *Let (M,d) be a proper, quasiconvex, Gromov hyperbolic space, o a point of M and μ a non-elementary Borel probability measure on the group $G := \mathrm{Isom}(M)$ with a finite second moment $\int_G \kappa(g)^2\,\mathrm{d}\mu(g) < \infty$. Let λ be the escape rate of μ.*
a) Then there exists a gaussian law N_μ on $\mathbb{R}$ such that, for every compactly supported continuous function F on $\mathbb{R}$, one has

$$\int_G F\left(\tfrac{\sigma(g,x)-n\lambda}{\sqrt{n}}\right)\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} F(t)\,\mathrm{d}N_\mu(t)\ , \tag{4.14}$$

uniformly for x in X, and

$$\int_G F\left(\tfrac{\kappa(g)-n\lambda}{\sqrt{n}}\right)\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} F(t)\,\mathrm{d}N_\mu(t)\ . \tag{4.15}$$

b) When μ is non-arithmetic, this gaussian law N_μ is non-degenerate.

As we already pointed out, the novelty here is in the fact that we do not assume a finite exponential moment for μ (see [5]). Even when G is the free group on two generators and d the distance on $M = G$ given by the length function, Theorem 4.7 was not known in this generality (see [24] and [22]).

We will use the following central limit theorem for martingale which is due to Brown in [7] (see also [17]).

Fact 4.8. *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, $\mathcal{B}_0 \subset \cdots \subset \mathcal{B}_n$ be sub-σ-algebras of $\mathcal{B}$. For $1 \le k \le n$, let $\varphi_{n,k} : \Omega \to \mathbb{R}$ be $\mathcal{B}_k$-measurable and square-integrable random variables such that*

$$\mathbb{E}(\varphi_{n,k} \mid \mathcal{B}_{k-1}) \;=\; 0. \tag{4.16}$$

Let N_Φ be a centered gaussian law on $\mathbb{R}$ with variance $\Phi \ge 0$. We assume that the random variables

$$W_n := \sum_{1\le k\le n} \mathbb{E}(\varphi_{n,k}^2 \mid \mathcal{B}_{k-1}) \ \text{ converge to } \Phi \text{ in probability,} \tag{4.17}$$

and that, for all $\varepsilon > 0$,

$$W_{\varepsilon,n} := \sum_{1\le k\le n} \mathbb{E}(\varphi_{n,k}^2 \mathbf{1}_{\{|\varphi_{n,k}|\ge\varepsilon\}} \mid \mathcal{B}_{k-1}) \xrightarrow[n\to\infty]{} 0 \text{ in probability.} \tag{4.18}$$

Then the sequence $S_n := \sum_{1\le k\le n} \varphi_{n,k}$ converges in law toward N_Φ.

Proof of Theorem 4.7. *a*) According to Proposition 3.2, if the limit (4.14) exists for a point x in X, then the sequence (4.15) converges toward the same limit and the limits (4.14) exist for all x in X, are all equal, and the convergence (4.14) is uniform for x in X.

Since the cocycle σ is centerable, one can write σ as the sum of two cocycles $\sigma = \sigma_0 + \sigma_1$ where σ_0 is given by (4.11) and where σ_1 is a coboundary which is uniformly bounded by the constant $2\|\psi\|_\infty$. In particular the cocycle σ_1 does not play any role in the limit (4.14). Hence we can replace σ by σ_0 in the limit (4.14).

As in the previous sections, let $(B, \mathcal{B}, \beta)$ be the associated Bernoulli space. We want to find x in X such that the laws of the random variables S_n on B given, for b in B, by

$$S_n(b) := \tfrac{1}{\sqrt{n}}(\sigma_0(b_n \cdots b_1, x) - n\lambda)$$

converge to some gaussian law N_μ.

We want to apply the martingale central limit theorem 4.8 to the sub-σ-algebras $\mathcal{B}_k$ spanned by $b_1, \ldots, b_k$ and to the triangular array of random variables $\varphi_{n,k}$ on B given by, for b in B,

$$\varphi_{n,k}(b) = \tfrac{1}{\sqrt{n}}(\sigma_0(b_k, b_{k-1} \cdots b_1 x) - \lambda) \ \ , \ \text{ for } \ 1 \le k \le n \ .$$

Since, by the cocycle property , one has $S_n = \sum_{1\le k\le n} \varphi_{n,k}$, we just have to check that the three assumptions of Fact 4.8 are satisfied with some constant $\Phi = \Phi_\mu \ge 0$. We keep the notations W_n and $W_{\varepsilon,n}$ of Fact 4.8.

First, since the function κ is square integrable, the functions $\varphi_{n,k}$ belong to $L^2(B, \beta)$, and, by Equation (4.12), the assumption (4.16) is satified: for β-almost all b in B,

$$\mathbb{E}(\varphi_{n,k} \mid \mathcal{B}_{k-1}) = \int_G (\sigma_0(g, b_{k-1} \cdots b_1 x) - \lambda)\, \mathrm{d}\mu(g) = 0 \ .$$

Second, we introduce the continuous function on X,

$$x \mapsto M(x) = \int_G (\sigma_0(g, x) - \lambda)^2\, \mathrm{d}\mu(g) \ .$$

and we compute, for β-almost all b in B,

$$W_n(b) = \tfrac{1}{n} \sum_{1\le k\le n} M(b_{k-1} ... b_1 x).$$

We fix a μ-ergodic μ-stationary Borel probability measure ν on X. According to Birkhoff ergodic theorem, for ν-almost all x in X, the sequence W_n converges to

$$\Phi_\mu := \int_X M(y)\,\mathrm{d}\nu(y) \tag{4.19}$$

in $L^1(B,\beta)$. We choose such a point x in X. In particular, the assumption (4.17) is satisfied.

Third, we introduce, for $T > 0$, the continuous function on X

$$x \mapsto M_T(x) = \int_G (\sigma_0(g,x) - \lambda)^2 \mathbf{1}_{\{|\sigma_0(g,x)-\lambda| \geq T\}}\,\mathrm{d}\mu(g)\ .$$

and the integral

$$I_T := \int_G \kappa_0(g)^2 \mathbf{1}_{\{\kappa_0(g)\geq T\}}\,\mathrm{d}\mu(g)\ ,$$

where $\kappa_0(g) := \kappa(g) + \lambda + 2\|\psi\|_\infty$, so that

$$M_T(x) \leq I_T \xrightarrow[T\to\infty]{} 0\,,$$

and we compute, for $\varepsilon > 0$ and β-almost all b in B,

$$W_{\varepsilon,n}(b) = \tfrac{1}{n}\textstyle\sum_{1\leq k\leq n} M_{\varepsilon\sqrt{n}}(b_{k-1}...b_1 x) \leq I_{\varepsilon\sqrt{n}} \xrightarrow[n\to\infty]{} 0\ .$$

In particular the sequence $W_{\varepsilon,n}$ converges to 0 in probability, i.e. the Lindeberg's condition (4.18) is satisfied. Hence, by Fact 4.8, the laws of S_n converge to the gaussian law N_μ with variance Φ_μ.

b) It remains to check that, when μ is non-arithmetic, the gaussian law N_μ is not a Dirac mass. The variance Φ_μ of this gaussian law is given by the formulas, for all $n \geq 1$,

$$\Phi_\mu = \tfrac{1}{n}\textstyle\int_{G\times X} (\sigma_0(g,x) - n\lambda)^2\,\mathrm{d}\mu^{*n}(g)\,\mathrm{d}\nu(x).$$

In particular since σ_1 is bounded by $2\,\|\psi\|_\infty$, and since σ is continuous, one gets, for all g in the support of μ^{*n} and all x in the support of ν,

$$|\sigma(g,x) - n\lambda| \leq 2\|\psi\|_\infty. \tag{4.20}$$

Using then Equation (4.3), one finds a constant $C > 0$ such that, for all $n \geq 1$ and all g in the support of μ^{*n},

$$|\kappa(g) - n\lambda| \leq C, \tag{4.21}$$

and hence, using Equation (2.13), one also has

$$\ell(g) = n\lambda, \tag{4.22}$$

which contradicts the non-arithmeticity of μ. □

Remark 4.9. *It follows from this proof that the variance Φ_μ given by Formula* (4.19) *does not depend on the choice of the μ-stationary measure ν on the Busemann boundary X.*

5. Log-regularity of the stationary measure

In this section we give an alternate proof for the $\log^p$-regularity of the μ-stationary measure on the Gromov boundary.

5.1. **Optimal log-regularity.** This alternate proof assumes that the group G of isometry of M acts cocompactly on M, but it only assumes the finiteness of the p^{th}-moment of the measure μ on G (instead of the $(p+1)^{\text{th}}$-moment in Proposition 4.2). Here is the precise alternate statement.

Proposition 5.1. *Let $p > 0$. Let (M, d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Assume that the group G of isometries of M acts cocompactly on M. Let μ be a non-elementary Borel probability measure on G such that $\int_G \kappa(g)^p \,\mathrm{d}\mu(g) < \infty$. Let ν be the μ-stationary Borel probability measure on the Gromov boundary ∂M. Then, one has*

$$\sup_{\eta\in\partial M} \int_{\partial M} (\xi|\eta)_o^p \,\mathrm{d}\nu(\xi) < \infty. \tag{5.1}$$

This alternate proof does not rely on martingales. It is a combination of three steps. The first step (Lemma 5.2) uses harmonic analysis on $L^2(G)$ via the spectral gap characterization of non-amenability. The second step (Lemma 5.3) relies on the geometric properties of hyperbolic spaces: all their geodesic triangles are δ-thin. The last step (Section 5.4) uses the interpretation of the stationary measure as the image of the Bernoulli measure by the boundary map. Here are the details.

5.2. **Spectral gap.** In order to prove Proposition 5.1, we may assume that e belongs to the support of μ. Indeed, ν is also μ'-stationary with $\mu' := \frac{1}{2}(\mu + \delta_e)$.

We first notice that since μ is a non-elementary probability measure on G, the group G_μ is non-amenable. Indeed, if G_μ were amenable, there would exist a G_μ-invariant probability measure on the Gromov boundary ∂M. By proposition 3.1, this would imply that the unique μ-stationary measure ν on ∂M is G_μ-invariant and hence a Dirac mass. This would contradict the fact that G_μ does not have fixed points on ∂M.

Hence we can apply the following lemma to our mesure μ.

Lemma 5.2. *Let (M,d) be a proper metric space. Assume that the group G of isometries of M acts cocompactly on M. Let μ be a non-elementary Borel probability measure on G whose support contains e and such that G_μ is non-amenable. Let $R>0$. Then, there exist $A_0>0$, $a_0<1$ such that, for all m, m' in M, for all $n\geq 1$, one has*

$$\mu^{*n}(\{g\in G \mid d(gm,m')\leq R\}) \leq A_0\, a_0^n. \tag{5.2}$$

For this lemma to be true, the cocompactness of the action of G is crucial, since one can always replace (M,d) by a space countaining isometric copies of all the homothetic metric spaces $(M,\frac{1}{k}d)$ for $k\geq 1$.

Proof. Let o be point of M. Since G acts cocompactly in M, any point of M is at bounded distance of the G-orbit Go. Hence we can assume that the points m and m' are on this G-orbit Go. We write $m=ho$ and $m'=h'o$ with h, h' in G.

Let μ_G be a left-invariant measure on G and λ_G be the left regular representation of G in $L^2(G)$ and $\lambda_G(\mu)$ be the contraction of $L^2(G)$ given by, for any φ in $L^2(G)$ and g in G,

$$\lambda_G(\mu)(\varphi) = \textstyle\int_G \varphi(g^{-1}.)\,\mathrm{d}\mu(g).$$

According to the spectral gap theorem due to Kesten, Derrienic-Guivarch, and Berg-Christensen (see [4, Thm 4] and also [21], [10]), since G_μ is non-amenable, the operator $\lambda_G(\mu)$ has a spectral gap, i.e. there exists $C_0>0$ and $a_0<1$ such that, for all $n\geq 1$, $\|\lambda_G(\mu)^n\|\leq C_0 a_0^n$. Let

$$B_R := \{g\in G \mid d(go,o)\leq R\}.$$

We want to bound $\mu^{*n}(h'B_Rh^{-1})$. We compute, for all $n\geq 1$, using the inclusion $B_RB_R\subset B_{2R}$, Cauchy-Schwartz inequality, and the spectral gap,

$$\begin{aligned} \mu_G(h'B_Rh^{-1})\,\mu^{*n}(h'B_Rh^{-1}) &\leq \langle \lambda_G(\mu)^n(\mathbf{1}_{hB_{2R}h^{-1}}), \mathbf{1}_{h'B_Rh^{-1}}\rangle_{L^2(G)} \\ &\leq C_0a_0^n\,\mu_G(hB_{2R}h^{-1})^{\frac12}\mu_G(h'B_Rh^{-1})^{\frac12} \end{aligned}$$

Hence, using the left invariance and the right semiinvariance of the Haar measure μ_G one deduces

$$\mu^{*n}(h'B_Rh^{-1}) \leq C_0a_0^n\,\mu_G(B_{2R})^{\frac12}\mu_G(B_R)^{-\frac12}$$

This proves the bound (5.2). □

5.3. **Thin triangles.**

Lemma 5.3. *Let $p > 0$. Let (M, d) be a proper quasiconvex Gromov hyperbolic space and $o \in M$. Assume that the group G of isometries of M acts cocompactly on M. Let μ be a non-elementary Borel probability measure on G such that $\int_G \kappa(g)^p \,\mathrm{d}\mu(g) < \infty$ and whose support contains e. Then there exists $A > 0$, $a < 1$ such that, for all ξ, η in ∂M, for all $n \geq 1$, one has*

$$\mu^{*n}(\{g \in G \mid (g\xi|\eta)_o \geq \kappa(g)\}) \leq Aa^n. \tag{5.3}$$

Moreover, when $p \geq 1$, one also has,

$$\int_G \kappa(g)^{p-1} \mathbf{1}_{\{(g\xi|\eta)_0 \geq \kappa(g)\}} \,\mathrm{d}\mu^{*n}(g) \leq Aa^n. \tag{5.4}$$

We first recall the properties of the triangles in M that we will use. Since the metric space (M, d) is proper, quasiconvex and hyperbolic, there exist constants $C > 1$ and $\delta > 0$ with the following properties : Every triple x_1, x_2, x_3 of points in M^* are the vertices of a C-geodesic triangle, i.e. a triangle whose sides are C-geodesics. Moreover every C-geodesic triangle is δ-thin, i.e. one can cut each of the three sides $[x_i, x_j]$ in two C-geodesic pieces, say $[x_i, x_j] = [x_i, m_{i,j}] \cup [m_{i,j}, x_j]$, such that the Hausdorff distance of the two geodesic pieces $[x_i, m_{i,j}]$ and $[x_i, m_{i,k}]$ starting from the same vertex x_i is bounded by δ.

Proof of Lemma 5.3. We will apply this property to the triple $(o, go, g\xi)$ and to the triple (o, go, η).

Let $(m_i)_{i\geq 1}$ be a C-geodesic between the points o and ξ. This means that $m_1 = o$, that $\lim\limits_{i\to\infty} m_i = \xi$, and that, for all $i < j$,

$$j - i - C \leq d(m_i, m_j) \leq j - i + C \ .$$

Similarly, let $(m'_j)_{j\geq 1}$ be a C-geodesic from o to η and let $(m''_k)_{k\geq 1}$ be a C-geodesic from o to $g\xi$.

a) We introduce the set

$$S_{\xi,\eta} := \{g \in G \mid (g\xi|\eta)_o \geq \kappa(g)\} \, .$$

We denote $R = 2\delta + 6C$ and choose $c > 1$ with $c^2 a_0 < 1$ where a_0 is the constant given by Lemma 5.2. Let $n \geq 1$. We will first prove that this set is included in the following union

$$S_{\xi,\eta} \subset S^0_{\xi,\eta} \cup S^1_{\xi,\eta} \, , \quad \text{where} \tag{5.5}$$

$$\begin{aligned} S^0_{\xi,\eta} &:= \{g \in G \mid \kappa(g) \geq c^n\} \, , \\ S^1_{\xi,\eta} &:= \{g \in G \mid \exists\, i, j \leq c^n \ \text{ with } \ d(gm_i, m'_j) \leq R\} \, . \end{aligned}$$

Let g be an element of $S_{\xi,\eta} \smallsetminus S^0_{\xi,\eta}$. We choose a C-geodesic between o and go. Applying first the above property to the C-geodesic triangle with vertices $(o, go, g\xi)$, since $d(o, go) = \kappa(g)$, one can find a point gm_i and a point m''_k such that

$$i \leq \kappa(g) \ \ , \ \ k \leq \kappa(g) \ \text{ and } \ d(gm_i, m''_k) \leq \delta + 3C.$$

Applying now the above property to the C-geodesic triangle with vertices (o, go, η), since $k \leq \kappa(g)$ and $(g\xi|\eta)_o \geq \kappa(g)$, one can find a point m'_j such that

$$j \leq \kappa(g) \ \text{ and } \ d(m'_j, m''_k) \leq \delta + 3C.$$

In particular, one has $d(gm_i, m'_j) \leq R$ and the element g belongs to $S^1_{\xi,\eta}$. This proves the inclusion (5.5).

We want to bound $\mu^{*n}(S_{\xi,\eta})$. On the one hand, using Chebyshev's inequality, and the finiteness of the p^{th}-moment, one computes,

$$\mu^{*n}(S^0_{\xi,\eta}) \leq c^{-np} \int_G \kappa(g)^p \, \mathrm{d}\mu^{*n}(g) \leq n^{p+1} c^{-np} \int_G \kappa(g)^p \, \mathrm{d}\mu(g)\,.$$

One the other hand, using the bound (5.2) for at most c^{2n} couples of points, one gets

$$\mu^{*n}(S^1_{\xi,\eta}) \leq A_0 c^{2n} a_0^n$$

This proves the bound (5.3) as soon as the constant $a < 1$ is chosen larger than c^{-p} and $a_0 c^2$.

b) The proof is very similar to point a). We assume now that $p > 1$ and choose $c > 1$ such that $c^{p+1} a_0 < 1$. We want to bound

$$\int_{S_{\xi,\eta}} \kappa(g)^{p-1} \, \mathrm{d}\mu^{*n}(g)\,.$$

As above, one has the bound

$$\begin{aligned} \int_{S^0_{\xi,\eta}} \kappa(g)^{p-1} \, \mathrm{d}\mu^{*n}(g) \ &\leq \ c^{-n} \int_G \kappa(g)^p \, \mathrm{d}\mu^{*n}(g) \\ &\leq \ n^p c^{-n} \int_G \kappa(g)^p \, \mathrm{d}\mu(g)\,, \end{aligned}$$

and the bound

$$\int_{S^1_{\xi,\eta}} \kappa(g)^{p-1} \, \mathrm{d}\mu^{*n}(g) \ \leq \ c^{(p-1)n} \, \mu^{*n}(S^1_{\xi,\eta}) \ \leq \ A_0 \, c^{(p+1)n} \, a_0^n\,.$$

This proves the bound (5.4) as soon as the constant $a < 1$ is chosen larger than c^{-1} and $a_0 c^{p+1}$. □

5.4. **Boundary map.** We can now conclude the proof.

Proof of Proposition 5.1. We assume first that $p \leq 1$. As in the previous sections, we let $(B, \mathcal{B}, \beta, S)$ be the associated Bernoulli system, and we denote by $b \mapsto \xi_b$ the boundary map introduced in Proposition

3.1. For $b=(b_1,b_2,\ldots)$ in B and $n\geq 0$, we set $\kappa_n(b)=\kappa(b_1\cdots b_n)$. We want to bound, uniformly for η in ∂M, the integral

$$I_{p,\eta} \;:=\; \int_{\partial M}(\xi|\eta)_o^p\,\mathrm{d}\nu(\xi)\,.$$

We compute

$$\begin{aligned} I_{p,\eta} &= \int_B(\xi_b|\eta)_o^p\,\mathrm{d}\beta(b)\\ &= \int_0^\infty \beta(\{b\mid (\xi_b|\eta)_o^p\geq t\})\,\mathrm{d}t\\ &= \sum_{n\geq 0}\int_0^\infty \beta(\{b\mid \kappa_n(b)^p\leq t<\kappa_{n+1}(b)^p\ ,\ (\xi_b|\eta)_o^p\geq t\})\,\mathrm{d}t\\ &\leq \sum_{n\geq 0}\int_B \max(\kappa_{n+1}(b)^p-\kappa_n(b)^p,0)\mathbf{1}_{\{(\xi_b|\eta)_o\geq\kappa_n(b)\}}\,\mathrm{d}\beta(b)\ . \end{aligned}$$

Since $p\leq 1$, for every $t>s>0$, one has the bound

$$t^p-s^p \;\leq\; (t-s)^p\,. \tag{5.6}$$

Hence, writing $g=b_1\cdots b_n$ and $b'=S^nb$ so that $b'_1=b_{n+1}$, one pursues the computation,

$$I_{p,\eta}\leq \sum_{n\geq 0}\int_B \kappa(b'_1)^p\mu^{*n}(\{g\in G\mid (g\xi_{b'}|\eta)_o\geq\kappa(g)\})\,\mathrm{d}\beta(b')\,. \tag{5.7}$$

Hence, using the bound (5.3), one gets

$$I_{p,\eta} \;\leq\; \sum_{n\geq 0}Aa^n\int_B\kappa(b'_1)^p\,\mathrm{d}\beta(b')\,.$$

This gives the final bound

$$I_{p,\eta} \;\leq\; \tfrac{A}{1-a}\int_G\kappa(g)^p\,\mathrm{d}\mu(g)\,.$$

When $p>1$, the same computation works except that we have to replace the bound (5.6), by the following bound, for every $t>s>0$,

$$t^p-s^p \;\leq\; 2^p(t-s)^p+2^ps^{p-1}(t-s)\,.$$

Hence an extra term occur in the right hand side, of (5.7) which is,

$$\sum_{n\geq 0}\int_B 2^p\kappa(b'_1)\int_G\kappa(g)^{p-1}\mathbf{1}_{\{g\in G\mid (g\xi_{b'}|\eta)_o\geq\kappa(g)\}}\,\mathrm{d}\mu^{*n}(g)\,\mathrm{d}\beta(b')\,.$$

And we bound this extra term thanks to the bound (5.4). All this gives the final inequality

$$I_{p,\eta} \;\leq\; \tfrac{2^pA}{1-a}\int_G(\kappa(g)^p+\kappa(g))\,\mathrm{d}\mu(g)\,,$$

and ends the proof of Proposition 5.1. □

5.5. **Free semigroup.**

> In this section we describe an example pointing out the optimality of Proposition 5.1.

We choose the group G to be the free group on two generators u, v. This group acts isometrically cocompactly on the corresponding Cayley graph (M, d) which is a regular tree of valence 4. We denote by $o = e$ its base point. The boundary ∂M is the space of infinite words $\xi = (\xi_1, \xi_2, \ldots)$ in the letters u, v, u^{-1}, v^{-1} which are reduced, i.e. $\xi_{i+1} \neq \xi_i^{-1}$ for all $i \geq 1$.

We choose the probability measure μ on G to be

$$\mu = \tfrac{1}{2}(\delta_v + \textstyle\sum_{n\geq 1} p_n \delta_{u^n})$$

with $\sum_{n\geq 1} p_n = 1$. The support of the unique μ-stationary probability measure ν on ∂M is included in the subset of infinite words ξ in the letters u and v.

We choose the point η on the boundary to be

$$\eta = (u, u, u, \ldots),$$

so that, for all $\xi = (\xi_1, \xi_2, \ldots)$ in ∂M, the Gromov product is given by

$$(\xi|\eta)_o = \inf\{i \geq 0 \mid \xi_{i+1} \neq u\}.$$

Let $p > 0$. We want to estimate the integral

$$I_{p,\eta} := \textstyle\int_{\partial M} (\xi|\eta)_o^p \, \mathrm{d}\nu(\xi)$$

Example 5.4. *In this case, one has the equivalence*

$$\textstyle\sum_{n\geq 1} p_n n^p < \infty \iff I_{p,\eta} < \infty\,. \tag{5.8}$$

This means that, in this case, the converse of Proposition 5.1 is true.

Proof. Indeed, in this case the Bernoulli space is the space B of sequences $b = (b_1, b_2, \ldots)$ with $b_i = v$ or u^n endowed with the Bernoulli measure $\beta = \mu^{\otimes \mathbb{N}^*}$, and the image ξ_b of b by the boundary map is the concatenation of the letters u, v occuring in b.

We denote by B_1 the set of element of B whose first letter is a power of u and whose second letter is v. Hence we have the following lower bound

$$\begin{array}{rcl} I_{p,\eta} & = & \int_B (\xi_b|\eta)_o^p \, \mathrm{d}\beta(b) \\ & \geq & \int_{B_1} (\xi_b|\eta)_o^p \, \mathrm{d}\beta(b) \;=\; \frac{1}{4} \sum_{n\geq 1} p_n n^p, \end{array}$$

which proves the converse implication in the claim (5.8). □

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