Percolation on the three dot system

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Abstract

Let $X \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ be the three dot system. Given a $\mathbb{Z}^2$-invariant ergodic probability measure on X, we study percolation properties on the set of 1's in a typical orbit. This gives us a strong dichotomy for such measures.

1 Introduction

The three dot system, which has been introduced by Ledrappier in [5], is the set X of all $(x_{k,l})_{(k,l)\in\mathbb{Z}^2}$ in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ such that, for any (k,l) in $\mathbb{Z}^2$, one has $x_{k,l}+x_{k+1,l}+x_{k,l+1}=0$ (in $\mathbb{Z}/2\mathbb{Z}$), equipped with the natural action of $\mathbb{Z}^2$ by coordinate translations. This action is spanned by the two commuting maps $T:(x_{k,l})\to(x_{k+1,l})$ and $S:(x_{k,l})\to(x_{k,l+1})$. We equip X with its natural compact topology: the group $\mathbb{Z}^2$ acts on X by homeomorphisms.

We can then seek for a classification of the Borel probability measures on X which are invariant under the action of $\mathbb{Z}^2$. This question is an analogue of the one, asked by Fürstenberg in [3], of the classification of the Borel probability measures on the circle which are simultaneously invariant by angle doubling and tripling. This problem has been studied by many authors who have adapted to the space X - and to other systems - the partial solutions to Fürstenberg's question: let us cite, for example, [2], [4], [6] and [7].

In this article, we are proposing a new approach to this problem, based on percolation properties we shall now describe.

Let us denote by Y the closed subset of X of all $(x_{k,l})$ in X with $x_{0,0}=1$. If $x=(x_{k,l})$ belongs to Y, one has $x_{1,0}+x_{0,1}=1$ and hence one and only one of the two elements Tx and Sx belongs to Y. This defines a continuous map $\sigma:Y\to Y$. Let μ be a Borel probability on X which is invariant by the $\mathbb{Z}^2$-action (that is one has $T_*\mu=S_*\mu=\mu$). One then has $\mu(Y)=0$ if and only if μ is the Dirac mass at the zero family. If $\mu(Y)>0$, the restriction of μ to the set Y is quasi-invariant by σ (that is, for any Borel subset B of Y with $\mu(B)=0$, one has $\mu(\sigma^{-1}(B))=0$). If x is a point of Y, we define the

σ-component of x as the set of points y in the $\mathbb{Z}^2$-orbit of x that belong to Y and for which there exists nonnegative integers p and q with $\sigma^p(x) = \sigma^q(y)$. In other terms, the σ-component of x is the set of points y of the $\mathbb{Z}^2$-orbit of x for which the paths drawn on the $\mathbb{Z}^2$-orbit of x $(\sigma^p(x))_{p\geq 0}$ and $(\sigma^q(y))_{q\geq 0}$ join each other. The σ-component of x possesses a natural tree structure: the neighbors of x are $\sigma(x)$ and the eventual elements of $\sigma^{-1}(x)$. We call the elements of $\bigcup_{n\in\mathbb{N}^*}\sigma^{-n}(x)$ the σ-antecedents of x.

Let Z be the set of elements of Y which admit an infinite number of σ-antecedents. The principal result of this article is the

Theorem. *Let μ be an atom free Borel probability on X which is invariant and ergodic under the action of $\mathbb{Z}^2$. Then one and only one of the following is true.*

(i) One has $\mu(Z) = 0$ and there exists a $\mathbb{Z}^2$-invariant Borel set B with $\mu(B) = 1$ such that, for any x in $Y \cap B$, the σ-component of x contains all the points of the $\mathbb{Z}^2$-orbit of x that belong to Y.

(ii) One has $\mu(Z) > 0$ and there exists a $\mathbb{Z}^2$-invariant Borel set B with $\mu(B) = 1$ having the following properties:

(a) for any x in $Y \cap B$, the σ-component of x contains points of Z.

(b) for any x in $Z \cap B$, the set $\sigma^{-1}(x)$ contains one and only one point of Z.

(c) for any x in $Y \cap B$, the $\mathbb{Z}^2$-orbit of x contains an infinite number of σ-components. More precisely, for any σ-component C of the $\mathbb{Z}^2$-orbit of x, there exists an integer k such that $T^kS^{-k}x$ belongs to $Z \cap C$ and the set $\{k \in \mathbb{Z} | T^kS^{-k}x \in Z\}$ is not bounded, neither from above nor from below.

We shall say that an atom free $\mathbb{Z}^2$-invariant and ergodic measure on X is of the tree type if it satisfies the first property of the theorem and that it is of the ribbon type if it satisfies the second property. Both these situations are pictured in figure 1. In this figure, on the left, we see, from a large distance, a generic orbit of the tree type: points have only a finite number of σ-antecedents and their trajectories under σ join each other. On the right, we see a generic orbit of the ribbon type: the points that have an infinite number of σ-antecedents form curves which are transverse to the lines of the

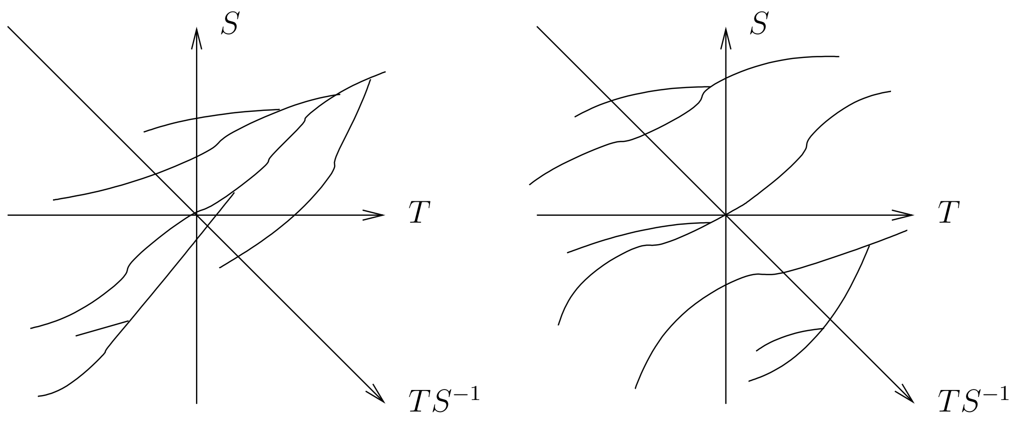

Figure 1: Orbits of the tree type and of the ribbon type

form $\{T^k S^{-k} x | k \in \mathbb{Z}\}$ and the trajectories under σ of points having only a finite number of σ-antecedents join these curves.

In figure 2, we give an example of a configuration, viewed from a small distance: 0's are pictured by a white dot and 1's are pictured by a black one. The values on the right and top axis have been chosen independantly under a Bernoulli law with parameter $\frac{1}{2}$.

In section 5, we will show that the Haar measure of X, viewed as a closed subgroup of the compact group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$, is of the tree type. In section 6, we will give an example of a ribbon type measure.

Before, in sections 2 and 3, we will establish a certain number of preliminary results on the topology of σ-components in $\mathbb{Z}^2$-orbits. We will finish the proof of the theorem in section 4. The results in section 2 rely on an argument which is analogous to the one employed by Burton and Keane in [1] to establish the uniqueness of the infinite component for independent percolation in $\mathbb{Z}^d$, with $d \geq 2$.

2 The σ-antecedents

Recall that Z denotes the set of elements in Y that admit an infinite number of σ-antecedents. Let Z_2 be the set of elements x in Z for which the set $\sigma^{-1}(x)$ contains two elements that both belong to Z. In this section we will prove the following

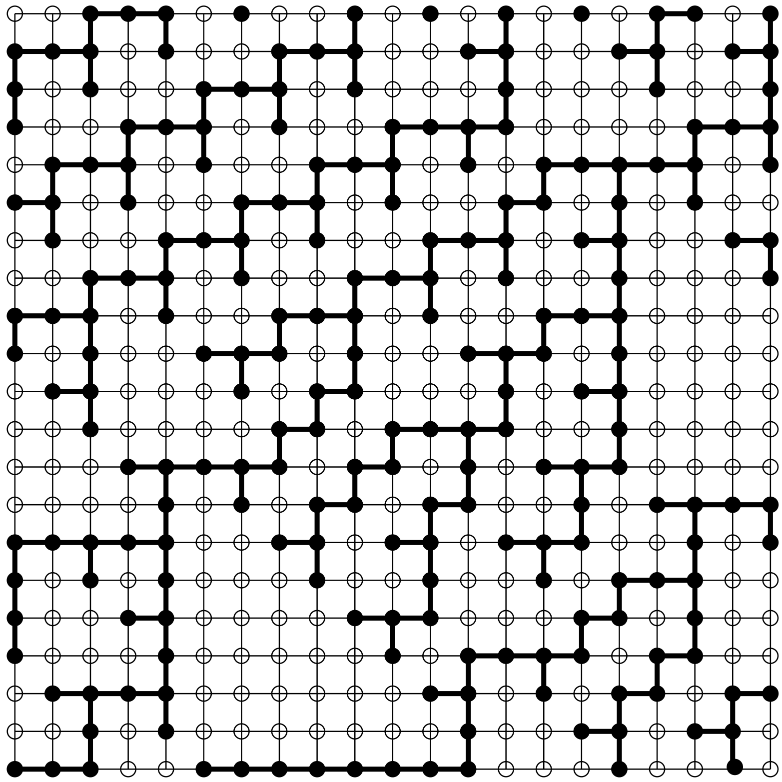

Figure 2: A typical configuration

Proposition 2.1. *Let μ be an atom free Borel probability on X which is invariant and ergodic under the action of $\mathbb{Z}^2$. One has $\mu(Z_2) = 0$.*

Let us introduce a few notations. Let x be a point of X and n a nonnegative integer. We set

$$\begin{aligned}
\Delta_n(x) &= \{T^k S^l x | k \geq 0, l \geq 0, k+l \leq n\} \\
\Delta_n^\circ(x) &= \{T^k S^l x | k > 0, l > 0, k+l \leq n\} \\
\partial\Delta_n(x) &= \{T^k x | 0 \leq k \leq n\} \cup \{S^l x | 0 \leq l \leq n\} \cup \{T^k S^{n-k} x | 0 \leq k \leq n\}.
\end{aligned}$$

That is, the set $\Delta_n(x)$ is a triangle with side length n and vertex x and $\partial\Delta_n(x)$ is its boundary.

If y is a point in $Y \cap \Delta_n(x)$, we define its (σ, x, n)-component as the set of elements z in $Y \cap \Delta_n(x)$ such that there exists nonnegative integers q and r with $\sigma^q(y) = \sigma^r(z) \in \Delta_n(x)$. By construction, each (σ, x, n)-component contains a unique point of $\{T^k S^{n-k} x | 0 \leq k \leq n\}$. The relevancy of these definitions comes from the following lemma, which is an analogue of Burton and Keane lemma in [1]:

Lemma 2.2. *Let x be a point in X and n be a nonnegative integer. If $C \subset \Delta_n(x)$ is a (σ, x, n)-component that contains points in $Z \cap \Delta_n^\circ(x)$, one has*

$$\sharp(Z_2 \cap C \cap \Delta_n^\circ(x)) \leq \sharp(\partial\Delta_n(x) \cap C) - 2.$$

The idea of the proof of this lemma is that C admits a natural tree structure for which points have two or three neighbours. A point z in $Z_2 \cap \Delta_n^\circ(x)$ has three neighbours and the connected components of $C - \{z\}$ all intersect $\partial\Delta_n(x)$.

Proof. As C contains points in $\Delta_n^\circ(x)$, C contains a unique point of the set $\{T^k S^{n-k} x | 1 \leq k \leq n-1\}$. Also, if y is a point in $C \cap Z$, at least one of the σ-antecedents of y belongs to $\partial\Delta_n(x)$. As this σ-antecedent does not belong to $\{T^k S^{n-k} x | 1 \leq k \leq n-1\}$, $C \cap \partial\Delta_n(x)$ contains at least two points and the result is established if $Z_2 \cap C \cap \Delta_n^\circ(x) = \emptyset$.

Else, if y is a point of $Z_2 \cap C \cap \Delta_n^\circ(x)$, at least two of the σ-antecedents of y belong to $\partial\Delta_n(x)$ and, hence, $C \cap \partial\Delta_n(x)$ contains at least three points. The set C admits a natural tree structure for which the neighbours of some element z in C are $\sigma(z)$, if it belongs to $\Delta_n(x)$, and the eventual elements of $\sigma^{-1}(z)$ that belong to $\Delta_n(x)$. If z is a point in $Z_2 \cap C \cap \Delta_n^\circ(x)$, $C - \{z\}$

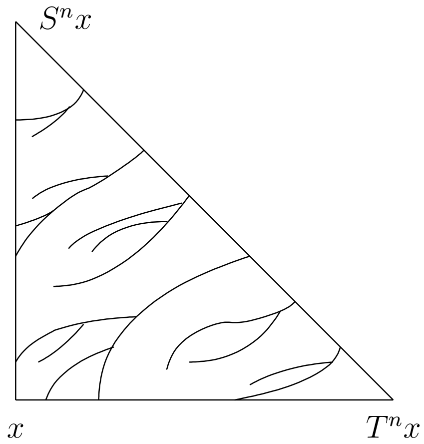

Figure 3: (σ, x, n)-components

then contains three connected components for the tree structure and these three connected component define a partition P_z of $\partial\Delta_n(x) \cap C$ into three non empty subsets. The map $z \mapsto P_z$ is one-to-one and the set $\mathcal{P} = \{P_z | z \in Z_2 \cap C \cap \Delta_n^\circ(x)\}$ is a compatible set of partitions, in the sense of Burton and Keane, that is if $P = \{P_1, P_2, P_3\}$ and $Q = \{Q_1, Q_2, Q_3\}$ belong to $\mathcal{P}$, after an eventual permutation of the indices, one has $Q_2 \cup Q_3 \subset P_1$. By Burton and Keane lemma in [1], one has $\sharp\mathcal{P} \leq \sharp(\partial\Delta_n(x) \cap C) - 2$ and we are done. $\square$

From this lemma, we immediately deduce the

Corollary 2.3. *For any x in X and for any nonnegative integer n, one has* $\mathrm{card}(Z_2 \cap \Delta_n(x)) \leq 5n + 1$.

We now use this corollary for the

Proof of proposition 2.1. By Birkhoff theorem, there exists a point x in X such that one has

$$\frac{\sharp(Z_2 \cap \Delta_n(x))}{\sharp(\Delta_n(x))} \xrightarrow[n\to\infty]{} \mu(Z_2).$$

Now, by corollary 2.3, this goes to 0. $\square$

Finally, we shall use the results of this section in the form of the

Corollary 2.4. *Let μ be an atom free Borel probability which is invariant and ergodic under the action of $\mathbb{Z}^2$ on X. Then, for μ-almost every x in X, for any σ-component C of the $\mathbb{Z}^2$-orbit of x, $C \cap Z$ contains at most one point of the form $T^k S^{-k} x$ for some k in $\mathbb{Z}$.*

Proof. Let U be the set of elements of X that do not satisfy the conclusion of the corollary and let x be in U. Then, there exists distincts integers k and l such that the points $y = T^k S^{-k} x$ and $z = T^l S^{-l} x$ are in Z and belong to the same σ-component. Let q be the smallest nonnegative integer such that $\sigma^q(y) = \sigma^q(z)$. Then, one has $\sigma^q(y) \in Z_2$. In other terms, we have just shown that one has $U \subset \bigcup_{i,j\in\mathbb{Z}} T^i S^j Z_2$. Therefore, by proposition 2.1, we have $\mu(U) = 0$. □

3 The σ-components

In this section, we establish a certain number of topological properties of σ-components.

Let x be a point of Y and C be its σ-component. We shall say that x is an extremal point of C if the other points of C are of the form $T^k S^l x$ with $k + l > 0$ or $k + l = 0$ and $l \geq 0$. A σ-component contains at most one extremal point. We let E denote the set of elements of Y which are extremal points of their σ-component. We have the following

Proposition 3.1. *Let μ be an atom free Borel probability which is invariant and ergodic under the action of $\mathbb{Z}^2$ on X. One has $\mu(E) = 0$.*

We keep the notations from section 2. The proof of proposition 3.1 now relies on the following

Lemma 3.2. *For any x in X and for any nonnegative integer n, one has $\sharp(E \cap \Delta_n(x)) \leq (n + 1)$.*

Proof. Let y be a point of $E \cap \Delta_n(x)$. Then, the (σ, x, n)-component of y contains a unique point of the set $\{T^k S^{n-k} x | 0 \leq k \leq n\}$. This defines a one-to-one map from $E \cap \Delta_n(x)$ into a set of cardinal $n + 1$, whence the result. □

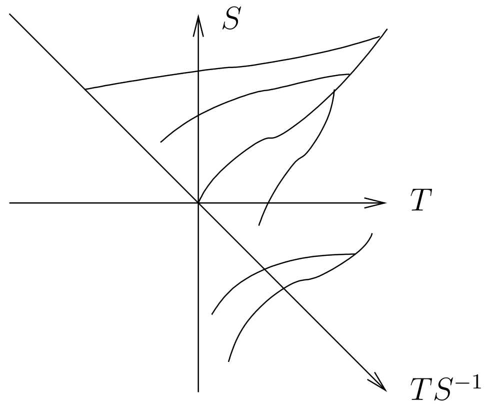

Figure 4: An extremal point

Let x be in Y and C be its σ-component. Set

$$K(x) = \{k \in \mathbb{Z} | T^k S^{-k} x \in C\}$$
$$L(x) = \{k \in \mathbb{Z} | T^k S^{-k} x \in Y, T^k S^{-k} x \notin C\}$$

Then, if $h \leq k \leq l$ are integers such that h and l belong to $K(x)$ and $T^k S^{-k} x$ belongs to Y, one has $k \in K(x)$.

Let F denote the set of points x of X whose $\mathbb{Z}^2$-orbit only contains one σ-component and G denote the set of x in Y such that $K(x)$ is finite. We have the following

Proposition 3.3. *Let μ be an atom free Borel probability which is invariant and ergodic under the action of $\mathbb{Z}^2$ on X. Suppose one has $\mu(F) = 0$. Then, one has $\mu(Y - G) = 0$. In particular, for μ-almost every x in X, the $\mathbb{Z}^2$-orbit of x contains an infinite number of components.*

Proof. Thanks to the remark above, to prove the proposition, it suffices to prove that, for μ-almost any x in Y, $L(x)$ contains both positive and negative elements.

Let us begin by showing that $L(x)$ is not empty. Indeed, as $\mu(F) = 0$, for μ-almost any x, there exists integers k and l such that the σ-component of $y = T^k S^l x$ does not contain x. If $k + l \leq 0$, the element $\sigma^{-k-l}(y)$ is of the form $T^h S^{-h} x$ for some integer h and we are done. If $k + l > 0$, the element $z = \sigma^{k+l}(x)$ is of the form $T^i S^j x$ with $i + j = k + l$. Suppose for example one

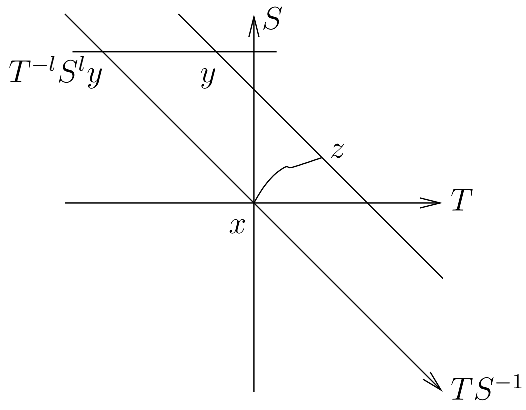

Figure 5: Proof of proposition 3.3

has $i > k$. The situation in the $\mathbb{Z}^2$-orbit of x of the different points involved there is then pictured in figure 5.

For any integer h, if $T^h S^{-h} y = T^{k+h} S^{l-h} x$ belongs to the σ-component of x, one has $h > 0$. Now, by Poincaré recurrence theorem, there exists an infinite number of positive integers m such that $T^{-m} S^m x$ belongs to Y. For sufficiently large m, the σ-component of $T^{-m} S^m x$ is therefore different from the one of x: hence, the set $L(x)$ is not empty.

Let us now consider, for example, the set H of x in Y such that $L(x)$ only contains nonnegative integers. Then, by the argument above, if H' denotes the set of x in H such that, for any $k > 0$, if $T^k S^{-k} x$ belongs to Y, $T^k S^{-k} x$ does not belong to the σ-component of x, one has $\mu \left(H - \bigcup_{k \geq 0} T^{-k} S^k H' \right) = 0$. Now, for any x in H', for any $k > 0$, one has $T^k S^{-k} x \notin H'$ and hence by Poincaré recurrence theorem, $\mu(H') = 0$. Therefore, we have $\mu(H) = 0$: in other terms, for μ-almost any x in Y, $L(x)$ contains negative elements. $\square$

From propositions 3.2 and 3.3, we deduce the following

Corollary 3.4. *Let μ be an atom free Borel probability which is invariant and ergodic under the action of $\mathbb{Z}^2$ on X. Suppose one has $\mu(F) = 0$. Then, for μ-almost any x in X, the σ-component of x contains a point of Z.*

Proof. Consider the set V of x in Y which σ-component does not contain points of Z. By proposition 3.3, for μ-almost any x in V, the σ-component C of x contains only a finite number of points of the form $T^k S^{-k} x$ for some k in

$\mathbb{Z}$. But, as each of these point only possesses a finite number of antecedents, C contains only a finite number of points of the form T^kS^lx with $k+l \leq 0$. In particular, C contains an extremal point. In other terms, one has $\mu\left(V - \bigcup_{k,l\in\mathbb{Z}} T^kS^lE\right) = 0$. Hence, by proposition 3.2, one has $\mu(V) = 0$. □

4 Proof of the theorem

Let us separate the two cases appearing in the theorem.

Proof of the theorem in case $\mu(Z) = 0$. We have to prove that, with notations from section 3, one has $\mu(F) = 1$. If this is not true, by ergodicity, one has $\mu(F) = 0$ and hence, by corollary 3.4, $\mu\left(\bigcup_{k,l\in\mathbb{Z}} T^kS^lZ\right) = 1$. Therefore, we do have $\mu(F) = 1$. □

Proof of the theorem in case $\mu(Z) > 0$. Let us begin by showing that one has $\mu(F) = 0$. Suppose the contrary, that is $\mu(F) = 1$. Then, by Poincaré recurrence theorem, for μ-almost any x in $Z \cap F$, there exists some integer $k \neq 0$ such that $y = T^kS^{-k}x$ belongs to Z. But x and y then belong to the same σ-component. Hence, by corollary 2.4, we have $\mu(Z \cap F) = 0$, which is a contradiction.

By proposition 3.3 and corollary 3.4, for μ-almost any x in X, the $\mathbb{Z}^2$-orbit of x contains an infinite number of σ-components and each of them contains a point of Z. In particular, each of them contains a point of Z which is of the form $T^kS^{-k}x$, for some k in $\mathbb{Z}$. By corollary 2.4, for μ-almost any x, for every σ-component of the $\mathbb{Z}^2$-orbit of x, this point is unique. Finally, by proposition 2.1, for μ-almost any x in Z, $\sigma^{-1}(x)$ contains a unique point of Z. □

5 A tree type measure

We now consider $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ as a compact group for the product law and X as a closed subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$. Then, $\mathbb{Z}^2$ acts on X by group automorphisms. In particular, this actions preserves the Haar measure μ_0 of X. This measure may be described more precisely. Let us denote by $P \subset \mathbb{Z}^2$ the union of the sets $P_- = \{(k,-k)|k \in \mathbb{Z}\}$ and $P_+ = \{(l,0)|l \in \mathbb{N}^*\}$. We immediately get the following

Lemma 5.1. *The map $X \to (\mathbb{Z}/2\mathbb{Z})^P, x \mapsto (x_{k,l})_{(k,l)\in P}$ is an isomorphism of compact groups.*

Thus, in probabilistic terms, to chose randomly an element x of X under the law μ_0, one choses randomly and independently the coordinates $(x_{k,l})_{(k,l)\in P}$ of x under a Bernoulli law with parameter $\frac{1}{2}$ and one completes step by step using the equation $x_{k,l} + x_{k+1,l} + x_{k,l+1} = 0$.

An elementary Fourier transform argument in the compact abelian group X allows to prove the following

Lemma 5.2. *The measure μ_0 is globally mixing for the action of $\mathbb{Z}^2$. More precisely, for any Borel subsets A and B in X, one has*

$$\mu_0(A \cap T^k S^l B) \xrightarrow[(k,l)\to\infty]{} \mu_0(A)\mu_0(B).$$

In particular, the measure μ_0 is ergodic. We shall prove the following

Proposition 5.3. *The measure μ_0 is of the tree type.*

The proof relies on the following extension argument:

Lemma 5.4. *Let $y = (y_{k,l})_{k+l\leq 0}$ be a family of elements of $\mathbb{Z}/2\mathbb{Z}$ such that, for any integers k and l with $k + l + 1 \leq 0$, one has $y_{k,l} + y_{k+1,l} + y_{k,l+1} = 0$. Suppose one has $y_{0,0} = 1$ and, for some $h \geq 0$, $y_{h,-h} = 1$. Then, there exists x in X such that, for any k, l with $k + l \leq 0$, $x_{k,l} = y_{k,l}$ and that x and $T^h S^{-h} x$ are in the same σ-component.*

Proof. The proof is established by induction on $h \geq 0$. If $h = 0$, the result is evident.

Suppose now $h \geq 1$ and the result has been proved for $h-1$. And suppose, to the contrary, that y does not admit any legal extension. Consider a family $z = (z_{k,l})_{k+l\leq 1}$ that extends y and such that, for any k and l with $k + l \leq 0$, one has $z_{k,l} + z_{k+1,l} + z_{k,l+1} = 0$. Then, z is completely determined by the data of $z_{1,0}$; the unique other extension of y to the set $\{k + l \leq 1\}$ satisfying the same condition having the value $z_{k,1-k} + 1$ at $(k, 1 - k)$ for any k in $\mathbb{Z}$. There are two cases: either the extension of y with value 1 at $(1, 0)$ has the value 1 at $(h, 1 - h)$ and, by induction, we then could extend this family in a legal way, what we have supposed to be impossible; or the extension of y with value 1 at $(1, 0)$ has the value 0 at $(h, 1-h)$. By iterating this argument, one can build, for any $1 \leq i \leq h$, a family $z^{(i)} = \left(z^{(i)}_{k+l}\right)_{k+l\leq i}$ of successive extensions of y such that, for any i and for any k,l with $k+l \leq i-1$, one has $z^{(i)}_{k,l} + z^{(i)}_{k+1,l} + z^{(i)}_{k,l+1} = 0$ and $z^{(i)}_{0,i} = z^{(i)}_{h,i-h} = 1$. But, for $i = h$, one then has $z^{(h)}_{h,0} = 1$ whereas, for any $0 \leq k \leq h$, $z^{(h)}_{0,k} = 1$, hence, for any $0 \leq k \leq h-1$, $z^{(h)}_{1,k} = z^{(h)}_{0,k} + z^{(h)}_{0,k+1} = 0$ which, by an easy induction, implies $z^{(h)}_{h,0} = 0$. We reach a contradiction, whence the result. □

Proof of proposition 5.3. Suppose to the contrary the measure μ_0 is of the ribbon type and let us chose some set B as in the theorem. For x in X and y in $(\mathbb{Z}/2\mathbb{Z})^{P_+}$, let us denote by $[x,y]$ the unique element z in X such that $z_{k,l} = x_{k,l}$ for $k+l \leq 0$ and $z_{k,0} = y_{k,0}$ for $k > 0$. As, by lemma 5.1, μ_0 may be seen as the product of the Haar measures μ_- of $(\mathbb{Z}/2\mathbb{Z})^{P_-}$ and μ_+ of $(\mathbb{Z}/2\mathbb{Z})^{P_+}$, there exists some element x of $B \cap Z$ such that, for μ_+-almost every y in $(\mathbb{Z}/2\mathbb{Z})^{P_+}$, the element $[x,y]$ belongs to B. In particular, as x belongs to B, there exists some $h > 0$ such that $T^h S^{-h} x$ belongs to Z. By lemma 5.4, there exists an integer $l > 0$ and elements $t_1, \ldots, t_l$ in $\mathbb{Z}/2\mathbb{Z}$ such that, for any y in $(\mathbb{Z}/2\mathbb{Z})^{P_+}$, if $y_{1,0} = t_1, \ldots, y_{l,0} = t_l$, the points $[x,y]$ and $T^h S^{-h}[x,y]$ belong to the same σ-component. But, if $[x,y]$ belongs to B, this then contradicts the fact that the σ-component of $[x,y]$ only contains one point of $Z \cap \{T^k S^{-k}[x,y] | k \in \mathbb{Z}\}$. □

6 A ribbon type measure

In this section, we shall construct a ribbon type measure.

Let us begin by considering the set $X_1 = \{x \in X | \forall k,l \in \mathbb{Z} \quad x_{2k,2l} = 0\}$: this is a closed subgroup of X which is stable under the action of $(2\mathbb{Z})^2$. For x in X, and for k and l in $\mathbb{Z}$, set $\theta_{2k,2l}(x) = 0$ and $\theta_{2k+1,2l}(x) = \theta_{2k,2l+1}(x) = \theta_{2k+1,2l+1}(x) = x_{k,l}$. One easily checks that the map $\theta : x \mapsto (\theta_{k,l}(x))_{(k,l)\in\mathbb{Z}^2}$ defines a group isomorphism from X onto X_1 and that, for any integers k and l, one has $\theta T^{2k} S^{2l} = T^k S^l \theta$. In particular, by lemma 5.2, the measure $\theta_* \mu_0$ is invariant and ergodic under the action of $(2\mathbb{Z})^2$ on X_1. We set $\mu_1 = \frac{1}{4}(\theta_*\mu_0 + T_*\theta_*\mu_0 + S_*\theta_*\mu_0 + (TS)_*\theta_*\mu_0)$: the measure μ_1 is invariant and ergodic under the action of $\mathbb{Z}^2$ on X and its support is the set $X_1 \cup TX_1 \cup SX_1 \cup TSX_1$. Finally, as $X_1 \cap TX_1 = X_1 \cap SX_1 = X_1 \cap TSX_1 = \{0\}$, there exists a map π from the support of μ_1 deprived from 0 into $(\mathbb{Z}/2\mathbb{Z})^2$ such that $\pi(X_1) = (0,0)$, $\pi(TX_1) = (1,0)$, $\pi(SX_1) = (0,1)$ and $\pi(TSX_1) = (1,1)$. In particular, the map π intertwines the action of $\mathbb{Z}^2$ on the support of μ_1 and the natural action of $\mathbb{Z}^2$ by translations on $(\mathbb{Z}/2\mathbb{Z})^2$.

Now, for k and l in $\mathbb{Z}$, set $u_{k,l} = 0$ if $k - l$ belongs to $3\mathbb{Z}$ and $u_{k,l} = 1$ otherwise. The family $u = (u_{k,l})$ then belongs to X and u is a periodic point for the action of $\mathbb{Z}^2$ on X: its stabilizer is precisely the set of (k, l) in $\mathbb{Z}^2$ such that $k-l$ belongs to $3\mathbb{Z}$. For v in $\{u, Tu, Su\}$, set $\kappa(v) = 0$ if $v = u$, $\kappa(v) = 1$ if $v = Tu$ and $\kappa(v) = 2$ if $v = Su$. The map $\kappa : \{u, Tu, Su\} \to \mathbb{Z}/3\mathbb{Z}$ then intertwines the action of $\mathbb{Z}^2$ on the $\mathbb{Z}^2$-orbit of u and the action of $\mathbb{Z}^2$ on $\mathbb{Z}/3\mathbb{Z}$ for which, for any (k, l) in $\mathbb{Z}^2$, (k, l) acts on $\mathbb{Z}/3\mathbb{Z}$ by translation by $k - l$. We denote by ν the invariant probability measure carried by the $\mathbb{Z}^2$-orbit of u, that is the measure $\frac{1}{3}(\delta_u + \delta_{Tu} + \delta_{Su})$, where δ stands for Dirac mass.

Finally, we set $\mu = \mu_1 * \nu$, the convolution product of the two probability measures μ_1 and ν on the compact group X, that is the image by the sum map of the measure $\mu_1 \otimes \nu$ on the cartesian product $X \times X$. As $\mathbb{Z}^2$ acts on X by group automorphisms, the measure μ is still invariant by this action.

Lemma 6.1. *The product map $(X \times X, \mu_1 \otimes \nu) \to (X, \mu)$ is a measurable isomorphism. The measure μ is ergodic under the action of $\mathbb{Z}^2$.*

Proof. Let us begin by proving the first statement. The dihedral group of order 6 acts in a natural way on our situation. One checks that, by symmetry, it suffices to prove that one has $\mu((u + X_1) \cap (u + TX_1)) = \mu((u + X_1) \cap (Tu + X_1)) = \mu((u + X_1) \cap (Tu + TX_1)) = 0$. As $X_1 \cap TX_1 = \{0\}$, one has $\mu((u+X_1)\cap(u+TX_1)) = 0$. As $(Tu-u)_{0,0} = 1$, one has $(u+X_1)\cap(Tu+X_1) = \emptyset$. Finally, let x be in $(u + X_1) \cap (Tu + TX_1)$. As x belongs to $u + X_1$, one checks that one has, for any l, for any k equalling 2 or 4 modulo 6, $x_{k,2l} = 1$. In the same way, as x belongs to $Tu + TX_1$, one checks that one has, for any l, for any k equalling 1 or 3 modulo 6, $x_{k,2l} = 1$. In particular, one then has $x_{1,0} = x_{2,0} = x_{3,0} = 1$, hence $x_{1,1} = x_{2,1} = 0$ and $x_{1,2} = 0$, which is a contradiction. Thus $(u + X_1) \cap (Tu + TX_1) = \emptyset$.

The system $(X, \mathbb{Z}^2, \mu)$ is therefore isomorphic to the product system $(X, \mathbb{Z}^2, \mu_1) \times (X, \mathbb{Z}^2, \nu)$. By lemma 5.2, the discrete spectrum of $(X, \mathbb{Z}^2, \mu_1)$ equals $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^2 \subset \mathbb{T}^2$. Now, by a direct computation, the discrete spectrum of $(X, \mathbb{Z}^2, \nu)$ equals $\{(1, 1), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})\} \subset \mathbb{T}^2$. As these subgroups only intersect at 1, by a classical ergodic theoretical argument, the product system is ergodic. □

We have the following

Proposition 6.2. *The measure μ is of the ribbon type.*

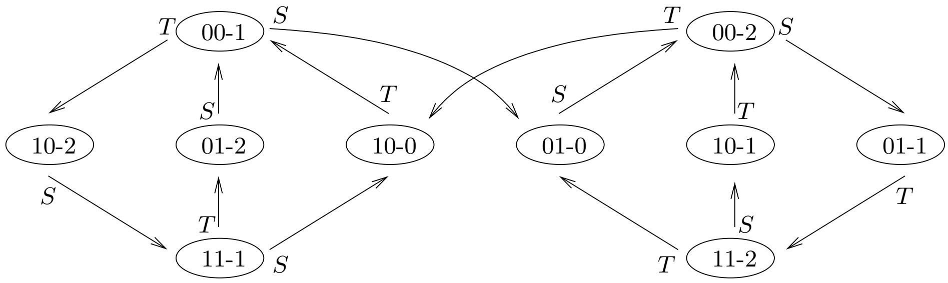

Figure 6: Contsraints on the successive values of $\varpi \circ \sigma^n$, $n \in \mathbb{N}$

Proof. By lemma 6.1, there exists a unique map $x \mapsto (y(x), v(x)), X \to X \times \{u, Tu, Su\}$ which is defined μ-almost everywhere and such that $y_*\mu = \mu_1$, $v_*\mu = \nu$ and, for μ-almost every x in X, one has $x = y(x) + v(x)$. We then set, for μ-almost any x in X, $\varpi(x) = (\pi(y(x)), \kappa(v(x))) \in (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$. We let 00, 10, 01 and 11 denote the four elements of $(\mathbb{Z}/2\mathbb{Z})^2$.

For μ-almost any x in Y, if $\varpi(x) = (a, b)$, one has $\varpi(\sigma(x)) = (a+10, b+1)$ or $\varpi(\sigma(x)) = (a + 01, b + 2)$, depending whether $\sigma(x) = Tx$ or $\sigma(x) = Sx$. There exists some (a, b) in $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$ such that, for μ-almost any x in Y, if $\varpi(x) = (a, b)$, one necessarily has $\sigma(x) = Tx$ or $\sigma(x) = Sx$. For example, if $\varpi(x) = (01, 0)$, by the definition of μ_1 and ν, one has $y_{1,0}(x) = y_{0,0}(x) = 1$ and $y_{0,1}(x) = 0$, whereas $v_{1,0}(x) = u_{1,0} = 1 = u_{0,1} = v_{0,1}(x)$, thus $x_{1,0} = 0$ and $x_{0,1} = 1$ and therefore $\sigma(x) = Sx$. In the same way, if $\varpi(x) = (10, 0)$, one has $\sigma(x) = Tx$. All these constraints have been pictured in figure 6: for any (a, b) in $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$, the values that $\varpi(\sigma(x))$ may take given $\varpi(x)$ are shown by one or two arrows. At the starting point of each arrow the letter T or S indicates whether the value is obtained when $\sigma(x) = Tx$ or $\sigma(x) = Sx$.

Let $\varphi : (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}$ be the map defined by

$$\begin{aligned}
\varphi(a, b) &= -2 \text{ for } (a, b) \in \{(10, 1), (01, 1)\} \\
\varphi(a, b) &= -1 \text{ for } (a, b) \in \{(00, 2), (11, 2)\} \\
\varphi(a, b) &= 0 \text{ for } (a, b) \in \{(10, 0), (01, 0)\} \\
\varphi(a, b) &= 1 \text{ for } (a, b) \in \{(00, 1), (11, 1)\} \\
\varphi(a, b) &= 2 \text{ for } (a, b) \in \{(10, 2), (01, 2)\}
\end{aligned}$$

and let $\psi = \varphi \circ \varpi$. One checks that, by figure 6, for μ-almost every x in Y, one has $\sigma(x) = Tx$ if $\psi(\sigma(x)) - \psi(x) = 1$ and $\sigma(x) = Sx$ if $\psi(\sigma(x)) - \psi(x) = -1$.

In other terms, for μ-almost any x in Y, one has

$$\sigma(x) = T^{\frac{1}{2}(1+\psi(\sigma(x))-\psi(x))} S^{\frac{1}{2}(1-\psi(\sigma(x))+\psi(x))} x.$$

By induction, one has, for any n in $\mathbb{N}$,

$$\sigma^n(x) = T^{\frac{1}{2}(n+\psi(\sigma^n(x))-\psi(x))} S^{\frac{1}{2}(n-\psi(\sigma^n(x))+\psi(x))} x$$

and hence, as ψ takes all it values in $[-2, 2]$, $\sigma^n(x)$ is of the form $T^k S^l x$ with $k + l = n$ and $|k - l| \leq 4$. Therefore, for μ-almost any x in Y, for any h in $\mathbb{Z}$ with $|h| > 8$, if $T^h S^{-h} x$ belongs to Y, the σ-components of x and of $T^h S^{-h} x$ do not intersect. The measure μ is thus of the ribbon type. $\square$

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