RANDOM WALKS ON FINITE VOLUME HOMOGENEOUS SPACES

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ABSTRACT. Extending previous results by A. Eskin and G. Margulis, and answering their conjectures, we prove that a random walk on a finite volume homogeneous space is always recurrent as soon as the transition probability has finite exponential moments and its support generates a subgroup whose Zariski closure is semisimple.

1. INTRODUCTION

In this introduction G is a real Lie group. Later on, in section 7, we will consider more generally products of real and p-adic Lie groups. We denote by $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{r}$ its maximal amenable ideal, $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ and by $\mathrm{Ad}_{\mathfrak{s}} : G \to \mathrm{Aut}(\mathfrak{s})$ the adjoint action on $\mathfrak{s}$. The Lie algebra $\mathfrak{s}$ is the largest semisimple quotient of $\mathfrak{g}$ with no compact factor.

Let Λ be a lattice in G, $X := G/\Lambda$ and $x_0 := \Lambda$ be the base point of X. Let $\mu \in \mathcal{P}(G)$ be a Borel probability measure on G and $\Gamma = \Gamma_\mu$ be the closed sub-semigroup of G generated by the support of μ. We denote by $H_\mu \subset \mathrm{Aut}(\mathfrak{s})$ the Zariski closure of $\mathrm{Ad}_{\mathfrak{s}}(\Gamma_\mu)$ and by H_μ^{nc} the non compact part of H_μ, i.e. the smallest normal Zariski closed subgroup of H_μ such that H_μ/H_μ^{nc} is compact. We assume that

$$\begin{array}{ll} \text{(1.1)} & \begin{array}{l} H_\mu^{nc} \text{ is semisimple and} \\ \mu \text{ has finite exponential moments in } \mathfrak{s}, \end{array} \end{array}$$

$$\text{i.e.} \quad \int_G \|\mathrm{Ad}_{\mathfrak{s}}(g)\|^\delta \,\mathrm{d}\mu(g) < \infty \text{ for some } \delta > 0.$$

In this paper, we study the random walk associated to μ on X that is the Markov chain with state space X and transition probabilities $\mu * \delta_x$, $x \in X$. In other terms, given $x \in X$, we focus on the sequence of probability measures $\mu^{*n} * \delta_x$, $n \in \mathbb{N}$. We address the recurrence properties of this random walk.

This topic has been studied in depth by Eskin and Margulis in [6] where they prove different kind of recurrence and uniform recurrence properties for this random walk. In [6, §2.5], Eskin and Margulis state two conjectures called $(R1)$ and (S) on the recurrence behavior of this random walk.

Our first theorem answers positively conjecture $(R1)$ of [6].

Theorem 1.1. *Let G be a real Lie group, Λ be a lattice in G, $X := G/\Lambda$, and μ be a probability measure on G with finite exponential moments in $\mathfrak{s}$ and such that H_μ^{nc} is semisimple.*

For any $\varepsilon > 0$, any x in X, there exists a compact set $M = M_{\varepsilon,x} \subset X$ such that for any $n \geq 0$, one has $\mu^{*n} * \delta_x(M) \geq 1 - \varepsilon$.

Moreover, the compact set $M = M_{\varepsilon,x}$ is uniform for x in a compact subset of X.

Remark For a linear group G i.e. a subgroup of $\mathrm{SL}(d, \mathbb{R})$, assumptions (1.1) are satisfied as soon as

$$\begin{array}{ll} & \Gamma_\mu \text{ has a semisimple Zariski closure and} \\ (1.2) & \\ & \mu \text{ has finite exponential moments in } \mathbb{R}^d. \end{array}$$

Here is a reformulation of Theorem 1.1.

Corollary 1.2. *Under the same assumptions, for any x in X, any weak limit ν_∞ of the sequence $\nu_n := \mu^{*n} * \delta_x$ in the space of finite measures on X is a probability measure, i.e. $\nu_\infty(X) = 1$.*

Conjecture $(R1)$ in [6] was stated in a slightly too optimistic way under the weaker hypothesis that the group H_μ^{nc} is generated by unipotent elements. However E. Breuillard constructed in [5, Prop. 10.4] a counter-example with $G = \mathrm{SL}(2, \mathbb{R})$, $\Lambda = \mathrm{SL}(2, \mathbb{Z})$ and μ a non-centered probability measure with compact support on some one-parameter unipotent subgroup of G.

In case μ has compact support, the recurrence properties proven in [6] have been used in [1] as the starting point for the classification of μ-stationary probability measures on X and of Γ_μ-invariant closed subsets on X when G is a simple group and Γ_μ is Zariski dense in G. Theorem 1.1 is now used in [3] to extend this classification to the case where $\mathrm{Ad}_{\mathfrak{g}}(\Gamma_\mu)$ is Zariski dense in a semisimple subgroup of $\mathrm{Aut}(\mathfrak{g})$ with no compact factor.

As in [6], one deduces from Theorem 1.1 the following

Corollary 1.3. *Let G be a real Lie group, Λ be a lattice in G, $X := G/\Lambda$, and Γ be a discrete subgroup of G such that the Zariski closure of $\mathrm{Ad}_{\mathfrak{s}}(\Gamma)$ is semisimple. Then any discrete Γ-orbit in G/Λ is finite.*

Note that the group $\Lambda_S := \mathrm{Ad}_{\mathfrak{s}}(\Lambda)$ is always a lattice in $\mathrm{Aut}(\mathfrak{s})$ (see Lemma 6.1). A parabolic subgroup $P \subset \mathrm{Aut}(\mathfrak{s})$ is said to be Λ_S-rational if the group Λ_S intersects the unipotent radical of P in a lattice.

Theorem 1.1 was proved in [6] under the additional assumption that no conjugate of H_μ^{nc} is contained in some proper Λ_S-rational parabolic subgroup of $\mathrm{Aut}(\mathfrak{s})$. The simplest case of Theorem 1.1 which is not covered by [6] is when $G = \mathrm{SL}(3,\mathbb{R})$, $\Lambda = \mathrm{SL}(3,\mathbb{Z})$ and Γ_μ is Zariski dense in the $\mathrm{SL}(2,\mathbb{R})$ sitting in the top left corner.

We will now describe an explicit subset of points $x \in X$ starting from which the random walk is recurrent inside a uniform compact set.

Our second theorem answers positively conjecture (S) of [6].

Theorem 1.4. *Let G be a real Lie group, Λ be a lattice in G, $X := G/\Lambda$, and μ be a probability measure on G with finite exponential moments in $\mathfrak{s}$ and such that the group H_μ^{nc} is semisimple.*

For any $\varepsilon > 0$, there exists a compact set $M \subset X$ such that, for any g in G, either $g^{-1}H_\mu^{nc}g$ is contained in some proper Λ_S-rational parabolic subgroup of $\mathrm{Aut}(\mathfrak{s})$*, or there exists $n_g \geq 0$ such that, for any $n \geq n_g$, one has $\mu^{*n} * \delta_{gx_0}(M) \geq 1 - \varepsilon$.*

Note that one can not replace H_μ^{nc} by H_μ in Theorem 1.4 : there exists a counterexample with $G = \mathrm{SL}(6,\mathbb{R})$, $\Lambda = \mathrm{SL}(6,\mathbb{Z})$ and Γ_μ Zariski dense in $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(\mathbb{R}^3 \otimes \mathbb{R}^2) \simeq G$. In this example, the action of Γ_μ on $\mathbb{R}^6$ is irreducible, hence the group H_μ is not included in any parabolic subgroup, while its non compact part H_μ^{nc} is included in a Λ_S-rational parabolic subgroup.

Let us sketch our strategy in a few words. As in [6] we prove the existence of proper functions f on X satisfying the so-called "Foster exponential recurrence criterion" (see [7], [12, Chapter 15] and [14]). The main new idea is to construct these functions f by using the representation theory of the semisimple group H_μ^{nc}. Since it avoids the use of Reduction Theory, this idea gives also a simpler proof of the main results of [6], even when G is simple and Γ_μ is Zariski dense in G. However in the case where $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$, our proof is the same as in [6].

Here is the structure of this paper.

In section 2, we explain how the recurrence of the random walk on X follows from the existence of proper functions f which are contracted by the random walk.

In section 3, we prove an inequality in the exterior algebra of a finite dimensional vector space, that we call the Mother Inequality, which is the main new technical tool of this paper.

In section 4, we show how the Mother Inequality allows us to construct functions φ on the exterior algebra which simultaneously satisfy a convexity property with respect to the exterior product and are contracted by the random walk.

In section 5, we use these functions φ to construct proper functions f on X which are contracted by the random walk when X is the space of covolume 1 lattices in $\mathbb{R}^d$.

In section 6, we reduce the general case to the previous one.

In section 7, we extend these results to lattices in products of real and p-adic Lie groups and show that the functions f grow exponentially.

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2. The contraction hypothesis

We present in this section a very general criterion implying the recurrence of a given random walk.

Let G be a second countable locally compact group, X be a second countable locally compact space and $(g, x) \mapsto gx$ be a continuous action of G on X.

Let μ be a Borel probability measure on G. Let us denote by $f \mapsto A_\mu f$ the averaging operator which, to a given nonnegative Borel function f on X, associates the function defined by, for any x in X,

$$A_\mu f(x) = \int_G f(gx) d\mu(g).$$

A Borel function $f : X \to [0, \infty]$ is said to be *proper* if for any $R < \infty$, $f^{-1}([0, R])$ is relatively compact in X. We denote by $D_f := \{x \in X \mid f(x) < \infty\}$ the domain of f.

We will say that the action of (G, μ) on X satisfies the contraction hypothesis for a Borel proper function f if

CH(f) there exist constants $a < 1$, $b > 0$ such that $A_\mu f \leq af + b$.

This condition is a very strong μ-subharmonicity property for f: it says roughly that the averaging operator strictly contracts f as soon as $f(x)$ is large enough.

For a subset $X' \subset X$, we will say that the action of (G, μ) on X satisfies the uniform recurrence property on X' if

S(X') For any $\varepsilon > 0$, there exists a compact set $M = M_\varepsilon \subset X$ such that, for any x in X', there exists $n_x \geq 0$ such that, for $n \geq n_x$, one has $\mu^{*n} * \delta_x(M) \geq 1 - \varepsilon$.

The following is a reformulation of [6, Lemma 3.1].

Lemma 2.1. *Assume that the action of (G, μ) on X satisfies the contraction hypothesis $\mathbf{CH}(f)$ for a proper Borel function $f : X \to [0, \infty]$, then it satisfies the uniform recurrence property $\mathbf{S}(D_f)$.*

Proof. Set $B = \frac{b}{1-a}$. Since f is proper, the closure M of the set

$$\{y \in X \mid f(y) \leq \tfrac{2B}{\varepsilon}\}$$

is compact. The characteristic function of M^c satisfies $\mathbf{1}_{M^c} \leq \frac{\varepsilon}{2B} f$.

According to the hypothesis $\mathbf{CH}(f)$, one has, for every $n \geq 1$

$$A_\mu^n f \leq a^n f + b(1 + \cdots + a^{n-1}) \leq a^n f + B$$

Hence, for any x in D_f, one has the following inequalities

$$\mu^{*n} * \delta_x(M^c) = A_\mu^n(\mathbf{1}_{M^c})(x) \leq \frac{\varepsilon}{2B} A_\mu^n(f)(x) \leq \frac{\varepsilon a^n}{2B} f(x) + \frac{\varepsilon}{2} \leq \varepsilon$$

as soon as n is large enough to have $f(x) \leq \frac{B}{a^n}$. $\square$

We will say that the action of (G, μ) on X satisfies the contraction hypothesis if

CH for every compact subset L of X, there exists a proper Borel function $f = f_L : X \to [0, \infty]$ which is uniformly bounded on L, and such that the action of (G, μ) on X satisfies the contraction hypothesis $\mathbf{CH}(f)$.

Hypothesis **CH** above is a variation of the contraction hypothesis of [12, Chap. 15] and [6]. This condition is shown in [12, Chap. 15] to be related to the existence of a finite exponential moment for the first return time in some bounded sets of X. We will not use this fact.

We will say that the action of (G, μ) on X satisfies the recurrence property if

R for any $\varepsilon > 0$, and any compact set $L \subset X$, there exists a compact set $M = M_\varepsilon \subset X$, such that for any x in L and $n \geq 0$, one has $\mu^{*n} * \delta_x(M) \geq 1 - \varepsilon$.

Corollary 2.2. *Assume that the (G, μ)-space X satisfies the contraction hypothesis **CH**, then it satisfies the recurrence property **R**.*

Proof. Let L be a compact subset of X and $f = f_L$ be the proper Borel function given by the hypothesis **CH**. Choose as a first trial the compact M given by Lemma 2.1 so that for n large enough, for any x in L, one has $\mu^{*n} * \delta_x(M^c) \leq \varepsilon$. Then, choose a compact set K of G such that, for the finitely many remaining values of n, $\mu^{*n}(K^c) \leq \varepsilon$ and replace M by $M \cup KL$. $\square$

3. The Mother Inequality

> In order to construct the function f of section 2 for the space $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$, we will need the following Mother Inequality.

Let $E = \mathbb{R}^d$ be the euclidean space with canonical basis $e_1, \ldots, e_d$. We denote by $\|.\|$ the euclidean norm on E and on its exterior algebra Λ^*E for which the family $e_{i_1} \wedge \ldots \wedge e_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq d$ form an orthonormal basis. An element u of Λ^*E is said to be *monomial* if it can be written as $u = u_1 \wedge \cdots \wedge u_r$ for some $u_1, \ldots, u_r$ in E.

Let $H \subset \mathrm{GL}(E)$ be a reductive algebraic subgroup, $A \subset H$ be a maximal split subtorus of H, $\Sigma = \Sigma(A, H)$ be the set of (restricted) roots, i.e. Σ is the set of non-zero weights of A in the Lie algebra $\mathfrak{h}$ of H. We choose a system $\Sigma^+ \subset \Sigma$ of positive roots. Let P be the set of algebraic characters of A. We endow P with the partial order given, for λ, μ in P, by

$$\lambda \leq \mu \Longleftrightarrow \mu - \lambda \text{ is a sum of positive roots.}$$

For any real algebraic irreducible representation of H, the set of weights of A in this representation has a unique maximal element λ called the (restricted) highest weight of the representation. Let P^+ be the set of all these highest weights. For any algebraic representation of H in a real finite dimensional vector space V, for λ in P^+ we denote by V^λ the sum of all the irreducible subrepresentations of V whose highest weight is equal to λ and $q_\lambda : V \to V$ the H-equivariant projection on V^λ. For instance, q_0 is the H-equivariant projection onto the subspace $V^{H^{nc}}$ of fixed points of H^{nc} in V.

Proposition 3.1. *Let $H \subset \mathrm{GL}(E)$ be a reductive algebraic subgroup. Then there exists $C_1 \geq 1$ such that, for any monomials u, v, w in Λ^*E, one has the inequality* **MI***:*

$$\|q_\lambda(u)\| \, \|q_\mu(u \wedge v \wedge w)\| \leq C_1 \max_{\substack{\nu,\rho \in P^+ \\ \nu+\rho \geq \lambda+\mu}} \|q_\nu(u \wedge v)\| \, \|q_\rho(u \wedge w)\|. \tag{3.1}$$

Remark 3.2. *Inequality* **MI** *will be a substitute for the following simpler inequality used in* [6]*: for any monomials u, v, w in Λ^*E, one* *has*

$$\|u\| \, \|u \wedge v \wedge w\| \leq \|u \wedge v\| \, \|u \wedge w\|. \tag{3.2}$$

To prove Proposition 3.1, we will need the following lemma.

Lemma 3.3. *Let H be a real algebraic reductive group, V be a real algebraic representation of H. For λ, μ in P^+, the kernel of the map $q_{\lambda+\mu} : V^\lambda \otimes V^\mu \to V^\lambda \otimes V^\mu$ does not contain non-zero pure tensors.*

Proof of Lemma 3.3. Let $x \in V^\lambda$ and $y \in V^\mu$ be two non-zero vectors such that $q_{\lambda+\mu}(x \otimes y) = 0$. We decompose x as a sum $x = \sum_{\alpha \in P} x_\alpha$ of weight vectors x_α of weight α. Similarly, we write $y = \sum_{\beta \in P} y_\beta$. Since any irreducible subrepresentation of V^λ, resp. V^μ, contains non-zero weight vectors of weight λ, resp. μ, and since we can replace both x and y by their images $h(x)$ and $h(y)$ by an element h of H, we may assume that both x_λ and y_μ are non-zero.

The latter vectors belong to the highest weight spaces $(V^\lambda)_\lambda$ and $(V^\mu)_\mu$ of the representations V^λ and V^μ. Since the component of weight $\lambda + \mu$ of $q_{\lambda+\mu}(x \otimes y)$ is equal to $x_\lambda \otimes y_\mu$, one has $x_\lambda \otimes y_\mu = 0$, hence a contradiction. □

Corollary 3.4. *Let H be a real algebraic reductive group, V be a real algebraic representation of H. Choose a euclidean norm on V. Then there exists $D_1 \geq 1$ such that, for λ, μ in P^+, for any x, y in V, one has*

$$\|q_\lambda(x)\| \, \|q_\mu(y)\| \leq D_1 \, \|q_{\lambda+\mu}(x \otimes y)\| \, .$$

Proof of Corollary 3.4. We may assume that the euclidean norm on $V \otimes V$ is chosen in a compatible way so that $\|x \otimes y\| = \|x\| \, \|y\|$ for any x, y in V and that the projectors q_λ, $\lambda \in P^+$, are orthogonal. We first note that the projector $q_{\lambda+\mu}$ preserves the decomposition $V \otimes V = \oplus_{\nu,\rho \in P^+} V^\nu \otimes V^\rho$. Hence there exists $D_1' \geq 1$ such that, for every x and y in V,

$$\|q_{\lambda+\mu}(q_\lambda(x) \otimes q_\mu(y))\| \leq D_1' \, \|q_{\lambda+\mu}(x \otimes y)\|. \tag{3.3}$$

Let $C_{\lambda,\mu} := \{x \otimes y \mid x \in V^\lambda, \ y \in V^\mu\}$ be the cone of pure tensors in $V^\lambda \otimes V^\mu$. According to Lemma 3.3, the intersection $C_{\lambda,\mu} \cap \mathrm{Ker}(q_{\lambda+\mu})$ is zero. Hence there exists $D_1'' \geq 1$ such that, for every x' in V^λ and y' in V^μ, one has

$$\|x'\| \, \|y'\| = \|x' \otimes y'\| \quad \leq \quad D_1'' \, \|q_{\lambda+\mu}(x' \otimes y')\|. \tag{3.4}$$

Our claim follows from inequalities (3.3) and (3.4). □

Proof of Proposition 3.1. Let r, s, t be non negative integers. According to Corollary 3.4, there exists $D_1 \geq 1$, such that, for any $u \in \Lambda^r E$, $v \in \Lambda^s E$ and $w \in \Lambda^t E$, one has

$$(3.5)\quad \|q_\lambda(u)\| \, \|q_\mu(u \wedge v \wedge w)\| \leq D_1 \, \|q_{\lambda+\mu}(u \otimes (u \wedge v \wedge w))\|.$$

We now introduce the linear map

$$\Psi_r : \Lambda^{r+s}E \otimes \Lambda^{r+t}E \to \Lambda^r E \otimes \Lambda^{r+s+t}E,$$

such that, for any monomial $x = x_1 \wedge \cdots \wedge x_{r+s}$ in $\Lambda^{r+s}E$ and any y in $\Lambda^{r+t}E$,

$$\Psi_r(x \otimes y) = \sum_{|I|=r} \varepsilon_I \, x_I \otimes (x_{I^c} \wedge y),$$

where the sum is taken over all the subsets $I \subset \{1, \ldots, r+s\}$ of size r. Let us explain each term of this sum:
– the element x_I is the exterior product $x_I = x_{i_1} \wedge \cdots \wedge x_{i_r}$, when one writes $I = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$.
– the element x_{I^c} is the exterior product $x_{I^c} = x_{i_{r+1}} \wedge \cdots \wedge x_{i_{r+s}}$, when one writes $I^c = \{i_{r+1}, \ldots, i_{r+s}\}$ with $i_{r+1} < \cdots < i_{r+s}$.
– the sign ε_I is the signature of the permutation of $\{1, \ldots, r+s\}$ sending k to i_k, $1 \leq k \leq r+s$.

The map Ψ_r is $\mathrm{GL}(E)$-equivariant. For any monomials u, v, w of degrees, respectively r, s and t, one has

$$(3.6)\qquad \Psi_r((u \wedge v) \otimes (u \wedge w)) = u \otimes (v \wedge u \wedge w).$$

Since the linear map $q_{\lambda+\mu} \circ \Psi_r$ is H-equivariant, and since all the weights of the tensor product $(\Lambda^* E)^\nu \otimes (\Lambda^* E)^\rho$ are smaller than $\nu + \rho$, the map $q_{\lambda+\mu} \circ \Psi_r$ is zero on $(\Lambda^* E)^\nu \otimes (\Lambda^* E)^\rho$ except if $\nu + \rho \geq \lambda + \mu$. Hence one has, with $D_1''' := \|q_{\lambda+\mu} \circ \Psi_r\|$,

$$(3.7)\quad \|q_{\lambda+\mu}(u \otimes (u \wedge v \wedge w))\| \leq D_1''' \max_{\substack{\nu,\rho \in P^+ \\ \nu+\rho \geq \lambda+\mu}} \|q_\nu(u \wedge v)\| \, \|q_\rho(u \wedge w)\|.$$

Our claim follows from inequalities (3.5) and (3.7). $\square$

To end this section, we state two elegant corollaries of Proposition 3.1 for which we have no simpler proof. We will not use them in this paper, but we think they might help the reader to understand the meaning of the Mother Inequality.

Let $(\Lambda^* E)^H$ be the set of fixed points of H in $\Lambda^* E$ and $q : \Lambda^* E \to \Lambda^* E$ be the unique H-equivariant projector whose kernel is $(\Lambda^* E)^H$.

Corollary 3.5. *Assume $H \subset \mathrm{GL}(E)$ is a connected semisimple subgroup with no compact factor. Then there exists $C_1' \geq 1$ such that, for any monomials u, v, w in $\Lambda^* E$, one has the inequality :*

$$\|q(u)\|\,\|q(u\wedge v\wedge w)\| \leq C_1'(\|q(u\wedge v)\|\,\|u\wedge w\| + \|u\wedge v\|\,\|q(u\wedge w)\|).$$

This corollary can not be extended to reductive groups. Indeed, if, for instance $H = \{\mathrm{diag}(t, t^{-1}, t^{-1}) \mid t \in \mathbb{R}^\times\}$, for $u = e_1$, $v = e_2$ and $w = e_3$, the monomials $u \wedge v$ and $u \wedge w$ are H-invariant but neither u nor $u \wedge v \wedge w$ is H-invariant.

Proof of Corollary 3.5. This inequality follows from Proposition 3.1 and the following two facts:
– since H has no compact factor, the projector q is the sum of all the projectors q_λ with $\lambda \neq 0$.
– since H is semisimple the sum $\lambda + \mu$ of two non-zero elements of P^+ is a non-zero element of P^+. □

Let us now state a second corollary to Proposition 3.1. Let $E := E_1 \oplus \cdots \oplus E_a$ be an orthogonal decomposition. For any multiindex $i = (i_1, \ldots, i_a) \in \mathbb{N}^a$, we denote by $q_i : \Lambda^* E \to \Lambda^* E$ the projector on the component $\Lambda^{i_1} E_1 \otimes \cdots \otimes \Lambda^{i_a} E_a$. We endow $\mathbb{N}^a$ with the partial order given by $i \leq j \iff j - i \in \mathbb{N}^a$.

Corollary 3.6. *There exists $C_1'' \geq 1$ such that, for any monomials u, v, w in $\Lambda^* E$, one has the inequality:*

$$\|q_i(u)\|\,\|q_j(u\wedge v\wedge w)\| \leq C_1'' \max_{\substack{k,\ell\,\in\mathbb{N}^a \\ k+\ell\,=\,i+j \\ \min(i,j)\,\leq\, k\,\leq\,\max(i,j)}} \|q_k(u\wedge v)\|\,\|q_\ell(u\wedge w)\|.$$

In this formula the element $m = \min(i,j) \in \mathbb{N}^a$ is the minimum for the partial order on $\mathbb{N}^a$, i.e. for $b = 1, \ldots, a$, its b^{th}-component is $m_b = \min(i_b, j_b)$. And similarly for the maximum.

Proof of Corollary 3.6. This inequality follows from Proposition 3.1 applied to the reductive group $H = \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_a)$. Indeed, let us assume to simplify $a = 2$. We set $d_1 = \dim E_1$. Let $e_1 = E_{1,1}, \ldots, e_d := E_{d,d}$ be the standard basis of the Lie algebra $\mathfrak{a}$ of diagonal matrices and let $e_1^*, \ldots, e_d^*$ be the dual basis.

We choose the positive roots of H to be the elements $e_p^* - e_q^*$ with either $1 \leq p < q \leq d_1$ or $d_1 < p < q \leq d$. The representation of H in $\Lambda^{i_1} E_1 \otimes \Lambda^{i_2} E_2$ is irreducible with highest weight

$$\lambda_i = e_1^* + \cdots + e_{i_1}^* + e_{d_1+1}^* + \cdots + e_{d_1+i_2}^*.$$

One has the equivalence, for non-zero projectors q_i, q_j, q_k, q_ℓ,

$$\lambda_k + \lambda_\ell \geq \lambda_i + \lambda_j \Longleftrightarrow (\ k + \ell = i + j \text{ and } \min(i,j) \leq k \leq \max(i,j)\ ).$$

□

4. The contraction hypothesis in vector spaces

We construct in this section functions on $\Lambda^*\mathbb{R}^d$ satisfying both a strong convexity property with respect to the exterior product and a strong contraction property with respect to averaging operators of probability measures on $\mathrm{SL}(d,\mathbb{R})$.

Let $E = \mathbb{R}^d$ and $H \subset \mathrm{GL}(E)$ be an algebraic subgroup with H^{nc} semisimple. We keep the notations A, P^+, q_λ,... from section 3. We choose a norm on E which is invariant by some maximal compact subgroup H^c of H. In order to construct the function φ_{ε_0}, we need to introduce two "exponents".

The first one $i \mapsto \delta_i$ is defined for any integer i with $0 \leq i \leq d$. It satisfies $\delta_0 = \delta_d = 0$ and has the following concavity property : for every integers r, s, t with $s > 0$ and $t > 0$,

$$\delta_{r+s} + \delta_{r+t} \geq \delta_r + \delta_{r+s+t} + 1. \tag{4.1}$$

For instance, one can choose to set $\delta_i := (d-i)i$.

The second one $\lambda \mapsto \delta_\lambda$ is defined for any highest weight $\lambda \in P^+$. It satisfies $\delta_\lambda = 0 \Longleftrightarrow \lambda = 0$ and, for any λ, μ in P^+,

$$\lambda \leq \mu \Longrightarrow \delta_\lambda \leq \delta_\mu \tag{4.2}$$

and it is invariant under the natural action of H/H^{nc} on P^+. For instance, one can choose to set $\delta_\lambda = \lambda(H_0)$ where H_0 is an element in the positive Weyl chamber of the Lie algebra of A whose image in all the non-zero simple ideals of $\mathfrak{h}$ is non-zero and which is H/H^{nc}-invariant.

Let $\varepsilon_0 > 0$. For v in $\Lambda^i E$, with $0 < i < d$, we define $\varphi_{\varepsilon_0}(v)$ to be the supremum of the set of real numbers $R \geq 0$ such that, for any $\lambda \in P^+$, one has

$$\|q_\lambda(v)\| < \varepsilon_0^{\delta_i} R^{-\delta_\lambda}.$$

More precisely

$$\begin{aligned} \varphi_{\varepsilon_0}(v) &= \min_{\lambda \in P^+ \smallsetminus 0} \varepsilon_0^{\frac{\delta_i}{\delta_\lambda}} \|q_\lambda(v)\|^{\frac{-1}{\delta_\lambda}} && \text{if } \|q_0(v)\| < \varepsilon_0^{\delta_i} \\ &= 0 && \text{otherwise.} \end{aligned} \tag{4.3}$$

For v in $\Lambda^i E$, with $i = 0$ or $i = d$, we do not define $\varphi_{\varepsilon_0}(v)$.

Remark 4.1. *For v in $\Lambda^i E$, with $0 < i < d$, one has the equivalence :*

$$\varphi_{\varepsilon_0}(v) = \infty \iff v \text{ is } H^{nc}\text{-invariant and } \|v\| < \varepsilon_0^{\delta_i}.$$

We will use the Mother Inequality through the following technical lemma 4.2 which states a convexity property for φ_{ε_0}.

We will need a few constants: the constant C_1 from Proposition 3.1, the constant $\kappa_1 = (\max_\lambda \delta_\lambda)^{-1}$, where the max is taken over all the highest weight λ of the irreducible subrepresentations of $\Lambda^* E$, and the constant $b_1 := \sup_{\|v\|\geq 1} \varphi_{\varepsilon_0}(v) < \infty$.

Lemma 4.2. *For any $0 < \varepsilon_0 < C_1^{-1}$, for any monomials u, v, w in $\Lambda^* E$ with respective degrees r, s, t with $r \geq 0$, $s > 0$ and $t > 0$ and such that $\varphi_{\varepsilon_0}(u \wedge v) \geq 1$ and $\varphi_{\varepsilon_0}(u \wedge w) \geq 1$, one has :*
i) If $r > 0$ and $r + s + t < d$, then

$$\min(\varphi_{\varepsilon_0}(u\wedge v), \varphi_{\varepsilon_0}(u\wedge w)) \leq (C_1\varepsilon_0)^{\frac{\kappa_1}{2}} \max(\varphi_{\varepsilon_0}(u), \varphi_{\varepsilon_0}(u\wedge v\wedge w)). \tag{4.4}$$

ii) If $r = 0$ and $r + s + t < d$, then

$$\min(\varphi_{\varepsilon_0}(v), \varphi_{\varepsilon_0}(w)) \leq (C_1\varepsilon_0)^{\frac{\kappa_1}{2}}\ \varphi_{\varepsilon_0}(v \wedge w). \tag{4.5}$$

iii) If $r > 0$, $r + s + t = d$, and $\|u \wedge v \wedge w\| \geq 1$, then

$$\min(\varphi_{\varepsilon_0}(u \wedge v), \varphi_{\varepsilon_0}(u \wedge w)) \leq (C_1\varepsilon_0)^{\frac{\kappa_1}{2}}\ \varphi_{\varepsilon_0}(u). \tag{4.6}$$

iv) If $r = 0$, $r + s + t = d$, and $\|v \wedge w\| \geq 1$, then

$$\min(\varphi_{\varepsilon_0}(v), \varphi_{\varepsilon_0}(w)) \leq b_1. \tag{4.7}$$

Proof of Lemma 4.2. The left-hand side of these inequalities is the supremum of the set of real numbers $R \geq 1$ such that for any ν, ρ in P^+, one has

$$\|q_\nu(u \wedge v)\| < \varepsilon_0^{\delta_{r+s}} R^{-\delta_\nu} \text{ and } \|q_\rho(u \wedge w)\| < \varepsilon_0^{\delta_{r+t}} R^{-\delta_\rho}. \tag{4.8}$$

We fix such a R, we set $S := (C_1\varepsilon_0)^{-\frac{\kappa_1}{2}} R$ and we distinguish the four cases :

i) If $r > 0$ and $r + s + t < d$.
We want to check that

$$\text{either } \|q_\lambda(u)\| < \varepsilon_0^{\delta_r} S^{-\delta_\lambda} \text{ for any } \lambda \text{ in } P^+ \tag{4.9}$$

$$\text{or } \|q_\mu(u \wedge v \wedge w)\| < \varepsilon_0^{\delta_{r+s+t}} S^{-\delta_\mu} \text{ for any } \mu \text{ in } P^+. \tag{4.10}$$

To this aim we compute, for every λ, μ in P^+,

$$\begin{aligned}
\|q_\lambda(u)\|\,\|q_\mu(u\wedge v\wedge w)\| &\leq C_1 \max_{\substack{\nu,\rho\in P^+\\ \nu+\rho\geq\lambda+\mu}} \|q_\nu(u\wedge v)\|\,\|q_\rho(u\wedge w)\| && \text{by (3.1)}\\
&< C_1 \max_{\substack{\nu,\rho\in P^+\\ \nu+\rho\geq\lambda+\mu}} \varepsilon_0^{\delta_{r+s}}\varepsilon_0^{\delta_{r+t}} R^{-\delta_\nu} R^{-\delta_\rho} && \text{by (4.8)}\\
&\leq C_1\, \varepsilon_0\, \varepsilon_0^{\delta_r} \varepsilon_0^{\delta_{r+s+t}} R^{-\delta_\lambda} R^{-\delta_\mu} && \text{by (4.1) and (4.2)}\\
&\leq \varepsilon_0^{\delta_r} S^{-\delta_\lambda}\ \varepsilon_0^{\delta_{r+s+t}} S^{-\delta_\mu}.
\end{aligned}$$

This proves that either (4.9) or (4.10) is true and ends the proof of (4.4).

ii) If $r=0$ and $r+s+t<d$.
By the same computation with $u=1$ one gets, for any μ in P^+,

$$\|q_\mu(v\wedge w)\| \ < \ \varepsilon_0^{\delta_{r+s+t}} S^{-\delta_\mu}.$$

This proves that (4.10) is true and ends the proof of (4.5).

iii) If $r>0$, $r+s+t=d$ and $\|u\wedge v\wedge w\|\geq 1$.
The same computation proves that, for any λ in P^+, one has

$$\|q_\lambda(u)\| \ < \ \varepsilon_0^{\delta_r} S^{-\delta_\lambda}.$$

This proves that (4.9) is true and ends the proof of (4.6).

iv) If $r=0$, $r+s+t=d$ and $\|v\wedge w\|\geq 1$.
One has either $\|v\|\geq 1$ or $\|w\|\geq 1$, hence either $\varphi_{\varepsilon_0}(v)\leq b_1$ or $\varphi_{\varepsilon_0}(w)\leq b_1$. $\square$

Let μ be a Borel probability measure on H with finite exponential moments and whose support spans a Zariski dense subgroup of H. Here are the functions which are contracted by averaging operators associated to random walks.

Lemma 4.3. *There exists $\delta_0>0$ such that, for every δ with $0<\delta<\delta_0$, for every $a_0>0$, there exists $n\geq 1$ such that, on every space $\Lambda^i E$ with $0<i<d$, one has*

$$A_\mu^n \varphi_{\varepsilon_0}^\delta \leq a_0 \varphi_{\varepsilon_0}^\delta \quad \text{for any } \varepsilon_0>0. \tag{4.11}$$

Proof of Lemma 4.3. Since the norm is H^c-invariant, for any h in H and v in $\Lambda^i E$, one has $\|q_0(hv)\| = \|q_0(v)\|$.

When $\|q_0(v)\|\geq \varepsilon_0^{\delta_i}$, one has $(A_\mu^n\varphi_{\varepsilon_0}^\delta)(v) = \varphi_{\varepsilon_0}^\delta(v) = 0$.

When $\|q_0(v)\| < \varepsilon_0^{\delta_i}$, one has $\varphi_{\varepsilon_0}(v) = \min_{\lambda \in P^+ \smallsetminus 0} \varepsilon_0^{\frac{\delta_i}{\delta_\lambda}} \|q_\lambda(v)\|^{\frac{-1}{\delta_\lambda}}$. Hence, since the mean of the minimum of a finite family of functions is bounded by the minimum of the family of means, our claim follows from the following lemma. $\square$

Lemma 4.4. ([6, Lemma 4.2]) *Let V be a real algebraic representation of H such that $V^{H^{nc}} = \{0\}$. Let φ be the function $\varphi : V \smallsetminus 0 \to \mathbb{R}^*; v \mapsto \|v\|^{-1}$. Then there exists $\delta_0 > 0$ such that, for every δ with $0 < \delta < \delta_0$, for every $a_0 > 0$, there exists $n_0 \geq 1$ such that, for $n \geq n_0$,*

$$A_\mu^n(\varphi^\delta) \leq a_0 \varphi^\delta \tag{4.12}$$

Proof of Lemma 4.4. This is a variation of [6, Lemma 4.2]. One uses an asymptotic expansion of order 2 of $e^{-\delta \log(\|hv\|/\|v\|)}$ and Furstenberg and Kesten's theorem on the positivity of the first Lyapunov exponent of the image of μ in $\mathrm{GL}(V)$ (see [4] for a proof at this level of generality) to find $a_0 < 1$ and $n \geq 1$ and get, uniformly for any non-zero v in V, $(A_\mu^n \varphi^\delta)(v) \leq a_0 \varphi^\delta(v)$. $\square$

5. The contraction hypothesis in the space of lattices

In this section, we assume $G = \mathrm{SL}(d, \mathbb{R})$, $\Lambda = \mathrm{SL}(d, \mathbb{Z})$ so that $X = G/\Lambda$ is the space of covolume 1 lattices in $\mathbb{R}^d$, and we construct proper Borel functions on X which are contracted by the random walk.

Given a Borel probability measure μ on G, we let H denote the Zariski closure of the group generated by the support of μ, and by H^{nc} the non compact part of H.

Proposition 5.1. *Let $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$. If the group H^{nc} is semisimple, the (G, μ)-space X satisfies the contraction hypothesis* **CH**.

For x in X, a non-zero monomial v of $\Lambda^i E$ is said to be *x-integral*, if either $i > 0$ and one can write v as $v_1 \wedge \cdots \wedge v_i$ where all the $v_1, \ldots, v_i$ belong to the lattice x, or if $i = 0$ and v belongs to $\mathbb{Z}$. It is then said to be *primitive* if it is not an integer multiple of any other x-integral monomial. For $\varepsilon_0 > 0$, we define $f_{\varepsilon_0} : X \to [0, \infty]$ to be the function given by, for any x in X

$$f_{\varepsilon_0}(x) = \max \varphi_{\varepsilon_0}(v) \tag{5.1}$$

where the max is taken over all the non-zero x-integral monomials $v \in \Lambda^i E$ for some i with $0 < i < d$, and where the functions $\varphi_{\varepsilon_0} : \Lambda^i E \to [0, \infty]$ are defined by (4.3). From Remark 4.1 and from Mahler's compactness criterion we get :

Remark 5.2. *The functions f_{ε_0} are lower semi-continuous, proper, H^c-invariant and, for x in X, one has the equivalence :*

$$f_{\varepsilon_0}(x) = \infty \iff \begin{array}{l}\text{there exists a non-zero } H^{nc}\text{-invariant} \\ \text{and } x\text{-integral monomial } v \in \Lambda^i E \text{ with} \\ 0 < i < d \text{ and } \|v\| < \varepsilon_0^{\delta_i}.\end{array}$$

Proposition 5.3. *Let $H \subset \mathrm{SL}(d, \mathbb{R})$ be an algebraic subgroup with H^{nc} semisimple and μ be Borel probability measure on H whose support spans a Zariski dense subgroup of H and which admits finite exponential moments. For $\delta > 0$ and $\varepsilon_0 > 0$ small enough, there exist $n \geq 1$, $a < 1$ and $b > 0$ such that,*

$$A_\mu^n f_{\varepsilon_0}^\delta \leq a f_{\varepsilon_0}^\delta + b. \tag{5.2}$$

Proof. For x in X, we want an upper bound of the integral $(A_\mu^n f_{\varepsilon_0}^\delta)(x)$. We may assume $f_{\varepsilon_0}^\delta(x) < \infty$.

According to Lemma 4.3, for δ small enough, there exists $a_0 < \frac{1}{d}$ and $n \geq 1$, such that,

$$A_\mu^n \varphi_{\varepsilon_0}^\delta \leq a_0 \varphi_{\varepsilon_0}^\delta$$

Let κ_1 be as in Lemma 4.2. Since μ^{*n} also has finite exponential moments, one can assume δ small enough to have

$$\int_H \|h^{-1}\|^{\kappa_1 \delta d} \, \mathrm{d}\mu^{*n}(h) < \infty.$$

and one can decompose μ^{*n} as the sum of two positive measures $\mu^{*n} = \mu_1 + \mu_2$ with μ_1 compactly supported and with

$$\int_H \|h^{-1}\|^{\kappa_1 \delta d} \, \mathrm{d}\mu_2(h) \leq \tfrac{1}{2}(1 - a_0 d). \tag{5.3}$$

Since $f_{\varepsilon_0}(hx) \leq \|h^{-1}\|^{\kappa_1 d} f_{\varepsilon_0}(x)$, for any h in H and x in X, one has

$$A_{\mu_2} f_{\varepsilon_0}^\delta \leq \tfrac{1 - a_0 d}{2} f_{\varepsilon_0}^\delta. \tag{5.4}$$

We set $E = \mathbb{R}^d$ and $c_0 = \sup\{\max(\|h\|, \|h^{-1}\|)^d \mid h \in \mathrm{supp}(\mu_1)\}$. Thus, for any x in X and for any non-zero x-integral monomial $v \in \Lambda^* E$, one has, for μ_1-almost every h,

$$c_0^{-\kappa_1} \varphi_{\varepsilon_0}(v) \leq \varphi_{\varepsilon_0}(hv) \leq c_0^{\kappa_1} \varphi_{\varepsilon_0}(v). \tag{5.5}$$

We introduce the finite set Ψ of primitive x-integral and monomial elements v of $\Lambda^* E$ with degree in $(0, d)$ such that

$$\varphi_{\varepsilon_0}(v) \geq c_0^{-2\kappa_1} f_{\varepsilon_0}(x). \tag{5.6}$$

We assume ε_0 is small enough to have

$$c_0^4 C_1 \varepsilon_0 < 1. \tag{5.7}$$

The proof then splits in two cases.

1st case: $f_{\varepsilon_0}(x) \leq \max(b_1, c_0^{2\kappa_1})$.
Then, by (5.5), for μ_1-almost every h, one has $f_{\varepsilon_0}(hx) \leq c_0^{\kappa_1} f_{\varepsilon_0}(x)$ and

$$(A_{\mu_1} f_{\varepsilon_0}^{\delta})(x) \leq b. \tag{5.8}$$

with $b = (c_0^{\kappa_1} \max(b_1, c_0^{2\kappa_1}))^{\delta}$.

2nd case: $f_{\varepsilon_0}(x) > \max(b_1, c_0^{2\kappa_1})$.
We claim that in this case
(5.9)
Ψ *contains at most one element up to sign change in each degree* i.

If not, assume for a while that, for some $0 < i < d$, the intersection $\Psi \cap \Lambda^i E$ contains two non-colinear elements v_0 and w_0. By (5.6), one has $\varphi_{\varepsilon_0}(v_0) \geq 1$ and $\varphi_{\varepsilon_0}(w_0) \geq 1$. Besides, since v_0 and w_0 are x-integral, one can write v_0 as $u \wedge v$ and w_0 as $u \wedge w$ where u, v, w are x-integral monomials, v and w have degree $j > 0$ and $u \wedge v \wedge w \neq 0$. The element $u \wedge v \wedge w$ is then a x-integral monomial with degree $i+j$. We distinguish four cases.

i) If $j < i$ and $j < d - i$. One has

$$\begin{aligned} f_{\varepsilon_0}(x) &\leq c_0^{2\kappa_1} \min(\varphi_{\varepsilon_0}(u \wedge v), \varphi_{\varepsilon_0}(u \wedge w)) && \text{by (5.6)} \\ &\leq (c_0^4 C_1 \varepsilon_0)^{\frac{\kappa_1}{2}} \max(\varphi_{\varepsilon_0}(u), \varphi_{\varepsilon_0}(u \wedge v \wedge w)) && \text{by (4.4)}, \end{aligned}$$

hence

$$f_{\varepsilon_0}(x) \leq (c_0^4 C_1 \varepsilon_0)^{\frac{\kappa_1}{2}} f_{\varepsilon_0}(x) \tag{5.10}$$

which contradicts inequality (5.7).

ii) If $j = i < d - i$. In this case $u = 1$. The same computation, using Lemma 4.2.ii), also gives (5.10) which still contradicts (5.7).

iii) If $j = d - i < i$. In this case $\|u \wedge v \wedge w\|$ is an integer. The same computation, using Lemma 4.2.iii), also gives (5.10) which still contradicts (5.7).

iv) If $j = i = d - i$. The same computation, using Lemma 4.2.iv), gives

$$f_{\varepsilon_0}(x) \leq b_1 \tag{5.11}$$

which contradicts our assumption.

This ends the proof of claim (5.9)

Now, by (5.5), for every non-zero x-integral monomial v in $\Lambda^* E$ with degree in $(0, d)$, and μ_1-almost every h, one has

$$\varphi_{\varepsilon_0}(hv) \leq \max_{w \in \Psi} \varphi_{\varepsilon_0}(hw)$$

and thus

$$(A_{\mu_1} f^\delta_{\varepsilon_0})(x) \;\leq\; \sum_{w\in\Psi} \int_G \varphi^\delta_{\varepsilon_0}(hw)\, \mathrm{d}\mu_1(h) \leq a_0 \sum_{w\in\Psi} \varphi^\delta_{\varepsilon_0}(w),$$

the second inequality following from Lemma 4.3. Hence, using (5.9), one has

$$(A_{\mu_1} f^\delta_{\varepsilon_0})(x) \;\leq\; a_0 d\, f^\delta_{\varepsilon_0}(x). \tag{5.12}$$

Finally, one gets (5.2) with $a := \frac{1+a_0 d}{2}$ by combining inequalities (5.8) and (5.12) with (5.4). □

Proof of Proposition 5.1. Let L be a compact subset of X. By Remark 5.2 and Mahler's compactness criterion, there exists $\varepsilon_0 > 0$ such that the function f_{ε_0} is bounded on L. As μ has finite exponential moments, so is the function $A^k_\mu f^\delta_{\varepsilon_0}$ for any nonnegative integer k, provided $\delta > 0$ is small enough. Now, by Proposition 5.3, one can suppose there exists $n \geq 1$, $0 < a < 1$ and $b > 0$ with $A^n_\mu f^\delta_{\varepsilon_0} \leq a f^\delta_{\varepsilon_0} + b$. By setting $f = \sum_{k=0}^{n-1} a^{1-\frac{k+1}{n}} A^k_\mu f^\delta_{\varepsilon_0}$ we get $A_\mu f \leq a^{\frac{1}{n}} f + b$, whence the result. □

Proof of Theorem 1.1 for $G = \mathrm{SL}(d,\mathbb{R})$ and $\Lambda = \mathrm{SL}(d,\mathbb{Z})$. This follows from Corollary 2.2 and Proposition 5.1. □

Proof of Theorem 1.4 for $G = \mathrm{SL}(d,\mathbb{R})$ and $\Lambda = \mathrm{SL}(d,\mathbb{Z})$. According to Lemma 2.1 and Proposition 5.3, is suffices to check that, if $f_{\varepsilon_0}(gx_0) = \infty$, then $g^{-1} H^{nc}_\mu g$ is contained in some Λ-rational parabolic subgroup of G. Here, the Λ-rational parabolic subgroups of G are the stabilizers of the vector subspaces of $\mathbb{R}^d$ which are defined over $\mathbb{Q}$. Hence this statement follows from Remark 5.2. □

6. Reduction steps

We explain now how to reduce Theorems 1.1 and 1.4 to the case we dealt with in the previous section.

Let G be a real Lie group, $\mathfrak{r}$ the largest amenable ideal of $\mathfrak{g}$, $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$, $S := \mathrm{Aut}(\mathfrak{s})$ and $R := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{s}})$ be the Kernel in G of the adjoint action in $\mathfrak{s}$. Let $\Lambda \subset G$ be a lattice and $X = G/\Lambda$. According to Auslander projection theorem and Borel density theorem, one has the following lemma (see [15] or [2] for a detailed proof) :

Lemma 6.1. *(i) The intersection $\Lambda \cap R$ is a cocompact lattice in R.*
(ii) The image group $\Lambda_S := \mathrm{Ad}_{\mathfrak{s}}(\Lambda)$ is a lattice in S.

Let $\mu \in \mathcal{P}(G)$ be a Borel probability measure on G with finite exponential moments in $\mathfrak{s}$, $H_\mu \subset S$ be the Zariski closure of the subgroup spanned by the support of μ in G, and

$$(6.1)\quad X' := \left\{ \begin{matrix} x = gx_0 \in X \mid g^{-1}H_\mu^{nc}g \text{ is not contained in any} \\ \text{proper } \Lambda_S\text{-rational parabolic subgroup of } S \end{matrix} \right\}$$

(recall that a parabolic subgroup of S is said to be Λ_S-rational if its unipotent radical intersects Λ_S in a lattice).

The following theorem is a restatement of Theorems 1.1 and 1.4.

Theorem 6.2. *Let G be a real Lie group, Λ be a lattice in G, $X := G/\Lambda$, and μ be a probability measure on G with finite exponential moments in $\mathfrak{s}$ and such that the group H_μ^{nc} is semisimple. Then,*
a) The action of (G,μ) on X satisfies the recurrence property **R**.
b) This action also satisfies the uniform recurrence property $\mathbf{S}(X')$.

Proof of Theorem 6.2.

1st case: G is semisimple and $\Lambda = G_{\mathbb{Z}}$.
More precisely, we assume here that G is a semisimple algebraic subgroup of $\mathrm{SL}(d,\mathbb{R})$ defined over $\mathbb{Q}$ and that Λ is the group $G \cap \mathrm{SL}(d,\mathbb{Z})$. In this case, according to [15, Chap. 1], the space $X = G/\Lambda$ is a closed subset of $X_0 := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. The recurrence property **R** of (G,μ) on X then follows from the recurrence property **R** on X_0.

By Lemma 2.1 and Proposition 5.3, to check the uniform recurrence property $\mathbf{S}(X')$, it suffices to check that for ε_0 small enough, one has the inclusion

$$(6.2)\qquad X' \subset D_{f_{\varepsilon_0}}.$$

To this aim, we first recall a few facts from Geometric Invariant Theory.

Let $\mathbf{V} = \mathbb{C}^D$ and $\mathbf{G} \subset \mathrm{GL}(\mathbf{V})$ be a reductive subgroup. A vector in $\mathbf{V}$ is said to be *stable* if its $\mathbf{G}$-orbit is closed. It is said to be *unstable* if 0 belongs to the Zariski closure of its $\mathbf{G}$-orbit. According to Geometric Invariant Theory in [13], a vector v is unstable if and only if for every $\mathbf{G}$-invariant polynomial F on $\mathbf{V}$, one has $F(v) = F(0)$.

Assume $\mathbf{G}$ is semisimple. According to Kempf in [10, Corol. 3.5], the stabilizer of an unstable vector $v \in \mathbf{V}$ is contained in a proper parabolic subgroup $\mathbf{P} \subsetneq \mathbf{G}$. Moreover, when $\mathbf{G}$ is defined over a subfield k of $\mathbb{C}$ and v belongs to k^D, one can choose the parabolic subgroup $\mathbf{P}$ to be defined over k ([10, Theorem 4.2], see also [16] when k is a number field).

Lemma 6.3. *Let $V = \mathbb{R}^D$ and $G \subset \mathrm{SL}(V)$ be a semisimple subgroup defined over $\mathbb{Q}$. Then there exists ε_0 such that every vector $v = gv_0$ with norm $\|v\| \leq \varepsilon_0$ which belongs to the G-orbit of some integral vector $v_0 \in \mathbb{Z}^D$ is unstable.*

Proof of Lemma 6.3. Let S be the set of G-invariant polynomials $F \in \mathbb{Z}[V]$ such that $F(0) = 0$. The set Z of unstable vectors in V is the set of zeroes of these polynomials. As the ring of G-invariant polynomials in $\mathbb{Z}[V]$ is finitely generated, it is Noetherian and Z is the set of zeroes of some finite subset $S_0 \subset S$. Choose ε_0 small enough to have, for any $v \in V$ and any $F \in S_0$, $\|v\| \leq \varepsilon_0 \Longrightarrow |F(v)| < 1$. If moreover v belongs to the G-orbit of some point $v_0 \in \mathbb{Z}^d$, $F(v) = F(v_0)$ is an integer. Hence $F(v) = 0$ and v is an unstable vector. □

We now can check inclusion (6.2). Indeed, let $x = gx_0$ be a point in X such that $f_{\varepsilon_0}(gx_0) = \infty$. According to Remark 5.2 there exists $0 < i < d$ and a H^{nc}-invariant x-integral monomial $v \in \Lambda^i E$ such that $\|v\| < \varepsilon_0^{\delta_i}$. Since v is x-integral, the vector $v_0 := g^{-1}v \in \Lambda^i E$ is integral. According to Lemma 6.3, if ε_0 is small enough, this vector is unstable. By Kempf's theorem quoted above, the stabilizer of v_0 is contained in some proper parabolic subgroup of G defined over $\mathbb{Q}$, hence the group $g^{-1}H_\mu^{nc}g$ is contained in some proper Λ-rational parabolic subgroup of G. This ends the proof of Theorem 6.2 in the first case.

2nd case: $G = \mathrm{Aut}(\mathfrak{g})$ with $\mathfrak{g}$ simple of real rank 1.
If the group H_μ^{nc} is non trivial, it is not contained in any proper parabolic subgroup of G. Hence our statements follow from Eskin, Margulis theorem in [6]. For the sake of completeness, we sketch a proof.

Let us construct a continuous finite and proper function f on X for which (G, μ) has the contraction property $\mathbf{CH}(f)$. As Λ is a lattice in G, by [9], there exists finitely many Λ-conjugacy classes of maximal unipotent subgroups which intersect Λ in a lattice. Pick representatives $U_1, \ldots, U_r$ of these Λ-conjugacy classes. Again by [9], if a sequence $x_n = g_n x_0$ goes to ∞ in X, after eventually extracting a subsequence, there exists $1 \leq i \leq r$ and a sequence (λ_n) in Λ such that, for any u in U_i, $g_n\lambda_n u\lambda_n^{-1}g_n^{-1}$ goes to e in U_i. Moreover, when n is large, $\lambda_n U_i \lambda_n^{-1}$ is uniquely defined by g_n. Now, let V be a faithful irreducible representation of G. If U is some maximal unipotent subgroup of G and v is some non-zero U-invariant vector in V, by Iwasawa decomposition, for any sequence (g_n) in G, $g_n v$ goes to 0 in V if and only if $g_n u g_n^{-1}$ goes to e in G for any u in U. Thus, if, for any $1 \leq i \leq r$, v_i is some non-zero U_i-invariant vector in V, the set $\Lambda v_1 \cup \ldots \cup \Lambda v_r$ is discrete in V and the function f defined by, for any $x = gx_0$ in X,

$$f(gx_0) := \max_{1\leq i\leq r} \max_{\lambda\in\Lambda} \|g\lambda v_i\|^{-1} \tag{6.3}$$

is continuous and proper. We claim that, for $\delta > 0$ small enough, f^δ satisfies the contraction property $\mathbf{CH}(f^\delta)$ with respect to some convolution power of μ. Indeed, this follows from Lemma 4.4, since the image in $\mathbb{P}(V)$ of the G-orbit Gv_i is compact and does not contain any H_μ^{nc}-invariant element.

3rd case: $G = \mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is semisimple without compact ideal. Replacing Λ by a finite index subgroup, we may write G as a finite product of groups G_i so that X is a finite product of spaces $X_i = G_i/\Lambda_i$ where Λ_i is an irreducible lattice in G_i. It is enough to prove Theorem 6.2 for each factor X_i.

Hence we may assume that Λ is an irreducible lattice of G. Thanks to the second case, we may assume that G has real rank at least two. According to Margulis' arithmeticity theorem in [11], there exist $d \geq 2$, a semisimple subgroup $G' \subset \mathrm{SL}(d, \mathbb{R})$ defined over $\mathbb{Q}$ and a Lie group morphism $\varphi : G' \to G$ with compact kernel, finite index image and such that $\varphi(\Lambda')$ and Λ are commensurable where $\Lambda' := G' \cap \mathrm{SL}(d, \mathbb{Z})$. We can choose φ to be surjective so that we can write $\mu = \varphi_* \mu'$ for some $\mathrm{Ker}(\varphi)$-invariant probability measure μ' on G'.

According to the first case, Theorem 6.2 is true for $X' := G'/\Lambda'$. Since $\mathrm{Ker}(\varphi)$ is a compact normal subgroup of the Zariski closure of $\Gamma_{\mu'}$, the functions f_{ε_0} constructed on X' are $\mathrm{Ker}(\varphi)$-invariant, hence can be seen as functions on $G/\varphi(\Lambda')$. This proves that Theorem 6.2 is also true for $G/\varphi(\Lambda')$.

The validity of the recurrence properties $\mathbf{R}$ and $\mathbf{S}(X')$ on $X = G/\Lambda$ only depends on the commensurability class of Λ. Hence Theorem 6.2 is also true for G/Λ.

4th case: General case.
Let $R := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{s}})$ be the Kernel in G of the adjoint action in $\mathfrak{s}$. Since $\mathfrak{s}$ is semisimple, the group G/R is a finite index subgroup of the group $\mathrm{Aut}(\mathfrak{s})$. According to Lemma 6.1, the intersection $\Lambda \cap R$ is a cocompact lattice in R and the image Λ_S of Λ in $\mathrm{Aut}(\mathfrak{s})$ is a lattice. According to the third case, Theorem 6.2 is true for S/Λ_S. Since the projection $G/\Lambda \to S/\Lambda_S$ is a proper map, Theorem 6.2 is also true for G/Λ. $\square$

7. Products of real and p-adic Lie groups

In this section we extend our results to finite products of real and p-adic Lie groups and more generally to S-adic Lie groups.

> We also check that the functions constructed above which are contracted by the random walk can be chosen to grow exponentially.

Let S be a finite subset of the set of prime numbers including ∞. We denote by $\mathbb{Q}_p$ the field of p-adic numbers and by $\mathbb{Q}_\infty = \mathbb{R}$ the field of real numbers or ∞-adic numbers.

Let G be an S-adic Lie group i.e. G is a locally compact group which contains an open subgroup U isomorphic to a group $(\prod_{p\in S} G_p)/N$ where, for each $p \in S$, G_p is a p-adic Lie group and N is a discrete normal subgroup of this product (see [2]). We denote by $\mathfrak{g}$ the Lie algebra of G i.e. the $\mathbb{Q}$-vector space $\mathfrak{g} := \oplus \mathfrak{g}_p$ which is the direct sum of the Lie algebras $\mathfrak{g}_p$ of G_p.

Let Λ be a lattice of G. For such a group G, we have to replace Lemma 6.1 by the following Lemma 7.1 which is the main Theorem of [2].

Lemma 7.1. *There exists a G-invariant ideal $\mathfrak{r}$ of $\mathfrak{g}$ with the following three properties. We let $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ and R be the kernel of the adjoint action* $\mathrm{Ad}_{\mathfrak{s}} : G \to \mathrm{Aut}(\mathfrak{s})$ *in $\mathfrak{s}$.*
(i) The Lie algebra $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ is semisimple.
(ii) The intersection $\Lambda \cap R$ is a cocompact lattice in R.
(iii) The image $\Lambda_S := \mathrm{Ad}_{\mathfrak{s}}(\Lambda) \simeq \Lambda/(\Lambda \cap R)$ is a lattice in $\mathrm{Aut}(\mathfrak{s})$.

We endow each of the Lie algebras $\mathfrak{s}_p$ with a norm $\|.\|_p$, and, for g in G, we let $\|\mathrm{Ad}_{\mathfrak{s}}(g)\| = \prod_{p\in S} \|\mathrm{Ad}_{\mathfrak{s}_p}(g)\|_p$ be the height of $\|\mathrm{Ad}_{\mathfrak{s}}(g)\|$.

Let $\mu \in \mathcal{P}(G)$ be a Borel probability measure on G. Let $\Gamma = \Gamma_\mu$ be the closed sub-semigroup generated by the support of μ. Let H_p be the Zariski closure of $\mathrm{Ad}_{\mathfrak{s}_p}(\Gamma)$ in $\mathrm{Aut}(\mathfrak{s}_p)$ and $H_\mu := \prod_{p\in S} H_p \subset \mathrm{Aut}(\mathfrak{s})$. Let H_p^{nc} be the smallest normal algebraic subgroup of H_p such that the image of Γ in H_p/H_p^{nc} is relatively compact, and $H_\mu^{nc} := \prod_{p\in S} H_p^{nc}$.

We assume that

$$\begin{array}{l} H_\mu^{nc} \text{ is semisimple and} \\ \mu \text{ has finite exponential moments in } \mathfrak{s}, \end{array} \tag{7.1}$$

i.e. the p-adic Lie groups H_p^{nc}, $p \in S$, are semisimple and

$$\int_G \|\mathrm{Ad}_{\mathfrak{s}}(g)\|^\delta \, \mathrm{d}\mu(g) < \infty \text{ for some } \delta > 0.$$

Since each factor $\|\mathrm{Ad}_{\mathfrak{s}_p}(g)\|_p$ is larger than 1, the latter assumption is equivalent to

$$\int_G N(\mathrm{Ad}_{\mathfrak{s}}(g))^{\delta'} \, \mathrm{d}\mu(g) < \infty \text{ for some } \delta' > 0,$$

where $N(\mathrm{Ad}_{\mathfrak{s}}(g)) := \max_{p \in S} \|\mathrm{Ad}_{\mathfrak{s}_p}(g)\|_p$.

As in the introduction, we note that, when G is a linear group i.e. a subgroup of $\prod_{p \in S} \mathrm{SL}(d, \mathbb{Q}_p)$ for some $d > 1$, assumptions (7.1) are satisfied as soon as, for every p in S, μ has finite exponential moments in $\mathbb{Q}_p^d$ and the Zariski closure of the image of Γ_μ in $\mathrm{SL}(d, \mathbb{Q}_p)$ is semisimple.

A subgroup P of $\mathrm{Aut}(\mathfrak{s})$ is called a *parabolic subgroup* if it is the product of parabolic subgroups P_p of $\mathrm{Aut}(\mathfrak{s}_p)$. The product U of the unipotent radicals U_p of P_p is called the unipotent radical of P. A parabolic subgroup $P \subset \mathrm{Aut}(\mathfrak{s})$ is said to be Λ_S-rational if the lattice Λ_S intersects the unipotent radical of P in a lattice. We can then again define the subset $X' \subset X$ by (6.1).

The following theorem is the extension of Theorem 6.2 to S-adic Lie groups. We keep the notations of Lemma 7.1.

Theorem 7.2. *Let G be an S-adic Lie group, Λ be a lattice in G, $X := G/\Lambda$ and μ be a probability measure on G with finite exponential moments in $\mathfrak{s}$ and such that the group H_μ^{nc} is semisimple. Then,*
a) The action of (G, μ) on X satisfies the recurrence property $\mathbf{R}$.
b) This action also satisfies the uniform recurrence property $\mathbf{S}(X')$.

As in the real case we will deduce this theorem from the existence of functions f satisfying contraction properties. For further use in [3], we will need to control the growth of these functions f.

Let x_0 be the base point of X. For x in X, we set

$$\|x\| := \min\{\|\mathrm{Ad}_{\mathfrak{s}}(g)\| \mid g \in G\ ,\ x = gx_0\}. \tag{7.2}$$

Definition 7.3. *A function $f : X \to [0, \infty]$ is said to grow exponentially, if there exists $c > 0$, $\kappa > 0$ such that, for any $x \in X$,*

$$f(x) \geq \|x\|^\kappa - c. \tag{7.3}$$

As above, it is equivalent to the existence of $c' > 0$, $\kappa' > 0$ such that, for any $x \in X$,

$$f(x) \geq N(x)^{\kappa'} - c'$$

where $N(x) := \min\{N(\mathrm{Ad}_{\mathfrak{s}}(g)) \mid g \in G\ ,\ x = gx_0\}$.

Proof of Theorem 7.2. It is a consequence of Lemma 2.1, Corollary 2.2 and of the following Proposition 7.4. □

Proposition 7.4. *Let G be a S-adic Lie groups, Λ be a lattice in G, $X := G/\Lambda$, and μ be a probability measure on G with finite exponential moments in $\mathfrak{s}$ and such that the group H_μ^{nc} is semisimple.*
a) For every compact set $L \subset X$, there exists a proper Borel function $f : X \to [0, \infty]$ which is uniformly bounded on L, and such that the

action of (G, μ) *on* X *satisfies the contraction hypothesis* $\mathbf{CH}(f)$.
b) Such a function f *can be chosen in such a way that* D_f *contains* X'.
c) Such a function f *can also be chosen to grow exponentially.*

Proof of Proposition 7.4. The proof is the same as for real groups. One just has to change a few definitions. Here are the main modifications.

In section 3, we first note that what we have done there goes through if one replaces the field $\mathbb{R}$ by a p-adic field $\mathbb{Q}_p$. The extension of the inequalities (3.1) we will need are nothing but product of inequalities (3.1) for various real and p-adic fields.

In a more precise way, we introduce, the locally compact algebra $\mathbb{Q}_S := \prod_{s\in S} \mathbb{Q}_p$. We endow this algebra with the height function given for $t = (t_p)_{p\in S}$ by $|t| := \prod_{p\in S} |t_p|_p$ where, for p finite, the absolute value $|.|_p$ on $\mathbb{Q}_p$ is normalized by $|p|_p = \frac{1}{p}$. We set $E_p := \mathbb{Q}_p^d$ and $E = \prod_{p\in S} E_p$, and we endow E with the *height* function given by $\|v\| := \prod_{p\in S} \|v_p\|_p$ and with the *norm* function $N(v) := \max\limits_{p\in S} \|v_p\|_p$, for $v = (v_p)_{p\in S} \in E$ where $\|.\|_p$ is a norm on $\mathbb{Q}_p^d$. The group H becomes $H = \prod_{p\in S} H_p$ where $H_p \subset \mathrm{SL}(d, \mathbb{Q}_p)$ is a reductive subgroup, and $A = \prod_{p\in S} A_p$ where $A_p \subset H_p$ is a maximal $\mathbb{Q}_p$-split subtorus. The group of *characters* of A is nothing but the product of the groups of characters of each A_p

One sets $\Lambda^i E = \prod_{p\in S} \Lambda^i E_p$. An element $u = (u_p)_{p\in S} \in \Lambda^i E$ is said to be *monomial* if, for all $p \in S$, u_p is *monomial* in $\Lambda^i \mathbb{Q}_p^d$.

With these notations, Proposition 3.1 is still true since (3.1) is the product of the analoguous inequalities in each of the factors.

In section 4, we keep the same formula (4.3) for φ_{ε_0}. We choose the norms $\|.\|_p$ to be H^c-invariant where H^c is a compact subgroup of H such that $\Gamma_\mu H^{nc} \subset H^c H^{nc}$.

With these notations, Lemmas 4.2 and 4.3 are still true. Note that the validity of Lemma 4.4 relies on the positivity of the Lyapunov exponent and hence on the fact that, by construction, the image of Γ in each simple factor of H_p^{nc} is not relatively compact (see [4]).

In section 5, the space X becomes G/Λ with $G = \mathrm{SL}(d, \mathbb{Q}_S)$ and $\Lambda = \mathrm{SL}(d, \mathbb{Z}_S)$, where $\mathbb{Z}_S := \mathbb{Z}[\frac{1}{p}, p \in S]$ is embedded diagonally in $\mathbb{Q}_S$ so that X can be identified with the set of discrete $\mathbb{Z}_S$-submodules of covolume 1 in $\mathbb{Q}_S^d$. We keep the same formula (5.1) for f_{ε_0}.

With these notations, Propositions 5.1 and 5.3 are still true. Indeed, in this context, Mahler compactness theorem is still valid: *a subset* $Y \subset X$ *is relatively compact if and only if one has* $\inf\limits_{x\in Y} \min\limits_{v\in x} \|v\| > 0$ or equivalently if $\inf\limits_{x\in Y} \min\limits_{v\in x} N(v) > 0$.

In section 6, there are still four cases:

1st case: In Lemma 6.3, V becomes $\mathbb{Q}_S^D$, v_0 is a vector in $\mathbb{Z}_S^D$ and $G \subset \mathrm{SL}_{\mathbb{Q}_S}(V)$ is the set of $\mathbb{Q}_S$-points of a $\mathbb{Q}$-group $\mathbf{G}$ defined by polynomial equations over $\mathbb{Q}$.

2nd case: In this case G is either a real or a p-adic semisimple Lie group of split rank one, with $p < \infty$. If G is a real Lie group, the proof goes the same; if not, by [17, Prop. 2], Λ is cocompact and there is nothing to prove.

3rd case: Margulis' Arithmeticity theorem still holds for irreducible lattices in product of real and p-adic semisimple groups as soon as the sum of the $\mathbb{Q}_p$-ranks of G_p is at least 2 (see [11]).

4th case: Same proof, replacing Lemma 6.1 by Lemma 7.1.

c) The bound (7.3) follows from the explicit formula for f. $\square$

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