HOW FAR ARE P-ADIC LIE GROUPS FROM ALGEBRAIC GROUPS?

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Abstract. We show that, in a weakly regular p-adic Lie group G, the subgroup G_u spanned by the one-parameter subgroups of G admits a Levi decomposition. As a consequence, there exists a regular open subgroup of G which contains G_u

Contents

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1. Introduction

1.1. **Motivations.** When studying the dynamics of the subgroups of a p-adic Lie group G on its homogeneous spaces, various assumptions can be made on G. For instance, one can ask G to be algebraic as in [10] i.e. to be a subgroup of a linear group defined by polynomial equations.

Another possible assumption on G is regularity. One asks G to satisfy properties that are well-known to hold for Zariski connected algebraic p-adic Lie groups: a uniform bound on the cardinality of the finite subgroups and a characterization of the center as the kernel of the adjoint representation (see Definition 2.5). Ratner's theorems in [13] are written under this regularity assumption.

A more natural and weaker assumption on G in this context is the weak regularity of G i.e. the fact that the one-parameter morphisms of G are uniquely determined by their derivative (see Definition 2.3). Our paper [1] is written under this weak-regularity assumption.

The aim of this text is to clarify the relationships between these three assumptions.

A key ingredient in the proof is Proposition 5.1. It claims

the finiteness of the center of the universal topological central extension of non-discrete simple p-adic Lie groups.

This is a fact which is due to Prasad and Raghunathan in [12].

1.2. **Main results.** We will first prove (Proposition 5.5):

In a weakly regular p-adic Lie group G, the subgroup G_u spanned by the one-parameter subgroups of G is closed and admits a Levi decomposition,

i.e. G_u is a semidirect product $G_u = S_u \ltimes R_u$ of a group S_u by a normal subgroup R_u where S_u is a finite cover of a finite index subgroup of an algebraic semisimple Lie group and where R_u is algebraic unipotent.

As a consequence, we will prove (Theorem 5.12):

In a weakly regular p-adic Lie group G there always exists an open subgroup G_Ω which is regular and contains G_u.

This Theorem 5.12 is useful since it extends the level of generality of Ratner's theorems in [13] (see [1, Th 5.15]). More precisely, Ratner's theorems for products of real and p-adic Lie groups in [13] are proven under the assumption that these p-adic Lie groups are *regular*. Ratner's theorems can be extended under the weaker assumption that these p-adic Lie groups are *weakly regular* thanks to our Theorem 5.12.

This Theorem 5.12 has been announced in [1, Prop. 5.8] and has been used in the same paper.

The strategy consists in proving first various statements for weakly regular p-adic Lie groups which were proven in [13] under the regularity assumption. To clarify the discussion we will reprove also the results from [13] that we need. But we will take for granted classical results on the structure of simple p-adic algebraic groups that can be found in [3], [6], [7] or [11].

1.3. **Plan.** In the preliminary Chapter 2 we recall a few definitions and examples.

Our main task in Chapters 3 and 4 is to describe, for a weakly regular p-adic Lie group G, the subset $\mathfrak{g}_G \subset \mathfrak{g}$ of derivatives of one-parameter morphisms of G.

In Chapter 3, the results are mainly due to Ratner. We first study the nilpotent p-adic Lie subgroups N of G spanned by one-parameter subgroups. We will see that they satisfy $\mathfrak{n} \subset \mathfrak{g}_G$ (Proposition 3.7). Those p-adic Lie groups N are called algebraic unipotent. We will simultaneously compare this set $\mathfrak{g}_G$ with the analogous set $\mathfrak{g}'_{G'}$ for a quotient group $G' = G/N$ of G when N is a normal algebraic unipotent subgroup (Lemma 3.6). This will allow us to prove that G contains a largest normal algebraic unipotent subgroup (Proposition 3.8).

In Chapter 4, we will then be able to describe precisely the set $\mathfrak{g}_G$ using a Levi decomposition of $\mathfrak{g}$ (Proposition 4.4). A key ingredient is a technic, borrowed from [1], for constructing one-parameter subgroups in a p-adic Lie group G (Lemma 4.1).

In the last Chapter 5, we will prove the main results we have just stated: Proposition 5.5 and Theorem 5.12, using Prasad–Raghunathan finiteness theorem in [12] (see Proposition 5.1). We will end this text by an example showing that, when a p-adic Lie group G with $G = G_u$ is not assumed to be weakly regular, it does not always contain a regular open subgroup H for which $H = H_u$ (Example 5.14).

2. Preliminary results

We recall here a few definitions and results from [13].

Let p be a prime number and $\mathbb{Q}_p$ be the field of p-adic numbers. When G is a p-adic Lie group (see [5]), we will always denote by $\mathfrak{g}$ the Lie algebra of G. It is a $\mathbb{Q}_p$-vector space. We will denote by $\mathrm{Ad}_{\mathfrak{g}}$ or Ad the adjoint action of G on $\mathfrak{g}$ and by $\mathrm{ad}_{\mathfrak{g}}$ or ad the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$. Any closed subgroup H of G is a p-adic Lie subgroup and its Lie algebra $\mathfrak{h}$ is a $\mathbb{Q}_p$-vector subspace of $\mathfrak{g}$ (see [13, Prop. 1.5]).

We choose a ultrametric norm $\|.\|$ on $\mathfrak{g}$ with values in $p^{\mathbb{Z}}$.

2.1. **One-parameter subgroups.**

Definition 2.1. *A one-parameter morphism φ of a p-adic Lie group G is a continuous morphism $\varphi : \mathbb{Q}_p \to G; t \mapsto \varphi(t)$. A one-parameter subgroup is the image $\varphi(\mathbb{Q}_p)$ of an injective one-parameter morphism.*

The derivative of a one-parameter morphism is an element X of $\mathfrak{g}$ for which $\mathrm{ad}X$ is nilpotent. This follows from the following Lemma from [13, Corollary 1.2].

Lemma 2.2. *Let $\varphi : \mathbb{Q}_p \to \mathrm{GL}(d, \mathbb{Q}_p)$ be a one-parameter morphism. Then there exists a nilpotent matrix X in $\mathfrak{gl}(d, \mathbb{Q}_p)$ such that $\varphi(t) = \exp(tX)$ for all t in $\mathbb{Q}_p$.*

Here exp is the exponential of matrices : $\exp(tX) := \sum_{n\geq 0} \frac{t^n X^n}{n!}$

Proof. First, we claim that, if K is a finite extension of $\mathbb{Q}_p$, any continuous one-parameter morphism $\psi : \mathbb{Q}_p \to K^*$ is constant. Indeed, since the modulus of ψ is a continuous morphism form $\mathbb{Q}_p$ to a discrete multiplicative subgroup of $(0, \infty)$, the kernel of $|\psi|$ contains $p^k\mathbb{Z}_p$, for some integer k. Since $\mathbb{Q}_p/p^k\mathbb{Z}_p$ is a torsion group and $(0, \infty)$ has no torsion, ψ has constant modulus, that is, ψ takes values in $\mathcal{O}^*$, where $\mathcal{O}$ is the integer ring of K. Now, on one hand, $\mathcal{O}^*$ is a profinite group, that is, it is a compact totally discontinuous group and hence it admits a basis of neighborhoods of the identity which are finite index subgroups (namely, for example, the subgroups $1 + p^k\mathcal{O}$, $k \geq 0$). On the other hand, every closed subgroup of $\mathbb{Q}_p$ is of the form $p^k\mathbb{Z}_p$, for some integer k, and hence, has infinite index in $\mathbb{Q}_p$ and therefore any continuous morphism from $\mathbb{Q}_p$ to a finite group is trivial. Thus, ψ is constant, which should be proved.

Let now φ be as in the lemma and let $X \in \mathfrak{gl}(d, \mathbb{Q}_p)$ be the derivative of φ. After having simultaneously reduced the commutative family of matrices $\varphi(t)_{t\in\mathbb{Q}_p}$, the joint eigenvalues give continuous morphisms $\mathbb{Q}_p \to K^*$ where K is a finite extension of $\mathbb{Q}_p$. By the remark above, these morphisms are constant, that is, there exists g in $\mathrm{GL}(d, \mathbb{Q}_p)$, such that, for any t in $\mathbb{Q}_p$, the matrix $g\varphi(t)g^{-1}$ is unipotent and upper triangular. We may assume $g = 1$. Then X is nilpotent and upper triangular and it remains to check that the map $\theta : t \mapsto \varphi(t)\exp(-tX)$ is constant. Since $\varphi(t)$ commutes with X, this map θ is a one-parameter morphism with zero derivative. Hence, the kernel of ψ is an open subgroup and the matrices $\theta(t)$, $t \in \mathbb{Q}_p$, have finite order. Since they are unipotent, they equal e, which should be proved. $\square$

2.2. **Weakly regular and regular p-adic Lie groups.**

Definition 2.3 (Ratner, [13])**.** *A p-adic Lie group G is said to be weakly regular if any two one-parameter morphisms $\mathbb{Q}_p \to G$ with the same derivative are equal.*

Note that any closed subgroup of a weakly regular p-adic Lie group is also weakly regular.

Example 2.4 ([13, Cor. 1.3 and Prop. 1.5]). Every closed subgroup of $\mathrm{GL}(d, \mathbb{Q}_p)$ is weakly regular.

Proof. This follows from Lemma 2.2. □

Definition 2.5 (Ratner, [13])**.** *A p-adic Lie group G is said to be* Ad*-regular if the kernel of the adjoint map* $\mathrm{Ker}(\mathrm{Ad}_{\mathfrak{g}})$ *is equal to the center $Z(G)$ of G. It is said to be regular if it is* Ad*-regular and if the finite subgroups of G have uniformly bounded cardinality.*

Note that any open subgroup of a regular p-adic Lie group is also regular.

This definition is motivated by the following example.

Example 2.6. *a*) The finite subgroups of a compact p-adic Lie group K have uniformly bounded cardinality.

b) The finite subgroups of a p-adic linear group have uniformly bounded cardinality.

c) The Zariski connected linear algebraic p-adic Lie groups G are regular.

Proof of Example 2.6. (see [13])

a) Since K contains a torsion free open normal subgroup Ω, for every finite subgroup F of K, one has the bound $|F| \leq |K/\Omega|$.

b) We want to bound the cardinality $|F|$ of a finite subgroup of a group $G \subset \mathrm{GL}(d, \mathbb{Q}_p)$. This follows from *a*) since F is included in a conjugate of the compact group $K = \mathrm{GL}(d, \mathbb{Z}_p)$.

c) It remains to check that G is Ad-regular. Let g be an element in the kernel of the adjoint map $\mathrm{Ad}_{\mathfrak{g}}$. This means that the centralizer Z_g of g in G is an open subgroup of G. This group Z_g is also Zariski closed. Hence it is Zariski open. Since G is Zariski connected, one deduces $Z_g = G$ and g belongs to the center of G. □

We want to relate the two notions "weakly regular" and "regular". We first recall the following implication in [13, Cor. 1.3].

Lemma 2.7. *Any regular p-adic Lie group is weakly regular.*

Proof. Let $\varphi_1 : \mathbb{Q}_p \to G$ and $\varphi_2 : \mathbb{Q}_p \to G$ be one-parameter morphisms of G with the same derivative. We want to prove that $\varphi_1 = \varphi_2$.

According to Lemma 2.2, the one-parameter morphisms $\mathrm{Ad}_{\mathfrak{g}}\varphi_1$ and $\mathrm{Ad}_{\mathfrak{g}}\varphi_2$ are equal. Since G is Ad-regular, this implies that, for all t in $\mathbb{Q}_p$, the element $\theta(t) := \varphi_1(t)^{-1}\varphi_2(t)$ is in the center of G. Hence θ is a one-parameter morphism of the center of G with zero derivative. Its image is either trivial or an infinite p-torsion group. This second case is excluded by the uniform bound on the finite subgroups of G. This proves the equality $\varphi_1 = \varphi_2$. $\square$

The aim of this text is to prove Theorem 5.12 which is a kind of converse to Lemma 2.7.

3. Algebraic unipotent p-adic Lie group

In this chapter, we study the algebraic unipotent subgroups of a weakly regular p-adic Lie group. The results in this chapter are mainly due to Ratner.

3.1. Definition and closedness.

We first focus on a special class of p-adic Lie groups.

We say that an element g of a p-adic Lie group G admits a logarithm if one has $g^{p^n} \xrightarrow[n\to\infty]{} e$: indeed, for such a g, the map $n \mapsto g^n$ extends as a continuous morphism $\mathbb{Z}_p \to G$ and one can define the logarithm $\log(g) \in \mathfrak{g}$ as being the derivative at 0 of this morphism.

Definition 3.1. *A p-adic Lie group N is called algebraic unipotent if its Lie algebra is nilpotent, if every element g in N admits a logarithm* $\log(g)$*, and if the logarithm map* $\log : N \to \mathfrak{n}$ *is a bijection.*

This implies that every non trivial element g of N belongs to a unique one-parameter subgroup of N. By definition these groups N are weakly regular. The inverse of the map log is denoted by exp. These maps exp and log are $\mathrm{Aut}(N)$-equivariant.

The following lemma is in [13, Prop. 2.1].

Lemma 3.2. *Let N be an algebraic unipotent p-adic Lie group. Then the map*

$$\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}; (X, Y) \mapsto \log(\exp X \exp Y)$$

is polynomial and is given by the Baker-Campbell-Hausdorff formula. In other words, a p-adic Lie group N is algebraic unipotent if and only if it is isomorphic to the group of $\mathbb{Q}_p$-points of a unipotent algebraic group defined over $\mathbb{Q}_p$.

These groups have been studied in [13, Sect.2] (where they are called *quasiconnected*).

In particular, we have

Corollary 3.3. *Let N be an algebraic unipotent p-adic Lie group. Then the exponential and logarithm maps of N are continuous.*

Proof of Lemma 3.2. In this proof, we will say that a p-adic Lie group is *strongly algebraic unipotent* if it is isomorphic to the group of $\mathbb{Q}_p$-points of a unipotent algebraic group defined over $\mathbb{Q}_p$. Such a group is always algebraic unipotent.

The aim of this proof is to check the converse. Let N be an algebraic unipotent p-adic Lie group. We want to prove that N is strongly algebraic unipotent. Its Lie algebra $\mathfrak{n}$ contains a flag $0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_r = \mathfrak{n}$ of ideals with $\dim \mathfrak{n}_i = i$. We will prove, by induction on $i \geq 1$, that the set $N_i := \exp(\mathfrak{n}_i)$ is a closed subgroup of N which is strongly algebraic unipotent.

By the induction assumption, the set N_{i-1} is a strongly algebraic unipotent closed subgroup. Since $\mathfrak{n}_{i-1}$ is an ideal, this subgroup N_{i-1} is normal. Let X_i be an element of $\mathfrak{n}_i \smallsetminus \mathfrak{n}_{i-1}$ and N_i' the semidirect product $N_i' := \mathbb{Q}_p \ltimes N_{i-1}$ where the action of $t \in \mathbb{Q}_p$ by conjugation on N_{i-1} is given by $t \exp(X) t^{-1} = \exp(e^{t\,\mathrm{ad} X_i} X)$, for all X in $\mathfrak{n}_{i-1}$. By construction this group N_i' is strongly algebraic unipotent and the map $\psi : N_i' \to N; (t, n) \mapsto \exp(tX_i)n$ is a group morphism. Since $N_i' = \exp(\mathfrak{n}_i')$, the set $N_i = \exp(\mathfrak{n}_i)$ is equal to the image $\psi(N_i')$. Hence, by Lemma 3.4 below, the set N_i is a closed subgroup which is isomorphic to N_i' and hence N_i is strongly algebraic unipotent. $\square$

The following lemma tells us that an algebraic unipotent Lie subgroup of a p-adic Lie group is always closed.

Lemma 3.4. *Let G be a totally discontinuous locally compact topological group, N be an algebraic unipotent p-adic Lie group and $\varphi : N \to G$ be an injective morphism. Then φ is a proper map. In particular, $\varphi(N)$ is a closed subgroup of G and φ is is an isomorphism of topological groups from N onto $\varphi(N)$.*

Proof. If φ was not proper, there would exist a sequence Y_n in the Lie algebra $\mathfrak{n}$ of N such that

$$\lim_{n\to\infty} \varphi(\exp(Y_n)) = e \quad \text{and} \quad \lim_{n\to\infty} \|Y_n\| = \infty.$$

We write $Y_n = p^{-k_n} X_n$ with integers k_n going to ∞ and $\|X_n\| = 1$. Since the group G admits a basis of compact open subgroups, one also has $\lim\limits_{n\to\infty} \varphi(\exp(X_n)) = e$. Let X be a cluster point of the sequence X_n. One has simultaneously, $\varphi(\exp(X)) = e$ and $\|X\| = 1$. This contradicts the injectivity of φ. $\square$

3.2. **Lifting one-parameter morphisms.**

We now explain how to lift one-parameter morphisms.

Lemma 3.5. *Let G be a p-adic Lie group, $N \subset G$ a normal algebraic unipotent closed subgroup, $G' := G/N$, and $\pi : G \to G'$ the projection. Then, for any one-parameter morphism φ' of G', there exists a one-parameter morphism φ of G which lifts φ', i.e. such that $\varphi' = \pi \circ \varphi$.*

If φ' has zero derivative, one can choose φ to have zero derivative.

Proof. According to Lemma 3.4 the image $\varphi'(\mathbb{Q}_p)$ is a closed subgroup of G'. Hence we can assume that $\varphi'(\mathbb{Q}_p) = G'$.

First Case : N is central in G. For $k \geq 1$, we introduce the subgroup Q'_k of G' spanned by the element $g'_k := \varphi'(p^{-k})$. Since Q'_k is cyclic and N is central, the group $Q_k := \pi^{-1}(Q'_k)$ is abelian. Since the increasing union of these groups Q_k is dense in G, the group G is also abelian. Since N is infinitely p-divisible, one can construct, by induction on $k \geq 0$, a sequence $(g_k)_{k\geq 0}$ in G such that

$$\pi(g_k) = g'_k \;\text{ and }\; p\, g_{k+1} = g_k\ .$$

We claim that $g_0^{p^k} \xrightarrow[k\to\infty]{} e$. Indeed, since $\pi(g_0)^{p^k} \xrightarrow[k\to\infty]{} e$ and since every element h of a p-adic Lie group that is close enough to the identity element satisfies $h^{p^k} \xrightarrow[k\to\infty]{} e$, one can find $\ell \geq 0$ and n in N such that $(g_0^{p^\ell} n^{-1})^{p^k} \xrightarrow[k\to\infty]{} e$. As N is algebraic unipotent, we have $n^{p^k} \xrightarrow[k\to\infty]{} e$, and the claim follows, since N is central.

Now, the formulae $\varphi(p^{-k}) = g_k$, for all $k \geq 0$, define a unique one-parameter morphism φ of G which lifts φ'.

Note that when φ' has zero derivative, one can assume, after a reparametrization of φ', that $\varphi'(\mathbb{Z}_p) = 0$ and choose the sequence g_k so that $g_0 = e$. Then the morphism φ has also zero derivative.

General Case : The composition of φ' with the action by conjugation on the abelianized group $N/[N,N] \simeq \mathbb{Q}_p^d$ is a one-parameter morphism $\psi : \mathbb{Q}_p \to \mathrm{GL}(d, \mathbb{Q}_p)$. According to Lemma 2.2, there exists a nilpotent matrix X such that $\psi(t) = \exp(tX)$ for all $t \in \mathbb{Q}_p$. The image of this matrix X corresponds to an algebraic unipotent subgroup N_1 with $[N,N] \subset N_1 \subsetneq N$ which is normal in G and such that N/N_1 is a central subgroup of the group G/N_1. According to the first case, the morphism φ' can be lifted as a morphism φ'_1 of G/N_1. By an induction argument on the dimension of N, this morphism φ'_1 can be lifted as a morphism of G. $\square$

Let G be a p-adic Lie group. We recall the notation

(3.1) $\mathfrak{g}_G := \{X \in \mathfrak{g}$ derivative of a one-parameter morphism of $G\}$.

Note that this set $\mathfrak{g}_G$ is invariant under the adjoint action of G.

The following lemma tells us various stability properties by extension when the normal subgroup is algebraic unipotent.

Lemma 3.6. *Let G be a p-adic Lie group, N a normal algebraic unipotent subgroup of G, and $G' := G/N$.*
a) One has the equivalence

$$G \text{ is algebraic unipotent} \iff G' \text{ is algebraic unipotent.}$$

b) Let X in $\mathfrak{g}$ and X' its image in $\mathfrak{g}' = \mathfrak{g}/\mathfrak{n}$. One has the equivalence

$$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_{G'}.$$

c) One has the equivalence

$$G \text{ is weakly regular} \iff G' \text{ is weakly regular.}$$

Later on in Corollary 5.7 we will be able to improve this Lemma.

Proof. We denote by $\pi : G \to G/N$ the natural projection.

a) The implication $\Rightarrow$ is well-known. Conversely, we assume that N and G/N are algebraic unipotent and we want to prove that G is algebraic unipotent. Arguing by induction on $\dim G/N$, we can assume $\dim G/N = 1$, i.e. that there exists an isomorphism $\varphi' : \mathbb{Q}_p \to G/N$. According to Lemma 3.5, one can find a one-parameter morphism φ of G that lifts φ'. By Lemma 3.4, the image $Q := \varphi(\mathbb{Q}_p)$ is closed and G is the semidirect product $G = Q \ltimes N$. By Lemma 2.2, the one-parameter morphism $t \mapsto \mathrm{Ad}_{\mathfrak{n}}\varphi(t)$ is unipotent, and hence the group G is algebraic unipotent.

b) The implication $\Rightarrow$ is easy. Conversely, we assume that X' is the derivative of a one-parameter morphism φ' of G'. When $X' = 0$, the element X belongs to $\mathfrak{n}$ and, since N is algebraic unipotent, X is the derivative of a one-parameter morphism φ of N. We assume now that $X' \neq 0$ so that the group $Q' := \varphi'(\mathbb{Q}_p)$ is algebraic unipotent and isomorphic to $\mathbb{Q}_p$. According to point *a*), the group $H := \pi^{-1}(Q')$ is algebraic unipotent. Since the element X belongs to the Lie algebra $\mathfrak{h}$ of H, it is the derivative of a one-parameter morphism φ' of H.

c) $\Rightarrow$ We assume that G is weakly regular. Let φ'_1 and φ'_2 be one-parameter morphisms of G' with the same derivative $X' \in \mathfrak{g}'$. We want to prove that $\varphi'_1 = \varphi'_2$. If this derivative X' is zero, by Lemma 3.5, we can lift both φ'_1 and φ'_2 as one-parameter morphisms of G with zero derivative. Since G is weakly regular, both φ_1 and φ_2 are trivial and $\varphi'_1 = \varphi'_2$. We assume now that the derivative X' is non-zero. As above, for $i = 1, 2$, the groups $Q'_i := \varphi'_i(\mathbb{Q}_p)$ and $H_i := \pi^{-1}(Q'_i)$ are algebraic unipotent. Since G is weakly regular, and its algebraic unipotent subgroups H_1 and H_2 have the same Lie algebra, one gets successively $H_1 = H_2$, $Q'_1 = Q'_2$, and $\varphi'_1 = \varphi'_2$.

$\Leftarrow$ We assume that G/N is weakly regular. Let φ_1 and φ_2 be one-parameter morphisms of G with the same derivative $X \in \mathfrak{g}$. We want to prove that $\varphi_1 = \varphi_2$. Since G/N is weakly regular the one-parameter morphisms $\pi \circ \varphi_1$ and $\pi \circ \varphi_2$ are equal and their image Q' is a unipotent algebraic subgroup of G'. According to point a), the group $H := \pi^{-1}(Q')$ is algebraic unipotent. Since φ_1 and φ_2 take their values in H, one has $\varphi_1 = \varphi_2$. $\square$

3.3. **Unipotent subgroups tangent to a nilpotent Lie algebra.**

> Proposition 3.7 below describes the nilpotent Lie subgroups of a weakly regular p-adic Lie group G which are spanned by one-parameter morphisms.

The following proposition is due to Ratner in [13, Thm. 2.1].

Proposition 3.7. *Let G be a weakly regular p-adic Lie group and $\mathfrak{n} \subset \mathfrak{g}$ be a nilpotent Lie subalgebra. Then the set $\mathfrak{n}_G := \mathfrak{n} \cap \mathfrak{g}_G$ is an ideal of $\mathfrak{n}$ and there exists an algebraic unipotent subgroup N_G of G with Lie algebra $\mathfrak{n}_G$.*

This group N_G is unique. It is a closed subgroup of G. By construction, it is the largest algebraic unipotent subgroup whose Lie algebra is included in $\mathfrak{n}$.

Proof. We argue by induction on $\dim \mathfrak{n}$. We can assume $\mathfrak{n}_G \neq 0$.

First case : $\mathfrak{n}$ is abelian. Let $X_1, \ldots, X_r$ be a maximal family of linearly indepent elements of $\mathfrak{n}_G$ and, for $i \leq r$, let φ_i be the one-parameter morphism with derivative X_i. Since G is weakly regular, the group spanned by the images $\varphi_i(\mathbb{Q}_p)$ is commutative and the map

$$\psi : \mathbb{Q}_p^d \to G; (t_1, \ldots, t_r) \mapsto \varphi_1(t_1) \ldots \varphi_r(t_r)$$

is an injective morphism. Its image is a unipotent algebraic subgroup N_G of G whose Lie algebra is $\mathfrak{n}_G$.

Second case : $\mathfrak{n}$ is not abelian. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{z}_2$ the ideal of $\mathfrak{n}$ such that $\mathfrak{z}_2/\mathfrak{z}$ is the center of $\mathfrak{n}/\mathfrak{z}$. If $\mathfrak{n}_G$ is included in the centralizer $\mathfrak{n}'$ of $\mathfrak{z}_2$, we can apply the induction hypothesis to $\mathfrak{n}'$. We assume now that $\mathfrak{n}_G$ is not included in $\mathfrak{n}'$, i.e. there exists

$$X \in \mathfrak{n}_G \text{ and } Y \in \mathfrak{z}_2 \text{ such that } [X, Y] \neq 0.$$

This element $Z := [X, Y]$ belongs to the center $\mathfrak{z}$.

We first check that Z belongs also to $\mathfrak{g}_G$. Indeed, let $\mathfrak{m}$ be the 2-dimensional Lie subalgebra of $\mathfrak{n}$ with basis X, Z. This Lie algebra is normalized by Y. For $\varepsilon \in \mathbb{Q}_p$ small enough, there exists a group morphism $\psi : \varepsilon\mathbb{Z}_p \to G$ whose derivative at 0 is Y, and one has $\mathrm{Ad}(\psi(\varepsilon))Y = e^{\varepsilon\,\mathrm{ad}Y}X = X - \varepsilon Z$. Since X belongs to $\mathfrak{g}_G$, the element $X - \varepsilon Z$ also belongs to $\mathfrak{g}_G$. By the first case applied to the abelian Lie subalgebra $\mathfrak{m}$, the element Z belongs to $\mathfrak{g}_G$.

This means that there exists a one-parameter subgroup U of G whose Lie algebra is $\mathfrak{u} = \mathbb{Q}_p Z$. Let C be the centralizer of U in G. According to Lemma 3.6, the quotient group C/U is also weakly regular. We apply our induction hypothesis to this group $C' := C/U$ and the nilpotent Lie algebra $\mathfrak{n}/\mathfrak{u}$. There exists a largest algebraic unipotent subgroup $N'_{C'}$ in C' whose Lie algebra is included in $\mathfrak{n}'$. Hence, using again Lemma 3.6, there exists a largest algebraic unipotent subgroup N_C of C whose Lie algebra is included in $\mathfrak{n}$. Since G is weakly regular, any one-parameter subgroup of G tangent to $\mathfrak{n}$ is included in C and N_C is also the largest algebraic unipotent subgroup of G whose Lie algebra is included in $\mathfrak{n}$. $\square$

3.4. **Largest normal algebraic unipotent subgroup.**

> We prove in this section that a weakly regular p-adic Lie group contains a largest normal algebraic unipotent subgroup.

Let G be a p-adic Lie group. We denote by $\overline{G}_u$ the closure of the subgroup G_u of G generated by all the one-parameter subgroups of G. This group $\overline{G}_u$ is normal in G. We denote by $\mathfrak{g}_u$ the Lie algebra of $\overline{G}_u$. It is an ideal of $\mathfrak{g}$.

We recall that the radical $\mathfrak{r}$ of $\mathfrak{g}$ is the largest solvable ideal of $\mathfrak{g}$ and that the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is the largest nilpotent ideal of $\mathfrak{g}$. The nilradical is the set of X in $\mathfrak{r}$ such that $\mathrm{ad}X$ is nilpotent and one has $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ (see [5]).

When G is weakly regular, we denote by R_u the largest algebraic unipotent subgroup of G whose Lie algebra is included in $\mathfrak{n}$. It exists by Proposition 3.7.

The following proposition is mainly in [13, Lem. 2.2].

Proposition 3.8. *Let G be a weakly regular p-adic Lie group.*
a) The group R_u is the largest normal algebraic unipotent subgroup of G.
b) Its Lie algebra $\mathfrak{r}_u$ is equal to $\mathfrak{r}_u = \mathfrak{n} \cap \mathfrak{g}_G = \mathfrak{r} \cap \mathfrak{g}_G$.
c) One has the inclusion $[\mathfrak{g}_u, \mathfrak{r}] \subset \mathfrak{r}_u$.
d) Let $G' = G/R_u$. Let X be in $\mathfrak{g}$ and X' be its image in $\mathfrak{g}' = \mathfrak{g}/\mathfrak{r}_u$.

One has the equivalence

$$X \in \mathfrak{g}_G \Longleftrightarrow X' \in \mathfrak{g}'_{G'}.$$

Proof. a) We have to prove that any normal algebraic unipotent subgroup U of G is included in R_u. Indeed the Lie algebra $\mathfrak{u}$ of U is a nilpotent ideal of $\mathfrak{g}$, hence it is included in $\mathfrak{n}$ and U is included in R_u.

b) We already know the equality $\mathfrak{r}_u = \mathfrak{n} \cap \mathfrak{g}_G$ from Proposition 3.7. It remains to check the inclusion $\mathfrak{r} \cap \mathfrak{g}_G \subset \mathfrak{n}$. Indeed, let X be an element in $\mathfrak{r} \cap \mathfrak{g}_G$. Since X is the derivative of a one-parameter morphism, by Lemma 2.2, the endomorphism $\mathrm{ad}X$ is nilpotent. Since X is also in the radical $\mathfrak{r}$, X has to be in the nilradical $\mathfrak{n}$.

c) We want to prove that the adjoint action of G_u on the quotient Lie algebra $\mathfrak{r}/\mathfrak{r}_u$ is trivial. That is, we want to prove that, for all

$$X \in \mathfrak{g}_G \text{ and } Y \in \mathfrak{r}\ , \text{ one has } [X, Y] \in \mathfrak{r}_u.$$

By Lemma 2.2, the endomorphism $\mathrm{ad}X$ is nilpotent. Since Y is in $\mathfrak{r}$, the bracket $[X, Y]$ belongs to $\mathfrak{n}$ and the vector space $\mathfrak{m} := \mathbb{Q}_p X \oplus \mathfrak{n}$ is a nilpotent Lie algebra normalized by Y. Hence, by Proposition 3.7, the set $\mathfrak{m} \cap \mathfrak{g}_G$ is a nilpotent Lie algebra. This set is normalized by Y since it is invariant by $e^{\varepsilon\,\mathrm{ad}Y}$ for $\varepsilon \in \mathbb{Q}_p$ small enough. In particular the element $[X, Y]$ belongs to $\mathfrak{g}_G$ and hence to $\mathfrak{r}_u$.

d) This is a special case of Proposition 3.7. □

4. Derivatives of one-parameter morphisms

In this chapter, we describe the set $\mathfrak{g}_G$ of derivatives of one-parameter morphisms of a weakly regular p-adic Lie group G.

4.1. **Construction of one-parameter subgroups.**

We explain first a construction of one-parameter morphisms of G borrowed from [1] and [2].

Lemma 4.1. *Let G be a p-adic Lie group and $g \in G$. Then the vector space $\mathfrak{g}_g^+ := \{v \in \mathfrak{g} \mid \lim_{n\to\infty} \mathrm{Ad}g^{-n} v = 0\}$ is included in $\mathfrak{g}_G$.*

Note that $\mathfrak{g}_g^+$ is a nilpotent Lie subalgebra of $\mathfrak{g}$.

The proof relies on the existence of compact open subgroups of G for which the exponential map satisfies a nice equivariant property. We need some classical definition (see [8]). A p-adic Lie group Ω is said to be a *standard* group if there exists a $\mathbb{Q}_p$-Lie algebra $\mathfrak{l}$ and a compact open sub-$\mathbb{Z}_p$-algebra O of $\mathfrak{l}$ such that the Baker-Campbell-Hausdorff series converges on O and Ω is isomorphic to the p-adic Lie group O equipped with the group law defined by this formula.

In this case, $\mathfrak{l}$ identifies canonically with the Lie algebra of Ω, every element of Ω admits a logarithm and the logarithm map induces an isomorphism $\Omega \to O$. If G is any p-adic Lie group, it admits a standard open subgroup (see [8, Theorem 8.29]). If Ω is such a subgroup and if O is the associated compact open sub-$\mathbb{Z}_p$-algebra of $\mathfrak{g}$, we denote by $\exp_\Omega : O \to \Omega$ the inverse diffeomorphism of the logarithm map $\Omega \to O$.

Note that if Ω and Ω' are standard open subgroups of G, the maps $\exp_\Omega$ and $\exp_{\Omega'}$ coincide in some neighborhood of 0 in $\mathfrak{g}$.

Lemma 4.2. *Let G be a p-adic Lie group, $\Omega \subset G$ a standard open subgroup and $\exp_\Omega : O \to \Omega$ the corresponding exponential map. For every compact subset $K \subset G$, there exists an open subset $O_K \subset \mathfrak{g}$ which is contained in O and in all the translates $\mathrm{Ad}g^{-1}(O)$, $g \in K$, and such that one has the equivariance property*

$$\exp_\Omega(\mathrm{Ad}g(v)) = g \exp_\Omega(v)\, g^{-1} \quad \textit{for any } v \in O_K,\ g \in K.$$

Proof. We may assume that K contains e. The intersection $\Omega_K := \cap_{g\in K} g^{-1}\Omega g$ is an open neighborhood of e in G. We just choose O_K to be the open set $O_K := \log(\Omega_K)$. $\square$

Proof of Lemma 4.1. This is [1, Lem. 5.4]. For the sake of completeness, we recall the proof. Fix $g \in G$. Let Ω be a standard open subgroup of G with exponential map $\exp_\Omega : O \to \Omega$. By Lemma 4.2, there exists an open additive subgroup $U \subset O \cap \mathfrak{g}_g^+$ such that $\mathrm{Ad}g^{-1}U \subset O$ and that

$$\exp_\Omega(u) = g \exp_\Omega(\mathrm{Ad}g^{-1}u)g^{-1} \text{ for any } u \text{ in } U.$$

After eventually replacing U by $\bigcap_{k\geq 0} \mathrm{Ad}g^k U$, we can assume $\mathrm{Ad}g^{-1}U \subset U$. Now, for $k \geq 0$, let $U_k := \mathrm{Ad}g^k U$ and define a continuous map $\psi_k : U_k \to G$ by setting

$$\psi_k(u) = g^k \exp_\Omega(\mathrm{Ad}g^{-k}u)g^{-k} \text{ for any } u \text{ in } U_k.$$

We claim that, for any k, one has $\psi_k = \psi_{k-1}$ on $U_{k-1} = \mathrm{Ad}g^{-1}U_k$. Indeed, let u be in U_k. As $u_k := \mathrm{Ad}g^{-k}u$ belongs to U, we have

$$\psi_{k+1}(u) = g^k(g \exp_\Omega(\mathrm{Ad}g^{-1}u_k)g^{-1})g^{-k} = g^k \exp_\Omega(u_k)g^{-k} = \psi_k(u).$$

Therefore, as $\mathfrak{g}_g^+ = \bigcup_{k\geq 0} U_k$, one gets a map $\psi : \mathfrak{g}_g^+ \to G'$ whose restriction to any U_k, $k \geq 0$, is ψ_k. For every v in $\mathfrak{g}_g^+$, the map $t \mapsto \psi(tv)$ is a one-parameter morphism of G whose derivative is equal to v. $\square$

4.2. The group G_{nc} and its Lie algebra $\mathfrak{g}_{nc}$.

We introduce in this section a normal subgroup G_{nc} of G which contains G_u.

Let G be a p-adic Lie group and $\mathfrak{g}_{nc}$ be the smallest ideal of the Lie algebra $\mathfrak{g}$ such that the Lie algebra $\mathfrak{s}_c := \mathfrak{g}/\mathfrak{g}_{nc}$ is semisimple and such that the adjoint group $\mathrm{Ad}_{\mathfrak{s}_c}(G)$ is bounded in the group $\mathrm{Aut}(\mathfrak{s}_c)$ of automorphisms of $\mathfrak{s}_c$. Let G_{nc} be the kernel of the adjoint action in $\mathfrak{s}_c$, i.e.

$$G_{nc} := \{g \in G \mid \text{for all } X \text{ in } \mathfrak{g}\,,\ \ \mathrm{Ad}g(X) - X \in \mathfrak{g}_{nc}\}$$

By construction G_{nc} is a closed normal subgroup of G with Lie algebra $\mathfrak{g}_{nc}$.

Lemma 4.3. *Let G be a p-adic Lie group. Any one-parameter morphism of G takes its values in G_{nc}*

In other words, the group G_u is included in G_{nc}.

Proof. Let φ be a one-parameter morphism of G. Then $\mathrm{Ad}_{\mathfrak{s}_c} \circ \varphi$ is a one-parameter morphism of $\mathrm{Aut}(\mathfrak{s}_c)$ whose image is relatively compact. By Lemma 2.2, this one-parameter morphism is trivial and φ takes its values in G_{nc}. ☐

4.3. Derivatives and Levi subalgebras.

We can now describe precisely which elements of $\mathfrak{g}$ are tangent to one-parameter subgroups of G.

We recall that an element X of a semisimple Lie algebra $\mathfrak{s}$ is said to be nilpotent if the endomorphism $\mathrm{ad}_{\mathfrak{s}}X$ is nilpotent. In this case, for any finite dimensional representation ρ of $\mathfrak{s}$, the endomorphism $\rho(X)$ is also nilpotent.

We recall that a Levi subalgebra $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ is a maximal semisimple Lie subalgebra, and that one has the Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$.

The following proposition is proven in [13, Th. 2.2] under the additional assumption that G is Ad-regular.

Proposition 4.4. *Let G be a weakly regular p-adic Lie group, $\mathfrak{r}$ be the radical of $\mathfrak{g}$, $\mathfrak{s}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$. One has the equality : $\mathfrak{g}_G = \{X \in \mathfrak{s}_u \oplus \mathfrak{r}_u \mid \mathrm{ad}X$ is nilpotent$\}$.*

It will follow from Lemma 4.7 that the Lie algebra $\mathfrak{s}_u \oplus \mathfrak{r}_u$ does not depend on the choice of $\mathfrak{s}$.

The key ingredient in the proof of Proposition 4.4 will be Lemma 4.1. We will begin by three preliminary lemmas. The first two lemmas are classical.

Lemma 4.5. *Let $V = \mathbb{Q}_p^d$ and G be a subgroup of $\mathrm{GL}(V)$ such that V is an unbounded and irreducible reprentation of G. Then G contains an element g with at least one eigenvalue of modulus not one.*

Proof. Let A be the associative subalgebra of $\mathrm{End}(V)$ spanned by G. Since V is irreducible, the associative algebra A is semisimple and the bilinear form $(a, b) \mapsto tr(ab)$ is non-degenerate on A (see [9, Ch. 17]). If all the eigenvalues of all the elements of G have modulus 1, this bilinear form is bounded on $G \times G$. Since A admits a basis included in G, for any a in A, the linear forms $b \mapsto tr(ab)$ on A are bounded on the subset G. Hence G is a bounded subset of A. □

Lemma 4.6. *Let $\mathfrak{s}_0$ be a simple Lie algebra over $\mathbb{Q}_p$ and $S_0 := \mathrm{Aut}(\mathfrak{s}_0)$. All open unbounded subgroups J of S_0 have finite index in S_0.*

Proof. Since J is unbounded, by Lemma 4.5, it contains an element g_0 with at least one eigenvalue of modulus not one. Since J is open, the unipotent Lie subgroups

$$U^+ = \{g \in S_0 \mid \lim_{n\to\infty} g_0^{-n} g g_0^n = e\} \text{ and } U^- = \{g \in S_0 \mid \lim_{n\to\infty} g_0^{n} g g_0^{-n} = e\}$$

are included in J. By [4, 6.2.v and 6.13], J has finite index in S_0. □

The third lemma contains the key ingredient.

Lemma 4.7. *Let G be a weakly regular p-adic Lie group, $\mathfrak{r}$ the radical of $\mathfrak{g}$, $\mathfrak{s}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$.*
a) One has the inclusion $[\mathfrak{s}_u, \mathfrak{r}] \subset \mathfrak{r}_u$.
b) Every nilpotent element X in $\mathfrak{s}_u$ belongs to $\mathfrak{g}_G$.

Note that Lemma 4.7.a does not follow from Proposition 3.8.c, since with the definitions of $\mathfrak{g}_u$ and $\mathfrak{s}_u$ that we have given, we do not know yet that $\mathfrak{s}_u = \mathfrak{s} \cap \mathfrak{g}_u$.

Proof. a) By Proposition 3.8, we can assume $\mathfrak{r}_u = 0$. We want to prove that $[\mathfrak{s}_u, \mathfrak{r}] = 0$. Let $\mathfrak{s}_i$, $i = 1, \ldots, \ell$ be the simple ideals of $\mathfrak{s}_u$. Replacing G by a finite index subgroup, we can also assume that the ideals $\mathfrak{s}_i \oplus \mathfrak{r}$ are G-invariant. Similarly, let $\mathfrak{r}_j$, $j = 1, \ldots, m$ be the simple subquotients of a Jordan-Hölder sequence of the G-module $\mathfrak{r}$.

On the one hand, by assumption, for all $i \leq \ell$, the group $\mathrm{Ad}_{\mathfrak{s}_i \oplus \mathfrak{r}/\mathfrak{r}}(G)$ is unbounded. Hence, by Lemma 4.5, there exists an element g_i in G and X_i in $\mathfrak{g}_{g_i}^+ \cap (\mathfrak{s}_i \oplus \mathfrak{r})$ whose image in $\mathfrak{g}/\mathfrak{r}$ is non zero. By Lemma 4.1, there exists a one-parameter morphism φ_i of G whose derivative is X_i.

On the other hand, since $\mathfrak{r}_u = 0$, by the same Lemma 4.1, for every g in G, all the eigenvalues of $\mathrm{Ad}_{\mathfrak{r}}(g)$ have modulus 1. By Lemma 4.5, for all $j \leq m$, the image $\mathrm{Ad}_{\mathfrak{r}_j}(G)$ of G in any simple subquotient

$\mathfrak{r}_j$ is bounded. In particular, the one-parameter morphisms $\mathrm{Ad}_{\mathfrak{r}_j} \circ \varphi_i$ are bounded. Hence, by Lemma 2.2, one has $\mathrm{ad}_{\mathfrak{r}_j}(X_i) = 0$. Since $\mathfrak{r}_j$ is a simple $\mathfrak{g}$-module, the Lie algebra $\mathrm{ad}_{\mathfrak{r}_j}(\mathfrak{g})$ is reductive and contains $\mathrm{ad}_{\mathfrak{r}_j}(\mathfrak{s}_i)$ as an ideal. Since $\mathfrak{s}_i$ is a simple Lie algebra, this implies $\mathrm{ad}_{\mathfrak{r}_j}(\mathfrak{s}_i) = 0$. Since the action of $\mathfrak{s}_u$ on $\mathfrak{r}$ is semisimple, this implies the equality $[\mathfrak{s}_u, \mathfrak{r}] = 0$.

b) As in *a*), we can assume that G preserves the ideals $\mathfrak{s}_i \oplus \mathfrak{r}$ and that $\mathfrak{r}_u = 0$. According to this point *a*), the Lie algebras $\mathfrak{s}_u$ and $\mathfrak{r}$ commute and hence $\mathfrak{s}_u$ is the unique Levi subalgebra of $\mathfrak{g}_{nc}$ (see [5, §6]). In particular,

$$\text{for all } g \text{ in } G, \text{ one has } \mathrm{Ad}g(\mathfrak{s}_u) = \mathfrak{s}_u. \tag{4.1}$$

Let X be a nilpotent element of $\mathfrak{s}_u$. We want to prove that X is the derivative of a one parameter morphism of G. By Jacobson-Morozov theorem, there exists an automorphism ψ of $\mathfrak{s}_u$ such that $\psi(X) = p^{-1}X$. Since, for every simple ideal $\mathfrak{s}_i$ of $\mathfrak{s}$, the subgroup $\mathrm{Ad}_{\mathfrak{s}_i \oplus \mathfrak{r}/\mathfrak{r}}(G_{nc}) \subset \mathrm{Aut}(\mathfrak{s}_i)$ is unbounded and open, this subgroup has finite index. Hence, remembering also (4.1), there exists $k \geq 1$ and g in G_{nc} such that $\mathrm{Ad}g(X) = p^{-k}X$. Then, by Lemma 4.1, X is the derivative of a one-parameter morphism of G. □

Proof of Proposition 4.4. We just have to gather what we have proved so far. By Proposition 3.8, we can assume $\mathfrak{r}_u = 0$. Let $X \in \mathfrak{g}$. We write $X = X_{\mathfrak{s}} + X_{\mathfrak{r}}$ with $X_{\mathfrak{s}} \in \mathfrak{s}$ and $X_{\mathfrak{r}} \in \mathfrak{r}$.

Proof of the inclusion $\subset$. Assume that X is the derivative of a one-parameter morphism φ of G. By Lemma 4.3, X belongs to $\mathfrak{g}_{nc}$ and hence $X_{\mathfrak{s}}$ belongs to $\mathfrak{s}_u$. By Lemma 2.2, the endomorphism $\mathrm{ad}_{\mathfrak{g}}X$ is nilpotent, and hence $X_{\mathfrak{s}}$ is a nilpotent element of the semisimple Lie algebra $\mathfrak{s}$. According to Lemma 4.7, $X_{\mathfrak{s}}$ and $X_{\mathfrak{r}}$ commute and $X_{\mathfrak{s}}$ is the derivative of a one-parameter morphism $\varphi_{\mathfrak{s}}$ of G. Then $X_{\mathfrak{r}}$ is also the derivative of a one-parameter morphism $\varphi_{\mathfrak{r}}$ of G, the one given by $t \mapsto \varphi_{\mathfrak{s}}(t)^{-1}\varphi(t)$. Hence $X_{\mathfrak{r}}$ belongs to $\mathfrak{r}_u$.

Proof of the inclusion $\supset$. Assume that $X_{\mathfrak{s}}$ belongs to $\mathfrak{s}_u$, $X_{\mathfrak{r}}$ belongs to $\mathfrak{r}_u$ and $\mathrm{ad}X$ is nilpotent. By Lemma 4.7, $X_{\mathfrak{s}}$ and $X_{\mathfrak{r}}$ commute and $X_{\mathfrak{s}}$ is the derivative of a one-parameter morphism $\varphi_{\mathfrak{s}}$ of G. By assumption $X_{\mathfrak{r}}$ is the derivative of a one-parameter morphism $\varphi_{\mathfrak{r}}$ of G. Hence X is also the derivative of a one-parameter morphism φ of G, the one given by $t \mapsto \varphi_{\mathfrak{s}}(t)\varphi_{\mathfrak{r}}(t)$. Hence X belongs to $\mathfrak{g}_G$. □

5. Groups spanned by unipotent subgroups

In this chapter, we prove the two main results Proposition 5.5 and Theorem 5.12 that we announced in the introduction.

5.1. **Semisimple regular p-adic Lie groups.**

We recall first a nice result due to Prasad-Raghunathan which is an output from the theory of congruence subgroups

Let $\mathfrak{s}$ be a semisimple Lie algebra over $\mathbb{Q}_p$, $\mathrm{Aut}(\mathfrak{s})$ the group of automorphisms of $\mathfrak{s}$ and $S_+ := \mathrm{Aut}(\mathfrak{s})_u \subset \mathrm{Aut}(\mathfrak{s})$ the subgroup spanned by the one-parameter subgroups of $\mathrm{Aut}(\mathfrak{s})$. We will say that $\mathfrak{s}$ is *totally isotropic* if $\mathfrak{s}$ is spanned by nilpotent elements. In this case S_+ is an open finite index subgroup of $\mathrm{Aut}(\mathfrak{s})$, see [4, 6.14]. Since $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$, the group S_+ is perfect i.e. $S_+ = [S_+, S_+]$. In particular this group admits a universal central topological extension $\widetilde{S}_+$, i.e. a group which is universal among the central topological extension of S_+. In [12] Prasad-Raghunathan were able to describe this group $\widetilde{S}_+$ (special cases were obtained before by Moore, Matsumoto and Deodhar). We will only need here the fact that this group is a finite extension of S_+.

Proposition 5.1. (Prasad–Raghunathan) *Let $\mathfrak{s}$ be a totally isotropic semisimple p-adic Lie algebra. Then the group S_+ admits a universal topological central extension*

$$1 \longrightarrow Z_0 \longrightarrow \widetilde{S}_+ \xrightarrow{\pi_0} S_+ \longrightarrow 1$$

and its center Z_0 is a finite group.

The word universal means that for all topological central extension

$$1 \longrightarrow Z_E \longrightarrow E \xrightarrow{\pi} S_+ \longrightarrow 1$$

where E is a locally compact group and Z_E is a closed central subgroup, there exists a unique continuous morphism $\psi : \widetilde{S}_+ \to E$ such that $\pi_0 = \pi \circ \psi$.

Proof. See [12, Theorem 10.4]. □

Remark 5.2. This result does not hold for real Lie groups: indeed, the center Z_0 of the universal cover of $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to $\mathbb{Z}$.

Corollary 5.3. *Let $\mathfrak{s}$ be a totally isotropic semisimple p-adic Lie algebra. For every topological central extension*

$$1 \longrightarrow Z_E \longrightarrow E \xrightarrow{\pi} S_+ \longrightarrow 1$$

with $E = \overline{[E, E]}$, the group Z_E is finite.

Proof of Corollary 5.3. Let $\psi : \widetilde{S}_+ \to E$ be the morphism given by the universal property. Since, by Proposition 5.1, the group Z_0 is finite, the projection $\pi_0 = \pi \circ \psi$ is a proper map, hence ψ is also a proper map and the image $\psi(\widetilde{S}_+)$ is a closed subgroup of E. Since $E = \psi(\widetilde{S}_+)Z_E$, one has the inclusion $[E, E] \subset \psi(\widetilde{S}_+)$, and the assumption $E = \overline{[E, E]}$ implies that the morphism ψ is onto. Hence the group $Z_E = \psi(Z_0)$ is finite. $\square$

Remark 5.4. For real Lie groups, the center Z_E might even be non discrete. Such an example is given by the quotient E of the product $\mathbb{R} \times \widetilde{\mathrm{SL}(2, \mathbb{R})}$ by a discrete subgroup of $\mathbb{R} \times Z_0$ whose projection on $\mathbb{R}$ is dense.

5.2. The Levi decomposition of G_u.

We prove in this section that in a weakly regular p-adic Lie group G, the subgroup G_u is closed and admits a Levi decomposition.

Let G be a weakly regular p-adic Lie group and $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$. We recall that $\mathfrak{g}_{nc}$ is the smallest ideal of $\mathfrak{g}$ containing $\mathfrak{r}$ such that the group $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{g}_{nc}}(G)$ is bounded. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$.

We recall that G_u is the subgroup of G spanned by all the one-parameter subgroups of G, that R_u is the subgroup of G spanned by all the one-parameter subgroups of G tangent to $\mathfrak{r}$, and we define S_u as the subgroup of G spanned by all the one-parameter subgroups of G tangent to $\mathfrak{s}$. Note that we don't know yet, but we will see it in the next proposition, that S_u is indeed a closed subgroup with Lie algebra equal to $\mathfrak{s}_u$.

Proposition 5.5. *Let G be a weakly regular p-adic Lie group.*
a) The group R_u is closed. It is the largest normal algebraic unipotent subgroup of G.
b) The group S_u is closed. Its Lie algebra is $\mathfrak{s}_u$, and the morphism $\mathrm{Ad}_{\mathfrak{s}_u} : S_u \to (\mathrm{Aut}\, \mathfrak{s}_u)_u$ is onto and has finite kernel.
c) The group G_u is closed. One has $G_u = S_u R_u$ and $S_u \cap R_u = \{e\}$.

Remark 5.6. In a real Lie group, the group tangent to a Levi subalgebra is not necessary closed, as for example, if $G = (S \times \mathbb{T})/Z$ where S is the universal cover of $\mathrm{SL}(2, \mathbb{R})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and Z is the cyclic subgroup spanned by (z_0, α_0) with z_0 a generator of the center of S and α_0 an irrational element of $\mathbb{T}$.

Proof of Proposition 5.5. *a)* This follows from Proposition 3.8.

b) Let $E := \overline{S}_u$ be the closure of S_u and $S_+ := (\mathrm{Aut}\,\mathfrak{s}_u)_u$. Note that this group E normalizes $\mathfrak{s}_u$. We want to apply Corollary 5.3 to the morphism

$$E \xrightarrow{\pi} S_+$$

where π is the adjoint action $\pi := \mathrm{Ad}_{\mathfrak{s}_u}$.

We first check that the assumptions of Corollary 5.3 are satisfied. Since E is weakly regular the kernel Z_E of this morphism π commutes with all the one-parameter subgroups tangent to $\mathfrak{s}_u$. Hence Z_E is equal to the center of E. Since S_+ is spanned by one-parameter subgroups, and since by Proposition 4.4 any nilpotent element X of $\mathfrak{s}_u$ is tangent to a one parameter subgroup φ of E, this morphism π is surjective. Now, by Jacobson Morozov Theorem, for any nilpotent element X of $\mathfrak{s}_u$, there exists an element H in $\mathfrak{s}_u$ such that $[H, X] = X$. Let $\varphi : \mathbb{Q}_p \to G$ be the one-parameter subgroup tangent to X, which exists by Proposition 4.4. Since G is weakly regular, one has, for t in $\mathbb{Q}_p$ and $g_\varepsilon := \exp(\varepsilon H)$ with ε small,

$$g_\varepsilon \varphi(t) g_\varepsilon^{-1} \varphi(t)^{-1} = \varphi(e^\varepsilon t)\varphi(-t) = \varphi((e^\varepsilon - 1)t).$$

This proves that $\varphi(\mathbb{Q}_p)$ is included in the derived subgroup $[E, E]$. In particular one has $E = \overline{[E, E]}$.

According to Corollary 5.3, the kernel Z_E is finite. In particular, one has $\dim E = \dim \mathfrak{s}_u$. Since $\mathfrak{s}_u$ is totally isotropic, one can find a basis of $\mathfrak{s}_u$ all of whose elements are nilpotent. By Lemma 4.7, all these elements are in $\mathfrak{g}_G$. Hence, by the implicit function theorem, the group S_u is open in E. Therefore, S_u is also closed and $S_u = E$.

c) Since the adjoint action of R_u on $\mathfrak{g}_{nc}/\mathfrak{r}$ is trivial, the intersection $S_u \cap R_u$ is included in the kernel Z_E of the adjoint map $\mathrm{Ad}_{\mathfrak{s}_u}$. Since, by *b*), this kernel is finite, and since the algebraic unipotent group R_u does not contain finite subgroups, one gets $S_u \cap R_u = \{e\}$.

It remains to check that $S_u R_u$ is closed and that $G_u = S_u R_u$. Thanks to Propositions 3.8 and 4.4, we can assume that $R_u = \{e\}$. In this case, we know from point *b*) that S_u is closed and from Proposition 4.4 that $G_u = S_u$. □

Here are a few corollaries. The first corollary is an improvement of Lemma 3.6.

Corollary 5.7. *Let G be a p-adic Lie group, H a normal weakly regular closed subgroup of G such that $H = H_u$, and $G' := G/H$.*
a) One has the equality $G'_u = G_u/H$.
b) Let X in $\mathfrak{g}$ and X' its image in $\mathfrak{g}' = \mathfrak{g}/\mathfrak{h}$. One has the equivalence

$$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_{G'}.$$

c) *One has the equivalence*

$$G \text{ is weakly regular} \Longleftrightarrow G' \text{ is weakly regular.}$$

Remark 5.8. The assumption that $H = H_u$ is important. For instance the group $G = \mathbb{Q}_p$ and its normal subgroup $H = \mathbb{Z}_p$ are weakly regular while the quotient G/H is not weakly regular.

Proof. We prove these three statements simultaneously. Since $H = H_u$, according to Proposition 5.5, the group H admits a Levi decomposition $H = SR$ where R is a normal algebraic unipotent Lie subgroup and where S is a Lie subgroup with finite center Z whose Lie algebra is semisimple, totally isotropic, and such that the adjoint map $\mathrm{Ad}_{\mathfrak{s}} : S \to \mathrm{Aut}(\mathfrak{s})_u$ is surjective. Note that R is also a normal subgroup of G. Using Lemma 3.6, we can assume that $R = \{e\}$.

Let C be the centralizer of $H = S$ in G. Since H is normal in G, since $H = H_u$ and H is weakly regular, C is also the kernel of the adjoint action of G on $\mathfrak{h} = \mathfrak{s}_u$. Therefore, by Proposition 5.5, the image of the group morphism

$$H \times C \to G; (h, c) \mapsto hc$$

has finite index in G. Its kernel is isomorphic to $H \cap C = Z$ and hence is finite. When this morphism $H \times C \to G$ is an isomorphism, our three statements are clear. The general case reduces to this one thanks to Lemma 5.9 below. □

Lemma 5.9. *Let G be a locally compact topological group, Z be a finite central subgroup of G and $\varphi : \mathbb{Q}_p \to G/Z$ be a continuous morphism. Then φ may be lifted as a continuous morphism $\widetilde{\varphi} : \mathbb{Q}_p \to G$.*

Proof. Let H be the inverse image of $\varphi(\mathbb{Z}_p)$ in G. Then H is totally discontinuous. In particular it contains an open compact subgroup U such that $U \cap Z = \{e\}$, so that U maps injectively in G/Z. Let ℓ be an integer such that $\varphi(p^\ell \mathbb{Z}_p) \subset UZ/Z$. After rescaling, we can assume that $\ell = 0$. We let g_0 be the unique element of U such that $\varphi(1) = g_0 Z$. Since U map injectively in G/Z, we have $g_0^{p^k} \xrightarrow[k \to \infty]{} e$.

Let X be the group of elements of p-torsion in Z and Y be the group of elements whose torsion is prime to p. For any $k \geq 0$ pick some g_k in G such that $\varphi(p^{\ell - k}) = g_k Z$ and let x_k and y_k be the elements of X and Y such that $g_k^{p^k} = g_0 x_k y_k$. We let z_k be the unique element of Y_k such that $z_k^{p^k} = y_k$. Replacing g_k by $g_k z_k^{-1}$, we can assume that $z_k = e$. Since x_k only takes finitely many values, we can find a x in X and an increasing sequence (k_n) such that, for any n, $x_{k_n} = x$. Now, since x is a central p-torsion element, one has $(g_0 x)^{p^k} \xrightarrow[k \to \infty]{} e$. Since, for any n,

$g_{k_n}^{p^n} = g_0 x$, there exists a unique morphism $\widetilde{\varphi} : \mathbb{Q}_p \to G$ such that, for any n, $\widetilde{\varphi}(p^{-k_n}) = g_{k_n}$ and $\widetilde{\varphi}$ clearly lifts φ. □

The second corollary is an improvement of Proposition 4.4.

Corollary 5.10. *Let G be a weakly regular p-adic Lie group. One has the equality : $\mathfrak{g}_G = \{X \in \mathfrak{g}_u \mid \mathrm{ad}X$ is nilpotent$\}$, where $\mathfrak{g}_u$ is the Lie algebra of G_u.*

Proof. This follows from Proposition 4.4 since, by Proposition 5.5, one has the equality $\mathfrak{g}_u = \mathfrak{s}_u \oplus \mathfrak{r}_u$. □

The last corollary tells us that a weakly regular p-adic Lie group G with $G = G_u$ is "almost" an algebraic Lie group.

Corollary 5.11. *Let G be a weakly regular p-adic Lie group such that $G = G_u$. Then there exists a Lie group morphism $\psi : G \to H$ with finite kernel and cokernel where H is the group of $\mathbb{Q}_p$-points of a linear algebraic group defined over $\mathbb{Q}_p$.*

Proof. According to Proposition 4.4, $G = G_u$ is a semidirect product $G_u = S_u \ltimes R_u$. We choose H to be the semi direct product $H := S' \ltimes R_u$ where S' is the Zariski closure of the group $\mathrm{Ad}(S_u)$ in $\mathrm{Aut}(\mathfrak{g}_u)$. Note that, since G is weakly regular, any automorphism of $\mathfrak{g}_u$ induces an automorphism of R_u. We define the morphism $\psi : G_u \to H$ by $\psi(g) = (\mathrm{Ad}(s), r)$ for $g = sr$ with $s \in S_u$, $r \in R_u$. Proposition 4.4 tells us also that this morphism ψ has finite kernel and cokernel. □

5.3. **Regular semiconnected component.**

We are now ready to prove the following theorem which was the main motivation of our paper.

Theorem 5.12. *Let G be a weakly regular p-adic Lie group. Then, there exists an open regular subgroup G_Ω of G which contains all the one-parameter subgroups of G.*

Remark 5.13. Let Ω be a *standard* open subgroup of G. We define the *Ω-semiconnected component of G* as its open subgroup $G_\Omega := \Omega G_u$ (see [13]). In this language, Theorem 5.12 states that, the Ω-semiconnected component of a weakly regular p-adic Lie group is regular, if the standard subgroup Ω is small enough.

Proof of Theorem 5.12. We will need some notations. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$, $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$, $\mathfrak{s}'$ the centralizer of $\mathfrak{s}_u$ in $\mathfrak{s}$, $\mathfrak{r}$ the radical of $\mathfrak{g}$, and $\mathfrak{r}'$ the centralizer of $\mathfrak{s}_u$ in $\mathfrak{r}$.

We can choose a standard subgroup Ω'_S of G with Lie algebra $\mathfrak{s}'$, and a standard subgroup Ω'_R of G with Lie algebra $\mathfrak{r}'$ such that Ω'_S

normalizes Ω'_R and the semidirect product $\Omega' := \Omega'_S\Omega'_R$ is a standard subgroup of G with Lie algebra $\mathfrak{s}' \oplus \mathfrak{r}'$. Since G is weakly regular, by shrinking Ω', we can assume that it commutes with S_u and normalizes R_u.

We claim that, if Ω' is small enough, the group

$$G_\Omega := \Omega' G_u$$

is an open regular subgroup of G.

First step : Openness. One has the equalities $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{s}_u$ and, according to Proposition 3.8, $\mathfrak{r} = \mathfrak{r}' + \mathfrak{r}_u$. Hence G_Ω is open in G.

Second step : Ad-regularity. Let $J \subset G_\Omega$ be the kernel of $\mathrm{Ad}_{\mathfrak{g}}$. We want to prove that J is the center of G_Ω. Since J acts trivially on the quotient $\mathfrak{g}/\mathfrak{r} \simeq \mathfrak{s}' \oplus \mathfrak{s}_u$, by Proposition 5.5, one has the inclusion $J \subset J' := Z_E\Omega'_R R_u$ where Z_E is a finite subgroup of S_u. One has $J_u = J \cap R_u$ and this group is algebraic unipotent. Hence, by Lemma 3.6, the quotient R_u/J_u is also algebraic unipotent. By Lemma 3.4, the group R_u/J_u is closed in the group J'/J. This means that R_uJ is closed in J', hence that J/J_u is closed in the compact group J'/R_u. In particular, J/J_u is compact. Therefore, there exists a compact set $K \subset J$ such that

$$J = KJ_u.$$

We can now prove that if Ω' is small enough the group J commutes with G_Ω. This is a consequence of the following three facts.

(i) Since J is the kernel of $\mathrm{Ad}_{\mathfrak{g}}$ and G is weakly regular, J commutes with G_u.

(ii) Since J_u is a subgroup of R_u whose Lie algebra $\mathfrak{j}_u$ is included in the center of $\mathfrak{g}$, if we choose Ω' small enough, one has $\mathrm{Ad}_{\mathfrak{j}_u}(\Omega') = \{e\}$, and the group J_u commutes with Ω'.

(iii) Since K is compact and $\mathrm{Ad}_{\mathfrak{g}}(K) = \{e\}$, by Lemma 4.2, if we choose Ω' small enough, the group K commutes with Ω'.

Third step : Size of finite subgroups. We want a uniform upper bound on the cardinality of the finite subgroups of G_Ω. This follows from the inclusions $R_u \subset G_u \subset G_\Omega$ of normal subgroups and from the following three facts.

(i) Since the group R_u is algebraic unipotent, it does not contain finite groups.

(ii) Since, by Proposition 5.5, the group G_u/R_u is a finite extension of a linear group, by Example 2.6.b, its finite subgroups have bounded cardinality.

(*iii*) Since the group G_Ω/G_u is a compact p-adic Lie group, by Example 2.6.a, its finite subgroups have bounded cardinality. $\square$

5.4. **Non weakly regular p-adic Lie groups.**

Not every p-adic Lie group is weakly regular. Here is a surprising example.

Example 5.14. There exists a p-adic Lie group G with $G = G_u$ which does not contain any open weakly regular subgroup H with $H = H_u$.

We will give the construction of such a group G with Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{Q}_p)$, but we will leave the verifications to the reader.

We recall that the group $G_0 := \mathrm{SL}(2, \mathbb{Q}_p)$ is an amalgamated product $G_0 = K \star_{I_0} K$ where $K := \mathrm{SL}(2, \mathbb{Z}_p)$ and $I_0 := \{k \in K \mid k_{21} \equiv 0 \bmod p\}$ is an Iwahori subgroup of K. We define G as the amalgamated product $G = K \star_I K$ where $I \subset I_0$ is an open subgroup such that $I \neq I_0$.

The morphism $G \to G_0$ is a non central extension. Using the construction in Lemma 4.1 one can check that G is spanned by one-parameter subgroups. However, one can check that the universal central extension $\widetilde{G}_0 \to G_0$ can not be lifted as a morphism $\widetilde{G}_0 \to G$.

References

[1] Y. Benoist, J.-F. Quint, Stationary measures and invariant subsets of homogeneous spaces (II), *Journ. Am. Math. Soc.* **26** (2013) 659-734.

[2] Y. Benoist, J.-F. Quint, Lattices in S-adic Lie groups, *Journal of Lie Theory* **24** (2014) 179-197.

[3] A. Borel, J. Tits, Groupes réductifs, *Publ. Math. IHES* **27** (1965) 55-150.

[4] A. Borel, J. Tits, Homomorphismes "abstraits" de groupes algébriques simples, *Ann. of Math.* **97** (1973) 499-571.

[5] N. Bourbaki, Groupes et Algèbres de Lie, chapitre 1, *CCLS* (1971).

[6] F. Bruhat, J. Tits *Groupes réductifs sur un corps local I*, Publ. IHES 41 (1972) p.5-252.

[7] F. Bruhat, J. Tits *Groupes réductifs sur un corps local II*, Publ. IHES 60 (1984) p.5-184.

[8] J. Dixon, M. duSautoy, A. Mann, D. Segal, Analytic Pro-p Groups, *CUP* (1991).

[9] S. Lang, Algebra, Addison-Wesley (1964).

[10] G. Margulis, G. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Invent. Math.* **116** (1994) 347-392.

[11] V. Platonov, A.Rapinchuk, Algebraic groups and number theory. Academic Press (1994).

[12] G. Prasad, M. Raghunathan, Topological central extensions of semi-simple groups over local fields, *Ann. of Math.* **119** (1984) 143-268.

[13] M. Ratner, Raghunathan's conjectures for cartesian products of real and p-adic Lie groups, *Duke Math. J.* **72** (1995) 275-382.

[14] J.P. Serre, Arbres, amalgames, SL2, *Asterisque* **46** (1972).

[15] J. Tits, Reductive groups over local fields, *Pr. Sym. P. Math.* **33** (1979) 22-69.

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