Harmonic analysis on the Pascal graph

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Abstract

In this paper, we completely determine the spectral invariants of an auto-similar planar 3-regular graph. Using the same methods, we study the spectral invariants of a natural compactification of this graph.

1 Introduction

In this whole article, we shall call Pascal graph the infinite, connected and 3-regular graph pictured in figure 1. We let Γ denote it. This graph may be constructed in the following way. One writes the Pascal triangle and one erases therein all the even values of binomial coefficients. In this picture, one links each point to its neighbors that have not been erased. One thus obtains a graph in which each point has three neighbors, except the vertex of the triangle which has only two. One then takes two copies of this graph and joins them by their vertices: one thus do get a 3-regular graph. This is the graph Γ.

Let φ be a complex valued function on Γ. For p in Γ, one sets $\Delta\varphi(p) = \sum_{q\sim p}\varphi(q)$. The linear operator Δ is self-adjoint with respect to the counting measure on Γ, that is, for any finitely supported functions φ and ψ, one has $\sum_{p\in\Gamma}\varphi(p)(\Delta\psi(p)) = \sum_{p\in\Gamma}(\Delta\varphi(p))\psi(p)$. In this article, we will completely determine the spectral invariants of the operator Δ in the space $\ell^2(\Gamma)$ of square-integrable functions on Γ.

To set our results, set $f:\mathbb{R}\to\mathbb{R}, x\mapsto x^2-x-3$. Let Λ be the Julia set of f, that is, in this case, the set of those x in $\mathbb{R}$ for which the sequence $(f^n(x))_{n\in\mathbb{N}}$ remains bounded. This is a Cantor set which is contained in the interval $[-2,3]$. More precisely, if one sets $I_{-2} = \left[-2, \frac{1-\sqrt{5}}{2}\right]$ and $I_3 = \left[\frac{1+\sqrt{5}}{2}, 3\right]$, for any $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ in $\{-2,3\}^{\mathbb{N}}$, there exists a unique x in Λ such that, for any n in $\mathbb{N}$, one has $f^n(x)\in I_{\varepsilon_n}$ and the thus defined map

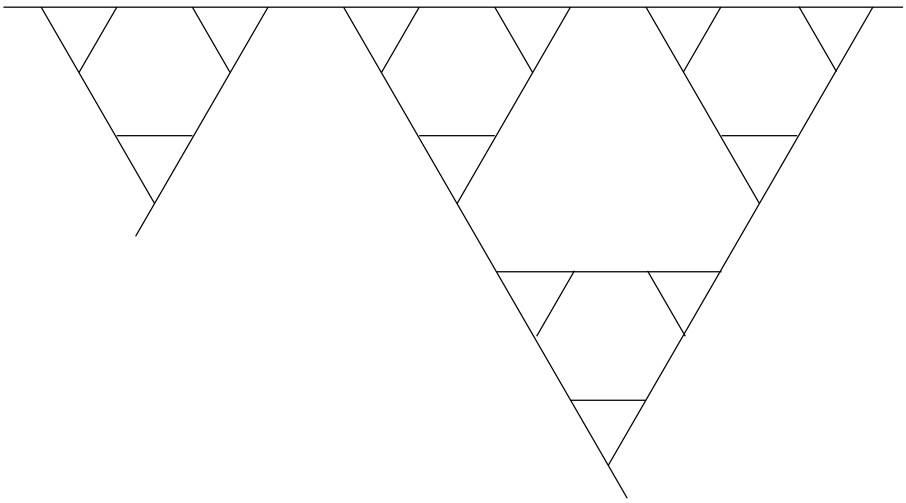

Figure 1: The Pascal graph

$\{-2,3\}^{\mathbb{N}} \to \Lambda$ is a bi-Hölder homeomorphism that conjugates f and the shift map in $\{-2,3\}^{\mathbb{N}}$. For x in Λ, set $\rho(x) = \frac{x}{2x-1}$ and, if φ is a continuous function on Λ, $L_\rho\varphi(x) = \sum_{f(y)=x} \rho(y)\varphi(y)$. One easily checks that one has $L_\rho(1) = 1$. Thus, by Ruelle-Perron-Frobenius theorem (see [13, § 2.2]), there exists a unique Borel probability measure ν_ρ on Λ such that $L_\rho^*\nu_\rho = \nu_\rho$. The measure ν_ρ is atom free and f-invariant. Finally, let us note that, if h denotes the function $\Lambda \to \mathbb{R}, x \mapsto 3-x$, one has $L_\rho h = 2$ and, therefore, $\int_\Lambda h \mathrm{d}\nu_\rho = 2$.

Let p_0 and $p_0^\vee$ denote the two vertices of the infinite triangles that have been glued to build the graph Γ. Let φ_0 be the function on Γ with value 1 at p_0, -1 at $p_0^\vee$ and 0 everywhere else. We have the following

Theorem 1.1. *The spectrum of Δ in $\ell^2(\Gamma)$ is the union of Λ and the set $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$. The spectral measure of φ_0 for Δ in $\ell^2(\Gamma)$ is the measure $h\nu_\rho$, the eigenvalues of Δ in $\ell^2(\Gamma)$ are the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$ and $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$ and the associate eigenspaces are spanned by finitely supported functions. Finally, the orthogonal complement of the sum of the eigenspaces of Δ in $\ell^2(\Gamma)$ is the cyclic subspace spanned by φ_0.*

The study of the Pascal graph is closely related to the one of the Sierpiński graph, pictured in figure 2. The spectral theory of the Sierpiński graph, and more generally the one of self-similar objects, has been intensively studied, since the original works of Rammal and Toulouse in [14] and Kigami in [10]

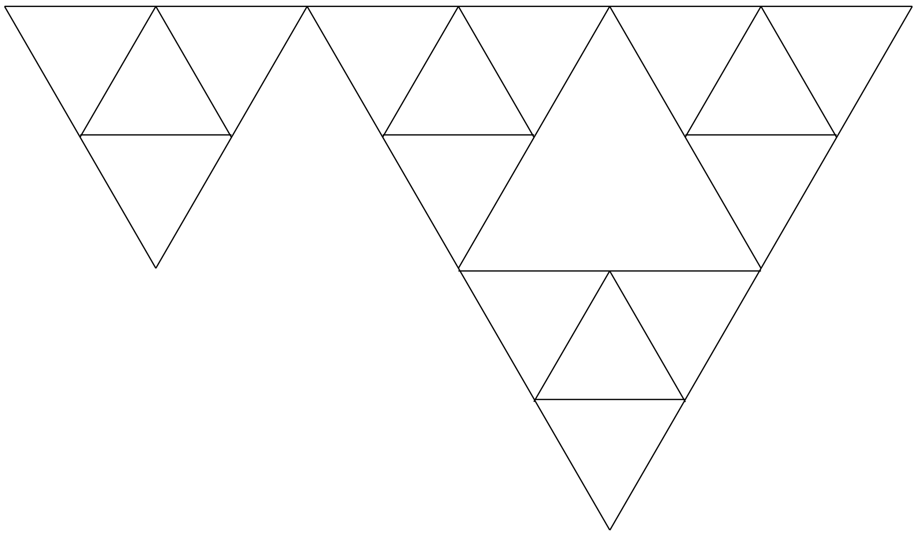

Figure 2: The Sierpiński graph

and [11]. These problems are attacked from a general viewpoint by Sabot in [16], where numerous references may be found; see also Krön [9]. Asymptotics of transition probabilities for the simple random walk on the Sierpiński have been computed by Jones in [8] and Grabner and Woess in [3]. The finitely supported eigenfunctions for the Laplace operator on the Sierpiński graph and the associate eigenvalues have been determined by Teplyaev in [18]. The Sierpiński graph may be seen as the edges graph of the Pascal graph, where two edges are linked when they have a common point. In particular, our description of the eigenvalues associate to finitely supported eigenfunctions on the Pascal graph for the operator Δ is a consequence of the work of Teplyaev. However, the exact determination of the cyclic components of Δ and of its continuous spectrum are new and answer the question asked by Teplyaev in [18, § 6.6]. In section 14, we shall precisely explain how to connect the study of the Sierpiński graph to the one of the Pascal graph.

The results cited above rely on the application of the so-called method of Schur complements. This method has recently been succesfully applied to get precise computations for numerous self-similar graphs, for example by Grigorchuk and Zuk [7], Bartholdi and Grigorchuk [1], Bartholdi and Woess [2] and Grigorchuk and Nekrashevych [6]. In [4] and [5], Grigorchuk and Šunik study finite analogues of the Pascal graph. The method we use in

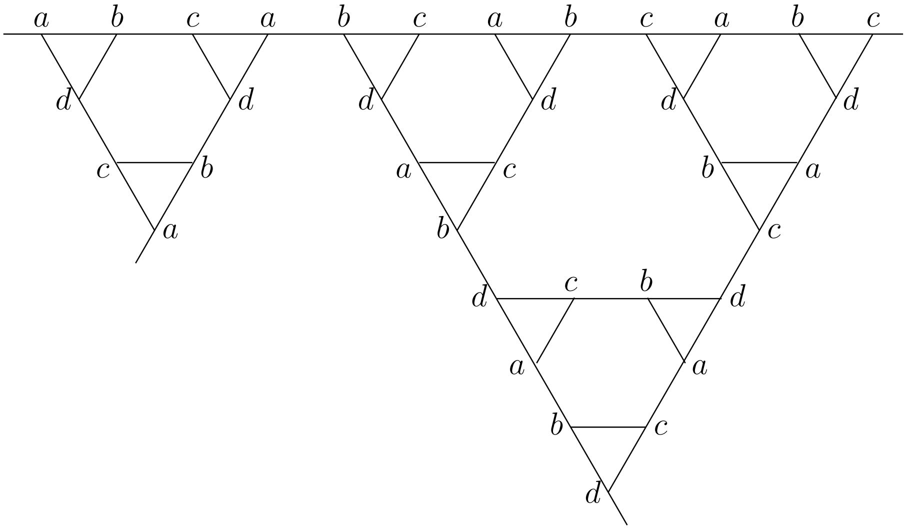

Figure 3: A covering map $\Gamma \to \Gamma_0$

this paper is different and rely on the existence of functional equations on the graph. It not only permits very quick computations of spectra, but also the computation of continuous spectral measures of some remarkable vectors, which is the key point in the proof of spectral decomposition theorems. It probably may be applied to self-similar objects satisfying strong homogeneity hypothesis as in [9], but this generalization seems quite difficult to handle.

This method allows to describe the spectral theory of other operators connected to the graph Γ. Let Γ_0 denote the complete graph with four vertices a, b, c and d. The graph Γ is a covering of Γ_0, as shown by the figure 3. Let us build, for any integer n, a finite graph in the following way: if the graph Γ_n has been built, the graph Γ_{n+1} is the graph which is obtained by replacing each point of Γ_n by a triangle (this process gets more formally detailed in section 2). We still let Δ denote the operator of summation over the neighbors, acting on functions on Γ_n.

Theorem 1.2. *For any nonnegative integers $m \geq n$, there exists covering maps $\Gamma_m \to \Gamma_n$ and $\Gamma \to \Gamma_n$. The characteristic polynomial of Δ in $\ell^2(\Gamma_n)$ is*

$$(X-3)(X+1)^3 \prod_{p=0}^{n-1} (f^p(X)-2)^3 (f^p(X))^{2.3^{n-1-p}} (f^p(X)+2)^{1+2.3^{n-1-p}}.$$

Analogous covering maps to those described by this theorem for the Sierpiński graph have been recently exhibited by Strichartz in [17]. Close computations of characteristic polynomials are made by Grigorchuk and Šunik in [4] and [5].

Let us now focus on the initial motivation of this article, which was the study of a phenomenum in dynamical systems. Let $X \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ be the three dot system, that is the set of families $(p_{k,l})_{(k,l)\in\mathbb{Z}^2}$ of elements of $\mathbb{Z}/2\mathbb{Z}$ such that, for any integers k and l, one has $p_{k,l}+p_{k+1,l}+p_{k,l+1} = 0$ (in $\mathbb{Z}/2\mathbb{Z}$). We equip X with the natural action of $\mathbb{Z}^2$, which is spanned by the maps $T : (x_{k,l}) \mapsto (x_{k+1,l})$ and $S : (x_{k,l}) \mapsto (x_{k,l+1})$. This system, which has been introduced by Ledrappier in [12], is an analogue of the natural extension of the angle doubling and tripling on the circle and, as in Fürstenberg conjecture, the problem of the classification of the Borel probabilities on X that are $\mathbb{Z}^2$-invariant is open. Let Y denote the set of p in X such that $p_{0,0} = 1$. If p is a point of Y, there then exists exactly three elements (k, l) in the set $\{(1, 0), (0, 1), (-1, 1), (-1, 0), (0, -1), (1, -1)\}$ such that T^kS^lx belongs to Y. This relation equips the set Y with a 3-regular graph structure (with multiple edges). If p is a point in Y, its connected component Y_p for this graph structure is exactly the set of points of the orbit of p under the action of $\mathbb{Z}^2$ that belong to Y, that is the equivalence class of p in the equivalence relation induced on Y by the action of $\mathbb{Z}^2$. For any continuous function φ on Y, one sets, for any p in Y,

$$\bar{\Delta}\varphi(p) = \sum_{\substack{(k,l)\in\{(1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1)\} \\ T^kS^lp\in Y}} \varphi(T^kS^lp).$$

If λ is a Borel probability which is invariant by the action of $\mathbb{Z}^2$ on X and such that $\lambda(Y) > 0$ (that is λ is not the Dirac mass at the zero family), the restriction μ of λ to Y satisfies $\bar{\Delta}^*\mu = 3\mu$ and $\bar{\Delta}$ is a self-adjoint operator in $\mathrm{L}^2(Y, \mu)$.

On the origin of this work, we wished to study the homoclinic intersections phenomena in X. Recall that, if ϕ is a diffeomorphism of a compact manifold M and if p is a hyperbolic fixed point of ϕ, a homoclinic intersection is an intersection point q of the stable leaf of p and of its unstable leaf. For such a point, one has $\phi^n(q) \xrightarrow[n\to\pm\infty]{} p$. This notion admits a symbolic dynamics analogue. Let M denote the space $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$, ϕ the shift map and p the point of M all of whose components are zero. If q is an element of M all but a finite number of whose components are zero, one has $\phi^n(q) \xrightarrow[n\to\pm\infty]{} p$. In particular, the point q such that $q_0 = 1$ and all other components are zero possesses this property. In our situation, one checks that there exists a unique element q in X such that one has $q_{0,0} = 1$, for any k in $\mathbb{Z}$ and $l \geq 1$, one has $q_{k,l} = 0$ and, for any $k \geq 1$ and $l \leq -1-k$, one has $q_{k,l} = 0$. There then exists a graph isomorphism from the graph Γ onto Y_q mapping p_0 on q: this is the origin of the planar representation of Γ pictured in figure 1. We shall now identify Γ with Y_q and p_0 with q. Note that, if p is the element of X all of whose components are zero, for any integers $l > k > 0$, one has $(T^{-k}S^l)^n q \xrightarrow[n\to\pm\infty]{} p$. Let $\bar{\Gamma}$ be the closure of Γ in Y: one can see $\bar{\Gamma}$ as a set of pointed planar graphs. Our goal is to determine the structures induced on $\bar{\Gamma}$ by the action of $\mathbb{Z}^2$ on X. For any p in $\bar{\Gamma}$, let Γ_p stand for Y_p. Note that, if Θ_p is the edges graph of Γ_p, the graphs Θ_p are exactly the graphs that are studied by Teplyaev in [18]. In particular, by [18, § 5.4], if Γ_p does not contain p_0 or one of its six images by the natural action of the dihedral group of order 6 on the space X, the spectrum of the operator Δ in $\ell^2(\Gamma_p)$ is discrete.

Every point p in $\bar{\Gamma}$ belongs to a unique triangle in Γ_p. Let a denote the set of elements p such that this triangle is $\{p, T^{-1}p, S^{-1}p\}$, b the one of points p for which it is of the form $\{p, Tp, TS^{-1}p\}$ and c the set of points p for which this triangle is $\{p, Sp, T^{-1}Sp\}$. The set $\bar{\Gamma}$ is the disjoint union of a, b and c. Let us denote by $\theta_1 : \bar{\Gamma} \to \{a, b, c\}$ the natural map associate to this partition. We shall say that a function φ from $\bar{\Gamma}$ into $\mathbb{C}$ is 1-triangular if it factors through θ_1. We let E_1 denote the space of 1-triangular functions φ such that $\varphi(a)+\varphi(b)+\varphi(c) = 0$: it identifies naturally with $\mathbb{C}^3_0 = \{(s,t,u) \in \mathbb{C}^3 | s+t+u = 0\}$. We equip $\mathbb{C}^3_0$ with the scalar product which equals one third of the canonical scalar product.

Let $\zeta : \Lambda \to \mathbb{R}^*_+, x \mapsto \frac{1}{3}\frac{(x+3)(x-1)}{2x-1}$ and, as above, let L_ζ denote the transfer operator associate with ζ for the dynamics of the polynomial f. As $L_\zeta(1) = 1$, there exists a unique Borel probability measure ν_ζ on Λ such that $L^*_\zeta \nu_\zeta = \nu_\zeta$. Then, if j designs the function $\Lambda \to \mathbb{R}, x \mapsto \frac{1}{3}\frac{3-x}{x+3}$, one has $L_\zeta(j) = 1$ and hence $\int_\Lambda j \mathrm{d}\nu_\zeta = 1$.

Theorem 1.3. *For any p in $\bar{\Gamma}$, the set Γ_p is dense in $\bar{\Gamma}$. There exists a unique Borel probability μ on $\bar{\Gamma}$ such that $\bar{\Delta}^*\mu = 3\mu$ and the operator $\bar{\Delta}$ is self-adjoint in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$. The spectrum of the operator $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ is the same as the one of Δ in $\ell^2(\Gamma)$. For any φ in E_1, the spectral measure of φ for $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ is $\|\varphi\|_2^2 j\nu_\zeta$ and the sum of the cyclic spaces spanned by* *the elements of* E_1 *is isometric to* $\mathrm{L}^2\left(j\nu_\zeta, \mathbb{C}_0^3\right)$. *The spectrum of* $\bar{\Delta}$ *in the orthogonal complement of this subspace is discrete and its eigenvalues are* 3, *which is simple, and the elements of* $\bigcup_{n\in\mathbb{N}} f^{-n}(0) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(-2)$.

The organization of the article is as follows.

Sections 2, 3, 4, 5 and 6 are devoted to the study of the graph Γ. In section 2, we precisely construct Γ and we establish some elementary properties of its geometry. In section 3, we determine the spectrum of Δ in $\ell^2(\Gamma)$ and, in section 4, we prove an essential result towards the computation of the spectral measures of the elements of this space. In section 5, we describe the structure of the eigenspaces of Δ in $\ell^2(\Gamma)$. Finally, in section 6, we apply all these preliminary results to the proof of theorem 1.1.

In section 7, we use the technics developed above to prove theorem 1.2.

In sections 8, 9, 10, 11, 12 and 13, we study the space $\bar{\Gamma}$. In section 8, we precisely describe the geometry of the space $\bar{\Gamma}$ and, in section 9, we introduce some remarkable spaces of locally constant functions on this space. Section 10 is devoted to the definition of the operator $\bar{\Delta}$ and to the proof of the uniqueness of its harmonic measure. Section 11 extends to $\bar{\Gamma}$ the properties proved for Γ in sections 3 and 4. In section 12, we study the eigenspaces of $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$. Finally, in section 13, we finish the proof of theorem 1.3.

In section 14, we explain shortly how to transfer our results on the Pascal graph to the Sierpiński graph.

2 Geometric preliminaries

In all the sequel, we shall call graph a set Φ equipped with a symmetric relation $\sim$ such that, for any p in Φ, one does not have $p \sim p$. For p in Φ, we call neighbors of p the set of elements q in Φ such that $p \sim q$. We shall say that Φ is k-regular if all the elements of Φ have the same number k of neighbors. We shall say that Φ is connected if, for any p and q in Φ, there exists a sequence $r_0 = p, \ldots, r_n = q$ of points of Φ such that, for any $1 \leq i \leq n$, one has $r_{i-1} \sim r_i$. We shall call such a sequence a path from p to q and the integer n the length of this path. If Φ is connected and φ is some function on Φ such that, for any points p and q in Φ with $p \sim q$, one has $\varphi(p) = \varphi(q)$, then φ is constant.

We shall say that a subset $\mathcal{T}$ of some graph Φ is a triangle if $\mathcal{T}$ contains exactly three points p, q and r and one has $p \sim q$, $q \sim r$ and $r \sim p$.

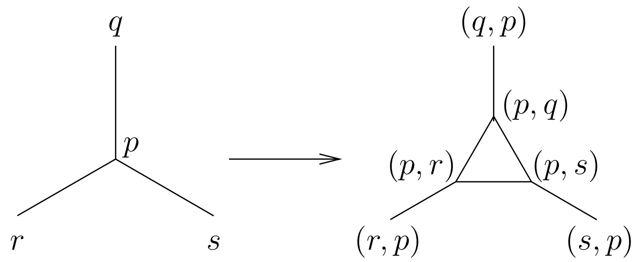

Figure 4: Construction of the graph $\hat{\Phi}$

Let Φ be a 3-regular graph. We let $\hat{\Phi}$ denote the set of ordered pairs (p,q) in Φ with $p \sim q$ equipped with the graph structure for which, if p is a point of Φ, with neighbors q, r and s, the neighbors of (p,q) are (q,p), (p,r) and (p,s). If Φ is connected, $\hat{\Phi}$ is connected. Geometrically, $\hat{\Phi}$ is the graph one obtains by replacing each point of Φ by a triangle. This process is pictured in figure 4. We let Π denote the map $\hat{\Phi} \to \Phi, (p,q) \mapsto p$.

Let $\ell^2(\Phi)$ denote the space of functions $\varphi : \Phi \to \mathbb{C}$ such that $\sum_{p\in\Phi} |\varphi(p)|^2 < \infty$, equipped with its natural structure of a Hilbert space $\langle .,. \rangle$. If Φ is 3-regular, the map Π induces a bounded linear map with norm $\sqrt{3}$, $\Pi^* : \varphi \mapsto \varphi \circ \Pi, \ell^2(\Phi) \to \ell^2\left(\hat{\Phi}\right)$. We still let Π denote the adjoint operator of Π^*: this is the bounded operator $\ell^2\left(\hat{\Phi}\right) \to \ell^2(\Phi)$ which, to some function φ in $\ell^2\left(\hat{\Phi}\right)$, associates the function whose value at some point p in Φ is $\sum_{q\sim p} \varphi(p,q)$. One has $\Pi\Pi^* = 3$.

Let extend the definition of the triangles graph to more general graphs. We shall say that a graph Φ is 3-regular with boundary if every point of Φ has two or three neighbors. In this case, we call the set of points of Φ that have two neighbors the boundary of Φ and we let $\partial\Phi$ denote it. If Φ is a 3-regular graph with boundary, we let $\hat{\Phi}$ denote the set which is the union of $\partial\Phi$ and of the set of ordered pairs (p,q) of elements of Φ with $p \sim q$. We equip $\hat{\Phi}$ with the graph structure for which, if p is a point of $\Phi - \partial\Phi$, with neighbors q, r and s, the neighbors of (p,q) are (q,p), (p,r) and (p,s) and, if p is a point of $\partial\Phi$, with neighbors q and r, the neighbors of p in $\hat{\Phi}$ are (p,q) and (p,r) and the neighbors of (p,q) are (q,p), p and (p,r). Thus, $\hat{\Phi}$ is itself a 3-regular graph with boundary and there exists a natural bijection between the boundary of $\hat{\Phi}$ and the one of Φ (see figure 5 for an example when Φ is a triangle). We still let Π denote the natural map $\hat{\Phi} \to \Phi$ and Π^* and Π the

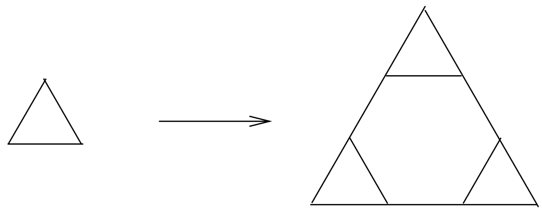

Figure 5: The graph $\hat{\Phi}$ when Φ is a triangle

associate bounded operators $\ell^2(\Phi) \to \ell^2\left(\hat{\Phi}\right)$ and $\ell^2\left(\hat{\Phi}\right) \to \ell^2(\Phi)$.

Lemma 2.1. *Let Φ be a 3-regular graph with boundary. Then, the triangles of $\hat{\Phi}$ are exactly the subsets of the form $\Pi^{-1}(p)$ where p is some point in Φ. In particular, every point of $\hat{\Phi}$ belongs to a unique triangle.*

Proof. If p is a point of Φ, the set $\Pi^{-1}(p)$ is clearly a triangle. Conversely, pick some point p in $\Phi - \partial\Phi$, with neighbors q, r and s. Then, the neighbors of (p, q) are (q, p), (p, r) and (p, s). By definition, as $p \neq q$, the point (q, p) can not be a neighbor of (p, r) or of (p, s). Therefore, the only triangle containing (p, q) is $\Pi^{-1}(p)$. In the same way, if p belongs to $\partial\Phi$, and if the neighbors of p are q and r, as p only has two neighbors in $\hat{\Phi}$, p belongs to only one triangle and, as $p \neq q$, no neighbor of (q, p) is also a neighbor of (p, q) and hence (p, q) belongs to only one triangle. $\square$

If Φ is a graph, we shall say that a bijection $\sigma : \Phi \to \Phi$ is an automorphism of the graph Φ if, for any p and q in Φ with $p \sim q$, one has $\sigma(p) \sim \sigma(q)$. The set of automorphisms of Φ is a subgroup of the permutation group of Φ that is denoted by $\operatorname{Aut}\Phi$. If Φ is 3-regular with boundary and if σ is an automorphism of Φ, one has $\sigma(\partial\Phi) = \partial\Phi$ and there exists a unique automorphism $\hat{\sigma}$ of $\hat{\Phi}$ such that $\Pi\hat{\sigma} = \sigma\Pi$.

Lemma 2.2. *Let Φ be a 3-regular graph with boundary. The map $\sigma \mapsto \hat{\sigma}, \operatorname{Aut}\Phi \to \operatorname{Aut}\hat{\Phi}$ is a group isomorphism.*

Proof. As this map is clearly a monomorphism, it suffices to prove that it is onto. Let thus τ be an automorphism of $\hat{\Phi}$. As τ exchanges triangles of $\hat{\Phi}$, by lemma 2.1, there exists a unique bijection $\sigma : \Phi \to \Phi$ such that $\Pi\hat{\sigma} = \sigma\Pi$. Let p and q be points of Φ such that $p \sim q$. One then as $(p, q) \sim (q, p)$, hence $\tau(p, q) \sim \tau(q, p)$ and, as these points of $\hat{\Phi}$ do not belong to a common triangle,

$\sigma(p) = \Pi\tau(p,q) \sim \Pi\tau(q,p) = \sigma(q)$. Therefore, σ is an automorphism of Φ and $\tau = \hat{\sigma}$, what should be proved. $\square$

We shall now define an important family of 3-regular graphs with boundary. If a, b and c are three distinct elements, we let $\mathcal{T}(a,b,c) = \mathcal{T}_1(a,b,c)$ denote the set $\{a,b,c\}$ equipped with the graph structure for which one has $a \sim b$, $b \sim c$ and $c \sim a$ and one says that $\mathcal{T}(a,b,c)$ is the triangle or the 1-triangle with vertices a, b and c. We consider it as a 3-regular graph with boundary. One then defines by induction a family of 3-regular graphs with boundary by setting, for any $n \geq 1$, $\mathcal{T}_{n+1}(a,b,c) = \widehat{\mathcal{T}_n(a,b,c)}$. For any $n \geq 1$, one calls $\mathcal{T}_n(a,b,c)$ the n-triangle with vertices a, b and c.

One let $\mathfrak{S}(a,b,c)$ denote the permutation group of the set $\{a,b,c\}$. By definition and by lemma 2.2, one has the following

Lemma 2.3. *Let a, b and c be three distinct elements. Then, for any $n \geq 1$, $\mathcal{T}_n(a,b,c)$ is a connected 3-regular graph with boundary and $\partial\mathcal{T}_n(a,b,c) = \{a,b,c\}$. The map that sends an automorphism of $\mathcal{T}_n(a,b,c)$ to its restriction to $\{a,b,c\}$ induces a group isomorphism from $\mathrm{Aut}\,\mathcal{T}_n(a,b,c)$ onto $\mathfrak{S}(a,b,c)$.*

If Φ is a graph and $n \geq 1$ an integer, we shall say that a subset $\mathcal{T}$ of Φ is a n-triangle if there exists points p, q and r in $\mathcal{T}$ such that the subset $\mathcal{T}$, endowed with the restriction of the relation $\sim$, is isomorphic to the graph $\mathcal{T}_n(p,q,r)$. By abuse of language, we shall call 0-triangles the points of Φ.

Let Φ be a 3-regular graph with boundary. Set $\hat{\Phi}^{(0)} = \Phi$, $\hat{\Phi}^{(1)} = \hat{\Phi}$ and, for any nonnegative integer n, $\hat{\Phi}^{(n+1)} = \widehat{\hat{\Phi}^{(n)}}$. By induction, for any nonnegative integer n, an automorphism σ of Φ induces a unique automorphism $\hat{\sigma}^{(n)}$ of $\hat{\Phi}^{(n)}$ such that $\Pi^n\hat{\sigma}^{(n)} = \sigma\Pi^n$.

By lemmas 2.1 and 2.2, one immediately deduces, by induction, the following

Lemma 2.4. *Let Φ be a 3-regular graph with boundary and n be a nonnegative integer. The n-triangles of $\hat{\Phi}^{(n)}$ are exactly the subsets of the form $\Pi^{-n}(p)$ where p is a point of Φ. In particular, every point of $\hat{\Phi}^{(n)}$ belongs to a unique n-triangle. The map $\sigma \mapsto \sigma^{(n)}, \mathrm{Aut}\,\Phi \to \mathrm{Aut}\,\hat{\Phi}^{(n)}$ is a group isomorphism.*

Corollary 2.5. *Let Φ be a 3-regular graph with boundary, $n \geq m \geq 1$ be integers, p be a point of $\hat{\Phi}^{(n)}$, $\mathcal{T}$ be the n-triangle containing p and $\mathcal{S}$ be the m-triangle containing p. One has $\mathcal{S} \subset \mathcal{T}$. If p is a vertex of $\mathcal{T}$ and if p does* *not belong to $\partial\hat{\Phi}^{(n)}$, the unique neighbor of p in $\hat{\Phi}^{(n)}$ that does not belong to the 1-triangle containing p is itself the vertex of some n-triangle of $\hat{\Phi}^{(n)}$.*

Proof. By lemma 2.4, one has $\mathcal{T} = \Pi^{-n}(\Pi^n p)$ and $\mathcal{S} = \Pi^{-m}(\Pi^m p)$, hence $\mathcal{S} \subset \mathcal{T}$. Suppose p belongs to $\partial\mathcal{T} - \partial\hat{\Phi}^{(n)}$. Then, p has a unique neighbor in $\hat{\Phi}^{(n)}$ that does not belong to $\mathcal{T}$. Let q be a neighbor of p and $\mathcal{R}$ be the n-triangle containing q. If q is not a vertex of $\mathcal{R}$, all the neighbors of q belong to $\mathcal{R}$ and hence $p \in \mathcal{R}$. As, by lemma 2.4, p belongs to a unique n-triangle of $\hat{\Phi}^{(n)}$, one has $\mathcal{T} = \mathcal{R}$ and hence q belongs to $\mathcal{T}$. Therefore, the neighbor of p that does not belong to $\mathcal{T}$ is a vertex of the n-triangle which it belongs to. $\square$

Corollary 2.6. *Let $n \geq 2$ be an integer and a, b and c be three distinct elements. Then, there exists unique elements ab, ba, ac, ca, bc and cb of $\mathcal{T}_n(a,b,c)$ such that $\mathcal{T}_n(a,b,c)$ is the union of the three $(n-1)$-triangles $\mathcal{T}_{n-1}(a,ab,ac)$, $\mathcal{T}_{n-1}(b,ba,bc)$ and $\mathcal{T}_{n-1}(c,ca,cb)$ and that one has $ab \sim ba$, $ac \sim ca$ and $bc \sim cb$.*

Proof. The corollary may be proven directly for $n = 2$. This case implies the general one by lemma 2.4. $\square$

Pick now an element a and two sequences of distinct elements $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ such that, for any $n \geq 1$, one has $b_n \neq a$, $c_n \neq a$ and $b_n \neq c_n$. By corollary 2.6, for any $n \geq 1$, one can identify $\mathcal{T}_n(a,b_n,c_n)$ to a subset of $\mathcal{T}_{n+1}(a,b_{n+1},c_{n+1})$ thanks to the unique graph isomorphism sending a to a, b_n to ab_{n+1} and c_n to ac_{n+1}. One then calls the set $\bigcup_{n\geq 1}\mathcal{T}_n(a,b_n,c_n)$, equipped with the graph structure that induces its n-triangle structure on each $\mathcal{T}_n(a,b_n,c_n)$, $n \geq 1$, the infinite triangle with vertex a and one let it be denoted by $\mathcal{T}_\infty(a)$. From the preceding results one deduces the following

Lemma 2.7. *Let a be an element. The graph $\mathcal{T}_\infty(a)$ is connected, 3-regular with boundary and $\partial\mathcal{T}_\infty(a) = \{a\}$. If b and c are the two neighbors of a in $\mathcal{T}_\infty(a)$, there exists a unique isomorphism from $\mathcal{T}_\infty(a)$ onto $\widehat{\mathcal{T}_\infty(a)}$ that sends a to (a,b) and c to (a,c). For any nonnegative integer n, this isomorphism induces a natural bijection between the points of $\mathcal{T}_\infty(a)$ and the n-triangles of $\mathcal{T}_\infty(a)$ and every point of $\mathcal{T}_\infty(a)$ belongs to a unique n-triangle. Finally, $\mathcal{T}_\infty(a)$ admits a unique non trivial automorphism; this automorphism is an involution that fixes a and that, for any $n \geq 1$, exchanges the two vertices of the n-triangle containing a that are different from a.*

In all this article, we fix two distincts elements p_0 and $p_0^\vee$. One calls Pascal graph the set $\mathcal{T}_\infty(p_0) \cup \mathcal{T}_\infty(p_0^\vee)$ endowed with the graph structure that induces the infinite triangle structure on $\mathcal{T}_\infty(p_0)$ and $\mathcal{T}_\infty(p_0^\vee)$ and for which $p_0 \sim p_0^\vee$. We let Γ denote the Pascal graph. From lemma 2.7, one deduces the following

Proposition 2.8. *The Pascal graph is an infinite, connected and 3-regular graph. If q_0 and r_0 are the two neighbors of p_0 in $\mathcal{T}_\infty(p_0)$ and $q_0^\vee$ and $r_0^\vee$ the two neighbors of $p_0^\vee$ in $\mathcal{T}_\infty(p_0^\vee)$, there exists a unique isomorphism from Γ onto $\hat{\Gamma}$ that sends p_0 to $(p_0, p_0^\vee)$, $p_0^\vee$ to $(p_0^\vee, p_0)$, q_0 to (p_0, q_0), r_0 to (p_0, r_0), $q_0^\vee$ to $(p_0^\vee, q_0^\vee)$ and $r_0^\vee$ to $(p_0^\vee, r_0^\vee)$. For any nonnegative integer n, this isomorphism induces a natural bijection between the points of Γ and the n-triangles of Γ and every point of Γ belongs to a unique n-triangle.*

A planar representation of the Pascal graph is given in figure 1.

One shall now identify Γ and $\hat{\Gamma}$ by the isomorphism described in proposition 2.8. In particular, from now on, one shall consider Π and Π^* as bounded endomorphisms in $\ell^2(\Gamma)$.

Let Θ denote the edges graph of Γ. More precisely, Θ is the set of non ordered pairs $\{p, q\}$ of elements of Γ with $p \sim q$, endowed with the relation for which, if p and q are two neighbor points in Γ, if r and s are the two other neighbors of p and t and u the two other neighbors of q, the neighbors of $\{p, q\}$ are $\{p, r\}$, $\{p, s\}$, $\{q, t\}$ and $\{q, u\}$. We call Θ the Sierpiński graph. It is an infinite, connected and 4-regular graph. A planar representation of it is given in figure 2.

If Φ is a k-regular graph, for any function φ from Φ into $\mathbb{C}$, one let $\Delta\varphi$ denote the function $p \mapsto \sum_{q \sim p} \varphi(q)$. Then, Δ induces a bounded self-adjoint operator of the space $\ell^2(\Phi)$ with norm $\leq k$. We call the spectrum of this operator the spectrum of Φ.

3 The spectrum of Γ

Let Φ be a 3-regular graph. In this section, we shall study the link between the spectral properties of Φ and those of $\hat{\Phi}$. Our study is based on the

Lemma 3.1. *One has $(\Delta^2 - \Delta - 3)\Pi^* = \Pi^*\Delta$ and $\Pi(\Delta^2 - \Delta - 3) = \Delta\Pi$.*

Proof. Let φ be a function on Φ, p be a point of Φ and q, r, s be the three neighbors of p. Suppose $\varphi(p) = a$, $\varphi(q) = b$, $\varphi(r) = c$ and $\varphi(s) = d$. Then, one has $\Pi^*\varphi(p,q) = a$, $\Delta\Pi^*\varphi(p,q) = 2a+b$ and $\Delta^2\Pi^*\varphi(p,q) = (2b+a) + (2a+c) + (2a+d) = 5a+2b+c+d$. We thus have $(\Delta^2 - \Delta - 3)\Pi^*\varphi(p,q) = b+c+d = \Pi^*\Delta\varphi(p,q)$. The second relation is obtained by switching to adjoint operators in the first one. $\square$

We shall now use lemma 3.1 to determine the spectrum of Δ in $\ell^2\left(\hat{\Phi}\right)$. We shall use elementary results from functional analysis.

Lemma 3.2. *Let E be a Banach space and T be a bounded linear operator in E. Suppose all elements of the spectrum of T have positive real part. Then, if $F \subset E$ is a subspace which is stable by T^2, F is stable by T.*

Proof. Let $0 < \alpha < \beta$ and $\gamma > 0$ be such that the spectrum S of T is contained in the interior of the rectangle $R = [\alpha,\beta] + [-\gamma,\gamma]i$ and $U \supset R$ and V be open subsets of $\mathbb{C}$ such that the map $\lambda \mapsto \lambda^2$ induces a biholomorphism from U onto V. There exists a holomorphic function r on V such that, for any λ in U, one has $r(\lambda^2) = \lambda$. As R is simply connected, by Runge theorem, there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of polynomials in $\mathbb{C}[X]$ that converges to r on R^2. As the spectrum of T^2 is S^2, which is contained in the interior of R^2, the sequence $r_n(T^2)$ then converges to T in the space of endomorphisms of E. For any integer n, $r_n(T^2)$ stabilizes F, hence T stabilizes F. $\square$

Lemma 3.3. *Let H be a Hilbert space, T be a bounded self-adjoint endomorphism and π be a real polynomial with degree 2. Suppose there exists a closed subspace K of H such that $\pi(T)K \subset K$ and K and TK span H. Then, the image by π of the spectrum of T in H equals the spectrum of $\pi(T)$ in K and, if one moreover has $T^{-1}K \cap K = \{0\}$, the spectrum of T in H is exactly the set of λ in $\mathbb{R}$ such that $\pi(\lambda)$ belongs to the spectrum of $\pi(T)$ in K.*

Proof. Once π has been written under its canonical form, one can suppose $\pi(X) = X^2$. Let E denote the spectral resolution of T: for any Borel subset B of $\mathbb{R}$, $E(B)$ is a projection of H that commutes with T. Let B be a Borel subset of $\mathbb{R}$ such that $B = -B$. Then, for any Radon measure μ on $\mathbb{R}$, in $\mathrm{L}^2(\mu)$, the indicator function of B is the limit of a sequence of even polynomials. One hence has $E(B)K \subset K$.

The spectrum of T^2 in H is exactly the set of squares of elements of the spectrum of T in H. As T^2 is self-adjoint and K is stable by T^2, the spectrum of T^2 in K is contained in its spectrum in H, and hence in the set of squares of elements of the spectrum of T. Conversely, suppose there exists elements of the spectrum of T whose square does not belong to the spectrum of T^2 in K. Then, there exists a symmetric open subset V of $\mathbb{R}$ such that V contains elements of the spectrum of T in H but that V^2 does not contain elements of the one of T^2 in K. One has $E(V)K \subset K$, but, as V^2 does not contain elements of the spectrum of T^2 in K, $E(V)K = 0$. Now, as K and TK span H, $E(V)K$ and $TE(V)K = E(V)TK$ span $E(V)H$. Hence, one has $E(V)H = 0$, which contradicts the fact that V contains spectral values of T. Therefore, the spectrum of T^2 in K is exactly the set of squares of elements of the spectrum of T in H.

Suppose now one has $T^{-1}K \cap K = \{0\}$. To conclude, it remains to prove that the spectrum of T is symmetric. Suppose this is not the case. Then, one can, after eventually having replaced T by $-T$, find real numbers $0 < \alpha < \beta$ such that $U =]\alpha, \beta[$ contains elements of the spectrum of T but that $-U$ does not contain any. But one then has $E(U) = E(U \cup (-U))$ and hence $E(U)K \subset K$. If L is the image of H by $E(U)$, one has therefore $T^2(K \cap L) \subset K \cap L$. As the spectrum of the restriction of T to L is contained in $\mathbb{R}_+^*$, one has, by lemma 3.2, $T(K \cap L) \subset K \cap L$ and hence, by the hypothesis, $K \cap L = 0$. As $E(U)K \subset K$, one thus has $E(U) = 0$ on K. As K and TK span H, one has $E(U) = 0$, which contradicts the fact that U contains elements of the spectrum of T. Therefore, the spectrum of T is symmetric. The lemma follows. $\square$

In order to apply these results to spaces of square integrable functions on graphs, we will need results on the geometry of graphs. Let Φ be a connected graph and P and Q be two disjoint subsets of Φ such that $\Phi = P \cup Q$. We shall say that Φ is split by the partition $\{P, Q\}$ if any neighbor of an element of P belongs to Q and any neighbor of an element of Q belongs to P. We shall say that Φ is splitable (or bipartite) if there exists a partition of Φ into two subsets that splits it. One easily checks that Φ is splitable if and only if, for any p and q in Φ, the paths joining p to q either all have even length or all have odd length. In particular, if Φ is splitable, the partition $\{P, Q\}$ that split Φ is unique, two points p and q belonging to the same atom if and only if they may be joined by a path with even length.

Lemma 3.4. *Let Φ be a connected graph and L be the space of functions φ on Φ such that, for any p in Φ, φ is constant on the neighbors of p. Then, if Φ is not splitable, L equals the space of constant functions. If Φ is split by the partition $\{P, Q\}$, L is spanned by constant functions and by the function $1_P - 1_Q$.*

Proof. Let φ be in L, p and q be points of Φ and $r_0 = p, r_1, \ldots, r_n = q$ a path from p to q. For any $1 \leq i \leq n-1$, one has $r_{i-1} \sim r_i \sim r_{i+1}$, hence $\varphi(r_{i-1}) = \varphi(r_{i+1})$ and, if n is even, $\varphi(p) = \varphi(q)$. Therefore, if $\varphi(p) \neq \varphi(q)$ and if $P = \{r \in \Phi | \varphi(r) = \varphi(p)\}$ and $Q = \{r \in \Phi | \varphi(r) = \varphi(q)\}$, the partition $\{P, Q\}$ splits Φ. The lemma easily follows. $\square$

We shall use this lemma in the following setting:

Lemma 3.5. *Let Φ be a connected 3-regular graph. Let φ be in $\ell^2(\Phi)$ such that $\Delta\varphi = 3\varphi$. If Φ is infinite, one has $\varphi = 0$ and, if Φ is finite, φ is constant. Let ψ be in $\ell^2(\Phi)$ such that $\Delta\psi = -3\psi$. If Φ is infinite or non splitable, one has $\psi = 0$ and, if Φ is finite and split by the partition $\{P, Q\}$, ψ is proportional to $1_P - 1_Q$.*

Proof. As φ is in $\ell^2(\Phi)$, the set $M = \{p \in \Phi | \varphi(p) = \max_\Phi \varphi\}$ is not empty. As $\Delta\varphi = 3\varphi$, for any p in M, the neighbors of p all belong to M and, hence, as Φ is connected, $M = \Phi$ and φ is constant. If Φ is infinite, as φ is in $\ell^2(\Phi)$, it is zero. In the same way, suppose $\psi \neq 0$ and set $P = \{p \in \Phi | \psi(p) = \max_\Phi \psi\}$ and $Q = \{q \in \Phi | \psi(q) = \min_\Phi \psi\}$. As $\Delta\psi = -3\psi$, one has $\min_\Phi \psi = -\max_\Phi \psi$ and the neighbors of the points of P belong to Q, whereas the neighbors of the points of Q belong to P. By connectedness, one has $P \cup Q = \Phi$, the graph Φ is splitable and ψ is proportional to $1_P - 1_Q$. Finally, as ψ is in $\ell^2(\Phi)$, the graph Φ is finite. $\square$

Recall we let f denote the polynomial $x^2 - x - 3$. From lemma 3.1, we deduce the following

Corollary 3.6. *Let Φ be a connected 3-regular graph and H be the closed subspace of $\ell^2\left(\hat{\Phi}\right)$ spanned by the image of Π^* and by the image of $\Delta\Pi^*$. Then H is stable by Δ and the spectrum of the restriction of Δ to H is,*

(i) if Φ is infinite, the inverse image by f of the spectrum of Δ in $\ell^2(\Phi)$.

(ii) if Φ is finite, but non splitable, the inverse image by f of the spectrum of Δ in $\ell^2(\Phi)$ deprived from -2.

(iii) if Φ is finite and splitable, the inverse image by f of the spectrum of Δ in $\ell^2(\Phi)$ deprived from -2 and 0.

Proof. Let K denote the image of Π^*. As $\frac{1}{\sqrt{3}}\Pi^*$ induces an isometry from $\ell^2(\Phi)$ onto K, by lemma 3.1, the spectrum of $f(\Delta)$ in K equals the spectrum of Δ in $\ell^2(\Phi)$. We will apply lemma 3.3 to the space H and the operator Δ. On this purpose, let us study the space $\Delta^{-1}K \cap K$. Let L be the space of φ in $\ell^2(\Phi)$ such that $\Delta\Pi^*\varphi$ belongs to K and let φ be in L. If p is a point of φ, with neighbors q, r and s, set $\varphi(p) = a$, $\varphi(q) = b$, $\varphi(r) = c$ and $\varphi(s) = d$. Then, one has $\Delta\Pi^*\varphi(p,q) = 2a + b$, $\Delta\Pi^*\varphi(p,r) = 2a + c$ and $\Delta\Pi^*\varphi(p,s) = 2a+d$. As $\Delta\Pi^*\varphi$ belongs to K, one has $b = c = d$. Conversely, if φ is an element of $\ell^2(\Phi)$ that, for any point p in Φ, is constant on the neighbors of p, φ belongs to L.

If Φ is infinite, by lemma 3.4, one has $L = \{0\}$ and one can apply lemma 3.3 to H. The spectrum of Δ in H is therefore the inverse image by f of the one of Δ in $\ell^2(\Phi)$. If Φ is finite, but non splitable, by lemma 3.4, L is the line of constant functions on Φ and we can apply lemma 3.3 to the orthogonal complement of the space of constant functions in H. We get the result, since $f(-2) = f(3) = 3$ and, by lemma 3.5, the constant functions are the only eigenfunctions with eigenvalue 3 in $\ell^2(\Phi)$. Finally, if Φ is finite and split by the partition $\{P, Q\}$, by lemma 3.4, L is spanned by constant functions and by the function $1_P - 1_Q$. Then, $\Pi^*(1_P - 1_Q)$ is an eigenvector with eigenvalue 1 in H. One applies lemma 3.3 to the orthogonal complement of the subspace of H spanned by the constant functions and by $\Pi^*(1_P - 1_Q)$. The result follows as $f(0) = f(1) = -3$ and, still by lemma 3.5, the eigenfunctions with eigenvalue -3 in $\ell^2(\Phi)$ are the multiples of $1_P - 1_Q$. □

We now have to determine the spectrum of Δ in the orthogonal complement of H. This is the aim of the following

Lemma 3.7. *Let Φ be a connected 3-regular graph and H be the closed subspace of $\ell^2\left(\hat{\Phi}\right)$ spanned by the image of Π^* and by the image of $\Delta\Pi^*$. The spectrum of Δ in the orthogonal complement of H is contained in $\{0, -2\}$. The eigenspace associate to the value 0 in $\ell^2\left(\hat{\Phi}\right)$ is the space of functions φ in $\ell^2\left(\hat{\Phi}\right)$ such that $\Pi\varphi = 0$ and that, for any p and q which are neighbors in Φ, one has $\varphi(p,q) = \varphi(q,p)$. The eigenspace associate to the value -2 in $\ell^2\left(\hat{\Phi}\right)$ is the space of functions φ in $\ell^2\left(\hat{\Phi}\right)$ such that $\Pi\varphi = 0$ and that, for any p and q which are neighbors in Φ, one has $\varphi(p,q) = -\varphi(q,p)$.*

Proof. Let φ be orthogonal to H and let p be a point of Φ, with neighbors

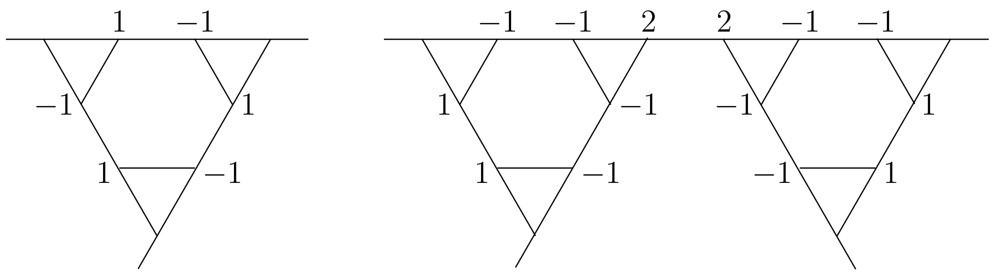

Figure 6: Eigenfunctions on $\hat{\Phi}^{(2)}$

q, r, s. Set $a = \varphi(p,q)$, $b = \varphi(p,r)$, $c = \varphi(q,p)$ and $d = \varphi(r,p)$. Finally, denote by ψ the indicator function of the set $\{p\}$ in Φ. As φ is orthogonal to $\Pi^*\psi$ and to $\Delta\Pi^*\psi$, one has $\varphi(p,s) = -a-b$ and $\varphi(s,p) = -c-d$. Thus $\Delta\varphi(p,q) = c-a$ and $\Delta^2\varphi(p,q) = (a-c)+(d-b)+(-c-d+a+b) = 2a-2c$. Hence, in the orthogonal complement of H, one has $\Delta^2 + 2\Delta = 0$ and, for φ in this subspace, one has $\Delta\varphi = 0$ if and only if, for any p an q which are neighbors in Φ, $\varphi(p,q) = \varphi(q,p)$ and $\Delta\varphi = -2\varphi$ if and only if, for any p an q which are neighbors in Φ, $\varphi(p,q) = -\varphi(q,p)$.

To finish the proof of the lemma, we have to prove that Δ does not have any eigenfunction with eigenvalue 0 or -2 in H. On this purpose, pick some φ in H such that $\Delta\varphi = -2\varphi$. By lemma 3.1, one then has $\Delta\Pi\varphi = \Pi(\Delta^2 - \Delta - 3)\varphi = 3\Pi\varphi$. If Φ is infinite, by lemma 3.5, $\Pi\varphi$ is zero. We thus have $\Pi\varphi = 0$ and $\Pi\Delta\varphi = -2\Pi\varphi = 0$. Therefore, as φ belongs to H which is spanned by the images of Π^* and of $\Delta\Pi^*$, one has $\varphi = 0$. If Φ is finite, still by lemma 3.5, $\Pi\varphi$ is constant. As φ is orthogonal to constant functions, one again has $\Pi\varphi = 0$ and $\Pi\Delta\varphi = 0$, which implies $\varphi = 0$.

If φ is now an element of H such that $\Delta\varphi = 0$, one has $\Delta\Pi\varphi = -3\Pi\varphi$. Again, by lemma 3.5, if Φ is infinite or non splitable, one has $\Pi\varphi = 0$ and, hence, $\varphi = 0$, whereas, if Φ is split by the partition $\{P, Q\}$, $\Pi\varphi$ is proportional to $1_P - 1_Q$. But $\Pi^*(1_P - 1_Q)$ is an eigenvector with eigenvalue 1 for Δ, hence $\langle \Pi\varphi, 1_P - 1_Q \rangle = \langle \varphi, \Pi^*(1_P - 1_Q) \rangle = 0$ and $\Pi\varphi = 0$, so that $\varphi = 0$. $\square$

Recall that, for any nonnegative integer, we let $\hat{\Phi}^{(n)}$ denote the graph obtained by replacing each point of Φ by a n-triangle. The space $\ell^2\left(\hat{\Phi}^{(2)}\right)$ contains finitely supported eigenfunctions with eigenvalue -2 and 0, as shown by figure 6, where only the non zero values of the functions have been represented.

We therefore have the following

Lemma 3.8. *For any $n \geq 2$, the space $\ell^2\left(\hat{\Phi}^{(n)}\right)$ contains eigenfunctions with eigenvalue -2 and 0.*

Recall we let Λ denote the Julia set of f. By applying corollary 3.6 and lemmas 3.7 and 3.8 to Γ, one gets the following

Corollary 3.9. *The spectrum of Γ is the union of Λ and of the set $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$.*

Proof. By proposition 2.8, $\hat{\Gamma}$ is isomorphic to Γ. Therefore, by corollary 3.6 and lemmas 3.7 and 3.8, the spectrum S of Δ satisfies $S = f^{-1}S \cup \{0, -2\}$. One easily checks that the set described in the setting of the corollary is the unique compact subset of $\mathbb{R}$ verifying this equation. □

4 The spectral measures of Γ

Let still Φ be a connected 3-regular graph. In this section, we will explain how to compute the spectral measures of some elements of $\ell^2\left(\hat{\Phi}\right)$. On this purpose, we use the following

Lemma 4.1. *One has $\Pi\Delta\Pi^* = 6 + \Delta$ and hence, for any φ and ψ in $\ell^2(\Phi)$,*

$$\langle \Delta\Pi^*\varphi, \Pi^*\psi\rangle = 6\langle\varphi,\psi\rangle + \langle\Delta\varphi,\psi\rangle = 2\langle\Pi^*\varphi,\Pi^*\psi\rangle + \frac{1}{3}\langle(\Delta^2 - \Delta - 3)\Pi^*\varphi, \Pi^*\psi\rangle.$$

Proof. Let φ be in $\ell^2(\Phi)$ and p be a point of Φ, with neighbors q, r, s. Set $a = \varphi(p)$, $b = \varphi(q)$, $c = \varphi(r)$ and $d = \varphi(s)$. Then, one has $\Delta\Pi^*\varphi(p,q) = 2a + b$ and hence $\Pi\Delta\Pi^*\varphi(p) = (2a+b)+(2a+c)+(2a+d) = 6a+(b+c+d)$, whence the first identity. The second one follows, by applying lemma 3.1 and the relation $\Pi\Pi^* = 3$. □

Let us now study the abstract consequences of this kind of identity.

Lemma 4.2. *Let H be a Hilbert space, T a bounded self-adjoint endomorphism of H, K a closed subspace of H and $\pi(x) = (x-u)^2 + m$ a real unitary polynomial with degree 2. Suppose one has $\pi(T)K \subset K$, K and TK span H and there exists real numbers a and b such that, for any v and w in K, one has $\langle Tv, w\rangle = a\langle v, w\rangle + b\langle\pi(T)v, w\rangle$. Then, for any $x \neq u$ in the spectrum of T, one has*

$$1 + \frac{a - u + b\pi(x)}{x - u} \geq 0.$$

Proof. Once π has been written under its canonical form, one may suppose one has $\pi(x) = x^2$. Let v be a unitary vector in K. Then, for any real number s, one has

$$\begin{aligned} 0 \leq \langle Tv + sv, Tv + sv\rangle &= \langle Tv, Tv\rangle + 2s\langle Tv, v\rangle + s^2 \\ &= \langle T^2 v, v\rangle + 2s(a + b\langle T^2 v, v\rangle) + s^2. \end{aligned}$$

By lemma 3.3, the squares of the elements of the spectrum of T in H belong to the spectrum of T^2 in K. If x is an element of the spectrum of T, there exists therefore unitary vectors v of K such that $\langle T^2 v, v\rangle$ is very close to x^2. Hence, by the remark above, for any real number s, one has $x^2 + 2s(a+bx^2) + s^2 \geq 0$. The discriminant of this polynomial of degree 2 is thus nonpositive, that is one has $x^2 - (a + bx^2)^2 \geq 0$. The lemma follows. $\square$

Let $\pi(x) = (x - u)^2 + m$ a real unitary polynomial with degree 2. Pick a Borel function θ on $\mathbb{R} - \{u\}$. Then, if α is a Borel function on $\mathbb{R} - \{u\}$, one sets, for any y in $]m, \infty[$, $L_{\pi,\theta}\alpha(y) = \sum_{\pi(x)=y} \theta(x)\alpha(x)$. Let μ be a Borel positive measure on $]m, \infty[$. If, for μ-almost all y in $]m, \infty[$, θ is nonnegative on the two inverse images of y by π, one let $L^*_{\pi,\theta}\mu$ denote the Borel measure ν on $\mathbb{R} - \{u\}$ such that, for any nonnegative Borel function α on $\mathbb{R} - \{u\}$, one has $\int_{\mathbb{R}-\{u\}} \alpha \mathrm{d}\nu = \int_{]m,\infty[} L_{\pi,\theta}\alpha \mathrm{d}\mu$.

We have the following

Lemma 4.3. *Let H be a Hilbert space, T a bounded self-adjoint endomorphism of H, K a closed subspace of H and $\pi(x) = (x-u)^2 + m$ a real unitary polynomial with degree 2. Suppose one has $\pi(T)K \subset K$, K and TK span H and there exists real numbers a and b such that, for any v and w in K, one has $\langle Tv, w\rangle = a\langle v, w\rangle + b\langle \pi(T)v, w\rangle$. Then, for any v in K, if μ is the spectral measure of v for $\pi(T)$ and ν its spectral measure for T, if $\mu(m) = 0$, one has $\nu(u) = 0$ and $\nu = L^*_{\pi,\theta}\mu$ where, for any $x \neq u$, one has*

$$\theta(x) = \frac{1}{2}\left(1 + \frac{a - u + b\pi(x)}{x - u}\right).$$

Proof. Note that, as $\mu(m) = 0$, by lemma 3.3, the measure μ is concentrated on $]m, \infty[$. Moreover, if w is some vector on H with $Tw = uw$, one has $\pi(T)w = mw$ and, by the hypothesis, $\langle v, w\rangle = 0$. Thus, one has $\nu(u) = 0$.

By lemma 4.2, the function θ is nonnegative on the spectrum of T, deprived from $\{0\}$. Let n be in $\mathbb{N}$. On one hand, one has

$$\int_{\mathbb{R}-\{u\}} \pi(x)^n \mathrm{d}\nu(x) = \langle \pi(T)^n v, v\rangle = \int_{]m,\infty[} y^n \mathrm{d}\mu(y).$$

On the other hand, for any $x \neq u$, one has $\theta(x) + \theta(2u - x) = 1$ and hence, for any y in $]m, \infty[$, $L_{\pi,\theta}\pi^n(y) = y^n$. Thus, one has $\int_{\mathbb{R}-\{u\}} \pi^n \mathrm{d}\nu = \int_{]m,\infty[} L_{\pi,\theta}\pi^n \mathrm{d}\mu$. In the same way, for any x in $\mathbb{R}$, set $\alpha(x) = x\pi(x)^n$. On one hand, one then has

$$\begin{aligned}
&\int_{\mathbb{R}-\{u\}} \alpha(x)\mathrm{d}\nu(x) = \langle T\pi(T)^n v, v\rangle \\
&= a\langle \pi(T)^n v, v\rangle + b\langle \pi(T)^{n+1} v, w\rangle = a\int_{]m,\infty[} y^n \mathrm{d}\mu(y) + b\int_{]m,\infty[} y^{n+1}\mathrm{d}\mu(y).
\end{aligned}$$

On the other hand, for any $x \neq u$, as $(2u - x) - u = u - x$, one has

$$\begin{aligned}
&\theta(x)\alpha(x) + \theta(2u - x)\alpha(2u - x) \\
&\quad = \left(\frac{1}{2}(x + (2u - x)) + \frac{1}{2}(x - (2u - x))\left(\frac{a - u + b\pi(x)}{x - u}\right)\right)\pi(x)^n \\
&\quad = a\pi(x)^n + b\pi(x)^{n+1}
\end{aligned}$$

and hence, for any y in $]m, \infty[$, $L_{\pi,\theta}\alpha(y) = ay^n + by^{n+1}$. Again, one has $\int_{\mathbb{R}-\{u\}} \alpha \mathrm{d}\nu = \int_{]m,\infty[} L_{\pi,\theta}\alpha \mathrm{d}\mu$. Therefore, for any polynomial α, one has $\int_{\mathbb{R}-\{u\}} \alpha \mathrm{d}\nu = \int_{]m,\infty[} L_{\pi,\theta}\alpha \mathrm{d}\mu$. In particular, the positive measure $L^*_{\pi,\theta}\mu$ is finite and hence, for any compactly supported continuous function α on $\mathbb{R} - \{u\}$, one still has $\int_{\mathbb{R}-\{u\}} \alpha \mathrm{d}\nu = \int_{]m,\infty[} L_{\pi,\theta}\alpha \mathrm{d}\mu$, so that $\nu = L^*_{\pi,\theta}\mu$. $\square$

By applying lemmas 4.1 and 4.3, one gets the following

Corollary 4.4. *Let φ be in $\ell^2(\Phi)$, μ be the spectral measure of φ for Δ in $\ell^2(\Phi)$ and ν be the spectral measure of $\Pi^*\varphi$ for Δ in $\ell^2\left(\hat{\Phi}\right)$. Then, one has $\nu\left(\frac{1}{2}\right) = 0$ and, if, for any $x \neq \frac{1}{2}$, one sets $\theta(x) = \frac{x(x+2)}{2x-1}$, one has $\nu = L^*_{f,\theta}\mu$.*

Proof. The minimal value on $\mathbb{R}$ of the polynomial f is $f\left(\frac{1}{2}\right) = -\frac{13}{4} < -3 \leq -\|\Delta\|_2$. Thus, one has $\mu\left(-\frac{13}{4}\right) = 0$ and the corollary follows from lemmas 4.1 and 4.3 by an elementary computation. $\square$

5 Eigenfunctions in $\ell^2(\Gamma)$

In this section, we shall complete the informations given by lemma 3.7 by describing more precisely the eigenspaces of Δ in $\ell^2(\Gamma)$ for the eigenvalues -2 and 0. We shall extend these results to the eigenvalues in $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$ and $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$ using the following

Lemma 5.1. *Let Φ be a 3-regular graph and H be the closed subspace of $\ell^2\left(\hat{\Phi}\right)$ spanned by the image of Π^* and by the one of $\Delta\Pi^*$. Then, for any x in $\mathbb{R} - \{0, -2\}$, x is an eigenvalue of Δ in H if and only if $y = f(x)$ is an eigenvalue of Δ in $\ell^2(\Phi)$. In this case, the map R_x which sends an eigenfunction φ with eigenvalue y in $\ell^2(\Phi)$ to $(x-1)\Pi^*\varphi + \Delta\Pi^*\varphi$ induces an isomorphism between the eigenspace associate to the eigenvalue y in $\ell^2(\Phi)$ and the eigenspace associate to the eigenvalue x in H and, for any φ, one has $\|R_x\varphi\|_2^2 = x(x+2)(2x-1)\|\varphi\|_2^2$.*

Proof. Let $\psi \neq 0$ be in H such that $\Delta\psi = x\psi$. As ψ is in H, one has $\Pi\psi \neq 0$ or $\Pi\Delta\psi \neq 0$. As $\Pi\Delta\psi = x\Pi\psi$, one has $\Pi\psi \neq 0$. By lemma 3.1, one has $\Delta\Pi\psi = y\Pi\psi$, hence y is an eigenvalue of Δ in $\ell^2(\Phi)$. In particular, as $f\left(\frac{1}{2}\right) = -\frac{13}{4} < -3 \leq -\|\Delta\|_2$, one has $x \neq \frac{1}{2}$. Conversely, if φ is an element of $\ell^2(\Phi)$ such that $\Delta\varphi = y\varphi$, one has, by lemma 3.1,

$$\begin{aligned}\Delta R_x\varphi &= \Delta((x-1)\Pi^*\varphi + \Delta\Pi^*\varphi)\\ &= (x-1)\Delta\Pi^*\varphi + \Pi^*\Delta\varphi + (\Delta+3)\Pi^*\varphi\\ &= x\Delta\Pi^*\varphi + (x^2-x)\Pi^*\varphi = xR_x\varphi.\end{aligned}$$

Now, by lemma 4.1, one has $\Pi\Delta\Pi^* = 6 + \Delta$, hence, if $\Delta\varphi = y\varphi$, one has, by a direct computation, $\Pi R_x\varphi = x(x+2)\varphi$ and, as we have supposed $x(x+2) \neq 0$, R_x is one-to-one and closed. It remains to prove that R_x is onto. In this aim, pick ψ in H such that $\Delta\psi = x\psi$ but ψ is orthogonal to the image of R_x. For any φ in $\ell^2(\Phi)$ such that $\Delta\varphi = y\varphi$, one has $\langle\psi, R_x\varphi\rangle = (x-1)\langle\Pi\psi,\varphi\rangle + \langle\Pi\Delta\psi,\varphi\rangle = (2x-1)\langle\Pi\psi,\varphi\rangle$ and hence, as $x \neq \frac{1}{2}$, $\langle\Pi\psi,\varphi\rangle = 0$. As $\Delta\Pi\psi = y\Pi\psi$, one has $\Pi\psi = 0$. As ψ is in H, one has $\psi = 0$. The operator R_x is thus an isomorphism. The computation of the norm is then direct, by using lemmas 3.1 and 4.1. □

Let us begin by looking at the eigenvalues in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$. Let $n \geq 1$ be an integer. Recall that, by corollary 2.5, if $\mathcal{T}$ is a n-triangle of Γ and if p is a vertex of $\mathcal{T}$, the neighbor of p that does not belong to $\mathcal{T}$ is the vertex of some n-triangle. We shall call the edges linking vertices of n-triangles exterior edges to n-triangles. We let Θ_n denote the set of edges which are exterior to n-triangles and we endow it with the graph structure for which two edges are neighbors if two of their end points are vertices of the same n-triangle. One easily checks that the graph Θ_n is naturally isomorphic to the Sierpiński graph, introduced in the end of section 2. We shall from now on

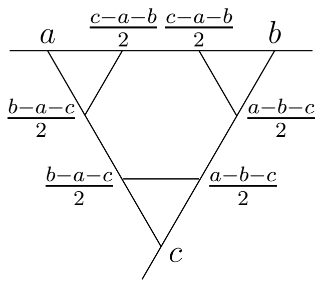

Figure 7: The eigenfunctions with eigenvalue 0

identify Θ_n and Θ. If φ is a function on Γ which is constant on edges which are exterior to n-triangles, we let $P_n\varphi$ denote the function on Θ whose value at one point of Θ is the value of φ on the associate edge which is exterior to n-triangles. Finally, let us recall that, as Θ is 4-regular, the norm of Δ in $\ell^2(\Theta)$ is ≤ 4.

By lemma 3.7, the eigenfunctions with eigenvalue 0 are constant on the edges which are exterior to 1-triangles. We have the following

Lemma 5.2. *The map P_2 induces a Banach spaces isomorphism from the eigenspace of $\ell^2(\Gamma)$ associate to the eigenvalue 0 onto $\ell^2(\Theta)$. Let Q_0 denote its inverse. For any ψ in $\ell^2(\Theta)$, one has $\|Q_0\psi\|^2_{\ell^2(\Gamma)} = 3\,\|\psi\|^2_{\ell^2(\Theta)} - \frac{1}{2}\langle\Delta\psi,\psi\rangle_{\ell^2(\Theta)}$.*

Proof. By using the characterization of lemma 3.7, one easily checks that, given three values a, b and c on the vertices of some 2-triangle, an eigenfunction with eigenvalue 0 taking these values at the three vertices must take in the interior of the triangle the values that are described in picture 7.

Recall that one has identified the graphs Θ and Θ_2. For any function ψ on Θ, let $Q_0\psi$ denote the function on Γ that, on each edge which is exterior to 2-triangles, is constant, with value the value of ψ at the point of Θ associate to this edge, and whose values in the interior of the 2-triangles are those described by figure 7. By an elementary computation, for any real numbers a, b and c, the sum of the squares of the values described in figure 7 is

$$\frac{3}{2}(a^2+b^2+c^2)-ab-ac-bc = \frac{1}{2}(3a^2-ab-ac)+\frac{1}{2}(3b^2-ab-bc)+\frac{1}{2}(3c^2-ac-bc),$$

so that, for any function ψ on Θ, one has $\|Q_0\psi\|^2_{\ell^2(\Gamma)} = 3\,\|\psi\|^2_{\ell^2(\Theta)} - \frac{1}{2}\langle\Delta\psi,\psi\rangle_{\ell^2(\Theta)}$. As $-4\,\|\psi\|^2_{\ell^2(\Theta)} \leq \langle\Delta\psi,\psi\rangle_{\ell^2(\Theta)} \leq 4\,\|\psi\|^2_{\ell^2(\Theta)}$, the function ψ belongs to $\ell^2(\Theta)$ if and only if $Q_0\psi$ belongs to $\ell^2(\Gamma)$. The lemma follows. $\square$

From lemmas 5.1 and 5.2, we shall deduce a description of the eigenspaces associate to the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$. For x in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$, let $n(x)$ denote the integer n such that $f^n(x) = 0$ and

$$\kappa(x) = \prod_{k=0}^{n(x)-1} \frac{f^k(x)(2f^k(x)-1)}{f^k(x)+2}.$$

We have the following

Proposition 5.3. *Let x be in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$. The eigenfunctions with eigenvalue x in $\ell^2(\Gamma)$ are constant on edges which are exterior to $(n(x)+1)$-triangles in Γ. The map $P_{n(x)+2}$ induces a Banach spaces isomorphism from the eigenspace of $\ell^2(\Gamma)$ associate to the eigenvalue x onto $\ell^2(\Theta)$. Let Q_x denote its inverse. Then, for any ψ in $\ell^2(\Theta)$, one has $\|Q_x\psi\|^2_{\ell^2(\Gamma)} = \kappa(x)\left(3\|\psi\|^2_{\ell^2(\Theta)} - \frac{1}{2}\langle\Delta\psi,\psi\rangle_{\ell^2(\Theta)}\right)$.*

Proof. We shall prove this result by induction on $n(x)$. The case $n(x) = 0$ has been dealt with in lemma 5.2. Suppose the lemma has been proved for $n(y)$ with $y = f(x)$. Pick φ in $\ell^2(\Gamma)$ such that $\Delta\varphi = x\varphi$. Then, as $n = n(x) \geq 1$, one has $x \notin \{-2, 0\}$ and hence, by lemma 3.7, φ belongs to H. By lemma 5.1, one thus has $\varphi = R_x\psi$, for some function ψ such that $\Delta\psi = y\psi$. By induction, ψ is constant on edges that exterior to n-triangles. Let p be a vertex of some $(n+1)$-triangle in Γ and q its exterior neighbor. The points Πp and Πq are vertices of some n-triangle in Γ. One hence has $\Pi^*\psi(p) = \psi(\Pi p) = \psi(\Pi q) = \Pi^*\psi(q)$ and $\Delta\Pi^*\psi(p) = 2\psi(\Pi p)) + \psi(\Pi q)) = 3\Pi^*\psi(p)$. Thus, $\varphi(p) = R_x\psi(p) = (x+2)\psi(p) = \varphi(q)$: the function ψ is constant on edges that are exterior to $(n+1)$-triangles and $P_{n+2}\varphi = (x+2)P_{n+1}\psi$. As, by induction, P_{n+1} induces an isomorphism from the eigenspace associate to the value y onto $\ell^2(\Theta)$, by lemma 5.1, P_{n+2} induces an isomorphism from the eigenspace associate to the value y onto $\ell^2(\Theta)$. The norm computation now follows from the induction and the formula $P_{n+2}R_x = (x+2)P_{n+1}$. $\square$

Corollary 5.4. *For any x in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$, the eigenspace associate to x in $\ell^2(\Gamma)$ has infinite dimension and is spanned by finitely supported functions.*

For the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$, there is no analogue of proposition 5.3. However, we will extend corollary 5.4. Let us begin by dealing with the eigenvalue -2. Recall we let p_0 and $p_0^\vee$ denote the vertices of the two infinite triangles in Γ.

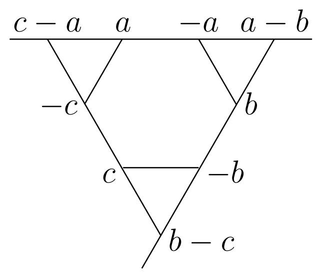

Figure 8: The eigenfunctions with eigenvalue -2

Lemma 5.5. *Let φ be an eigenfunction with eigenvalue -2 in $\ell^2(\Gamma)$. Then, for any $n \geq 1$, the sum of the values of φ on the vertices of each n-triangle of Γ is zero and $\varphi(p_0) = \varphi(p_0^\vee) = 0$. The eigenspace associate to the eigenvalue -2 has infinite dimension and is spanned by finitely supported functions.*

Proof. An immediate computation using lemma 3.7 shows that the values of φ on some 2-triangle satisfy the rules described by figure 8. In particular, the sum of these values on the vertices of each 2-triangle is zero. By induction, using corollary 2.6, it follows that, for any $n \geq 1$, the sum of these values on the vertices of each n-triangle is zero. Let then, for any $n \geq 1$, p_n and q_n denote the two other vertices of the n-triangle with vertex p_0. As φ is square integrable, one has $\varphi(p_n) \xrightarrow[n\to\infty]{} 0$ and $\varphi(q_n) \xrightarrow[n\to\infty]{} 0$. Thus $\varphi(p_0) = 0$ and, in the same way, $\varphi(p_0^\vee) = 0$. In particular, for any $n \geq 1$, $\varphi(p_n) + \varphi(q_n) = 0$.

Let us now prove that φ is the limit of some sequence of finitely supported functions. Let still, as in section 2, $\mathcal{T}_\infty(p_0)$ denote the infinite triangle with vertex p_0. Then, as $\varphi(p_0) = 0$, $\varphi 1_{\mathcal{T}_\infty(p_0)}$ is still an eigenvector with eigenvalue -2 and one can suppose $\varphi = 0$ on $\mathcal{T}_\infty(p_0^\vee)$. For any nonnegative integer n, let φ_n denote the function on Γ that is zero outside the $(n+1)$-triangle $\mathcal{T}_{n+1}(p_0, p_{n+1}, q_{n+1})$, that equals φ_n on the n-triangle $\mathcal{T}_n(p_0, p_n, q_n)$ and that is invariant by the action of the elements of signature 1 of the group $\mathfrak{S}(p_0, p_{n+1}, q_{n+1})$ on the $(n+1)$-triangle $\mathcal{T}_{n+1}(p_0, p_{n+1}, q_{n+1})$. In view of corollary 2.6, the values of φ_n on the vertices of n-triangles are those described by figure 9. Then, by lemma 3.7, for any $n \geq 1$, φ_n is an eigenvector with eigenvalue -2 and one has $\|\varphi_n\|_2^2 \leq 3\,\|\varphi\|_2^2$. The sequence (φ_n) converges weakly to φ in $\ell^2(\Gamma)$. The function φ belongs to the weak closure of the subspace spanned by finitely supported eigenfunctions with eigenvalue -2 and hence to its strong closure.

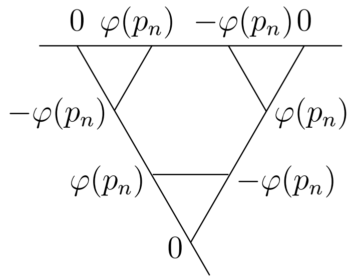

Figure 9: The function φ_n

Finally, the space of eigenfunctions with eigenvalue -2 has infinite dimension since, by figure 6, every 2-triangle contains the support of some eigenfunction with eigenvalue -2. $\square$

For x in $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$, let $n(x)$ denote the integer n such that $f^n(x) = -2$. By an induction based on lemma 5.1, one can deduce from lemma 5.5 the following

Corollary 5.6. *Let x be in $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$ and φ be an eigenfunction with eigenvalue x in $\ell^2(\Gamma)$. Then, the values of φ on the edges that are exterior to $(n(x)+1)$-triangles are opposite, for any $n \geq n(x)+1$ the sum of the values of φ on the vertices of each n-triangle is zero and $\varphi(p_0) = \varphi(p_0^\vee) = 0$. The eigenspace associate to the eigenvalue x has infinite dimension and is spanned by finitely supported functions.*

6 Spectral decomposition of $\ell^2(\Gamma)$

Let φ_0 denote the function on Γ that has value 1 at p_0, -1 at $p_0^\vee$ and 0 everywhere else. In this paragraph, we will prove that $\ell^2(\Gamma)$ is the direct sum of the eigenspaces associate to the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$ and the cyclic subspace spanned by φ_0. Let us begin by studying the latter. By a direct computation, one gets the following

Lemma 6.1. *One has $\Pi^*\varphi_0 = (\Delta + 2)\varphi_0$.*

This relation and corollary 4.4 will allow the determination of the spectral measure of φ_0. On this purpose, let us recall the properties of transfer operators we will have to use: they follow from the version of Ruelle-Perron-Frobenius theorem given in [13, § 2.2]. If κ is a Borel function on Λ, one let L_κ stand for $L_{f,\kappa}$.

Lemma 6.2. *Let $\kappa : \Lambda \to \mathbb{R}_+^*$ be a Hölder continuous function. Equip the space $\mathcal{C}^0(\Lambda)$ with the uniform convergence topology. Then, if $\lambda_\kappa > 0$ is the spectral radius of the operator L_κ in $\mathcal{C}^0(\Lambda)$, there exists a unique Borel probability ν_κ on Λ and a unique continuous positive function l_κ on Λ such that one has $L_\kappa l_\kappa = \lambda_\kappa l_\kappa$, $L_\kappa^* \nu_\kappa = \lambda_\kappa \nu_\kappa$ and $\int_\Lambda l_\kappa \mathrm{d}\nu_\kappa = 1$. The spectral radius of L_κ in the space of functions with zero integral with respect to ν_κ is $< \lambda_\kappa$ and, in particular, for any g in $\mathcal{C}^0(\Lambda)$, the sequence $\left(\frac{1}{\lambda_\kappa^n} L_\kappa^n(g)\right)_{n\in\mathbb{N}}$ uniformly converges to $\int_\Lambda g \mathrm{d}\nu_\kappa$. The measure ν_κ is atom free and its support is Λ.*

For any x in $\mathbb{R}$, set $h(x) = 3-x$, $k(x) = x+2$ and, for $x \neq \frac{1}{2}$, $\rho(x) = \frac{x}{2x-1}$. One has $h \circ f = hk$. From lemma 6.1, we deduce, thanks to corollary 4.4, the following

Corollary 6.3. *Let ν_ρ be the unique Borel probability on Λ such that $L_\rho^* \nu_\rho = \nu_\rho$. The spectral measure of φ_0 is $h\nu_\rho$.*

Proof. Let μ be the spectral measure of φ_0. For $x \neq \frac{1}{2}$, set $\theta(x) = \frac{x(x+2)}{2x-1} = k(x)\rho(x)$. The spectral measure of $(\Delta+2)\varphi_0$ is $k^2\mu$. Therefore, by corollary 4.4 and lemma 6.1, one has $\mu(\frac{1}{2}) = 0$ and $k^2\mu = L_\theta^*\mu$. Now, by lemma 5.5, if φ is an eigenfunction with eigenvalue -2 in $\ell^2(\Gamma)$, one has $\varphi(p_0) = \varphi(p_0^\vee) = 0$ and hence $\langle \varphi, \varphi_0 \rangle = 0$, so that $\mu(-2) = 0$. Therefore, one has $L_{\frac{1}{k}\rho}^* \mu = \mu$.

Besides, by lemma 3.5, one has $\mu(3) = 0$. Therefore, as $h \circ f = hk$, one has $L_\rho^*(\frac{1}{h}\mu) = \frac{1}{h} L_{\frac{1}{k}\rho}^*(\mu) = \frac{1}{h}\mu$.

The Borel measure $\frac{1}{h}\mu$ on $\mathbb{R}$ is concentrated on the spectrum of Δ. Now, by proposition 5.3, for any x in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$ and φ in $\ell^2(\Gamma)$ such that $\Delta\varphi = x\varphi$, one has $\varphi(p_0) = \varphi(p_0^\vee)$ and hence $\langle \varphi, \varphi_0 \rangle = 0$. Therefore, one has $\mu\left(\bigcup_{n\in\mathbb{N}} f^{-n}(0)\right) = 0$ and, by corollary 3.9, μ is concentrated on Λ.

The function ρ is Hölder continuous and positive on Λ and one has $L_\rho(1) = 1$ on Λ. By lemma 6.2, there exists a unique Borel probability measure ν_ρ on Λ such that $L_\rho^*(\nu_\rho) = \nu_\rho$ and, for any continuous function g on Λ, the sequence $(L_\kappa^n(g))_{n\in\mathbb{N}}$ uniformly converges to the constant function with value $\int_\Lambda g \mathrm{d}\nu_\rho$. Let us prove that the positive Borel measure $\frac{1}{h}\mu$ is finite and hence proportional to ν_ρ. Pick some continuous nonnegative function g on Λ, which is zero in the neighborhood of 3 and such that one has $0 < \int_\Lambda \frac{1}{h} g \mathrm{d}\mu < \infty$. There exists an integer n and a real number $\varepsilon > 0$ such that, for any x in Λ, one has $L_\rho^n(g)(x) \geq \varepsilon$. As $\int_\Lambda \frac{1}{h} g \mathrm{d}\mu = \int_\Lambda \frac{1}{h} L_\rho^n(g) \mathrm{d}\mu$, one has $\int_\Lambda \frac{1}{h} \mathrm{d}\mu < \infty$. Therefore, the measure $\frac{1}{h}\mu$ is a multiple of ν_ρ. Now, one has $\mu(\Lambda) = \|\varphi_0\|_2^2 = 2$ and, by a direct computation, $L_\rho(h) = 2$, so that $\int_\Lambda h \mathrm{d}\nu_\rho = 2$. We hence do have $\mu = h\nu_\rho$. □

For any polynomial p in $\mathbb{C}[X]$, let $\hat{p}$ denote the function $p(\Delta)\varphi_0$ on Γ. By definition, the map $g \mapsto \hat{g}$ extends to an isometry from $\mathrm{L}^2(h\nu_\rho)$ onto the cyclic subspace Φ of $\ell^2(\Gamma)$ spanned by φ_0. Let l denote the function $x \mapsto x$ on Λ. On has the following

Proposition 6.4. *The subspace Φ is stable by the operators Δ, Π and Π^*. For any g in $\mathrm{L}^2(h\nu_\rho)$, one has*

$$\Delta\hat{g} = \widehat{lg}$$
$$\Pi\hat{g} = \widehat{L_\rho g}$$
$$\Pi^*\hat{g} = k\widehat{(g \circ f)}.$$

Proof. By definition, Φ is stable by Δ and one has the formula concerning Δ.

By a direct computation, one proves that $L_\rho(l) = 1$. Let n be in $\mathbb{N}$. One has $L_\rho(f^n) = l^n L_\rho(1) = l^n$ and $L_\rho(f^n l) = l^n L_\rho(l) = l^n$. Now, by lemma 3.1, one has $\Pi(f(\Delta)^n\varphi_0) = \Delta^n\Pi\varphi_0$ and $\Pi(f(\Delta)^n\Delta\varphi_0) = \Delta^n\Pi\Delta\varphi_0$ and hence, as $\Pi\varphi_0 = \Pi\Delta\varphi_0 = \varphi_0$, the subspace Φ is stable by Π and, for any p in $\mathbb{C}[X]$, $\Pi\hat{p} = \widehat{L_\rho p}$. Finally, by convexity, for any measurable function g on Λ, one has $|L_\rho(g)|^2 \leq L_\rho\left(|g|^2\right)$, so that, for g in $\mathrm{L}^2(h\nu_\rho)$, one has

$$\begin{aligned}\int_\Lambda |L_\rho(g)|^2 h \mathrm{d}\nu_\rho \leq \int_\Lambda L_\rho\left(|g|^2\right) h \mathrm{d}\nu_\rho &= \int_\Lambda |g|^2 (h \circ f) \mathrm{d}\nu_\rho \\ &= \int_\Lambda |g|^2 kh \mathrm{d}\nu_\rho \leq 5 \int_\Lambda |g|^2 h \mathrm{d}\nu_\rho,\end{aligned}$$

hence the operator L_ρ is continuous in $\mathrm{L}^2(h\nu_\rho)$ and, by density, for any g in $\mathrm{L}^2(h\nu_\rho)$, $\Pi\hat{g} = \widehat{L_\rho g}$.

Finally, by lemmas 3.1 and 6.1, for any polynomial p in $\mathbb{C}[X]$, one has $\Pi^*(p(\Delta)\varphi_0) = p(f(\Delta))\Pi^*\varphi_0 = p(f(\Delta))(\Delta + 2)\varphi_0$. Therefore, the subspace Φ is stable par Π^* and, for any p in $\mathbb{C}[X]$, $\Pi^*\hat{p} = k\widehat{(p \circ f)}$. Now, for any g in

$\mathrm{L}^2(h\nu_\rho)$, one has

$$\int_\Lambda |k(g\circ f)|^2\, h\mathrm{d}\nu_\rho = \int_\Lambda k\,|g\circ f|^2\,(h\circ f)\mathrm{d}\nu_\rho$$
$$= \int_\Lambda L_\rho(k)\,|g|^2\, h\mathrm{d}\nu_\rho = 3\int_\Lambda |g|^2\, h\mathrm{d}\nu_\rho$$

and hence, by density, for any g in $\mathrm{L}^2(h\nu_\rho)$, $\Pi^*\hat{g} = \widehat{k(g\circ f)}$. □

In order to determine the complete spectral structure of Δ, we will analyse other remarkable elements of $\ell^2(\Gamma)$. Let us begin by letting ψ_0 denote the function on Γ that takes the value 1 at p_0 and $p_0^\vee$ and 0 everywhere else. We have the following

Lemma 6.5. *One has* $\Pi^*\psi_0 = \Delta\psi_0$.

From this, we deduce the following

Corollary 6.6. *The spectral measure of ψ_0 is discrete. More precisely, the function ψ_0 is contained in the direct sum of the eigenspaces of Δ associate to the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$.*

Proof. Let μ be the restriction of the spectral measure of ψ_0 to Λ. By corollary 3.9, it suffices to prove that $\mu = 0$.

For $x \notin \left\{0, \frac{1}{2}\right\}$, set $\tau(x) = \frac{(x+2)}{x(2x-1)}$ and $\sigma(x) = \frac{1}{x(2x-1)}$. The function σ is Hölder continuous and positive on Λ. Proceeding as in the proof of corollary 6.3, one proves that, as $0 \notin \Lambda$, one has, by lemma 6.5, $L_\tau^*\mu = \mu$. Now, as $h\circ f = hk$, for any x in $\mathbb{R}$, for any integer n, one has $L_\tau^n(h) = hL_\sigma^n(1)$. Let λ_σ denote the spectral radius of L_σ and ν_σ its equilibrium state, as in lemma 6.2. By a direct computation one shows that, for any x in Λ, one has $L_\sigma(1)(x) = \frac{1}{x+3}$. In particular, for $x \neq -2$, one has $L_\sigma(1)(x) < 1$, so that $\lambda_\sigma = \int_\Lambda L_\sigma(1)\mathrm{d}\nu_\sigma < 1$ and hence the sequence $(L_\sigma^n(1))_{n\in\mathbb{N}}$ uniformly converges to 0 on Λ. Therefore, the sequence $(L_\tau^n(h))_{n\in\mathbb{N}}$ uniformly converges to 0 on Λ and one has $\int_\Lambda h\mathrm{d}\mu = 0$, so that $\mu(\Lambda - \{3\}) = 0$. By lemma 3.5, one has $\mu(3) = 0$ and hence $\mu = 0$. □

Note that, by using lemma 6.5, one could establish a formula giving, for any x in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$, the value of the norm of the projection of ψ_0 on the space of eigenfunctions with eigenvalue x.

Let us now study the spectral invariants of a last element of $\ell^2(\Gamma)$. For this purpose, let q_0 and r_0 denote the two neighbors of p_0 that are different from $p_0^\vee$ and χ_0 the function on Γ that takes the value 1 at q_0, -1 at r_0 and 0 everywhere else. In the same way, one let $q_0^\vee$ and $r_0^\vee$ denote the two neighbors of $p_0^\vee$ that are different from p_0 and $\chi_0^\vee$ the function on Γ that takes the value 1 at q_0, -1 at r_0 and 0 everywhere else. One could again note that one has $\Pi^*\chi_0 = (\Delta^2 + 2\Delta)\chi_0$ and study the spectral measure of χ_0 by using the same methods as in corollaries 6.3 and 6.6. We shall follow another approach, analogous to the one of the proof of lemma 5.5.

By lemma 2.7, there exists a unique automorphism ι of the graph Γ such that $\iota(q_0) = r_0$ and $\iota(q_0^\vee) = r_0^\vee$ and ι is an involution. Let H denote the space of elements φ in $\ell^2(\Gamma)$ such that $\iota(\varphi) = -\varphi$ and K (resp. $K^\vee$) the subspace of H consisting of those elements which are zero on the infinite triangle with vertex $p_0^\vee$ (resp. p_0). One has $H = K \oplus K^\vee$, $\chi_0 \in K$, $\chi_0^\vee \in K^\vee$ and the subspaces K and $K^\vee$ are stable by the endomorphisms Δ, Π and Π^*. For any $n \geq 1$, let $\mathcal{T}_n$ denote the n-triangle with vertex p_0 and $\mathcal{T}_n^\vee$ the n-triangle with vertex $p_0^\vee$. The permutation groups $\mathfrak{S}(\partial\mathcal{T}_n)$ and $\mathfrak{S}(\partial\mathcal{T}_n^\vee)$ act on the triangles $\mathcal{T}_n$ and $\mathcal{T}_n^\vee$. One let K_n (resp. $K_n^\vee$) denote the space of functions φ on $\mathcal{T}_n$ (resp. $\mathcal{T}_n^\vee$) such that, for any s in $\mathfrak{S}(\partial\mathcal{T}_n)$ (resp. in $\mathfrak{S}(\partial\mathcal{T}_n^\vee)$), one has $s\varphi = \varepsilon(s)\varphi$, where ε is the signature morphism. One identifies K_n and $K_n^\vee$ with finite dimensional subspaces of K and $K^\vee$. One then has $\Delta K_n \subset K_n$, $\Pi^* K_n \subset K_{n+1}$ and, if $n \geq 2$, $\Pi K_n \subset K_{n-1}$, and the analogous identities in $K^\vee$.

We have the following

Lemma 6.7. *The spaces K and $K^\vee$ are topologically spanned by the sets $\bigcup_{n\geq 1} K_n$ and $\bigcup_{n\geq 1} K_n^\vee$.*

Proof. Let φ be a function in K. For any integer $n \geq 2$, one let φ_n denote the unique element of K_n that equals φ on $\mathcal{T}_{n-1}$. One has $\|\varphi_n\|_2 \leq \sqrt{3}\,\|\varphi\|_2$. Then, for any φ, the sequence (φ_n) weakly converges to φ in $\ell^2(\Gamma)$. Hence, the set $\bigcup_{n\geq 1} K_n$ is weakly dense in H and the vector subspace it spans is therefore strongly dense. The result for $K^\vee$ follows, by symmetry. □

Corollary 6.8. *The spectrum of Δ in H is discrete. Its eigenvalues are exactly the elements of the set $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$.*

Proof. As, for any n, the subspaces K_n and $K_n^\vee$ are stable by Δ and finite dimensional, the fact that the spectrum of Δ in H is discrete immediately follows from lemma 6.7. The exact determination of the eigenvalues is obtained as in section 3. A formula for the characteristic polynomial of Δ in K_n is given in proposition 13.6. $\square$

We can now finish the proof of theorem 1.1 with the following

Proposition 6.9. *Let $\Phi^\perp$ be the orthogonal complement in $\ell^2(\Gamma)$ of the cyclic space Φ spanned by φ_0. Then, the spectrum of Δ in $\Phi^\perp$ is discrete and the set of its eigenvalues is exactly $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$.*

Before to proving this proposition, let us establish a preliminary result. For φ and ψ in $\ell^2(\Gamma)$, let $\mu_{\varphi,\psi}$ denote the unique complex Borel measure on $\mathbb{R}$ such that, for any polynomial p in $\mathbb{C}[X]$, one has $\int_{\mathbb{R}} p \mathrm{d}\mu_{\varphi,\psi} = \langle p(\Delta)\varphi, \psi\rangle$. One has the following

Lemma 6.10. *For any φ and ψ in $\ell^2(\Gamma)$, one has $\mu_{\Pi\varphi,\psi} = f_*\mu_{\varphi,\Pi^*\psi}$.*

Proof. For any p in $\mathbb{C}[X]$, one has, by lemma 3.1,

$$\int_{\mathbb{R}} p \mathrm{d}\mu_{\Pi\varphi,\psi} = \langle p(\Delta)\Pi\varphi, \psi\rangle = \langle p(f(\Delta))\varphi, \Pi^*\psi\rangle = \int_{\mathbb{R}} (p\circ f) \mathrm{d}\mu_{\varphi,\Pi^*\psi}.$$

$\square$

Proof of proposition 6.9. By corollaries 5.4 and 5.6, the eigenspaces associate to the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$ are non zero. Let P denote the orthogonal projection onto $\Phi^\perp$ in $\ell^2(\Gamma)$. By proposition 6.4, the operator P commutes with Δ, Π and Π^*. To prove the proposition, it suffices to establish that, for any finitely supported φ, for any ψ in $\ell^2(\Gamma)$, the measure $\mu_{P\varphi,\psi}$ is atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$.

Let still q_0, r_0, $q_0^\vee$ and $r_0^\vee$ be the neighbors of p_0 and $p_0^\vee$ and, for any integer n, $\mathcal{T}_n$ be the n-triangle containing p_0 and $\mathcal{T}_n^\vee$ be the n-triangle containing $p_0^\vee$. One let L_n denote the space of functions on Γ whose support is contained in the union of $\mathcal{T}_n$, $\mathcal{T}_n^\vee$ and of the neighbors of the vertices of $\mathcal{T}_n$ and of $\mathcal{T}_n^\vee$. One has, for $n \geq 1$, $\Pi L_n \subset L_{n-1}$ and $\Pi\Delta L_n \subset L_{n-1}$. Let us show, by induction on n, that, for any function φ in L_n, for any ψ in $\ell^2(\Gamma)$, the measure $\mu_{P\varphi,\psi}$ is atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$.

For $n = 0$, L_0 is the space of functions which are zero outside the set $\{p_0, q_0, r_0, p_0^\vee, q_0^\vee, r_0^\vee\}$. One easily checks that this space is spanned by the functions φ_0, $\Delta\varphi_0$, ψ_0, $\Delta\psi_0$, χ_0 and $\chi_0^\vee$. In this case, the description of the spectral measures follows immediately from corollaries 6.3, 6.6 and 6.8.

If the result is true for some integer n, let us pick some φ in L_{n+1}. Then, the functions $\Pi\varphi$ and $\Pi\Delta\varphi$ are in L_n and, by induction, for any ψ in $\ell^2(\Gamma)$, the measures $\mu_{\Pi P\varphi,\psi} = \mu_{P\Pi\varphi,\psi}$ and $\mu_{\Pi\Delta P\varphi,\psi} = \mu_{P\Pi\Delta\varphi,\psi}$ are atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$. By lemma 6.10, the measures $\mu_{P\varphi,\Pi^*\psi}$ and $\mu_{\Delta P\varphi,\Pi^*\psi} = \mu_{P\varphi,\Delta\Pi^*\psi}$ are thus atomic and concentrated on the set $\bigcup_{n\geq 1} f^{-n}(-2) \cup \bigcup_{n\geq 1} f^{-n}(0)$. Now, by lemma 3.7, the spectrum of Δ in the orthogonal complement of the subspace of $\ell^2(\Gamma)$ spanned by the image of Π^* and by the one of $\Delta\Pi^*$ equals $\{-2, 0\}$. Therefore, for any ψ in $\ell^2(\Gamma)$, the measure $\mu_{P\varphi,\psi}$ is atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$. The result follows. $\square$

7 Finite quotients of Γ

In this section, we apply the previous methods to the description of the spectrum of certain finite graphs which are strongly related to Γ.

Let Φ and Ψ be graphs. We shall say that a map $\varpi : \Phi \to \Psi$ is a covering map if, for any p in Φ, the map ϖ induces a bijection from the set of neighbors of p onto the set of neighbors of $\varpi(p)$. The composition of two covering maps is a covering map. If Φ and Ψ are 3-regular graphs and if $\varpi : \Phi \to \Psi$ is a covering map, there exists a unique covering map $\hat{\varpi} : \hat{\Phi} \to \hat{\Psi}$ such that $\Pi\hat{\varpi} = \varpi\Pi$. Conversely, proceeding as in lemma 2.2, one proves that every covering map $\hat{\Phi} \to \hat{\Psi}$ is of this form.

Let us fix four distinct elements a, b, c and d. Let Γ_0 denote the graph obtained by endowing the set $\{a, b, c, d\}$ with the relation that links every pair of distinct points: this is a 3-regular graph. Its automorphism group equals the permutation group $\mathfrak{S}(a, b, c, d)$ of the set $\{a, b, c, d\}$.

Lemma 7.1. *Let Φ be a 3-regular graph and $\varpi : \Phi \to \Gamma_0$ be a covering map. Then, the map $\tilde{\varpi} : \hat{\Phi} \to \Gamma_0, (p, q) \mapsto \varpi(q)$ is a covering map. The map $\varpi \mapsto \tilde{\varpi}$ is a $\mathfrak{S}(a, b, c, d)$-equivariant bijection from the set of covering maps $\Phi \to \Gamma_0$ onto the space of covering maps $\hat{\Phi} \to \Gamma_0$.*

The construction of the covering $\tilde{\varpi}$ is pictured in figure 10.

Démonstration. Let p be a point of Φ and let q, r and s be the neighbors of p. After an eventual permutation of the elements of $\{a, b, c, d\}$, suppose one has $\varpi(p) = a$, $\varpi(q) = b$, $\varpi(r) = c$ and $\varpi(s) = d$. Then, one has $\tilde{\varpi}(p, q) = b$,

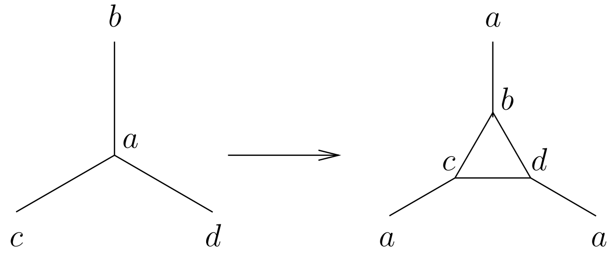

Figure 10: Construction of the covering map $\tilde{\varpi}$

$\tilde{\varpi}(q,p) = a$, $\tilde{\varpi}(p,r) = c$ and $\tilde{\varpi}(p,s) = d$ and hence $\tilde{\varpi}$ is really a covering map.

Conversely, let $\omega : \hat{\Phi} \to \Gamma_0$ be a covering map. Let still p be a point of Φ, with neighbors q, r and s. Again, after an eventual permutation, suppose one has $\omega(p,q) = b$, $\omega(p,r) = c$ and $\omega(p,s) = d$. Then, as ω is a covering map, one necessarily has $\omega(q,p) = \omega(r,p) = \omega(s,p) = a$. Thus, there exists a map $\varpi : \Phi \to \Gamma_0$ such that, for any p and q in Φ with $p \sim q$, one has $\omega(q,p) = \varpi(p)$. By construction, ϖ is a covering map and one has $\tilde{\varpi} = \omega$. $\square$

For any nonnegative integer n, let $\Gamma_n = \hat{\Gamma}^{(n)}$ be the graph obtained by replacing each point of Γ_0 by a n-triangle. By lemma 2.2, the automorphism group of Γ_n naturally identifies with $\mathfrak{S}(a,b,c,d)$. From lemma 7.1, one deduces the following

Corollary 7.2. *For any nonnegative integers $n \leq m$, there exists covering maps $\Gamma_m \to \Gamma_n$. The group $\mathfrak{S}(a,b,c,d)$ acts simply transitively on the set of these covering maps.*

Proof. Covering maps $\Gamma_0 \to \Gamma_0$ are simply bijections of Γ_0 and the corollary is thus true for $m = n = 0$. By induction, by lemma 7.1, the corollary is still true for any nonnegative integer m and $n = 0$. Finally, as, if Φ and Ψ are 3-regular graphs, there exists a natural bijection between the sets of covering maps $\Phi \to \Psi$ and $\hat{\Phi} \to \hat{\Psi}$, again, by induction, the corollary is true for any nonnegative integers $m \geq n$. $\square$

Let us now go back to Γ. Let q_0 and r_0 be the two neighbors of p_0 that are distinct from $p_0^\vee$ and $q_0^\vee$ and $r_0^\vee$ be the two neighbors of $p_0^\vee$ that are distinct from p_0. We have the following

Lemma 7.3. *There exists a unique covering map $\varpi : \Gamma \to \Gamma_0$ such that $\varpi(p_0) = a$, $\varpi(q_0) = c$, $\varpi(r_0) = d$, $\varpi(p_0^\vee) = b$, $\varpi(q_0^\vee) = c$ and $\varpi(r_0^\vee) = d$.*

This covering map is pictured in figure 3.

Proof. Let $n \geq 1$ be an integer or the infinity and $\mathcal{T}$ be a n-triangle. Let $\varpi : \mathcal{T} \to \Gamma_0$. Say that ϖ is a quasi-covering map if, for any point p in $\mathcal{T} - \partial\mathcal{T}$, ϖ induces a bijection from the set of neighbors of p onto $\Gamma_0 - \{\varpi(p)\}$ and if, for any point p in $\partial\mathcal{T}$, the values of ϖ on the neighbors of p are distinct elements of $\Gamma_0 - \{\varpi(p)\}$. In this case, one still let $\tilde{\varpi}$ denote the map $\hat{\mathcal{T}} \to \Gamma_0$ such that, for any p and q in $\mathcal{T}$ with $p \sim q$, one has $\tilde{\varpi}(p, q) = \varpi(q)$ and that, for any p in $\partial\mathcal{T} = \partial\hat{\mathcal{T}}$, if the neighbors of p in $\mathcal{T}$ are q and r, $\tilde{\varpi}(p)$ is the unique element of $\Gamma_0 - \{\varpi(p), \varpi(q), \varpi(r)\}$. Proceeding as in the proof of lemma 7.1, one easily checks that the map $\varpi \mapsto \tilde{\varpi}$ is a $\mathfrak{S}(a, b, c, d)$-equivariant bijection from the set of quasi-covering maps $\mathcal{T} \to \Gamma_0$ onto the set of quasi-covering maps $\hat{\mathcal{T}} \to \Gamma_0$.

Therefore, for any $n \geq 1$, if $\mathcal{T}_n$ is the n-triangle containing p_0 in Γ, there exists a unique quasi-covering map ϖ_n from $\mathcal{T}_n$ into Γ_0 such that $\varpi_n(p_0) = a$, $\varpi_n(q_0) = c$ and $\varpi_n(r_0) = d$. By uniqueness, ϖ_n an ϖ_{n+1} coincide on $\mathcal{T}_n$. Hence, there exists a unique quasi-covering map ϖ_∞ from the infinite triangle $\mathcal{T}_\infty$ with vertex p_0 into Γ_0 such that $\varpi_\infty(p_0) = a$, $\varpi_\infty(q_0) = c$ and $\varpi_\infty(s_0) = d$. In the same way, if $\mathcal{T}_\infty^\vee$ is the infinite triangle with vertex $p_0^\vee$, there exists a unique quasi-covering map $\varpi_\infty^\vee$ from $\mathcal{T}_\infty^\vee$ into Γ_0 such that $\varpi_\infty^\vee(p_0^\vee) = b$, $\varpi_\infty^\vee(q_0^\vee) = c$ and $\varpi_\infty^\vee(s_0^\vee) = d$. The map $\varpi : \Gamma \to \Gamma_0$ whose restriction to $\mathcal{T}_\infty$ is ϖ_∞ and whose restriction to $\mathcal{T}_\infty^\vee$ is $\varpi_\infty^\vee$ is therefore the unique covering map from Γ into Γ_0 enjoying the required properties. □

Again, from lemmas 7.1 and 7.3, one deduces the following

Corollary 7.4. *For any nonnegative integer n, there exists covering maps $\Gamma \to \Gamma_n$. The group $\mathfrak{S}(a, b, c, d)$ acts simply on the set of these covering maps. This action admits two orbits: on one hand, the set of covering maps ϖ such that $\varpi(q_0) = \varpi(q_0^\vee)$, on the other hand, the set of covering maps ϖ such that $\varpi(q_0) = \varpi(r_0^\vee)$.*

We shall now describe, for any integer n, the spectral theory of the graph Γ_n. Let still f denote the polynomial $X^2 - X - 3$. The methods from sections 3 and 5 allow to prove the following

Proposition 7.5. *For any nonnegative integer n, the characteristic polynomial of Δ in $\ell^2(\Gamma_n)$ is*

$$(X-3)(X+1)^3\prod_{p=0}^{n-1}(f^p(X)-2)^3(f^p(X))^{2.3^{n-1-p}}(f^p(X)+2)^{1+2.3^{n-1-p}}.$$

Recall that, in section 3.9, we have defined splitable graphs. The proof uses the following

Lemma 7.6. *Let Φ be a 3-regular connected graph. The graph $\hat{\Phi}$ is non splitable. In particular, for any nonnegative integer n, the graph Γ_n is non splitable.*

Proof. As every point of $\hat{\Phi}$ is contained in a 1-triangle, every point may be joined to itself by a path with odd length and hence $\hat{\Phi}$ is non splitable. In the same way, every point of Γ_0 may be joined to itself by a path with odd length. □

Proof of proposition 7.5. We shall prove this result by induction on n. For $n=0$, the space $\ell^2(\Gamma_0)$ has dimension 4 and, for the natural action of the group $\mathfrak{S}(a,b,c,d)$, it is the sum of two irreducible non isomorphic subspaces, the space of constant functions and the space of functions φ such that $\varphi(a)+\varphi(b)+\varphi(c)+\varphi(d)=0$. The operator Δ commutes with the action of $\mathfrak{S}(a,b,c,d)$ and hence stabilizes both these spaces. In the first one, it acts by multiplication by 3 and, in the second one, by multiplication by -1. Its characteristic polynomial is therefore $(X-3)(X+1)^3$.

Suppose the result has been proved for n. By lemma 7.6, Γ_n is non splitable. Therefore, if H is the subspace of $\ell^2(\Gamma_n)$ spanned by the image of Π^* and by the one of $\Delta\Pi^*$, by corollary 3.6 and lemma 5.1, the characteristic polynomial of Δ in the orthogonal complement in H of the constant functions is

$$(f(X)+1)^3\prod_{p=0}^{n-1}(f^{p+1}(X)-2)^3(f^{p+1}(X))^{2.3^{n-1-p}}(f^{p+1}(X)+2)^{1+2.3^{n-1-p}}$$

and hence, as $f(X)+1=(X+1)(X-2)$, the characteristic polynomial of Δ in H may be written

$$(X-3)(X+1)^3\prod_{p=0}^{n}(f^p(X)-2)^3\prod_{p=1}^{n}(f^p(X))^{2.3^{n-p}}(f^p(X)+2)^{1+2.3^{n-p}}.$$

It remains to determine the dimensions of the eigenspaces associate to the eigenvalues 0 and -2 in the orthogonal complement of H in $\ell^2(\Gamma_{n+1})$. They are described by lemma 3.7. Now, if $n \geq 1$, the 2-triangles in Γ_{n+1} are the inverse images by Π^2 of the points in Γ_{n-1} and every point in Γ_{n+1} belongs to a unique 2-triangle. Proceeding as in lemma 5.2, one shows that the eigenspace associate to the eigenvalue 0 in $\ell^2(\Gamma_{n+1})$ is isomorphic to the space of functions on the edges of Γ_{n-1}. As Γ_{n-1} is a 3-regular graph containing 4.3^{n-1} points, it has 2.3^n edges and the eigenspace associate to the eigenvalue 0 has dimension 2.3^n. If $n = 0$, by using the characterization of lemma 3.7, one checks through a direct computation the eigenspace associate to the eigenvalue 0 in $\ell^2(\Gamma_1)$ has dimension 2. Then, as, by corollary 3.6 and lemma 5.1, H has dimension $2 \dim \ell^2(\Gamma_n) - 1 = 8.3^n - 1$, the orthogonal complement of the sum of H and of the eigenspace associate to the eigenvalue 0 has dimension $4.3^{n+1} - (8.3^n - 1) - 2.3^n = 2.3^n + 1$. By lemma 3.7, this space is the eigenspace associate to the eigenvalue -2 of Δ and the characteristic polynomial of Δ in $\ell^2(\Gamma_{n+1})$ has the form which is given in the setting. $\square$

8 The planar compactification of Γ

We now consider the set X of those elements $(p_{k,l})_{(k,l)\in\mathbb{Z}^2}$ of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ such that, for any integers k and l, one has $p_{k,l} + p_{k+1,l} + p_{k,l+1} = 0$ in $\mathbb{Z}/2\mathbb{Z}$. This is a compact topological space for the topology induced by the product topology. We let T and S denote the two maps from X into X such that, for any p in X, one has $Tp = (p_{k+1,l})_{(k,l)\in\mathbb{Z}^2}$ and $Sp = (p_{k,l+1})_{(k,l)\in\mathbb{Z}^2}$. The homeomorphisms T and S span the natural action of $\mathbb{Z}^2$ on X.

For p in X and k and l in $\mathbb{Z}$, one has $p_{k,l} + p_{k-1,l+1} + p_{k-1,l} = 0$ and $p_{k,l} + p_{k,l-1} + p_{k+1,l-1} = 0$. Now, the finite subgroup $\mathfrak{S}$ of $\mathrm{GL}_2(\mathbb{Z})$ which is spanned by the matrices $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ exchanges the three pairs of vectors $\{(1,0),(0,1)\}$, $\{(-1,1),(-1,0)\}$ and $\{(0,-1),(1,-1)\}$ of $\mathbb{Z}^2$. In particular, the group $\mathfrak{S}$ acts on X in a natural way: for any p in X and s in $\mathfrak{S}$, for any k and l in $\mathbb{Z}^2$, one has $(sp)_{k,l} = p_{s^{-1}(k,l)}$. The action of $\mathfrak{S}$ on the three pairs of vectors $\{(1,0),(0,1)\}$, $\{(-1,1),(-1,0)\}$ and $\{(0,-1),(1,-1)\}$ identifies $\mathfrak{S}$ with the permutation group of this set with 3 elements.

Let Y be the set of points p in X such that $p_{0,0} = 1$. The set Y is stable by the action of $\mathfrak{S}$. For any p in Y, one let Y_p denote the set of points in the orbit of p under the action of $\mathbb{Z}^2$ that belong to Y. If p is some point in

Y, one has $p_{1,0} + p_{0,1} = 1$ in $\mathbb{Z}/2\mathbb{Z}$ and hence one and only one of the points Tp and Sp belongs to Y. In the same way, one and only one of the points $T^{-1}p$ and $T^{-1}Sp$ belongs to Y and one and only one of the points $S^{-1}p$ and $TS^{-1}p$ belongs to Y. For p and q in Y, let us write $p \sim q$ if q belongs to the set $\{Tp, Sp, T^{-1}Sp, T^{-1}p, S^{-1}p, TS^{-1}p\}$. This relation is symmetric and $\mathfrak{S}$-invariant.

In the same way, if p belongs to Y, one let $\tilde{Y}_p$ denote the set of (k,l) in $\mathbb{Z}^2$ such that $p_{k,l} = 1$ and, for any (i,j) and (k,l) in $\tilde{Y}_p$, one writes $(i,j) \sim (k,l)$ if $(i-k, j-l)$ belongs to the set $\{(1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1)\}$. Then, $\tilde{Y}_p$ is a 3-regular graph. If the stabilizer of p in $\mathbb{Z}^2$ is trivial, Y_p is a 3-regular graph and the natural map $\tilde{Y}_p \to Y_p$ is a graph isomorphism.

Let u denote the unique element of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ such that $u_{k,l} = 0$ if and only if $k - l$ equals 0 modulo 3. The element u is periodic under the action of $\mathbb{Z}^2$ and its stabilizer is the set of (k,l) in $\mathbb{Z}^2$ such that $k - l$ equals 0 modulo 3. One checks that u belongs to X. Its orbit under the action of $\mathbb{Z}^2$ equals $\{u, Tu, Su\}$ and is stable under the action of $\mathfrak{S}$: the elements with signature 1 in $\mathfrak{S}$ fix u, Tu and Su and the elements with signature -1 fix u and exchange Tu and Su. We have the following

Lemma 8.1. *Let p be in Y. The graph $\tilde{Y}_p$ is connected. If p is different from Tu and Su, the set Y_p, endowed with the relation $\sim$, is a connected 3-regular graph and the natural map $\tilde{Y}_p \to Y_p$ is a covering map.*

Proof. Let us show that $\tilde{Y}_p$ is connected. Let (k,l) be in Y_p. After an eventual permutation under the group $\mathfrak{S}$ and an exchange of the roles of $(0,0)$ and (k,l), one can suppose k and l are nonnegative. In this case, let us prove by induction on $k+l$ that (k,l) belongs to the same connected component of $\tilde{Y}_p$ as $(0,0)$. If $k+l = 0$, this is trivial. Suppose now $k+l > 0$ and consider $p_{h,k+l-h}$, for $0 \leq h \leq k+l$. Would all these elements of $\mathbb{Z}/2\mathbb{Z}$ equal 1, one would have, for any nonnegative integers i and j with $i+j \leq k+l-1$, $p_{i,j} = 0$, which is impossible, since $p_{0,0} = 1$. After another eventual permutation by $\mathfrak{S}$, one can therefore suppose there exists some integer $0 \leq i \leq l-1$ such that, for any $0 \leq j \leq i$, one has $p_{k+j,l-j} = 1$, but $p_{k+i+1,l-i-1} = 0$. This situation is pictured in figure 11. Then, the points $(k+i, l-i)$ and (k,l) belong to the same connected component in $\tilde{Y}_p$ and, as $p_{k+i+1,l-i-1} = 0$, one has $p_{k+i,l-i-1} = 1$ and $(k+i, l-i-1)$ belongs to $\tilde{Y}_p$. As $(k+i)+(l-i-1) = k+l-1$, the result follows by induction.

As the natural map $\tilde{Y}_p \to Y_p$ is onto, to conclude, it remains to prove that, for $p \notin \{Tu, Su\}$, the points of the set $\{p, Tp, Sp, T^{-1}Sp, T^{-1}p, S^{-1}p, TS^{-1}p\}$

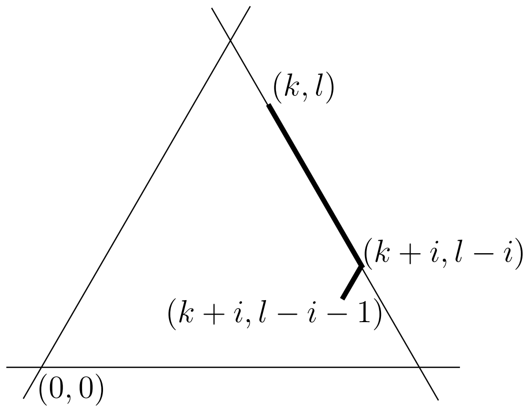

Figure 11: Connectedness of $\tilde{Y}_p$

are distinct. Set $V_p = \{Tp, Sp, T^{-1}Sp, T^{-1}p, S^{-1}p, TS^{-1}p\}$ and let us begin by supposing that p belongs to V_p. Then, after an eventual action of the group $\mathfrak{S}$, one can suppose one has $p = Tp$ and $p_{0,-1} = 1$ and hence $p_{1,-1} = p_{0,-1} + p_{0,0} = 0$, which contradicts the fact that $Tp = p$. One thus has $p \notin V_p$. Suppose now two elements of the set V_p equal each other. Again, after an eventual action of the group $\mathfrak{S}$, one can suppose one has $Tp = Sp$, $Tp = T^{-1}p$ or $Tp = T^{-1}Sp$. If $Tp = Sp$, one has $S^{-1}Tp = p$ and we just have proved it to be impossible. If $Tp = T^{-1}p$, one has $T^2p = p$ and the family $q = (p_{2k,2l})_{(k,l)\in\mathbb{Z}^2}$ belongs to Y and satisfies $Tq = q$: again, we just have proved it to be impossible. Finally, if $T^{-2}Sp = p$, still suppose, after an eventual permutation, one has $p_{0,-1} = 1$. Then, one has $p_{1,-1} = 0$, and hence, as $T^{-2}Sp = p$, $p_{-1,0} = 0$. In the same way, one has $p_{-2,0} = p_{0,-1} = 1$ and $p_{-1,-1} = p_{0,-1} + p_{-1,0} = 1$. Again, this implies $p_{-3,0} = p_{-1,-1} = 1$, $p_{-2,-1} = p_{-1,-1} + p_{-2,0} = 0$ and, finally, $p_{-3,-1} = p_{-2,-1} + p_{-3,0} = 1$, so that the point $q = T^{-3}p$ again satisfies $T^{-2}Sq = q$ and $q_{0,0} = q_{0,-1} = 1$. By induction, one deduces that, for any integer $k \leq 0$, one has $p_{k,0} = 1$ if k equals 0 or 1 modulo 3 and that $p_{k,0} = 0$ if k equals 2 modulo 3. Proceeding in the same way, one shows that $p_{-1,1} = p_{-1,0} + p_{0,0} = 1$ and that, as $T^2S^{-1}p = p$, $p_{1,0} = 1$. Thus, one has $p_{2,0} = p_{0,1} = p_{0,0} + p_{1,0} = 0$, hence $p_{3,0} = p_{1,1} = p_{1,0} + p_{2,0} = 1$ and $p_{3,-1} = p_{1,0} = 1$. The point $r = T^3p$ therefore also satisfies $T^{-2}Sr = r$ and $r_{0,0} = r_{0,-1} = 1$, so that, for any k in $\mathbb{Z}$, one has $p_{k,0} = 0$ if and only if k equals 2 modulo 3. In particular, the sequence $(p_{k,0})_{k\in\mathbb{Z}}$ is 3-periodic. As $T^{-2}Sp = p$, for any l in $\mathbb{Z}$, the sequence $(p_{k,l})_{k\in\mathbb{Z}}$ is 3-periodic and hence

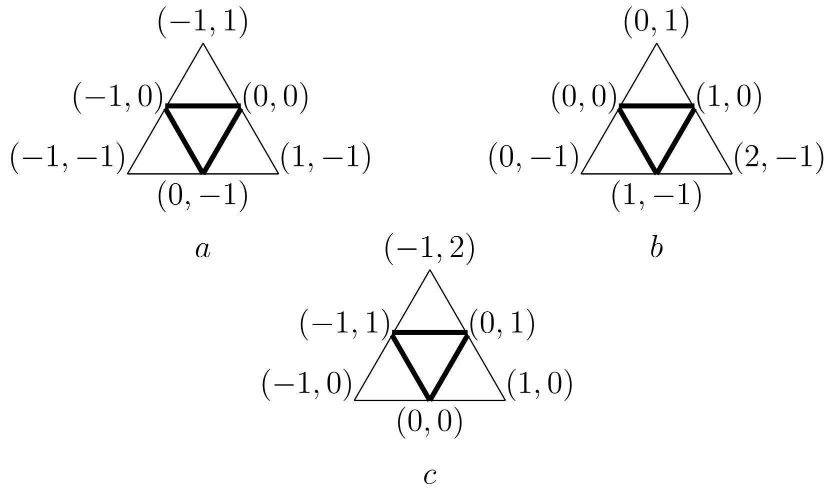

Figure 12: The sets Y^a, Y^b and Y^c

$T^3 p = p$. Therefore, for any k and l in $\mathbb{Z}$, if $k-l$ equals 0 modulo 3, one has $T^k S^l p = p$. As one has $p_{0,0} = u_{1,0}$, $p_{-1,0} = u_{-1,1}$ and $p_{-1,1} = u_{-1,2}$, one has $p = Tu$. Therefore, if p does not belong to $\{Tu, Su\}$, the relation $\sim$ induces a 3-regular graph structure on the set Y_p. By definition, the natural map $\tilde{Y}_p \to Y_p$ is then a covering map. In particular, Y_p is connected. $\square$

Let ε and η be in $\{0,1\}$. Let $X^{(\varepsilon,\eta)}$ denote the set of elements p in X such that, for any k and l in $\mathbb{Z}$, if (k,l) equals (ε,η) in $(\mathbb{Z}/2\mathbb{Z})^2$, one has $p_{k,l} = 0$. If p belongs to $X^{(\varepsilon,\eta)}$, for any k and l in $\mathbb{Z}$, one has $p_{2k+1+\varepsilon,2l+1+\eta} = p_{2k+\varepsilon,2l+1+\eta} = p_{2k+1+\varepsilon,2l+\eta}$. In particular, for $(\varepsilon',\eta') \neq (\varepsilon,\eta)$, one has $X^{(\varepsilon,\eta)} \cap X^{(\varepsilon',\eta')} = \{0\}$ and, if p is a point in $Y^{(\varepsilon,\eta)} = Y \cap X^{(\varepsilon,\eta)}$ (one then has $(\varepsilon,\eta) \neq (0,0)$), the point p belongs to a triangle in the graphe Y_p.

The groupe $\mathfrak{S}$ acts on $(\mathbb{Z}/2\mathbb{Z})^2$ in a natural way and, for any s in $\mathfrak{S}$, for any (ε,η) in $(\mathbb{Z}/2\mathbb{Z})^2$, one has $X^{s(\varepsilon,\eta)} = sX^{(\varepsilon,\eta)}$. From now on, we set $a = (1,1)$, $b = (0,1)$, $c = (1,0)$ and $\mathcal{T}_1 = \{a,b,c\}$. We shall consider $\mathcal{T}_1$ as a 1-triangle. The groupe $\mathfrak{S}$ may be identified with the permutation group $\mathfrak{S}(a,b,c)$. One sets $\hat{Y} = Y^a \cup Y^b \cup Y^c$: this is a disjoint union and the set $\hat{Y}$ is $\mathfrak{S}$-invariant. The elements of Y^a, Y^b and Y^c are described by figure 12.

Let p be a point of Y. We shall let $\hat{p}^a$, $\hat{p}^b$ and $\hat{p}^c$ denote the elements of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ such that, for any k and l in $\mathbb{Z}$, one has

(i) $\hat{p}^a_{2k,2l} = \hat{p}^a_{2k-1,2l} = \hat{p}^a_{2k,2l-1} = p_{k,l}$ and $\hat{p}^a_{2k-1,2l-1} = 0$.

(ii) $\hat{p}^b_{2k,2l} = \hat{p}^b_{2k+1,2l} = \hat{p}^b_{2k+1,2l-1} = p_{k,l}$ and $\hat{p}^b_{2k+2,2l-1} = 0$.

(iii) $\hat{p}^c_{2k,2l} = \hat{p}^c_{2k,2l+1} = \hat{p}^c_{2k-1,2l+1} = p_{k,l}$ and $\hat{p}^c_{2k-1,2l+2} = 0$.

One checks that, by construction, one has $\hat{p}^a = T\hat{p}^b = S\hat{p}^c$ and that, for any s in $\mathfrak{S}$, for any d in $\mathcal{T}_1$, one has $\widehat{sp}^{sd} = s\left(\hat{p}^d\right)$.

Lemma 8.2. *Let d be in $\mathcal{T}_1$. The map $p \mapsto \hat{p}^d$ induces a homomorphism from Y onto Y^d. Conversely, a point p of $Y - \{Tu, Su\}$ belongs to $\hat{Y}$ if and only if, for any q in Y_p, q belongs to some triangle contained in Y_p. In this case, there exists a unique d in $\mathcal{T}_1$ and a unique point r of Y such that $p = \hat{r}^d$ and the triangle containing p is $\{\hat{r}^a, \hat{r}^b, \hat{r}^c\}$.*

The proof uses the following

Lemma 8.3. *Let p be a point of $Y - \{Tu, Su\}$ such that each point of Y_p is contained in a triangle. Then the triangle containing p is either $\{p, T^{-1}p, S^{-1}p\}$ or $\{p, Tp, TS^{-1}p\}$ or $\{p, Sp, T^{-1}Sp\}$. If it is of the form $\{p, T^{-1}p, S^{-1}p\}$, the third neighbor q of p is either Tp or Sp. Finally, if $q = Tp$, the triangle containing q is $\{q, Tq, TS^{-1}q\}$ and if $q = Sp$, the triangle containing q is $\{q, Sq, T^{-1}Sq\}$.*

Proof. Let p be as in the setting. After an eventual action of $\mathfrak{S}$, one can suppose that the triangle containing p contains the point $T^{-1}p$. Then, by definition, the only possible common neighbors of p and $T^{-1}p$ are $T^{-1}Sp$ and $S^{-1}p$. Now, as $T^{-1}p$ belongs to Y, one has $p_{-1,0} = 1$, hence $p_{-1,1} = p_{0,0} + p_{-1,0} = 0$ and $T^{-1}Sp \notin Y$. Therefore, the triangle containing p is $\{p, T^{-1}p, S^{-1}p\}$. The other cases follow, by letting $\mathfrak{S}$ act on the situation.

In case the triangle containing p is $\{p, T^{-1}p, S^{-1}p\}$, the third neighbor q of p is, by construction, necessarily in $\{Tp, Sp\}$. Suppose now, still after an eventual permutation under $\mathfrak{S}$, one has $q = Tp$. Then, one has $p_{0,1} = 0 = p_{1,-1}$ and hence $T^{-1}Sq$ and $S^{-1}q$ do not belong to Y. The triangle containing q is thus $\{q, Tq, TS^{-1}q\}$. The other case follows, by symmetry. $\square$

Proof of lemma 8.2. One easily checks that, for d in $\mathcal{T}_1$, the point $\hat{p}^d$ belongs to Y^d and that the thus defined map induces a homeomorphism from Y onto Y^d.

Conversely, let p be a point of $Y - \{Tu, Su\}$ such that every element of Y_p is contained in a triangle of Y_p. Then, by definition and by lemma 8.1, every point of $\tilde{Y}_p$ is contained in a triangle of $\tilde{Y}_p$. Let (k, l) be a point of $\tilde{Y}_p$. By lemma 8.3, the triangle containing (k,l) is of the form $\{(k,l),(k-1,l),(k,l-1)\}$, $\{(k,l),(k+1,l),(k+1,l-1)\}$ or $\{(k,l),(k,l+1),(k-1,l+1)\}$. Let $(\varepsilon(k,l),\eta(k,l))$ denote the unique element of $(\mathbb{Z}/2\mathbb{Z})^2$ that does not equal one of the elements of this triangle modulo $(2\mathbb{Z})^2$.

Let us prove that, for any (i,j) and (k,l) in $\tilde{Y}_p$ with $(i,j)\sim(k,l)$, one has $(\varepsilon(i,j),\eta(i,j))=(\varepsilon(k,l),\eta(k,l))$. If (i,j) and (k,l) belong to the same triangle, this is clear. Else, after an eventual action of $\mathfrak{S}$, by lemma 8.3, one can suppose that the triangle containing (k,l) is $\{(k,l),(k-1,l),(k,l-1)\}$, that $(i,j)=(k+1,l)$ and hence that the triangle containing (i,j) is $\{(k+1,l),(k+2,l),(k+2,l-1)\}$. Then, by definition, one has $(\varepsilon(i,j),\eta(i,j))=(\varepsilon(k,l),\eta(k,l))$.

As, by lemma 8.1, the graph $\tilde{Y}_p$ is connected, the function (ε,η) is constant. By definition, for any integers k and l, one has $p_{2k+\varepsilon,2l+\eta}=0$, hence p belongs to $Y^{(\varepsilon,\eta)}$. The property on triangles immediately follow from the definition of the objects. $\square$

Let p be a point of $\hat{Y}$. One let $\bar{\Pi}p$ and $\theta_1(p)$ denote the unique elements of Y and $\mathcal{T}_1$ for which one has $\widehat{\bar{\Pi}p}^{\theta_1(p)}=p$. By construction, one has $\theta_1(p)=a$ (resp. b, resp. c) if and only if the triangle containing p is $\{p,T^{-1}p,S^{-1}p\}$ (resp. $\{p,Tp,TS^{-1}p\}$, resp. $\{p,Sp,T^{-1}Sp\}$). The maps $\bar{\Pi}$ and θ_1 are $\mathfrak{S}$-equivariant. The map $\bar{\Pi}$ is continuous and θ_1 is locally constant. For any p in $\hat{Y}$, θ_1 induces a bijection from the 1-triangle containing p onto $\mathcal{T}_1$.

Lemma 8.4. *Let p be in $Y-\{Tu,Su\}$. There exists a unique graph isomorphism $\sigma:\hat{Y}_p\to\bar{\Pi}^{-1}(Y_p)$ such that $\bar{\Pi}\sigma=\sigma\Pi$.*

Démonstration. By lemma 2.2, such an isomorphism is necessarily unique. Let us prove that it exists. Let q be the neighbor of p belonging to $\{Tp,Sp\}$, r its neighbor in $\{T^{-1}p,T^{-1}Sp\}$ and s its neighbor in $\{S^{-1}p,TS^{-1}p\}$. One sets $\sigma(p,q)=\hat{p}^a$, $\sigma(p,r)=\hat{p}^b$ and $\sigma(p,s)=\hat{p}^c$. Then, the three points $\sigma(p,q)$, $\sigma(p,r)$ and $\sigma(p,s)$ are neighbors in Y_p. Let us check, for example, that $\sigma(p,q)$ is a neighbor of $\sigma(q,p)$. After an eventual action of $\mathfrak{S}$, one can suppose one has $q=Tp$. Then, one has $p=T^{-1}q$ and hence $\sigma(q,p)=\hat{q}^b$. By construction, one then has $T\hat{p}^a=\hat{q}^b$, hence $\sigma(q,p)=T\sigma(p,q)$, what should be proved. $\square$

From now on, we shall, for any p in $\hat{Y}$, identify the graphs $\hat{Y}_p$ and $\bar{\Pi}^{-1}(Y_p)$.

We will now construct an element p of Y for which the graph Y_p is isomorphic to the Pascal graph. Set, for any $k, l \geq 0$, $p_{-k,-l} = p_{k+l+1,-l} = \binom{l+k}{k}$ in $\mathbb{Z}/2\mathbb{Z}$ and, for any k, l in $\mathbb{Z}$ with either $l > 0$ or $k \geq 1$ and $k + l \geq 0$, $p_{k,l} = 0$. One easily checks that p belongs to X and hence to Y since $p_{0,0} = 1$. We have the following

Proposition 8.5. *The point p belongs to $\hat{Y}$ and one has $\bar{\Pi}p = p$ and $\theta_1(p) = a$. There exists an isomorphism from the Pascal graph Γ onto Y_p sending p_0 to p and $p_0^\vee$ to Tp.*

This planar representation of the Pascal graph appears in figure 1. The proof uses the following

Lemma 8.6. *Let $0 \leq k \leq n$ be integers. Then, the integers $\binom{n}{k}$, $\binom{2n}{2k}$, $\binom{2n+1}{2k}$ and $\binom{2n+1}{2k+1}$ equal each other modulo 2.*

Proof. Let A and B be indeterminates. In the characteristic 2 ring $\mathbb{Z}/2\mathbb{Z}[A, B]$, one has $(A+B)^n = \sum_{k=0}^n \binom{n}{k} A^k B^{n-k}$ and hence $(A+B)^{2n} = \sum_{k=0}^n \binom{n}{k} A^{2k} B^{2n-2k}$. Therefore, by uniqueness, for any $0 \leq k \leq n$, one has, in $\mathbb{Z}/2\mathbb{Z}$, $\binom{2n}{2k} = \binom{n}{k}$ and $\binom{2n}{2k-1} = \binom{2n}{2k+1} = 0$. By the classical identity, one then has $\binom{2n+1}{2k} = \binom{2n}{2k-1} + \binom{2n}{2k} = \binom{2n}{2k}$ and $\binom{2n+1}{2k+1} = \binom{2n}{2k} + \binom{2n}{2k+1} = \binom{2n}{2k}$. □

Proof of proposition 8.5. By using lemma 8.6, one checks that one has $p = \hat{p}^a$. Therefore, p belongs to $\hat{Y}$, $\bar{\Pi}p = p$ and $\theta_1(p) = a$. By induction, using lemma 8.4, one deduces that, for any integer n, p is the vertex of a n-triangle contained in Y_p. In the same way, one has $\bar{\Pi}(Tp) = Tp$, $\theta_1(Tp) = b$ and Tp is the vertex of a n-triangle contained in Y_p. By lemma 8.1, the graph Y_p is connected and hence it equals the union of both of these infinite triangles. The existence of the isomorphism in question follows. □

From now on, we shall identify p with p_0, Tp with $p_0^\vee$ and Γ with Y_p. One let $\bar{\Gamma}$ denote the closure of Γ in Y and, for any p in $\bar{\Gamma}$, one sets $\Gamma_p = Y_p$. One has $\bar{\Pi}\bar{\Gamma} = \bar{\Gamma}$.

We will now describe the set $\bar{\Gamma}$ in a more detailed way. For this purpose, let us introduce a partition of $\hat{Y}$ into six subsets that refines the partition $\hat{Y} = Y^a \cup Y^b \cup Y^c$. Let b be a point of $\hat{Y}$. Then, by lemma 8.3, the set of neighbors of p is either $\{Tp, T^{-1}p, S^{-1}p\}$ or $\{Sp, T^{-1}p, S^{-1}p\}$ or $\{T^{-1}p, Tp, TS^{-1}p\}$ or $\{T^{-1}Sp, Tp, TS^{-1}p\}$ or $\{S^{-1}p, Sp, T^{-1}Sp\}$ or $\{TS^{-1}p, Sp, T^{-1}Sp\}$. Let us call the set of q in $\hat{Y}$ for which one has $\{(k,l) \in \mathbb{Z}^2 | T^k S^l p \sim p\} = \{(k,l) \in \mathbb{Z}^2 | T^k S^l q \sim q\}$ the keel of p. The keels are six closed subsets of $\hat{Y}$ on which the group $\mathfrak{S}$ act simply transitively. We let B_0 denote the keel of p_0, i the element $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ of $\mathfrak{S}$ and r the element $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. The element i identifies with the transposition (ab) of $\{a, b, c\}$ and r with the cycle (cba).

For any integer n, set $\hat{Y}^{(n)} = \bar{\Pi}^{-n} Y$. Then, by a direct induction, by lemmas 8.2 and 8.4, for any integer n, $\hat{Y}^{(n)}$ is the set of elements p in $Y - \{Tu, Su\}$ for which every point of Y_p belongs to a n-triangle in Y_p. One hence has $\Gamma \subset \bigcap_{n \in \mathbb{N}} \hat{Y}^{(n)}$.

Lemma 8.7. *Let n be an integer and p and q be in $\hat{Y}^{(n+1)}$ such that, for any $0 \leq m \leq n$, $\bar{\Pi}^m p$ and $\bar{\Pi}^m q$ belong to the same keel. Then, for any k and l in $\mathbb{Z}$ with $k \geq -2^n$, $l \geq -2^n$ and $k + l \leq 2^n$, one has $p_{k,l} = q_{k,l}$.*

Proof. Let us prove this result by induction on n. For $n = 0$, suppose, after an eventual action of $\mathfrak{S}$, one has $p, q \in B_0$. Then, one has $p_{-1,0} = p_{0,0} = p_{1,0} = p_{0,-1} = 1$, so that $p_{-1,-1} = p_{-1,0} + p_{0,-1} = 0$ and, in the same way, $p_{1,-1} = p_{-1,1} = p_{0,1} = p_{-1,2} = 0$ and $p_{2,-1} = 1$. As this is also true for q, the lemma is true for $n = 0$.

Suppose now $n \geq 1$ and the lemma has been proved for $n-1$. Pick p and q as in the setting. Then, as p and q are in the same keel, one has $\theta_1(p) = \theta_1(q)$. After an eventual action of $\mathfrak{S}$, one can suppose one has $\theta_1(p) = a$, so that $p = \widehat{\bar{\Pi} p}^a$ and $q = \widehat{\bar{\Pi} q}^a$. The result now follows by induction and by the definition of the map $r \mapsto \hat{r}^a$. $\square$

Lemma 8.8. *Let p be in $\hat{Y}$ such that the keel of p is B_0. Then, the keel of $\hat{p}^a$ is B_0, the one of $\hat{p}^b$ is iB_0 and the one of $\hat{p}^c$ is rB_0 and $\hat{p}^a$ is the vertex of a 2-triangle in $\bar{\Gamma}$. If q and r are two points in $\hat{Y}^{(2)}$ such that $\bar{\Pi} q$ and $\bar{\Pi} r$ belong to the same keel, there exists r' in the 1-triangle containing r in Y_r such that q and r' belong to the same keel.*

Proof. The first point follows directly from the construction of the objects. Pick q and r as in the setting. After an eventual action of $\mathfrak{S}$, one can suppose that the keel of $\bar{\Pi} q$ and of $\bar{\Pi} r$ is B_0. The first part of the lemma now clearly implies the setting. $\square$

Let Σ be the set of sequences $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathfrak{S}$ such that, for any integer n, one has $s_n \in \{s_{n+1}, s_{n+1} i, s_{n+1} r\}$. We equip Σ with the topology induced by the product topology and we let $\sigma : \Sigma \to \Sigma$ denote the shift map. One let $\mathfrak{S}$ act on Σ by left multiplication on all the components. The set $\bar{\Gamma}$ is described by the following

Proposition 8.9. *One has $\bar{\Gamma} = \bigcap_{n\in\mathbb{N}} \hat{Y}^{(n)}$. For any p in $\bar{\Gamma}$, for any integer n, let $s_n(p)$ be the unique element of $\mathfrak{S}$ such that the keel of $\bar{\Pi}^n p$ is $s_n(p)B_0$. The thus defined map s induces a $\mathfrak{S}$-equivariant homeomorphism from $\bar{\Gamma}$ onto Σ and one has $\sigma s = s\bar{\Pi}$. The image of the point p_0 by s is the constant sequence with value e and the fixed points of $\bar{\Pi}$ in $\bar{\Gamma}$ are exactly the six images of p_0 by the action of the group $\mathfrak{S}$. Finally, for any p in $\bar{\Gamma}$, the set Γ_p is dense in $\bar{\Gamma}$.*

Proof. As the set Γ is included in $\bigcap_{n\in\mathbb{N}} \hat{Y}^{(n)}$, so is the set $\bar{\Gamma}$. Conversely, let us note that each of the six points $T^{-1}p_0$, $T^{-2}p_0$, $T^{-2}S^{-1}p_0$, $T^{-1}S^{-2}p_0$, $S^{-2}p_0$ and $S^{-1}p_0$ belong to a different keel. Therefore, if p is a point of $\bigcap_{n\in\mathbb{N}} \hat{Y}^{(n)}$, there exists a point q in Γ such that p and q belong to the same keel. By induction, using lemma 8.8, one deduces that, for any integer n, there exists a point q_n of Γ such that, for any $0 \leq m \leq n$, $\bar{\Pi}^m p$ and $\bar{\Pi}^m q_n$ belong to the same keel. By lemma 8.7, one then has $q_n \xrightarrow[n\to\infty]{} p$ and p belongs to $\bar{\Gamma}$.

Let p be in $\bar{\Gamma}$. As we have just seen, the point p is completely determined by the sequence $s(p) = (s_n(p))_{n\in\mathbb{N}}$. The map s is clearly continuous and $\mathfrak{S}$-equivariant and, by definition, one has $s\bar{\Pi} = \sigma s$. Besides, by lemma 8.2, if p is a point of $\bar{\Gamma}$, it admits exactly three antecedents by $\bar{\Pi}$ and, by lemma 8.8, the keels of these antecedents are $s_0(p)B_0$, $s_0(p)iB_0$ and $s_0(p)rB_0$. It follows that the map s takes its values in Σ and that it induces a homeomorphism from $\bar{\Gamma}$ onto Σ.

By construction, one has $s_0(p_0) = e$ and, as, by proposition 8.5, $\bar{\Pi}p_0 = p_0$, for any nonnegative integer n, $s_n(p_0) = e$. In particular, the other fixed points of $\bar{\Pi}$ are the images of p_0 by the action of $\mathfrak{S}$.

Finally, if p is a point of $\bar{\Gamma}$, by lemmas 2.4 and 8.4, for any nonnegative integer n, the n-triangle containing p in Γ_p is the set $\bar{\Pi}^{-n}\left(\bar{\Pi}^n p\right)$. As r and i span the group $\mathfrak{S}$, one easily checks that the subshift of finite type (Σ, σ) is transitive, so that, for any t in Σ, the set $\bigcup_{n\in\mathbb{N}} \sigma^{-n}(\sigma^n t)$ is dense in Σ. Therefore, for any p in $\bar{\Gamma}$, Γ_p is dense in $\bar{\Gamma}$. □

9 Triangular functions and integration on $\bar{\Gamma}$

In this section, we study a particular class of locally constant functions on $\bar{\Gamma}$. We use these functions to determine some properties of a remarkable Radon measure on $\bar{\Gamma}$. For p in $\bar{\Gamma}$, let still, as in section 8, $(s_n(p)B_0)_{n\in\mathbb{N}}$ denote the associate keel sequence.

Let $n \geq 1$ be an integer and $\mathcal{T}$ be a n-triangle in $\bar{\Gamma}$. Then, by lemma

8.4, the set $\bar{\Pi}^{n-1}\mathcal{T}$ is a 1-triangle of $\bar{\Gamma}$. Therefore, the map $\theta_1 \circ \bar{\Pi}^{n-1}$ induces a bijection from the set of vertices of $\mathcal{T}$ onto $\mathcal{T}_1 = \{a, b, c\}$. One let a_n (resp. b_n, resp. c_n) denote the set of vertices p of n-triangles of $\bar{\Gamma}$ such that $\theta_1(\bar{\Pi}^{n-1}p) = a$ (resp. b, resp. c) and θ_n the map which sends a vertex p of a n-triangle of $\bar{\Gamma}$ to the element of $\{a_n, b_n, c_n\}$ to which it belongs. Let $\mathcal{T}_n$ be the n-triangle $\mathcal{T}_n(a_n, b_n, c_n)$. By lemma 2.3, the map θ_n extends in a unique way to a map $\bar{\Gamma} \to \mathcal{T}_n$, still denoted by θ_n, that, on each n-triangle $\mathcal{T}$ of $\bar{\Gamma}$, induces a graph isomorphism from $\mathcal{T}$ onto $\mathcal{T}_n$. This map is locally constant. For $n = 1$, this definition is coherent with the notations of section 8, provided one identifies a with a_1, b with b_1 and c with c_1. By abuse of language, we will sometimes consider $\mathcal{T}_0$ as a set containing only one element and θ_0 as the constant map $\bar{\Gamma} \to \mathcal{T}_0$.

For any $n \geq 1$, the group $\mathfrak{S}$ act on $\mathcal{T}_n$ and identifies with $\mathfrak{S}(a_n, b_n, c_n)$. We shall identify $\mathcal{T}_{n+1}$ and $\hat{\mathcal{T}}_n$ through the $\mathfrak{S}$-equivariant bijection from $\{a_n, b_n, c_n\}$ onto $\{a_{n+1}, b_{n+1}, c_{n+1}\}$ that sends a_n to a_{n+1}, b_n to b_{n+1} and c_n to c_{n+1}. In particular, one let $\Pi : \mathcal{T}_{n+1} \to \mathcal{T}_n$ denote the triangle contracting map coming from this identification and Π^* and Π the associate operators $\ell^2(\mathcal{T}_n) \to \ell^2(\mathcal{T}_{n+1})$ and $\ell^2(\mathcal{T}_{n+1}) \to \ell^2(\mathcal{T}_n)$.

Let, for any $n \geq 2$, a_nb_n, a_nc_n, b_na_n, b_nc_n, c_na_n and c_nb_n be the points of $\mathcal{T}_n$ defined by corollary 2.6. The principal properties of the maps θ_n, $n \geq 1$, we shall use in the sequel are described by the following

Lemma 9.1. *Let $n \geq 1$ be an integer. One has $\Pi\theta_{n+1} = \theta_n\bar{\Pi}$. If p and q are points of $\bar{\Gamma}$ such that $\theta_{n+1}(p) = \theta_{n+1}(q)$, one has $\theta_n(p) = \theta_n(q)$. In particular, one has $\theta_n(p) = a_n$ if and only if $\theta_{n+1}(p)$ is a_{n+1}, $b_{n+1}a_{n+1}$ or $c_{n+1}a_{n+1}$. Let p and q be in $\bar{\Gamma}$, such that $\theta_n(p) = \theta_n(q)$. For any $0 \leq m \leq n$, if $\theta_n(p)$ is not contained in a m-triangle which admits one of the vertices of $\mathcal{T}_n$ as a vertex, one has $s_m(p) = s_m(q)$.*

Proof. Let $\mathcal{T}$ be a $(n+1)$-triangle of $\bar{\Gamma}$. The map θ_{n+1} induces an isomorphism from $\mathcal{T}$ onto $\mathcal{T}_{n+1}$ and the map θ_n induces an isomorphism from the n-triangle $\bar{\Pi}\mathcal{T}$ onto $\mathcal{T}_n$. As, by definition, the maps $\theta_n\bar{\Pi}$ and $\Pi\theta_{n+1}$ coincide on the set $\partial\mathcal{T}_n$, one has, by lemma 2.2, $\Pi\theta_{n+1} = \theta_n\bar{\Pi}$.

Let p, q and r be the vertices of $\mathcal{T}$, so that $\theta_{n+1}(p) = a_{n+1}$, $\theta_{n+1}(q) = b_{n+1}$ and $\theta_{n+1}(r) = c_{n+1}$. By definition, one has $\theta_n(p) = a_n$. Let us prove that $\theta_n(qp) = \theta_n(rp) = a_n$. This amounts to proving that one has $\theta_1\left(\bar{\Pi}^{n-1}qp\right) = \theta_1\left(\bar{\Pi}^{n-1}rp\right) = a_1$. Now, as above, one has $\theta_2\left(\bar{\Pi}^{n-1}qp\right) = \Pi^{n-1}\theta_{n+1}(qp) = b_2a_2$ and $\theta_2\left(\bar{\Pi}^{n-1}rp\right) = \Pi^{n-1}\theta_{n+1}(rp) = c_2a_2$, so that we only have to deal with the case where $n = 1$. Then, with the notations of section 8, if $s = \bar{\Pi}^2 p$, one checks that one has $qp = \widehat{\hat{s}^b}^a$ and $rp = \widehat{\hat{s}^c}^a$, whence the result.

In particular, if $\mathcal{S}$ is some n-triangle in $\bar{\Gamma}$, the restriction of θ_n to $\partial\mathcal{S}$ is completely determined by the restriction of θ_{n+1} to $\partial\mathcal{S}$. By definition, the values of θ_n are thus determined by those of θ_{n+1}. Finally, let p and q be such that $\theta_n(p) = \theta_n(q)$ and let us show the assumption of the lemma by induction on $n \geq 1$. For $n = 1$, this assumption is empty. Suppose $n \geq 2$ and the assumption has been established for $n - 1$. Then, one has $\theta_{n-1}\left(\bar{\Pi}p\right) = \Pi\theta_n(p) = \Pi\theta_n(q) = \theta_{n-1}\left(\bar{\Pi}q\right)$ and, for any integer m with $1 \leq m \leq n$, if $\theta_n(p)$ does not belong to the m-triangle which admits one of the vertices of $\mathcal{T}_n$ as a vertex, $\theta_{n-1}(\bar{\Pi}p)$ does not belong to the $(m-1)$-triangle which admits one of the vertices of $\mathcal{T}_{n-1}$ as a vertex and hence, by induction, $s_m(p) = s_{m-1}\left(\bar{\Pi}p\right) = s_{m-1}\left(\bar{\Pi}q\right) = s_m(q)$. It remains to handle the case where $m = 0$. Suppose thus $\theta_n(p)$ is not a vertex of $\mathcal{T}_n$ and let us prove that $s_0(p) = s_0(q)$. Note that, by the first part of the proof, one has $\theta_1(p) = \theta_1(q)$. After an eventual action of $\mathfrak{S}$, suppose $\theta_1(p) = a_1$. Then, let $\mathcal{T}$ be the n-triangle of $\bar{\Gamma}$ containing p and p' be the neighbor of p that does not belong to the 1-triangle containing p. As p is not a vertex of $\mathcal{T}$, p' belongs to $\mathcal{T}$ and, by lemma 8.3, $s_0(p)$ is either B_0 or riB_0, following $\theta_1(p')$ is b_1 or c_1. Now, as θ_n induces an isomorphism from $\mathcal{T}$ onto $\mathcal{T}_n$, $\theta_n(p')$ only depends on $\theta_n(p)$ and hence, still by the first part of the lemma, the value of θ_1 at p' is completely determined by the value of θ_n at p. Therefore, the value of s_0 at p is determined by the value of θ_n at p, what should be proved. $\square$

For any integer $n \geq 1$, by proposition 8.5, one has $\theta_n(p_0) = a_n = \theta_n(rip_0)$, so that the coding of $\bar{\Gamma}$ by the maps θ_n, $n \geq 1$ is ambiguous. This ambiguities are described by the following

Corollary 9.2. *Let p and q be in $\bar{\Gamma}$ such that, for any integer n, one has $\theta_n(p) = \theta_n(q)$. Then, if $p \neq q$, there exists s in $\mathfrak{S}$ such that p belongs to the infinite triangle with vertex sp_0 in $s\Gamma$ and q belongs to the infinite triangle with vertex $srip_0$ in $sri\Gamma$.*

Proof. Suppose one has $p \neq q$. Then, by proposition 8.9, there exists some natural integer m such that $s_m(p) \neq s_m(q)$. By lemma 9.1, for any integer $n \geq m$, the point $\theta_n(p) = \theta_n(q)$ belongs to the m-triangle which admits one of the vertices of $\mathcal{T}_n$ as a vertex. Set $p' = \bar{\Pi}^m p$ and $q' = \bar{\Pi}^m q$. By lemma 9.1, for any integer n, one has $\theta_n(p') = \Pi^m\theta^{n+m}(p) = \Pi^m\theta^{n+m}(q) = \theta_n(q')$ and this point is one of the vertices of $\mathcal{T}_n$. As, for any integer $n \geq 1$, one has

$\theta_n(a_{n+1}) = a_n$, $\theta_n(b_{n+1}) = b_n$ and $\theta_n(c_{n+1}) = c_n$, one can suppose, after an eventual action of $\mathfrak{S}$, one has, for any integer $n \geq 1$, $\theta_n(p') = \theta_n(q') = a_n$. As $\theta_1(p') = a_1$, the keel of p' is B_0 or riB_0. After another action of $\mathfrak{S}$, suppose this keel is B_0. Then, by lemma 8.8, the keel of $\bar{\Pi}p'$ is B_0, iB_0 or $r^{-1}B_0$. As $\theta_1(\bar{\Pi}p') = \Pi\theta_1(p) = a_1$, the keel of $\bar{\Pi}p'$ is B_0 and, by induction, for any integer n, one has $s_n(p') = e$, so that, by proposition 8.9, $p' = p_0$ and p belongs to the infinite triangle $\mathcal{T}_\infty(p_0)$ with vertex p_0 in Γ. In the same way, one has $q' = p_0$ or $q' = rip_0$ and q belongs to the infinite triangle with vertex p_0 in Γ or to the infinite triangle with vertex rip_0 in $ri\Gamma$. For any integer n, the map θ_n induces a bijection from the n-triangle with vertex p_0 in Γ onto $\mathcal{T}_n$. Therefore, if p'' is some point in $\mathcal{T}_\infty(p_0)$ such that, for any integer n, one has $\theta_n(p'') = \theta_n(p)$, one has $p'' = p$. As we supposed $p \neq q$, q belongs to the infinite triangle with vertex rip_0 in $ri\Gamma$, what should be proved. $\square$

Let n be an integer. We shall say that a function $\varphi : \bar{\Gamma} \to \mathbb{C}$ is n-triangular if it may be written $\varphi = \psi \circ \theta_n$, for some function ψ on $\mathcal{T}_n$. When there is no ambiguity, to simplify notations, we shall identify φ and ψ. By lemma 9.1, a n-triangular function is $(n+1)$-triangular. In particular, triangular functions constitute a subalgebra of the algebra of locally constant functions on $\bar{\Gamma}$. As, for any triangular function φ, one has $\varphi(p_0) = \varphi(rip_0)$, this subalgebra is not dense in $\mathcal{C}^0\left(\bar{\Gamma}\right)$ for the topology of uniform convergence.

From now on, we let μ denote the Borel probability measure on $\bar{\Gamma}$ whose image under the coding map of proposition 8.9 is the maximal entropy measure for σ on Σ. In other terms, μ is the unique measure such that, for any sequence $t_0, \ldots, t_n$ of elements of $\mathfrak{S}$, if, for any $0 \leq m \leq n-1$, one has $t_m \in \{t_{m+1}, t_{m+1}i, t_{m+1}r\}$, then $\mu\left(t_0B_0 \cap \bar{\Pi}^{-1}t_1B_0 \cap \ldots \cap \bar{\Pi}^{-n}t_nB_0\right) = \frac{1}{6.3^n}$. By definition, the measure μ is $\bar{\Pi}$-invariant and $\mathfrak{S}$-invariant.

For any Borel function φ on $\bar{\Gamma}$, one sets $\bar{\Pi}^*\varphi = \varphi \circ \bar{\Pi}$. For any $1 \leq p \leq \infty$, the operator $\bar{\Pi}^*$ preserves the norm of $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$. One let $\bar{\Pi}$ denote its adjoint, that is, for any Borel function φ on $\bar{\Gamma}$, one has $\bar{\Pi}\varphi(p) = \frac{1}{3}\sum_{\bar{\Pi}(q)=p}\varphi(q)$. One has $\bar{\Pi}1 = 1$ and, for any $1 \leq p \leq \infty$, the operator $\bar{\Pi}$ is bounded with norm 1 in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$. Finally, one has $\bar{\Pi}\bar{\Pi}^* = 1$.

The integral of triangular functions with respect to the measure μ may be computed in a natural way:

Lemma 9.3. *Let n be a nonnegative integer and φ be a n-triangular function. One has $\int_{\bar{\Gamma}} \varphi \mathrm{d}\mu = \frac{1}{3^n}\sum_{p \in \mathcal{T}_n}\varphi(p)$.*

In other terms, the image measure of μ by θ_n is the normalized counting measure on $\mathcal{T}_n$.

Proof. Let us prove the result by induction on n. If $n = 0$, φ is constant and the lemma is evident. If $n \geq 1$, as, by lemma 9.1, one has $\Pi\theta_n = \theta_{n-1}\bar{\Pi}$, the function $\bar{\Pi}\varphi$ is $(n-1)$-triangular and one has, by induction, $\int_{\bar{\Gamma}} \varphi \mathrm{d}\mu = \int_{\bar{\Gamma}} \bar{\Pi}\varphi \mathrm{d}\mu = \frac{1}{3^{n-1}} \sum_{p\in\mathcal{T}_{n-1}} \frac{1}{3} \sum_{\Pi(q)=p} \varphi(q) = \frac{1}{3^n} \sum_{p\in\mathcal{T}_n} \varphi(p)$, whence the result. $\square$

From now on, for any integer n, one shall identify θ_n and the associate partition of the measure space $(\bar{\Gamma}, \mu)$. By lemma 9.1, this sequence of partitions is increasing. As μ is atom free, one has $\mu\left(\bigcup_{s\in\mathfrak{S}} s\Gamma\right) = 0$ and, by corollary 9.2, for any p and q in the total measure set $\bar{\Gamma} - \bigcup_{s\in\mathfrak{S}} s\Gamma$, if, for any integer n, one has $\theta_n(p) = \theta_n(q)$, one has $p = q$. For any φ in $\mathrm{L}^1\left(\bar{\Gamma}, \mu\right)$, for any integer n, one let $\mathbb{E}(\varphi|\theta_n)$ denote the conditional expectation of φ knowing θ_n, that is, for any p in $\mathcal{T}_n$, one has $\mathbb{E}(\varphi|\theta_n)(p) = \frac{1}{\mu(\theta_n^{-1}(p))} \int_{\theta_n^{-1}(p)} \varphi \mathrm{d}\mu$.

Lemma 9.4. *For any $1 \leq p < \infty$, for any φ in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$, one has $\mathbb{E}(\varphi|\theta_n) \xrightarrow[n\to\infty]{} \varphi$ in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$. In particular, the space of triangular functions is dense in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$. In the same way the space of triangular functions that take the value zero at the vertices of their definition triangle is dense in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$. Finally, for any integer n, for any φ in $\mathrm{L}^1\left(\bar{\Gamma}, \mu\right)$, one has $\mathbb{E}\left(\bar{\Pi}^*\varphi|\theta_{n+1}\right) = \Pi^*\mathbb{E}(\varphi|\theta_n)$, $\mathbb{E}\left(\bar{\Pi}\varphi|\theta_n\right) = \frac{1}{3}\Pi\mathbb{E}(\varphi|\theta_{n+1})$ and, for any p in $\mathcal{T}_n$,*

$$\mathbb{E}(\varphi|\theta_n)(p) = \frac{1}{3} \sum_{\substack{q\in\mathcal{T}_{n+1} \\ \theta_n(q)=p}} \mathbb{E}(\varphi|\theta_{n+1})(q).$$

Démonstration. The convergence in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$ follows from the discussion above and general properties of probability spaces. The density of triangular functions that are zero at the vertices of their definition triangles follows, since, by lemma 9.4, for any $n \geq 1$, the measure of the set of elements of $\bar{\Gamma}$ that are the vertex of some n-triangle is 3^{n-1}. Finally, the formulae linking conditional expectations knowing θ_{n+1} and θ_n follow from lemma 9.1 and the fact that, by lemma 9.3, the image measure of μ by θ_n is the normalized counting measure on $\mathcal{T}_n$. $\square$

Let us finally describe a homeomorphism $\bar{\Gamma} \to \bar{\Gamma}$ that will be useful in the sequel. For any p in $\bar{\Gamma}$, let $\alpha(p)$ denote the unique neighbor of p that does not belong to the triangle containing p. The map α is a fixed point free involution. By corollary 2.5, for any $n \geq 1$, α stabilizes the set of points in $\bar{\Gamma}$ that are vertex of some n-triangle. For p in $\mathcal{T}_n - \partial\mathcal{T}_n$, let still $\alpha_n(p)$ denote the unique neighbor of p in $\mathcal{T}_n$ that does not belong to the triangle containing p.

Lemma 9.5. *For any integer n, for any p in $\bar{\Gamma}$, if p is not a vertex of some n-triangle of $\bar{\Gamma}$, one has $\theta_n(\alpha(p)) = \alpha_n(\theta_n(p))$. The map α preserves the measure μ and, for any $n \geq 1$, φ in $\mathrm{L}^1\left(\bar{\Gamma}, \mu\right)$ and p in $\mathcal{T}_n - \partial\mathcal{T}_n$, one has $\mathbb{E}\left(\varphi \circ \alpha | \theta_n\right)(p) = \mathbb{E}(\varphi|\theta_n)(\alpha_n(p))$.*

As triangular functions are not dense in $\mathcal{C}^0\left(\bar{\Gamma}\right)$, to check that α preserves the measure μ, we shall use the following

Lemma 9.6. *Let X be a compact metric space and let A be a complex uniformly closed and conjugation stable subalgebra of $\mathcal{C}^0(X)$. Let Y be the set of elements x in X for which there exists $y \neq x$ in X such that, for any φ in A, $\varphi(y) = \varphi(x)$. The set Y is Borel and, if λ is a Borel complex measure on X such that $\lambda_{|Y} = 0$ and that, for any φ in A, $\int_X \varphi \mathrm{d}\lambda = 0$, one has $\lambda = 0$.*

Proof. Let S be the spectrum of the commutative C*-algebra A and $\pi : X \to S$ the surjective continuous map which is dual to the natural injection from A into $\mathcal{C}^0(X)$. By the hypothesis, the complex measure $\pi_*\lambda$ is zero on S. Let p denote the projection onto the first component $X \times X \to X$ and set $D = \{(x,x)|x \in X\} \subset X \times X$ and $E = \{(x,y) \in X \times X | \pi(x) = \pi(y)\}$. One has $Y = p(E - D)$. As E and D are closed subsets of the compact metrizable space X, the space $E - D$ is a countable union of compact sets, hence so are Y and $\pi(Y)$. In particular, these subsets are Borel and π induces a Borel isomorphism from $X - Y$ onto $S - \pi(Y)$. Therefore, the restriction of λ to $X - Y$ is zero. As its restriction to Y is zero, one has $\lambda = 0$. □

Proof of lemma 9.5. The first part of the lemma follows from the definition of α and the fact that θ_n induces a graph isomorphism from the n-triangle containing p onto $\mathcal{T}_n$.

Let $n \geq 1$. As α exchanges the points of $\bar{\Gamma}$ that are vertex of some n-triangle, if φ is a n-triangular function which is zero at the vertices of $\mathcal{T}_n$, one has, by lemma 9.3, $\int_{\bar{\Gamma}} \varphi \circ \alpha \mathrm{d}\mu = \int_{\bar{\Gamma}} \varphi \mathrm{d}\mu$. As, again by lemma 9.3, for any integer $n \geq 1$, the measure of the set points of $\bar{\Gamma}$ that are vertex of some n-triangle is 3^{n-1}, one deduces that, for any triangular function φ, one has $\int_{\bar{\Gamma}} \varphi \circ \alpha \mathrm{d}\mu = \int_{\bar{\Gamma}} \varphi \mathrm{d}\mu$.

Let $A \subset \mathcal{C}^0\left(\bar{\Gamma}\right)$ be the uniform closure of the algebra of triangular functions. By corollary 9.2, the set of elements p in $\bar{\Gamma}$ for which there exists $q \neq p$ such that, for any φ in A, one has $\varphi(p) = \varphi(q)$ is $\bigcup_{s \in \mathfrak{S}} s\Gamma$. As μ is atom free, this set has zero measure for μ and for $\alpha_* \mu$. For any φ in A, one has $\int_{\bar{\Gamma}} \varphi \circ \alpha \mathrm{d}\mu = \int_{\bar{\Gamma}} \varphi \mathrm{d}\mu$. By lemma 9.6, one hence has $\alpha_* \mu = \mu$. $\square$

Finally, let $\bar{\Theta}$ denote the quotient of $\bar{\Gamma}$ by the map α, endowed with the measure λ, which is the image of μ by the natural projection. The space $\bar{\Theta}$ may be seen in a natural way as the set of edges of $\bar{\Gamma}$ and may be equipped with a 4-regular graph structure. The image of Γ in $\bar{\Theta}$ then identifies in a natural way with the Sierpiński graph Θ and is a dense subset of $\bar{\Theta}$. We call triangular functions on $\bar{\Theta}$ the functions coming from triangular functions on $\bar{\Gamma}$ that are zero on the vertices of their definition triangle and α-invariant. Then, the results of this section transfer into analogous results on $\bar{\Theta}$.

10 The operator $\bar{\Delta}$ and its harmonic measures

We shall now study an operator $\bar{\Delta}$ on $\bar{\Gamma}$ which is an analogue of the operator Δ on Γ.

Let φ be a Borel function on $\bar{\Gamma}$. For any p in $\bar{\Gamma}$, one sets $\bar{\Delta}\varphi(p) = \sum_{q \sim p} \varphi(q)$. To study the properties of this operator on triangular functions, set moreover, for any nonnegative integer n, for any function φ on $\mathcal{T}_n$, for any p in $\mathcal{T}_n - \partial\mathcal{T}_n$, $\Delta\varphi(p) = \sum_{q \sim p} \varphi(q)$ and, for any p in $\partial\mathcal{T}_n$, $\Delta\varphi(p) = \varphi(p) + \sum_{q \sim p} \varphi(q)$.

Lemma 10.1. *For any integer $n \geq 1$, the operator Δ is self-adjoint in $\ell^2(\mathcal{T}_n)$. If φ is a n-triangular function that is constant on $\partial\mathcal{T}_n$, one has $\Delta\varphi = \bar{\Delta}\varphi$.*

Proof. One checks easily that $\bar{\Delta}$ is self-adjoint. Besides, as θ_n induces graph isomorphisms from n-triangles of $\bar{\Gamma}$ onto $\mathcal{T}_n$, for any p in $\mathcal{T}_n - \partial\mathcal{T}_n$, one has

$$\bar{\Delta} 1_{\theta_n^{-1}(p)} = \sum_{q \sim p} 1_{\theta_n^{-1}(q)} = \Delta 1_{\theta_n^{-1}(p)}.$$

In the same way, as any point p of $\bar{\Gamma}$ that is a vertex of some n-triangle, the unique neighbor of p that does not belong to this n-triangle is itself a vertex of some n-triangle, one has $\bar{\Delta} \sum_{p \in \partial\mathcal{T}_n} 1_{\theta_n^{-1}(p)} = \Delta \sum_{p \in \partial\mathcal{T}_n} 1_{\theta_n^{-1}(p)}$ and hence, for any n-triangular function φ that is constant on $\partial\mathcal{T}_n$, $\bar{\Delta}\varphi = \Delta\varphi$. $\square$

We can now set the principal properties of $\bar{\Delta}$ in the following

Proposition 10.2. *The operator $\bar{\Delta}$ commutes with the action of $\mathfrak{S}$. It is continuous with norm 3 in the space of continuous functions on $\bar{\Gamma}$ and one has $\bar{\Delta}^*\mu = 3\mu$. For any $1 \leq p \leq \infty$, the operator $\bar{\Delta}$ is continuous with norm 3 in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$ and, for $\frac{1}{p} + \frac{1}{q} = 1$, for any φ in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$ and ψ in $\mathrm{L}^q\left(\bar{\Gamma}, \mu\right)$, one has $\langle \bar{\Delta}\varphi, \psi \rangle = \langle \varphi, \bar{\Delta}\psi \rangle$.*

Proof. The first assumption is evident. As $\bar{\Delta}$ is positive and $\bar{\Delta}1 = 3$, $\bar{\Delta}$ is continuous with norm 3 in the space of continuous functions.

Recall one let α denote the map $\bar{\Gamma} \to \bar{\Gamma}$ that sends some point p to its unique neighbor that does not belong to the 1-triangle containing p. For φ in $\mathcal{C}^0\left(\bar{\Gamma}\right)$, one has $\bar{\Delta}\varphi = 3\bar{\Pi}^*\bar{\Pi}\varphi + \varphi \circ \alpha - \varphi$. As the operators $\bar{\Pi}$ and $\bar{\Pi}^*$ preserve μ and, by lemma 9.5, the homeomorphism α preserves μ, one has $\bar{\Delta}^*\mu = 3\mu$.

For any $1 \leq p \leq \infty$, the positive operator $\bar{\Delta}$ thus acts on $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$ and it is bounded with norm 3 in this space. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. By lemma 9.4, triangular functions that are zero on the vertices of their definition triangles are dense in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$ and $\mathrm{L}^q\left(\bar{\Gamma}, \mu\right)$. By lemma 9.4, one hence has, for any φ in $\mathrm{L}^p\left(\bar{\Gamma}, \mu\right)$ and ψ in $\mathrm{L}^q\left(\bar{\Gamma}, \mu\right)$, $\langle \bar{\Delta}\varphi, \psi \rangle = \langle \varphi, \bar{\Delta}\psi \rangle$. As the operators appearing in this identity are continuous in $\mathrm{L}^1\left(\bar{\Gamma}, \mu\right)$ and $\mathrm{L}^\infty\left(\bar{\Gamma}, \mu\right)$, it is still true for $p = 1$ and $q = \infty$. $\square$

We shall now prove that the measure μ is, up to scalar multiplication, the unique Borel complex measure λ on $\bar{\Gamma}$ such that $\bar{\Delta}^*\lambda = 3\lambda$. Let us begin by handling the case where λ is $\mathfrak{S}$-invariant.

Lemma 10.3. *Let λ be a $\mathfrak{S}$-invariant Borel complex measure on $\bar{\Gamma}$ with $\bar{\Delta}^*\lambda = 3\lambda$. One has $\lambda = \lambda\left(\bar{\Gamma}\right)\mu$.*

Proof. Let $\varphi : p \mapsto \lambda(p), \Gamma \to \mathbb{C}$. Then, φ belongs to $\ell^1(\Gamma)$ and one has $\Delta\varphi = 3\varphi$. By the maximum principle, one hence has $\varphi = 0$. Therefore, the restriction of φ to $\bigcup_{s \in \mathfrak{S}} s\Gamma$ is zero. By corollary 9.2 and lemma 9.6, it thus suffices to check that λ is proportional to μ on the space of triangular functions. Let $n \geq 1$ be an integer and, for any p in $\mathcal{T}_n$, let $\varphi_n(p) = \lambda(\theta_n^{-1}(p)) = \int_{\bar{\Gamma}} 1_{\theta_n^{-1}(p)} \mathrm{d}\lambda$. By lemma 10.1, if p is not a vertex of $\mathcal{T}_n$, one has $\bar{\Delta}1_{\theta_n^{-1}(p)} = \Delta 1_{\theta_n^{-1}(p)}$ and hence, as $\bar{\Delta}^*\lambda = 3\lambda$, $\Delta\varphi_n(p) = 3\varphi_n(p)$. Moreover, as λ is $\mathfrak{S}$-invariant, φ_n is constant on $\partial\mathcal{T}_n$. By the maximum principle, φ_n is constant. As $\sum_{p \in \mathcal{T}_n} \varphi_n(p) = \lambda\left(\bar{\Gamma}\right)$, one has, for any p in $\mathcal{T}_n$, $\varphi_n(p) = \frac{1}{3^n}\lambda\left(\bar{\Gamma}\right)$, whence the result, by lemma 9.3. $\square$

Let us now study the eigenspace associate to the eigenvalue 1 in $\mathrm{L}^1\left(\bar{\Gamma},\mu\right)$. We will need the following

Lemma 10.4. *Let $n \geq 1$ be an integer, φ be in $\mathrm{L}^1\left(\bar{\Gamma},\mu\right)$ and p be in $\mathcal{T}_n - \partial\mathcal{T}_n$. One has $\Delta\mathbb{E}(\varphi|\theta_n)(p) = \mathbb{E}\left(\bar{\Delta}\varphi|\theta_n\right)(p)$.*

Proof. Let still α and α_n be as in lemma 9.5. One has $\bar{\Delta}\varphi = 3\bar{\Pi}^*\bar{\Pi}\varphi + \varphi\circ\alpha - \varphi$ and hence, by lemmas 9.4 and 9.5,

$$\begin{aligned}\mathbb{E}\left(\bar{\Delta}\varphi|\theta_n\right)(p) = \Pi^*\Pi\mathbb{E}\left(\varphi|\theta_n\right)(p) + \mathbb{E}\left(\varphi|\theta_n\right)(\alpha_n(p)) - \mathbb{E}\left(\varphi|\theta_n\right)(p)\\ = \Delta\mathbb{E}\left(\varphi|\theta_n\right)(p),\end{aligned}$$

what should be proved. □

For any $n \geq 1$, let H_n denote the space of functions φ on $\mathcal{T}_n$ such that, for any p in $\mathcal{T}_n$ that is not a vertex, one has $\Delta\varphi(p) = 3\varphi(p)$. By lemma 10.4, for any φ in $\mathrm{L}^1\left(\bar{\Gamma},\mu\right)$, if $\bar{\Delta}\varphi = 3\varphi$, one has $\mathbb{E}(\varphi|\theta_n) \in H_n$. One identifies $\mathbb{C}^3$ and the space of complex valued function on $\mathcal{T}_1$ by considering $(1,0,0)$ (resp. $(0,1,0)$, resp. $(0,0,1)$) as the characteristic function of the singleton $\{a_1\}$ (resp. $\{b_1\}$, resp. $\{c_1\}$) and one let η_n denote the $\mathfrak{S}$-equivariant linear map $H_n \to \mathbb{C}^3, \varphi \mapsto (\varphi(a_n), \varphi(b_n), \varphi(c_n))$. Besides, one let $(s_n)_{n\geq 1}$ denote the real sequence such that $s_1 = 1$ and, for any $n \geq 1$, $s_{n+1} = \frac{3s_n}{3s_n+5}$. One easily shows that one has $s_n \xrightarrow[n\to\infty]{} 0$. Let $\mathbb{C}^3_0$ be the set of elements in $\mathbb{C}^3$ the sum of whose coordinates is zero. We have the following

Lemma 10.5. *Let $n \geq 1$. For any φ in H_n and p in $\mathcal{T}_n$, one has $|\varphi(p)| \leq \max\{|\varphi(a_n)|, |\varphi(b_n)|, |\varphi(c_n)|\}$. In particular, the map η_n is an isomorphism. Suppose $n \geq 2$. Let φ be in H_n such that $\eta_n(\varphi)$ belongs to $\mathbb{C}^3_0$ and $\psi = \mathbb{E}(\varphi|\theta_{n-1})$. Then ψ belongs to H_{n-1} and one has $\eta_{n-1}(\psi) = \frac{2}{3s_{n-1}+5}\eta_n(\varphi)$.*

Proof. The bound follows from the maximum principle, applied to the operator $\frac{1}{3}\Delta$. It implies that, for any $n \geq 1$, the operator η_n is injective. Let φ be a function on $\mathcal{T}_n$ and, for any p in $\mathcal{T}_n$, set $\delta_n\varphi(p) = \Delta\varphi(p) - 3\varphi(p)$. If p is not a vertex and $\delta_n\varphi(p) = \varphi(p)$ if p is a vertex. As η_n is injective, so is δ_n, and hence it is an isomorphism; in particular, η_n is onto, hence it is an isomorphism.

For any $n \geq 2$, set $t_n = \frac{3s_{n-1}+2}{3s_{n-1}+5}$ and $u_n = \frac{1}{3s_{n-1}+5}$. Recall that, as in corollary 2.6, if $\mathcal{S}$ is a n-triangle and if p and q are two vertices of $\mathcal{S}$, one let pq denote the unique point of $\mathcal{S}$ belonging to a $(n-1)$-triangle containing p

and admitting a neighbor belonging to the $(n-1)$-triangle containing q. Let d_n and e_n be the two neighbors of a_n in $\mathcal{T}_n$. Let us prove by induction on $n \geq 2$ that, for any φ in H_n, one has $\varphi(d_n) + \varphi(e_n) = s_n(\varphi(b_n) + \varphi(c_n)) + 2(1 - s_n)\varphi(a_n)$ and $\varphi(a_n b_n) = t_n \varphi(a_n) + u_n(2\varphi(b_n) + \varphi(c_n))$. For $n = 2$, this is an immediate computation. If $n \geq 3$ and the formula is true for $n - 1$, pick some function φ in H_n. Then, as $\Delta\varphi(a_n b_n) = 3\varphi(a_n b_n)$, by applying the induction to the restriction of φ to the $(n-1)$-triangle containing a_n, one has

$$s_{n-1}(\varphi(a_n) + \varphi(a_n c_n)) + 2(1 - s_{n-1})\varphi(a_n b_n) + \varphi(b_n a_n) = 3\varphi(a_n b_n).$$

As η_n is an isomorphism, there exists a unique (x, y, z) in $\mathbb{C}^3$ such that, for any φ in H_n, one has $\varphi(a_n b_n) = x\varphi(a_n) + y\varphi(b_n) + z\varphi(c_n)$. As η_n is $\mathfrak{S}$-equivariant, one has, for any φ in H_n, $\varphi(b_n a_n) = x\varphi(b_n) + y\varphi(a_n) + z\varphi(c_n)$ and $\varphi(a_n c_n) = x\varphi(a_n) + y\varphi(c_n) + z\varphi(b_n)$. Thus

$$\begin{aligned} s_{n-1}(1 + x) + 2(1 - s_{n-1})x + y &= 3x \\ s_{n-1}z + 2(1 - s_{n-1})y + x &= 3y \\ s_{n-1}y + 2(1 - s_{n-1})z + z &= 3z. \end{aligned}$$

By solving this system, one gets $x = t_n$, $y = 2u_n$ and $z = u_n$. Finally, by induction, one has

$$\begin{aligned} \varphi(d_n) + \varphi(e_n) &= s_{n-1}(\varphi(a_n b_n) + \varphi(a_n c_n)) + 2(1 - s_{n-1})\varphi(a_n) \\ &= 3s_{n-1}u_n(\varphi(b_n) + \varphi(c_n)) + 2(1 - s_{n-1} + s_{n-1}t_n)\varphi(a_n), \end{aligned}$$

whence the result, since $3s_{n-1}u_n = s_n = s_{n-1}(1 - t_n)$.

Then, if $\psi = \mathbb{E}(\varphi|\theta_{n-1})$, one has, by lemma 9.4, for any p in $\mathcal{T}_{n-1}$, $\psi(p) = \frac{1}{3}\sum_{\theta_{n-1}(q)=p} \varphi(q)$. As θ_{n-1} induces a graph isomorphism from each of the $(n-1)$-triangles of $\mathcal{T}_n$ onto $\mathcal{T}_{n-1}$, one deduces that ψ belongs to H_{n-1} and that, in particular, by lemma 9.1, if $\eta_n(\varphi)$ is in $\mathbb{C}_0^3$, one has

$$\begin{aligned} \psi(a_n) &= \frac{1}{3}(\varphi(a_n) + \varphi(b_n a_n) + \varphi(c_n a_n)) \\ &= \frac{1}{3}\left((1 + 4u_n)\varphi(a_n) + (t_n + u_n)\left(\varphi(b_n) + \varphi(c_n)\right)\right) \\ &= \frac{1 + 4u_n - t_n - u_n}{3}\varphi(a_n) = \frac{2}{3s_{n-1} + 5}\varphi(a_n), \end{aligned}$$

where, for the penultimate equality, one has used the relation $\varphi(a_n) + \varphi(b_n) + \varphi(c_n) = 0$. By $\mathfrak{S}$-equivariance, one has the analogous formula at the two other vertices of $\mathcal{T}_n$ and hence $\eta_{n-1}(\psi) = \frac{2}{3s_{n-1}+5}\eta_n(\varphi)$. $\square$

Corollary 10.6. *Let φ be in $\mathrm{L}^\infty\left(\bar{\Gamma},\mu\right)$ such that $\bar{\Delta}\varphi = 3\varphi$ and that $\sum_{s\in\mathfrak{S}}\varphi\circ s = 0$. One has $\varphi = 0$.*

Démonstration. For any integer $n \geq 1$, set $\varphi_n = \mathbb{E}(\varphi|\theta_n)$. By lemma 10.4, one has $\varphi_n \in H_n$. As $\sum_{s\in\mathfrak{S}}\varphi\circ s = 0$, one has $\eta_n(\varphi_n) \in \mathbb{C}_0^3$. Therefore, if $n \geq 2$, by lemma 10.5, as $\varphi_{n-1} = \mathbb{E}(\varphi_n|\theta_{n-1})$, one has $\eta_{n-1}(\varphi_{n-1}) = \frac{2}{3s_{n-1}+5}\eta_n(\varphi_n)$. Now, for any $n \geq 1$, one has $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$ and, as $s_n \xrightarrow[n\to\infty]{} 0$, $\prod_{n=1}^\infty \frac{2}{3s_n+5} = 0$. Therefore, one necessarily has, for any $n \geq 1$, $\eta_n(\varphi_n) = 0$, hence, by lemma 10.5, $\varphi_n = 0$, and, by lemma 9.4, $\varphi = 0$. □

We can now describe the eigenvectors with eigenvalue 3 in $\mathrm{L}^1\left(\bar{\Gamma},\mu\right)$:

Lemma 10.7. *Let φ be in $\mathrm{L}^1\left(\bar{\Gamma},\mu\right)$ such that $\bar{\Delta}\varphi = 3\varphi$. The function φ is constant μ-almost everywhere.*

Proof. One can suppose that φ takes only real values. Let us prove that it suffices to handle the case where φ is nonnegative. Indeed, as $\bar{\Delta}$ is positive, one has $\bar{\Delta}\left|\varphi\right| \geq \left|\bar{\Delta}\varphi\right| = 3\left|\varphi\right|$ and hence, as $\bar{\Delta}$ has norm 3, $\bar{\Delta}\left|\varphi\right| = 3\left|\varphi\right|$. By studying the functions $|\varphi| - \varphi$ and $|\varphi| + \varphi$, one can suppose one has $\varphi \geq 0$. Set $\psi = \sum_{s\in\mathfrak{S}}\varphi\circ s$. The measure $\lambda = \psi\mu$ is $\mathfrak{S}$-invariant and one has $\bar{\Delta}^*\lambda = 3\lambda$. By lemma 10.3, λ is proportional to μ, that is ψ is constant μ-almost everywhere. In particular, ψ is in $\mathrm{L}^\infty\left(\bar{\Gamma},\mu\right)$. As one has $0 \leq \varphi \leq \psi$, φ is in $\mathrm{L}^\infty\left(\bar{\Gamma},\mu\right)$. Then, by corollary 10.6, on a $\varphi - \frac{1}{6}\psi = 0$. □

In order to extend this result to all complex measures on $\bar{\Gamma}$, we shall use a general, surely classical lemma. Let X be a compact metric space. Equip the space $\mathcal{C}^0(X)$ with the uniform convergence topology. If λ is a complex Borel measure on X, recall the total variation $|\lambda|$ of λ is the finite positive Borel measure on X such that, for any continuous nonnegative function g on X, one has

$$\int_X g\mathrm{d}\left|\lambda\right| = \sup_{\substack{h\in\mathcal{C}^0(X)\\ \|h\|_\infty\leq 1}}\left|\int_X gh\mathrm{d}\lambda\right|$$

(one may refer to [15, Chapter 6]). In particular, $|\lambda|$ is the smallest positive Radon measure such that, for any continuous nonnegative function g on X, one has $\left|\int_X g\mathrm{d}\lambda\right| \leq \int_X g\mathrm{d}\left|\lambda\right|$.

Lemma 10.8. *Let X be a compact metric space and P a positive operator with norm 1 on the space of continuous functions on X. For any Borel* *complex measure λ on X, one has $|P^*\lambda| \leq P^* |\lambda|$. In particular, if $P^*\lambda = \lambda$, one has $P^* |\lambda| = |\lambda|$.*

Proof. For any nonnegative continuous function g on X, one has $Pg \geq 0$, hence $\left|\int_X Pg\mathrm{d}\lambda\right| \leq \int_X Pg\mathrm{d}|\lambda|$. As the measure $P^* |\lambda|$ is positive, one thus has $|P^*\lambda| \leq P^* |\lambda|$. If $P^*\lambda = \lambda$, one has $|\lambda| \leq P^* |\lambda|$, whence the equality, since P has norm 1. □

We finally deduce the following

Proposition 10.9. *Let μ be complex Borel measure on $\bar{\Gamma}$ such that $\bar{\Delta}^*\lambda = 3\lambda$. One has $\lambda = \lambda\left(\bar{\Gamma}\right)\mu$.*

Proof. One can suppose λ takes real values. By applying lemma 10.8 to the operator $\frac{1}{3}\bar{\Delta}$, one has $\bar{\Delta}^* |\lambda| = 3|\lambda|$, so that, by studying the measures $|\lambda| - \lambda$ and $|\lambda| + \lambda$, one can suppose λ is positive. Then, by lemma 10.3, the measure $\sum_{s \in \mathfrak{S}} s_*\lambda$ is proportional to μ. As one has $0 \leq \lambda \leq \sum_{s \in \mathfrak{S}} s_*\lambda$, λ is absolutely continuous with respect to μ. By lemma 10.7, λ is thus proportional to μ. □

11 Spectrum and spectral measures of $\bar{\Gamma}$

We shall now come to the spectral study of the operator $\bar{\Delta}$. Let us begin by noting that, as in lemma 3.1, one has the following

Lemma 11.1. *One has $(\bar{\Delta}^2 - \bar{\Delta} - 3)\bar{\Pi}^* = \bar{\Pi}^*\bar{\Delta}$ and $\bar{\Pi}(\bar{\Delta}^2 - \bar{\Delta} - 3) = \bar{\Delta}\bar{\Pi}$.*

Recall we let α denote the map that sends a point p of $\bar{\Gamma}$ to the neighbor of p that does not belong to the 1-triangle containing p. As in section 3, from lemma 11.1, one deduces the following

Corollary 11.2. *The spectrum of $\bar{\Delta}$ is the union of Λ and of the set $\bigcup_{n \in \mathbb{N}} f^{-n}(0)$. The eigenspace associate to the eigenvalue -2 is the space of functions φ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ such that $\bar{\Pi}\varphi = 0$ and $\varphi \circ \alpha = -\varphi$. The eigenspace associate to the eigenvalue 0 is the space of functions φ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ such that $\bar{\Pi}\varphi = 0$ and $\varphi \circ \alpha = \varphi$.*

Proof. Let, as in corollary 3.6, $K = \bar{\Pi}^*\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ and H be the closed subspace of $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ spanned by K and by $\bar{\Delta}K$. By lemma 11.1, one has $f\left(\bar{\Delta}\right)K \subset K$ and, as $\bar{\Pi}^*$ is an isometry from $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ onto K, the spectrum of $f\left(\bar{\Delta}\right)$ in K equals the spectrum of $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$. We will seek to apply lemma 3.3 to the operator $\bar{\Delta}$ in H. On this purpose, let us prove that $\bar{\Delta}^{-1}K \cap K$ only contains constant functions. Let φ and ψ be in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ such that $\bar{\Delta}\bar{\Pi}^*\varphi = \bar{\Pi}^*\psi$. For any integer $n \geq 1$, set $\varphi_n = \mathbb{E}(\varphi|\theta_n)$ and $\psi_n = \mathbb{E}(\psi|\theta_n)$. By lemmas 9.4 and 10.4, for any p in $\mathcal{T}_{n+1} - \partial\mathcal{T}_{n+1}$, one has $\Delta\Pi^*\varphi_n(p) = \Pi^*\psi_n(p)$. By proceeding as in the proof of corollary 3.6, one deduces that, for any q in $\mathcal{T}_n$, φ_n is constant on the neighbors of q. As any point of $\mathcal{T}_n$ is contained in a triangle and $\mathcal{T}_n$ is connected, φ_n is constant. As, by lemma 9.4, one has $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$, φ is constant. Thus, the space $\bar{\Delta}^{-1}K \cap K$ equals the line of constant functions. By lemmas 3.3 and 11.1, the spectrum of $\bar{\Delta}$ in H thus equals the union of $\{3\}$ and of the inverse image by f of the spectrum of $\bar{\Delta}$ in the space of functions with zero integral in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$.

Besides, proceeding as in lemma 3.7, one sees that the orthogonal complement L of H in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ is the direct sum of the space L_{-2} of the elements φ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ such that $\bar{\Pi}\varphi = 0$ and $\varphi \circ \alpha = -\varphi$ and of the space L_0 of the elements φ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ such that $\bar{\Pi}\varphi = 0$ and $\varphi \circ \alpha = \varphi$. One has $\bar{\Delta} = -2$ on L_{-2} and $\bar{\Delta} = 0$ on L_0. Proceeding as in lemma 3.8 and using lemma 10.1, one sees that these subspaces are not reduced to $\{0\}$, since they contain triangular functions. As in the proof of corollary 3.9, one deduces that the spectrum of $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ equals the union of Λ and $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$.

Finally, as in the proof of lemma 3.7, it remains to prove that L_{-2} and L_0 are exactly the eigenspaces of $\bar{\Delta}$ associate to the eigenvalues -2 and 0, that is $\bar{\Delta}$ does not admit the eigenvalue -2 or 0 in H. Let φ be in H such that $\bar{\Delta}\varphi = -2\varphi$. By lemma 11.1, one has $\bar{\Delta}\bar{\Pi}\varphi = 3\bar{\Pi}\varphi$ and hence by lemma 10.7, $\bar{\Pi}\varphi$ is constant. As φ is orthogonal to constant functions, one has $\bar{\Pi}\varphi = 0$ and $\bar{\Pi}\bar{\Delta}\varphi = -2\bar{\Pi}\varphi = 0$. As φ is in H, one thus has $\varphi = 0$. In the same way, if φ is in H and if $\bar{\Delta}\varphi = 0$, one has $\bar{\Delta}\bar{\Pi}\varphi = -3\bar{\Pi}\varphi$. Now, by an immediate computation, -3 does not belong to the spectrum of $\bar{\Delta}$. Thus $\bar{\Pi}\varphi = 0$ and hence $\varphi = 0$, what should be proved. $\square$

We also have an analogue of lemma 4.1:

Lemma 11.3. *One has* $\bar{\Pi}\bar{\Delta}\bar{\Pi}^* = 2 + \frac{1}{3}\bar{\Delta}$ *and hence, for any* φ *and* ψ *in* $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$,

$$\begin{aligned}\left\langle \bar{\Delta}\bar{\Pi}^*\varphi, \bar{\Pi}^*\psi \right\rangle &= 2\left\langle \varphi, \psi \right\rangle + \frac{1}{3}\left\langle \bar{\Delta}\varphi, \psi \right\rangle \\ &= 2\left\langle \bar{\Pi}^*\varphi, \bar{\Pi}^*\psi \right\rangle + \frac{1}{3}\left\langle \left(\bar{\Delta}^2 - \bar{\Delta} - 3\right)\bar{\Pi}^*\varphi, \bar{\Pi}^*\psi \right\rangle.\end{aligned}$$

As in section 4, one deduces the following

Corollary 11.4. *Let φ be in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$, μ be the spectral measure of φ for $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ and ν be the spectral measure of $\bar{\Pi}^*\varphi$ for $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. Then, one has $\nu(\frac{1}{2})=0$ and, if, for any $x\neq\frac{1}{2}$, one sets $\tau(x)=\frac{x(x+2)}{3(2x-1)}$, one has $\nu=L^*_{f,\tau}\mu$.*

12 Eigenfunctions in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$

In this section, we shall follow the plan of section 5, in order to describe the eigenspaces of $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. As in section 5, by using lemmas 11.1 and 11.3, one proves the following analogue of lemma 5.1:

Lemma 12.1. *Let H be the closed subspace of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ spanned by the image of $\bar{\Pi}^*$ and by the one of $\bar{\Delta}\bar{\Pi}^*$. Then, for any x in $\mathbb{R}-\{0,-2\}$, x is an eigenvalue of $\bar{\Delta}$ in H if and only if $y=f(x)$ is an eigenvalue of $\bar{\Delta}$ in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. In this case, the map $\bar{R}_x$ which sends an eigenfunction φ with eigenvalue y in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ to $(x-1)\bar{\Pi}^*\varphi+\bar{\Delta}\bar{\Pi}^*\varphi$ induces an isomorphism between the eigenspace associate to the eigenvalue y in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ and the eigenspace associate to the eigenvalue x in H and, for any φ, one has $\left\|\bar{R}_x\varphi\right\|_2^2=\frac{1}{3}x(x+2)(2x-1)\left\|\varphi\right\|_2^2$.*

To describe the eigenfunctions with eigenvalue in $\bigcup_{n\in\mathbb{N}}f^{-n}(0)$, we shall proceed as in section 5. On this purpose, note again that, for any integer $n\geq 1$, the space of edges that are exterior to n-triangles of $\bar{\Gamma}$ may be identified in a natural way with $\bar{\Theta}$. If φ is a function on $\bar{\Gamma}$ which is constant on edges which are exterior to n-triangles, we shall let $\bar{P}_n\varphi$ denote the function on $\bar{\Theta}$ whose value at one point of $\bar{\Theta}$ is the value of φ on the associate edge which is exterior to n-triangles of $\bar{\Gamma}$. Besides, one still let $\bar{\Delta}$ denote the operator that sends a function ψ on $\bar{\Theta}$ to the function whose value at some point p of $\bar{\Theta}$ is $\sum_{q\sim p}\psi(q)$. This operator satisfies $\bar{\Delta}^*\lambda=4\lambda$ and it is self-adjoint with norm 4 in $\mathrm{L}^2\left(\bar{\Theta},\lambda\right)$, where λ is the measure on $\bar{\Theta}$ that has been introduced at the end of section 9.

Lemma 12.2. *The map $\bar{P}_2$ induces a Banach spaces isomorphism from the eigenspace of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ associate to the eigenvalue 0 onto $\mathrm{L}^2\left(\bar{\Theta},\lambda\right)$. Let $\bar{Q}_0$ denote its inverse. For any ψ in $\mathrm{L}^2\left(\bar{\Theta},\lambda\right)$, one has $\left\|\bar{Q}_0\psi\right\|^2_{\mathrm{L}^2\left(\bar{\Gamma},\mu\right)}=\frac{1}{2}\left\|\psi\right\|^2_{\mathrm{L}^2\left(\bar{\Theta},\lambda\right)}-\frac{1}{12}\langle\Delta\psi,\psi\rangle_{\mathrm{L}^2\left(\bar{\Theta},\lambda\right)}$.*

Proof. One proceeds as in lemma 5.2 by using the characterization of eigenfunctions with eigenvalue 0 given in corollary 11.2. The formula may be easily checked on triangular functions that are zero at the vertices of their definition triangle and the general case follow by density. □

Recall that, for x in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$, one let $n(x)$ denote the integer n such that $f^n(x) = 0$ and

$$\kappa(x) = \prod_{k=0}^{n(x)-1} \frac{f^k(x)(2f^k(x)-1)}{f^k(x)+2}.$$

From lemmas 12.1 and 12.2, one deduces the following analogue of proposition 5.3:

Proposition 12.3. *Let x be in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$. The eigenfunctions with eigenvalue x in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ are constant on edges which are exterior to $(n(x)+1)$-triangles in $\bar{\Gamma}$. The map $\bar{P}_{n(x)+2}$ induces a Banach spaces isomorphism from the eigenspace of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ associate to the eigenvalue x onto $\mathrm{L}^2\left(\bar{\Theta},\lambda\right)$. Let $\bar{Q}_x$ denote its inverse. Then, for any ψ in $\mathrm{L}^2\left(\bar{\Theta},\lambda\right)$, one has $\|Q_x\psi\|^2_{\mathrm{L}^2\left(\bar{\Gamma},\mu\right)} = \kappa(x)\left(3\|\psi\|^2_{\mathrm{L}^2\left(\bar{\Theta},\lambda\right)} - \frac{1}{2}\langle\Delta\psi,\psi\rangle_{\mathrm{L}^2\left(\bar{\Theta},\lambda\right)}\right)$.*

Corollary 12.4. *For any x in $\bigcup_{n\in\mathbb{N}} f^{-n}(0)$, the eigenspace associate to x in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ has infinite dimension and is spanned by triangular functions that are zero at the vertices of their definition triangle.*

As in section 5, the description of the eigenvalues associate to the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$ is less precise.

Let us begin by the case of the eigenvalue -2. We shall need supplementary informations on triangular functions that are eigenvectors with eigenvalue -2. On this purpose, pick some integer $n \geq 1$ and some n-triangle $\mathcal{S}$ and let $E_{\mathcal{S}}$ denote the space of functions φ on $\mathcal{S}$ such that $\Pi\varphi = 0$ and, for any point p of $\mathcal{S}$ that is not a vertex of $\mathcal{S}$, if q is the neighbor of p that does not belong to the triangle containing p, one has $\varphi(q) = -\varphi(p)$. If $\mathcal{S}$ is $\mathcal{T}_n$, one let E_n stand for $E_{\mathcal{T}_n}$. By lemmas 9.4 and 9.5 and corollary 11.2, if φ is an element of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ such that $\bar{\Delta}\varphi = -2\varphi$, for any integer $n \geq 1$, one has $\mathbb{E}(\varphi|\theta_n) \in E_n$. Proceeding as in lemma 5.5, one proves the following

Lemma 12.5. *Let $n \geq 1$, $\mathcal{S}$ be a n-triangle with vertices p, q and r and φ be in $E_{\mathcal{S}}$. One has $\varphi(p) + \varphi(q) + \varphi(r) = 0$.*

The space $\mathbb{C}^3_0 = \{(s,t,u) \in \mathbb{R}^3 | s+t+u=0\}$ is stable under the action of $\mathfrak{S}$ on $\mathbb{C}^3$. We endow it with the $\mathfrak{S}$-invariant hermitian norm $\|.\|_0$ such that, for any (s,t,u) in $\mathbb{C}^3_0$, one has $\|(s,t,u)\|_0^2 = \frac{1}{3}\left(|s|^2+|t|^2+|u|^2\right)$. For any $n \geq 1$, one let ρ_n denote the $\mathfrak{S}$-equivariant linear map $E_n \to \mathbb{C}^3_0, \varphi \mapsto (\varphi(a_n), \varphi(b_n), \varphi(c_n))$, F_n the kernel of ρ_n and G_n the orthogonal complement of F_n in E_n for the norm of $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$. By lemma 10.1, the elements of F_n are eigenvectors with eigenvalue -2 of $\bar{\Delta}$.

Lemma 12.6. *Let $n \geq 1$. One has $\dim F_n = \frac{1}{2}(3^{n-1}-1)$. The map ρ_n is onto and, for any φ in G_n, one has $\|\varphi\|^2_{\mathrm{L}^2(\bar{\Gamma},\mu)} = \left(\frac{5}{9}\right)^{n-1} \|\rho_n(\varphi)\|_0^2$. Finally, if $n \geq 2$ and if $\psi = \mathbb{E}(\varphi|\theta_{n-1})$, ψ belongs to G_{n-1} and $\rho_{n-1}(\psi) = \frac{2}{3}\rho_n(\varphi)$.*

Proof. Let $n \geq$, $\mathcal{S}$ be a n-triangle and p and q be distinct vertices of $\mathcal{S}$. Define a function $\varphi^{p,q}_{\mathcal{S}}$ on $\mathcal{S}$ in the following way. If $n=1$, one sets $\varphi^{p,q}_{\mathcal{S}}(p) = 1$, $\varphi^{p,q}_{\mathcal{S}}(q) = -1$ and one says that $\varphi^{p,q}_{\mathcal{S}}$ is zero at the third point of $\mathcal{S}$. If $n \geq 2$, let still pq and qp denote the points defined in corollary 2.6: the point pq belongs to the $(n-1)$-triangle $\mathcal{P}$ containing p in $\mathcal{S}$, the point qp belongs to the $(n-1)$-triangle $\mathcal{Q}$ containing q in $\mathcal{S}$ and the points pq and qp are neighbors. One defines $\varphi^{p,q}_{\mathcal{S}}$ as the function whose restriction to $\mathcal{P}$ is $\varphi^{p,pq}_{\mathcal{P}}$, whose restriction to $\mathcal{Q}$ is $\varphi^{qp,q}_{\mathcal{Q}}$ and whose restriction to the third $(n-1)$-triangle of $\mathcal{S}$ is zero. One easily checks that $\varphi^{p,q}_{\mathcal{S}}$ belongs to $E_{\mathcal{S}}$. If $\mathcal{S} = \mathcal{T}_n$, one let $\varphi^{p,q}_n$ stand for $\varphi^{p,q}_{\mathcal{T}_n}$. As one has $\rho_n(\varphi^{a_n,b_n}_n) = (1,-1,0)$ and $\rho_n(\varphi^{a_n,c_n}_n) = (1,0,-1)$, the map ρ_n is onto.

For $n \geq 2$, let ψ_n denote the function on $\mathcal{T}_n$ whose restriction to the $(n-1)$-triangle $\mathcal{A}_n$ (resp. $\mathcal{B}_n$, resp. $\mathcal{C}_n$) containing a_n (resp. b_n, resp. c_n) equals φ^{a_n,b_n}_n (resp. φ^{b_n,c_n}_n, resp. φ^{c_n,a_n}_n). Then, one easily checks that ψ_n belongs to F_n.

These functions are pictured in figure 13.

Let us now establish by induction on $n \geq 1$ the formulae of the lemma on the dimension of F_n and the norm of the elements of G_n. For $n=1$, one has $F_1 = \{0\}$ and the map ρ_1 is an isomorphism, so that the formula on norms follows from lemma 9.3. Let thus suppose $n \geq 2$ and the formulae have been proved for $n-1$. We will explicitely construct the inverse map of ρ_n, depending on the one of ρ_{n-1}. For any triangle $\mathcal{S}$, let $F_{\mathcal{S}}$ be the set of elements of $E_{\mathcal{S}}$ that are zero at the vertices of $\mathcal{S}$ and $G_{\mathcal{S}}$ be the orthogonal complement of $F_{\mathcal{S}}$ with respect to the natural scalar product on $\ell^2(\mathcal{S})$. For any (s,t,u) in $\mathbb{C}^3_0$, let $\tau(s,t,u)$ be the unique function on $\mathcal{T}_n$ that takes the value s at a_n, t at b_n, u at c_n, $\frac{t-s}{3}$ at a_nb_n, $\frac{u-s}{3}$ at a_nc_n, $\frac{s-t}{3}$ at b_na_n, $\frac{u-t}{3}$ at

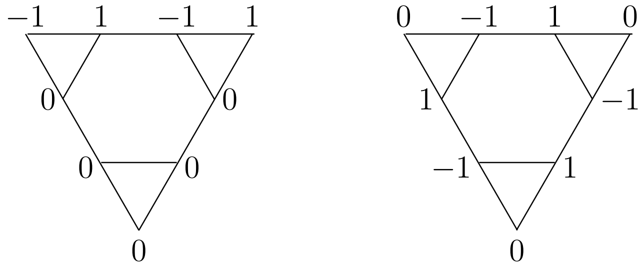

Figure 13: The functions $\varphi_2^{a_2,b_2}$ and ψ_2

b_nc_n, $\frac{s-u}{3}$ at c_na_n and $\frac{t-u}{3}$ at c_nb_n and whose restriction to $\mathcal{A}_n$ (resp. $\mathcal{B}_n$, resp. $\mathcal{C}_n$) belongs to $G_{\mathcal{A}_n}$ (resp. $G_{\mathcal{B}_n}$, resp. $G_{\mathcal{C}_n}$). Then $\tau(s,t,u)$ clearly belongs to E_n and $\rho_n(\tau(s,t,u)) = (s,t,u)$. Besides, one has, by lemma 9.3 and by induction,

$$\begin{aligned}\langle \tau(s,t,u), \psi_n\rangle_{\mathrm{L}^2(\bar{\Gamma},\mu)} &= \frac{1}{3^n}\Bigg(\left\langle \tau(s,t,u), \varphi_{\mathcal{A}_n}^{a_nb_n,a_nc_n}\right\rangle_{\ell^2(\mathcal{A}_n)} \\ &\quad + \left\langle \tau(s,t,u), \varphi_{\mathcal{B}_n}^{b_nc_n,b_na_n}\right\rangle_{\ell^2(\mathcal{B}_n)} \\ &\quad + \left\langle \tau(s,t,u), \varphi_{\mathcal{C}_n}^{c_na_n,c_nb_n}\right\rangle_{\ell^2(\mathcal{C}_n)}\Bigg) \\ &= \frac{1}{3}\frac{5^{n-2}}{9^{n-1}}\left((t-s)-(u-s)+(u-t)-(s-t)\right. \\ &\quad \left.+(s-u)-(t-u)\right) = 0\end{aligned}$$

Conversely, one easily checks, by an analogous scalar product computation, that, if φ is an element of E_n that is orthogonal to ψ_n and whose restriction to $\mathcal{A}_n$ (resp. $\mathcal{B}_n$, resp. $\mathcal{C}_n$) is in $G_{\mathcal{A}_n}$ (resp. $G_{\mathcal{B}_n}$, resp. $G_{\mathcal{C}_n}$), then φ belongs to the image of τ. As both these spaces have dimension 2, they coincide and τ is the inverse map of ρ_n. In particular, F_n is spanned by ψ_n and the elements that are zero at the vertices of $(n-1)$-triangles, so that $\dim F_n = 3\dim F_{n-1} + 1$, whence the dimension computation, by induction. Besides, again by lemma 9.3 and by induction, for any φ in G_n, if $\rho_n(\varphi) = (s,t,u)$, one has, by the definition of τ,

$$\begin{aligned}\|\varphi\|^2_{\mathrm{L}^2(\bar\Gamma,\mu)} &= \frac{5^{n-2}}{9^{n-1}}\left(|s|^2+|t|^2+|u|^2+2\left|\frac{s-t}{3}\right|^2+2\left|\frac{t-u}{3}\right|^2+2\left|\frac{s-u}{3}\right|^2\right)\\ &= \frac{1}{3}\frac{5^{n-1}}{9^{n-1}}\left(|s|^2+|t|^2+|u|^2\right)\end{aligned}$$

(taking in account, for the last equality, that $s+t+u=0$). The formula on norms follows, by induction.

Finally, for $n \geq 2$, pick φ in G_n and set $\psi = \mathbb{E}(\varphi|\theta_{n-1})$. As in the proof of lemma 10.5, one deduces from lemma 9.4 and the fact that θ_{n-1} induces graph isomorphisms between the $(n-1)$-triangles of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ that, as φ belongs to E_n, ψ belongs to E_{n-1}. As the elements of F_{n-1} are zero at the vertices of the $(n-1)$-triangles, they belong to F_n too, hence they are orthogonal to φ, so that ψ belongs to G_{n-1}. By lemmas 9.1 and 9.4, one has $\psi(a_n) = \frac{1}{3}(\varphi(a_n)+\varphi(b_na_n)+\varphi(c_na_n))$ and hence, by the formulae above, if $\rho_n(\varphi) = (s,t,u)$, one has $\psi(a_n) = \frac{1}{3}(s+\frac{s-t}{3}+\frac{s-u}{3}) = \frac{2}{3}s$ and $\rho_{n-1}(\psi) = \frac{2}{3}\rho_n(\varphi)$. ◻

We can now describe the eigenspace of $\bar\Delta$ associate to the eigenvalue -2:

Lemma 12.7. *The eigenspace associate to the eigenvalue -2 of $\bar\Delta$ has infinite dimension and is spanned by triangular functions that are zero at the vertices of their definition triangle.*

Proof. As, by lemma 12.6, for any $n \geq 1$, the space F_n has dimension $\frac{1}{2}(3^{n-1}-1)$ and, by lemma 10.1, its elements are eigenfunctions with eigenvalue -2, the eigenspace associate to the eigenvalue -2 has infinite dimension.

Let φ be an eigenfunction with eigenvalue -2 in $\mathrm{L}^2\left(\bar\Gamma,\mu\right)$ that is orthogonal to the eigenfunctions that are triangular and zero at the vertices of their definition triangle. Let us prove that φ is zero. For any integer $n \geq 1$, let $\varphi_n = \mathbb{E}(\varphi|\theta_n)$. By corollary 11.2, one has $\bar\Pi\varphi = 0$ and $\varphi\circ\alpha = -\varphi$ and hence, by lemmas 9.4 and 9.5, for any $n \geq 1$, φ_n belongs to E_n. As φ is orthogonal to the elements of F_n, φ_n belongs to G_n. If $n \geq 2$, as $\varphi_{n-1} = \mathbb{E}(\varphi_n|\theta_{n-1})$, by lemma 12.7, one has $\rho_{n-1}(\varphi_{n-1}) = \frac{2}{3}\rho_n(\varphi_n)$. Hence, there exists v in $\mathbb{C}^3_0$ such that, for any $n \geq 1$, one has $\rho_n(\varphi_n) = \left(\frac{3}{2}\right)^{n-1}v$, so that, again by lemma 12.7,

$$\|\varphi_n\|^2_{\mathrm{L}^2(\bar\Gamma,\mu)} = \left(\frac{5}{9}\right)^{n-1}\|\rho_n(\varphi_n)\|^2_0 = \left(\frac{5}{4}\right)^{n-1}\|v\|^2_0\,.$$

Figure 14: Values of φ and of $\bar{\Delta}\varphi$

As, by lemma 9.4, one has $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$, one thus has necessarily $v=0$, hence, for any $n\geq 1$, $\varphi_n=0$ and $\varphi=0$, what should be proved. $\square$

From lemmas 12.1 and 12.7, one deduces by induction the following

Corollary 12.8. *For any x in $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)$, the eigenspace associate to the eigenvalue x has infinite dimension and is spanned by triangular functions that are zero at the vertices of their definition triangle.*

13 Spectral decomposition of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$

In this section, we shall prove that $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ is the orthogonal direct sum of the space of constant functions, the eigenspaces associate to the elements of the set $\bigcup_{n\in\mathbb{N}} f^{-n}(-2)\cup\bigcup_{n\in\mathbb{N}} f^{-n}(0)$ and the cyclic subspaces spanned by 1-triangular functions φ such that $\bar{\Pi}\varphi=0$. Let us begin by describing these cyclic subspaces.

Lemma 13.1. *Let φ be in E_1. One has $\left(\bar{\Delta}+2\right)\varphi=\left(\bar{\Delta}-1\right)\bar{\Pi}^*\varphi$ and $\bar{\Pi}\bar{\Delta}\varphi=\left(1+\frac{1}{3}\bar{\Delta}\right)\varphi$.*

Proof. Let $(s,t,u)=(\varphi(a_1),\varphi(b_1),\varphi(c_1))$. One has, by definition, $s+t+u=0$. Let p be in $\bar{\Gamma}$. After an eventual action of the group $\mathfrak{S}$, one can suppose that $\bar{\Pi}p$ belongs to the keel B_0 from section 8. Then, the values of φ and of $\bar{\Delta}\varphi$ on the 1-triangle containing p and on its neighbors are those described by figure 14. In the same way, the values of $\bar{\Pi}^*\varphi$ and of $\bar{\Delta}\bar{\Pi}^*\varphi$ on the 1-triangle containing p and on its neighbors are those described by figure 15. If $\theta_1(p)=a_1$ or $\theta_1(p)=b_1$, one hence has $\left(\bar{\Delta}+2\right)\varphi(p)=2s+2t+u=s+t=\left(\bar{\Delta}-1\right)\bar{\Pi}^*\varphi(p)$; if $\theta_1(p)=c_1$, one has $\left(\bar{\Delta}+2\right)\varphi(p)=2s+t+2u=s+u=\left(\bar{\Delta}-1\right)\bar{\Pi}^*\varphi(p)$. Thus, we do have $\left(\bar{\Delta}+2\right)\varphi=\left(\bar{\Delta}-1\right)\bar{\Pi}^*\varphi$.

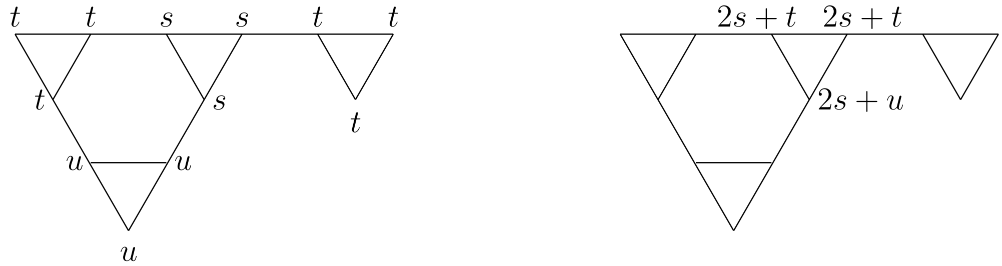

Figure 15: Values of $\bar{\Pi}^*\varphi$ and of $\bar{\Delta}\bar{\Pi}^*\varphi$

By definition, one has $\bar{\Pi}\varphi = 0$, so that, by applying $\bar{\Pi}$ to the preceding identity, one gets $\bar{\Pi}\bar{\Delta}\varphi = \bar{\Pi}\left(\bar{\Delta}+2\right)\varphi = \bar{\Pi}\bar{\Delta}\bar{\Pi}^*\varphi - \varphi = \left(1+\frac{1}{3}\bar{\Delta}\right)\varphi$, where, for the last equality, we made use of lemma 11.3. $\square$

Thanks to lemma 13.1, we shall proceed as in section 6 to determine the spectral measures of the elements of E_1. Let us begin by proving that these measures do not give mass to the points -2 and 0.

Lemma 13.2. *Let φ be in E_1 and ψ be an eigenfunction with eigenvalue -2 or 0 in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. One has $\langle\varphi,\psi\rangle = 0$.*

Proof. Suppose ψ is an eigenvector with eigenvalue 0. By corollary 11.2, one has $\bar{\Pi}\psi = 0$ and $\psi\circ\alpha = \psi$ and, by corollary 12.4, one can suppose that, for a certain integer $n \geq 2$, ψ is n-triangular, with value 0 at the vertices of $\mathcal{T}_n$. Let p, q and r be the vertices of a 2-triangle $\mathcal{S}$ of $\mathcal{T}_n$ and let pq, qp, pr, rp, qr and rq be the other points of $\mathcal{S}$, with the convention from corollary 2.6. Then, one has $\psi(qp) = \psi(pq)$ and $\psi(rp) = \psi(pr)$, hence $\psi(p)+\psi(qp)+\psi(rp) = 0$ and, by using the analogous identities on the other 1-triangles of $\mathcal{S}$, by lemma 9.1, as φ is 1-triangular, one has

$$\sum_{s\in\mathcal{S}}\overline{\varphi(s)}\psi(s) = \overline{\varphi(p)}\left(\psi(p)+\psi(qp)+\psi(rp)\right)$$
$$+\overline{\varphi(q)}\left(\psi(q)+\psi(pq)+\psi(rq)\right)+\overline{\varphi(r)}\left(\psi(r)+\psi(pr)+\psi(qr)\right) = 0$$

and hence, by lemma 9.3, $\langle\varphi,\psi\rangle = 0$.

Let us now handle the case of the eigenvalue -2. For any $n \geq 1$, let E_n and F_n be as in section 12. Let $(s,t,u) = (\varphi(a_1),\varphi(b_1),\varphi(c_1))$. Let us prove by induction on n that, if ψ belongs to E_n, one has

$$\sum_{p\in\mathcal{T}_n}\varphi(p)\psi(p)=2^{n-1}(s\psi(a_n)+t\psi(b_n)+u\psi(c_n)).$$

For $n=1$, the result is trivial. Suppose $n\geq 2$ and the result has been established for n. Then, by applying the induction to the restriction of ψ to $(n-1)$-triangles of $\mathcal{T}_n$, one gets, as φ is 1-triangular, by lemma 9.1,

$$\begin{aligned}\sum_{p\in\mathcal{T}_n}\varphi(p)\psi(p)&=2^{n-2}(\psi(a_n)s+\psi(a_nb_n)t+\psi(a_nc_n)u\\&\quad+\psi(b_n)t+\psi(b_na_n)s+\psi(b_nc_n)u+\psi(c_n)u+\psi(c_na_n)s+\psi(c_nb_n)t).\end{aligned}$$

Now, one has $\psi(a_nb_n)+\psi(b_na_n)=0$ and, by lemma 12.5, $\psi(a_n)+\psi(a_nb_n)+\psi(a_nc_n)=0$, so that $\psi(b_na_n)+\psi(c_na_n)=\psi(a_n)$. By using this identity and the analogous formulae at the other vertices of $\mathcal{T}_n$, we get

$$\sum_{p\in\mathcal{T}_n}\varphi(p)\psi(p)=2^{n-1}(s\psi(a_n)+t\psi(b_n)+u\psi(c_n)),$$

what should be proved. In particular, for ψ in F_n, one has, by lemma 9.3, $\langle\varphi,\psi\rangle=0$ and hence, by lemma 12.7, this is still true for any eigenvector ψ with eigenvalue -2. □

Corollary 13.3. *Let φ be in E_1 and ψ be an eigenfunction with eigenvalue in $\bigcup_{n\in\mathbb{N}}f^{-n}(-2)\cup\bigcup_{n\in\mathbb{N}}f^{-n}(0)$. One has $\langle\varphi,\psi\rangle=0$.*

Proof. By corollary 11.2 and lemmas 12.1 and 13.2, it suffices to prove that, for x in $\mathbb{R}$, if ψ is an eigenvector with eigenvalue x and if $\langle\varphi,\psi\rangle=0$, one has $\left\langle\varphi,\bar{\Pi}^*\psi\right\rangle=\left\langle\varphi,\bar{\Delta}\bar{\Pi}^*\psi\right\rangle=0$. Now, by definition, one has $\bar{\Pi}\varphi=0$, hence $\left\langle\varphi,\bar{\Pi}^*\psi\right\rangle=0$. Besides, by lemma 13.1, one has

$$\left\langle\varphi,\bar{\Delta}\bar{\Pi}^*\psi\right\rangle=\left\langle\bar{\Pi}\bar{\Delta}\varphi,\psi\right\rangle=\left\langle\varphi,\left(1+\frac{1}{3}\bar{\Delta}\right)\psi\right\rangle=\left(1+\frac{x}{3}\right)\langle\varphi,\psi\rangle=0,$$

what should be proved. □

Set, for any $x\neq-3$, $j(x)=\frac{1}{3}\frac{3-x}{x+3}$ and, for $x\neq\frac{1}{2}$, $\zeta(x)=\frac{1}{3}\frac{(x+3)(x-1)}{2x-1}$. As for corollary 6.3, we deduce from lemma 13.1 and corollary 11.4 the following

Corollary 13.4. *Let ν_ζ the unique Borel probability on Λ such that one has $L_\zeta^*\nu_\zeta=\nu_\zeta$. For any φ in E_1, the spectral measure of φ is $\|\varphi\|_2^2\,j\nu_\zeta$.*

Proof. As the proof of this result is analogous to the one of corollary 6.3, we just give its big steps. Let λ be the spectral measure of φ. By lemma 13.2, one has $\lambda(-2) = 0$. Set, for $x \notin \left\{-2, \frac{1}{2}\right\}$, $\theta(x) = \frac{x(x-1)^2}{3(x+2)(2x-1)}$. One has $\theta = \frac{j}{j \circ f}\zeta$ and, by corollary 11.4 and lemma 13.1, $\lambda = L_\theta^* \lambda$. By lemma 9.3, φ is orthogonal to constant functions. Therefore, by lemma 10.7, one has $\lambda(3) = 0$. Moreover, by corollaries 11.2 and 13.3, the measure λ is concentrated on Λ.

The function ζ is positive on Λ and $L_\zeta(1) = 1$. By lemma 6.2, there exists a unique Borel probability ν_ζ on Λ such that $L_\zeta^* \nu_\zeta = \nu_\zeta$. Proceeding as in the proof of corollary 6.3, one proves that the measures λ and $j\nu_\zeta$ are proportional. As one has $L_\zeta j = 1$, one gets $\lambda = \|\varphi\|_2^2 \, j\nu_\zeta$. $\square$

Let l denote the function $x \mapsto x$ on Λ and set, for $x \neq 1$, $m(x) = \frac{x+2}{x-1}$ and, for $x \neq \frac{1}{2}$, $\xi(x) = \frac{1}{3}\frac{x(x-1)}{2x-1}$. Let $\bar{\Phi}$ denote the closed subspace of $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ spanned by the elements of E_1 and their images by the powers of $\bar{\Delta}$ and, as in section 12, let ρ_1 design the $\mathfrak{S}$-equivariant isomorphism from E_1 onto $\mathbb{C}_0^3$. Still endow $\mathbb{C}_0^3$ with the hermitian norm that equals one third of the canonical norm and denote by $\langle ., . \rangle_0$ the associate scalar product. By lemma 9.3, the map ρ_1 is an isometry. Identify the Hilbert spaces $\mathrm{L}^2\left(j\nu_\zeta, \mathbb{C}_0^3\right)$ and $\mathrm{L}^2\left(j\nu_\zeta\right) \otimes \mathbb{C}_0^3$ and, for any polynomial p in $\mathbb{C}[X]$ and for any v in $\mathbb{C}_0^3$, set $\widehat{p \otimes v} = p\left(\bar{\Delta}\right) \rho_1^{-1}(v)$. We have an analogue of proposition 6.4:

Proposition 13.5. *The map $g \mapsto \hat{g}$ induces a $\mathfrak{S}$-equivariant isometry from $\mathrm{L}^2\left(j\nu_\zeta, \mathbb{C}_0^3\right)$ onto $\bar{\Phi}$. The subspace $\bar{\Phi}$ is stable by the operators $\bar{\Delta}$, $\bar{\Pi}$ and $\bar{\Pi}^*$. For any g in $\mathrm{L}^2\left(j\nu_\zeta, \mathbb{C}_0^3\right)$, one has*

$$\begin{aligned}
\bar{\Delta}\hat{g} &= \widehat{lg} \\
\bar{\Pi}\hat{g} &= \widehat{L_\xi g} \\
\bar{\Pi}^*\hat{g} &= \widehat{m(g \circ f)}.
\end{aligned}$$

Proof. Let p be in $\mathbb{C}[X]$. The map

$$\begin{aligned}
\mathbb{C}_0^3 \times \mathbb{C}_0^3 &\to \mathbb{C} \\
(v, w) &\mapsto \left\langle p\left(\bar{\Delta}\right) \rho_1^{-1}(v), \rho_1^{-1}(w)\right\rangle_{\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)}
\end{aligned}$$

is a $\mathfrak{S}$-invariant sesquilinear form. As the representation of $\mathfrak{S}$ on $\mathbb{C}_0^3$ is irreducible, this sesquilinear form is proportional to the scalar product $\langle ., . \rangle_0$.

By lemma 9.3 and corollary 13.4, for v in $\mathbb{C}_0^3$, one has

$$\left\langle p\left(\bar{\Delta}\right)\rho_1^{-1}(v),\rho_1^{-1}(v)\right\rangle_{\mathrm{L}^2\left(\bar{\Gamma},\mu\right)}=\|v\|_0^2\int_{\bar{\Gamma}}pj\mathrm{d}\nu_\zeta,$$

therefore, for any p and q in $\mathbb{C}[X]$, for any v and w in $\mathbb{C}_0^3$, one has

$$\left\langle\widehat{p\otimes v},\widehat{q\otimes w}\right\rangle_{\mathrm{L}^2\left(\bar{\Gamma},\mu\right)}=\langle v,w\rangle_0\langle p,q\rangle_{\mathrm{L}^2(j\nu\zeta)}$$

and hence the map $g\mapsto\hat{g}$ induces an isometry from $\mathrm{L}^2\left(j\nu_\zeta,\mathbb{C}_0^3\right)$ onto a closed subspace of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. As this subspace is spanned by the elements of E_1 and their images by the powers of $\bar{\Delta}$, by definition, it equals $\bar{\Phi}$.

The rest of the proof is analogous to the one of proposition 6.4.

The stability of $\bar{\Phi}$ by $\bar{\Delta}$ and the formula for $\bar{\Delta}$ result from the very definition of the objects in question.

A direct computation shows that $L_\xi(1)=0$ and that $L_\xi(l)=1+\frac{1}{3}l$, so that, for any nonnegative integer n, one has $L_\xi(f^n)=0$ and $L_\xi(f^nl)=l^n\left(1+\frac{1}{3}l\right)$. Now, by lemmas 11.1 and 13.1, for any φ in E_1, one has $\bar{\Pi}\left(f\left(\bar{\Delta}\right)^n\varphi\right)=0$ and $\bar{\Pi}\left(f\left(\bar{\Delta}\right)^n\bar{\Delta}\varphi\right)=\bar{\Delta}^n\left(1+\frac{1}{3}\bar{\Delta}\right)\varphi$. The space $\bar{\Phi}$ is thus stable by $\bar{\Pi}$ and, for any p in $\mathbb{C}[X]$ and v in $\mathbb{C}_0^3$, one has $\bar{\Pi}\widehat{p\otimes v}=\widehat{L_\xi(p)\otimes v}$. As ζ is positive on Λ, there exists a real number $c>0$ such that, for any x in Λ, one has $|\xi(x)|\leq c\zeta(x)$, so that, for any Borel function g on Λ, one has $|L_\xi(g)|\leq cL_\zeta\left(|g|\right)$. Proceeding as in the proof of proposition 6.4, one proves that L_ζ is bounded in $\mathrm{L}^2(j\nu_\zeta)$. One deduces that L_ξ is bounded and the identity concerning $\bar{\Pi}$ follows, by density.

Finally, by lemmas 11.1 and 13.1, for any p in $\mathbb{C}[X]$ and φ in E_1, one has $\left(\bar{\Delta}-1\right)\bar{\Pi}^*\left(p\left(\bar{\Delta}\right)\varphi\right)=p\left(f\left(\bar{\Delta}\right)\right)\left(\bar{\Delta}+2\right)\varphi$. By corollary 11.2, 1 does not belong to the spectrum of $\bar{\Delta}$, so that, by density, for any rational function p whose poles do not belong to the spectrum of $\bar{\Delta}$, one has $\bar{\Pi}^*\left(p\left(\bar{\Delta}\right)\varphi\right)=\left(m(p\circ f)\right)\left(\bar{\Delta}\right)\varphi$ and hence the space $\bar{\Phi}$ is stable by $\bar{\Pi}^*$. Moreover, as, for any x in Λ, one has $m(x)^2\frac{j(x)}{j(f(x))}=\frac{x(x+2)}{(x-1)(x+3)}$, one gets, by an elementary computation, $L_\zeta\left(m^2\frac{j}{j\circ f}\right)=1$ and, for any g in $\mathrm{L}^2(j\nu_\zeta)$,

$$\int_\Lambda|m(g\circ f)|^2j\mathrm{d}\nu_\zeta=\int_\Lambda\left(m^2\frac{j}{j\circ f}\right)|g\circ f|^2(j\circ f)\mathrm{d}\nu_\zeta=\int_\Lambda|g|^2j\mathrm{d}\nu_\zeta.$$

The formula for $\bar{\Pi}^*$ follows, by density. □

Let us now deal with the other $\mathfrak{S}$-isotypic components of the space $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. Let $\varepsilon:\mathfrak{S}\to\{-1,1\}$ denote the signature morphism. We shall say that a function φ on $\bar{\Gamma}$ is $(\mathfrak{S},\varepsilon)$-semi-invariant if, for any s in $\mathfrak{S}$, one has $\varphi\circ s=\varepsilon(s)\varphi$.

Proposition 13.6. *For any integer $n\geq 1$, the space of $\mathfrak{S}$-invariant n-triangular functions on $\bar{\Gamma}$ is stable by $\bar{\Delta}$ and the characteristic polynomial of $\bar{\Delta}$ in this space is*

$$(X-3)\prod_{p=0}^{n-2}(f^p(X))^{\frac{3^{n-2-p}+2n-2p-1}{4}}(f^p(X)+2)^{\frac{3^{n-2-p}-2n+2p+3}{4}}.$$

For any integer $n\geq 2$, the space of $(\mathfrak{S},\varepsilon)$-semi-invariant n-triangular functions on $\bar{\Gamma}$ is stable by $\bar{\Delta}$ and the characteristic polynomial of $\bar{\Delta}$ in this space is

$$\prod_{p=0}^{n-2}(l\circ f^p(X))^{\frac{3^{n-2-p}-2n+2p+3}{4}}(k\circ f^p(X))^{\frac{3^{n-2-p}+2n-2p-1}{4}}.$$

Proof. These spaces are stable by lemma 10.1. The computation of the characteristic polynomials is obtained proceeding as in the proof of proposition 7.5. □

From this proposition, we deduce, using lemma 9.4, the following:

Corollary 13.7. *The spectrum of $\bar{\Delta}$ in the space of $\mathfrak{S}$-invariant elements of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ is discrete. The eigenvalues of $\bar{\Delta}$ in this space are 3, which is simple, and the elements of $\bigcup_{n\in\mathbb{N}}f^{-n}(-2)\cup\bigcup_{n\in\mathbb{N}}f^{-n}(0)$. The spectrum of $\bar{\Delta}$ in the space of $(\mathfrak{S},\varepsilon)$-semi-invariant elements of $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$ is discrete. The eigenvalues of $\bar{\Delta}$ in this space are the elements of $\bigcup_{n\in\mathbb{N}}f^{-n}(-2)\cup\bigcup_{n\in\mathbb{N}}f^{-n}(0)$.*

The proof of theorem 1.3 ends with the following

Proposition 13.8. *Let $\bar{\Phi}^\perp$ be the orthogonal complement of $\bar{\Phi}$ in $\mathrm{L}^2\left(\bar{\Gamma},\mu\right)$. The spectrum of $\bar{\Delta}$ in $\bar{\Phi}^\perp$ is discrete. Its eigenvalues in this space are 3, which is simple, and the elements of $\bigcup_{n\in\mathbb{N}}f^{-n}(-2)\cup\bigcup_{n\in\mathbb{N}}f^{-n}(0)$.*

The proof of this proposition is analogous to the one of proposition 6.9. It needs us to introduce objects that will play the role of the spaces L_n, $n\in\mathbb{N}$, of this proof.

Let us use the notations from section 8 and recall that, by construction, if p is a point of $\bar{\Gamma}$ such that $\theta_1(p) = a_1$, the keel of p is B_0 or riB_0. For any integer $n \geq 1$, let $\mathcal{B}_n$ be the set which is the union of $\mathcal{T}_n - \partial\mathcal{T}_n$ and of the set of the six pairs of the form (d, B) where d belongs to $\partial\mathcal{T}_n$ and B is one of the two keels for which there exists points p of B with $\theta_n(p) = d$. Denote by τ_n the locally constant map $\bar{\Gamma} \to \mathcal{B}_n$ such that, for any p in $\bar{\Gamma}$, if p is not a vertex of some n-triangle, one has $\tau_n(p) = \theta_n(p)$ and, if p is a vertex of some n-triangle, $\tau_n(p)$ is the pair $(\theta_n(p), B)$ where B is the keel containing p. Finally, let say that a function φ on $\bar{\Gamma}$ is τ_n-measurable if one has $\varphi = \psi \circ \tau_n$, where ψ is some function defined on $\mathcal{B}_n$. The interest of this definition comes from the following

Lemma 13.9. *Let $n \geq 1$ be an integer and φ be a τ_{n+1}-mesurable function on $\bar{\Gamma}$. Then, the functions $\bar{\Pi}\varphi$ and $\bar{\Pi}\bar{\Delta}\varphi$ are τ_n-mesurable.*

Proof. Let p be a point of $\bar{\Gamma}$. If p is not a vertex of some n-triangle, the triangle $\bar{\Pi}^{-1}p$ does not contain a vertex of some $(n+1)$-triangle. In the same way, none of the neighbors of $\bar{\Pi}^{-1}p$ is a vertex of some $(n+1)$-triangle. Thus, for any of the points q appearing in the computation of $\bar{\Pi}\varphi(p)$ and $\bar{\Pi}\bar{\Delta}\varphi(p)$, one has $\tau_n(q) = \theta_n(q)$. Therefore, by the definition of θ_n and by lemma 9.1, $\bar{\Pi}\varphi(p)$ and $\bar{\Pi}\bar{\Delta}\varphi(p)$ only depend on $\theta_n(p)$.

If p is now a vertex of some n-triangle, the neighbor q of p that does not belong to this n-triangle is itself a vertex of some n-triangle and, by lemma 8.3, the keel of q is determined by the one of p. In particular, $\tau_n(q)$ is determined by $\tau_n(p)$. Only one of the antecedents of the point p by the map $\bar{\Pi}$ is a vertex of some $(n+1)$-triangle. By lemma 8.8, it is the one whose keel equals the one of p. In particular, the image by τ_{n+1} of this point r is determined by $\tau_n(p)$. In the same way, the image by τ_{n+1} of the unique antecedent s of q that is a vertex of some $(n+1)$-triangle only depends on $\tau_n(q)$, and so on $\tau_n(p)$. The point s is the neighbor of r that does not belong to $\bar{\Pi}^{-1}p$. Finally, the two other points of $\bar{\Pi}^{-1}p$ and their neighbors that do not belong to $\bar{\Pi}^{-1}p$ are not vertices of some $(n+1)$-triangle and hence their image by τ_{n+1} is their image by θ_{n+1} that only depend on $\theta_n(p)$. Again, $\bar{\Pi}\varphi(p)$ and $\bar{\Pi}\bar{\Delta}\varphi(p)$ only depend on $\theta_n(p)$. □

Proof of proposition 13.8. By lemma 10.7, the eigenvalue 3 of $\bar{\Delta}$ is simple. By corollaries 12.4 and 12.8, the eigenspaces associate to the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$ are non zero. Let $\bar{P}$ denote the orthogonal projector from $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$ onto $\bar{\Phi}^{\perp}$ and, for any φ and ψ in $\mathrm{L}^2\left(\bar{\Gamma}, \mu\right)$, denote by

$\lambda_{\varphi,\psi}$ the unique complex Borel measure on $\mathbb{R}$ such that, for any polynomial p in $\mathbb{C}[X]$, one has $\int_{\mathbb{R}} p \mathrm{d}\lambda_{\varphi,\psi} = \langle p(\bar{\Delta}) \varphi, \psi \rangle$. By proposition 13.5, the operator $\bar{P}$ commutes with $\bar{\Delta}$, $\bar{\Pi}$ and $\bar{\Pi}^*$. By lemma 9.4, to prove the proposition, it suffices to establish that, for any integer $n \geq 1$, for any τ_n-mesurable function φ, for any ψ in $\mathrm{L}^2(\bar{\Gamma}, \mu)$, the measure $\lambda_{\bar{P}\varphi,\psi}$ is atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(3) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$. Let us prove this result by induction on n.

For $n = 1$, the τ_1-mesurables functions are those functions that only depend on the keel. One easily checks that this space is spanned by the constant functions, a line of $(\mathfrak{S}, \varepsilon)$-semi-invariant functions, the elements of E_1 and their images by $\bar{\Delta}$. In this case, the description of spectral measures immediately follows from corollaries 13.4 and 13.7.

If the result is true for an integer $n \geq 1$, let us pick some τ_{n+1}-measurable function φ. Then, by lemma 13.9, the functions $\bar{\Pi}\varphi$ and $\bar{\Pi}\bar{\Delta}\varphi$ are τ_n-measurable and, by induction, for any ψ in $\mathrm{L}^2(\bar{\Gamma}, \mu)$, the measures $\lambda_{\bar{\Pi}\bar{P}\varphi,\psi} = \lambda_{\bar{P}\bar{\Pi}\varphi,\psi}$ and $\lambda_{\bar{\Pi}\bar{\Delta}\bar{P}\varphi,\psi} = \lambda_{\bar{P}\bar{\Pi}\bar{\Delta}\varphi,\psi}$ are atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(3) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$. Proceeding as in lemma 6.10, one deduces that the measures $\lambda_{\bar{P}\varphi,\bar{\Pi}^*\psi}$ and $\lambda_{\bar{\Delta}\bar{P}\varphi,\bar{\Pi}^*\psi} = \lambda_{\bar{P}\varphi,\bar{\Delta}\bar{\Pi}^*\psi}$ are atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(3) \cup \bigcup_{n\geq 1} f^{-n}(0)$. Now, by corollary 11.2, the spectrum of $\bar{\Delta}$ in the orthogonal complement of the subspace of $\mathrm{L}^2(\bar{\Gamma}, \mu)$ spanned by the image of $\bar{\Pi}^*$ and by the one of $\bar{\Delta}\bar{\Pi}^*$ equals $\{-2, 0\}$. Therefore, for any ψ in $\mathrm{L}^2(\bar{\Gamma}, \mu)$, the measure $\lambda_{\bar{P}\varphi,\psi}$ is atomic and concentrated on the set $\bigcup_{n\in\mathbb{N}} f^{-n}(3) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(0)$. The result follows. $\square$

14 The Sierpiński graph

In this section, we will quickly explain how the results that have been obtained in this article for the Pascal graph Γ may be transfered to the Sierpiński graph Θ pictured in figure 2. As explained in section 2, the graph Θ identifies with the edges graph of Γ. If φ is some function on Γ, one let $\Xi^*\varphi$ denote the function on Θ such that, for any neighbor points p and q in Γ, the value of $\Xi^*\varphi$ on the edge associate to p and q is $\varphi(p) + \varphi(q)$. One let Ξ denote the adjoint of Ξ^* and one immediately verifies the following

Lemma 14.1. *One has $(\Delta - 1)\Xi^* = \Xi^*\Delta$ and $\Xi\Xi^* = 3 + \Delta$. The restriction of Δ to the orthogonal complement of the image of Ξ^* in $\ell^2(\Theta)$ is the operator of multiplication by -2.*

Through this lemma, all the results of this article transfer from the Pascal graph to the Sierpiński graph. They could also be obtained directly in the Sierpiński graph, by considering the suitable operators in $\ell^2(\Theta)$. We will only describe the continuous spectrum of Θ and translate theorem 1.1: this answers the question asked by Teplyaev in [18, § 6.6].

For x in $\mathbb{R}$, set $k(x) = x + 2$ and $t(x) = x + 1$. From lemma 14.1, one deduces the following

Lemma 14.2. *Let φ be in $\ell^2(\Gamma)$, μ be the spectral measure of φ for Δ in $\ell^2(\Gamma)$ and λ the spectral measure of $\Xi^*\varphi$ for Δ in $\ell^2(\Theta)$. Then, one has $\lambda = k(t_*\mu)$.*

For any x in $\mathbb{R}$, set $g(x) = x^2 - 3x = f(x-1)+1$. One let $\Sigma = t(\Lambda)$ denote the Julia set of g. For any x in $\mathbb{R}$, let $c(x) = (x+2)(4-x) = k(x)h(x-1)$ and, for $x \neq \frac{3}{2}$, $\gamma(x) = \frac{x-1}{2x-3} = \rho(x-1)$. One let $\nu_\gamma = t_*\nu_\rho$ denote the unique $L_{g,\gamma}$-invariant probability measure on Σ.

Let still φ_0 denote the function on Γ that appears in section 6 and set $\theta_0 = \Xi^*\varphi_0$ (this is the function which is denoted by $1_{\partial\partial V}$ in [18, § 6]). From theorem 1.1 and lemmas 14.1 and 14.2, one deduces the following theorem, that completes the description of the spectrum of Θ by Teplyaev in [18]:

Theorem 14.3. *The spectrum of Δ in $\ell^2(\Theta)$ is the union of Σ and the set $\bigcup_{n\in\mathbb{N}} g^{-n}(-2)$. The spectral measure of θ_0 for Δ in $\ell^2(\Theta)$ is the measure $c\nu_\gamma$, the eigenvalues of Δ in $\ell^2(\Theta)$ are the elements of $\bigcup_{n\in\mathbb{N}} f^{-n}(-2) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(-1)$ and the associate eigenspaces are spanned by finitely supported functions. Finally, the orthogonal complement of the sum of the eigenspaces of Δ in $\ell^2(\Theta)$ is the cyclic subspace spanned by θ_0.*

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