

LATTICES IN S -ADIC LIE GROUPS

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ABSTRACT. We show that any finite volume quotient of an S -adic Lie group admits a fibration with compact fibers over some finite volume quotient of a product of algebraic semisimple p -adic Lie groups.

We also prove a similar decomposition for lattices in a solvable locally compact group.

1. INTRODUCTION

This text can be seen as a short survey of elementary results about lattices Λ in a real Lie group G . However, its main purpose is the extension of some of these results to the context of lattices Λ in more general locally compact groups G . In particular, when G is an S -adic Lie group i.e. a group which is locally the product of real and p -adic Lie groups (see Definition 4.1), we prove in Theorem 6.6 a decomposition theorem of Λ with respect to the adjoint action of G on a suitable semisimple quotient \mathfrak{s} of its Lie algebra \mathfrak{g} .

With the same methods we prove also in Proposition 3.4 a similar decomposition for lattices in a solvable locally compact group. The proof relies on a property of certain minimal actions of \mathbb{R}^d that we call *strong minimality*.

Our motivation to prove the decomposition theorem 6.6 comes from our paper [4] : In this paper we prove, when G is a real Lie group, some recurrence properties of random walks on G/Λ which were conjectured in [9]. Our decomposition theorem 6.6 is then the key ingredient which allows us, in the last section of [4], to extend these recurrence properties from the framework of real Lie groups to the one of S -adic Lie groups. These recurrence properties will be used in [5] to extend the results of [3].

Here is the structure of the paper :

- Section 2 : General facts about minimal actions of abelian groups.
- Section 3 : General facts about lattices in locally compact groups and decomposition of lattices in solvable locally compact groups.
- Section 4 : General facts about S -adic Lie groups and Borel density

theorem.

- Section 5 : A cocompactness criterion for lattices in S -adic Lie groups.
- Section 6 : The decomposition theorem for lattices in S -adic Lie groups.

We thank U. Bader, P.E. Caprace, T. Gelander and S. Mozes for showing us their unpublished example 3.5 related to [1].

2. MINIMAL ACTIONS OF \mathbb{R}^d

In this section we give a criterion for a locally compact space X , equipped with a continuous minimal action of \mathbb{R}^d , to be compact.

Let $A = \mathbb{R}^d$ and X be a locally compact A -space, i.e. a space endowed with a continuous action of A . We denote this action by $(a, x) \mapsto ax$.

An orbit Ax is said to be *strongly dense* if, for every non-empty open convex cone $C \subset A$, the set Cx is dense.

We recall that the A -space X is minimal if all its A -orbits are dense. The A -space X is said to be *strongly minimal* if all its A -orbits are strongly dense.

Proposition 2.1. *Let X be a minimal locally compact \mathbb{R}^d -space.*

- a) *If the action preserves a Borel probability measure μ on X , then there exists at least one strongly dense orbit in X .*
- b) *If the space X is compact, the action is strongly minimal.*
- c) *Conversely, if the action is strongly minimal, X is compact.*

By reading the proof, it is worth keeping in mind the following two examples.

Example 2.2. *There exists a minimal action of \mathbb{R}^d which does not contain any strongly dense orbit.*

Proof. The action by translations of \mathbb{R} on itself, or the product action of \mathbb{R}^2 on $\mathbb{R} \times X'$ where X' is a minimal \mathbb{R} -space. \square

Example 2.3. *There exists a continuous action of \mathbb{R} on a non-compact locally compact space X which is minimal and preserves a Borel probability measure μ on X .*

Proof. Our example is a suspension over an irrational rotation of the circle. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ be the circle, $\alpha \in \mathbb{T}$ an irrational element, $d\theta$ the Lebesgue probability on \mathbb{T} and $f : \mathbb{T} \rightarrow (0, \infty]$ a continuous function such that $\int_{\mathbb{T}} f(\theta) d\theta = \frac{1}{2}$ and $f^{-1}(\infty) = \{0\}$. We set

$$Y := \{(\theta, t) \in \mathbb{T} \times \mathbb{R} \mid -f(\theta) \leq t \leq f(\theta + \alpha)\}$$

and $X = Y/\sim$ the quotient space for the identifications

$$(\theta, -f(\theta)) \sim (\theta - \alpha, f(\theta)), \text{ for all } \theta \neq 0.$$

The space X is locally compact but not compact. There exists a continuous flow $s \mapsto \varphi_s$ on X such that $\varphi_s(\theta, t) = (\theta, t + s)$ as soon as both (θ, t) and $(\theta, t + s)$ are in Y . This flow is minimal and it preserves the probability measure $\mu = d\theta \otimes dt$. This flow is not strongly minimal since the only limit point of the half orbit \mathbb{R}_+x_0 of the point $x_0 := (-\alpha, 0)$ is the point $x_\infty := (-\alpha, \infty)$. \square

Proof of Proposition 2.1. We endow \mathbb{R}^d with the usual euclidean norm $\|\cdot\|$. Let \mathcal{C} be the set of open convex cones $C \subset \mathbb{R}^d$.

For $C \in \mathcal{C}$ and $x \in X$, we define the ω_C -limit set of x to be

$$\omega_C(x) := \bigcap_{a \in A} \overline{(a + C)x}.$$

By definition this set is closed and A -invariant. Since the action is minimal, this set is either empty or equal to X .

a) We will check that

$$(2.1) \quad \mu(\{x \in X \mid \forall C \in \mathcal{C}, \omega_C(x) = X\}) = 1.$$

For $\varepsilon > 0$, choose a compact set $K \subset X$ with $\mu(K) > 1 - \varepsilon$. By Poincaré recurrence theorem, for μ -almost every x in K , for every rational vector a in \mathbb{R}^d , infinitely many translates $(na)x$, $n \in \mathbb{N}$, belong to K . We note that the interior of any $C \in \mathcal{C}$ contains a rational vector a . Hence for such a point x , for every $C \in \mathcal{C}$, one has $\omega_C(x) \cap K \neq \emptyset$ and thus, since the action is minimal, one has $\omega_C(x) = X$. Since ε is arbitrarily small, this proves (2.1). In particular, the set given in (2.1) is non empty. We conclude since every point in this set has a strongly dense orbit.

b) Since X is compact, all the sets $\omega_C(x)$ are non empty. Since the action is minimal, they are equal to X . Hence the action is strongly minimal.

c) We can choose a constant $\varepsilon_0 > 0$ and $d+1$ open convex cones C_0, \dots, C_d of \mathbb{R}^d such that, for every family of $d+1$ vectors v_0, \dots, v_d with $\|v_i\| = 1$ and $v_i \in C_i$,

$$(2.2) \quad \text{the ball } B(0, \varepsilon_0) \text{ is contained in the convex hull of } v_0, \dots, v_d.$$

Let U be a non-empty open subset of X with compact closure K . For $0 \leq i \leq d$, let $T_i : X \rightarrow [0, \infty]$ be the ‘‘hitting time of U in the direction C_i ’’ :

$$(2.3) \quad T_i(x) = \inf\{\|c\| \geq 1 \mid c \in C_i, cx \in U\}.$$

Since the action is strongly minimal, the function T_i is finite everywhere. Since U is open, the function T_i is upper semi-continuous. Hence, since K is compact, the constant

$$M_0 := \sup\{T_i(x) \mid 0 \leq i \leq d, x \in K\}$$

is finite. For every $x \in X$, we introduce the set of “hitting times”

$$A_x := \{a \in \mathbb{R}^d \mid ax \in K\}.$$

Since the action is minimal the closed set A_x is non-empty. Let a be an element of A_x with minimal norm. We claim that

$$(2.4) \quad \|a\| \leq \frac{M_0}{2\varepsilon_0}.$$

Indeed, for every $0 \leq i \leq d$, one can find $c_i \in C_i$ with $1 \leq \|c_i\| \leq M_0$ and $a + c_i \in A_x$. Since the element a has minimal norm in A_x , one has, for all i ,

$$\|a + c_i\| \geq \|a\|.$$

Thus, for all i , if we set $v_i := \frac{1}{\|c_i\|}c_i$, we get

$$2 \langle a, v_i \rangle \geq -M_0.$$

Therefore, using (2.2), for any v in $B(0, \varepsilon_0)$, one has also

$$2 \langle a, v \rangle \geq -M_0.$$

Choosing $v := -\frac{\varepsilon_0}{\|a\|}a$, we get the expected bound (2.4).

This bound (2.4) proves that, for all x in X , there exists a in $B(0, \frac{M_0}{2\varepsilon_0})$ and y in K such that $x = a^{-1}y$. In particular, it proves that the space X is compact. \square

We conclude this section by noting that these results can easily be adapted to actions of the group $A = \mathbb{Z}^d$.

3. LATTICES IN LOCALLY COMPACT GROUPS

We give elementary properties of lattices and we prove a decomposition result for lattices in a locally compact solvable group.

Let G be a locally compact group and H be a closed subgroup of G . We shall say that H has *finite covolume* in G if the quotient G/H admits a *finite* G -invariant Borel measure. For instance, a *lattice* is by definition a discrete finite covolume subgroup.

Let us state some elementary properties of finite covolume subgroups.

Lemma 3.1. *Let G be a locally compact group and H_1, H_2 be two closed subgroups such that $H_1 \subset H_2$. Then H_1 has finite covolume in G if and only if simultaneously H_1 has finite covolume in H_2 and H_2 has finite covolume in G .*

Proof. This is classical (see [13, Lemma 1.6]). Recall (see [16]) that a quotient G/H admits a G -invariant Radon measure if and only if the modular function of H is the restriction to H of the modular function of G .

If H_1 has finite covolume in H_2 and H_2 has finite covolume in G , the transitivity formula for integration on homogeneous spaces (see [16]) proves that G/H_1 supports a G -invariant measure with total mass 1.

Conversely, if H_1 has finite covolume in G , then the image in G/H_2 of the G -invariant probability measure on G/H_1 is also G -invariant with total mass 1. The same transitivity formula proves that H_1/H_2 supports also a H_1 -invariant measure with total mass 1. \square

Lemma 3.2. *Let G be a locally compact group, H be a finite covolume closed subgroup of G , G' be an open subgroup of G and $H' := H \cap G'$.*

- a) *The group H' has finite covolume in G' .*
- b) *If H is cocompact in G then H' is cocompact in G' .*
- c) *Conversely, if H' is cocompact in G' and G' is normal in G then H is cocompact in G .*

Proof. a) The restriction of the G -invariant probability on G/H to the G' -orbit G'/H' is a non-zero finite G' -invariant measure.

b) The G' -orbits in the compact space G/H are open hence closed. In particular G'/H' is compact.

c) Since the group $G'H$ is open, the space $G/G'H$ is discrete. This space admits a finite measure which is invariant under the transitive action of the group G/G' . Hence, this space is finite, that is G/H is a finite union of G' -orbits. As each of these orbits is compact, G/H is compact. \square

From these results, we at once get the following

Lemma 3.3. *Let N be a nilpotent locally compact group. Then any finite covolume closed subgroup $H \subset N$ is cocompact.*

Proof. Let Z be the center of N and $N' := \overline{HZ}$. By lemma 3.1, N' has finite covolume in N and H has finite covolume in N' .

Now, by an induction argument on the length of the central series of N , the finite covolume subgroup N'/Z of N/Z is cocompact. Hence N' is cocompact in N .

Besides, one has

$$[N', N'] \subset \overline{[HZ, HZ]} \subset \overline{[H, H]} \subset H.$$

Hence H is normal in N' and the quotient N'/H is a group. As its Haar measure is finite, it is compact and H is cocompact in N . \square

The proof of the following proposition will be much more delicate.

Proposition 3.4. *Let G be a solvable locally compact group, G_e its connected component, and H a finite covolume closed subgroup of G . Then H is cocompact in the group $\overline{HG_e}$.*

In particular, as was proven by Mostow, when G is solvable and connected, any lattice H in G is cocompact in G .

If we set G_d for the totally discontinuous quotient $G_d = G/G_e$ and H_d for the finite covolume closed subgroup $H_d := \overline{HG_e}$, Proposition 3.4 tells us that *the finite volume quotient G/H fibers over the totally discontinuous finite volume homogeneous space G_d/H_d with compact fibers.*

Example 3.5. (Bader, Caprace, Gelander, Mozes) *There exist metabelian locally compact groups containing non-cocompact lattices.*

Proof of Example 3.5. The group G is a semidirect product $G = A \rtimes K$ of a commutative discrete group A by a commutative compact group K , where A is the direct sum of finite fields \mathbb{F}_p for an infinite set S of primes p such that $\sum_{p \in S} p^{-1} < \infty$, and where K is the corresponding product of multiplicative groups \mathbb{F}_p^* .

Let H be the subgroup of G generated by the following elements h_p for $p \in S$. The only non-trivial component of h_p is the p^{th} coordinate which is $(1 - k_p, k_p) \in \mathbb{F}_p \rtimes \mathbb{F}_p^*$ where k_p is a generator of \mathbb{F}_p^* . The orbits of H in $A \simeq G/K$ are the sets $A_I := \{a \in A \mid p \in I \Leftrightarrow a_p = 1\}$, for I finite subset of S . The cardinality N_I of the stabilizer of a point $a \in A_I$ is $N_I = \prod_{p \in I} (p - 1)$. Since $\sum_I N_I^{-1} < \infty$, H is a lattice in G . \square

Proof of Proposition 3.4. We may assume that $G = \overline{HG_e}$. We argue by induction on the length of the derived series of G_e . Let A be the closure of the last non-trivial term of the derived series of G_e and $H' := \overline{HA}$. According to Lemma 3.1, the group H'/A has finite covolume in G/A and the group H has finite covolume in H' .

By the induction hypothesis, the group H'/A is cocompact in G/A hence H' is cocompact in G . By Lemma 3.6 below, H is cocompact in H' . Hence H is cocompact in G . \square

Lemma 3.6. *Let G be a locally compact group, $H \subset G$ a finite covolume closed subgroup and $A \subset G$ a normal closed connected abelian subgroup. If $G = \overline{HA}$, then H is cocompact in G .*

Proof. According to [12, §2.21], as A is a connected locally compact abelian group, it contains a largest compact subgroup K_A and the quotient group A/K_A is isomorphic to \mathbb{R}^d , for some $d \geq 0$. By uniqueness, K_A is normal in G . Thus, after replacing G by G/K_A and H by HK_A/K_A , we may assume $A = \mathbb{R}^d$.

We will apply Proposition 2.1 to the action of A on the locally compact space $X := G/H$. Since H has finite covolume, this action preserves a Borel probability measure. Since $G = \overline{HA}$, this action is minimal. Hence, by Proposition 2.1.a, there exists at least one orbit Ax_0 in X which is strongly dense. Since G acts transitively on X and normalizes A , all the orbits Ax in X are strongly dense. Hence, by Proposition 2.1.c, the space X is compact. \square

More generally, the same argument proves the following

Proposition 3.7. *Let G be a locally compact group, R a closed normal connected amenable subgroup of G , and H a finite covolume closed subgroup of G . Then H is cocompact in the group \overline{HR} .*

Proof. We use the following two facts: every connected locally compact group is a compact extension of a connected Lie group (see [12]); every amenable connected Lie group is a compact extension of its solvable radical (see [17]). Hence the group R is a compact extension of a connected solvable Lie group R_1 which is normal in G . We follow then the same proof as for Proposition 3.4 by induction on the length of the derived series of R_1 . \square

4. S -ADIC LIE GROUPS

This section contains elementary definitions and facts about S -adic Lie groups. It also contains a version of the Borel density theorem for S -adic Lie groups.

We recall that \mathbb{Q}_p is the field of p -adic numbers and $\mathbb{Q}_\infty = \mathbb{R}$ is the field of real numbers or ∞ -adic numbers. Let S be a finite subset of the set of prime numbers including ∞ .

Definition 4.1. *An S -adic Lie group G is a locally compact group which contains an open subgroup U isomorphic to a group of the form $(\prod_{p \in S} G_p)/N$ where, for each $p \in S$, G_p is a p -adic Lie group and N is a discrete normal subgroup of this product.*

Let G be an S -adic Lie group. The \mathbb{Q} -vector space $\mathfrak{g} := \bigoplus_{p \in S} \mathfrak{g}_p$ which is the direct sum of the Lie algebras \mathfrak{g}_p of G_p does not depend on the choices and is called the Lie algebra of G . This Lie algebra is an S -adic Lie algebra i.e. a direct sum of p -adic Lie algebras with p in S . The real Lie subalgebra \mathfrak{g}_∞ is called the real factor of \mathfrak{g} . We say that \mathfrak{g} is *non-archimedean* if $\mathfrak{g}_\infty = 0$. The Lie subalgebra $\mathfrak{g}_f := \bigoplus_{p \neq \infty} \mathfrak{g}_p$ is called the non-archimedean factor of \mathfrak{g} . We will denote by $\text{Ad}_{\mathfrak{g}_p}, \text{Ad}_{\mathfrak{g}}, \dots$ the adjoint action of G in $\mathfrak{g}_p, \mathfrak{g}, \dots$

Here are the first properties of S -adic Lie groups:

- Real Lie groups and p -adic Lie groups are S -adic Lie groups.
- A product of two S -adic Lie groups is an S -adic Lie group.
- A closed subgroup of an S -adic Lie group is an S -adic Lie group (see [14, Prop. 1.5]).
- The quotient of an S -adic Lie group by a closed normal subgroup is an S -adic Lie group.

An S -adic Lie group can be connected, even if its Lie algebra admits a nontrivial non-archimedean factor, as, for example, the solenoid $(\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[\frac{1}{p}]$, where $\mathbb{Z}[\frac{1}{p}]$ is embedded diagonally in $\mathbb{R} \times \mathbb{Q}_p$, or the group $(\text{SL}(2, \mathbb{R}) \times \mathbb{Z}_p)/\mathbb{Z}$, where \mathbb{Z} is embedded diagonally as a central subgroup in $\text{SL}(2, \mathbb{R}) \times \mathbb{Z}_p$.

The following proposition is a version of the classical Borel density theorem in the framework of S -adic Lie groups (see [13, Chap. 5], [17, Chap. 3] or [11, §2.4]).

Proposition 4.2. *Let G be an S -adic Lie group, $p \in S$, $H \subset G$ a finite covolume closed subgroup, $\pi : G \rightarrow \text{GL}(d, \mathbb{Q}_p)$ a continuous morphism.*

- a) *For any H -invariant line $x_0 \in \mathbb{P}(\mathbb{Q}_p^d)$, the G -orbit Gx_0 is compact.*
- b) *If G has no proper cocompact normal subgroups, then any H -invariant line $x_0 \in \mathbb{P}(\mathbb{Q}_p^d)$ is G -invariant.*
- c) *If G has no proper cocompact normal subgroups, then the Zariski closures of $\pi(H)$ and $\pi(G)$ are equal.*

Example 4.3. *In point a), there does not always exist a cocompact normal subgroup of G stabilizing x_0 .*

Proof of 4.3. We give an example with $p = \infty$ and G real connected. We denote by $r_\theta \in \text{SO}(2, \mathbb{R}) \subset \text{SO}(3, \mathbb{R})$ the rotation of angle θ and we fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We choose

$$\begin{aligned} G &:= \text{SO}(3, \mathbb{R}) \times \mathbb{R}, \\ H &:= \{(r_\theta, \theta) \mid \theta \in 2\alpha\pi\mathbb{Z}\}, \\ K &:= \text{SO}(3, \mathbb{R}) \times \text{SO}(2, \mathbb{R}) \times \text{SO}(2, \mathbb{R}), \quad \text{and} \end{aligned}$$

$$\pi : G \rightarrow K; (k, \theta) \mapsto (k, r_\theta, r_{\theta/\alpha}),$$

and we set $M := \overline{\pi(H)} = \{(r_\theta, r_\theta, 1) \mid \theta \in \mathbb{R}\}$. The subgroup H is a cocompact lattice in G but, for no $\beta \in \mathbb{R}$, does the cocompact normal subgroup $G' = \{(1, \theta) \mid \theta \in \beta\mathbb{Z}\}$ fix the base point $x_0 \in K/M$. \square

Proof of Proposition 4.2. a) After replacing G by a finite index subgroup, we may assume that $\pi(G)$ is Zariski connected. We may also assume that the orbit Gx_0 spans \mathbb{Q}_p^d .

We claim that

$$(4.1) \quad \text{the closure } K := \overline{\pi(G)} \text{ is a compact subgroup of } \mathrm{PGL}(d, \mathbb{Q}_p).$$

Since x_0 is H -invariant, one has a G -equivariant map

$$i : G/H \rightarrow \mathbb{P}(\mathbb{Q}_p^d) \text{ given by } i(gH) = gx_0.$$

The probability measure $i_*(\mu)$ on $\mathbb{P}(\mathbb{Q}_p^d)$, which is the image of the G -invariant probability measure μ on G/H , is K -invariant. Lemma 4.4 below shows that the group K is compact, as claimed in (4.1).

Since we have not assumed G and π to be algebraic, the group $\pi(G)$ might not be closed as in Example 4.3. This will make the proof a little bit longer. Let $C \subset G$ be an open relatively compact subset so that one has

$$i_*(\mu)(Cx_0) > 0.$$

We can assume π to be injective. The group G is then a p -adic Lie group. Since the continuous morphism π is \mathbb{Q}_p -analytic, the set Cx_0 is a \mathbb{Q}_p -submanifold of Kx_0 . Since $i_*(\mu)$ is the K -invariant probability measure on Kx_0 , one has then

$$\dim_{\mathbb{Q}_p} Cx_0 = \dim_{\mathbb{Q}_p} Kx_0$$

and the orbit Gx_0 is open in Kx_0 . Since $\pi(G)$ is dense in K , every G -orbit in Kx_0 is dense hence meets the open set Gx_0 . This proves the equality $Gx_0 = Kx_0$ and this ends the proof of a).

b) We assume now that G does not admit any proper cocompact normal subgroup. In particular, the group $\pi(G)$ is Zariski connected. We may again assume that the orbit Gx_0 spans \mathbb{Q}_p^d . We want to prove that this orbit is a singleton or, equivalently, that

$$(4.2) \quad \text{the group } K := \overline{\pi(G)} \text{ is trivial.}$$

We may again assume that π is injective. We note first that the group K is connected : indeed for any open normal subgroup K' in K , the group $G' := \pi^{-1}(K')$ is a finite index normal subgroup of G .

Therefore, we may assume $p = \infty$ and G is a real Lie group. Now, the connected component G_e of G is an open subgroup of G . Thus,

$G_e H$ being an open finite index subgroup of G , it contains an open normal finite index subgroup and we get

$$G = G_e H.$$

Let $S := [K, K]$ and let T be the connected component of the center of K , so that $K = ST$ and $S \cap T$ is finite. We let \mathfrak{k} , \mathfrak{s} and \mathfrak{t} denote the Lie algebras of the real compact Lie groups K , S and T .

Let L be the immersed subgroup $\pi(G_e)$ in K and \mathfrak{l} be its Lie algebra. As G_e is normal in G and $\pi(G)$ is dense in K , \mathfrak{l} is an ideal in \mathfrak{k} , and one has

$$\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{s}) \oplus (\mathfrak{l} \cap \mathfrak{t}).$$

Set $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{s}$ and let L' be the closed normal connected subgroup of K with Lie algebra \mathfrak{l}' . This group L' is included in L and the group $G' = \pi^{-1}(L')$ is included in G_e . As G_e is a connected Lie group, π induces a homeomorphism from compact subsets of G_e onto their images and G' is a compact connected normal subgroup of G . Now, we have $\text{Ad}_\nu(K) = \text{Ad}_\nu(L')$, hence

$$G = Z_G(G')G',$$

where $Z_G(G')$ denotes the centralizer of G' in G . In particular, $Z_G(G')$ is a normal cocompact subgroup of G . Therefore, $Z_G(G') = G$ and L' is a central subgroup of K , that is

$$L \subset T$$

Let M be the closure of $\pi(H)$. As $G = G_e H$, we have

$$K = TM.$$

As Gx_0 spans \mathbb{Q}_p^d , the group K acts faithfully on the orbit $Kx_0 = Gx_0$ and we have

$$(4.3) \quad \bigcap_{k \in K} kMk^{-1} = \{e\}$$

Thus we have $M = \{e\}$, that is $H = \{e\}$. Now, the group G has finite Haar measure, hence is compact. Since G does not admit any proper cocompact normal subgroup, G is trivial and K is trivial too, what we claimed in (4.2).

c) Now, let \mathbf{G} be the Zariski closure of $\pi(G)$ in $\text{GL}(d, \mathbb{Q}_p)$ and \mathbf{H} be the Zariski closure of $\pi(H)$. According to Chevalley Theorem (see [2, 5.1]) there exists an algebraic representation $\rho : \mathbf{G} \rightarrow \text{PGL}(m, \mathbb{Q}_p)$ and a line $x_0 \in \mathbb{P}(\mathbb{Q}_p^m)$ whose stabilizer is \mathbf{H} . By point b) applied to this representation ρ , we get $\pi(G) \subset \mathbf{H}$ hence $\mathbf{G} = \mathbf{H}$. \square

Lemma 4.4. *Let ν be a probability measure on $\mathbb{P}(\mathbb{Q}_p^d)$. Suppose that, for any two proper subspaces $E_1, E_2 \subsetneq \mathbb{Q}_p^d$ with $\dim E_1 + \dim E_2 \leq d$, the support of ν is not contained in the union $\mathbb{P}(E_1) \cup \mathbb{P}(E_2)$. Then the stabilizer $S := \{g \in \mathrm{PGL}(\mathbb{Q}_p^d) \mid g_*\nu = \nu\}$ of ν is compact.*

Proof. This lemma due to Furstenberg is proven in [17, §3.2]. \square

Corollary 4.5. *Let G be an S -adic Lie group and H be a finite covolume closed subgroup of G , with Lie algebra \mathfrak{h} .*

a) *Then the normalizer $N_G(\mathfrak{h})$ is cocompact in G .*

b) *If G has no proper cocompact normal subgroup, G normalizes \mathfrak{h} .*

Example 4.6. *In point a), there does not always exist a cocompact normal subgroup of G normalizing \mathfrak{h} .*

Proof of 4.6. We give an example with $p < \infty$ and G nilpotent with an exact sequence $1 \rightarrow G_0 \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$ where G_0 is an open compact subgroup. Let K be the group of upper triangular unipotent 4×4 -matrices u with coefficients $u_{i,j}$ in \mathbb{Z}_p , for $1 \leq i < j \leq 4$. The group G is immersed in K as

$$G := \{u \in K \mid u_{1,2} \in \mathbb{Z}\}.$$

The group H is the closed subgroup of G isomorphic to $\mathbb{Z} \times \mathbb{Z}_p$,

$$H := \{u \in G \mid u_{1,3} = u_{2,3} = u_{1,4} = u_{2,4} = 0\}.$$

One computes the normalizer $N_G(\mathfrak{h}) = \{u \in G \mid u_{1,3} = u_{2,3} = 0\}$ and the group $\bigcap_{g \in G} gN_G(\mathfrak{h})g^{-1} := \{u \in G \mid u_{1,2} = u_{1,3} = u_{2,3} = 0\}$ is not cocompact in G . \square

Proof of Corollary 4.5. a) Applying, for any p in S , Proposition 4.2 to the adjoint representation of G in $\Lambda^{d_p} \mathfrak{g}_p$, where $d_p := \dim_{\mathbb{Q}_p}(\mathfrak{h}_p)$, we get that the G -orbit of the line $x_p := \Lambda^{d_p} \mathfrak{h}_p$ is compact. But the G -orbits in the product $\prod_{p \in S} Gx_p$ are open and hence closed. Thus the stabilizer $N_G(\mathfrak{h})$ of the point $x := (x_p)_{p \in S}$ is cocompact in G .

b) Since H normalizes \mathfrak{h} , by Proposition 4.2, G normalizes \mathfrak{h} too. \square

Corollary 4.7. *For any p in S , let G_p be the group of \mathbb{Q}_p -points of a \mathbb{Q}_p -algebraic semisimple group with no anisotropic factor and set $G = \prod_{p \in S} G_p$. If H is a finite covolume closed subgroup of G , for any p in S , the image of H in G_p has finite index Zariski closure.*

Proof. For any p in S , let G_p^+ be the subgroup of G_p which is spanned by unipotent one-parameter subgroups in G_p . As G_p does not have anisotropic factors, G_p^+ is open with finite index in G_p and every cocompact normal subgroup of G_p contains G_p^+ . The result now follows from Proposition 4.2 applied to the group $G = \prod_{p \in S} G_p^+$. \square

Corollary 4.8. *Let G be as above and Λ be a lattice in G , then Λ has finite index in its normalizer $N_G(\Lambda)$.*

Proof. Let $N := N_G(\Lambda)$ be the normalizer of Λ and \mathfrak{n} its Lie algebra. By noetherianity there exists a finitely generated subgroup $\Lambda_0 \subset \Lambda$ whose centralizer in \mathfrak{g} is the same as the one of Λ . Since Λ is discrete, the elements of N which are small enough commute with Λ_0 . Hence the group Λ centralizes \mathfrak{n} . By corollary 4.7, the centralizer of \mathfrak{n} in G has finite index. Hence \mathfrak{n} is a central ideal of \mathfrak{g} . Such an ideal is trivial, hence the group N is discrete. Since $\Lambda \subset N$ and Λ is a lattice in G , this group Λ has finite index in N . \square

5. COCOMPACTNESS OF LATTICES

We give a sufficient criterion for an S -adic Lie group G to admit only cocompact lattices.

We say that an S -adic Lie algebra \mathfrak{g} is *amenable* if it is the Lie algebra of some amenable S -adic Lie group, that is if \mathfrak{g}_∞ does not admit any noncompact semisimple Lie algebra or, equivalently, if \mathfrak{g} does not contain a copy of $\mathfrak{sl}(2, \mathbb{R})$. In particular, every non-archimedean S -adic Lie algebra is amenable.

Proposition 5.1. *Let G be an S -adic Lie group whose Lie algebra \mathfrak{g} is amenable. Then any finite covolume closed subgroup $H \subset G$ is cocompact.*

We begin the proof of Proposition 5.1 by a special case :

Lemma 5.2. *Let G be a non-archimedean S -adic Lie group. Then any finite covolume closed subgroup $H \subset G$ is cocompact.*

We note that Lemma 5.2 can not be extended to any locally compact totally discontinuous group G . Indeed, for example, if k is the non-archimedean local field with positive characteristic $\mathbb{F}_q((T))$, the group $\mathrm{SL}(2, \mathbb{F}_q[[T^{-1}]])$ is a non-cocompact lattice in $\mathrm{SL}(2, k)$.

When H is a lattice in G , Lemma 5.2 is [14, Prop. 2]. In this case the proof is very short: Just choose a torsion free compact open subgroup Ω of G , and note successively that the action of Ω on G/H is free, that all the Ω -orbits have same volume, that there are only finitely many Ω orbits and that G/H is compact.

In order to adapt this proof to non discrete groups H , we recall a few facts on standard groups and on invariant measures.

A p -adic Lie group G_p with Lie algebra \mathfrak{g}_p is said to be *standard* if there exists a compact open subgroup O_p of \mathfrak{g}_p which is invariant by

the Lie bracket and such that the exponential map $O_p \rightarrow G_p$ is well-defined and is a bijection onto G_p (see [7]). A non-archimedean S -adic Lie group is said to be *standard* if it is a product of standard p -adic Lie groups. By [15, Prop. 1.1], if G is standard, every closed subgroup H of G with Lie algebra \mathfrak{h} is contained in $\exp(\mathfrak{h})$. Every non-archimedean S -adic Lie group contains a standard open subgroup.

Let G be an S -adic Lie group. Then the tangent bundle of G identifies G -equivariantly on the right with $G \times \mathfrak{g}$ and, through this identification, the left action of G reads as the map

$$G \times (G \times \mathfrak{g}) \rightarrow G \times \mathfrak{g}; (g, h, v) \mapsto (gh, \text{Ad}_{\mathfrak{g}}(g)v).$$

Besides, if λ is a right Haar measure on G , there exists a Haar measure ω on \mathfrak{g} such that λ is the measure associated to the constant field $g \mapsto \omega$ on G . In particular, the modular function of G is the function $g \mapsto |\det(\text{Ad}_{\mathfrak{g}}(g))| := \prod_{p \in S} |\det(\text{Ad}_{\mathfrak{g}_p}(g))|$ and, if H is a closed subgroup of G , the space G/H admits a G -invariant measure if and only if, for any h in H , $|\det(\text{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))| = 1$.

Proof of Lemma 5.2. According to Lemma 3.1, the group H has also finite covolume in the normalizer $N_G(\mathfrak{h})$ of \mathfrak{h} in G . According to Corollary 4.5, $N_G(\mathfrak{h})$ is cocompact in G . Hence, it is enough to show that H is cocompact in $N_G(\mathfrak{h})$. Thus, replacing G by $N_G(\mathfrak{h})$, we may assume that the Lie algebra \mathfrak{h} is $\text{Ad}_{\mathfrak{g}}(G)$ -invariant. In this case, the tangent bundle of $X := G/H$ identifies with $X \times \mathfrak{g}/\mathfrak{h}$ and the action of G on this bundle can be read as the map

$$G \times (X \times \mathfrak{g}/\mathfrak{h}) \rightarrow G \times \mathfrak{g}/\mathfrak{h}; (g, x, v) \mapsto (gx, \text{Ad}_{\mathfrak{g}/\mathfrak{h}}(g)v).$$

In the same way, if x_0 is the base point of X and λ_X is the G -invariant probability measure, then λ_X comes from the field

$$x = gx_0 \mapsto |\det(\text{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))|\omega$$

where ω is some fixed Haar measure on $\mathfrak{g}/\mathfrak{h}$ (note that, by the remark above, for any h in H , $|\det(\text{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))| = 1$). Now, we claim that

$$(5.1) \quad |\det(\text{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))| = 1 \text{ for all } g \text{ in } G.$$

Indeed the character $\chi : G \rightarrow \mathbb{R}_+^*; g \mapsto |\det(\text{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))|$ is trivial on H and the probability measure $\chi_*\lambda_X$ on \mathbb{R}_+^* is $\chi(G)$ -invariant, hence $\chi(G) = 1$. Thus, λ_X comes from the constant field $x \mapsto \omega$ on X .

We will then prove that, for any compact open subgroup, Ω one has

$$(5.2) \quad \inf_{x \in X} \lambda_X(\Omega x) > 0.$$

To do this, we may assume that $\Omega = \prod_{p \in S} \Omega_p$ is a *standard* open compact subgroup of G . We write $\Omega = \exp(O)$ where $O = \prod_{p \in S} O_p$ and \exp is the componentwise exponential map. Now, since Ω is standard, the set $H_0 := \exp(\mathfrak{h} \cap O)$ is the largest subgroup of Ω with Lie algebra \mathfrak{h} . For every $x := gx_0$ in X , the Ω -orbit map at x gives rise to an embedding $\Omega x \simeq \Omega/(\Omega \cap H_x) \hookrightarrow X$, where $H_x := gHg^{-1}$ is the stabilizer of x in G , and λ_X restricts on Ωx as the measure coming from the constant field $y \mapsto \omega$. Since G normalizes \mathfrak{h} , the Lie algebra of H_x is equal to \mathfrak{h} and $H_x \cap \Omega$ is contained in H_0 as a subgroup of finite index d_x . Thus, we get

$$\lambda_X(\Omega x) = \frac{d_x}{d_{x_0}} \lambda_X(\Omega x_0)$$

and this quantity is bounded below by $\frac{\lambda_X(\Omega x_0)}{d_{x_0}}$, whence Equation (5.2).

Since X has finite volume, this implies that Ω has only finitely many orbits in X and X is compact. \square

Proof of Proposition 5.1. We proceed by induction on the dimension of the largest solvable ideal \mathfrak{r}_∞ of \mathfrak{g}_∞ .

1st case : $\mathfrak{r}_\infty = 0$.

In this case, the connected immersed subgroup G_∞ of G with Lie algebra \mathfrak{g}_∞ is compact. According to Lemma 3.1, the group HG_∞/G_∞ has finite covolume in the non-archimedean S -adic Lie group G/G_∞ . Hence according to Lemma 5.2, the group H is cocompact in G .

2nd case : $\mathfrak{r}_\infty \neq 0$.

We argue here exactly as in the proof of Proposition 3.4. Let R_∞ be the connected immersed real Lie group with Lie algebra \mathfrak{r}_∞ . Let A be the closure of the last non trivial term of the derived series of R_∞ and $H' := \overline{HA}$. According to Lemma 3.1, the group H'/A has finite covolume in G/A and the group H has finite covolume in H' . By the induction hypothesis, the group H'/A is cocompact in G/A hence H' is cocompact in G . By Lemma 3.6, the group H is cocompact in H' . Hence the group H is cocompact in G . \square

6. PROJECTIONS OF LATTICES

The aim of this section is to prove, for any lattice Λ in any S -adic Lie group G , a decomposition theorem of Λ with respect to the adjoint action on a suitable semisimple quotient \mathfrak{s} of the Lie algebra of G (Theorem 6.6).

Proposition 6.1. *Let G be an S -adic Lie group. Let \mathfrak{r}_0 be the smallest ideal of \mathfrak{g} such that the Lie algebra $\mathfrak{s}_0 := \mathfrak{g}/\mathfrak{r}_0$ is semisimple and such*

that, for any non zero G -invariant ideal \mathfrak{s}_1 of \mathfrak{s}_0 , the group $\text{Ad}_{\mathfrak{s}_1}(G)$ is an unbounded subgroup of $\text{Aut}(\mathfrak{s}_1)$. Let $R_0 := \text{Ker}(\text{Ad}_{\mathfrak{s}_0})$ be the kernel of the adjoint map in \mathfrak{s}_0 .

Then, for any lattice Λ in G , the Lie algebra \mathfrak{l} of the group $L := \overline{\Lambda R_0}$ is amenable.

We postpone the proof of Proposition 6.1 to the end of this section.

If $S = \{\infty\}$, the group ΛR_0 is closed (see lemma 6.4 below). In the general setting, our statement is optimal, as shown by the following two examples.

Example 6.2. *The S -adic Lie group L may contain Lie subgroups isomorphic to $\text{PSL}(2, \mathbb{Q}_p)$ with p finite.*

Proof. Let G_0 be a non compact simple p -adic Lie group, for instance $G_0 = \text{PSL}(2, \mathbb{Q}_p)$. We will prove, more generally, that L may contain Lie subgroups isomorphic to G_0 . Let $Y := G_0/K_0$ where K_0 is a compact open subgroup of G_0 and R_0 be the free group on Y , i.e. the discrete free non-abelian group with infinitely many generators e_y with y in Y . We define G to be the semidirect product $G := G_0 \ltimes R_0$ where the action by conjugation of G_0 on R_0 is given by $g_0 e_y g_0^{-1} = e_{g_0 y}$ for all g_0 in G_0 and all y in Y . Every element g of G can be written in a unique way as $g = r_g s_g$ with r_g in R_0 and s_g in G_0 .

Let us now construct a lattice Λ in G . We first construct a discrete subgroup F of G . For y in Y , one chooses an element a_y in G_0 which fixes y . We assume that a_y has infinite order. Let F be the group generated by the elements

$$f_y := e_y a_y = a_y e_y.$$

We claim that

$$(6.1) \quad \begin{array}{l} \text{the group } F \text{ is a free discrete subgroup of } G \\ \text{and the map } F \rightarrow R_0; f \mapsto r_f \text{ is a bijection.} \end{array}$$

Indeed, the R_0 component r_w of a word

$$w = f_{y_1}^{n_1} \cdots f_{y_\ell}^{n_\ell} \text{ with } y_i \in Y \text{ and } n_i \in \mathbb{Z}$$

is equal to

$$r_w = e_{z_1}^{n_1} \cdots e_{z_\ell}^{n_\ell} \text{ with } z_j := a_1^{n_1} \cdots a_{j-1}^{n_{j-1}}(y_j).$$

This proves (6.1).

Let Λ_0 be a torsion free cocompact lattice in G_0 . Since Λ_0 acts freely on Y , we can choose the a_y , $y \in Y$, in such a way that, for every λ_0 in Λ_0 and y in Y , one has $a_{\lambda_0 y} = \lambda_0 a_y \lambda_0^{-1}$. This ensures that Λ_0

normalizes the group F . We choose Λ to be the group $\Lambda := \Lambda_0 F$. This group is discrete and cocompact in G but its projection on G/R_0 is dense. Hence, in this case, the group L is equal to $G = G_0 \times R_0$. \square

Example 6.3. *When Λ is a discrete subgroup of G which is not assumed to be a lattice, the Lie algebra \mathfrak{l} may contain Lie subalgebras isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.*

Proof. Let G_0 be a simple real Lie group, for instance $G_0 = \mathrm{PSL}(2, \mathbb{R})$, and Γ be a dense subgroup of G_0 . Let G be the cartesian product $G := G_0 \times R_0$ where R_0 is a copy of Γ endowed with the discrete topology. The discrete subgroup $\Lambda := \{(\gamma, \gamma) \mid \gamma \in \Gamma\}$ is a discrete subgroup of G whose projection in G_0 is dense. Proposition 6.1 tells us that this can not happen if Λ is a lattice in G . \square

For real Lie groups G , Proposition 6.1 is a consequence of the following

Lemma 6.4. *Let G be a real Lie group and Λ be a lattice in G . Let \mathfrak{r} be the largest amenable ideal of \mathfrak{g} , $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ and R be the kernel of the adjoint action $\mathrm{Ad}_{\mathfrak{s}} : G \rightarrow \mathrm{Aut}(\mathfrak{s})$ in \mathfrak{s} . Then, the intersection $\Lambda \cap R$ is a cocompact lattice in R and the image $\mathrm{Ad}_{\mathfrak{s}}(\Lambda)$ is a lattice in $\mathrm{Aut}(\mathfrak{s})$.*

Proof. **1st case :** G is a connected real Lie group.

Let R_e be the connected component of R . The group R_e is the largest closed connected normal amenable subgroup of G . Since R_e is a compact extension of a solvable group, according to Auslander projection theorem in [13, Chap. 8], the group $L := \overline{\Lambda R_e}$ has an amenable Lie algebra \mathfrak{l} . Since \mathfrak{l} is normalized by Λ and \mathfrak{s} has no compact factors, by Borel density theorem ([13, Chap. 5] or Corollary 4.7), \mathfrak{l} is an ideal of \mathfrak{g} . Hence $\mathfrak{l} = \mathfrak{r}$, and the group ΛR_e is closed. By Lemma 3.1, $\Lambda \cap R_e$ is a lattice in R_e . Since R_e is amenable, this lattice is cocompact ([13, Chap. 4] or Proposition 5.1). Replacing G by G/R_e , we can assume that G is semisimple connected with no compact factor.

The group R is then discrete and is the center of G . Since the group $L := \overline{\Lambda R}$ has a discrete derived subgroup, its Lie algebra \mathfrak{l} is abelian. Again by Borel density theorem, one gets $\mathfrak{l} = 0$. Hence L is discrete and, by Lemma 3.1, the group $L/\Lambda \simeq R/(\Lambda \cap R)$ is finite. Thus the image $\Lambda_{\mathfrak{s}} = \mathrm{Ad}_{\mathfrak{s}}(\Lambda) \simeq \Lambda/(\Lambda \cap R)$ is a lattice in the adjoint group $\mathrm{Ad}_{\mathfrak{s}}(G)$. Since this adjoint group has finite index in $\mathrm{Aut}(\mathfrak{s})$, $\Lambda_{\mathfrak{s}}$ is also a lattice in $\mathrm{Aut}(\mathfrak{s})$.

2nd case : G is any real Lie group.

Since the connected component G_e of G is an open subgroup of G , by Lemma 3.2, the intersection $\Lambda_0 := \Lambda \cap G_e$ is a lattice in G_e . According

to the first case, the group $\Lambda_{0,S} := \text{Ad}_{\mathfrak{g}}(\Lambda_0)$ is a lattice in the group $\text{Aut}(\mathfrak{s})$. Since the group $\Lambda_{\mathfrak{g}} := \text{Ad}_{\mathfrak{g}}(\Lambda)$ normalizes $\Lambda_{0,S}$, it is discrete by Corollary 4.8. Hence $\Lambda_{\mathfrak{g}}$ is a lattice. As a consequence, the intersection $\Lambda_R := \Lambda \cap R$ is a lattice in R by Lemma 3.1. According also to the first case, the group $\Lambda_0 \cap R$ is cocompact in $G_e \cap R$, hence, by Lemma 3.2, the lattice Λ_R is also cocompact in R . \square

Lemma 6.5. *Let G be a non-archimedean S -adic Lie group, \mathfrak{r} be the largest solvable ideal of \mathfrak{g} and $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$. Assume that the image $\text{Ad}_{\mathfrak{g}}(G)$ of G in $\text{Aut}(\mathfrak{s})$ is a compact group. Then there exists an increasing sequence*

$$G_1 \subset \cdots \subset G_i \subset \cdots \subset G$$

of open subgroups whose union $G' := \cup_i G_i$ is a normal open subgroup of G and such that, for all integers $i \geq 1$, the group $\text{Ad}_{\mathfrak{g}}(G_i)$ is compact.

In order to help the reader to understand the following technical proof, we suggest him to keep in mind the example where $G := (\Gamma \times \mathbb{Q}_p^*) \times \mathbb{Q}_p^2$ where Γ is the group $\text{SL}(2, \mathbb{Z}[\frac{1}{p}])$ with the discrete topology and where \mathbb{Q}_p^* acts diagonally on \mathbb{Q}_p^2 .

Proof of Lemma 6.5. We first note that it is enough to prove this lemma for a finite index subgroup of G .

We will need some notations. Let $\mathfrak{g} = \oplus \mathfrak{g}_p$ be the Lie algebra of G and \mathfrak{s}_p^0 be a maximal semisimple Lie subalgebra of \mathfrak{g}_p . Let

- $\text{Aut}(\mathfrak{g}_p)$ be the group of automorphisms of the Lie algebra \mathfrak{g}_p ,
- \mathbf{A}_p the Zariski connected component of $\text{Aut}(\mathfrak{g}_p)$,
- \mathbf{S}_p a maximal semisimple Zariski connected subgroup of \mathbf{A}_p ,
- \mathbf{R}_p the maximal solvable Zariski connected normal subgroup of \mathbf{A}_p ,
- \mathbf{U}_p the maximal unipotent normal subgroup of \mathbf{A}_p ,

and $\mathfrak{a}_p, \mathfrak{s}_p, \mathfrak{r}_p$ and \mathfrak{u}_p their Lie algebras. We may assume that \mathfrak{s}_p contains the Lie algebra $\mathfrak{s}_p^1 := \text{ad}(\mathfrak{s}_p^0)$. Since the image $\text{ad}(\mathfrak{g}_p)$ is an ideal of \mathfrak{a}_p , \mathfrak{s}_p^1 is an ideal of \mathfrak{s}_p . We denote by \mathfrak{s}_p^2 the complementary ideal of \mathfrak{s}_p^1 in \mathfrak{s}_p . We have the decomposition

$$\mathfrak{s}_p^1 \oplus \mathfrak{s}_p^2 \oplus \mathfrak{r}_p = \mathfrak{a}_p$$

We set \mathbf{S}_p^1 and \mathbf{S}_p^2 for the Zariski closed and Zariski connected semisimple subgroups of \mathbf{A}_p with Lie algebras respectively \mathfrak{s}_p^1 and \mathfrak{s}_p^2 . The group

$$\mathbf{S}_p^1 \mathbf{S}_p^2 \mathbf{R}_p \subset \mathbf{A}_p$$

is a finite index subgroup.

Let Ω_p be an open compact subgroup of $\mathbf{S}_p^1 \mathbf{R}_p$. By the compactness assumption in Lemma 6.5, and since we are allowed to replace G by a

finite index subgroup, we may assume that

$$\mathrm{Ad}_{\mathfrak{g}_p}(g) \in \Omega_p \mathbf{S}_p^2 \mathbf{R}_p \text{ for all } g \in G \text{ and for all } p.$$

On the one hand, since $\mathrm{ad}(\mathfrak{g}_p) \subset \mathfrak{s}_p^1 \oplus \mathfrak{r}_p$, the group

$$G' := \{g \in G \mid \mathrm{Ad}_{\mathfrak{g}_p}(g) \in \Omega_p \mathbf{U}_p \text{ for all } p\}.$$

is an open subgroup of G .

On the other hand, since $[\mathbf{A}_p, \mathbf{R}_p] \subset \mathbf{U}_p$, the group $\Omega_p \mathbf{U}_p$ is normal in $\Omega_p \mathbf{S}_p^2 \mathbf{R}_p$ and the group G' is normal in G .

To conclude we just apply the following fact with $H := \mathrm{Ad}_{\mathfrak{g}_p}(G')$.

Let $H = \Omega U$ be a non-archimedean linear p -adic Lie group which is generated by a compact subgroup Ω and a normal unipotent subgroup U , then there exists an increasing sequence $(H_i)_{i \geq 1}$ of compact open subgroups of H whose union is equal to H .

We now check this fact. We first notice that this fact is true for the groups U_d of p -adic upper triangular unipotent $d \times d$ matrices. Since any unipotent group U is isomorphic to a closed subgroup of U_d , for some $d \geq 1$, this fact is also true for the group U . Hence, any subgroup of U generated by two compact open subgroups is still an open compact subgroup. Therefore any compact subgroup of U is included in a compact subgroup invariant by conjugation by Ω . Our claim follows. \square

Proof of Proposition 6.1. Let $\mathfrak{g} = \mathfrak{g}_f \oplus \mathfrak{g}_\infty$ and $\mathfrak{s}_0 = \mathfrak{s}_{0,f} \oplus \mathfrak{s}_{0,\infty}$ be the decompositions of \mathfrak{g} and \mathfrak{s}_0 as a sum of a non-archimedean ideal and a real one and let G_∞ be the connected immersed subgroup of G with Lie algebra \mathfrak{g}_∞ . We first begin by a special case.

1st case : The group $\mathrm{Ad}_{\mathfrak{g}_f}(G)$ is compact.

Let $D := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{g}})$ be the Kernel of the adjoint action. Since the Lie algebras \mathfrak{g}_f and \mathfrak{g}_∞ commute, there exists a compact subgroup G_f of G with Lie algebra \mathfrak{g}_f which commutes with G_∞ . Since $\mathrm{Ad}_{\mathfrak{g}_f}(G)$ is compact, the group $G_\infty G_f D$ is a finite index open subgroup of G . Hence we may assume that

$$G = G_\infty G_f D.$$

For any compactly generated subgroup D_1 of D , the centralizer of D_1 in G_f has finite index in G_f . In particular, D is the union of its G_f -invariant compactly generated subgroups D_1 .

We want to prove that the group $L := \overline{\Lambda R_0}$ has an amenable Lie algebra.

We proceed by contradiction. Assume this is not the case. Since every dense subgroup of a real connected Lie group contains a finitely generated dense subgroup, there would exist a finitely generated subgroup Λ_0 of Λ such that the group $\overline{\Lambda_0 R_0}$ has a non-amenable Lie algebra. Let D_1 be an open compactly generated G_f -invariant subgroup of D such that the group $G_1 := G_\infty G_f D_1$ contains Λ_0 . By Lemma 3.2, the intersection $\Lambda_1 := \Lambda \cap G_1$ is a lattice in G_1 .

It is enough to prove our claim for (G_1, Λ_1) . Hence we can assume D to be compactly generated. But then, after replacing G by an open finite index subgroup, we can assume that G_f and D commute. The quotient group $G' := G/G_f$ is then a real Lie group and the image Λ' of Λ in G' is a lattice. Since one has $G_f R_0 = \text{Ker}(\text{Ad}_{\mathfrak{g}_{0,\infty}})$, according to Lemma 6.4 applied to (G', Λ') , the group $\Lambda G_f R_0$ is closed. Hence the group $L := \overline{\Lambda R_0}$ has an amenable Lie algebra, whence a contradiction.

2nd case : General case.

Let K be a compact open subgroup of $\text{Aut}(\mathfrak{g}_{0,f})$. Since the group

$$(6.2) \quad G_K := \{g \in G \mid \text{Ad}_{\mathfrak{g}_{0,f}}(g) \in K\}$$

is an open subgroup of G , $\Lambda \cap G_K$ is a lattice in G_K by Lemma 3.2. Since G_K contains R_0 , the groups $\overline{\Lambda R_0}$ and $(\Lambda \cap G_K) R_0$ both have Lie algebra \mathfrak{l} . Hence we may assume that $G = G_K$.

According to equality (6.2) and the definition of \mathfrak{g}_0 , the adjoint group $\text{Ad}_{\mathfrak{g}_f}(G)$ is compact. Hence we can apply Lemma 6.5 to the group $G/\overline{G_\infty}$: there exists an increasing sequence

$$G_1 \subset \cdots \subset G_i \subset \cdots \subset G$$

of open subgroups of G containing G_∞ whose union $G' := \cup_i G_i$ is normal in G and such that, for $i \geq 1$, the group $\text{Ad}_{\mathfrak{g}_f}(G_i)$ is compact.

Again by Lemma 3.2, for all $i \geq 1$, the group $\Lambda_i := \Lambda \cap G_i$ is a lattice in G_i . We denote by

$$\Lambda_{\mathfrak{g}_0,i} := \text{Ad}_{\mathfrak{g}_{0,\infty}}(\Lambda_i)$$

its image in the group $\text{Aut}(\mathfrak{g}_{0,\infty})$ and we set

$$R_{0,\infty} := \text{Ker}(\text{Ad}_{\mathfrak{g}_{0,\infty}}).$$

According to the first case applied to (G_i, Λ_i) , the group $\overline{\Lambda_i R_{0,\infty}}/R_{0,\infty}$ has an amenable Lie algebra. Since the group $\text{Aut}(\mathfrak{g}_{0,\infty})$ is semisimple with no compact factor, by the Borel density theorem (Corollary 4.7), this Lie algebra is an ideal of $\mathfrak{g}_{0,\infty}$, hence it is trivial. Therefore, $\Lambda_i R_{0,\infty}$ is closed and, by Lemma 3.1, the group $\Lambda_{\mathfrak{g}_0,i}$ is a lattice in $\text{Aut}(\mathfrak{g}_{0,\infty})$. Since any increasing sequence of lattices in a semisimple real Lie group

is stationary (see [10]), there exists $i_0 \geq 1$ such that $\Lambda_{\mathfrak{s}_0, i} = \Lambda_{\mathfrak{s}_0, i_0}$ for all $i \geq i_0$.

We set $\Lambda_{\mathfrak{s}_0}$ and $\Lambda'_{\mathfrak{s}_0}$ for the images of Λ and $\Lambda' := \Lambda \cap G'$ in $\text{Aut}(\mathfrak{s}_{0, \infty})$. We have just proven that $\Lambda'_{\mathfrak{s}_0} = \Lambda_{\mathfrak{s}_0, i_0}$ is a lattice in $\text{Aut}(\mathfrak{s}_{0, \infty})$. Since $\Lambda_{\mathfrak{s}_0}$ normalizes $\Lambda'_{\mathfrak{s}_0}$, we obtain, by Corollary 4.8, that $\Lambda_{\mathfrak{s}_0}$ is also a lattice in $\text{Aut}(\mathfrak{s}_{0, \infty})$. Hence the group $\Lambda R_{0, \infty}$ is closed in G . This proves that the Lie algebra \mathfrak{l} is amenable. \square

Let us now state the main result of this paper.

Theorem 6.6. *Let G be an S -adic Lie group and Λ be a lattice in G . Then, there exists a G -invariant ideal \mathfrak{r} of \mathfrak{g} with the following properties. Let $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ and R be the kernel of the adjoint map $\text{Ad}_{\mathfrak{s}} : G \rightarrow \text{Aut}(\mathfrak{s})$.*

- (i) *$\text{Aut}(\mathfrak{s})$ is a semisimple S -adic Lie group with no compact factor.*
- (ii) *The group $\text{Ad}_{\mathfrak{s}}(G)$ is a finite index subgroup in $\text{Aut}(\mathfrak{s})$.*
- (iii) *The group $\Lambda_{\mathfrak{s}} := \text{Ad}_{\mathfrak{s}}(\Lambda)$ is a lattice in $\text{Aut}(\mathfrak{s})$.*
- (iv) *The intersection $\Lambda \cap R$ is a cocompact lattice in R .*

See [8, Thm 9.5] where a related projection theorem is proven for a general locally compact group G and a normal amenable subgroup R . We note that in our Theorem 6.6, the group R may not be amenable (see Example 6.2).

Proof. Let $L = \overline{\Lambda R_0}$ be as in Proposition 6.1 and let $\mathfrak{r} = \mathfrak{l}$ be the Lie algebra of L . Note that, by [6, 6.14 and 9.10], for any simple real or p -adic Lie algebra \mathfrak{h} , any non-compact open subgroup of $\text{Aut}(\mathfrak{h})$ has finite index. Hence, G/R_0 is a finite index subgroup of $\text{Aut}(\mathfrak{s}_0)$. Besides, by Lemma 3.1, L is a finite covolume closed subgroup of G . Therefore, by Corollary 4.7, the Lie algebra \mathfrak{r} is an ideal of \mathfrak{g} . In particular, the quotient Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semisimple. We thus get (i) and (ii).

Let R be the kernel of the map $\text{Ad}_{\mathfrak{s}}$ in G and let us prove that $L \cap R$ has finite index in R . As R and $L \cap R$ have Lie algebra \mathfrak{r} , through the map $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(G)$, R/R_0 and $L \cap R/R_0$ identify with open subgroups of $\text{Aut}(\mathfrak{r}/\mathfrak{r}_0)$ which are normalized by $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(L)$. Now, by assumption, the group $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(G)$ is a finite index open subgroup of $\text{Aut}(\mathfrak{r}/\mathfrak{r}_0)$ and, since L is a finite covolume subgroup of G and since the Lie algebra of L is \mathfrak{r} , the group $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(L)$ is a finite index open subgroup of $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(G)$. Thus, $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(R)$ and $\text{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(L \cap R)$ are also finite index open subgroup of $\text{Aut}(\mathfrak{r}/\mathfrak{r}_0)$ and $L \cap R$ has finite index in R .

In particular, the group LR is closed and has Lie algebra \mathfrak{r} . Now, as $R \subset \overline{\Lambda R} \subset LR$, $\overline{\Lambda R}$ also has Lie algebra R , that is ΛR is closed. By Lemma 3.1, ΛR has finite covolume in G/R and $\Lambda \cap R$ has finite covolume in R . By Proposition 6.1, the Lie algebra of R being amenable,

$\Lambda \cap R$ is cocompact in R . Thus, G/R being a finite index open subgroup of $\text{Aut}(\mathfrak{s})$, we get (iii) and (iv). \square

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