RANDOM WALKS ON PROJECTIVE SPACES

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Abstract. Let G be a connected real semisimple Lie group, V be a finite dimensional representation of G, and μ be a probability measure on G whose support spans a Zariski dense subgroup. We prove that the set of ergodic μ-stationary probability measures on the projective space $\mathbb{P}(V)$ is in one-to-one correspondance with the set of compact G-orbits in $\mathbb{P}(V)$. When V is strongly irreducible, we prove the existence of limits for the empirical measures.

We prove related results over local fields as the finiteness of the set of ergodic μ-stationary measures on the flag variety of G.

Contents

1. Introduction

1.1. **Random walks on $\mathbb{P}(V)$.** Let $\mathbb{K}$ be a local field of characteristic 0, i.e. $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. Let V be the $\mathbb{K}$-vector space $V = \mathbb{K}^d$, X be the projective space $X = \mathbb{P}(V)$ and μ be a probability measure on the linear group $\mathrm{GL}(V)$. In this text, "probability measure" will stand for "Borel probability measure". We set Γ_μ for the smallest closed subsemigroup of $\mathrm{GL}(V)$ such that $\mu(\Gamma_\mu) = 1$ and G_μ for the Zariski closure of Γ_μ in $\mathrm{GL}(V)$.

We assume that the action of Γ_μ on V is semisimple i.e. every Γ_μ-invariant vector subspace of V admits a Γ_μ-invariant complementary subspace. Equivalently, the algebraic group G_μ is reductive.

A Borel probability measure ν on X is said to be μ-stationary if $\mu * \nu = \nu$. It is said to be μ-ergodic if it is extremal among the μ-stationary probability measures. We denote by $F_\nu = \mathrm{supp}(\nu)$ the support of ν.

A closed subset $F \subset X$ is said to be Γ_μ-invariant if $gF \subset F$ for all g in Γ_μ. It is said to be Γ_μ-minimal if it is minimal for the inclusion among the non-empty Γ_μ-invariant closed subsets. If ν is a μ-stationary Borel probability measure on X, its support F_ν is a Γ_μ-invariant closed subset.

The aim of this text is to describe the asymptotic properties of the random walk on $\mathbb{P}(V)$ associated to μ. We will also describe the μ-ergodic μ-stationary Borel probability measures on $\mathbb{P}(V)$ and check that they are in one-to-one correspondence with the Γ_μ-minimal subsets of $\mathbb{P}(V)$.

This paper extends previous works of Furstenberg, Guivarc'h and Raugi.

1.2. **Empirical measures on $\mathbb{P}(V)$.** Let $V = \mathbb{K}^d$, $X = \mathbb{P}(V)$ and $x \in X$. Our first result describes the asymptotic behavior in law at time n of the random walk induced by μ on $\mathbb{P}(V)$ starting from x. This behavior is given by the probability measure $\mu^{*n} * \delta_x$. We want to prove the existence of a limit for this sequence in the set of probability measures on X endowed with the $*$-weak topology. We will assume that Γ_μ is *strongly irreducible* i.e. that the only Γ_μ-invariant finite union of vector subspaces of V is $\{0\}$ or V.

Theorem 1.1. (Asymptotic law) *Let $\mathbb{K}$ be a local field of characteristic* 0*, $X := \mathbb{P}(\mathbb{K}^d)$, μ be a probability measure on* $\mathrm{GL}(\mathbb{K}^d)$ *such that the action of Γ_μ on $\mathbb{K}^d$ is strongly irreducible.*
(*i*) *Then for every x in X, the limit probability measure*

$$\nu_x := \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n \mu^{*k} * \delta_x \tag{1.1}$$

exists, is μ-stationary and depends continuously on x.
(*ii*) *When $\mathbb{K} = \mathbb{R}$ and the Zariski closure of Γ_μ is semisimple, one has*

$$\nu_x = \lim_{n\to\infty} \mu^{*n} * \delta_x. \tag{1.2}$$

Remarks 1.2. 1. Theorem 1.1 is due to Guivarc'h and Raugi when X is an "isometric extension" of a flag variety of G and $\mathbb{K} = \mathbb{R}$ (see [16] and [14]).

2. Theorem 1.1.*ii* can not be extended to any local field $\mathbb{K}$. For instance when $\mathbb{K} = \mathbb{Q}_p$, and when the support $\mathrm{Supp}(\mu)$ is included in the compact open group $K = \mathrm{SL}(d, \mathbb{Z}_p)$ and is equal to a translate of a small open normal subgroup of K, Equation (1.2) may not be satisfied.

3. When $\mathbb{K} = \mathbb{R}$, semisimplicity of the Zariski closure of Γ_μ is necessary for Theorem 1.1.*ii* to be true. For instance when μ is a Dirac mass supported by an irrational rotation of $\mathbb{R}^2$, Equation (1.2) is not satisfied.

Our second result describes, when V is strongly irreducible, the asymptotic behavior of the trajectories of the random walk induced by μ on $\mathbb{P}(V)$ starting from x. We denote $\mathbb{N}^* = \{1, 2, \ldots\}$. This behavior is given by the empirical measures $\frac{1}{n}\sum_{k=1}^n \delta_{b_k \cdots b_1 x}$ for a sequence $(b_n)_{n\geq 1}$ of elements of $\mathrm{GL}(\mathbb{K}^d)$ chosen independently with law μ i.e. for β-almost all such sequences where $\beta = \mu^{\otimes \mathbb{N}^*}$.

Theorem 1.3. (Empirical measures) *Let $\mathbb{K}$ be a local field of characteristic* 0*, $X := \mathbb{P}(\mathbb{K}^d)$, μ be a probability measure on* $\mathrm{GL}(\mathbb{K}^d)$ *such that the action of Γ_μ on $\mathbb{K}^d$ is strongly irreducible. Then, for every x in X, for β-almost all sequences $b = (b_n)_{n\geq 1}$, the limit of the empirical probability measures*

$$\nu_{x,b} = \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n \delta_{b_k \cdots b_1 x} \tag{1.3}$$

exists and is a μ-ergodic μ-stationary probability measure on X. Moreover one has

$$\nu_x = \int \nu_{x,b} \, \mathrm{d}\beta(b).$$

Remarks 1.4. 1. We note that, the assumption "V is strongly irreducible" is crucial for Theorems 1.1 and 1.3 to be true (see Example 3.3).

2. Even when $\mathbb{K}=\mathbb{R}$, the limit measures ν_x might be non μ-ergodic and hence the limit measures $\nu_{x,b}$ might not be equal to ν_x. See Remark 1.9 and Example 2.11 (for a Markov chain which does not come from a group action).

1.3. **Stationary measures on $\mathbb{P}(V)$.** In our third result, we do not assume that the representation V is irreducible, and we describe the μ-stationary probability measures on $\mathbb{P}(V)$.

Theorem 1.5. (Stationary measures) *Let $\mathbb{K}$ be a local field of characteristic* 0*, $X := \mathbb{P}(\mathbb{K}^d)$, μ be a probability measure on* $\mathrm{GL}(\mathbb{K}^d)$ *such that the action of Γ_μ on $\mathbb{K}^d$ is semisimple. Then the map $\nu \mapsto \mathrm{supp}(\nu)$ is a bijection between the sets*

$$\{\mu\text{-}ergodic\ probability\ on\ X\} \longleftrightarrow \{\Gamma_\mu\text{-}minimal\ subset\ of\ X\}\,. \tag{1.4}$$

Remarks 1.6. 1. Theorem 1.5 is due to Furstenberg when the action of Γ_μ on $\mathbb{K}^d$ is strongly irreducible, and "proximal" i.e. when there exists a sequence g_n in Γ_μ such that the sequence $\frac{g_n}{\|g_n\|}$ converges to a rank-one endomorphism π in $\mathrm{End}(V)$. In this case, $\mathbb{P}(\mathbb{K}^d)$ supports a unique μ-stationary probability measure called "Furstenberg measure" (see the book [7]).

2. When $\mathbb{K}=\mathbb{R}$ and the action of Γ_μ on $\mathbb{K}^d$ is strongly irreducible, Theorem 1.5 can also be seen as a corollary of the main result of Y. Guivarc'h and A. Raugi in [16] where a bijection like (1.4) is obtained for "isometric extensions" X of flag varieties.

3. We note also that even for a deterministic topological dynamical system on a compact space X, the support of an ergodic probability measure is not always minimal. For instance the Lebesgue probability measure on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is ergodic for the map $t\mapsto 2t$. It might also happen that X is minimal without being uniquely ergodic (see [10, p. 585]).

4. We note that, when the action of Γ_μ is not supposed to be semisimple, the support of a μ-ergodic probability measure is not always Γ_μ-minimal. Here is an example with $V=\mathbb{R}^2$ and μ the finitely supported measure

$$\mu = \tfrac{1}{2}(\delta_{a_0}+\delta_{a_1}) \ \text{ where } \ a_0 = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \ \text{ and } \ a_1 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

In this case one has $\mathbb{P}(V)=\mathbb{R}\cup\{\infty\}$ and $\{\infty\}$ is the only minimal Γ_μ-invariant subset of $\mathbb{P}(V)$ while there exists a μ-ergodic μ-stationary probability on $\mathbb{P}(V)$ whose support is $[0,\infty]$.

When $\mathbb{K}=\mathbb{R}$ and $X=\mathbb{P}(\mathbb{R}^d)$, applying the following Theorem 1.7 with $\Gamma=\Gamma_\mu$, one can describe more precisely the μ-ergodic probability measures on X (see Proposition 5.5).

Theorem 1.7. (Minimal subsets) *Let $\Gamma \subset \mathrm{GL}(\mathbb{R}^d)$ be a subsemigroup whose action on $\mathbb{R}^d$ is semisimple and let G be the Zariski closure of Γ. Every minimal Γ-invariant subset F of $X := \mathbb{P}(\mathbb{R}^d)$ is supported by a compact G-orbit O_F, and the map $F \mapsto O_F$ is a bijection between the sets*

$$\{\ \Gamma\text{-}minimal\ subset\ of\ X\} \longleftrightarrow \{compact\ G\text{-}orbit\ in\ X\}\,. \tag{1.5}$$

Remark 1.8. It is easy to describe the set of compact orbits of this real reductive group G. Indeed, let MAN be a minimal parabolic subgroup of G, AN its maximal $\mathbb{R}$-split solvable subgroup and X^{AN} the set of fixed point of AN in X. Then, the map $O \mapsto O \cap X^{AN}$ is a bijection between the sets

$$\{compact\ G\text{-}orbit\ in\ X\} \longleftrightarrow \left\{M\text{-}orbit\ in\ X^{AN}\right\} \tag{1.6}$$

(see Lemma 4.15). In particular, one recovers the well-known fact due to Furstenberg, Guivarc'h, Raugi, Goldsheid and Margulis ([11], [13], [15]) : *there exists a unique μ-stationary probability measure $\nu_{\mathcal{P}}$ on the flag variety $\mathcal{P}$ of G.* This measure is called *Furstenberg measure.*

Remark 1.9. Even when V is strongly irreducible the sets (1.6) may be uncountable. For instance for $G := \mathrm{SO}(n,1)$ acting on $V := \Lambda^3\mathbb{R}^{n+1}$ with $n \geq 5$. In this case the compact group M is isomorphic to $O(n-1)$, the set X^{AN} is $\mathbb{P}(W)$ where $W = \Lambda^2\mathbb{R}^{n-1}$ and M has uncountably many orbits in X^{AN}.

1.4. **Stationary measures on the flag variety.** Let p be a prime number. When $\mathbb{K} = \mathbb{Q}_p$ there may exist more than one μ-stationary probability measure on the flag variety $\mathcal{P}$ of G (see Section 4.1 for the definition of $\mathcal{P}$). However one has the following finiteness result. We recall that the expression $\mathbb{K}$-*group* is a shortcut for *algebraic group defined over* $\mathbb{K}$.

Theorem 1.10. (Finiteness) *Let G be the group of $\mathbb{Q}_p$-points of a reductive $\mathbb{Q}_p$-group, μ be a probability measure on G such that Γ_μ is Zariski dense in G. Then there exist only finitely many μ-ergodic μ-stationary probability measures on the flag variety $\mathcal{P}$ of G.*

1.5. **Strategy of proofs.** In order to prove Theorems 1.1 and 1.3, we will introduce the averaging operator

$$P_\mu : \mathcal{C}^0(X) \to \mathcal{C}^0(X)\ ;\ \varphi \mapsto P_\mu(\varphi) = \textstyle\int_{\Gamma_\mu} \varphi(gx)\, \mathrm{d}\mu(g), \tag{1.7}$$

and prove in Proposition 3.1 that, as soon as Γ_μ acts strongly irreducibly on V, this Markov-Feller operator is equicontinuous (see [23, 22] and section 2 below for definitions ; this strategy is inspired by the work of Guivarc'h and Raugi [16]). When $\mathbb{K} = \mathbb{R}$ and Γ_μ has semisimple Zariski closure, the only eigenvalue of modulus 1 of this operator P_μ is 1 (Lemma 5.6). Then Theorems 1.1 and 1.3 will occur as special cases of statements about equicontinuous Markov-Feller operators.

For Theorem 1.1, we will use well-known decomposition theorems for operators in Banach spaces spanning a compact semigroup (Propositions 2.2 and 2.3), that we will recall in section 2.1, and that we will apply to equicontinuous Markov Feller operators (Proposition 2.9).

For Theorem 1.3, we will use a general fact due to Raugi [22] about equicontinuous Markov-Feller operators P : for such operators the empirical measures converge almost surely toward a P-ergodic probability measure (Proposition 2.9.e).

In the setting of Theorem 1.5 and 1.7, the Markov-Feller operator P_μ might be non-equicontinuous. Hence we have to develop new tools (Lemmas 5.3 and 5.4) to be able to describe the algebraic homogeneous G-spaces which support a μ-stationary probability measure. When $\mathbb{K} = \mathbb{R}$, those homogeneous spaces are exactly the compact ones, and each of them supports a unique μ-stationary probability measure (Proposition 5.5). When $\mathbb{K}$ is any local field, those homogeneous space are exactly those containing a Γ_μ-invariant compact subset (Proposition 5.1). The description of these homogeneous spaces (Proposition 4.2) occupies most of Chapter 4. An important tool that we have to introduce is a compact group M_Γ that we associate to any Zariski dense subsemigroup Γ and that we call the *limit group of* Γ (Propositions 4.5 and 4.9).

In the setting of Theorem 1.10, the Markov-Feller operator P_μ is again equicontinuous. We can use directly Proposition 2.9 and we only have to check that there exist only finitely many Γ_μ-minimal subsets in the flag variety (Proposition 4.17) using again the limit group M_Γ.

2. Equicontinuous operators

The aim of this chapter is to recall decomposition theorems for bounded operators on Banach spaces spanning a compact semigroup (Propositions 2.2 and 2.3), to recall Breiman law of large numbers for Markov-Feller operators on a compact space (Proposition 2.4) and the results of Raugi about equicontinuous Markov-Feller operators (Propositions 2.7 and 2.9).

2.1. **Decomposition theorems.**

We begin by recalling the JLG decomposition theorem for bounded operators spanning a compact semigroup.

Let $(E, \|.\|)$ be a Banach space. We endow the space $\mathcal{L}(E)$ of bounded linear operators with the strong topology: a sequence P_n in $\mathcal{L}(E)$ converges strongly towards P in $\mathcal{L}(E)$ if and only if, for any f in E, one has $\lim\limits_{n\to\infty} \|P_n f - Pf\| = 0$.

Definition 2.1. *We say that an operator P in $\mathcal{L}(E)$ spans a strongly compact semigroup if P belongs to a semigroup of $\mathcal{L}(E)$ which is compact for the strong topology. Equivalently, the operators P^n have uniformly bounded norms: $\sup_{n\geq 1} \|P^n\| < \infty$, and for every f in E, the orbit $(P^n f)_{n\geq 1}$ is strongly relatively compact in E.*

We endow the dual Banach space E^* with the $*$-weak topology: a sequence ν_n in E^* converges $*$-weakly towards ν in E^* if and only if, for any f in E, one has $\lim\limits_{n\to\infty} \nu_n(f) = \nu(f)$. For any operator P in $\mathcal{L}(E)$, we will write $\nu \mapsto \nu P$ for the adjoint operator of P in E^* , E^P for the set of P-invariant vectors and $(E^*)^P$ for the set of P-invariant linear forms:

$$E^P := \{f \in E \mid Pf = f\} \ , \ (E^*)^P := \{\nu \in E^* \mid \nu P = \nu\}. \tag{2.1}$$

We also introduce the Banach subspaces

$$E_P := \{f \in E \mid \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n P^k f = 0 \ \text{ strongly}\} \tag{2.2}$$

$$(E^*)_P := \{\nu \in E^* \mid \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n \nu P^k = 0 \ \ *\text{-weakly}\}. \tag{2.3}$$

The following proposition is known as "von Neumann functional ergodic theorem".

Proposition 2.2. *Let E be a Banach space and $P \in \mathcal{L}(E)$ be an operator spanning a strongly compact semigroup. Then*
a) The restriction map $\nu \mapsto \nu|_{E^P}$ is an isomorphism $(E^*)^P \simeq (E^P)^*$.
b) One has the decomposition $E = E^P \oplus E_P$.
c) One also has the decomposition $E^* = (E^*)^P \oplus (E^*)_P$.

Sketch of proof. a) To prove injectivity, we start with a linear form ν on E which is zero on E^P. Since, by [24, Th. 3.20.c)], the convex hull of a compact subset of E is relatively compact, for any f in E, we can choose a cluster point y_∞ of the sequence $\frac{1}{n}\sum_{k=1}^n P^k f$. This point is P-invariant and one has $\nu(f) = \nu(y_\infty) = 0$. Hence, one has $\nu = 0$.

To prove surjectivity, we start with a linear form on E^P extend it by Hahn-Banach Theorem to a linear form ν on E and notice that any cluster point of the weakly relatively compact sequence $\frac{1}{n}\sum_{k=1}^n \nu P^k$ is P-invariant and has same restriction to E^P as ν.

b) Again, for f in E, the sequence $\frac{1}{n}\sum_{k=1}^{n} P^k f$ is relatively compact. If y_∞ is a cluster point of it, for any P-invariant linear form ν, one has $\nu(y_\infty) = \nu(f)$. Hence the sequence $\frac{1}{n}\sum_{k=1}^{n} P^k f$ admits a unique cluster point, that is it converges to some $\pi_P f \in E^P$. The map $\pi_P : E \to E$ is then a P-invariant projector whose image is E^P and whose kernel is E_P.

c) This follows from *b*). The map $\nu \mapsto \nu\pi_P$ is a P-invariant projector $E^* \to E^*$ whose image is $(E^*)^P$ and whose kernel is $(E^*)_P$. □

The following JLG decomposition is a strong improvement of Proposition 2.2. We will only use it to prove Theorem 1.1.*ii*.

For any complex number χ we consider the eigenspace

$$E_\chi := \{f \in E \mid Pf = \chi f\},$$

set E_r for the linear closure of $\oplus_{|\chi|=1} E_\chi$ and E_s for the space

$$E_s := \{f \in E \mid \lim_{n\to\infty} P^n f = 0 \ \text{ strongly}\} \tag{2.4}$$

Proposition 2.3. (Jacobs, de Leeuw, Glicksberg) *Let E be a Banach space and $P \in \mathcal{L}(E)$ be an operator spanning a strongly compact semigroup. Then*
a) One has the decomposition $E = E_r \oplus E_s$.
b) In particular, if 1 is the only eigenvalue of P with modulus 1, then the following limits exist:
(*i*) *for every f in E, $\lim_{n\to\infty} P^n f = \pi_P f$ strongly,*
(*ii*) *for every ν in E^*, $\lim_{n\to\infty} \nu P^n = \nu\pi_P$ $*$-weakly.*

Sketch of proof. Let S be the closure of the semigroup spanned by P in $\mathcal{L}(E)$ for the strong topology. Then one easily checks that S is compact and that the composition map $S \times S \to S$ is continuous. We let T be a non-empty minimal closed subset of S such that $ST \subset T$. One checks that one has $T = Su$ where u is an idempotent element and that the composition map induces a group structure on T with identity element u. We have $E = \ker u \oplus \operatorname{im} u$ and since S is abelian, both these subspaces are S-invariant. Since the image of S in $\mathcal{L}(\operatorname{im} u)$ is a strongly compact abelian group, we have $\operatorname{im} u \subset E_r$ and it only remains to prove one has $\ker u \subset E_s$. Indeed, if f belongs to $\ker u$, as u belongs to the strong closure of the sequence P^n, there exists a sequence n_k of integers with $\|P^{n_k} f\| \to 0$. Now, by Banach-Steinhaus Theorem, the sequence $\|P^n\|$ is bounded and hence $\|P^n f\| \to 0$, what should be proved. □

For a detailed proof, see [9, Chap. 12].

2.2. **Empirical measures for Markov-Feller operators.** For further quotation, we recall in this section Breiman law of large numbers.

Let X be a compact metrizable space, $E = \mathcal{C}^0(X)$ be the Banach space of continuous functions on X endowed with the supremum norm. Its dual space E^* is the space $\mathcal{M}(X)$ of complex measures on X. We denote by $\underline{X}$ the compact set $\underline{X} = X^{\mathbb{N}}$ of infinite sequences $\underline{x} = (x_0, x_1, x_2, \ldots)$.

Let $P : \mathcal{C}^0(X) \to \mathcal{C}^0(X)$ be a Markov-Feller operator i.e. a bounded operator such that $\|P\| \leq 1$, $P1 = 1$ and such that $Pf \geq 0$ for all function $f \geq 0$. Such a Markov-Feller operator can be seen alternatively as a continuous map $x \mapsto P_x$ from X to the set of probability measures on X, where P_x is defined by $P_x(f) = (Pf)(x)$ for all f in $\mathcal{C}^0(X)$. We denote by $\mathbb{P}_x$ the Markov probability measure on $\underline{X}$ which gives the law of the trajectories of the Markov chain starting from x associated to P.

For any trajectory $\underline{x} \in \underline{X}$, and $n \geq 1$ the probability measures $\nu_{\underline{x},n} := \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}$ are called *empirical measures*. Heuristically this sequence of measures tells us where the trajectories spend a positive proportion of their time. We want to understand the behavior of this sequence of measures. The first result in that direction is Breiman's law of large numbers:

Proposition 2.4. (Breiman) *Let X be a compact metrizable space and P be a Markov-Feller operator on X. Then, for every point x in X, for $\mathbb{P}_x$-almost every trajectory $\underline{x} \in \underline{X}$, every $*$-weak cluster point ν_∞ of the sequence of empirical measures $\frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}$ is P-invariant.*

In particular, if P is uniquely ergodic i.e. admits a unique P-invariant probability measure ν on X , then, for every point x in X, for $\mathbb{P}_x$-almost every trajectory $\underline{x} \in \underline{X}$, one has

$$\lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^{n} \delta_{x_k} = \nu. \tag{2.5}$$

For a proof, see [8] or [4].

Example 2.5. When P is not uniquely ergodic, the limit (2.5) does not always exists. This is already the case for deterministic operators : for example, if P is the Markov-Feller operator on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ such that $P_x = \delta_{2x}$, $x \in \mathbb{T}$.

2.3. **Equicontinuous Markov-Feller operators.** We now recall the description of P-invariant measures of an equicontinuous Markov-Feller operator P.

Definition 2.6. *We say that the Markov-Feller operator P is equicontinuous if, for every f in $\mathcal{C}^0(X)$, the family of functions $(P^n f)_{n\geq 1}$ is equicontinuous.*

Equivalently, by Ascoli Theorem, this means that P spans a strongly compact semigroup in $\mathcal{L}(\mathcal{C}^0(X))$.

Let P be a Markov-Feller operator on X. A closed subset $F \subset X$ is said to be P-*invariant* if, for all x in X, one has $P_x(F) = 1$. A P-invariant subset F is said to be P-*minimal* if it is minimal among the non-empty closed P-invariant subsets of X.

We shall now describe the structure of the P-minimal subsets of X. We recall from [24, Th. 11.12] that if A is a commutative C*-algebra, there exists a unique compact space Z such that A is isomorphic to the algebra $\mathcal{C}^0(Z)$. The space Z is called the spectrum of A. If $\varphi : A_1 \to A_2$ is a morphism of commutative C*-algebras and if Z_1 and Z_2 are the spectra of A_1 and A_2, then there exists a unique continuous map $\theta : Z_2 \to Z_1$ such that, for any f in A_1, one has $\varphi(f) = f \circ \theta$.

Proposition 2.7. *Let X be a compact metrizable space and P be an equicontinuous Markov-Feller operator on X. Let Y be the closure of the union of P-minimal subsets of X. Then*
a) The restriction map

$$\mathcal{C}^0(X)^P \to \mathcal{C}^0(Y)^P \tag{2.6}$$

is an isometry of Banach spaces.
b) More generally, when $|\chi| = 1$, the restriction map between the eigenspaces

$$\mathcal{C}^0(X)_\chi \to \mathcal{C}^0(Y)_\chi \tag{2.7}$$

is an isometry of Banach spaces.
*c) Each P-invariant function $f \in \mathcal{C}^0(X)^P$ is constant on the P-minimal subsets and hence $\mathcal{C}^0(Y)^P$ is a Banach sub-*C**-algebra of $\mathcal{C}^0(Y)$.*

Let Z be the spectrum of $\mathcal{C}^0(X)^P$ and $\pi : Y \to Z$ be the surjective continuous map associated with the inclusion $C^0(Y)^P \to \mathcal{C}^0(Y)$.
d) For any z in Z, the set $\pi^{-1}(z)$ is P-invariant and contains a unique P-minimal subset F_z.

This result is essentially due to Raugi [22, Th. 2.6].

Example 2.8. It might happen that, for some z in Z, the preimage $\pi^{-1}(z)$ is not minimal. This is the case when $X = \overline{\mathbb{N}^*} \cup \{0, 1\}$, where $\overline{\mathbb{N}^*} = \mathbb{N}^* \cup \{\infty\}$ is the one-point compactification of $\mathbb{N}^*$, and P is the

Markov operator such that, for any n in $\mathbb{N}^*$, one has

$$P_{(n,0)} = P_{(n,1)} = \tfrac{1}{n}\delta_{n,0} + (1-\tfrac{1}{n})\delta_{n,1}$$

and $P_{(\infty,0)} = P_{(\infty,1)} = \delta_{\infty,1}$. Then one has $Y = X$, $Z = \overline{\mathbb{N}^*}$ and π is the map $(n,u) \mapsto n$. In particular $\pi^{-1}(\infty) = \{(\infty,0),(\infty,1)\}$ and this set is not P-minimal.

Proof of Proposition 2.7. a)&b) We first prove that this restriction map is injective. Let f be a continuous function on X such that $Pf = \chi f$ with $|\chi| = 1$. Assume that the restriction of f to Y is zero. We want to prove that $f = 0$. The function $g := |f|$ satisfies $Pg \geq g$. Let $M := \sup_{x\in X} g(x)$. The set $g^{-1}(M)$ is then a closed P-invariant subset of X and hence contains a P-minimal subset. This proves that $M = 0$ as required.

We now prove that this restriction map is a surjective isometry. Let g be a continuous function on Y such that $Pg = \chi g$. This function can be extended as a continuous function h on X with $\|h\|_{\mathcal{C}^0(X)} = \|g\|_{\mathcal{C}^0(Y)}$. Since g is an eigenfunction of P, g is also the restriction of the functions $h_n := \frac{1}{n}\sum_{k=1}^{n} \chi^{-k} P^k h$, for $n \geq 1$. This sequence is equicontinuous and admits a cluster value f in $\mathcal{C}^0(X)$. By construction, this function f belongs to the eigenspace $\mathcal{C}^0(X)_\chi$, the restriction of f to Y is equal to g and one has $\|f\|_{\mathcal{C}^0(X)} = \|g\|_{\mathcal{C}^0(Y)}$ as required.

c) Let f be a P-invariant continuous function with real values, $F \subset X$ be a closed P-minimal subset and $M = \sup_F f$. Then the set $f^{-1}(M) \cap F$ is closed and P-invariant, hence $f = M$ on F, what should be proved.

d) Equip Z with a distance which defines its topology, fix z in Z and, for y in Y, set $f(y) = d(\pi(y), z)$. By definition of π, f is P-invariant, so that, if $f(x) = 0$, one has $f(y) = 0$ for P_x-almost any y, that is the set $\pi^{-1}(z)$ is P-invariant.

Let $F_1 \neq F_2$ be closed P-minimal subsets. Then, as $F_1 \cap F_2$ is closed and P-minimal, one has $F_1 \cap F_2 = \emptyset$. Let f be in $\mathcal{C}^0(Y)$ with $f = 0$ on F_1 and $f = 1$ on F_2 and set $g = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k f$. Then g belongs to $\mathcal{C}^0(X)^P$ and g does not take the same value on F_1 and F_2, so that $\pi(F_1) \neq \pi(F_2)$, what should be proved. □

Recall a probability measure ν on X is said to be *P-invariant*[1] if $\nu P = \nu$. It is then said to be *P-ergodic* if it is an extremal point of the compact convex set of P-invariant probability measures on X.

[1]Many synonyms for the word “invariant” have been used in the litterature like “stationary”, “harmonic” or even “regular” in [25].

For an equicontinuous Markov-Feller operator P, one can describe the P-invariant probability measures. For x in X, we denote by δ_x the Dirac mass at x and $\delta_x P^k$ its image by the transpose of P^k.

We have the following results by Raugi [22, Prop. 3.2 and 3.3].

From Propositions 2.2 and 2.7, we get

Proposition 2.9. *Let X be a compact metrizable space, P be an equicontinuous Markov-Feller operator on X, Y be the closure of the union of the P-minimal subsets of X and Z be the spectrum of the Banach algebra $\mathcal{C}^0(Y)^P$.*
a) Any P-ergodic P-invariant probability measure on X has P-minimal compact support and any P-minimal closed subset F of X carries a unique P-invariant probability measure ν_F. The set of P-ergodic P-invariant probability measures on X is compact for the weak-$*$ topology.
b) The map

$$\mathcal{M}(Z) \to \mathcal{M}(X)^P ; \alpha \mapsto \int_Z \nu_{F_z} \,\mathrm{d}\alpha(z) \tag{2.8}$$

is an isomorphism.
c) For every x in X, the limit probability measure

$$\nu_x := \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n \delta_x P^k \tag{2.9}$$

exists, is P-invariant and depends continuously on x.
d) Seeing these ν_x as measures on Z, the map

$$\mathcal{C}^0(Z) \to \mathcal{C}^0(X)^P ; \varphi \mapsto \big(x \mapsto \int_Z \varphi(z)\,\mathrm{d}\nu_x(z)\big) \tag{2.10}$$

is a Banach spaces isomorphism.
e) For every x in X, for $\mathbb{P}_x$-almost every trajectory $\underline{x} \in \underline{X}$, the limit

$$\nu_{\underline{x}} := \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n \delta_{x_k} \tag{2.11}$$

exists, is P-invariant and P-ergodic, and one has the equality

$$\nu_x = \int_{\underline{X}} \nu_{\underline{x}} \,\mathrm{d}\mathbb{P}_x(\underline{x}). \tag{2.12}$$

Remarks 2.10. 1. In particular, any limit of a sequence of P-ergodic probability measures on X is also P-ergodic.

2. Formula (2.10) is a kind of Poisson formula expressing harmonic functions thanks to continuous functions on a “boundary”.

Proof of Proposition 2.9. *a*), *b*), *c*) and *d*) directly follow from Propositions 2.2 and 2.7. For *a*), note that, if $\pi : Y \to Z$ is the natural map, necessarily, for any z in Z, the set $\pi^{-1}(z)$ carries a unique P-invariant probability measure ν. Since, by definition, $F_z \subset \pi^{-1}(z)$ and F_z also carries a P-invariant probability measure, one has $\nu(F_z) = 1$, and *a*) follows.

Let us prove e). We fix x in X. By Breiman's proposition 2.4, we already know that the cluster points of the sequence of empirical measures $\nu_{\underline{x},n}$ are P-invariant probability measures. Hence, by Proposition 2.2.a and since X is metrizable, to prove convergence in (2.11), we only have to check that for every P-invariant function f on X, for $\mathbb{P}_x$-almost all trajectories $\underline{x}$ in $\underline{X}$, the sequence $\nu_{\underline{x},n}(f)$ converges. For that, we note that, since f is P-invariant, the sequence of functions $\Phi_n : \underline{x} \mapsto f(x_n)$ is a bounded martingale on $\underline{X}$, with respect to the natural filtration. Hence, by Doob's martingale theorem, for $\mathbb{P}_x$-almost all $\underline{x}$ in $\underline{X}$, the sequence $f(x_n)$ converges. Therefore the Cesaro average $\nu_{\underline{x},n}(f)$ converges too.

It remains to check that, for $\mathbb{P}_x$-almost all trajectories $\underline{x}$ in $\underline{X}$, the limit $\nu_{\underline{x}}$ is P-ergodic. Indeed, for any P-invariant continuous function f on X, for $\mathbb{P}_x$-almost all trajectories $\underline{x}$ in $\underline{X}$, the sequence $f(x_n)$ converges to $\ell = \nu_{\underline{x}}(f)$. Hence, all the cluster points in X of the trajectory $\underline{x}$ belong to the level set $f^{-1}(\ell)$ and the support of ν_x is contained in this level set. In particular the set $\pi(\mathrm{supp}\nu_{\underline{x}})$ is a singleton z and ergodicity follows from a). Formula (2.12) is obvious. $\square$

Example 2.11. Here is an example where the limits of empirical measures $\nu_{\underline{x}}$ given in (2.11) are not equal to ν_x. Choose $X := \mathbb{Z} \cup \{-\infty, \infty\}$ to be the two points compactification of $\mathbb{Z}$ and P to be the Markov Feller operator on X such that

$$P_{\pm\infty} = \delta_{\pm\infty} \text{ and } P_n = a_n \delta_{n-1} + (1 - a_n)\delta_{n+1} \quad (n \in \mathbb{Z})$$

with $a_n = \frac{1}{3}$ and $a_{-n} = \frac{2}{3}$ for $n > 0$, and $a_0 = \frac{1}{2}$. This operator P is equicontinuous and P has two ergodic measures $\delta_{-\infty}$ and δ_∞. One computes using (2.10) that for x in $\mathbb{Z}$ the limit probability measure ν_x in (2.12) is given by

$$\begin{aligned} \nu_x &= (1 - 2^{x-1})\delta_{-\infty} + 2^{x-1}\delta_\infty && \text{for } x \leq 0, \\ \nu_x &= 2^{-x-1}\delta_{-\infty} + (1 - 2^{-x-1})\delta_\infty && \text{for } x \geq 0, \end{aligned}$$

and hence ν_x is not P-ergodic.

A very similar example is obtained by choosing $P = P_\mu$ to be the averaging operator of a Zariski dense probability measure μ on the group $SO(2,1)$ acting on the projective sphere $X = \mathbb{S}^2$ of $\mathbb{R}^3$. In this case, P is equicontinuous and there exists exactly two P-ergodic measures on X, ν_+ and ν_- and two extremal dual P-invariant continuous functions φ_+ and φ_- on X.

Another very similar example (in the setting of Theorem 1.3) can be obtained by choosing $P = P_\mu$ to be the averaging operator of a probability measure μ on the group $G = SO(5,1)$ with $\Gamma_\mu = G$ acting on the projective space $X = \mathbb{P}(V)$ for the irreducible representation $V = \Lambda^3\mathbb{R}^6$ of G introduced in Remark 1.9, and by choosing a point $x = \mathbb{R}v$ in $\mathbb{P}(V)$ for which the orbit closure $\overline{Gx}$ contains uncountably many compact G-orbits. For instance $v = v_1 + wv_2$ where v_1 and v_2 are non zero N-invariant vectors in V belonging to distinct MA-orbits and where w is the non trivial element of the Weyl group.

Example 2.12. When P has a unique P-ergodic probabilty measure ν, Equation (2.5) gives us an information on the statistical behavior of a typical trajectory starting from x. In particular this trajectory spends most of the time near the support of ν. However, even when P is equicontinuous, the limit set of $(x_k)_{k\geq 1}$ may be strictly larger than $\mathrm{Supp}(\nu)$. Here is an example: choose $X := \mathbb{Z}\cup\{\infty\}$ to be the one point compactification of $\mathbb{Z}$ and P to be the Markov Feller operator on X for which $P_x = \mu * \delta_x$ where μ is the probability measure $\mu := \frac{1}{2}(\delta_{-1} + \delta_1)$, $x \neq \infty$, and $P_\infty = \delta_\infty$. The operator P is equicontinuous and is uniquely ergodic with invariant measure δ_∞, but, for all x in $\mathbb{Z}$, $\mathbb{P}_x$-almost all trajectories visit infinitely often every point in $\mathbb{Z}$.

3. Linear random walks

In this chapter, we use the results of Chapter 2 in order to prove Theorems 1.1.*i* and 1.3.

3.1. **Equicontinuity on the projective spaces.**

The main step will be to understand when the Markov-Feller operator P_μ in (1.7) is equicontinuous (see Proposition 3.1).

Let $\mathbb{K}$ be a local field of characteristic 0, $V = \mathbb{K}^d$, $X = \mathbb{P}(V)$ and μ be a probability measure on the linear group $\mathrm{GL}(V)$. We set Γ_μ for the smallest closed subsemigroup of $\mathrm{GL}(V)$ such that $\mu(\Gamma_\mu) = 1$.

We recall the averaging operator that we introduced in (1.7): this operator is the Markov-Feller operator $P = P_\mu : \mathcal{C}^0(X) \to \mathcal{C}^0(X)$ whose transition probabilities are given by $P_x = \mu * \delta_x$ for all x in X.

We set $(B, \mathcal{B}, \beta, T)$ to be the one-sided Bernoulli shift with alphabet (Γ_μ, μ). This means that B is the set of sequences $b = (b_1, \ldots, b_n, \ldots)$ with b_n in $\mathrm{GL}(V)$, $\mathcal{B}$ is its Borel σ-algebra, β is the product probability measure $\beta = \mu^{\otimes\mathbb{N}^*}$ and T is the shift: $Tb := (b_2, b_3, \ldots)$.

For every x in X, the Markov measure $\mathbb{P}_x$ is the image of β by the map

$$B \to \underline{X}\,;\, b \mapsto (x, b_1x, b_2b_1x, b_3b_2b_1x, \ldots).$$

Proposition 3.1. *Let $\mathbb{K}$ be a local field of characteristic 0, $V = \mathbb{K}^d$, $X := \mathbb{P}(V)$, μ be a probability measure on $\mathrm{GL}(V)$ such that the action* *of Γ_μ on V is strongly irreducible. Then the Markov-Feller operator P_μ on X is equicontinuous.*

We will need the following lemma.

We introduce a distance on $\mathbb{P}(V)$. We fix a norm $\|.\|$ on V: we choose it to be euclidean when $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, and to be ultrametric when $\mathbb{K}$ is non-archimedean. We endow $\Lambda^2 V$ with a compatible norm also denoted $\|.\|$. The formula

$$d(x,y) = \tfrac{\|v\wedge w\|}{\|v\|\,\|w\|}\ , \quad \text{for } x = \mathbb{K}v \text{ and } y = \mathbb{K}w \text{ in } \mathbb{P}(V),$$

defines a distance on $\mathbb{P}(V)$ which induces the usual compact topology.

Lemma 3.2. *Let $V = \mathbb{K}^d$ and μ be a probability measure on $\mathrm{GL}(V)$ such that the action of Γ_μ on V is strongly irreducible. For all $\varepsilon > 0$,*
a) there exists $c_\varepsilon > 0$ such that, for all v in $V \smallsetminus \{0\}$, one has

$$\beta(\{b \in B \mid \inf_{n\geq 1} \tfrac{\|b_n\cdots b_1 v\|}{\|b_n\cdots b_1\|\,\|v\|} \geq c_\varepsilon\}) \geq 1-\varepsilon, \tag{3.1}$$

b) there exists $M_\varepsilon > 0$ such that, for all x, y in $\mathbb{P}(V)$, one has

$$\beta(\{b \in B \mid \sup_{n\geq 1} d(b_n\cdots b_1 x, b_n\cdots b_1 y) \leq M_\varepsilon\, d(x,y)\}) \geq 1-\varepsilon. \tag{3.2}$$

We recall that the *proximal dimension* of a subsemigroup $\Gamma \subset \mathrm{GL}(V)$ is the smallest integer $r \geq 1$ for which there exists an endomorphism π in $\mathrm{End}(V)$ of rank r such that $\pi = \lim_{n\to\infty} \lambda_n\, g_n$ with λ_n in $\mathbb{K}$ and g_n in $\mathbf{G}$. The semigroup Γ is *proximal* if and only if $r = 1$.

Proof of Lemma 3.2. a) By [7, Th. 3.1], we know that there exists a Borel map $b \mapsto W_b$ from B to the Grasmannian variety $\mathrm{Gr}_{d-r}(V)$, where r is the proximal dimension of Γ_μ in V, such that, for β-almost all b in B, W_b is the kernel of all the matrices $\pi \in \mathrm{End}(V)$ which are cluster points of the sequence $\frac{b_n\cdots b_1}{\|b_n\cdots b_1\|}$. By [7, Pr. 2.3], we also know that, for all x in $\mathbb{P}(V)$, one has

$$\beta(\{b \in B \mid x \in \mathbb{P}(W_b)\}) = 0.$$

Hence, for all $\varepsilon > 0$, there exists $\alpha_\varepsilon > 0$, such that, for all x in $\mathbb{P}(V)$,

$$\beta(\{b \in B \mid d(x, \mathbb{P}(W_b)) \geq \alpha_\varepsilon\}) \geq 1-\varepsilon/2. \tag{3.3}$$

By definition of W_b, for all $\alpha > 0$, for β-almost all b in B, there exists $c_{\alpha,b} > 0$ such that, for all non-zero vector v in V with $d(\mathbb{K}v, \mathbb{P}(W_b)) \geq \alpha$, one has

$$\inf_{n\geq 1} \tfrac{\|b_n\cdots b_1 v\|}{\|b_n\cdots b_1\|\,\|v\|} \geq c_{\alpha,b}. \tag{3.4}$$

We choose then the constant $c_\varepsilon > 0$ such that

$$\beta(\{b \in B \mid c_{\alpha_\varepsilon, b} \geq c_\varepsilon\}) \geq 1 - \varepsilon/2. \tag{3.5}$$

Then (3.1) follows from (3.3), (3.4) and (3.5).

b) For $p_n = b_n \cdots b_1$, v in x and w in y, we have

$$\tfrac{d(p_n x, p_n y)}{d(x,y)} = \tfrac{\|p_n v \wedge p_n w\|}{\|v \wedge w\|} \tfrac{\|v\|}{\|p_n v\|} \tfrac{\|w\|}{\|p_n w\|} \leq \tfrac{\|p_n\| \|v\|}{\|p_n v\|} \tfrac{\|p_n\| \|w\|}{\|p_n w\|},$$

hence (3.2) follows from (3.1) with $M_\varepsilon = (c_{\varepsilon/2})^{-2}$. □

Proof of Proposition 3.1. Let φ be a continuous function on X. We want to prove that the family of functions $(P^n\varphi)_{n\geq 1}$ is equicontinuous. We can assume $\|\varphi\|_\infty \leq 1$. We fix $\varepsilon > 0$. By uniform continuity of φ, there exists $\eta_\varepsilon > 0$ such that, for all x', y' in $\mathbb{P}(V)$,

$$d(x', y') \leq \eta_\varepsilon \Longrightarrow |\varphi(x') - \varphi(y')| \leq \varepsilon.$$

Let x, y be in $\mathbb{P}(V)$ such that $d(x,y) \leq \eta_\varepsilon / M_\varepsilon$ where M_ε is as in Lemma 3.2. We know from this lemma that the set

$$B_{\varepsilon,x,y} := \{b \in B \mid \sup_{n\geq 1} d(b_n \cdots b_1 x, b_n \cdots b_1 y) \leq M_\varepsilon\, d(x,y)\}$$

satisfies $\beta(B^c_{\varepsilon,x,y}) \leq \varepsilon$. We compute then by decomposing the following integral into two pieces,

$$\begin{aligned} |(P^n_\mu\varphi)(x) - (P^n_\mu\varphi)(y)| &\leq \textstyle\int_B |\varphi(b_n \cdots b_1 x) - \varphi(b_n \cdots b_1 y)|\, \mathrm{d}\beta(b) \\ &\leq \varepsilon\, \beta(B_{\varepsilon,x,y}) + 2\, \beta(B^c_{\varepsilon,x,y}) \leq 3\,\varepsilon. \end{aligned}$$

Since this upperbound does not depend on n, this computation proves that the family $(P^n\varphi)_{n\geq 1}$ is equicontinuous. □

Example 3.3. Lemma 3.2 and Proposition 3.1 are not always true when V is a semisimple representation of Γ_μ which is not strongy irreducible. For instance, when $V = W \oplus \mathbb{K}$ is a direct sum of an irreducible proximal representation of Γ_μ and the trivial representation, then the operator P_μ on $\mathbb{P}(V)$ is not equicontinuous. Indeed, in this case there are only two P_μ-ergodic probability measures on $\mathbb{P}(V)$: ν which is supported by $\mathbb{P}(W)$ and the Dirac mass δ_{x_0} where x_0 is the Γ_μ-invariant point in $\mathbb{P}(V)$. For every $x \neq x_0$, one has $\nu_x = \lim_{n\to\infty} \mu^n * \delta_x = \nu$ while $\nu_{x_0} = \delta_{x_0}$. Hence the map $x \mapsto \nu_x$ is not continuous and, according to Proposition 2.9.*c*, the operator P_μ is not equicontinuous.

However Proposition 3.1 is also true under a slightly more general assumption than strong irreducibility. This fact will be useful in the proof of Proposition 5.1.

Corollary 3.4. *Let $\mathbb{K}$ be a local field of characteristic* 0*,* $V = \mathbb{K}^d$*,* $X := \mathbb{P}(V)$*,* μ *be a probability measure on* $\mathrm{GL}(V)$*. We assume that* V *is a direct sum of strongly irreducible representations* V_i *of* Γ_μ *such that*

$$\sup_{g\in\Gamma_\mu} \tfrac{\|g|_{V_i}\|}{\|g|_{V_j}\|} < \infty, \quad \text{for all } i, j. \tag{3.6}$$

Then the Markov-Feller operator P_μ *is equicontinuous.*

Remark 3.5. One can prove that the converse is also true: when P_μ is equicontinuous, condition (3.6) is satisfied.

Proof of Corollary 3.4. This is a corollary of the proofs of Proposition 3.1, Lemma 3.2 which are true with the same proof under this assumption (3.6). □

3.2. **Limit law on projective spaces.**

We can now prove part of the first two theorems of the introduction :

Proof of Theorem 1.1.i. By Proposition 3.1, P_μ is equicontinuous. Our statement follows then from Proposition 2.9. □

Proof of Theorem 1.3. Just apply Propositions 2.9.e and 3.1. □

Remark 3.6. When V is not irreducible, the limit (1.1) in Theorem 1.1 does not always exist. Indeed, an example can be constructed with $V = \mathbb{R}^2$ and μ a probability measure (with infinite moments) on the group of diagonal matrices $\Gamma := \{\mathrm{diag}(e^t, e^{-t}) \mid t \in \mathbb{R}\}$.

4. Compact minimal subsets in homogeneous spaces

In this chapter G will be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group and Γ a Zariski-dense subsemigroup of G. Our main goal is to describe the compact Γ-minimal subsets on an algebraic homogeneous space G/H (Proposition 4.2) and, in particular when $\mathbb{K} = \mathbb{R}$, to prove Theorem 1.7.

Studying the compact Γ-minimal subsets on algebraic homogeneous spaces is equivalent to studying the Γ-minimal subsets on projective spaces. Indeed, by Chevalley theorem, every algebraic homogeneous space G/H can be realized as an orbit in the projective space $\mathbb{P}(V)$ of an algebraic representation V of G. Conversely, since the G-orbits in the projective space $\mathbb{P}(V)$ of an algebraic representation of G are locally closed, any compact Γ-minimal subset on $\mathbb{P}(V)$ is supported by a G-orbit i.e. by an algebraic homogeneous space G/H.

4.1. **Zariski-dense subsemigroups.** In this section we recall well-known definitions and properties of reductive groups and their Zariski dense subsemigroups.

Let $\mathbb{K}$ be a local field of characteristic 0, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group $\mathbf{G}$, and $\mathfrak{g}$ be the Lie algebra of G. Let A be a maximal $\mathbb{K}$-split torus of G, Z be the centralizer of A in G, and $Z \to \mathfrak{a}; z \mapsto z^\omega$ the universal morphism of Z in a real vector space. Since A is central and cocompact in Z, any continuous morphism $A \to \mathbb{R}$ extends in a unique way as a continuous morphism $Z \to \mathbb{R}$ and hence defines a linear form on $\mathfrak{a}$. Thus, for any algebraic character χ of A, we let χ^ω be the unique linear form on $\mathfrak{a}$, such that, for any z in A, $|\chi(z)| = e^{\chi^\omega(z^\omega)}$. Let Σ be the set of restricted roots of A in Z. The set Σ^ω is a root system in the real vector space $\mathfrak{a}^*$. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be a closed Weyl chamber, $Z^+ := \{z \in Z \mid z^\omega \in \mathfrak{a}^+\}$, Π be the corresponding set of simple restricted roots, N be the corresponding maximal unipotent subgroup of G, $P := ZN$ be the corresponding minimal parabolic subgroup, $\mathcal{P} \simeq G/P$ be the full flag variety and $x_\Pi \in \mathcal{P}$ be the base point whose stabilizer is P.

Let K be a "good" maximal compact subgroup of G with respect to $\mathfrak{a}$, so that one has the Cartan decomposition $G = KZ^+K$ and the Iwasawa decomposition $G = KZN$. Every element g of G can be written as

$$g = k_{g,1} z_g^+ k_{g,2} \text{ with } k_{g,1} \in K,\ z_g^+ \in Z^+,\ k_{g,2} \in K. \tag{4.1}$$

The element $\kappa(g) := (z_g^+)^\omega \in \mathfrak{a}^+$ is uniquely defined and called the *Cartan projection of* g.

For every g in G and $x = kx_\Pi$ in $\mathcal{P}$ with k in K, there exists an element z_{gk} in Z such that

$$gk \in K z_{gk} N.$$

The element $\sigma(g, x) := (z_{gk})^\omega \in \mathfrak{a}$ is uniquely defined and this map $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is a cocycle which is called the *Iwasawa cocycle*

For any set $\Theta \subset \Pi$ of simple restricted roots, we let A_Θ be the centralizer in A of the sum of the root spaces associated to the elements of Θ^c, we let Z_Θ be the centralizer in G of A_Θ, we let N_Θ be the smallest unipotent normal subgroup of N whose Lie algebra contains the root spaces associated to the elements of Θ, we let $P_\Theta = Z_\Theta N_\Theta$ be the normalizer in G of N_Θ, we let $\mathcal{P}_\Theta = G/P_\Theta$ be the associated partial flag variety and $x_\Theta \in \mathcal{P}_\Theta$ be the base point whose stabilizer is P_Θ. In particular when $\Theta = \Pi$, one has

$$A_\Pi = A,\ Z_\Pi = Z,\ N_\Pi = N,\ P_\Pi = P,\ \mathcal{P}_\Pi = \mathcal{P}.$$

Let Γ be a Zariski-dense semigroup in G. Let $\Theta = \Theta_\Gamma \subset \Pi$ be the set of simple restricted roots α for which the set $\alpha(\kappa(\Gamma)) \subset \mathbb{R}$ is unbounded. Since the action of Γ on $\mathcal{P}_\Theta$ is proximal, there exists a unique Γ-minimal subset $\Lambda_\Gamma \subset \mathcal{P}_\Theta$: it is called the limit set of Γ in $\mathcal{P}_\Theta$ (see [2, 3.6]). For a suitable choice of torus A and Weyl chamber $\mathfrak{a}^+$, we may assume that

$$\text{the base point } x_\Theta \text{ belongs to the limit set } \Lambda_\Gamma. \tag{4.2}$$

Let A_Γ be the smallest subtorus A' of A such that

$$\kappa(\Gamma) \text{ stays at bounded distance from } \omega(A'). \tag{4.3}$$

and let H_Γ be the following solvable subgroup of G

$$H_\Gamma := A_\Gamma N_\Theta. \tag{4.4}$$

Let Z_Γ be the group

$$Z_\Gamma := Z_\Theta / A_\Gamma. \tag{4.5}$$

The following G-equivariant fibration

$$Y_\Gamma = G/H_\Gamma \longrightarrow \mathcal{P}_\Theta = G/P_\Theta \tag{4.6}$$

is a principal Z_Γ-bundle. This homogeneous space Y_Γ will play a crucial role in our analysis. We will denote by y_{Γ} the base point of Y_Γ.

Remark 4.1. When $\mathbb{K} = \mathbb{R}$, according to [13], one has $\Theta_\Gamma = \Pi$ and according to [2], one has $A_\Gamma = A$, and hence $H_\Gamma = AN$ is a maximal $\mathbb{R}$-split solvable algebraic subgroup of G and the principal bundle (4.6) is $Y_\Gamma = G/AN \longrightarrow \mathcal{P}_\Pi = G/P$.

4.2. **Minimal subsets in homogeneous spaces.** The following Proposition 4.2, describes exactly which algebraic homogeneous spaces support a compact Γ-invariant subset.

Proposition 4.2. *Let $\mathbb{K}$ be a local field of characteristic* 0*, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, Γ be a Zariski-dense subsemigroup of G, H be an algebraic subgroup of G and $X = G/H$. Then the following two assertions are equivalent :*
(i) There exists a compact Γ-invariant subset in X,
(ii) H contains a conjugate of the group $H_\Gamma := A_\Gamma N_{\Theta_\Gamma}$.

We will need the following Lemma which does not involve Zariski dense subsemigroups and which describes the cluster points of a G-orbit in a projective space.

Lemma 4.3. *Let $\mathbb{K}$ be a local field of characteristic* 0*, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, (V, ρ) be an algebraic representation of G and $\Theta \subset \Pi$ a subset of restricted simple roots. Let g_k be a sequence in G such that*

$$\text{for all } \alpha \text{ in } \Theta, \text{ one has } \alpha^\omega(\kappa(g_k)) \xrightarrow[k\to\infty]{} \infty. \tag{4.7}$$

and π be a non zero limit point in $\mathrm{End}(V)$ *of a sequence $\lambda_k \rho(g_k)$ with λ_k in $\mathbb{K}$.*
a) For all x in $\mathbb{P}(V) \smallsetminus \mathbb{P}(\ker \pi)$, the limit $\lim_{k\to\infty} g_k x$ exists and belongs to the projective space $\mathbb{P}(\mathrm{im}\, \pi)$.
b) This space $\mathbb{P}(\mathrm{im}\, \pi)$ is included in the set of fixed points of a conjugate of the unipotent group N_Θ.
c) More precisely, let $A' \subset A$ be the smallest subtorus of A such that $\sup_k d(\kappa(g_k), A') < \infty$*. This space $\mathbb{P}(\mathrm{im}\, \pi)$ is included in the set of fixed points of a conjugate of the solvable group $A'N_\Theta$.*

Proof of Lemma 4.3. a) The endomorphism π induces a well-defined map from $\mathbb{P}(V) \smallsetminus \mathbb{P}(\ker \pi)$ to $\mathbb{P}(V)$ and the sequence g_k converges toward π uniformly on compact subsets of $\mathbb{P}(V) \smallsetminus \mathbb{P}(\ker \pi)$.

b) and *c)* Using the Cartan decomposition $G = KZ^+K$ and using the compactness of the quotient Z/A, we may assume that the sequence g_k is in A^+. We may also assume that, for any pair of weights χ_1, χ_2 of A in V, the sequence $\chi_1^\omega(\kappa(g_k)) - \chi_2^\omega(\kappa(g_k))$ converges to a limit $\ell_{\chi_1,\chi_2} \in \mathbb{R} \cup \{\pm\infty\}$. Let S be the non-empty set of weights of A in V such that, for all χ_1 in S, when χ_2 is also in S, the limit ℓ_{χ_1,χ_2} is finite and, when χ_2 is not in S, the limit ℓ_{χ_1,χ_2} is $+\infty$. The image of π is then the direct sum $\mathrm{im}\, \pi = \oplus_{\chi \in S} V_\chi$ of the weight spaces V_χ of A in V such that χ is in S. By definition of Θ, if χ belongs to S and $\alpha \in \Sigma^+$ is a positive root whose decomposition into simple roots contains elements of Θ, the character $\chi + \alpha$ is not a weight of V. This proves that $\mathrm{im}\, \pi$ is included in the space V^{N_Θ} of fixed points of N_Θ. Moreover, by definition, all the characters of S coincide on A', hence this subtorus acts by a character on $\mathrm{im}\, \pi$. □

Proof of Proposition 4.2. We first want to prove $(i) \Rightarrow (ii)$. As the limit cone ℓ_Γ of $\kappa(\Gamma)$ in $\mathfrak{a}$ is convex (see [2, § 4]), there exists a sequence g_k in Γ such that, for any weight χ of A that is non trivial on A_Γ, one has $|\chi^\omega(\kappa(g_k))| \xrightarrow[k\to\infty]{} \infty$. Now, by Chevalley Theorem [6, 5..1], there exists an algebraic representation (V, ρ) of G and a point y_0 in $\mathbb{P}(V)$ such that the stabilizer of y_0 in G is equal to H. We may assume that the G-orbit Gy_0 spans the $\mathbb{K}$-vector space V. After extraction, we may assume that, for some λ_k in $\mathbb{K}$, the sequence $\lambda_k \rho(g_k)$ has a non-zero limit π in $\mathrm{End}(V)$.

By assumption there exists a point y on the G-orbit Gy_0 such that the orbit closure $\overline{\Gamma y}$ is a compact subset of Gy_0. As Gy_0 spans V and Γ is Zariski dense in G, we can assume $y \notin \mathbb{P}(\ker \pi)$. According to Lemma 4.3, the limit $hy = \lim_{k\to\infty} g_k y$ exists and is invariant by a conjugate of $A_\Gamma N_\Theta$. Since this point πy is still on the G-orbit Gy_0, this proves that the group $A_\Gamma N_\Theta$ is contained in a conjugate of H.

Implication $(ii) \Rightarrow (i)$ follows from the following more precise Proposition 4.5. □

Remark 4.4. The reader who is only interested in real Lie groups may avoid the next three Sections 4.3, 4.4 and 4.6 and go directly to Section 4.5. Indeed, when $\mathbb{K} = \mathbb{R}$, one has $\Theta = \Pi$ and $A_\Gamma = A$, so that the whole space $Y_\Gamma = G/AN$ is compact and implication $(ii) \Rightarrow (i)$ is trivial.

4.3. **Minimal subsets in Y_Γ.** We will now describe the set of compact Γ-minimal subsets of the homogeneous space $Y_\Gamma = G/H_\Gamma$. The main point will be to prove that this set is non empty.

We recall that y_{Γ} is the base point of Y_Γ, that Y_Γ is endowed with a left-action of G and a commuting free right-action of Z_Γ, and that the set of N_Θ-fixed points $Y_\Gamma^{N_\Theta}$ is equal to the fiber $\pi^{-1}(x_\Theta) = Z_\Theta y_{\Gamma} = y_{\Gamma} Z_\Gamma$ of the principal Z_Γ-bundle $Y_\Gamma \xrightarrow{\pi} \mathcal{P}_\Theta$.

Proposition 4.5. *Let $\mathbb{K}$ be a local field of characteristic 0, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, Γ be a Zariski-dense subsemigroup of G and $\Theta = \Theta_\Gamma$. Let $H_\Gamma = A_\Gamma N_\Theta$, $Y_\Gamma := G/H_\Gamma$ and y be a point of Y_Γ whose image $\pi(y)$ in $\mathcal{P}_\Theta$ is in the limit set Λ_Γ.*
a) The orbit closure $\overline{\Gamma y}$ is compact and Γ-minimal.
b) The set $M_y := \{z \in Z_\Gamma \mid yz \in \overline{\Gamma y}\}$ is a compact subgroup of Z_Γ.
c) For any y' in $\overline{\Gamma y}$, one has $M_{y'} = M_y$.
d) For every z in Z_Γ, one has $M_{yz} = z^{-1} M_y z$. When $y = y_{\Gamma}$, the group $M_\Gamma := M_{y_\Gamma}$ is called the limit group of Γ.
e) The map $F \mapsto \{z \in Z_\Gamma \mid y_\Gamma z \in F\}$ is a bijection between the sets

$$\{\textit{compact } \Gamma\textit{-minimal subset in } Y_\Gamma\} \longleftrightarrow M_\Gamma \backslash Z_\Gamma$$

In case $\mathbb{K} = \mathbb{R}$, the limit group was introduced by Benoist [3] and Proposition 4.5 was proved by Guivarc'h and Raugi [16].

We will need a few lemmas. First, to exhibit compact orbits on non compact homogeneous spaces, we will use Lemma 4.6 below, which, in a given linear representation, produces subspaces where Γ almost acts by similarities:

Lemma 4.6. *Let $\mathbb{K}$ be a local field of characteristic* 0*,* $V = \mathbb{K}^d$*,* Γ *be a subsemigroup of* $\mathrm{GL}(V)$ *and* r *be its proximal dimension. There exists* $C > 1$ *such that, for every* γ *in* Γ*,* π *in* $\overline{\mathbb{K}\Gamma}$ *with rank* r *and* $v, v' \neq 0$ *in* $W = \operatorname{im} \pi$*, one has*

$$\frac{\|\gamma v'\|}{\|v'\|} \leq C \frac{\|\gamma v\|}{\|v\|}. \tag{4.8}$$

Proof. First, note that, for any $\varepsilon > 0$, there exists $\alpha > 0$ such that, for any $x \in \mathbb{P}(V)$ and π in $\overline{\mathbb{K}\Gamma}$ with rank r, if $d(x, \mathbb{P}(\ker \pi)) \geq \varepsilon$, one has $\|\pi w\| \geq \alpha \|\pi\| \|w\|$. Indeed, if this were not the case, one could find a sequence of elements of $\overline{\mathbb{K}\Gamma}$ with rank r but with a non zero cluster point of rank $< r$.

Using the compactness of the Grassmann varieties, we pick $\varepsilon > 0$ such that, for any U in $\mathbb{G}_{n-r}(V)$ and U' in $\mathbb{G}_{n-r+1}(V)$, there exists x in $\mathbb{P}(U')$ with $d(x, \mathbb{P}(U)) \geq \varepsilon$, and we let α be as above. For γ in Γ, $W = \operatorname{im} \pi$ in Λ^r_Γ and $v \neq 0$ in W, we can find w in V such that $\pi w = v$ and $d(\mathbb{K}w, \mathbb{P}(\ker \pi)) \geq \varepsilon$. We get

$$\alpha \|\pi\| \|w\| \leq \|v\| \leq \|\pi\| \|w\|$$
$$\alpha \|\gamma\pi\| \|w\| \leq \|\gamma v\| \leq \|\gamma\pi\| \|w\|$$

hence

$$\alpha \frac{\|\gamma\pi\|}{\|\pi\|} \leq \frac{\|\gamma v\|}{\|v\|} \leq \frac{1}{\alpha} \frac{\|\gamma\pi\|}{\|\pi\|}.$$

(4.8) follows immediately. □

Now, the following Lemma constructs a representation that is adapted to the setting of Proposition 4.5.

Lemma 4.7. *Let $\mathbb{K}$ be a local field of characteristic* 0*, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, Γ be a Zariski-dense subsemigroup of G and H be an algebraic subgroup containing the group* $H_\Gamma := A_\Gamma N_\Theta$.
a) Then there exists an algebraic representation V of G and a point x in $\mathbb{P}(V)$ whose stabilizer in G is equal to H and whose orbit spans V.
b) For such a representation V, the group A_Γ acts by a character on the space V^{N_Θ}.
c) There exists $C > 1$ such that, for every γ in Γ, and v, v' non-zero in V^{N_Θ}, one has

$$\frac{\|\gamma v'\|}{\|v'\|} \leq C \frac{\|\gamma v\|}{\|v\|}. \tag{4.9}$$

Proof of Lemma 4.7. *a*) This is a special case of Chevalley Theorem [6, 5.1].

b) We write $x = \mathbb{K}v$ and $V = \bigoplus_i V_i$, where each V_i is an irreducible subrepresentation with highest weight χ_i. We have $V^{N_\Theta} = \bigoplus_i V_i^{N_\Theta}$

and, for any i, $V_i^{N_\Theta}$ is the sum of the weight spaces $V_{i,\chi'}$ of V_i associated to characters χ' of A such that $\chi_i - \chi'$ is a sum of elements of Θ^c. In particular, since $A_\Gamma \subset A_\Theta$, A_Γ acts by a character on $V_i^{N_\Theta}$. Now, write $v = \sum_i v_i$. Since Gv spans V, for any i, we have $v_i \neq 0$. As A_Γ fixes $\mathbb{K}v$, A_Γ acts by a character on this line, hence all the characters χ_i have the same restriction to A_Γ, what should be proved.

c) Let us prove that the proximal dimension of $\rho(\Gamma)$ is the dimension of V^{N_Θ} and that, due to (4.2), V^{N_θ} is the image of an element of $\overline{\mathbb{K}\rho(\Gamma)}$: this and Lemma 4.6 will imply the result.

Indeed, let g_k be a sequence in Γ and assume, for some λ_k in $\mathbb{K}$, the sequence $\lambda_k\rho(g_k)$ converges towards a non zero endomorphism π of V. For any k, let $g_k = h_k z_k \ell_k$ be a Cartan decomposition of g_k with h_k, ℓ_k in K and z_k in Z^+. After extracting a subsequence, we may assume $\lambda_k\rho(z_k)$ converges towards a non zero endomorphism ϖ of V and π and ϖ have the same rank. Since ϖ is not zero, we must have

$$\sup_k |\log \|\rho(z_k)\| - \log|\lambda_k|| < \infty.$$

Now, since A is cocompact in Z and acts by characters on the weight spaces of V, we have

$$\sup_{z\in Z^+} \left|\log \|\rho(z)\| - \max_i \chi_i^\omega(z^\omega)\right| < \infty.$$

As, for any k, $z_k^\omega = \kappa(g_k)$ and all the characters χ_i have the same restriction to A_Γ, we get

$$\sup_{i,k} |\chi_i^\omega(z_k^\omega) - \log|\lambda_k|| < \infty.$$

Finally, for any i, we let X_i be the set of characters of A such that $\chi_i - \chi'$ is a sum of elements of Θ^c. By definition of Θ, and still since $z_k^\omega = \kappa(g_k)$, we get

$$\sup_{i,k} \sup_{\chi'\in X_i} |(\chi')^\omega(z_k^\omega) - \log|\lambda_k|| < \infty.$$

Hence, we have $\bigoplus_{i,\chi'\in X_i} V_{i,\chi'} = V^{N_\Theta} \subset \operatorname{im}\varpi$ and ϖ has rank $\geq \dim V^{N_\Theta}$. Conversely, since the limit cone ℓ_Γ of $\kappa(\Gamma)$ in $\mathfrak{a}$ is convex (see [2, § 4]), we can chose g_k in such a way that, for any α in Θ, one has $\alpha^\omega(\kappa(g_k)) \to \infty$. Since by (4.2) x_Π belongs to the inverse image of Λ_Γ in $\mathcal{P}$, we can assume $g_k V^{N_\Theta} \to V^{N_\Theta}$. Then, we get $\operatorname{im}\pi = \operatorname{im}\varpi = V^{N_\Theta}$ and we are done. □

Proof of Proposition 4.5. We will first prove that *the orbit closure* $\overline{\Gamma y_\Gamma}$ *in* Y_Γ *is compact.* We pick a representation V of G as in Lemma 4.7

with $H = H_\Gamma$ and we let $d = \dim V^{N_\Theta}$. We set

$$\mathcal{R} = \{(x_1, \ldots, x_{d+1}) \in \mathbb{P}(V)^{d+1} \text{ invariant by a conjugate of } N_\Theta \\ \text{and } d \text{ by } d \text{ linearly independant}\}.$$

We claim that *the G-orbit of any element* $\mathbf{x} = (x_1, \ldots, x_{d+1}) \in \mathcal{R}$ *is closed in* $\mathcal{R}$. Indeed, we will check that the stabilizer $G_{\mathbf{x}}$ of such an element $\mathbf{x}$ is conjugate to H_Γ. We can assume $\mathbf{x}$ to be N_Θ-invariant. But then, $G_{\mathbf{x}}$ acts trivially on $\mathbb{P}\left(V^{N_\theta}\right)$. Since by assumption $\mathbb{P}\left(V^{N_\theta}\right)$ contains a point whose stabilizer in G is exactly H_Γ, and since H_Γ acts trivially on $\mathbb{P}\left(V^{N_\theta}\right)$, we get $G_{\mathbf{x}} = H_\Gamma$. This proves our claim.

By Lemma 4.7.*c*, if $(x_1, \ldots, x_{d+1})$ is in $\mathcal{R}$ and $x_1, \ldots, x_{d+1}$ belong to V^{N_Θ}, the Γ-orbit of $(x_1, \ldots, x_{d+1})$ in $\mathcal{R}$ has compact closure. Hence the orbit closure $\overline{\Gamma y_\Gamma}$ in $Y_\Gamma = G/H_\Gamma$ is compact.

The remaining statements follow from the following Lemma 4.8, applied to the principal Z_Γ-bundle $\pi^{-1}(\Lambda_\Gamma) \xrightarrow{\pi} \Lambda_\Gamma$. □

Lemma 4.8. *Let Γ and Z be locally compact topological groups. Let Y be a locally compact topological space, equipped with a continuous left-action of Γ and a continuous right-action of Z that commute to each other, such that the action of Z is proper and cocompact and the action of Γ on $X = Y/Z$ is minimal. Assume that there exists a point y_0 in Y such that the orbit closure $\overline{\Gamma y_0}$ is compact. Then, for all y in Y,*
a) The orbit closure $\overline{\Gamma y}$ is also compact and is Γ-minimal.
b) The set $M_y := \{z \in Z \mid yz \in \overline{\Gamma y}\}$ is a compact subgroup of Z.
c) For any y' in $\overline{\Gamma y}$, one has $M_{y'} = M_y$.
d) For every z in Z, one has $M_{yz} = z^{-1} M_y z$.
e) The map $F \mapsto \{z \in Z | y_0 z \in F\}$ is a bijection between the sets

$$\{\textit{compact } \Gamma\textit{-minimal subset in } Y\} \longleftrightarrow M_{y_0}\backslash Z.$$

Proof of Lemma 4.8. *a*) Since $F_0 = \overline{\Gamma y_0}$ contains a Γ-minimal closed subset, we may assume it is Γ-minimal. Since X is Γ-minimal, one has $\pi(F_0) = X$. Hence for every y in Y, there exists z in Z such that y belongs to $F_0 z$. Since the actions of Γ and Z commute the set $F_0 z$ is Γ-invariant and Γ-minimal and the orbit closure $\overline{\Gamma y}$ is equal to $F_0 z$.

b) Since $\overline{\Gamma y}$ is Γ-minimal, the set M_y can also be defined as

$$M_y = \{z \in Z \mid \overline{\Gamma y}\, z = \overline{\Gamma y}\}. \tag{4.10}$$

Hence M_y is a compact subgroup of Z.

c), *d*) and *e*) follow also from (4.10). □

4.4. **Limit group of a Zariski dense semigroup.** In this section we give another definition of the *limit group* M_Γ of a Zariski dense subgroup that will be useful for the proof of Theorem 1.10. This definition is similar to the one which has been introduced for real Lie groups in the appendix of [3, Th. 8.2].

Let $\mathbb{K}$ be a local field of characteristic 0, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group and Γ be a Zariski dense subsemigroup of G. We keep the notations

$$\Theta = \Theta_\Gamma, Z_\Theta, N_\Theta, \mathcal{P}_\Theta, x_\Theta, Z_\Gamma, Y_\Gamma, y_\Gamma, \dots$$

from Section 4.1. Let C_Γ be the center of Z_Γ. By construction this group C_Γ is compact modulo A_Θ/A_Γ.

Let N_Θ^- be the A-invariant unipotent subgroup of G opposite to P_Θ. According to the Bruhat decomposition [6, 21.15], the set

$$U_\Theta = N_\Theta^- Z_\Theta N_\Theta \tag{4.11}$$

is a Zariski open subset of G and every element g of U_Θ can be written in a unique way as a product $g = n_g^- z_g n_g$ with n_g^-, z_g and n_g in N_Θ^-, Z_Θ and N_Θ respectively. We introduce the Bruhat projection m as the map

$$m : U_\Theta \to Z_\Gamma \; ; \; g \mapsto m(g) := z_g A_\Gamma = \text{image of } z_g \text{ in } Z_\Gamma. \tag{4.12}$$

By definition of $\Theta = \Theta_\Gamma$, we can find a semisimple element γ_0 of Γ whose action on $\mathcal{P}_\Theta$ is proximal (see [2, 3.6]). Hence, for a suitable choice of a torus A and Weyl chamber $\mathfrak{a}^+$ we may assume a stronger condition than (4.2), namely that

$$\text{there exists } \gamma_0 \in Z_\Theta \cap \Gamma \text{ with } x_\Theta \text{ as attractive fixed point.} \tag{4.13}$$

Here are the alternative definition and the main properties of the limit group M_Γ.

Proposition 4.9. *Let $\mathbb{K}$ be a local field of characteristic 0, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, Γ be a Zariski-dense subsemigroup of G. We choose A and $\mathfrak{a}^+$ satisfying (4.13).*
a) The limit group M_Γ is equal to the closure $M_\Gamma = \overline{m(\Gamma \cap U_\Theta)}$.
b) This group M_Γ is a Zariski dense and compact subgroup of Z_Γ.
c) Moreover, if $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, the group $C_\Gamma M_\Gamma$ is open in Z_Γ.

Remark 4.10. By reasoning as in the proof of [20, 1.3], one could also prove that if $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$ the group M_Γ is open in Z_Γ.

We need the following Lemma

Lemma 4.11. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, G be the group of $\mathbb{K}$-points of a connected semisimple $\mathbb{K}$-group and H be a compact Zariski-dense subgroup of G. Then H is open in G.*

An example of such a group is $H = \mathrm{SL}(d, \mathbb{Z}_p)$ in $G = \mathrm{SL}(d, \mathbb{Q}_p)$.

Proof of Lemma 4.11. Since $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, the Lie algebra $\mathfrak{h}$ of H is a $\mathbb{K}$-subspace of the Lie algebra $\mathfrak{g}$ of G. Since H is Zariski dense in G, $\mathfrak{h}$ is $\mathrm{Ad}G$-invariant and hence $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Let H' be the Kernel of the adjoint action in $\mathfrak{g}/\mathfrak{h}$. This group H' is an algebraic subgroup of G with Lie algebra $\mathfrak{h}$. Since H is compact, and since $H \cap H'$ is open in H, the group $H \cap H'$ has finite index in H. Since H is Zariski dense in G, $H \cap H'$ and also H' are Zariski dense in G. Hence one has $\mathfrak{h} = \mathfrak{g}$. □

Proof of Proposition 4.9. a) We set $M'_\Gamma = \overline{m(\Gamma \cap U_\Theta)}$. We want to prove that $M_\Gamma = M'_\Gamma$. We only have to check

$$\overline{\Gamma y_{\Gamma}} \cap Y_\Gamma^{N_\Theta} = y_{\Gamma} M'_\Gamma \tag{4.14}$$

We first prove the inclusion $\subset$ in (4.14). Let g_k be a sequence in Γ such that the limit $y_\infty = \lim_{k\to\infty} g_k y_{\Gamma}$ exists and belongs to the fiber $y_{\Gamma} Z_\Gamma$. We want to prove that y_∞ belongs to the set $y_{\Gamma} M_\Gamma$. We first notice that, for k large, g_k belongs to U_Θ and we write as in (4.11) $g_k = n^-_{g_k} z_{g_k} n_{g_k}$. Since y_∞ belongs to the fiber $y_{\Gamma} Z_\Gamma$, we must have $\lim_{k\to\infty} n^-_{g_k} = e$ and the sequence $m(g_k)$ must converge to some $m_\infty \in M_\Gamma$. But then, one has the equality

$$y_\infty = \lim_{k\to\infty} z_{g_k} y_{\Gamma} = \lim_{k\to\infty} y_{\Gamma} m(g_k) = y_{\Gamma} m_\infty$$

and y_∞ belongs to $y_{\Gamma} M'_\Gamma$.

Finally, we prove the inverse inclusion $\supset$ in (4.14). By construction the image $m(\gamma_0)$ of γ_0 in Z_Γ is an elliptic element. In particular, there exists a sequence $k_i \to \infty$ such that

$$\lim_{i\to\infty} m(\gamma_0)^{k_i} = e. \tag{4.15}$$

Because of (4.13), the Bruhat decomposition (4.11) is related to the element γ_0 by the formulas

$$N^-_\Theta := \{g \in G \mid \lim_{k\to\infty} \gamma_0^k g \gamma_0^{-k} = e\}, \tag{4.16}$$

$$Z_\Theta := \{g \in G \mid \lim_{i\to\infty} \gamma_0^{k_i} g \gamma_0^{-k_i} = \lim_{i\to\infty} \gamma_0^{-k_i} g \gamma_0^{k_i} = g\}, \tag{4.17}$$

$$N_\Theta := \{g \in G \mid \lim_{k\to\infty} \gamma_0^{-k} g \gamma_0^{k} = e\}. \tag{4.18}$$

In particular, for g in $\Gamma \cap U_\Theta$, one has $y_\Gamma m(g) = z_g y_\Gamma = \lim_{i\to\infty} \gamma_0^{k_i} g y_\Gamma$, hence $y_\Gamma M'_\Gamma \subset \overline{\Gamma y_\Gamma}$.

b) By Proposition 4.5, M_Γ is a compact subgroup of Z_Γ. Since Γ is Zariski dense in G and m is a rational map, it follows from *a*), that M_Γ is Zariski dense in Z_Γ.

c) Since the quotient group Z_Γ/C_Γ is a finite index subgroup in the group of $\mathbb{K}$-points of a semisimple $\mathbb{K}$-group and since the image of M_Γ in this quotient is compact and Zariski dense, our claim follows from Lemma 4.11. □

This ends the proof of Proposition 4.2.

4.5. Minimal subsets and compact orbits for real groups.

In this section one has $\mathbb{K} = \mathbb{R}$ and we prove Theorem 1.7.

Proof of Theorem 1.7. Since the G-orbits in $\mathbb{P}(V)$ are locally closed, any Γ-minimal closed subset of $\mathbb{P}(V)$ is contained in a G-orbit and Theorem 1.7 follows from Proposition 4.12 below. □

Proposition 4.12 strengthens Proposition 4.2 when $\mathbb{K} = \mathbb{R}$.

Proposition 4.12. *Let G be the group of real points of a connected reductive $\mathbb{R}$-group, Γ be a Zariski-dense subsemigroup of G, H be an algebraic subgroup of G and $X = G/H$.*
a) X contains a compact Γ-minimal subset if and only if X is compact.
b) In this case, there exists a unique Γ-minimal subset in X.

Proof. *a*) If X is compact, it contains a Γ-minimal subset. Conversely, if X contains a compact Γ-minimal subset, by Proposition 4.2, we can assume $A_\Gamma N_{\Theta_\Gamma} \subset H$. Since $\mathbb{K} = \mathbb{R}$, one has $\Theta_\Gamma = \Pi$, $A_\Gamma = A$ and $N_\Theta = N$. As $P = ZN$ is cocompact in G and A is cocompact in Z, AN is cocompact in G and X is compact.

b) If the homogeneous space $X = G/H$ is compact, by Proposition 4.2 applied to $\Gamma = G$, the algebraic group H contains a conjugate of AN. The last statement then follows from Lemma 4.13 below. □

Lemma 4.13. *Let G be the group of real points of a connected reductive $\mathbb{R}$-group, $H = AN$ be a maximal $\mathbb{R}$-split solvable algebraic subgroup of G and Γ be a Zariski-dense subsemigroup of G. Then there exists a unique Γ-minimal subset F in G/AN.*

This Lemma is a special case of a result of Guivarc'h and Raugi in [16, Th. 2] relying on the appendix of [3].

Remark 4.14. Since A is an $\mathbb{R}$-split torus, the number of connected components of A is $2^{\dim A}$. There may exist more than one Γ-minimal subset in G/A_eN where A_e is the connected component of A. For instance, when $G = \mathrm{SL}(3,\mathbb{R})$ and Γ preserves a properly convex subset $\Omega \subset \mathbb{P}(\mathbb{R}^3)$, there are exactly four Γ-minimal subsets in G/A_eN. See [16] for more details.

Proof of Lemma 4.13. By Proposition 4.5, this amounts to proving that $M_\Gamma = Z_\Gamma = Z/A$. Now, by definition, M_Γ is a compact subgroup of Z_Γ, so that, by Godement Theorem, it is Zariski closed. The result follows since, by Proposition 4.9, it is also Zariski dense. □

To conclude this section, we will establish bijection (1.6). This will follow from Proposition 4.2 applied to $\Gamma = G$ and the following

Lemma 4.15. *Let G be the group of real points of a connected reductive $\mathbb{R}$-group, $P = MAN$ a minimal parabolic subgroup, H an algebraic subgroup containing AN and $X = G/H$. Then, the set X^{AN} of fixed points of AN in X is an M-orbit.*

This will be a consequence of the following classical

Lemma 4.16. *Let $\mathbb{K}$ be a field, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group and P be the group of $\mathbb{K}$-points of a minimal parabolic $\mathbb{K}$-subgroup. Then, for any g in G, g belongs to the subgroup of G spanned by P and gPg^{-1}.*

Proof. We let A be the group of $\mathbb{K}$-points of a maximal $\mathbb{K}$-split torus contained in P, Σ be the set of restricted roots of A in the Lie algebra of G, Σ^+ be the set of positive roots associated to the choice of P, Π be the basis of Σ^+ and $W = N_G(A)/Z_G(A)$ be the Weyl group of A. For w in W, let us prove by induction on $\ell_w = \sharp(\Sigma^+ \cap w(-\Sigma^+))$ that w may be written as a product of reflections s_α associated to elements α of $\Pi \cap w(-\Sigma^+)$.

Indeed, if $\ell_w = 0$, there is nothing to prove. If $\ell_w > 0$, we have necessarily $\Pi \cap w(-\Sigma^+) \neq \emptyset$. We pick $\alpha \in \Pi \cap w(-\Sigma^+)$. For any $\beta \in \Sigma^+ \smallsetminus \mathbb{R}\alpha$, since $s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ may be written as linear combination of elements of Π in which either all coefficients are ≥ 0 or all coefficients are ≤ 0, we have $s_\alpha(\beta) \in \Sigma^+$. Thus s_α permutes the elements of $\Sigma^+ \smallsetminus \mathbb{R}\alpha$ and, if $w' = s_\alpha w$, we have

$$\Sigma^+ \cap w'(-\Sigma^+) = \Sigma^+ \cap w(-\Sigma^+) \smallsetminus \mathbb{R}\alpha.$$

The result follows by induction.

Now, let g be in G and let us prove g belongs to the subgroup Q spanned by P and gPg^{-1}. By Bruhat decomposition, we can assume g normalizes A. Set $w = gZ_G(A) \in W$. By construction, for any α in $\Sigma^+ \cap w(-\Sigma^+)$, $N_Q(A)/Z_G(A)$ contains the reflection s_α associated to

α. Since we have proved that w may be written as the product of such reflections, we get $w \in N_Q(A)/Z_G(A)$, hence $g \in Q$. □

Proof of Lemma 4.15. Let x and $x' = gx$ be two points of X^{AN}. Still by Bruhat decomposition, we can assume g normalizes A and hence M. We get $P = MAN$ and $g^{-1}Pg = MA(g^{-1}Ng)$. As $x' = gx$ is N-invariant, x is $g^{-1}Ng$-invariant and $Mx = Px = g^{-1}Pgx$. Since, by Lemma 4.16, g belongs to the subgroup spanned by P and $g^{-1}Pg$, we get $gMx = Mx$, hence $x' \in Mx$. □

4.6. **Minimal subsets on the flag variety.** In this section $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$ for a prime number p. We prove that the flag variety $\mathcal{P} = G/P$ supports only finitely many Γ-minimal subsets. This result is easier to prove when $\mathbb{K} = \mathbb{R}$ since in this case there exists only one Γ-minimal subset on the flag variety.

Proposition 4.17 (Finiteness)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group, Γ be a Zariski dense subsemigroup in G and P be a minimal parabolic subgroup of G. Then there exists only finitely many Γ-minimal subsets in the flag variety $\mathcal{P} = G/P$.*

Remarks 4.18. 1. When the field $\mathbb{K}$ is $\mathbb{R}$, or more generally when the set Θ_Γ is the whole set Π of simple restricted roots, the action on the full flag variety is proximal and *there exists only one Γ-minimal subset in $\mathcal{P}$.*

2. When the field $\mathbb{K}$ is $\mathbb{C}$, *there exists also only one Γ-minimal subset in $\mathcal{P}$.* Indeed the Zariski closure H of Γ in G for the real Zariski topology is a reductive group which contains a real form of G. Such a group H has only one compact orbit in the flag variety $\mathcal{P}$ and this orbit is a partial flag variety H/Q of H. Hence our claim follows from the first remark combined with Proposition 4.12.

3. When the field $\mathbb{K}$ is $\mathbb{Q}_p$, *there may exist more than one Γ-minimal subset in $\mathcal{P}$.* This is the case when Γ is a small open compact subgroup of G.

4. When the field $\mathbb{K}$ is an extension of $\mathbb{Q}_p$, *there may exist uncountably many Γ-minimal subsets in $\mathcal{P}$.* This is the case, when $G = \mathrm{SL}(2, \mathbb{K})$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z}_p)$ as soon as $\mathbb{K}$ is an extension of $\mathbb{Q}_p$ of degree $d \geq 4$, because, in this example, $\dim_{\mathbb{Q}_p} \mathcal{P} = d > \dim_{\mathbb{Q}_p} \Gamma = 3$.

Proof of Proposition 4.17. We set $\Theta = \Theta_\Gamma$ and we use freely the notations from the previous sections. We consider the fibrations

$$Y_\Gamma = G/A_\Gamma N_\Theta \xrightarrow{\pi} \mathcal{P} = G/P \xrightarrow{\varpi} \mathcal{P}_\Theta = G/P_\Theta.$$

Let x be in $\mathcal{P} = G/P$ be such $\overline{\Gamma x}$ is minimal. Then by uniqueness of the Γ-minimal subset in G/P_Θ, we get $\varpi(\overline{\Gamma x}) = \Lambda_\Gamma$ and we can assume $\varpi(x) = x_\Theta$. Note that the left action of Z_Θ on the fibers $\varpi^{-1}(x)$ and $(\pi\varpi)^{-1}(x)$ factors as an action of Z_Γ. Pick y in Y_Γ such that $\pi(y) = x$. By Proposition 4.5.*e*, we have $\overline{\Gamma y} \cap Z_\Gamma y = M_\Gamma y$, hence $\overline{\Gamma x} \cap Z_\Gamma x$ contains $M_\Gamma x$. Now, by Proposition 4.9.*c*, the group M_Γ has open orbits in $\varpi^{-1}(x_\Theta)$ which is a compact set. The result follows. □

5. Finite stationary measures on homogeneous spaces

In this chapter we describe the stationary probability measures on projective spaces and prove Theorems 1.1.*ii*, 1.5 and 1.10. More precisely we describe exactly which algebraic homogeneous spaces support a stationary probability measure. Those are the ones that support a compact minimal subset and that were described in Chapter 4.

We keep the notations of Chapter 4. Let μ be a Zariski-dense probability measure on G i.e. a probability measure such that the semigroup $\Gamma = \Gamma_\mu$ is Zariski-dense in G. We will shorten the notations, writing

$$\Theta_\mu = \Theta_{\Gamma_\mu}.$$

5.1. **Stationary measures on homogeneous spaces.** Studying μ-ergodic probability measures on projective spaces is equivalent to studying μ-ergodic probability measures on homogeneous algebraic spaces. Indeed, by Chevalley Theorem [6, 5.1], every algebraic homogeneous space G/H can be realized as an orbit in the projective space $\mathbb{P}(V)$ of an algebraic representation V of G. Conversely, since the G-orbits in the projective space $\mathbb{P}(V)$ of an algebraic representation of G are locally closed, any μ-ergodic probability measure on $\mathbb{P}(V)$ is supported by a G-orbit i.e. by an algebraic homogeneous space G/H.

Proof of Theorem 1.5. According to the previous discussion, Theorem 1.5 follows from Proposition 5.1 below. □

Proposition 5.1. *Let $\mathbb{K}$ be a local field of characteristic* 0*, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, μ be a Zariski-dense probability measure on G, H be an algebraic subgroup of G and $X = G/H$.*
a) The following three assertions are equivalent :
(i) There exists a μ-stationary probability measure on X,
(ii) There exists a compact Γ_μ-invariant subset in X,
(iii) H contains a conjugate of the group $H_{\Gamma_\mu} = A_{\Gamma_\mu} N_{\Theta_\mu}$.
b) Every μ-ergodic probability measure on G/H has compact support.
c) The map $\nu \mapsto \mathrm{supp}(\nu)$ is a bijection between the sets

$$\{\mu\text{-ergodic probability on } X\} \longleftrightarrow \{\Gamma_\mu\text{-minimal compact subset of } X\}.$$

Remark 5.2. When $\mathbb{K} = \mathbb{R}$, one can improve the statement of Proposition 5.1: see Proposition 5.5.

The proof of Proposition 5.1 will occupy the next three sections.

5.2. N_{Θ_μ} **is in the stabilizer.** The aim of this section is to prove part of the implication $(i) \Rightarrow (iii)$ in Proposition 5.1.*a*. More precisely, we will check that a conjugate of N_{Θ_μ} is included in H or equivalently we will prove the following

Lemma 5.3. *Let $\mathbb{K}$ be a local field of characteristic 0, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, μ be a Zariski-dense probability measure on G, $V = \mathbb{K}^d$ be an algebraic representation of G and ν be a μ-stationary probability measure on $\mathbb{P}(V)$. Let Y be the set of points of $\mathbb{P}(V)$ wich are invariant by a conjugate of N_{Θ_μ}. Then we have $\nu(Y) = 1$.*

Proof. We can assume ν to be μ-ergodic and, by induction on the dimension of V, for any proper subspace W of V, one has $\nu(\mathbb{P}(W)) < 1$. Let us prove this implies, for any such W, one has $\nu(\mathbb{P}(W)) = 0$. This is a variation on a classical argument due to Furstenberg.

Indeed, let r be the smallest integer > 0 such that there exists an r-dimensional subspace W of V with $\nu(\mathbb{P}(W)) > 0$. For any $W \neq W'$ in $\mathbb{G}_r(V)$, one has $\nu(\mathbb{P}(W) \cap \mathbb{P}(W')) = 0$, hence, if W_i is a finite or countable family of distinct elements of $\mathbb{G}_r(V)$,one has

$$\nu(\textstyle\bigcup_i \mathbb{P}(W_i)) = \sum_i \nu(\mathbb{P}(W_i)).$$

Thus, if, for any subset E of $\mathbb{G}_r(V)$, we set $\nu'(E) = \sum_{W \in E} \nu(\mathbb{P}(W))$, the function ν' is a finite measure defined on all the subsets of $\mathbb{G}_r(V)$. Moreover, the measure ν' is atomic and μ-stationary. Hence, it may be written as a countable sum of invariant measures carried by finite orbits of Γ_μ in $\mathbb{G}_r(V)$. (see for example [7, Prop 2.3] or [5]). Since ν is ergodic, ν' is ergodic, hence it is supported on a unique finite Γ_μ-orbit $\mathcal{W} \subset \mathbb{G}_r(V)$. Now, as Γ_μ is Zariski dense in G, $\mathcal{W}$ is also G-invariant, and, as G is Zariski connected, $\mathcal{W}$ is a singleton $\{W\}$. In other terms, there exists a G-invariant subspace $W \in \mathbb{G}_r(V)$ with $\nu(\mathbb{P}(W)) > 0$. By ergodicity of ν, we get $\nu(\mathbb{P}(W)) = 1$, hence by assumption, $W = V$, that is $r = d$ and we are done.

Let $B = G^{\mathbb{N}^*}$ and $\beta = \mu^{\otimes\mathbb{N}^*}$. According to a result of Furstenberg and Guivarc'h-Raugi , for β-almost any b in B, for any α in Θ_μ, one has $\alpha(\kappa(b_1 \dots b_n)) \xrightarrow[n\to\infty]{} \infty$ (see [7, Prop. 3.2] or [5]). Thus, by Lemma 4.3.*b*, for β-almost all b in B, the image $\mathbb{P}(\operatorname{im}\pi)$ of any non-zero limit point π in $\operatorname{End}(V)$ of a sequence $\lambda_k b_1 \dots b_{n_k}$ with λ_k in $\mathbb{K}$ is contained in Y. Now, according to another result of Furstenberg and Guivarc'h-Raugi, for β-almost any b in B, the measure $(b_1 \cdots b_n)_* \nu$ converges towards a probability measure ν_b on $\mathbb{P}(V)$ and $\nu = \int_B \nu_b \, \mathrm{d}\beta(b)$ (see [7, Lem. 2.1]). If π is as above, since $\nu(\ker \pi) = 0$, we get $\nu_b(\operatorname{im} \pi) = 1$, hence $\nu_b(Y) = 1$. Thus $\nu(Y) = 1$ and we are done. □

5.3. A_{Γ_μ} **is in the stabilizer.** The aim of this section is to prove the second half of the implication $(i) \Rightarrow (iii)$ in Proposition 5.1.a, namely, that a conjugate of A_{Γ_μ} is contained in H.

Proof of Proposition 5.1.a. The equivalence $(ii) \Leftrightarrow (iii)$ follows from Proposition 4.2. The implication $(ii) \Rightarrow (i)$ is clear since any compact Γ_μ-invariant set supports a μ-stationary probability measure.

It only remains to prove the implication $(i) \Rightarrow (iii)$. By Lemma 5.3, we can assume that H contains N_{Θ_μ}. Since every algebraic subgroup H of G contains a cocompact algebraic subgroup which is $\mathbb{K}$-split solvable, we can assume that H is $\mathbb{K}$-split solvable. Since AN is a maximal $\mathbb{K}$-split solvable subgroup of G, after conjugation, we may assume that $H = A'N'$ with N' a unipotent subgroup such that $N_{\Theta_\mu} \subset N' \subset N$ and A' a subtorus of A normalizing N'. Enlarging H, we may assume that $N \subset H$.

Now, according to Lemma 5.4 below, the torus A' contains A_{Γ_μ} and we are done. □

In this proof, we used the following

Lemma 5.4. *Let $\mathbb{K}$ be a local field of characteristic 0, G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, μ be a Zariski-dense probability measure on G. Let A be a maximal $\mathbb{K}$-split torus of G, N be a maximal unipotent subgroup normalized by A, A' be a subtorus of A and $H = A'N$. If G/H supports a μ-ergodic μ-stationary probability measure ν then ν has compact support and the torus A' contains A_{Γ_μ}.*

Proof of Lemma 5.4. We let $\mathfrak{a}' = \omega(A')$ and $Z' = \omega^{-1}(\mathfrak{a}')$, so that A' is a cocompact subgroup of Z'. We consider the action of G on $\mathcal{P} \times \mathfrak{a}/\mathfrak{a}'$ such that, for any g in G, x in $\mathcal{P}$ and t in $\mathfrak{a}/\mathfrak{a}'$, one has

$$g(x, t) = (gx, t + \overline{\sigma}(g, x)),$$

where $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is the Iwasawa cocycle and $\overline{\sigma}$ denotes its composition with the natural map $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}'$.

We claim the stabilizer of $(x_\Pi, 0)$ for this action is $Z'N$ and the orbit map $G/Z'N \to \mathcal{P} \times \mathfrak{a}/\mathfrak{a}'$ is proper. Indeed, if, for some g in G, one has $g(x_\Pi, 0) = (x_\Pi, 0)$, then $g = zn$ belongs to $P = ZN$ and $\omega(z) = \sigma(g, x_\Pi) \in \mathfrak{a}'$. Now, if g_n is a sequence in G such that $g_n Z'N$ leaves every compact subset of $G/Z'N$, since G/P is compact, we can assume g_n belongs to ZN. Since N is normal in Z, we can assume $g_n = z_n$ belongs to Z and z_n leaves every compact subset of Z. Now, since ω is a proper morphism $Z \to \mathfrak{a}$, the image of $\omega(z_n)$ in $\mathfrak{a}/\mathfrak{a}'$ leaves every compact subset and we are done.

We let ν' be the image of ν under the maps

$$G/A'N \to G/Z'N \to \mathcal{P} \times \mathfrak{a}/\mathfrak{a}',$$

so that ν' is a μ-ergodic μ-stationary probability measure on $\mathcal{P} \times \mathfrak{a}/\mathfrak{a}'$ and we will prove ν' has compact support. Since A' is cocompact in Z' and the orbit map $G/Z'N \to \mathcal{P} \times \mathfrak{a}/\mathfrak{a}'$ is proper, this will imply ν has compact support too.

The dynamical system

$$\begin{aligned} &B \times (\mathcal{P} \times \mathfrak{a}/\mathfrak{a}') \to B \times (\mathcal{P} \times \mathfrak{a}/\mathfrak{a}') \\ &\qquad (b,(x,t)) \mapsto (Tb, b_1(x,t)) = (Tb,(b_1x, t+\overline{\sigma}(b_1,x))) \end{aligned}$$

preserves the probability measure $\beta \otimes \nu'$ and is ergodic. Hence by Birkhoff ergodic theorem, for all $M > 0$, for ν'-almost all (x,t) in $\mathcal{P} \times \mathfrak{a}/\mathfrak{a}'$, for β-almost all b in B, one has

$$\lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^{n} \mathbf{1}_{\{\|t+\overline{\sigma}(b_k\cdots b_1,x)\| \le M\}} = \nu(\mathcal{P} \times B(0,M)),$$

where $B(0,M)$ is the ball of radius M and center 0 in $\mathfrak{a}/\mathfrak{a}'$.

We will need the following fact which is an intrinsic reformulation of (3.1), and relates the Iwasawa cocycle and the Cartan projection for a random trajectory (see also [5]): *for all $\varepsilon > 0$, there exists $M_\varepsilon > 0$, such that, for all x in $\mathcal{P}$,*

$$\beta(\{b \in B \mid \sup_{n\ge 1} \|\sigma(b_n\cdots b_1,x) - \kappa(b_n\cdots b_1)\| \le M_\varepsilon\}) \ge 1-\varepsilon. \tag{5.1}$$

Fix $\varepsilon > 0$. One can find $M_1 > 0$ such that, for ν'-almost all (x,t) in $\mathcal{P} \times \mathfrak{a}/\mathfrak{a}'$,

$$\liminf_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^{n} \beta(\{b \in B \mid \|t+\overline{\sigma}(b_k\cdots b_1,x)\| \le M_1\}) \ge 1-\varepsilon. \tag{5.2}$$

Then, using (5.1), one can find $M_2 > 0$ such that,

$$\liminf_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^{n} \beta(\{b \in B \mid \|\overline{\kappa}(b_k\cdots b_1)\| \le M_2\}) \ge 1-2\,\varepsilon \tag{5.3}$$

(where $\overline{\kappa}$ denotes the image of the Cartan projection in $\mathfrak{a}/\mathfrak{a}'$). Using again (5.1), one can find $M_3 > 0$ such that, for all x in $\mathcal{P}$,

$$\liminf_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^{n} \beta(\{b \in B \mid \|\overline{\sigma}(b_k\cdots b_1,x)\| \le M_3\}) \ge 1-3\,\varepsilon. \tag{5.4}$$

If $\mathrm{supp}(\nu')$ were not compact, the number t in (5.2) could be chosen to be arbitrarily large. This would contradict (5.4) when $\varepsilon < \frac{1}{4}$ and

$t > M_1 + M_3$. Hence ν' has compact support and so does ν as remarked above.

In particular, Γ_μ-preserves a compact subset in $G/A'N$, so that, by Proposition 4.2, $A_{\Gamma_\mu} N_{\Theta_\mu}$ fixes a point in $G/A'N$. Since, by Bruhat decomposition, the set of fixed points of N_{Θ_μ} in $\mathcal{P} = G/ZN$ is $Z_{\Theta_\mu} x_\Pi$, we get $A_{\Gamma_\mu} \subset A'$, what should be proved. □

5.4. **Equicontinuity on homogeneous spaces.** In this section we finish the proof of the classification of μ-stationary probability measures on a homogeneous space G/H by using a compactification of G/H for which the Markov-Feller operator P_μ is equicontinuous.

Proof of Proposition 5.1.b and c. By point a), one can assume that H contains H_{Γ_μ}. By Lemma 4.7, the homogeneous space G/H occurs as a G-orbit in a projective space $\mathbb{P}(V)$ where (ρ, V) is a representation of G which is the direct sum of strongly irreducible representations (ρ_i, V_i) with highest weight χ_i, such that all the χ_i have the same restriction to A_{Γ_μ}. By Cartan decomposition, for any i, there exists $C_i > 0$ such that, for any g in G, one has

$$\tfrac{1}{C_i} \left\|\rho_i(g)\right\| \leq \exp(\chi_i^\omega(\kappa(g))) \leq C_i \left\|\rho_i(g)\right\|.$$

Thus, the assumptions of Corollary 3.4 are satisfied and hence the Markov-Feller operator P_μ on $\mathbb{P}(V)$ is equicontinuous. Our statement then follows from Proposition 2.9.a. □

We end this chapter by discussing a few properties of stationary measures which are different over the real numbers and over the non-archimedean local fields: we conclude the proof of Theorem 1.1.ii, Theorem 1.7 and Theorem 1.10.

5.5. **Stationary measures for real groups.** We strengthen here Proposition 5.1 when $\mathbb{K} = \mathbb{R}$.

Proposition 5.5. *Let G be the group of real points of a connected reductive $\mathbb{R}$-group, μ be a Zariski-dense probability measure on G, H be an algebraic subgroup of G and $X = G/H$.*
a) There exists a μ-stationary probability measure on X if and only if X is compact.
b) In this case, (i) the Markov-Feller operator P_μ on X is equicontinous, (ii) there exists a unique μ-stationary probability measure on X.

Proof of Proposition 5.5. a) Since $\mathbb{K} = \mathbb{R}$, one knows that $\Theta_\mu = \Pi$, $A_{\Gamma_\mu} = A$ and $N_{\Theta_\mu} = N$ and our claims follow from Proposition 5.1 and the compactness of G/AN.

b) When the homogeneous space $X = G/H$ is compact, the algebraic group H contains a conjugate of AN. By Lemma 4.7 and Corollary 3.4, the Markov-Feller operator P_μ on X is equicontinuous. The last statement then follows from Proposition 2.9.*a* and Lemma 4.13. □

5.6. **Eigenvalues of P_μ.** In this section one has $\mathbb{K} = \mathbb{R}$ and we end the proof of Theorem 1.1.

Proof of Theorem 1.1.ii. Our statement will follow from Proposition 2.3.*b* and the following Lemma 5.6. □

Lemma 5.6. *Let $X = \mathbb{P}(\mathbb{R}^d)$ and μ be a probability measure on $\mathrm{GL}(\mathbb{R}^d)$ such that the action of Γ_μ on $\mathbb{R}^d$ is strongly irreducible and the Zariski closure of Γ_μ is semisimple. Then, the only eigenvalue of modulus 1 of the averaging operator P_μ in $\mathcal{C}^0(X)$ is 1.*

Proof of Lemma 5.6. Let φ be a non-zero continuous function on X such that $P_\mu\varphi = \chi\varphi$ with $\chi \in \mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. We want to prove that $\chi = 1$. According to Proposition 2.7.*b*, there exists a Γ_μ-minimal subset of X on which φ is non-zero. By Theorem 1.7, this minimal subset is supported by a compact orbit G/H of the Zariski closure G of Γ_μ. By Lemma 4.15, H contains a conjugate of the maximal $\mathbb{R}$-split solvable subgroup AN of G.

We construct this way a non-zero continuous function ψ on $Y = G/AN$ such that $P_\mu\psi = \chi\psi$ with $\chi \in \mathbb{S}^1$. We want to prove that $\chi = 1$. This space Y is then an isometric extension of the flag variety $\mathcal{P}$ and this statement is due to Guivarc'h and Raugi in [16, Th. 3]. Here is a short proof of it.

We assume first that χ is a n^{th}-root of unity. We note that $P_{\mu^{*n}}\psi = \psi$ and, since G is semisimple, that the probability measure μ^{*n} is still Zariski dense in G. Hence, by Propositions 2.7.*c* and 4.12, the $P_{\mu^{*n}}$-invariant function ψ is constant and $\chi = 1$.

We assume now that χ is not a root of unity. We introduce the probability measure

$$\mu' := \mu \otimes \delta_\chi \ \text{ on } \ G' := G \times \mathbb{S}^1.$$

Since G is semisimple and since χ is not a root of unity, the probability measure μ' is Zariski dense in the real algebraic reductive group G'. We also introduce the continuous function ψ' on $Y' := G'/AN \simeq Y \times \mathbb{S}^1$ given by

$$\psi'(y, z) = z^{-1}\psi(y) \ \text{ for all } y \text{ in } Y,\ z \text{ in } \mathbb{S}^1.$$

This function ψ' is $P_{\mu'}$-invariant since one has

$$\begin{aligned} P_{\mu'}\psi'\,(y,z) &= \textstyle\int_G \psi'(gy,\chi z)\,\mathrm{d}\mu(g) \\ &= z^{-1}\chi^{-1}\,P_\mu\psi(y) = z^{-1}\psi(y) = \psi'(y,z). \end{aligned}$$

Hence, by Propositions 2.7.*c* and 4.12, the $P_{\mu'}$-invariant function ψ' is constant. Contradiction. □

5.7. **Stationary measures on the flag variety.** In this section $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$. We prove Theorem 1.10 which says that the flag variety $\mathcal{P} = G/P$ supports only finitely many μ-stationary measures. This statement is interesting only when $\mathbb{K}$ is non-archimedean since, when $\mathbb{K} = \mathbb{R}$, one knows that there exists only one μ-stationary measure on the flag variety (see for instance Proposition 5.5).

Proof of Theorem 1.10. By Proposition 5.1, the set of μ-ergodic probability measures on X is in bijection with the set of Γ_μ-minimal subsets of X. According to Proposition 4.17 this set is finite. □

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