ADDITIVE REPRESENTATIONS OF TREE LATTICES 1. QUADRATIC FIELDS AND DUAL KERNELS

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ABSTRACT. Given a discrete cofinite group of isometries Γ of a locally finite tree X, we study certain Γ-invariant quadratic forms on distribution spaces on the boundary ∂X of X which are defined by singular integrals. Their kernels are constructed from certain cohomology classes of functions on the space of parametrized geodesic lines of $\Gamma \backslash X$, equipped with the time shift dynamics. We develop a structure theory for these quadratic forms when they are non-negative.

CONTENTS

1. Introduction

1.1. **Motivations.** This article is the first of a series of papers which aim at studying certain phenomena in the representation theory and harmonic analysis on non-abelian countable groups. This study is motivated by questions from homogeneous dynamics and geometric probability theory. In homogeneous dynamics, one studies actions of subgroups of a Lie group G on the homogeneous spaces of G. In geometric probability theory, one studies random walks on the homogeneous spaces of G defined by Borel probability measures on G.

In both fields, numerous striking equidistribution results were obtained in the last decades. The question of the speed of those equidistribution results is still open in many cases. Often, understanding this speed amounts to proving a spectral bound for a certain linear operator acting on a Banach space.

Inspired in particular by the work of Bourgain [7] on absolute continuity of stationary measures, and by his own joint contribution with Benoist to the subject [6], the author was lead to believe that it might be possible to describe part of the spectral structure of operators P defined as follows: let G be a semisimple Lie group and Δ be a finitely generated (Zariski dense) subgroup of G. Choose an irreducible unitary representation of G on a Hilbert space H and let $\rho : \Delta \to \mathrm{U}(H)$ be its restriction to Δ. Let S be a finite set which generates Δ and P be the self-adjoint operator of H defined by

$$P = \frac{1}{2|S|} \sum_{g \in S} \rho(g) + \rho(g^{-1}).$$

We will not solve the question of describing the spectral invariants of P in this article, but we will start to build a structure theory for certain unitary representations of the abstract free group generated by S which share some analogy with ρ.

1.2. **Special representation of** $\mathrm{SL}_2(\mathbb{R})$**.** To motivate the introduction of this theory, let us focus on the case where $G = \mathrm{SL}_2(\mathbb{R})$. One can define a unitary representation of G in the following way. Let H_0 be the space of all distributions T in the Sobolev space $H = \mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$ such that $\langle T, \mathbf{1} \rangle = 0$, where $\mathbf{1}$ is the constant function with value 1. The group G acts on H and H_0 in a natural way. Let us construct an invariant scalar product for this action. For $\xi \neq \eta$ in $\mathbb{P}^1_{\mathbb{R}}$, and p in the hyperbolic plane $\mathbb{H}$, let $(\xi|\eta)_p$ denote the Gromov product of ξ and η viewed from p. Equivalently, if $\xi = \mathbb{R}v$ and $\eta = \mathbb{R}w$ for some non-zero vectors v and w in $\mathbb{R}^2$, we set

$$(\xi|\eta)_p = -\frac{1}{4} \log \left(\frac{\|v \wedge w\|_p}{\|v\|_p \|w\|_p} \right),$$

where $\|.\|_p$ stands for the Euclidean norms associated to p on $\mathbb{R}^2$ and $\wedge^2\mathbb{R}^2$. Then the symmetric bilinear form defined formally by

$$\Phi_p : (\varphi, \theta) \mapsto \int_{\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}} (\xi|\eta)_p \mathrm{d}\varphi(\xi) \mathrm{d}\theta(\eta)$$

is bounded on $\mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$. It is positive definite and defines the usual topology of $\mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$. Now the restriction of Φ_p to H_0 does not depend on p. This follows from the following additive property of the Gromov product

$$(1.1)\quad (\xi|\eta)_p - (\xi|\eta)_q = \frac{1}{2}(b_\xi(p,q) + b_\eta(p,q)) \quad p,q \in \mathbb{H} \quad \xi \neq \eta \in \mathbb{P}^1_{\mathbb{R}},$$

where $b : \mathbb{H} \times \mathbb{H} \times \mathbb{P}^1_{\mathbb{R}} \to \mathbb{R}$ is the Busemann cocycle which can be defined in this instance by

$$b_\xi(p,q) = \frac{1}{2}\log\left(\frac{\|v\|_p}{\|v\|_q}\right),$$

for p, q in $\mathbb{H}$ and ξ the vector line spanned by the non-zero vector v in $\mathbb{R}^2$.

This representation is irreducible if one considers distributions with coefficients in $\mathbb{R}$; if one considers distributions with coefficients in $\mathbb{C}$, it is the sum of two irreducible components. In this case, these irreducible components are discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ which are complex conjugate to each other (see [26]).

Moreover, the very definition of the space of this representation as a hyperplane in $H = \mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$ says that the representation H_0 appears in an exact sequence

$$0 \to H_0 \to H \to \mathbb{C} \to 0,$$

where H is a representation of $\mathrm{SL}_2(\mathbb{R})$ by bounded automorphisms which is not uniformly bounded. This defines a natural non trivial 1-cohomology class of H_0.

Bargmann's classification of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ is given in [26]. One can check that they all can be constructed from the representation in H_0 by abstract algebraic operations, such as taking tensor products (as in the work of Repka [31]) or exponentials defined by the natural 1-cohomology class (this latter procedure is described in Section 2.11 of the book of Bekka, de la Harpe and Valette [4]).

To summarize, the additive property (1.1) allows to define an invariant symmetric bilinear form on certain function spaces on the boundary of $\mathbb{H}$. This bilinear form is positive definite and most of the representation theory of $\mathrm{SL}_2(\mathbb{R})$ can be recovered from the unitary representation defined by this data. For this reason, and by analogy with the case of automorphisms of trees explained below, let us call this unitary representation the special representation of $\mathrm{SL}_2(\mathbb{R})$.

1.3. **Special representations of automorphisms of trees.** We will now recall an analogue construction for trees. Let X be a regular tree with valence $d \geq 3$, that is, every vertex x in X has exactly d neighbours. Later, X will be the tree associated to a free group. The group G of automorphisms of X comes with a natural locally compact topology. The study of the unitary irreducible representations of G was initiated by Cartier [12] and a full classification of them was given by Olshanski [29]. This classification is described in detail in the book of Figà-Talamanca and Nebbia [15], and it shares some deep analogy with the representation theory of $\mathrm{SL}_2(\mathbb{R})$.

In particular, let ∂X be the boundary of X, $\partial^2 X$ be the set of pairs of different points in ∂X and $\omega : X \times \partial^2 X \to \mathbb{Z}$ be the Gromov product. Then, there is a certain Hilbert space H^ω of distributions on ∂X whose scalar product is formally defined as the symmetric bilinear form

$$(\varphi, \theta) \mapsto \int_{\partial X \times \partial X} \omega_x(\xi, \eta) \mathrm{d}\varphi(\xi) \mathrm{d}\theta(\eta),$$

where x is a fixed vertex of X. Again, if we set H_0^ω to be the space of those T in H^ω with $\langle T, \mathbf{1} \rangle = 0$, the restriction of this scalar product to H_0^ω does not depend on x, which is due to the relation

(1.2)

$$\omega_x(\xi, \eta) - \omega_y(\xi, \eta) = \frac{1}{2}(b_\xi(x, y) + b_\eta(x, y)) \quad x, y \in X \quad \xi \neq \eta \in \partial X,$$

where $b : X \times X \times \partial X \to \mathbb{Z}$ again is the Busemann cocycle. The construction of the Hilbert spaces H^ω and H_0^ω is recalled in Subsection 3.1.

1.4. **Pull-back of the special representation.** We will now show how certain unitary representations ρ as in Section 1.1 can be defined directly by looking at functions on a tree.

Let Δ and S be as in Subsection 1.1 and let Γ be the abstract free group generated by S, so that Δ may be seen as the image of Γ under a homomorphism θ. The data of the system of generators determines a transitive action of Γ on a d-regular tree X, where $d = 2|S|$. Fix x in X and equip the boundary ∂X of X with the harmonic measure ν_x associated to x. Boundary theory gives a Γ-equivariant measurable map $F : \partial X \to \mathbb{P}^1_{\mathbb{R}}$ which is defined ν_x-almost everywhere. It follows from the construction of this map by means of probability theory (see [5, Sec. 9, 13]) that the $\nu_x \otimes \nu_x$-almost everywhere defined function

$$\Omega_x : (\xi, \eta) \mapsto (F(\xi)|F(\eta))_p$$

is $\nu_x \otimes \nu_x$-integrable. Thus, it defines a symmetric bilinear form Ψ_x on the space $\mathfrak{M}^\infty(\partial X, \nu_x)$ of Borel signed measures on ∂X which are absolutely continuous with respect to ν_x with a bounded Radon derivative. As the bilinear form Φ_p of Subsection 1.2 is non-negative, so is

Ψ_x. Now, from (1.1), one can get a relation of the form

(1.3)
$$\Omega_x(\xi,\eta)-\Omega_y(\xi,\eta)=\frac{1}{2}(B_\xi(x,y)+B_\eta(x,y))\quad x,y\in X\quad \xi\neq\eta\in\partial X,$$

which implies again that the restriction of Ψ_x to the space $\mathfrak{M}_0^\infty(\partial X,\nu_x)$ of signed measures ρ with $\rho(\mathbf{1})=0$ is Γ-invariant. The completion of this space with respect to this Γ-invariant non-negative bilinear form defines a representation of Γ which is a subrepresentation of $\rho\circ\theta$. Thus, at least part of the representation ρ may be seen as being obtained from the data of the function Ω_x and the additive relation (1.1).

1.5. **Objectives of the article.** Functions Ω which are Γ-invariant and satisfy a relation as (1.3) will be called additive kernels in this article. They are rather common in negatively curved geometry (see Ledrappier's survey paper [27], whose results are adapted to our framework in Section 2). What remains mysterious (at least for the author) is the fact that, as in the above construction, the bilinear forms defined by such functions may turn out to be non-negative (and hence to define unitary representations of Γ).

In Subsection 1.4, the additive kernel Ω is only measurable.

In Subsection 1.3, the additive kernel ω is smooth, that is, locally constant on ∂^2X. One can show that the representation of Γ on H_0^ω may be embedded in a finite product of copies of the regular representation of Γ. In particular, if $r=|S|$, the spectrum of the operator $P=\frac{1}{2r}\sum_{s\in S}(s+s^{-1})$ in H_0^ω was computed by Kesten [24]: this is the interval $[-\frac{1}{2r}\sqrt{2r-1},\frac{1}{2r}\sqrt{2r-1}]$.

We plan to get a better understanding of the spectrum of the operator P in the measurable case by a careful study of the smooth case.

Therefore, the purpose of this first paper is to study the set of *smooth* Γ-invariant additive kernels $\Omega: X\times\partial^2X\to\mathbb{R}$ such that the associated bilinear forms are non-negative (a first step being to give a precise definition of this non-negativity phenomenon). We will give a description of all such smooth additive kernels as the images under linear maps of certain explicit cones in some finite-dimensional vector spaces. The construction of these vector spaces and of these linear maps will occupy most of the article. It turns out that these objects possess a rather rich structure theory, which we will develop in the slightly more general framework of discrete groups acting on trees with a finite quotient.

Our later objective is to use this structure theory in order to build approximation schemes of measurable non-negative additive kernels by smooth ones, along which schemes part of the spectral properties of the operator P are preserved.

1.6. **Related works.** The group $\mathrm{SL}_2(\mathbb{R})$ or the group of automorphisms of a regular tree are type I groups. In other words, the space of all their irreducible unitary representations may be parametrized by a standard Borel space. Such a parametrization can not be obtained for discrete groups with infinite conjugacy classes. These notions and facts are explained in [3, Chap. 6, 7]. In other words, there is no hope of getting a classification of the irreducible unitary representations of Γ. Nevertheless one can try and describe special examples of such representations.

As far as the author knows, the study of unitary representations of discrete groups acting on negatively curved spaces can be traced back to the work of of Figà-Talamanca and Picardello [16] who proved that the restriction to a free group of a spherical irreducible representation of the group of automorphisms of its tree stays irreducible. For Lie groups, Cowling and Steger [14] determined under which condition the restriction to a lattice of a unitary irreducible representation of a semisimple Lie group remains irreducible.

The construction, from the geometry of its tree, of unitary representations of a free group that are not necessarily representations of the full group of automorphisms was initiated by Kuhn and Steger [25]. This work was recently pursued by Iozzi, Kuhn and Steger [22]. The

A main development of the field was the proof of the irreducibility of the quasi-regular representation associated with the Patterson-Sullivan measure of the fundamental group of a compact negatively curved manifold by Bader and Muchnik [1]. This result was extended to groups of isometries of CAT(-1)-spaces by Boyer [8] and then to a wider class of quasi-regular representations associated to Gibbs measures by Boyer and Garncarek [9]. In this latter work, there appears a strong relation between the unitary representation theory of the fundamental group and the thermodynamic formalism of the geodesic flow. This connection also exists in the present paper.

We notice that the representations that are studied by these authors may be seen as analogues or deformations of the principal series representations of $\mathrm{SL}_2(\mathbb{R})$ or of the group of isometries of a regular tree. In particular, the Hilbert spaces on which they are defined are easily constructed: they are the L^2 spaces associated with a certain quasi-invariant measure on the boundary of the group. Our point of view is different in as much as the representations that we build are analogues of the special representations mentioned in Subsections 1.2 and 1.3. In particular, the definition of the associated Hilbert spaces is more intricate. One could say that we study the additive representation theory of Γ, whereas the above mentioned authors study the multiplicative representation theory of Γ. Both theories are related through the exponentiation process associated to a certain 1-cohomology class in the additive representations (see again [4, Sec. 2.11]). The precise study of this relation will also be the subject of a later work, as will be the use of this exponentiation process to build generalized complementary series.

1.7. **Structure of the paper.** We now give a sketch of the contents of the different parts of the paper.

Section 2 introduces precisely the language of trees X equipped with a cofinite action of a group Γ and the one of smooth additive kernels, that are locally constant Γ-invariant functions Ω satisfying (1.2). This is mostly a translation of the material in [27] from the language of Hadamard manifolds. We show in particular how smooth additive kernels are defined by (cohomology classes of) Γ-invariant symmetric functions w on a space $X_k = \{(x,y) \in X \times X | d(x,y) = k\}$ for some $k \geq 1$. Note that [27] was already extended to trees in [10] for Hölder continuous objects.

Section 3 introduces the space H^ω of distributions on the boundary ∂X which we mentioned in Subsection 1.3 and which is the analogue of the Sobolev space $\mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$ from Subsection 1.2. In case the tree is regular, the space H^ω_0 of distributions in H^ω which kill the constant functions is the skew-symmetric special representation of the group of automorphisms of X which is studied in [15]. We prove that the bilinear form Φ_w defined formally by the additive kernel associated to the function w is bounded on H^ω. Its restriction to H^ω_0 is Γ-invariant. In the sequel of the article, we will give an alternate construction of this bilinear form.

In Section 4, we study bilinear forms on the space

$$\overline{\mathcal{D}}(\partial X) = \mathcal{D}(\partial X)/\mathbb{R}$$

which is the quotient of the space $\mathcal{D}(\partial X)$ of locally constant functions on ∂X by the line of constant functions. We show how these bilinear forms may be defined in terms of certain functions called quadratic type functions on $X_* = \{(x,y) \in X \times X | x \neq y\}$. We introduce quadratic fields, which are one of the main objects of study of this article. More precisely, for any integer $k \geq 2$, there is a notion of a k-quadratic field. When k is even, $k = 2\ell$, a quadratic field p is the data, for any x in X, of a symmetric bilinear form p_x on the space of functions on the sphere $S^\ell(x)$ with radius ℓ and center x, with $\mathbf{1}$ in the null space of p_x, and with a compatibility relation between p_x and p_y for $x \sim y$. When the associated bilinear forms are positive definite (modulo the constant functions), a k-quadratic field is called a k-Euclidean field. From a k-Euclidean field, one can build a $(k+1)$-Euclidean field by a process called orthogonal extension, and, by induction, one eventually gets a symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. The set of all Γ-invariant k-Euclidean fields is denoted by $\mathcal{P}_k$. It is an open subset of the finite-dimensional vector space of Γ-invariant k-quadratic fields.

In Section 5, we define a dual notion to the one of a k-quadratic field, namely the one of a k-dual kernel. The finite-dimensional vector space of all Γ-invariant k-dual kernels is denoted by $\mathcal{K}_k$ and there is a natural linear embedding $\mathcal{K}_k \to \mathcal{K}_{k+1}$ which is also called orthogonal extension. Euclidean fields may be embedded into dual kernels and orthogonal extension of Euclidean fields is the same as orthogonal extension of the associated dual kernels. Still, the definition of dual kernels and their orthogonal extensions is rather intricate, and we hope our choice of order for the exposition will help the reader to get a better understanding of the objects. We define a closed convex cone of k-dual kernels in $\mathcal{K}_k$ which are called non-negative k-dual kernels. A Euclidean field, when viewed as a dual kernel, is non-negative. To each such non-negative k-dual kernel, we can associate a Γ-invariant space of distributions equipped with a Γ-invariant non-negative symmetric bilinear form. Our goal now will be to show that these bilinear forms may be defined by means of an additive kernel.

The purpose of the technical Section 6 is to define this additive kernel. More precisely, we build there a linear map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ from the space of Γ-invariant k-dual kernels towards the space of cohomology classes of Γ-invariant symmetric functions on X_k. This map W_k is called the weight map.

In Section 7, we draw the connection between the language of additive kernels and the one of dual kernels. Indeed, we prove that the Hilbert space associated to a non-negative dual kernel always contains the above constructed space H_0^ω (up to a quotient) and that the bilinear form induced by the dual kernel on H_0^ω is of the form Φ_w, where w is a function defined from the dual kernel through the weight map. Conversely to every function w such that Φ_w is non-negative on H_0^ω, we can associate a non-negative dual kernel which is called the image kernel of w. In the rest of the paper, we will study the set of all image dual kernels.

As a preliminary, in Section 8, we prove that the weight map is surjective and we describe its null space[1]. This leads to the introduction

[1]There is a problem with terminology here. In linear algebra, the kernel of a linear map is the space where it cancels. In functional analysis, the kernel of a

of a new family of objects which are called k-pseudokernels, where $k \geq 1$ is an integer. The vector space of all Γ-invariant k-pseudokernels is denoted by $\mathcal{L}_k$. When $k \geq 2$, there is a natural embedding of $\mathcal{L}_{k-1}$ into the space $\mathcal{K}_k$ of Γ-invariant k-dual kernels whose range is exactly the null space of the weight map.

In Section 9, we give a geometric description of the set of image k-dual kernels as a subset of the finite-dimensional space $\mathcal{K}_k$.

In the rest of the paper, we will study the image dual kernels coming from bilinear forms Φ_w which are not only non-negative but coercive, that is, which define the topology of H_0^ω. These kernels are actually associated to certain Euclidean fields which are called admissible Euclidean fields. In Section 10, we give a criterion for a Euclidean field to be admissible. This criterion involves a natural linear operator associated to a k-Euclidean field and acting on the space of $(k-1)$-pseudokernels, which we call the transfer operator by analogy with the theory of hyperbolic dynamical systems.

In Section 11, we build a natural Riemannian structure on the space $\mathcal{P}_k^{\mathrm{ad}}$ of all admissible Γ-invariant k-Euclidean fields. It is an analogue of the Riemannian structure on the space of all scalar products of a finite-dimensional vector space (see [21]). The orthogonal extension map injects $\mathcal{P}_k^{\mathrm{ad}}$ smoothly into $\mathcal{P}_{k+1}^{\mathrm{ad}}$ and the Riemannian structure of $\mathcal{P}_k^{\mathrm{ad}}$ is the pull-back of the one of $\mathcal{P}_{k+1}^{\mathrm{ad}}$.

The building of these Euclidean norms on spaces of Euclidean fields is a first step towards building approximation schemes of non smooth additive kernels by smooth ones.

1.8. **Miscellaneous notation.** When speaking of a function, we shall always mean a function with values in $\mathbb{R}$. All vector spaces considered in this paper and in particular all Hilbert spaces will be defined over $\mathbb{R}$.

If V is a vector space, we shall denote its algebraic dual space by V^*. If W is another vector space and $\theta : V \to W$ is a linear map, we write $\theta^* : W^* \to V^*$ for the adjoint linear map. If V (resp. W) is equipped with a scalar product p (resp. q), we write $\theta^{\dagger pq} : W \to V$ for the adjoint linear map of T with respect to these Euclidean structures. When the choices of p and q are clear from the context, we simply write $\theta^\dagger$ for $\theta^{\dagger pq}$. The null space of θ is denoted by $\ker \theta$.

The space of all symmetric bilinear forms on V is denoted by $\mathcal{Q}(V)$. The space of non-negative (resp. positive definite) forms is denoted

bilinear form on a space of functions is a function with two variables. To avoid confusions, we will only use the word kernel with the latter meaning and speak of the null space of a linear map.

by $\mathcal{Q}_+(V)$ (resp. $\mathcal{Q}_{++}(V)$). If q is a symmetric bilinear form on W, then $\theta^\star q$ stands for the pull-back of q under θ, that is, the bilinear form $(v,w) \mapsto q(Tv,Tw)$ on V. Thus, θ^* and $\theta^\star$ are both pull-back maps, but they don't act on the same spaces. Unfortunately, at some point, we will need to write $(\theta^*)^\star$.

If U is a totally discontinuous locally compact topological space, we say that a function φ on U is smooth if it is locally constant. The space of all compactly supported smooth functions on U will be denoted by $\mathcal{D}(U)$. A distribution on U is a linear functional on this space. The space of all distributions on U is therefore the algebraic dual space of $\mathcal{D}(U)$. We denote it by $\mathcal{D}^*(U)$. This notion of a distribution and its use in the representation theory of groups acting on totally discontinuous spaces can be traced back to [11].

If φ is in $\mathcal{D}(U)$ and T is in $\mathcal{D}^*(U)$, we write $\langle T, \varphi\rangle$ for the evaluation of T on φ. We write φT for the distribution $\psi \mapsto \langle T, \varphi\psi\rangle$.

If U is compact, we set $\overline{\mathcal{D}}(U) = \mathcal{D}(U)/\mathbb{R}$ to be the quotient of the space of smooth functions by the line of constant functions. Its dual space can be identified with the space of distributions which kill the function $\mathbf{1}$. It is denoted by $\mathcal{D}_0^*(U)$.

If V is another totally discontinuous locally compact topological space, for φ in $\mathcal{D}(U)$ and ψ in $\mathcal{D}(V)$, we write $\varphi \otimes \psi$ for the function $(u,v) \mapsto \varphi(u)\psi(v)$ on $U \times V$, which belongs to $\mathcal{D}(U \times V)$. This identifies $\mathcal{D}(U \times V)$ with the algebraic tensor product $\mathcal{D}(U) \otimes \mathcal{D}(V)$. In particular, if ρ is a distribution in $\mathcal{D}^*(U)$ and θ is a distribution in $\mathcal{D}^*(V)$, we define a distribution $\rho \otimes \theta$ in $\mathcal{D}^*(U \times V)$ by setting

$$\langle \rho \otimes \theta, \varphi \otimes \psi\rangle = \langle \rho, \varphi\rangle\langle \theta, \psi\rangle, \quad \varphi \in \mathcal{D}(U), \quad \psi \in \mathcal{D}(V).$$

The characteristic function of a subset V in a set U is denoted by $\mathbf{1}_V^U$ or more simply by $\mathbf{1}_V$ when there is no ambiguity.

Let Γ be a group acting on a set X and $S \subset X$ be a system of representatives of the elements of $\Gamma\backslash X$. If φ is a Γ-invariant function on X which is summable on S, we write $\sum_{x \in \Gamma\backslash X} \varphi(x)$ for $\sum_{x \in S} \varphi(x)$.

2. Tree lattices and additive kernels

2.1. **Trees.** In all the article, the letter X will stand for a locally finite tree. We start by giving a precise definition of the version of this notion that we will use.

We let X be countable set equipped with a symmetric relation $\sim$ such that for any x in X, the set of neighbours of x, that is, the set $S^1(x)$ of y in X with $x \sim y$, is finite (X is locally finite) and does not contain x. We let $d(x)$ denote its number of elements. To avoid technicalities, we assume that $d(x) \geq 3$ for any x in X.

We assume that $(X, \sim)$ is connected, that is, for every x, y in X there exists a sequence $z_0 = x, z_1, \ldots, z_n = y$ of elements of X such that $z_{h-1} \sim z_h$ for $1 \leq h \leq n$ and $z_{h-1} \neq z_{h+1}$ for $1 \leq h \leq n-1$. Such a sequence will be called a geodesic path from x to y. The integer n is called the length of the path.

We assume that $(X, \sim)$ is simply connected, that is, for every x, y in X there exists a unique geodesic path $z_0 = x, z_1, \ldots, z_n = y$ from x to y. The set $\{z_0, \ldots, z_n\}$ is called the geodesic segment between x and y and is denoted by $[xy]$. The length of this path is called the distance between x and y and denoted by $d(x, y)$. For any $n \geq 0$, we write $S^n(x)$ for the sphere with radius n and center x for this distance.

A sequence $(x_h)_{h \geq 0}$ of elements of X is called a geodesic ray if, for all $H \geq 0$, the sequence $(x_h)_{0 \leq h \leq H}$ is a geodesic path. The element x_0 is called the origin of the geodesic ray.

Two geodesic rays $(x_h)_{h \geq 0}$ and $(y_h)_{h \geq 0}$ are said to be equivalent if there exists a relative integer $k \in \mathbb{Z}$ with $x_{h+k} = y_h$ for any large enough h. This is an equivalence relation among geodesic rays and the set of equivalence classes is called the boundary of X and is denoted by ∂X. For any x in X and ξ in ∂X, there exists a unique geodesic ray $(x_h)_{h \geq 0}$ with origin x in the equivalence class defined by ξ. Note that the parametrization of the set $\{x_h | h \geq 0\}$ which makes it into a geodesic ray is unique. By abuse of notations, we shall identify the geodesic ray $(x_h)_{h \geq 0}$ and the set $\{x_h | h \geq 0\}$ and denote both of them by $[x\xi)$. The elements ξ is called the endpoint of the ray $[x\xi)$.

Fix x in X. The set of geodesic rays with origin x embeds naturally as a subset of the product set $\prod_{h \geq 0} S^h(x)$, which is closed for the product topology of the discrete topologies on the spheres. We equip this set with the induced topology which is compact. The image of this topology on ∂X does not depend on x. We shall henceforward equip ∂X with this topology. It is compact and totally discontinuous.

Let $\partial^2 X$ denote the set of pairs of different points in ∂X.

A sequence $(x_h)_{h \in \mathbb{Z}}$ of elements of X is called a parametrized geodesic line if, for all $H \geq 0$, the sequence $(x_h)_{|h| \leq H}$ is a geodesic path. Let $\mathscr{S}$ be the set of all parametrized geodesic lines of X.

Let $s = (x_h)_{h \in \mathbb{Z}}$ be in $\mathscr{S}$. The point x_0 is called the base point of s and denoted by $\pi(s)$. The sequence $(x_{h+1})_{h \in \mathbb{Z}}$ (resp. $(x_{-h})_{h \in \mathbb{Z}}$) is again a parametrized geodesic line. It is denoted by Ts (resp. ιs). The maps $T : \mathscr{S} \to \mathscr{S}$ and $\iota : \mathscr{S} \to \mathscr{S}$ are called the time shift and the time reversal.

If $s = (x_h)_{h\in\mathbb{Z}}$ is in $\mathscr{S}$, the endpoints ξ and η of the geodesic rays $(x_{-h})_{h\geq 0}$ and $(x_h)_{h\geq 0}$ are different. They are respectively called the origin and endpoint of s and denoted by s_- and s_+.

Conversely, given $\xi \neq \eta$ in $\partial^2 X$, there exists a parametrized geodesic line $(x_h)_{h\in\mathbb{Z}}$ with origin ξ and endpoint η. The set $\{x_h | h \in \mathbb{Z}\}$ only depends on ξ and η and is denoted by $(\xi\eta)$. It is called the geodesic line between ξ and η. The parametrizations of this geodesic line are unique up to time shift and time reversal.

The map $\mathscr{S} \to X \times \partial^2 X, s \mapsto (\pi(s), s_-, s_+)$ is injective and its range is a closed subset of $X \times \partial^2 X$ (where X is equipped with the discrete topology). We equip $\mathscr{S}$ with the topology induced by this injection. This makes $\mathscr{S}$ a locally compact totally discontinuous space and the maps T and ι homeomorphisms of $\mathscr{S}$.

An automorphism of X is a map $g : X \to X$ such that, for any x, y in X, one has $gx \sim gy$ if and only if $x \sim y$. A group Γ of automorphisms of X is said to be discrete if it acts properly on X. It is said to be cofinite if the quotient $\Gamma\backslash X$ is finite. A cofinite lattice of X is a discrete cofinite group of automorphisms of X. In the sequel we fix a cofinite lattice Γ.

Below are two examples of such a tree lattice that the reader may keep in mind along the article.

Example 2.1. Let A be a finite set with at least three elements and set $d = |A|$ to be the cardinality of A. We assume that X is d-regular (that is, $d(x) = d$ for every x in X). We fix a map $w : X_1 \to A$ from the set $X_1 = \{(x, y) \in X^2 | x \sim y\}$ of edges of X towards A which is symmetric (that is, $w(x, y) = w(y, x)$ for $x \sim y$ in X) and such that, for every x in X, the map $y \mapsto w(x, y)$ is a bijection from the set $S^1(x)$ of neighbours of x onto A. We then let Γ be the group of automorphisms of X which preserve the map w. Then Γ is a cofinite lattice, which as an abstract group is the free product of d copies of $\mathbb{Z}/2\mathbb{Z}$.

Example 2.2. Let A be a finite set with at least two elements. Set $d = 2|A|$ and let X be a d-regular tree. We now fix a map $w : X_1 \to A \times \{-1, 1\}$ which is skew-symmetric (in the sense that, for every $x \sim y$ in X, if $w(x, y) = (a, \epsilon)$, then $w(y, x) = (a, -\epsilon)$) and again such that, for every x in X, the map $y \mapsto w(x, y)$ is a bijection from the set $S^1(x)$ of neighbours of x onto $A \times \{-1, 1\}$. We then let Γ be the group of automorphisms of X which preserve the map w. Then Γ is a cofinite lattice, which as an abstract group is the free product of d copies of $\mathbb{Z}$: this is the classical construction of the tree of a free group.

More generally, trees appear naturallly as universal covers of finite graphs and tree lattices as their fundamental groups. We refer the reader to [2] for more on tree lattices.

2.2. **Dynamical properties.** The action of Γ on $\mathscr{S}$ is proper and the space $\Gamma\backslash\mathscr{S}$ is compact. Since the action of Γ on $\mathscr{S}$ commutes with the maps T and ι, the latter induce homeomorphisms of the compact space $\Gamma\backslash\mathscr{S}$. By abuse of notation, we still denote these maps by T and ι. The map T may be seen as an analogue of the geodesic flow for the quotient of X by Γ. In this Subsection, we prove that this geodesic flow is topologically transitive. This property will be used in the sequel to prove uniqueness of the solutions of certain functional equations.

Proposition 2.3. *The map T admits dense orbits on $\Gamma\backslash\mathscr{S}$. Equivalently, the group Γ admits dense orbits on $\partial^2 X$.*

This rather standard result will follow from classical arguments of hyperbolic dynamical systems and hyperbolic geometry as in [13]. These arguments will not be used elsewhere in the paper. Most of the steps of the proof could be deduced from general properties of hyperbolic groups as in [17]. As these properties are much easier to prove in our particular case, we include a sketch of their proofs here.

We start with an easy consequence of the fact that $\Gamma\backslash X$ is finite.

Lemma 2.4. *Let x, y, z be in X with $x \sim y$. Then there exists g in Γ with $x \notin [y(gz)]$.*

Proof. Indeed, the set $\{t \in X | x \notin [yt]\}$ is unbounded whereas, as $\Gamma\backslash X$ is finite, one has $\sup_{t\in X} \inf_{g\in\Gamma} d(t, gz) < \infty$. $\square$

An automorphism g of X will be called hyperbolic if there exists a geodesic line $(\xi\eta)$ such that $g(\xi\eta) = (\xi\eta)$ and the restriction of g to $(\xi\eta)$ is a non-trivial translation. More precisely, there exists $k \neq 0$ such that, if $(x_h)_{h\in\mathbb{Z}}$ is a parametrization of $(\xi\eta)$ with origin ξ and endpoint η, one has $gx_h = x_{h+k}$, $h \in \mathbb{Z}$. Up to reversing the roles of ξ and η, one can assume $k > 0$. In that case, for every $\zeta \neq \xi$ in ∂X, one has $g^n\zeta \xrightarrow[n\to\infty]{} \eta$. In particular, the fixed points ξ and η of g and the positive integer k are uniquely determined by g. They are respectively called the repulsive fixed point, the attractive fixed point and the translation length of g. The geodesic line $(\xi\eta)$ is called the axis of g.

Let us give an easy criterion for an automorphism to be hyperbolic. This is a version of the closing Lemma from hyperbolic dynamics (see [20]).

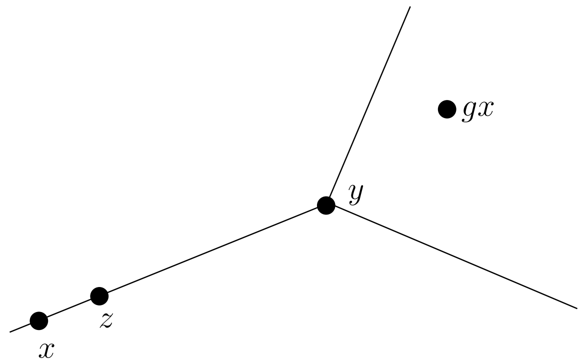

FIGURE 1. Proof of Lemma 2.6

Lemma 2.5. *Let x be in X and g be an automorphism of X. Assume $gx \neq x$. Let y be the neighbour of x on $[x(gx)]$. Then, if gy does not belong to the segment $[x(gx)]$, g is hyperbolic with translation length $k = d(x, gx)$ and x belongs to the axis of g.*

Proof. Let $x_0 = x, x_1 = y, \dots, x_k = gx$ be the parametrization of the segment $[x(gx)]$. For any h in $\mathbb{Z} \smallsetminus [0, k]$, if $h = \ell k + m$, $\ell \in \mathbb{Z}$, $0 \leq m \leq k-1$, we set $x_h = g^k x_m$. Then one easily checks that $(x_h)_{h\in\mathbb{Z}}$ is a parametrized geodesic line and that $gx_h = x_{h+k}$, $h \in \mathbb{Z}$. Thus, g is hyperbolic and we are done. □

From this, we deduce that Γ contains hyperbolic elements.

Lemma 2.6. *The group Γ contains hyperbolic elements. More precisely, the set of attractive fixed points of hyperbolic elements of Γ is dense in ∂X.*

Proof. Fix $x \neq y$ in X and let us build a hyperbolic element γ of Γ whose attractive fixed point ξ is such that $y \in [x\xi)$.

By Lemma 2.4, we can find an element g in Γ with $[xy] \subset [x(gx)]$. We let z be the neighbour of x on $[x(gx)]$. Then, if gz does not belong to $[x(gx)]$, by Lemma 2.5, g is hyperbolic and x belongs to the axis of g. In particular, as $[xy] \subset [x(gx)]$, we can set $\gamma = g$.

If not, again by Lemma 2.4, we can find h in Γ with $hx \neq x$ and $[x(gx)] \cap [x(hx)] = \{x\}$. We let t be the neighbour of x on $[x(hx)]$.

Assume $ht \notin [x(hx)]$. Then, still by by Lemma 2.5, h is hyperbolic and its attractive fixed point η satisfies $[x\eta) \cap [x(gx)] = \{x\}$. As t is the neighbour of x on $[x\eta)$, gt is the neighbour of gx on $[(gx)(g\eta))$. Since $t \neq z$ and gz is the neighbour of gx on $[x(gx)]$, we have $[xy] \subset [x(gx)] \subset [x(g\eta))$ and we can set $\gamma = ghg^{-1}$.

Finally, if ht belongs to $[x(hx)]$, we claim that gh^{-1} is hyperbolic. Indeed, by construction the geodesic segment $[(hx)(gx)]$ is equal to the union $[x(gx)] \cup [x(hx)]$. Now, the neighbour of hx on this segment is ht and the one of gx is gz. Since by assumption, $z \neq t$, we have $gt = (gh^{-1})ht \neq gz$, hence gt does not belong to $[(hx)(gx)]$. Thus, again by Lemma 2.5, gh^{-1} is hyperbolic and its attractive fixed point ζ satisfies $[xy] \subset [(hx)(gx)] \subset [(hx)\zeta)$, so that we can set $\gamma = gh^{-1}$. □

We recall the definition of the Busemann cocycle: this is a a first example of a smooth boundary cocycle, a notion which will play a key role in this article.

Let x and y be in X and ξ be in ∂X. The set $[x\xi) \cap [y\xi)$ is a geodesic ray. The number $d(x, z) - d(y, z)$ does not depend on z when z varies in $[x\xi) \cap [y\xi)$. We denote it by $b_\xi(x, y)$. The map $b : \partial X \times X \times X \to \mathbb{R}$ is smooth and is invariant under all automorphisms of X. It satisfies the cocycle relation:

$$b_\xi(x, z) = b_\xi(x, y) + b_\xi(y, z), \quad x, y, z \in X, \quad \xi \in \partial X.$$

For ξ in ∂X, we let Γ_ξ be the stabilizer of ξ in Γ. Fix x in X. By the cocycle property, the map $g \mapsto b_\xi(x, gx)$ is a homomorphism from Γ_ξ to $\mathbb{Z}$ which does not depend on x. We denote this homomorphism by χ_ξ.

Set $U_\xi = \partial X \smallsetminus \{\xi\}$. We fix x in X. We define a ultrametric distance on U_ξ by setting, for $\eta \neq \zeta$ in U_ξ, $D^\xi_x(\eta, \zeta) = e^{b_\xi(x,z)}$ where z in X is such that $(\xi\eta) \cap (\xi\zeta) = (\xi z]$. This is a proper distance, meaning that the associated balls are compact. It defines the locally compact topology of U_ξ, viewed as a subset of ∂X.

For x, y in X, one has $D^\xi_x = e^{b_\xi(x,y)} D^\xi_y$ and for g in Γ_ξ and η, ζ in U_ξ, one has $D^\xi_x(g\eta, g\zeta) = e^{\chi_\xi(g)} D^\xi_x(\eta, \zeta)$. Thus, a fixed point argument gives:

Lemma 2.7. *Let ξ be in ∂X and g be in Γ with $\chi_\xi(g) < 0$. Then g is hyperbolic and its repulsive fixed point is ξ.*

Let Γ^0_ξ be the kernel of χ_ξ in Γ_ξ.

Lemma 2.8. *Let ξ be in ∂X. The action of the group Γ^0_ξ on the locally compact space $U_\xi = \partial X \smallsetminus \{\xi\}$ is proper.*

Proof. We need to prove that for any compact subset K of U_ξ, the set of g in Γ^0_ξ with $gK \cap K \neq \emptyset$ is finite.

For x in X, define $K_{\xi x}$ as the set of those η in U_ξ such that x belongs to $(\xi\eta)$. These sets are the balls of the above introduced distances on U_ξ. In particular $K_{\xi x}$ is a compact subset of U_ξ and every compact subset of U_ξ is contained in $K_{\xi x}$ for some x. Thus, to check that the action is proper, we can assume that K above is of the form $K_{\xi x}$.

Now, for $x \neq y$ in X with $b_\xi(x, y) = 0$, we have $K_{\xi x} \cap K_{\xi y} = \emptyset$. Therefore, we get

$$\{g \in \Gamma_\xi^0 | gK_{\xi x} \cap K_{\xi x} \neq \emptyset\} = \{g \in \Gamma_\xi^0 | gx = x\},$$

and the latter is finite by assumption. □

The group Γ_ξ can not be too large:

Lemma 2.9. *Let ξ be in ∂X. If $\chi_\xi \neq 0$ on Γ_ξ, then Γ_ξ fixes a point in U_ξ. In particular, we always have $\Gamma_\xi \neq \Gamma$.*

Proof. Assume $\chi_\xi = 0$. Then, for every x in X, we have

$$\Gamma_\xi x \subset \{y \in X | b_\xi(x, y) = 0\},$$

hence $\Gamma_\xi \neq \Gamma$ by Lemma 2.4.

Assume $\chi_\xi \neq 0$, that is, $\chi_\xi(\Gamma_\xi)$ is a non trivial subgroup of $\mathbb{Z}$. Let g be in Γ_ξ such that $\chi_\xi(g) < 0$ and $\chi_\xi(\Gamma_\xi) = \chi_\xi(g)\mathbb{Z}$. By Lemma 2.7, the automorphism g is hyperbolic with repulsive fixed point ξ. Let $\eta \in U_\xi$ be its attractive fixed point. We claim that η is fixed by Γ_ξ. Indeed, as by Lemma 2.8, the action of Γ_ξ^0 on U_ξ is proper, there exists a neighborhood V of η in U_ξ such that $V \cap \Gamma_\xi^0 \eta = \{\eta\}$. As η is the attractive fixed point of g, for every ζ in U_ξ, there exists $n \geq 0$ with $g^n \zeta \in V$. Since Γ_ξ^0 is normal in Γ_ξ and $g\eta = \eta$, we have $g\Gamma_\xi^0 \eta = \Gamma_\xi^0 \eta$. Thus, we get $\Gamma_\xi^0 \eta = \{\eta\}$. By assumption, we have $\Gamma_\xi = g^{\mathbb{Z}} \Gamma_\xi^0$ and therefore η is a fixed point of Γ_ξ. In particular, for x in $(\xi\eta)$, we have $\Gamma_\xi x \subset (\xi\eta)$, hence again $\Gamma_\xi \neq \Gamma$ by Lemma 2.4. □

The action of Γ on ∂X is minimal.

Lemma 2.10. *Let ξ be in ∂X. Then $\Gamma\xi$ is dense in ∂X.*

Proof. Let g be a hyperbolic element of Γ with attractive fixed point ζ and repulsive fixed point η. If $\xi \neq \eta$, we have $g^n \xi \xrightarrow[n\to\infty]{} \zeta$. If $\xi = \eta$, by Lemma 2.9, we have $\Gamma_\xi \neq \Gamma$, that is, we can find h in Γ with $h\xi \neq \xi$. Then, $g^n h\xi \xrightarrow[n\to\infty]{} \zeta$. In both cases, ζ belongs to the closure of $\Gamma\xi$, hence $\Gamma\xi$ is dense by Lemma 2.6. □

We can now finish the proof of Proposition 2.3. This relies on the classical shadowing argument from hyperbolic dynamics (see [20]).

Proof of Proposition 2.3. First, the two statements in the Proposition are equivalent. Indeed, for s in $\mathscr{S}$, saying that the image of s in $\Gamma\backslash\mathscr{S}$

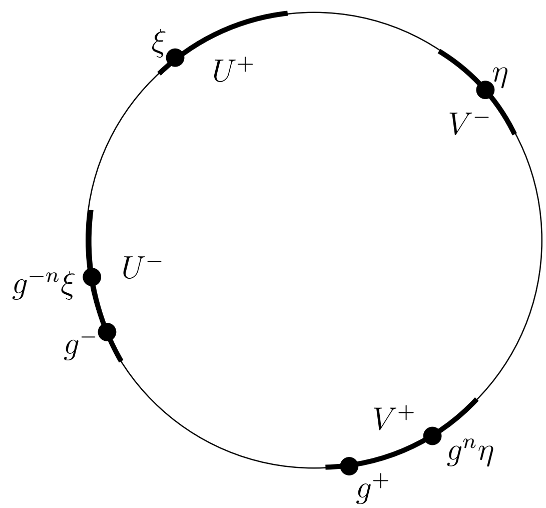

FIGURE 2. Proof of Proposition 2.3

has dense orbit under T is saying that s has dense orbit under the $(\Gamma \times \mathbb{Z})$-action on $\mathscr{S}$ defined by

$$(\gamma, n) \cdot s = \gamma(T^n s) = T^n(\gamma s), \quad \gamma \in \Gamma, \quad n \in \mathbb{Z}, \quad s \in \mathscr{S}.$$

Now, the surjective map $\mathscr{S} \to \partial^2 X, s \mapsto (s_-, s_+)$ identifies $\partial^2 X$ with the quotient of $\mathscr{S}$ by the $T^{\mathbb{Z}}$-action, so that saying that s has dense $(\Gamma \times \mathbb{Z})$-orbit in $\mathscr{S}$ is saying that (s_-, s_+) has dense Γ-orbit in $\partial^2 X$.

We will now show that the action of Γ on $\partial^2 X$ admits dense orbits. As $\partial^2 X$ is a locally compact topological space with a countable basis, it suffices to prove that, for every non empty open subsets U and V of $\partial^2 X$, there exists γ in Γ with $\gamma U \cap V \neq \emptyset$ (existence of a dense set of points with dense orbits then follows from a Baire category argument).

Now, one can assume that U (resp. V) is of the form $U^- \times U^+$ (resp. $V^- \times V^+$) where U^-, U^+, V^- and V^+ are non empty open subsets of ∂X and $U^- \cap U^+ = V^- \cap V^+ = \emptyset$. We fix a hyperbolic element g in Γ (which exists by Lemma 2.6). Let g^- be the repulsive fixed point of g and g^+ be its attractive fixed point. By Lemma 2.10, there exists γ in Γ with $\gamma g^+ \in V^+$. Thus, up to replacing V by its image by γ^{-1}, we can assume that g^+ belongs to V^+. In the same way, we can assume that g^- belongs to U^-. Now, we fix ξ in U^+ and η in V^-. We can find an integer $n \geq 0$ such that one has $g^n\xi \in V^+$ and $g^{-n}\eta \in U^-$. In particular, we have $(g^{-n}\eta, \xi) \in U^- \times U^+$ whereas $(\eta, g^n\xi) \in V^- \times V^+$, hence $g^n U \cap V \neq \emptyset$ and the result follows. $\square$

2.3. **Boundary cocycles.** We now introduce smooth boundary cocycles: they are generalizations of the Busemann cocycle. Recall that, given a locally compact topological space U, we let $\mathcal{D}(U)$ denote the space of smooth functions with compact support on U. We will freely identify functions in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ with Γ-invariant smooth functions on $\mathscr{S}$.

Two smooth functions f and g in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ are said to be cohomologuous if there exists some smooth function h such that $f - g = h \circ T - h$. Note that in particular, f is cohomologuous with $f \circ T$. By Proposition 2.3, the smooth function h such that $f - g = h \circ T - h$ is unique up to the addition of a constant. A smooth function f is said to be even if f is cohomologuous to $f \circ \iota$.

By a smooth boundary cocycle, we shall mean a set-theoretic cocycle $X \times X \to \mathcal{D}(\partial X)$. More precisely, such a cocycle B is a map $\partial X \times X \times X \to \mathbb{R}, (\xi, x, y) \mapsto B_\xi(x, y)$ such that, for any ξ in ∂X and x, y, z in X,

$$B_\xi(x, z) = B_\xi(x, y) + B_\xi(y, z)$$

and that, for any x, y in X, the function $\xi \mapsto B_\xi(x, y)$ is locally constant on ∂X. The group of automorphisms of X acts in a natural way on the space of smooth boundary cocycles and in the sequel, we shall only consider Γ-invariant smooth boundary cocyles, that is, we require that for any γ in Γ, ξ in ∂X and x, y in X,

$$B_{\gamma\xi}(\gamma x, \gamma y) = B_\xi(x, y).$$

Two Γ-invariant smooth boundary cocycles B and C are said to be cohomologuous if there exists a Γ-invariant smooth function F on $X \times \partial X$ such that, for any ξ in ∂X and x, y in X,

$$B_\xi(x, y) - C_\xi(x, y) = F(x, \xi) - F(y, \xi).$$

Example 2.11. The Busemann cocycle is a smooth boundary cocycle which is invariant under all automorphisms of X.

There is a general philosophy, coming from [27], that under some regularity assumptions, there is a bijection between cohomology classes of functions on $\Gamma\backslash\mathscr{S}$ and cohomology classes of Γ-invariant boundary cocycles on X. We shall make it explicit in the case of smooth objects (see [10] for the case of Hölder continuous objects).

To begin with, let us give an alternate definition of a smooth boundary cocycle. This is a kind of generalization of the construction of the Busemann cocycle.

Lemma 2.12. *For any smooth function f on $X \times \partial X$, there exists a unique smooth boundary cocycle B such that, for any x in X and ξ in*

∂X, if x_1 is the unique neighbour of x on $[x\xi)$, we have

$$B_\xi(x, x_1) = f(x, \xi).$$

Proof. Let x and y be in X, ξ be in ∂X and z be a point in $[x\xi) \cap [y\xi)$. We denote by $x_0 = x, x_1, \dots, x_n = z$ the geodesic path from x to z and by $y_0 = y, y_1, \dots, y_p = z$ the geodesic path from y to z. The number

$$\sum_{h=0}^{n-1} f(x_h, \xi) - \sum_{h=0}^{p-1} f(y_h, \xi)$$

does not depend on z. We denote it by $B_\xi(x, y)$. One easily checks that B is then the unique smooth boundary cocycle satisfying the requirements of the lemma. □

Let us now focus on Γ-invariant objects. Consider f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ and a Γ-invariant smooth boundary cocycle B. We shall say that f is a potential for B if f is cohomologuous to the smooth function $s \mapsto B_{s_+}(\pi(s), \pi(Ts))$.

Proposition 2.13. *The map which sends a Γ-invariant smooth boundary cocycle to the set of its potentials induces a bijection between the set of cohomology classes of Γ-invariant smooth boundary cocycles and the set of cohomology classes of smooth functions on $\Gamma\backslash\mathscr{S}$.*

Before proving Proposition 2.13, let us give a lemma which will allow us to get surjectivity of the involved map between cohomology classes.

There exists a natural surjective map $\mathscr{S} \to X \times \partial X$, namely the map $s \mapsto (\pi(s), s_+)$. For f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$, if f factors through a smooth function on $X \times \partial X$, then by Lemma 2.12 we can associate to it a Γ-invariant smooth boundary cocycle. To extend this to any f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$, let us describe more precisely the fibers of the map $\mathscr{S} \to X \times \partial X$.

For s in $\mathscr{S}$, define M_s as the set of those t in $\mathscr{S}$ such that $\pi(s) = \pi(t)$ and $s_+ = t_+$. This is a compact subset of $\mathscr{S}$ and we have $TM_s \subset M_{Ts}$. We say that a function f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ is M-invariant if for any s in $\mathscr{S}$, for any t in M_s, we have $f(s) = f(t)$.

From the dynamical point of view, M_s plays the role of a local stable leaf for s. Thus, we get

Lemma 2.14. *Let f be in $\mathcal{D}(\Gamma\backslash\mathscr{S})$. There exists $k \geq 0$ such that $f \circ T^k$ is M-invariant.*

Proof. Heuristically, M_s being a piece of the stable leaf of s with respect to the transformation T, if t belongs to M_s, the points $T^k\Gamma s$ and $T^k\Gamma t$ must get closer and closer in $\Gamma\backslash\mathscr{S}$. The conclusion follows since f is locally constant. Let us make this argument more precise.

For k in $\mathbb{N}$ and s in $\mathscr{S}$, set $M_s^k = T^k M_{T^{-k}s}$. One has $\bigcap_{k\geq 0} M_s^k = \{s\}$. We let D be the diagonal in $(\Gamma\backslash\mathscr{S})^2$ and $D^k \subset (\Gamma\backslash\mathscr{S})^2$ be the image of the set $\{(s,t)\in\mathscr{S}|t\in M_s^k\}$ under the natural map $\mathscr{S}^2 \to (\Gamma\backslash\mathscr{S})^2$.

We claim that we have $\bigcap_{k\geq 0} D^k = D$ in $\Gamma\backslash\mathscr{S}$. Indeed, let s and t be in $\mathscr{S}$ and assume that, for every $k\geq 0$, one has $(\Gamma s, \Gamma t)\in D^k$. We need to prove that s belongs to Γt. By assumption, for every $k\geq 0$, there exists γ_k in Γ with $\gamma_k t \in M_s^k$. We can assume that $\gamma_0 = e$ the identity element. Then, let $x = \pi(s) = \pi(t)$ and $\xi = s_+ = t_+$ be the common base point and the common endpoint of s and t. For any $k\geq 0$, as $\gamma_k t$ belongs to $M_s^k \subset M_s$, we also have

$$\gamma_k x = \pi(\gamma_k t) = \pi(s) = x \text{ and } \gamma_k\xi = (\gamma_k t)_+ = s_+ = \xi.$$

Thus, γ_k belongs to the group Γ_ξ^0 defined in Subsection 2.2. Let $\eta = s_-$ and $\zeta = t_-$ be the origins of s and t. By Lemma 2.8, the action of Γ_ξ^0 on $U_\xi = \partial X \smallsetminus \{\xi\}$ is proper, hence $\Gamma_\xi^0\zeta$ is a closed subset of U_ξ. As, for every $k\geq 0$, $\gamma_k t$ belongs to M_s^k, we have $\gamma_k\zeta \xrightarrow[k\to\infty]{} \eta$. Therefore, there exists $k\geq 0$ with $\gamma_k\zeta = \eta$. As $\gamma_k x = x$ and $\gamma_k\xi = \xi$, we get $\gamma_k t = s$ and we are done.

Now, let (U_i) be an open cover of $\Gamma\backslash\mathscr{S}$ such that f is constant on each of the U_i. The set $U = \bigcup_i U_i \times U_i$ is open in $(\Gamma\backslash\mathscr{S})^2$ and contains D. By compactness, there exists $k\geq 0$ with $D^k \subset U$.

Let s be in $\mathscr{S}$ and t be in M_s, and let $\overline{s}$ and $\overline{t}$ be their images in $\Gamma\backslash\mathscr{S}$. By definition we have $(T^k\overline{s}, T^k\overline{t}) \in D^k$, hence $(T^k\overline{s}, T^k\overline{t}) \in U$. Now, from the definition of U, we get $f(T^k\overline{s}) = f(T^k\overline{t})$, which should be proved. ☐

To prove injectivity of the map between cohomology classes, we will need

Lemma 2.15. *Let f be an M-invariant function in $\mathcal{D}(\Gamma\backslash\mathscr{S})$. If f is cohomologuous to 0, then any function h in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ such that $f = h - h\circ T$ is M-invariant.*

Proof. Let us first construct an M-invariant function h with $f = h - h\circ T$. Let h_1 be in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ with $f = h_1 - h_1\circ T$. By Lemma 2.14, there exists $k\geq 0$ such that $h_2 = h_1\circ T^k$ is M-invariant. We have

$$f - h_2 + h_2\circ T = f - f\circ T^k = \sum_{j=0}^{k-1} f\circ T^j - \sum_{j=0}^{k-1} f\circ T^{j+1}$$

and we can set $h = h_2 + \sum_{j=0}^{k-1} f\circ T^j$.

Now, by Proposition 2.3, if h' is any other smooth function with $f = h' - h'\circ T$, then $h - h'$ is constant, hence h' also is M-invariant. ☐

Proof of Proposition 2.13. By Lemmas 2.12 and 2.14, every smooth function on $\Gamma\backslash\mathscr{S}$ is the potential of some Γ-invariant boundary cocycle. To conclude the proof it only remains to prove that such a cocycle B is cohomologuous to 0 if and only if its potentials are cohomologuous to 0.

Assume first that B is cohomologuous to 0. Then there exists a smooth Γ-invariant function h on $X \times \partial X$ such that, for any x, y in X and ξ in ∂X, we have

$$B_\xi(x, y) = h(x, \xi) - h(y, \xi).$$

For s in $\mathscr{S}$, we get

$$B_{s_+}(\pi(s), \pi(Ts)) = h(\pi(s), s_+) - h(\pi(Ts), s_+),$$

hence the potentials of B are cohomologuous to 0.

Conversely, assume that the potentials of B are cohomologuous to 0. By Lemma 2.15, there exists a Γ-invariant smooth function h on $X \times \partial X$ such that, for any x in X and ξ in ∂X,

$$B_\xi(x, x_1) = h(x, \xi) - h(x_1, \xi),$$

where x_1 is the unique neighbour of x on $[x\xi)$. By the uniqueness part in Lemma 2.12, we get that B is a coboundary. $\square$

2.4. **Additive kernels.** Still following the main lines of [27], one can associate to a cohomology class on $\Gamma\backslash\mathscr{S}$ a family of smooth functions on $\partial^2 X$. In case the cohomology class is even, the associated functions on $\partial^2 X$ are symmetric. This fact will be crucial later when using these functions to define symmetric bilinear forms.

Proposition 2.16. *Let B be a Γ-invariant smooth boundary cocycle. Assume that the potentials of B are even. Then, there exists a smooth Γ-invariant function Ω on $X \times \partial^2 X$ such that, for any x in X, the function $(\xi, \eta) \mapsto \Omega_x(\xi, \eta)$ is symmetric and that, for any x, y in X and (ξ, η) in $\partial^2 X$, one has*

$$\Omega_x(\xi, \eta) - \Omega_y(\xi, \eta) = \frac{1}{2}(B_\xi(x, y) + B_\eta(x, y)). \tag{2.1}$$

The function Ω is unique up to a constant.

Definition 2.17. Such a map Ω will be called an additive kernel associated to B. More generally, we will speak of the additive kernels associated to cocycles which are cohomologuous to B as the additive kernels associated to the cohomology class of B.

Example 2.18. When $B = b$, the Busemann cocycle, let ω be the Gromov product, that is, for any x in X and (ξ, η) in ∂X, $\omega_x(\xi, \eta)$ is the distance from x to the geodesic line $(\xi\eta)$. Then ω satisifies the conclusions of the Proposition.

Proof. Let us first define $\Omega_x(\xi, \eta)$ when x belongs to the geodesic line $(\xi\eta)$. We let f be the smooth function $s \mapsto B_{s_+}(\pi(s), \pi(Ts))$ on $\Gamma\backslash\mathscr{S}$. By assumption, the functions f and $f \circ \iota$ are cohomologuous. Hence, the functions f and $f \circ \iota T$ are cohomologuous. We chose a smooth function h such that

$$f - f \circ \iota T = h - h \circ T.$$

We claim that h is then invariant under ι, that is, $h \circ \iota = h$. Indeed, we have

$$\begin{aligned} h \circ \iota - h \circ \iota T &= (h - h \circ T^{-1}) \circ \iota = (h \circ T - h) \circ T^{-1}\iota \\ &= (f \circ \iota T - f) \circ T^{-1}\iota = f - f \circ \iota T = h - h \circ T. \end{aligned}$$

By Proposition 2.3, $h - h \circ \iota$ is a constant function. Let c be its value. As ι^2 is the identity map, we have $h = h \circ \iota + c = h + 2c$, hence $c = 0$, and h is ι-invariant.

If (ξ, η) is in $\partial^2 X$ and if x belongs to the geodesic line $(\xi\eta)$, we set

$$\Omega_x(\xi, \eta) = \frac{1}{2}h(s),$$

where s is the unique parametrized geodesic line such that $s_- = \xi$, $s_+ = \eta$ and $\pi(s) = x$. As h is ι-invariant, we have $\Omega_x(\xi, \eta) = \Omega_x(\eta, \xi)$.

Let us check that (2.1) holds on $(\xi\eta)$. We let y be the unique neighbour of x on $[x\eta)$. By definition we have, on one hand,

$$\Omega_y(\xi, \eta) = \frac{1}{2}h(Ts)$$

and, on the other hand,

$$B_\eta(x, y) = f(s) \text{ and } B_\xi(x, y) = -B_\xi(y, x) = -f(\iota Ts).$$

Hence (2.1) holds for any two neighbouring points x and y on $(\xi\eta)$. By the cocycle identity, it holds for any two points.

Now, if x is any element in X, we set

$$\Omega_x(\xi, \eta) = \Omega_y(\xi, \eta) + \frac{1}{2}(B_\xi(x, y) + B_\eta(x, y)),$$

where y is on the geodesic line $(\xi\eta)$. As (2.1) holds on $(\xi\eta)$, this does not depend on y. One easily checks that (2.1) holds everywhere.

Uniqueness follows from the fact that Γ has a dense orbit on $\partial^2 X$ (see Proposition 2.3). $\square$

The additive kernels determine the cocycle.

Lemma 2.19. *Let B be an even smooth Γ-invariant boundary cocycle. Assume the cohomology class of B admits 0 as an additive kernel. Then B is cohomologuous to 0.*

Proof. Indeed, up to replacing B by a cohomologuous cocycle, one can assume that, for any x, y in X and $\xi \neq \eta$ in ∂X, one has $B_\xi(x, y) + B_\eta(x, y) = 0$. Let ξ, η, ζ be three different points in ∂X, which exist due to our assumptions on X. We get $B_\xi(x, y) = -B_\eta(x, y) = B_\zeta(x, y) = -B_\xi(x, y)$, hence $B_\xi(x, y) = 0$ and we are done. □

Let still ω be the Gromov product, as in Example 2.18. To prove that certain formulae make sense, we shall use

Lemma 2.20. *Let Ω be an additive kernel associated to a Γ-invariant smooth even boundary cocycle B. Then there exists $C \geq 0$ such that, for any x in X and $\xi \neq \eta$ in ∂X, one has*

$$|\Omega_x(\xi, \eta)| \leq C(1 + \omega_x(\xi, \eta)).$$

Proof. The Γ-invariant function $h : s \mapsto \Omega_{\pi(s)}(s_-, s_+)$ on $\mathscr{S}$ is smooth. As $\Gamma\backslash\mathscr{S}$ is compact, it is bounded. In the same way, the smooth Γ-invariant function $f : s \mapsto B_{s_+}(\pi(s), \pi(Ts))$ on $\mathscr{S}$ is bounded. We chose $C \geq 0$ with $C \geq \max(\|f\|_\infty, \|h\|_\infty)$ and the result follows by the defining properties of Ω. □

2.5. **Normalized smooth functions.** We will give a formula for the function Ω in a particular case that will play an important role in the article.

For any $k \geq 1$, we denote by X_k the set

$$X_k = \{(x, y) \in X^2 | d(x, y) = k\}.$$

Note that, for $k = 1$, the set X_1 is the set of oriented edges of X.

Given a Γ-invariant symmetric function w on X_k, we can define an even smooth function f on $\Gamma\backslash\mathscr{S}$ by setting, for $s = (x_h)_{h\in\mathbb{Z}}$ in $\mathscr{S}$,

$$f(s) = w(x_0, x_k).$$

We then say that f is a normalized even function.

Lemma 2.21. *Any smooth even function on $\Gamma\backslash\mathscr{S}$ is cohomologuous to a normalized even function.*

Proof. By Lemma 2.14, we can assume that f is M-invariant. Again, by Lemma 2.14, applied to the function $f \circ \iota$, there exists $k \geq 0$ such that, for any $s = (x_h)_{h\in\mathbb{Z}}$ and $t = (y_h)_{h\in\mathbb{Z}}$ in $\mathscr{S}$, if $x_h = y_h$ for any $h \geq k$, then $f(s) = f(t)$. In other words, there exists a Γ-invariant function v on X_k, such that, for any $s = (x_h)_{h\in\mathbb{Z}}$, $f(s) = v(x_0, x_k)$. This gives also, $f(\iota T^k s) = v(x_k, x_0)$, hence

$$\frac{1}{2}(f(s) + f(\iota T^k s)) = \frac{1}{2}(v(x_0, x_k) + v(x_k, x_0)).$$

Now, f being even, it is cohomologuous to $\frac{1}{2}(f + f \circ \iota T^k)$ and we are done. □

For normalized functions, we have an explicit formula for the associated additive kernel.

Proposition 2.22. *Let w be a Γ-invariant symmetric function on X_k for some $k \geq 1$. Let f be the associated normalized smooth even function on $\Gamma\backslash\mathscr{S}$ and B be the smooth Γ-invariant boundary cocycle defined as in Lemma 2.12. Then, let us give a formula for the function Ω from Proposition 2.16. For x in X and (ξ, η) in $\partial^2 X$, we let $(y_i)_{i\geq 0}$ and $(z_i)_{i\geq 0}$ denote the geodesic rays $[x\xi)$ and $[x\eta)$. Let $j = \omega_x(\xi, \eta)$ be the distance from x to the geodesic line $(\xi\eta)$ so that $y_0 = z_0, \dots, y_j = z_j$ but $y_{j+1} \neq z_{j+1}$. Then one can set*

$$\Omega_x(\xi, \eta) = \frac{1}{2}\sum_{h=0}^{j-1}(w(y_h, y_{h+k}) + w(z_h, z_{h+k})) - \frac{1}{2}\sum_{h=1}^{k-1} w(y_{j+h}, z_{j+k-h}).$$

The proof is straightforward.

Definition 2.23. If w is a Γ-invariant symmetric function on X_k for some $k \geq 1$, we say that the additive kernels Ω associated to the cohomology class of the normalized function

$$s = (x_h)_{h\in\mathbb{Z}} \mapsto w(x_0, x_k)$$

on $\mathscr{S}$ are the additive kernels associated to w.

3. Normalized kernels and bilinear forms

In this section, we will associate to every bounded symmetric function w on X_k, $k \geq 1$, a bounded symmetric bilinear form on a certain Hilbert space H_0^ω. The definition of this bilinear form will then be related to the formula in Proposition 2.22.

Recall that, for $k \geq 1$, X_k stands for the set of pairs (a, b) in X^2 with $d(a, b) = k$. We equip the countable set X_k with its counting measure and, for $1 \leq p < \infty$, we let $\ell^p(X_k)$ denote the associated space of p-integrable functions, equipped with its natural norm. In other words, a function $\theta : X_k \to \mathbb{R}$ is in $\ell^p(X_k)$ if and only if one has $\sum_{\substack{a,b\in X\\ d(a,b)=k}} |\theta(a, b)|^p < \infty$ and the latter number is then $\|\theta\|_p^p$.

3.1. **The Hilbert spaces H^ω and H_0^ω.** We start by building the space H_0^ω. In case X is homogeneous (that is, for any x in X, its number of neighbourgs $d(x)$ is independent of x), the space H_0^ω is the skew-symmetric special representation of the group of automorphisms of X as built in [29] and described in [15].

We fix x in X which will play the role of an origin and we associate to it a smooth function χ_x on $X_1 \times \partial X$ as follows: for any $a \sim b$ in X and ξ in ∂X, we set

$$\chi_x(a, b, \xi) = 1 \text{ if } [ab] \subset [x\xi) \text{ and } [xa] \cap [b\xi) = \emptyset.$$
$$\chi_x(a, b, \xi) = -1 \text{ if } [ab] \subset [x\xi) \text{ and } [xb] \cap [a\xi) = \emptyset.$$
$$\chi_x(a, b, \xi) = 0 \text{ else.}$$

In other words, $\chi_x(., ., \xi)$ may be seen as an oriented characteristic function of the geodesic ray $[x\xi)$. In particular, note that $\chi_x(a, b, \xi)$ is skew-symmetric in (a, b).

Let us precisely describe how this map depends on x. From a direct computation, we get

Lemma 3.1. *For any x, y and $a \sim b$ in X, the function $\chi_x(a, b, .) - \chi_y(a, b, .)$ is constant on ∂X. Its value $\delta_{xy}(a, b)$ is given by*

$$\begin{aligned} \delta_{xy}(a, b) &= 1 && \textit{if } [ab] \subset [xy] \textit{ and } [xa] \cap [by] = \emptyset. \\ \delta_{xy}(a, b) &= -1 && \textit{if } [ab] \subset [xy] \textit{ and } [xb] \cap [ay] = \emptyset. \\ \delta_{xy}(a, b) &= 0 && \textit{else.} \end{aligned}$$

In particular, δ_{xy} is a finitely supported function on X_1.

Now, here comes the key observation which will allow us to relate the scalar product on a certain subspace of $\ell^2(X_1)$ to integral formulae on the boundary. Recall that ω is the Gromov product of X, that is, for any x in X and $\xi \neq \eta$ in ∂X, $\omega_x(\xi, \eta)$ is the distance from x to the geodesic line $(\xi\eta)$ (see Example 2.18). The proof of the following is immediate.

Lemma 3.2. *Fix x in X and $\xi \neq \eta$ in ∂X. Let y in X be such that $[x\xi) \cap [x\eta) = [xy]$. Then, for $a \sim b$ in X, we have*

$$\begin{aligned} \chi_x(a, b, \xi)\chi_x(a, b, \eta) &= 1 && \textit{if } [ab] \subset [xy] \\ &= 0 && \textit{else.} \end{aligned}$$

In particular, the function $(a, b) \mapsto \chi_x(a, b, \xi)\chi_x(a, b, \eta)$ is finitely supported on X_1 and we have

$$\sum_{a \sim b \in X} \chi_x(a, b, \xi)\chi_x(a, b, \eta) = 2\omega_x(\xi, \eta).$$

Recall that, if U is a totally discontinuous compact topological space, the space of distributions on U is denoted by $\mathcal{D}^*(U)$: this is the dual space to the space $\mathcal{D}(U)$ of smooth functions on U. Also, $\mathcal{D}^*_0(U)$ denotes the space of distributions T on U such that $\langle T, \mathbf{1}\rangle = 0$, which we view as the dual space to $\overline{\mathcal{D}}(U) = \mathcal{D}(U)/\mathbb{R}$.

As $\chi_x(a, b, \xi)$ depends smoothly on ξ, we can use it to define a linear map from distributions to functions on X_1. Fix x in X. If T is in $\mathcal{D}(\partial X)$, we let $\mathcal{P}_x T$ be the skew-symmetric function on X_1 such that, for any $a \sim b$ in X,

$$\mathcal{P}_x T(a, b) = \langle T, \chi_x(a, b, .)\rangle.$$

From Lemma 3.1 we immediately get

Lemma 3.3. *For x, y in X, T in $\mathcal{D}^*(\partial X)$ and $a \sim b$ in X, we have*

$$\mathcal{P}_x T(a, b) - \mathcal{P}_y T(a, b) = \delta_{xy}(a, b)\langle T, \mathbf{1}\rangle \text{ and } \langle T, \mathbf{1}\rangle = \sum_{z\sim x} \mathcal{P}_x T(x, z).$$

In particular $\mathcal{P}_x T - \mathcal{P}_y T$ is a finitely supported function on X_1 and, if T is in $\mathcal{D}^*_0(\partial X)$, the function $\mathcal{P}_x T = \mathcal{P}T$ does not depend on x.

Remark 3.4. Since the map $\mathcal{P}$ on $\mathcal{D}^*_0(\partial X)$ does not depend on the choice of x, it commutes with the natural action of the group of automorphisms of X. This fact will be instrumental in our next constructions.

We can describe the spaces $\mathcal{P}_x\mathcal{D}^*(\partial X)$ and $\mathcal{P}\mathcal{D}^*_0(\partial X)$.

Lemma 3.5. *For any x in X, the map $\mathcal{P}_x$ establishes a linear isomorphism between the space $\mathcal{D}^*(\partial X)$ and the space of skew-symmetric functions θ on X_1 such that, for any $a \neq x$ in X, one has $\sum_{b\sim a} \theta(a, b) = 0$.*

The map $\mathcal{P}$ establishes a linear isomorphism between the space $\mathcal{D}^*_0(\partial X)$ and the space of skew-symmetric functions θ on X_1 such that, for any a in X, one has $\sum_{b\sim a} \theta(a, b) = 0$.

In the proof, we shall need the following notation which will also be used later in the article. If $x \neq y$ are in X, we let U_{xy} be the closed open subset in ∂X defined by

$$U_{xy} = \{\xi \in \partial X | [xy] \cap [y\xi) = \{y\}\}.$$

In the language of hyperbolic geometry, the set U_{xy} is the shadow of y viewed from x (see for example [32]). By definition, for any x in X, the open sets U_{xy}, $y \in X \smallsetminus \{x\}$, generate the topology of ∂X. From this, we get

Lemma 3.6. *Fix x in X. Let φ be in $\mathcal{D}(\partial X)$. There exists $\ell \geq 1$ and a function f on $S^\ell(x)$ such that $\varphi = \sum_{y\in S^\ell(x)} f(y)\mathbf{1}_{U_{xy}}$.*

Proof. By definition, for every ξ in ∂X, there exists $y \neq x$ such that $\xi \in U_{xy}$ and f is constant on U_{xy}. By compactness, there exists finitely many $y_1, \dots, y_n$ in $X \smallsetminus \{x\}$ such that, for $1 \leq i \leq n$, f is constant on U_{xy_i} and these open subset cover ∂X. The result follows by taking $\ell = \max_{1 \leq i \leq n} d(x, y_i)$. □

Proof of Lemma 3.5. Note that the second part of the Lemma follows from the first and the formula for $\langle T, \mathbf{1} \rangle$, $T \in \mathcal{D}^*(\partial X)$, from Lemma 3.3.

Now, let us prove the first part. Let θ be a skew-symmetric function on X_1 such that, for any $a \neq x$ in X, one has $\sum_{b \sim a} \theta(a, b) = 0$ and let us build T in $\mathcal{D}^*(\partial X)$ with $\mathcal{P}_x T = \theta$. For y in X, $y \neq x$, we let y_- be the unique neighbour of y on $[xy]$.

Pick φ and chose, as in Lemma 3.6, some $\ell \geq 1$ and a function f on $S^\ell(x)$ such that $\varphi = \sum_{y \in S^\ell(x)} f(y) \mathbf{1}_{U_{xy}}$. We claim that the number $u_\ell = \sum_{y \in S^\ell(x)} f(y) \theta(y_-, y)$ does not depend on ℓ. Indeed, for any y in $S^\ell(x)$, we have

$$\theta(y_-, y) = \sum_{\substack{z \sim y \\ z \neq y_-}} \theta(y, z),$$

hence $u_\ell = u_{\ell+1}$. As this number clearly depends linearly on φ, we can define a distribution T by setting $\langle T, \varphi \rangle = u_\ell$ for ℓ large enough.

Let us check that we have $\theta = \mathcal{P}_x T$. To this aim, we pick (a, b) in X_1. With no loss of generality, we can assume that we have $d(x, b) = \ell \geq 1$ and $a \in [xb]$. Then, by construction, we have $\chi_x(a, b, .) = \mathbf{1}_{U_{xb}}$, hence $\mathcal{P}_x T(a, b) = \langle T, \mathbf{1}_{U_{xb}} \rangle = \theta(a, b)$ and the description of $\mathcal{P}_x \mathcal{D}^*(\partial X)$ follows. □

We can now use the map $\mathcal{P}$ to define a remarkable Hilbert space. We let H^ω denote the space of distributions ρ in $\mathcal{D}^*(\partial X)$ such that, for some x in X, the function $\mathcal{P}_x \rho$ belongs to $\ell^2(X_1)$. By Lemma 3.3, this condition does not depend on x. We equip it with the norm induced by this embedding: the restriction of the norm to $H_0^\omega = \mathcal{D}_0^*(\partial X) \cap H^\omega$ is independent of x. As, by Lemma 3.5, $\mathcal{P}\mathcal{D}^*(\partial X) \cap \ell^2(X_1)$ is a closed subspace of $\ell^2(X_1)$, the space H_0^ω is a Hilbert space.

Let ν be a Borel probability measure on ∂X. For any $1 \leq p \leq \infty$, we let $\mathfrak{M}^p(\nu) \subset \mathcal{D}(\partial X)$ denote the space of signed Borel measures on ∂X whose density is p-integrable with respect to ν. We write $\mathfrak{M}_0^p(\nu) = \mathfrak{M}^p(\nu) \cap \mathcal{D}_0^*(\partial X)$ for the set of those ρ in $\mathfrak{M}^p(\nu)$ with $\rho(\mathbf{1}) = 0$.

Assume ν is atom-free, so that for any x in X, ω_x is defined $\nu \otimes \nu$-almost everywhere. For $p \geq 1$, we say that ω is ν-p-integrable if ω_x is $\nu \otimes \nu$-p-integrable: this condition does not depend on the choice of x since, for any x, y in X, one has $|\omega_x - \omega_y| \leq d(x, y)$.

These properties are closely related to the structure of the space H^ω by the following Proposition which is a rather straightforward consequence of Lemma 3.2:

Proposition 3.7. *Let ν be an atom-free Borel probability measure on ∂X and x be in X. Then ω is ν-integrable if and only if ν belongs to H^ω. In this case, we have*

$$\|\mathcal{P}_x\nu\|_2^2 = 2\int_{\partial^2 X} \omega_x \mathrm{d}(\nu\otimes\nu).$$

In the same way, for $1 \leq p < \infty$ and $1 < p' \leq \infty$ with $\frac{1}{p}+\frac{1}{p'}=1$, if ω is ν-p-integrable then $\mathfrak{M}^{p'}(\nu)$ is contained in H^ω. In this case, for every ρ in $\mathfrak{M}^{p'}(\nu)$, we have

$$\|\mathcal{P}_x\rho\|_2^2 = 2\int_{\partial^2 X} \omega_x \mathrm{d}(\rho\otimes\rho).$$

Proof. For y in X, $y \neq x$, let y_- denote the neighbour of y on $[xy]$. For any symmetric function θ in $\ell^1(X_1)$, we have

$$\sum_{a\sim b\in X} \theta(a,b) = 2\sum_{y\in X\smallsetminus\{x\}} \theta(y_-,y). \tag{3.1}$$

For any $k \geq 0$ the function $\omega_x^k = \min(\omega_x, k)$ is smooth on $\partial X \times \partial X$. Let ρ be any element of $\mathcal{D}^*(\partial X)$. By Lemma 3.2, we have

$$\sum_{\substack{y\in X\smallsetminus\{x\}\\ d(x,y)\leq k}} \mathcal{P}_x\rho(y_-,y)^2 = \rho\otimes\rho(\omega_x^k), \tag{3.2}$$

where $\rho\otimes\rho$ is the tensor square distribution of ρ when we use the natural identification $\mathcal{D}(\partial X\times\partial X) \simeq \mathcal{D}(\partial X)\otimes\mathcal{D}(\partial X)$. By (3.1), the left hand-side of (3.2) is increasing, with finite limit if and only if ρ belongs to H^ω. If $\rho = \nu$ is a Borel probability measure, by the Monotone Convergence Theorem, the right hand-side is increasing with finite limit if and only if ω is ν-integrable. The first part of the proposition follows by taking the limit as $k \to \infty$ in (3.2).

Assume now ω is ν-p-integrable and assume the ρ in (3.2) belongs to $\mathfrak{M}^{p'}(\nu)$. Then the right hand-side of (3.2) converges to $\int_{\partial^2 X}\omega_x \mathrm{d}(\rho\otimes\rho)$. Hence the left hand-side has a finite limit, that is, $\mathcal{P}_x\rho$ belongs to $\ell^2(X_1)$. The computation of the norm follows by taking limits in (3.2). □

3.2. **Bilinear forms on H^ω.** We will now define symmetric bilinear forms on H^ω for which an analogue of Proposition 3.7 will be true, where ω will be replaced by an additive kernel as in Proposition 2.22.

Fix $k \geq 1$. If θ is a function on X_1, we define $\theta_{2,k}$ as the function on X_k such that, for any a, b in X with $d(a,b) = k$, one has $\theta_{2,k}(a,b) = \theta(a,a_1)\theta(b_1,b)$, where a_1 is the neighbour of a on $[ab]$ and b_1 the neighbour of b on $[ab]$. Note that if θ is skew-symmetric, then $\theta_{2,k}$ is symmetric.

If $k = 1$, we have $\theta_{2,1} = \theta^2$. In general, we easily get

Lemma 3.8. *For any θ in $\ell^2(X_1)$, the function $\theta_{2,k}$ belongs to $\ell^1(X_k)$ and we have $\|\theta_{2,k}\|_1 \leq (D-1)^{k-1}\|\theta\|_2^2$, where $D = \sup_{x \in X} d(x)$.*

Let w be a bounded symmetric function on X_k. We associate to w the symmetric bilinear form Φ_w on $\ell^2(X_1)$ such that, for any θ in $\ell^2(X_1)$,

$$\Phi_w(\theta,\theta) = \frac{1}{2}\sum_{(a,b)\in X_k} w(a,b)\theta_{2,k}(a,b). \tag{3.3}$$

By Lemma 3.8 above, this bilinear form is well-defined and bounded.

For any x in X and $\xi \neq \eta$ in ∂X, we also set, as in Proposition 2.22,

$$\Omega_x^w(\xi,\eta) = \frac{1}{2}\sum_{h=0}^{j-1}(w(y_h,y_{h+k}) + w(z_h,z_{h+k})) - \frac{1}{2}\sum_{h=1}^{k-1} w(y_{j+h}, z_{j+k-h}),$$

where $(y_h)_{h\geq 0}$ and $(z_h)_{h \geq 0}$ are the geodesic rays $[x\xi)$ and $[x\eta)$ and $j = \omega_x(\xi,\eta)$. Note that one has $|\Omega_x^w| \leq (\omega_x + (k-1))\|w\|_\infty$, so that integrability properties for ω imply the same for Ω^w.

We now have an analogue of Proposition 3.7 for the bilinear form Φ_w.

Proposition 3.9. *Fix $k \geq 1$. Let w be a symmetric bounded function on X_k and ν be an atom-free Borel probability on ∂X such that ω is ν-integrable. Then, for any x in X, we have*

$$\Phi_w(\mathcal{P}_x\nu, \mathcal{P}_x\nu) = \int_{\partial^2 X} \Omega_x^w \mathrm{d}(\nu\otimes\nu).$$

In the same way, for $1 \leq p < \infty$ and $1 < p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, if ω is ν-p-integrable then, for any x in X and ρ, θ in $\mathfrak{M}^{p'}(\nu)$, we have

$$\Phi_w(\mathcal{P}_x\rho, \mathcal{P}_x\theta) = \int_{\partial^2 X} \Omega_x^w \mathrm{d}(\rho\otimes\theta).$$

As for Proposition 3.7, the proof of Proposition 3.9 relies on an elementary computation, which is a generalization of Lemma 3.2:

Lemma 3.10. *Let $k \geq 1$. Fix x in X and $\xi \neq \eta$ in ∂X. Let $(y_h)_{h\geq 0}$ and $(z_h)_{h\geq 0}$ be the geodesic rays $[x\xi)$ and $[x\eta)$ and $j = \omega_x(\xi,\eta)$. Then,* *if a, b are in X with $d(a, b) = k$ and a_1 and b_1 are the neighbours of a and b on $[ab]$, we have*

$$\chi_x(a, a_1, \xi)\chi_x(b_1, b, \eta) = \sum_{h=0}^{j-1}(\mathbf{1}_{(b,a)=(y_h,y_{h+k})} + \mathbf{1}_{(a,b)=(z_h,z_{h+k})}) - \sum_{h=1}^{k-1}\mathbf{1}_{(a,b)=(y_{j+h},z_{j+k-h})}.$$

The above formula has to be understood as an equality of numbers, where the left hand side is either -1, 0 or 1.

Proof of Proposition 3.9. Again, this is a straightforward consequence of Lemma 3.10. Let us be more precise.

We start by defining a truncated version of Ω_x^w. Fix $\ell \geq 0$ and pick ξ, η in ∂X. We let again $(y_h)_{h\geq 0}$ and $(z_h)_{h\geq 0}$ be the geodesic rays $[x\xi)$ and $[x\eta)$. Now, if $\xi \neq \eta$ and $\omega_x(\xi, \eta) < \ell$, we set

$$\Omega_x^{w,\ell}(\xi, \eta) = \frac{1}{2}\sum_{\substack{0\leq h\leq j-1\\ h+k\leq \ell}}(w(y_h, y_{h+k}) + w(z_h, z_{h+k})) - \frac{1}{2}\sum_{\substack{1\leq h\leq k-1\\ j+h\leq \ell\\ j+k-h\leq \ell}} w(y_{j+h}, z_{j+k-h}).$$

Else, we just set

$$\Omega_x^{w,\ell}(\xi, \eta) = \frac{1}{2}\sum_{h=0}^{\ell-k} w(y_h, y_{h+k}) + \frac{1}{2}\sum_{h=0}^{\ell-k} w(z_h, z_{h+k}).$$

Then, $\Omega_x^{w,\ell}$ is a smooth function on $\partial X \times \partial X$ and, by Lemma 3.10, for any ρ in $\mathcal{D}^*(\partial X)$, we have

$$\frac{1}{2}\sum_{\substack{(a,b)\in X_k\\ d(x,a)\leq \ell\\ d(x,b)\leq \ell}}(\mathcal{P}_x\rho)_{2,k}(a, b) = (\rho \otimes \rho)(\Omega_x^{w,\ell}). \tag{3.4}$$

Now, on one hand, by definition, if ρ is in H^ω, the left hand-side of (3.4) goes to $\Phi_w(\mathcal{P}_x\rho, \mathcal{P}_x\rho)$ as $\ell \to \infty$.

On the other hand, assume that ω is ν-integrable so that, by Proposition 3.7, ν belongs to H^ω. We have $\left|\Omega_x^{w,\ell}\right| \leq (\omega_x + (k-1)) \|w\|_\infty$ and for any $\xi \neq \eta$ in ∂X, $\Omega_x^{w,\ell}(\xi, \eta) \xrightarrow[\ell\to\infty]{} \Omega_x^w(\xi, \eta)$. Hence, by the Dominated Convergence Theorem, for $\rho = \nu$, the right hand-side of

(3.4) goes to $2\int_{\partial^2 X}\Omega_x^w \mathrm{d}(\nu\otimes\nu)$ as $\ell\to\infty$ and the first part of the Proposition follows.

The second part is proved in the same way. □

3.3. **Bilinear forms on H_0^ω.** We will now focus on the case where w is Γ-invariant and prove that the restriction of Φ_w to H_0^ω only depends on the cohomology class of the normalized smooth function associated to w.

Proposition 3.11. *Let f be a smooth even Γ-invariant function on $\mathscr{S}$, $k\geq 1$ and w be a Γ-invariant symmetric function on X_k such that the normalized smooth function associated to w is cohomologuous to f. Then the symmetric bilinear form $(\rho,\theta)\mapsto\Phi_w(\mathcal{P}\rho,\mathcal{P}\theta)$ on H_0^ω is Γ-invariant. This bilinear form only depends on the cohomology class of f; in particular it does not depend on the choices of k and w as soon as f is cohomologuous to the normalized smooth function associated to w.*

Let Ω be an additive kernel associated to the cohomology class of f. Then, for $1\leq p<\infty$ and $1<p'\leq\infty$ with $\frac{1}{p}+\frac{1}{p'}=1$, if ν is an atom-free Borel probability measure on ∂X such that ω is ν-p-integrable, for any x in X and ρ,θ in $\mathfrak{M}_0^{p'}(\nu)$, we have

$$\Phi_w(\mathcal{P}\rho,\mathcal{P}\theta)=\int_{\partial^2 X}\Omega_x \mathrm{d}(\rho\otimes\theta).$$

Remark 3.12. Let ν be as above. It easy to check that the formula in Proposition 3.11 defines a bilinear form on $\mathfrak{M}_0^{p'}(\nu)$ which does not depend on the choice of x nor of the one of Ω.

Indeed, pick a smooth boundary B cocycle as in (2.1). For any x,y in X, we have, for ρ in $\mathfrak{M}_0^{p'}(\nu)$,

$$\int_{\partial X\times\partial X}(\Omega_x(\xi,\eta)-\Omega_y(\xi,\eta))\mathrm{d}\rho(\xi)\mathrm{d}\rho(\eta)$$
$$=\int_{\partial X}B_\xi(x,y)\mathrm{d}\rho(\xi)\int_{\partial X}\mathrm{d}\rho(\eta)=0$$

since $\rho(\mathbf{1})=0$. In the same way, by the uniqueness statement in Proposition 2.16, if f is cohomologuous to 0, Ω is of the form

$$(x,\xi,\eta)\mapsto F(x,\xi)+F(y,\eta)$$

for some smooth function F on $X\times\partial X$ and $\int_{\partial X\times\partial X}\Omega_x\mathrm{d}(\rho\otimes\rho)=0$ by the same argument.

Therefore, one way of proving the independence statement in Proposition 3.11 would be to exhibit a Borel probability measure ν on ∂X

such that $\mathfrak{M}^\infty(\nu)$ is dense in H^ω, as will be done later in Corollary 7.3. Here, we will chose an other more direct approach.

For normalized smooth even functions, we have a criterion for cohomology:

Lemma 3.13. *Let $k \geq k' \geq 1$ and $w : X_k \to \mathbb{R}$ and $w' : X_{k'} \to \mathbb{R}$ be Γ-invariant symmetric functions. Then the smooth normalized functions associated to w and w' are cohomologuous if and only if there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that, for any x, y in X with $d(x, y) = k$, one has*

$$w(x,y) = \frac{1}{k-k'+1} \sum_{h=0}^{k-k'} w'(x_h, x_{h+k'}) + v(x, y_{k-1}) - v(x_1, y), \tag{3.5}$$

where $x_0 = x, x_1, \dots, x_k = y$ is the geodesic path from x to y.

When $k = 1$, by convention, a skew-symmetric function on X_0 is 0.

By abuse of language, when w and w' are as above, we shall say that they are cohomologuous and, when w is cohomologuous to 0, we shall say that w is a coboundary.

Proof. First, one easily checks that the normalized smooth function associated to the function defined by the right hand-side of (3.5) is cohomologuous to the normalized smooth function defined by w'. This gives the “if” part of the statement and reduces the proof of the “only if”part to the case where $k' = k$.

In other words, to conclude, we need to show that, if for some Γ-invariant symmetric function w on X_k, the associated normalized smooth function is cohomologuous to 0, then there exists a skew-symmetric function v on X_{k-1} such that, for any (x, y) in X_k, one has $w(x, y) = v(x, y_1) - v(x_1, y)$, where x_1 and y_1 are the neighbours of x and y on $[xy]$. To do this, we will use the language of Subsection 2.3.

Indeed, by assumption, there exists a Γ-invariant smooth function h on $\mathscr{S}$ such that, for any $s = (x_h)_{h\in\mathbb{Z}}$, one has $w(x_0, x_k) = h(s) - h(Ts)$. Now, by Lemma 2.15, the function h is M-invariant, that is, it does not depend on the coordinates $(x_h)_{h<0}$. In the same way, for any such s, one has $w(x_0, x_{-k}) = h(\iota s) - h(T\iota s)$, hence

$$w(x_k, x_0) = h(\iota T^k s) - h(T\iota T^k s) = h(\iota T^k s) - h(\iota T^{k-1} s),$$

so that, again by Lemma 2.15, the function $h \circ \iota T^{k-1}$ is M-invariant, that is, h does not depend on the coordinates $(x_h)_{h\geq k}$. In other words, there exists a Γ-invariant function v on X_{k-1} such that, for any s as above, one has $h(s) = v(x_0, x_{k-1})$. We get, for any (x, y) in X_k,

$w(x,y) = v(x,y_1) - v(x_1,y)$, where x_1 and y_1 are the neighbours of x and y on $[xy]$ and it only remains to prove that one can chose v to be skew-symmetric.

Let still x, y, x_1, y_1 be as above. We have

$$v(x,y_1) - v(x_1,y) = w(x,y) = w(y,x) = v(y,x_1) - v(y_1,x),$$

hence $v(x,y_1) + v(y_1,x) = v(x_1,y) + v(y,x_1)$. In other words, the Γ-invariant smooth function on $\mathscr{S}$,

$$s = (x_h)_{h\in\mathbb{Z}} \mapsto v(x_0,x_{k-1}) + v(x_{k-1},x_0)$$

is T-invariant. By Proposition 2.3, this function is constant, that is, there exists c such that, for any (x,y) in X_{k-1}, one has $v(x,y) + v(y,x) = c$. The result follows by replacing v with $v - \frac{c}{2}$. $\square$

By using Lemma 3.13, we can split the proof of independence in Proposition 3.11 into two cases.

Lemma 3.14. *Let $k \geq 1$ and w be a bounded symmetric function on X_k. Assume that there exists a bounded skew-symmetric function v on X_{k-1} such that, for any (x,y) in X_k,*

$$w(x,y) = v(x,y_1) - v(x_1,y),$$

where x_1 and y_1 are the neighbours of x and y on $[xy]$. Then the bilinear form Φ_w is zero on the space $\mathcal{P}H_0^\omega$.

Proof. By Lemma 3.5, we have to prove that, if θ is a skew-symmetric function in $\ell^2(X_1)$ such that, for any x in X, one has $\sum_{y\sim x}\theta(x,y) = 0$, then $\Phi_w(\theta,\theta) = 0$.

Indeed, if $k = 1$, then $v = 0$ and there is nothing to prove. If $k \geq 2$, we have

$$\Phi_w(\theta,\theta) = \sum_{x\in X}\sum_{x_1\sim x}\sum_{\substack{d(y_1,x)=k-1\\ x_1\in[xy_1]}}\sum_{\substack{y\sim y_1\\ y\notin[xy_1]}}\theta(x,x_1)\theta(y_1,y)v(x,y_1).$$

Now, if x and y_1 are as under the sum, we have

$$\sum_{\substack{y\sim y_1\\ y\notin[xy_1]}}\theta(y_1,y) = \theta(y,y_2),$$

where y_2 is the neighbour of y_1 on $[xy_1]$. Thus, we get

$$\Phi_w(\theta,\theta) = -\sum_{(a,b)\in X_{k-1}}\theta_{2,k-1}(a,b)v(a,b),$$

where $\theta_{2,k-1}$ is the same as in Subsection 3.2. As $\theta_{2,k-1}$ is symmetric and v is skew-symmetric, the latter sum is zero and we are done. $\square$

Lemma 3.15. *Let $k \geq k' \geq 1$ and w and w' be bounded symmetric functions on X_k and $X_{k'}$. Assume that, for any (x, y) in X_k,*

$$w(x,y) = \frac{1}{k-k'+1} \sum_{h=0}^{k-k'} w'(x_h, x_{h+k'}),$$

where $x_0 = x, x_1, \ldots, x_k = y$ is the geodesic path from x to y. Then the bilinear forms Φ_w and $\Phi_{w'}$ are equal to each other on the space $\mathcal{P}H_0^\omega$.

Proof. Again, it suffices to prove that, if θ is a skew-symetric function in $\ell^2(X_1)$ with $\sum_{y \sim x} \theta(x, y) = 0$ for any x in X, one has $\Phi_w(\theta, \theta) = \Phi_{w'}(\theta, \theta)$. For such a θ, we have

$$\begin{aligned} (3.6) \quad & 2\Phi_w(\theta,\theta) \\ &= \frac{1}{k-k'+1} \sum_{h=0}^{k-k'} \sum_{(x,y)\in X_{k'}} \sum_{\substack{d(a,y)=h+k' \\ x\in[ay]}} \sum_{\substack{d(b,x)=k-h \\ y\in[xb]}} w'(x,y)\theta(a,a_-)\theta(b_-,b), \end{aligned}$$

where, if a and b are as under the sum, a_- and b_- are the neighbours of a and b in $[ay]$ and $[xb]$. Now, for any x, y in X with $x \sim y$, an easy induction argument shows that, for $h \geq 0$,

$$\sum_{\substack{d(a,y)=h+1 \\ x\in[ay]}} \theta(a,a_-) = \theta(x,y),$$

hence, for (x, y) in $X_{k'}$ and $0 \leq h \leq k - k'$,

$$\sum_{\substack{d(a,y)=h+k' \\ x\in[ay]}} \theta(a,a_-) \sum_{\substack{d(b,x)=k-h \\ y\in[xb]}} \theta(b_-,b) = \theta_{2,k'}(x,y).$$

By (3.6), we get $\Phi_w(\theta, \theta) = \Phi'_w(\theta, \theta)$ as required. □

Proof of Proposition 3.11 . The fact that the definition of the bilinear form is independent on w follows from Lemmas 3.13, 3.14 and 3.15.

As the linear map $\mathcal{P} : H_0^\omega \to \ell^2(X_1)$ and the quadratic maps $\theta \mapsto \theta_{2,k}$, $k \geq 0$, commute with the action of the group of automorphisms of X, the bilinear form Φ_w is Γ-invariant on $\mathcal{P}H_0^\omega$ as soon as w is Γ-invariant.

Finally, the integral formula follows from the one in Proposition 3.9 and from Remark 3.12. □

In the sequel of the paper, we will study those Γ-invariant functions w such that the associated symmetric bilinear form Φ_w is non-negative on H_0^ω. This will require us to introduce first several notions related to non-negative bilinear forms on vector spaces associated with X.

4. Bilinear forms on smooth functions

In this section, we build scalar products on spaces of smooth functions. The topological dual spaces of these spaces with respect to these scalar products will later turn out to be defined by additive kernels.

4.1. **Quadratic type functions.** Recall that $\overline{\mathcal{D}}(\partial X) = \mathcal{D}(\partial X)/\mathbb{R}$ is the quotient space of $\mathcal{D}(\partial X)$ by the space of constant functions on ∂X. We will give algebraic constructions of symmetric bilinear forms on the space $\overline{\mathcal{D}}(\partial X)$.

Recall that, for any $k \geq 1$, we let X_k stand for the set of (x, y) in X^2 with $d(x, y) = k$. We set

$$X_* = \{(x, y) \in X^2 | x \neq y\}.$$

A function $\varphi : X_* \to \mathbb{R}$ is said to be of quadratic type if it is symmetric and if, for every (x, y) in X_*, we have

$$\varphi(x, y) = \sum_{\substack{z \sim y \\ z \notin [xy]}} \varphi(x, z).$$

Recall also that, for (x, y) in X_*, we let U_{xy} be the closed open subset of ∂X

$$U_{xy} = \{\xi \in \partial X | [xy] \cap [y\xi) = \{y\}\}.$$

Let p be a symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. We associate to p a symmetric function φ_p on X_* by setting, for any (x, y) in X_*,

$$\varphi_p(x, y) = -p(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{yx}}).$$

We get the following characterization of quadratic type functions, which will be used throughout the article:

Proposition 4.1. *The map $p \mapsto \varphi_p$ is a linear isomorphism between the space of symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ and the space of quadratic type functions on X_*.*

The proof of this result will follow from a truncated version of it which we will now give. We first define quadratic type functions on X_k, $k \geq 1$.

Definition 4.2. Let $k \geq 1$. If $k = 1$, a function $\varphi : X_1 \to \mathbb{R}$ is said to be of quadratic type if it is symmetric. If $k \geq 2$, a function $\varphi : X_k \to \mathbb{R}$ is said to be of quadratic type if it is symmetric and if the function

$$\varphi^- : (x, y) \mapsto \sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi(z, y), X_{k-1} \to \mathbb{R} \tag{4.1}$$

is symmetric. The function φ^- is called the reduction of φ.

The reduction of a quadratic type function is of quadratic type.

Lemma 4.3. *Let $k \geq 2$ and φ be a quadratic type function on X_k. Then φ^- is of quadratic type and, for (x,y) in X_k, we have*

$$\sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi(z,y) = \varphi^-(x,y) = \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(x,t).$$

Proof. We first prove the formula. As both φ and φ^- are symmetric, by (4.1), we have, for (x,y) in X_k,

$$\varphi^-(x,y) = \varphi^-(y,x) = \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(t,x) = \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(x,t).$$

Now, about the first statement, if $k = 2$, there is nothing to prove. If $k \geq 3$, for $(x,y) \in X_{k-1}$, we have

$$\sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi^-(z,y) = \sum_{\substack{z \sim x \\ z \notin [xy]}} \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(z,t),$$

which is clearly symmetric in (x,y). Thus, φ^- is of quadratic type. $\square$

In particular, any quadratic type function φ on X_k defines in a natural way a quadratic type function on all the X_h, $1 \leq h \leq k$. By abuse of notation, this function will be sometimes again denoted by φ.

4.2. **Quadratic fields.** We will now give an interpretation of this notion in terms of symmetric bilinear forms on certain spaces.

If $\ell \geq 0$ is an integer, recall that, for any x in X, we denote by $S^\ell(x)$, the sphere with center x and radius ℓ in X, that is, the set of y in X with $d(x,y) = \ell$. We let $V^\ell(x)$ denote the space of real-valued functions on $S^\ell(x)$ and $\overline{V}^\ell(x) = V^\ell(x)/\mathbb{R}$ denote its quotient by the line of constant functions.

If x and y are neighboring elements of X (that is, $x \sim y$), we let $S^\ell(xy)$ denote the set

$$S^\ell(xy) = \{z \in S^\ell(x) | y \notin [xz]\} \cup \{z \in S^\ell(y) | x \notin [yz]\}.$$

We let $V^\ell(xy)$ denote the space of real-valued functions on $S^\ell(xy)$ and $\overline{V}^\ell(xy) = V^\ell(xy)/\mathbb{R}$ denote its quotient by the line of constant functions.

For any $\ell \geq 0$ and any x, y in X with $x \sim y$, we define linear maps

$$\begin{aligned} I^\ell_{xy} &: V^\ell(xy) \to V^{\ell+1}(x) \\ J^\ell_{xy} &: V^\ell(x) \to V^\ell(xy) \end{aligned}$$

as follows.

If f is in $V^\ell(xy)$, then $I^\ell_{xy}f$ is the function on $S^{\ell+1}(x)$ such that, for any z in $S^{\ell+1}(x)$, one has

$I^\ell_{xy}f(z) = f(z)$ if y is on $[xz]$ (and hence $d(y,z) = \ell$).

$I^\ell_{xy}f(z) = f(w)$ if y is not on $[xz]$ and w is the neighbor of z on $[xz]$.

If f is in $V^\ell(x)$, then $J^\ell_{xy}f$ is the function on $S^\ell(xy)$ such that, for any z in $S^\ell(xy)$, one has

$J^\ell_{xy}f(z) = f(z)$ if y is not on $[xz]$ (and hence $d(x,z) = \ell$).

$J^\ell_{xy}f(z) = f(w)$ if y is on $[xz]$ and w is the neighbor of z on $[xz]$.

These maps are injections and they send constant functions to constant functions. In particular, they induce linear injections $\overline{V}^\ell(xy) \to \overline{V}^{\ell+1}(x)$ and $\overline{V}^\ell(x) \to \overline{V}^\ell(xy)$ which we still denote by I^ℓ_{xy} and J^ℓ_{xy}.

Finally, for any $\ell \geq 0$ and x in X, we let $M^\ell_x : V^\ell(x) \to V^{\ell+1}(x)$ be the map that sends a function f in $V^\ell(x)$ towards the function $M^\ell_x f$ such that, for any z in $S^{\ell+1}(x)$,

$$M^\ell_x f(z) = f(w) \text{ where } w \text{ is the neighbor of } z \text{ on } [xz].$$

In the same way, if x, y are in X and $x \sim y$, we let $M^\ell_{xy} : V^\ell(xy) \to V^{\ell+1}(xy)$ be the map that sends a function f in $V^\ell(xy)$ towards the function $M^\ell_x f$ such that, for any z in $S^{\ell+1}(xy)$,

$$M^\ell_{xy} f(z) = f(w) \text{ where } w \text{ is the neighbor of } z \text{ on } [xz].$$

Again, we still denote by M^ℓ_x and by M^ℓ_{xy} the associated injections $\overline{V}^\ell(x) \to \overline{V}^{\ell+1}(x)$ and $\overline{V}^\ell(xy) \to \overline{V}^{\ell+1}(xy)$.

An immediate computation gives

Lemma 4.4. *For any $\ell \geq 0$ and x, y in X with $x \sim y$, we have*

$$I^\ell_{xy} J^\ell_{xy} = M^\ell_x$$
$$J^{\ell+1}_{xy} I^\ell_{xy} = M^\ell_{xy}.$$

Let us describe how these maps allow to split the spaces into smaller ones.

Proposition 4.5. *For any $\ell \geq 1$ and x in X, the space $\overline{V}^\ell(x)$ is spanned by the subspaces*

$$I^{\ell-1}_{xy}\overline{V}^{\ell-1}(xy), \quad y \sim x.$$

For y, z in X with $y \sim x$, $z \sim x$ and $y \neq z$, we have

$$I^{\ell-1}_{xy}\overline{V}^{\ell-1}(xy) \cap I^{\ell-1}_{xz}\overline{V}^{\ell-1}(xz) = M^{\ell-1}_x \overline{V}^{\ell-1}(x).$$

If $\ell \geq 2$, we have

$$\overline{V}^{\ell}(x)/M_x^{\ell-1}\overline{V}^{\ell-1}(x) = \bigoplus_{y\sim x} I_{xy}^{\ell-1}\overline{V}^{\ell-1}(xy)/M_x^{\ell-1}\overline{V}^{\ell-1}(x).$$

Proposition 4.6. *For any $\ell \geq 1$ and x, y in X with $x \sim y$, the space $\overline{V}^{\ell}(xy)$ is spanned by the subspaces*

$$J_{xy}^{\ell}\overline{V}^{\ell}(x) \text{ and } J_{yx}^{\ell}\overline{V}^{\ell}(y).$$

The intersection of these two subspaces is

$$J_{xy}^{\ell}\overline{V}^{\ell}(x) \cap J_{yx}^{\ell}\overline{V}^{\ell}(y) = M_{xy}^{\ell-1}\overline{V}^{\ell-1}(xy).$$

The proofs are immediate.

For $k \geq 1$, we will now define the notion of a k-quadratic field. The definition depends on the parity of k.

Definition 4.7. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. A k-quadratic field is a family $(p_x)_{x\in X}$ where, for any x in X, p_x is a symmetric bilinear form on $\overline{V}^{\ell}(x)$, such that, for any x, y in X with $x \sim y$, we have

$$(I_{xy}^{\ell-1})^{\star}p_x = (I_{yx}^{\ell-1})^{\star}p_y.$$

This bilinear form on $\overline{V}^{\ell-1}(xy)$ is denoted by p_{xy}^{-}.

Definition 4.8. (k odd) Let k be an odd integer, $k = 2\ell+1$, $\ell \geq 0$. A k-quadratic field is a family $(p_{xy})_{x\sim y\in X}$ where, for any x, y in X with $x \sim y$, $p_{xy} = p_{yx}$ is a symmetric bilinear form on $\overline{V}^{\ell}(xy)$, such that, for any x in X, the bilinear forms

$$(J_{xy}^{\ell})^{\star}p_{xy}, \quad y \sim x,$$

are all equal to each other. This bilinear form on $\overline{V}^{\ell}(x)$ is denoted by p_x^{-}.

From the combinatorial properties of the spaces of functions on spheres, we have

Proposition 4.9. *Let $k \geq 2$ and let p be a k-quadratic field. Then p^{-} is a $(k-1)$-quadratic field.*

We call p^{-} the reduction of p.

Proof. First assume k is even, $k = 2\ell$, $\ell \geq 1$. For x in X and y with $y \sim x$, we need to prove that the bilinear form

$$(J_{xy}^{\ell-1})^{\star}p_{xy}^{-}$$

does not depend on y. By definition, we have

$$p^-_{xy} = (I^{\ell-1}_{xy})^\star p_x,$$

hence

$$(J^{\ell-1}_{xy})^\star p^-_{xy} = (I^{\ell-1}_{xy} J^{\ell-1}_{xy})^\star p_x.$$

Now, by Lemma 4.4,

$$I^{\ell-1}_{xy} J^{\ell-1}_{xy} = M^{\ell-1}_x$$

and the result follows.

Assume now k is odd, $k = 2\ell + 1$, $\ell \geq 1$. For x, y in X with $x \sim y$, we need to prove that the bilinear forms

$$(I^{\ell-1}_{xy})^\star p^-_x \text{ and } (I^{\ell-1}_{yx})^\star p^-_y$$

are equal to each other. Again, by definition, we have

$$p^-_x = (J^\ell_{xy})^\star p_{xy}$$

hence

$$(I^{\ell-1}_{xy})^\star p^-_x = (J^\ell_{xy} I^{\ell-1}_{xy})^\star p_{xy}.$$

Still by Lemma 4.4, we have

$$J^\ell_{xy} I^{\ell-1}_{xy} = M^{\ell-1}_{xy}$$

and the result follows. □

Remark 4.10. If $k = 1$ the compatibility condition in the definition of a 1-quadratic field is empty. In particular, such a field is simply the data of the symmetric function $(x, y) \mapsto p_{xy}(\mathbf{1}_x, \mathbf{1}_y)$ on X_1. We shall extend this correspondance to higher k.

4.3. **Fields and quadratic type functions.** Let $k \geq 1$ and p be a k-quadratic field. We will associate to p a symmetric function φ_p on X_k as follows. Pick x, y in X and let $z_0 = x, z_1, \dots, z_k = y$ be the geodesic path from x to y. If k is even, $k = 2\ell$, $\ell \geq 1$, we set

$$\varphi_p(x, y) = -p_{z_\ell}(\mathbf{1}_x, \mathbf{1}_y).$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, we set

$$\varphi_p(x, y) = -p_{z_\ell z_{\ell+1}}(\mathbf{1}_x, \mathbf{1}_y).$$

Proposition 4.11. *Fix $k \geq 1$. The map $p \mapsto \varphi_p$ is a linear isomorphism between the space of k-quadratic field and the one of quadratic type functions on X_k.*

Remark 4.12. From Definition 4.7 and Definition 4.8, we see that the interpretation of a quadratic type function as a field of quadratic forms depends on the parity of k. When k is even, one has to take the point of view of vertices, whereas when k is odd, one should take the point of view of edges. From now on and until the end of the paper, we will often need to split the proofs according to the parity of k.

We will prove Proposition 4.11 in several steps.

Lemma 4.13. *For any $k \geq 1$, if p is a k-quadratic field, then the function φ_p is of quadratic type. If $k \geq 2$, one has $\varphi_{p^-} = (\varphi_p)^-$.*

Proof. If $k = 1$, this has already been noticed in Remark 4.10.

Assume $k \geq 2$. Recall that, by Proposition 4.9, p^- is a $(k-1)$-quadratic field. Let us prove that, for any (x, y) in X_{k-1}, we have

$$\varphi_{p^-}(x,y) = \sum_{\substack{z \sim x \\ z \notin [x,y]}} \varphi_p(z,y). \tag{4.2}$$

Let $z_0 = x, z_1, \ldots, z_{k-1} = y$ be the geodesic path from x to y.

If k is even, $k = 2\ell$, $\ell \geq 1$, we have

$$-\varphi_{p^-}(x,y) = p^-_{z_{\ell-1}z_\ell}(\mathbf{1}_x, \mathbf{1}_y) = p_{z_{\ell-1}}(I^{\ell-1}_{z_{\ell-1}z_\ell}\mathbf{1}_x, I^{\ell-1}_{z_{\ell-1}z_\ell}\mathbf{1}_y).$$

Now, by definition, we have

$$I^{\ell-1}_{z_{\ell-1}z_\ell}\mathbf{1}_y = \mathbf{1}_y \text{ and } I^{\ell-1}_{z_{\ell-1}z_\ell}\mathbf{1}_x = \sum_{\substack{z \sim x \\ z \neq z_1}} \mathbf{1}_z$$

and the result follows.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we have

$$-\varphi_{p^-}(x,y) = p^-_{z_\ell}(\mathbf{1}_x, \mathbf{1}_y) = p_{z_\ell z_{\ell-1}}(J^\ell_{z_\ell z_{\ell-1}}\mathbf{1}_x, J^\ell_{z_\ell z_{\ell-1}}\mathbf{1}_y).$$

Again, by definition, we have

$$J^\ell_{z_\ell z_{\ell-1}}\mathbf{1}_y = \mathbf{1}_y \text{ and } J^\ell_{z_\ell z_{\ell-1}}\mathbf{1}_x = \sum_{\substack{z \sim x \\ z \neq z_1}} \mathbf{1}_z$$

and we are done.

In particular, as φ_{p^-} is symmetric, φ_p is of quadratic type. By comparing (4.1) with (4.2), we get $\varphi_{p^-} = (\varphi_p)^-$. □

We will now prove that the map $p \mapsto \varphi_p$ is injective. Recall that, for any quadratic type function φ on X_k, we still denote by φ its natural extension to $\bigcup_{1 \leq \ell \leq k} X_\ell$. We get the following easy formula for recovering p from φ_p.

Lemma 4.14. *Let $k \geq 1$ and p be a k-quadratic field.*

If k is even, $k = 2\ell$, $\ell \geq 1$, for any x in X and $z \neq t$ in $S^\ell(x)$, we have

$$p_x(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_p(z,t). \tag{4.3}$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any x, y in X with $x \sim y$ and any $z \neq t$ in $S^\ell(xy)$, we have

$$p_{xy}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_p(z,t). \tag{4.4}$$

Proof. We prove this by induction on k. If $k = 1$, this is obvious.

Assume $k \geq 2$ and the result is true for $k - 1$.

Assume k is even, $k = 2\ell$, $\ell \geq 1$ and pick x in X and $z \neq t$ in $S^\ell(x)$. If x belongs to $[zt]$, (4.3) follows from the definition of φ_p. Else, there exists a neighbour y of x such that z and t belong to $S^{\ell-1}(y)$. We then get

$$\mathbf{1}_z = I_{xy}^{\ell-1}(\mathbf{1}_z) \text{ and } \mathbf{1}_t = I_{xy}^{\ell-1}(\mathbf{1}_t),$$

hence

$$p_x(\mathbf{1}_z, \mathbf{1}_t) = p_{xy}^-(\mathbf{1}_z, \mathbf{1}_t).$$

Now, by the induction assumption, the latter is equal to $\varphi_p(z,t)$ and we are done.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we pick x, y in X with $x \sim y$ and $z \neq t$ in $S^\ell(xy)$. If $[xy] \subset [zt]$, again, we have (4.4) by definition. Else, up to exchanging the roles of x and y, we can assume $z, t \in S^\ell(x)$, hence

$$\mathbf{1}_z = J_{xy}^\ell(\mathbf{1}_z) \text{ and } \mathbf{1}_t = J_{xy}^\ell(\mathbf{1}_t)$$

and

$$p_{xy}(\mathbf{1}_z, \mathbf{1}_t) = p_x^-(\mathbf{1}_z, \mathbf{1}_t).$$

Again, the result now follows from the induction assumption. □

Surjectivity will follow from the following elementary

Lemma 4.15. *Let A be a finite set. Let V be the space of real-valued functions on A and $\overline{V} = V/\mathbb{R}$ be its quotient by the space of constant functions. Set $A_2 = \{(a,b) \in A^2 | a \neq b\}$. If p is a symmetric bilinear form on $\overline{V}$, let φ_p be the function on A_2 defined by*

$$\varphi_p(a,b) = -p(\mathbf{1}_a, \mathbf{1}_b), \quad a \neq b.$$

Then the map $p \mapsto \varphi_p$ is a linear isomorphism between the space of symmetric bilinear form on $\overline{V}$ and the space of symmetric real-valued functions on A_2.

We are now ready to give the full

Proof of Proposition 4.11. Let $k \geq 1$. By Lemma 4.13, the map $p \mapsto \varphi_p$ sends k-quadratic fields to quadratic type functions on X_k. By Lemmas 4.14 and 4.15, this map is injective. It remains to prove that it is surjective. Fix φ a quadratic type function on X_k and let us construct p such that $\varphi = \varphi_p$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any x in X, by Lemma 4.15, there exists a unique symmetric bilinear form p_x on $\overline{V}^{\ell}(x)$ such that

$$p_x(\mathbf{1}_z, \mathbf{1}_w) = -\varphi(z, w), \quad z \neq w \in S^{\ell}(x).$$

Let us show that the family $p = (p_x)_{x \in X}$ is a k-quadratic field. We claim that, for any $x \sim y$ in X, for any $z \neq t$, in $S^{\ell-1}(xy)$ we have

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_t) = -\varphi(z, t),$$

which, by Lemma 4.15, implies that $(I_{xy}^{\ell-1})^{\star} p_x = (I_{yx}^{\ell-1})^{\star} p_y$. Indeed, if z and t are in $S^{\ell-1}(y)$, we have

$$I_{xy}^{\ell-1}(\mathbf{1}_z) = \mathbf{1}_z \text{ and } I_{xy}^{\ell-1}(\mathbf{1}_t) = \mathbf{1}_t,$$

hence by definition

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_t) = -\varphi(z, t).$$

If z is in $S^{\ell-1}(x)$ and t is in $S^{\ell-1}(y)$, we have

$$I_{xy}^{\ell-1}(\mathbf{1}_z) = \sum_{\substack{z' \sim z \\ z' \notin [xz]}} \mathbf{1}_{z'} \text{ and } I_{xy}^{\ell-1}(\mathbf{1}_t) = \mathbf{1}_t,$$

hence

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_t) = -\sum_{\substack{z' \sim z \\ z' \notin [xz]}} \varphi(z', t) = -\varphi(z, t).$$

Finally, if z and t are in $S^{\ell-1}(x)$, we have

$$I_{xy}^{\ell-1}(\mathbf{1}_z) = \sum_{\substack{z' \sim z \\ z' \notin [xz]}} \mathbf{1}_{z'} \text{ and } I_{xy}^{\ell-1}(\mathbf{1}_t) = \sum_{\substack{t' \sim t \\ t' \notin [xz]}} \mathbf{1}_{t'},$$

and again

$$\begin{aligned} p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_z) &= -\sum_{\substack{z' \sim z \\ z' \notin [xz]}} \sum_{\substack{t' \sim t \\ t' \notin [xz]}} \varphi(z', t') \\ &= -\sum_{\substack{z' \sim z \\ z' \notin [xz]}} \varphi(z', t) = -\varphi(z, t). \end{aligned}$$

If k is odd, $k = 2\ell$, $\ell \geq 0$, still by Lemma 4.15, for any $x \sim y$ in X, there exists a unique symmetric bilinear form p_{xy} on $\overline{V}^\ell(xy)$ such that

$$p_{xy}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi(z,t), \quad z \neq t \in S^\ell(xy).$$

We also show that the family $p = (p_{xy})_{x \sim y \in X}$ is a k-quadratic field. This will now follow from the fact that, for any $x \sim y$ in X, for any $z \neq w$, in $S^\ell(x)$,

$$p_{xy}(J^\ell_{xy}\mathbf{1}_z, J^\ell_{xy}\mathbf{1}_t) = -\varphi(z,t),$$

which we prove as above. □

4.4. **Fields and bilinear forms on smooth functions.** We will now give the proof of Proposition 4.1. To this aim, let us introduce a new set of linear operators. For x in X and $\ell \geq 0$, we let

$$N^\ell_x : V^\ell(x) \to \mathcal{D}(\partial X)$$

be the linear operator such that, for any f in $V^\ell(x)$, y in $S^\ell(x)$ and ξ in U_{xy}, one has

$$N^\ell_x f(\xi) = f(y).$$

Again, one still denotes by N^ℓ_x the induced operator $\overline{V}^\ell(x) \to \overline{\mathcal{D}}(\partial X)$.

We will use the easy

Lemma 4.16. *For any x in X, one has*

$$\mathcal{D}(\partial X) = \bigcup_{\ell \geq 0} N^\ell_x V^\ell(x)$$

and, for any $\ell \geq 0$,

$$N^{\ell+1}_x M^\ell_x = N^\ell_x.$$

Proof. The first part is a rewriting of Lemma 3.6. The second part follows from a straightforward computation. □

Proof of Proposition 4.1. Let p be a symmetric bilinear form on the space $\overline{\mathcal{D}}(\partial X)$. Then, since for any $x \neq y$ in X, the closed open set U_{yx} is the disjoint union of the U_{yz}, $z \sim x$, $z \notin [xy]$, we have

$$\varphi_p(x,y) = \sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi_p(z,y),$$

hence φ_p is of quadratic type.

If φ_p is 0, we claim that p is 0. Indeed, as the characteristic functions of the closed open subsets U_{xy}, $x \sim y \in X$, span $\mathcal{D}(\partial X)$, it suffices to check that for any $x \sim y$ and $z \sim w$ in X, we have $p(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{zw}}) = 0$. As $\mathbf{1}_{U_{xy}} + \mathbf{1}_{U_{yx}} = \mathbf{1}$ and $\mathbf{1}$ is in the null space of p, we can assume that $y \neq w$ and x and z belong to $[yw]$. We then have $U_{xy} = U_{wy}$ and $U_{zw} = U_{yw}$ hence

$$p(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{zw}}) = -\varphi_p(y, w) = 0,$$

and we are done.

Finally, if φ is a quadratic type function on X_*, for any $k \geq 1$, let φ_k be the restriction of φ to X_k which is a quadratic type function on X_k. By Proposition 4.11, there exists a unique k-quadratic field p^k such that $\varphi_{p^k} = \varphi_k$. By Lemma 4.13, for $k \geq 2$, one has $(p^k)^- = p^{k-1}$. Fix x in X. We get, for any $\ell \geq 1$,

$$(M_x^\ell)^\star p^{2(\ell+1)} = p^{2\ell}.$$

By Lemma 4.16, this tells us that there exists a unique symmetric bilinear form p on $\overline{\mathcal{D}}(\partial X)$ such that, for any $\ell \geq 1$,

$$(N_x^\ell)^\star p = p^{2\ell}.$$

One verifies that $\varphi = \varphi_p$. $\square$

4.5. **Orthogonal extension of Euclidean fields.** We just saw how a globally defined quadratic type function on X_* gives rise to quadratic type functions on X_k for any $k \geq 1$. We will now introduce a reverse operation in the Euclidean case. It will rely on the following lemma, which we will apply to the decompositions of spaces of functions on spheres from Proposition 4.5 and Proposition 4.6.

Lemma 4.17. *Let X be a finite-dimensional real vector space, $d \geq 2$ be an integer and $X_1, \ldots, X_d$ be subspaces of X. We assume that there exists a subspace X_0 of X such that, for any $1 \leq i \neq j \leq d$, $X_i \cap X_j = X_0$ and $X/X_0 = \bigoplus_i X_i/X_0$. Let $p_0, p_1, \ldots, p_d$ be positive definite symmetric bilinear forms on $X_0, X_1, \ldots, X_d$ such that, for any $1 \leq i \leq d$, $p_{i|X_0} = p_0$. For $1 \leq i \leq d$, let $Y_i \subset X_i$ be the orthogonal complement of X_0 in X_i with respect to p_i. Then, there exists a unique positive definite symmetric bilinear form p on X such that, for any $1 \leq i \leq d$, $p_{|X_i} = p_i$ and, for any $1 \leq i \neq j \leq d$, the spaces Y_i and Y_j are orthogonal with respect to p. The form p is called the orthogonal extension of $p_1, \ldots, p_d$ to X.*

Definition 4.18. Let $k \geq 1$. If p is a k-quadratic field, we shall say that p is a k-Euclidean field if the associated symmetric bilinear forms are positive definite.

If $k \geq 2$, we will build an orthogonal extension of these fields which is a $(k+1)$-Euclidean field.

Definition 4.19. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. If p is a k-Euclidean field, for any $x \sim y$ in X, we let p^+_{xy} denote the orthogonal extension of p_x and p_y to $\overline{V}^\ell(xy)$, where $\overline{V}^\ell(x)$ and $\overline{V}^\ell(y)$ are identified to subspaces of $\overline{V}^\ell(xy)$ through the maps J^ℓ_{xy} and J^ℓ_{yx}. The family $p^+ = (p^+_{xy})_{x\sim y\in X}$ is called the orthogonal extension of p.

Definition 4.20. (k odd) Let k be an odd integer, $k = 2\ell+1$, $\ell \geq 1$. If p is a k-Euclidean field, for any x in X, we let p^+_x denote the orthogonal extension of $(p_{xy})_{y\sim x}$ to $\overline{V}^{\ell+1}(x)$, where the spaces $\overline{V}^\ell(xy)$, $y \sim x$, are identified to subspaces of $\overline{V}^{\ell+1}(x)$ through the maps I^ℓ_{xy}, $y \sim x$. The family $p^+ = (p^+_x)_{x\in X}$ is called the orthogonal extension of p.

The orthogonal extension is again a quadratic field. More precisely, we have the following result, whose proof directly follows from the definitions:

Proposition 4.21. *Let $k \geq 2$ and p be a k-Euclidean field. Then its orthogonal extension p^+ is a $(k+1)$-Euclidean field and $(p^+)^- = p$.*

Recall that the reduction p^- of p was introduced in Definition 4.7 and Definition 4.8.

4.6. **The Hilbert space of a Euclidean field.** Let p be a k-Euclidean field with $k \geq 2$. The successive orthogonal extensions of p allow to define a quadratic type function φ^∞_p on X_*, or equivalently, by Proposition 4.1, a symmetric bilinear form p^∞ on $\overline{\mathcal{D}}(\partial X)$, which clearly turns out to be positive definite. We let H^p be the completion of $\overline{\mathcal{D}}(\partial X)$ with respect to p^∞ and we call it the Hilbert space of p.

If p is Γ-invariant, so is φ^∞_p, hence p^∞ is Γ-invariant and H^p comes with a natural action of Γ which makes it a unitary representation.

In the next section we will study the topological dual space of H^p.

5. Hilbert spaces of distributions

5.1. **Dual kernels.** We will now introduce dual notions to the ones studied above. We start with a dual statement to Lemma 4.15.

Lemma 5.1. *Let A be a finite set. Let V be the space of real-valued functions on A and*

$$V_0 = \{f \in V | \sum_{a\in A} f(a) = 0\}.$$

If q is a symmetric bilinear form on V_0, let K^q be the function on $A \times A$ defined by

$$K^q(a,b) = q(\mathbf{1}_a - \mathbf{1}_b, \mathbf{1}_a - \mathbf{1}_b), \quad a,b \in A.$$

Then the map $q \mapsto K^q$ is a linear isomorphism between the space of symmetric bilinear forms on V_0 and the space of symmetric real-valued functions on $A \times A$ which are zero on the diagonal. If K is such a function and q is the associated bilinear form, for any f, g in V_0, we have

$$q(f,g) = -\frac{1}{2} \sum_{(a,b)\in A^2} K(a,b)f(a)f(b).$$

In the sequel, if A is a finite set and V is the space of real-valued functions on A, we always identify the space V with its dual space through the positive definite bilinear form on V

$$(f,g) \mapsto \sum_{a\in A} f(a)g(a).$$

Let V_0 be the space of functions in V with zero sum,

$$V_0 = \{f \in V | \sum_{a\in A} f(a) = 0\}.$$

The space V_0 may now be seen as the dual space of $\overline{V} = V/\mathbb{R}$.

In particular, for any $\ell \geq 0$, for any x in X, let $V_0^\ell(x)$ denote the set of real-valued functions on $S^\ell(x)$ with zero sum, and, for any $x \sim y$ in X, let $V_0^\ell(xy)$ denote the set of real-valued functions on $S^\ell(xy)$ with zero sum. We regard these spaces as the dual spaces of $\overline{V}^\ell(x)$ and $\overline{V}^\ell(xy)$.

Recall that if V is a finite-dimensional real vector space, to any non-degenerate symmetric bilinear form p on V, we can associate its dual bilinear form q on the dual space V^* of V. The form q is defined as the image of p by the linear isomorphism from V to V^* associated to the form p.

Let p be a k-Euclidean field for some $k \geq 1$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any x in X, we let q_x be the dual symmetric bilinear form to p_x on $V_0^\ell(x)$. If z and w are in $S^\ell(x)$, we set

$$K_x^p(z,w) = q_x(\mathbf{1}_z - \mathbf{1}_w, \mathbf{1}_z - \mathbf{1}_w).$$

If ξ, η are in ∂X, we write

$$K_x^p(\xi,\eta) = K_x^p(z,w),$$

where z (resp. w) is the intersection point of the geodesic ray $[x\xi)$ (resp. $[x\eta)$) with $S^\ell(x)$.

If k is odd, $k = 2\ell+1$, $\ell \geq 0$, for any $x \sim y$ in X, we let q_{xy} be the dual symmetric bilinear form to p_{xy} on $V_0^\ell(xy)$. If z and w are in

$S^\ell(xy)$, we set

$$K^p_{xy}(z,w) = q_{xy}(\mathbf{1}_z - \mathbf{1}_w, \mathbf{1}_z - \mathbf{1}_w).$$

If ξ, η are in ∂X, we write

$$K^p_{xy}(\xi,\eta) = K^p_{xy}(z,w),$$

where z (resp. w) is the intersection point of the geodesic ray $[x\xi)$ (resp. $[x\eta)$) with $S^\ell(xy)$.

By Lemma 5.1, the Euclidean field p is completely determined by the data of K^p. We have a nice way of computing K^{p^+} from K^p and K^{p^-}. Recall that, for x in X, $d(x)$ is the number of neighbours of x.

Proposition 5.2. *Let p be a k-Euclidean field for some $k \geq 2$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have, as functions on $\partial X \times \partial X$,

$$K^{p^+}_{xy} = K^p_x + K^p_y - K^{p^-}_{xy}.$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any x in X, we have, as functions on $\partial X \times \partial X$,

$$K^{p^+}_x = \sum_{y \sim x} K^p_{xy} - (d(x) - 1) K^{p^-}_x.$$

Proof. The proof is a direct translation of Lemma 5.3 below. □

Lemma 5.3. *Let $X, X_0, X_1, \ldots, X_d$ and $p_0, p_1, \ldots, p_d$ be as in Lemma 4.17. Let p be the orthogonal extension of $p_1, \ldots, p_d$ to X. Equip the dual spaces $X^*, X_0^*, X_1^*, \ldots, X_d^*$ of $X, X_0, X_1, \ldots, X_d$ with the bilinear forms $q, q_0, q_1 \ldots, q_d$ which are dual to $p, p_0, p_1, \ldots, p_d$. Then, for every φ, ψ in X^*, we have*

$$q(\varphi,\psi) = q_1(\varphi_{|X_1}, \psi_{|X_1}) + \cdots + q_d(\varphi_{|X_d}, \psi_{|X_d}) - (d-1) q_0(\varphi_{|X_0}, \psi_{|X_0}).$$

Proof. For $1 \leq i \leq d$, let $Y_i \subset X_i$ be the orthogonal complement of X_0 in X_i. Set u_i to be the vector in X_i which represents $\varphi_{|X_i}$ with respect to p_i, that is, such that $\varphi(x_i) = p_i(u_i, x_i)$ for x_i in X_i. Write $u_i = v_i + w_i$, with v_i in X_0 and w_i in Y_i. By definition, we have $q_i(\varphi_{|X_i}, \varphi_{|X_i}) = p_i(u_i, u_i) = p_i(v_i, v_i) + p_i(w_i, w_i)$.

We claim that $v_1, \ldots, v_d$ are equal to each other. Indeed, for $1 \leq i \leq d$, we have, for any x_0 in X_0, $p_0(v_i, x_0) = p_i(v_i, x_0) = p_i(u_i, x_0) = \varphi(x_0)$, which does not depend on i, hence v_i does not depend on i since p_0 is positive definite.

Set $v = v_1 = \cdots = v_d$ and $u = v + w_1 + \cdots + w_d$. We claim that the vector u represents the linear functional φ on X with respect to p.

Indeed, for x in X, write $x = x_0 + x_1 + \cdots + x_d$, with x_0 in X_0 and x_i in Y_i, $1 \leq i \leq d$. We have

$$\begin{aligned}\varphi(x) &= \varphi(x_0) + \varphi(x_1) + \cdots + \varphi(x_d)\\ &= p_0(v, x_0) + p_1(u_1, x_1) + \cdots + p_d(u_d, x_d)\\ &= p_0(v, x_0) + p_1(w_1, x_1) + \cdots + p_d(w_d, x_d) = p(u, x),\end{aligned}$$

where the latter equality follows from the definition of p in Lemma 4.17. We get, still by this definition,

$$\begin{aligned}q(\varphi, \varphi) = p(u, u) &= p_0(v, v) + p_1(w_1, w_1) + \cdots + p_d(w_d, w_d)\\ &= p_1(u_1, u_1) + \cdots + p_d(u_d, u_d) - (d-1)p_0(v, v),\end{aligned}$$

and the result follows. $\square$

We will now axiomatize the relations which appear in Proposition 5.2.

Definition 5.4. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. A k-dual prekernel is a family $(K_x)_{x \in X}$ where, for any x in X, K_x is a symmetric function on $S^\ell(x) \times S^\ell(x)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(x)$ associated to K_x by Lemma 5.1 is denoted by q_x^K.

Definition 5.5. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$. A k-dual prekernel is a family $(K_{xy})_{x \sim y \in X}$ where, for any $x \sim y$ in X, $K_{xy} = K_{yx}$ is a symmetric function on $S^\ell(xy) \times S^\ell(xy)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(xy)$ associated to K_{xy} by Lemma 5.1 is denoted by q_{xy}^K.

As above, depending on the context, we may also consider dual prekernels as families of locally constant functions on $\partial X \times \partial X$.

Definition 5.6. Let $k \geq 2$ be an integer. Then a k-dual kernel is a pair (K, K^-) where K is a k-dual prekernel and K^- is a $(k-1)$-dual prekernel.

Dual kernels admit orthogonal extensions which behave as in Proposition 5.2.

Definition 5.7. Let $k \geq 2$ be an integer and (K, K^-) be a k-dual kernel.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, set

$$K_{xy}^+ = K_x + K_y - K_{xy}^-.$$

Then K^+ is a $(k+1)$-dual prekernel.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any x in X, set

$$K_x^+ = \sum_{y \sim x} K_{xy} - (d(x) - 1)K_x^-.$$

Then K^+ is a $(k+1)$-dual prekernel.

In both cases, the $(k+1)$-dual kernel (K^+, K) is called the orthogonal extension of the k-dual kernel (K, K^-). More generally, for any $j \geq k$, we denote by (K^j, K^{j-1}) the $(j-k)$-th orthogonal extension of (K, K^-).

Remark 5.8. The orthogonal extension map $(K, K^-) \mapsto (K^+, K)$ is a linear embedding from the vector space of k-dual kernels into the vector space of $(k+1)$-dual kernels.

5.2. **Large extensions of dual kernels.** As an example of the use of these notions, let us give formulae for the K^j, $j \geq k+1$. For $h \geq 0$ and x in X, we set $B^h(x) = \bigcup_{0 \leq \ell \leq h} S^\ell(x)$ to be the ball with center x and radius h in X. In the same way, for $x \sim y$ in X, we set $B^h(x) = \bigcup_{0 \leq \ell \leq h} S^\ell(xy)$. Successive orthogonal extensions are defined by summing the kernels on points and edges in these sets.

Lemma 5.9. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. The orthogonal extensions of (K, K^-) may be defined by the following formulae. Fix $h \geq 1$ and $x \sim y$ in X. If k is even, we have*

$$K_x^{k+2h} = \sum_{z \in B^h(x)} K_z - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^-$$

$$\text{and } K_{xy}^{k+2h-1} = \sum_{z \in B^{h-1}(xy)} K_z - \frac{1}{2} \sum_{\substack{z,t \in B^{h-1}(xy) \\ z \sim t}} K_{zt}^-.$$

If k is odd, we have

$$K_x^{k+2h-1} = \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt} - \sum_{z \in B^{h-1}(x)} (d(z) - 1)K_z^-$$

$$\text{and } K_{xy}^{k+2h} = \frac{1}{2} \sum_{\substack{z,t \in B^h(xy) \\ z \sim t}} K_{zt} - \sum_{z \in B^{h-1}(xy)} (d(z) - 1)K_z^-.$$

Proof. We fix $j \geq 3$ and we prove the formula for K^j when (K, K^-) is a k-dual kernel by descending induction on k with $2 \leq k \leq j - 1$.

If $k = j - 1$, the formula is the same as in Definition 5.7.

Now, assume that $k \leq j - 2$ and that the formula holds for $k + 1$. We will prove it for k. We need to split the discussion according to the parities of j and k. Assume j is even, $j = 2m$, $m \geq 2$.

If k is, $k = 2\ell$, $\ell \geq 1$, we set $h = m - \ell \geq 1$. By the induction assumption, applied to the $(k+1)$-dual kernel (K^+, K), we have

$$K_x^j = \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^+ - \sum_{z \in B^{h-1}(x)} (d(z) - 1) K_z.$$

By Definition 5.7, we get

$$K_x^j = \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} (K_z + K_t - K_{zt}^-) - \sum_{z \in B^{h-1}(x)} (d(z) - 1) K_z,$$

which equals

$$\sum_{z \in B^h(x)} |S^1(z) \cap B^h(x)| K_z - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^- - \sum_{z \in B^{h-1}(x)} (d(z) - 1) K_z,$$

where $|.|$ is the cardinality of finite sets. For z in $B^{h-1}(x)$, we have $S^1(z) \subset B^h(x)$, hence $|S^1(z) \cap B^h(x)| = d(z)$. For z in $S^h(x)$, we have $|S^1(z) \cap B^h(x)| = 1$. Thus,

$$K_x^j = \sum_{z \in B^h(x)} K_z - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^-,$$

which should be proved.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we set $h = m - \ell - 1 \geq 1$. The induction assumption and Definition 5.7 now give

$$\begin{aligned} K_x^j &= \sum_{z \in B^h(x)} K_z^+ - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt} \\ &= \sum_{z \in B^h(x)} \left(\sum_{t \sim z} K_{zt} - (d(z) - 1) K_z^- \right) - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}. \end{aligned}$$

Note that

$$\sum_{z \in B^h(x)} \sum_{t \sim z} K_{zt} = \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt} + \sum_{z \in S^{h+1}(x)} K_{zz_-},$$

where, for z in $S^{h+1}(x)$, z_- is the neighbour of z on $[xz]$. Thus, we get

$$K_x^j = \frac{1}{2} \sum_{\substack{z,t \in B^{h+1}(x) \\ z \sim t}} K_{zt} - \sum_{z \in B^h(x)} (d(z) - 1) K_z^-,$$

as required.

The proofs in case j is odd are analoguous. $\square$

5.3. **Non-negative dual kernels.** We will introduce a non-negativity property that is satisfied by the dual kernels of the form (K^p, K^{p^-}), where p is a Euclidean field. When this property holds, we can associate a Hilbert space to a dual kernel.

We first start by introducing a natural notion for prekernels.

Definition 5.10. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and K be a k-dual prekernel. We say that K is non-negative if, for any x in X, the bilinear form q_x^K is non-negative.

Definition 5.11. (k odd) Let $k \geq 1$ be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$, and K be a k-dual prekernel. We say that K is non-negative if, for any $x \sim y$ in X, the bilinear form q_{xy}^K is non-negative.

Let us now define a related notion for dual kernels. We need some more notation. For $x \sim y$ in X and $\ell \geq 0$, the adjoint maps of the maps I_{xy}^ℓ and J_{xy}^ℓ will be denoted by $I_{xy}^{\ell,*}$ and $J_{xy}^{\ell,*}$. In other words, for any $x \sim y$ in X, we have linear maps

$$\begin{aligned} I_{xy}^{\ell,*} :& V^{\ell+1}(x) \to V^\ell(xy) \\ J_{xy}^{\ell,*} :& V^\ell(xy) \to V^\ell(x) \end{aligned}$$

defined as follows.

If f is in $V^{\ell+1}(x)$, then $I_{xy}^{\ell,*} f$ is the function on $S^\ell(xy)$ such that, for any z in $S^\ell(xy)$, one has

$$\begin{aligned} I_{xy}^{\ell,*} f(z) &= f(z) \text{ if } y \text{ is on } [xz]. \\ I_{xy}^{\ell,*} f(z) &= \sum_{\substack{w \sim z \\ w \notin [xz]}} f(w) \text{ if } y \text{ is not on } [xz]. \end{aligned}$$

If f is in $V^\ell(xy)$, then $J_{xy}^{\ell,*} f$ is the function on $S^\ell(x)$ such that, for any z in $S^\ell(x)$, one has

$$\begin{aligned} J_{xy}^{\ell,*} f(z) &= f(z) \text{ if } y \text{ is not on } [xz]. \\ J_{xy}^{\ell,*} f(z) &= \sum_{\substack{w \sim z \\ w \notin [xz]}} f(w) \text{ if } y \text{ is on } [xz]. \end{aligned}$$

Let V and W be real vector spaces and let $\pi : V \to W$ be a surjective linear map. If q is a non-negative symmetric bilinear form on V, we define in Appendix A the Euclidean image $\pi_\star q$ of q: this is a non-negative symmetric bilinear form on W. For any w in W, we have

$$\pi_\star q(w, w) = \inf_{\substack{v \in V \\ \pi(v) = w}} q(v, v).$$

Definition 5.12. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and (K, K^-) be a k-dual kernel.

We say that (K, K^-) is non-negative if the dual prekernels K and K^- are non-negative and if, for any $x \sim y$ in X, we have

$$q_x^K \geq (I_{xy}^{\ell-1,*})^\star q_{xy}^{K^-}.$$

We say that (K, K^-) is exact if it is non-negative and, for any $x \sim y$ in X, we have

$$(I_{xy}^{\ell-1,*})_\star q_x^K = q_{xy}^{K^-}.$$

We say that (K, K^-) is Euclidean if it is exact and, for any x in X, the bilinear form q_x^K is positive definite.

Definition 5.13. (k odd) Let $k \geq 2$ be an odd integer, $k = 2\ell + 1$, $\ell \geq 1$, and (K, K^-) be a k-dual kernel.

We say that (K, K^-) is non-negative if the dual prekernels K and K^- are non-negative and if, for any $x \sim y$ in X, we have

$$q_{xy}^K \geq (J_{xy}^{\ell,*})^\star q_x^{K^-}.$$

We say that (K, K^-) is exact if it is non-negative and, for any $x \sim y$ in X, we have

$$(J_{xy}^{\ell,*})_\star q_{xy}^K = q_x^{K^-}.$$

We say that (K, K^-) is Euclidean if it is exact and, for any $x \sim y$ in X, the bilinear form q_{xy}^K is positive definite.

As $\Gamma\backslash X$ is finite, the vector space of Γ-invariant k-dual kernels has finite dimension. We denote it by $\mathcal{K}_k$ and we set $\mathcal{K}_k^+ \subset \mathcal{K}_k$ to be the set of Γ-invariant non-negative k-dual kernels. Elementary properties of Euclidean images give

Proposition 5.14. *Let $k \geq 2$ and (H, H^-) and (K, K^-) be non-negative k-dual kernels. Then $(H + K, H^- + K^-)$ is non-negative.*

The set of non-negative k-dual kernels is a convex cone in the vector space of all k-dual kernels.

The set $\mathcal{K}_k^+$ of Γ-invariant non-negative k-dual kernels is a closed convex cone with non-empty interior inside the vector space $\mathcal{K}_k$ of Γ-invariant k-dual kernels.

For x in X, we let Γ_x be the stabilizer of x in Γ, which by assumption is a finite subgroup.

Proof. The fact that, for (H, H^-) and (K, K^-) as above, the dual kernel $(H + K, H^- + K^-)$ is non-negative follows from Lemma A.5. As the set of non-negative k-dual kernels is clearly stable by multiplication by non-negative real numbers, it is a convex cone.

The set $\mathcal{K}_k^+$ is closed in $\mathcal{K}_k$ as being defined by a set of closed inequalities. It remains to prove that it has non-empty interior.

We let $S \subset X$ be a finite set of representatives for the Γ-action on vertices of X, that is, $X = \Gamma S$ and, for every x in S, $\Gamma x \cap S = \{x\}$. In the same way, we let $\overline{X}_1 = \{\{x,y\}|(x,y) \in X_1\}$ be the set of non oriented edges of X and $T \subset \overline{X}_1$ be a finite set of representatives for the Γ-action on $\overline{X}_1$. Now, we first define Γ-invariant dual prekernels as follows.

Fix $\ell \geq 1$. For x in S, we chose a Γ_x-invariant positive definite symmetric bilinear form $p_x^{2\ell}$ on $V_0^\ell(x)$. For x in X, we set $p_x^{2\ell} = (g^{-1})^\star p_{gx}^{2\ell}$ where g in Γ is such that gx is in S. We let $H_x^{2\ell}$ be the associated function on $S^\ell(x) \times S^\ell(x)$ as in Lemma 5.1.

Fix $\ell \geq 0$. For $\{x,y\}$ in T, we set $\Gamma_{xy} = \{g \in \Gamma | g\{x,y\} = \{x,y\}\}$ and we chose a Γ_{xy}-invariant positive definite symmetric bilinear form $p_{xy}^{2\ell+1}$ on $V_0^\ell(xy)$. For $x \sim y$ in X, we set $p_{xy}^{2\ell+1} = (g^{-1})^\star p_{(gx)(gy)}^{2\ell+1}$ where g in Γ is such that $\{gx, gy\}$ is in T. We let $H_{xy}^{2\ell+1}$ be the associated function on $S^\ell(xy) \times S^\ell(xy)$ as in Lemma 5.1.

Now, let k be even, $k = 2\ell$, $\ell \geq 1$. For $x \sim y$ in X, we set $K_x = H_x^k + \sum_{z \sim x} H_{xz}^{k-1}$ and $K_{xy}^- = H_{xy}^{k-1}$. Then (K, K^-) is a Γ-invariant k-dual kernel which clearly lies in the interior of $\mathcal{K}_k^+$.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we set $K_{xy} = H_{xy}^k + H_x^{k-1} + H_y^{k-1}$ and $K_x^- = H_x^{k-1}$. Again, (K, K^-) is an interior point of $\mathcal{K}_k^+$. □

Euclidean kernels are in one-to-one correspondance with Euclidean fields.

Proposition 5.15. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Then (K, K^-) is Euclidean if and only if there exists a k-Euclidean field p such that $(K, K^-) = (K^p, K^{p^-})$.*

The notions we have defined behave well with respect to orthogonal extension.

Proposition 5.16. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. If (K, K^-) is non-negative (resp. exact, resp. Euclidean), so is the $(k+1)$-dual kernel (K^+, K).*

Proof. This is a direct consequence of the abstract lemma below, which we apply to the structures that are dual to the ones in Proposition 4.5 and Proposition 4.6. □

Lemma 5.17. *Let $W_0, W_1, \ldots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces, and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective* *linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. Assume $q_0, q_1, \dots q_d$ to be non-negative symmetric bilinear forms on $W_0, W_1, \dots, W_d$ with $q_i \geq \varpi_i^\star q_0$, $1 \leq i \leq d$, and set

$$q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d - (d-1)\pi_0^\star q_0.$$

Then,
(i) *the symmetric bilinear form q is non-negative and $q \geq \pi_i^\star q_i$ for $1 \leq i \leq d$.*
(ii) *if we have $(\varpi_i)_\star q_i = q_0$ for $1 \leq i \leq d$, then we also have $(\pi_i)_\star q = q_i$ for $1 \leq i \leq d$.*
(iii) *if the forms $q_1, \dots q_d$ are positive definite, the form q is positive definite.*

Proof. (i) Pick $1 \leq i \leq d$. We have

$$q = \pi_i^\star q_i + \sum_{j \neq i} (\pi_j^\star q_j - \pi_0^\star q_0) = \pi_i^\star q_i + \sum_{j \neq i} \pi_j^\star (q_j - \varpi_j^\star q_0),$$

hence $q \geq \pi_i^\star q_i$. In particular, q is non-negative.

(ii) Still fix $1 \leq i \leq d$ and let w_i be a vector in W_i. We set $w_0 = \varpi_i(w_i)$. For $j \neq i$, as $(\varpi_j)_\star q_j = q_0$, we can find w_j in W_j with $\varpi_j(w_j) = w_0$ and $q_j(w_j, w_j) = q_0(w_0, w_0)$. Now the vector $w = (w_1, \dots, w_d)$ belongs to W and by construction, we have $\pi(w) = w_i$ and $q(w, w) = q_i(w_i, w_i)$.

(iii) Let $w = (w_1, \dots, w_d)$ be in W with $q(w, w) = 0$. By *(i)*, for $1 \leq i \leq d$, we have $q_i(w_i, w_i) \leq q(w, w)$, hence $w_i = 0$. We get $w = 0$. $\square$

5.4. **The Hilbert space of a non-negative dual kernel.** Recall that, by definition, the space $\mathcal{D}^*(\partial X)$ of distributions on X is the dual space of the space $\mathcal{D}(\partial X)$ of smooth functions on X and that $\mathcal{D}_0^*(\partial X)$ is the set of distributions T with $\langle T, \mathbf{1} \rangle = 0$, which we freely identify with the dual space of $\overline{\mathcal{D}}(U) = \mathcal{D}(\partial X)/\mathbb{R}$.

Recall also that we have defined natural linear operators, for x in X and $\ell \geq 0$,

$$N_x^\ell : V^\ell(x) \to \mathcal{D}(\partial X).$$

Again, we let

$$N_x^{\ell,*} : \mathcal{D}^*(U) \to V^\ell(x)$$

denote the adjoint operator of N_x^ℓ.

To a non-negative kernel, we will associate a natural Hilbert space of distributions by using the results in Appendix B.

Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Recall that $(K^j)_{j \geq k}$ denote the successive predual kernels obtained from (K, K^-) by orthogonal extension. As above, for any even $j \geq k-1$, $j = 2\ell$, $\ell \geq 1$, for any x in X, we associate to K^j_x a symmetric bilinear form $q^{K^j}_x$ on $V^\ell_0(x)$. When there is no ambiguity, we shall write q^j_x for $q^{K^j}_x$. In the same way, for any odd $j \geq k-1$, $j = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we let $q^{K^j}_{xy}$ or q^j_{xy} denote the symmetric bilinear form associated to K^j_{xy} on $V^\ell_0(xy)$.

Proposition 5.18. *Let $k \geq 2$ and (K, K^-) be a non-negative k-dual kernel. Fix x in X and let L^{K,K^-} denote the set of distributions θ in $\mathcal{D}_0(\partial X)$ such that*

$$\sup_{\ell \geq \frac{k}{2}} q^{2\ell}_x(N^{\ell,*}_x \theta, N^{\ell,*}_x \theta) < \infty.$$

Then L^{K,K^-} is a vector subspace of $\mathcal{D}_0(\partial X)$ and the map

$$\theta \mapsto \sup_{\ell \geq \frac{k}{2}} q^{2\ell}_x(N^{\ell,*}_x \theta, N^{\ell,*}_x \theta)$$

is a non-negative quadratic form on L^{K,K^-}. Let q^{K,K^-} be its polar form. Both L^{K,K^-} and q^{K,K^-} do not depend on the choice of x. The space $H^{K,K^-} = L^{K,K^-} / \ker q^{K,K^-}$, equipped with the positive definite bilinear form induced by q^{K,K^-} is complete.

From the definition of the Hilbert space associated to a Euclidean field, we get

Corollary 5.19. *If (K, K^-) is Euclidean and p is the k-Euclidean field such that $(K, K^-) = (K^p, K^{p^-})$, then $L^{K,K^-} = H^{K,K^-}$ is exactly the space of distributions which are bounded linear functional for the scalar product p^∞ on $\overline{\mathcal{D}}(\partial X)$. In particular, H^{K,K^-} may be seen as the topological dual space of the Hilbert space H^p associated to p.*

The space H^{K,K^-}, equipped with its natural scalar product, will be called the Hilbert space associated to the dual kernel (K, K^-).

Proof of Proposition 5.18. Let us check that the definition of the objects is independent on x. To this aim, let $x \sim y$ be in X. For any $\ell \geq 0$, the linear operator $I^\ell_{xy} J^\ell_{yx}$ embeds $V^\ell(y)$ as a subspace in $V^{\ell+1}(x)$. One easily checks that one has $N^{\ell+1}_x I^\ell_{xy} J^\ell_{yx} = N^\ell_y$. Hence, if $2\ell \geq k$, we get

$$(N^{\ell,*}_y)^\star q^{2\ell}_y = (N^{\ell+1,*}_x)^\star (I^{\ell,*}_{xy})^\star (J^{\ell,*}_{yx})^\star q^{2\ell}_y.$$

By Proposition 5.16, we have

$$(J^{\ell,*}_{yx})^\star q^{2\ell}_y \leq q^{2\ell+1}_{xy} \text{ and } (I^{\ell,*}_{xy})^\star q^{2\ell+1}_{xy} \leq q^{2\ell+2}_x.$$

Thus, we have

$$(N_y^{\ell,*})^\star q_y^{2\ell} \leq (N_x^{\ell+1,*})^\star q_x^{2\ell+2}.$$

In particular, for any θ in $\mathcal{D}_0(\partial X)$, we have

$$\sup_{\ell\geq 1} q_x^{2\ell}(N_x^{\ell,*}\theta, N_x^{\ell,*}\theta) = \sup_{\ell\geq 1} q_y^{2\ell}(N_y^{\ell,*}\theta, N_y^{\ell,*}\theta).$$

By connectedness of X, the latter equality holds for any x, y in X, hence the constructions in the Proposition do not depend on the choice of x.

The rest of the proof directly follows from Lemma 4.16, Proposition 5.16 and Lemma B.3: indeed, the family $(V_0^\ell(x), M_x^{\ell+1,*}, q_x^{2\ell})_{\ell\geq\frac{k}{2}}$ is a non-negative projective system in the sense of Definition B.1, whose algebraic projective limit may be identified with $\mathcal{D}_0(\partial X)$ (see Appendix B for more details). $\square$

Note that for the moment, we don't know wether L^{K,K^-} is not reduced to 0. We will later prove that, when (K, K^-) is Γ-invariant, L^{K,K^-} contains the space H_0^ω from Section 3. We will first show how this phenomenon appears on a particular example.

5.5. **Examples of dual kernels.** In this Subsection, we give two examples of non-negative dual kernels. Their constructions are based on the elementary

Lemma 5.20. *Let A be a finite set and V be the space of real-valued functions on A and V_0 be the subspace of functions f with $\sum_{a\in A} f(a) = 0$. We let q be the scalar product $(f,g) \mapsto \sum_{a\in A} f(a)g(a)$ on V_0 and, for a in A, we let e_a denote the evaluation linear functional $f \mapsto f(a)$. Then if q^* is the scalar product dual to q on the dual space of V_0, for $a \neq b$ in A, we have $q^*(e_a, e_a) = \frac{n-1}{n}$ and $q^*(e_a, e_b) = -\frac{1}{n}$, where $n = |A|$ is the cardinality of A.*

Let us define a 2-dual kernel (χ, χ^-). For any $x \sim y$ in X, we set $\chi_{xy}^-(x, y) = 1$. For any x in X and any neighbours $y \neq z$ of x, we set $\chi_x(y, z) = 2\frac{d(x)-1}{d(x)}$. We call (χ, χ^-) the harmonic kernel.

Proposition 5.21. *The harmonic kernel is a Euclidean kernel.*

Proof. Let $x \sim y$ be in X and q_x and q_{xy}^- be the symmetric bilinear forms associated with χ and χ^- on the spaces $V_0^1(x)$ and $V_0^0(xy)$. By construction (see Lemma 5.1), one has $q_x(f,f) = \frac{d(x)-1}{d(x)} \sum_{z\sim x} f(z)^2$ for any f in $V_0^1(x)$ and $q_{xy}^-(\mathbf{1}_x - \mathbf{1}_y, \mathbf{1}_x - \mathbf{1}_y) = 1$. We must check that we have $q_{xy}^- = (I_{xy}^{0,*})_\star q_x$. Now, for any f in $V_0^1(x)$, we have $I_{xy}^{0,*} f =$

$f(y)(\mathbf{1}_y - \mathbf{1}_x)$, so that, by Lemma 5.20 and Lemma A.10, we have

$$(I^{0,*}_{xy})_\star q_x(\mathbf{1}_y - \mathbf{1}_x, \mathbf{1}_y - \mathbf{1}_x) = \frac{1}{p_x(\mathbf{1}_y, \mathbf{1}_y)} = 1,$$

where p_x is the scalar product dual to q_x on $\overline{V}^1(x)$. □

We shall pursue the study of the harmonic kernel in Subsections 9.6 and 10.5. In particular, we will prove in Proposition 10.13 that the Hilbert space of distributions H^{χ,χ^-} associated to (χ, χ^-) in Proposition 5.18 is exactly the Hilbert space H_0^ω that has been studied in Section 3.

By changing slightly the construction, we can build another dual kernel, that is not exact any more, but for which the computations are easier. We define the Busemann kernel $(\kappa, \kappa^-) = (\kappa^2, \kappa^1)$ as follows. For any $x \sim y$ in X, we set $\kappa^1_{xy}(x, y) = 1$. For any x in X and any neighbours $y \neq z$ of x, we set $\kappa^2_x(y, z) = 2$. We denote by $(\kappa^k)_{k\geq 1}$ the dual prekernels obtained from (κ^2, κ^1) by successive orthogonal extensions as in Definition 5.7. For the Busemann kernel, all the objects that have been introduced in Section 6 can be computed explicitely.

Proposition 5.22. *The Busemann kernel (κ^2, κ^1) is a non-negative 2-dual kernel. Let $k \geq 1$. If k is even $k = 2\ell$, $\ell \geq 1$, for any x in X and z, t in $S^\ell(x)$, one has $\kappa^k_x(z, t) = d(z, t)$. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X and z, t in $S^\ell(xy)$, one has $\kappa^k_{xy}(z, t) = d(z, t)$.*

Proof. Let still (χ, χ^-) be the harmonic kernel. For any x in X, we have $\kappa_x = \frac{d(x)}{d(x)-1}\chi_x$ and, for $x \sim y$ in X, we have $\kappa^-_{xy} = \chi^-_{xy}$. As, for any x, $\frac{d(x)}{d(x)-1} \geq 1$ and as, by Proposition 5.21, the harmonic kernel is Euclidean, the k-dual kernel $(\kappa - \chi, \kappa^- - \chi^-) = (\kappa - \chi, 0)$ is non-negative, hence (κ, κ^-) is non-negative by Proposition 5.14.

Let us compute κ^k, $k \geq 3$. Assume k is even, $k = 2\ell$, $\ell \geq 2$. Fix x in X. For z, t in $S^\ell(x)$, Lemma 5.9 and the definition of (κ, κ^-) give

$$\begin{aligned}\kappa^k_x(z, t) = 2|B^{\ell-1}(x) \cap [zt]| \\ - \frac{1}{2}|\{(u, v) \in B^{\ell-1}(x) | u \sim v \text{ and } [uv] \subset [zt]\}|.\end{aligned}$$

If $z = t$, all these numbers are 0. Else, one has $d(z, t) \geq 2$ and

$$|B^{\ell-1}(x) \cap [zt]| = d(z, t) - 1$$
$$|\{(u, v) \in B^{\ell-1}(x) | u \sim v \text{ and } [uv] \subset [zt]\}| = 2(d(z, t) - 2)$$

and the result follows. The proof in the odd case is analoguous. □

Corollary 5.23. *The Hilbert space of distributions $H^{\kappa,\kappa^-} \subset \mathcal{D}_0(\partial X)$ associated to the Busemann kernel is exactly the space H_0^ω.*

Proof. For x, y, z in X, let t be such that $[xy] \cap [xz] = [xt]$. We set $\omega_x(y,z) = d(x,t)$. This extends the definition of the Gromov product (see Example 2.18). If $d(x,y) = d(x,z) = \ell$, we get

$$d(y,z) = 2\ell - 2\omega_x(y,z). \tag{5.1}$$

Let x be in X, $\ell \geq 1$ and $q_x^{2\ell}$ be the symmetric bilinear form associated to $\kappa_x^{2\ell}$ on $V_0^\ell(x)$. By Lemma 5.1, Proposition 5.22 and (5.1), for f in $V_0^\ell(x)$, we have

$$\begin{aligned} q_x^{2\ell}(f,f) &= \sum_{y,z \in S^\ell(x)} (\omega_x(y,z) - \ell) f(y) f(z) \\ &= \sum_{y,z \in S^\ell(x)} \omega_x(y,z) f(y) f(z) = \sum_{k=1}^{\ell} \sum_{y,z \in S^\ell(x)} \mathbf{1}_{\omega_x(y,z) \geq k} f(y) f(z). \end{aligned}$$

Fix T in $\mathcal{D}_0(\partial X)$. We get

$$\begin{aligned} q_x^{2\ell}(N_x^{\ell,*}T, N_x^{\ell,*}T) &= \sum_{\substack{t \in X \\ 1 \leq d(x,t) \leq \ell}} \sum_{\substack{y,z \in S^\ell(x) \\ t \in [xy] \cap [xz]}} \langle T, \mathbf{1}_{U_{xy}} \rangle \langle T, \mathbf{1}_{U_{xz}} \rangle \\ &= \sum_{\substack{t \in X \\ 1 \leq d(x,t) \leq \ell}} \langle T, \mathbf{1}_{U_{xt}} \rangle^2 = \frac{1}{2} \sum_{\substack{(u,v) \in X_1 \\ d(x,u) \leq \ell \\ d(x,v) \leq \ell}} \mathcal{P}_x T(u,v)^2, \end{aligned} \tag{5.2}$$

where $\mathcal{P}_x$ is as in Subsection 3.1. By Proposition 5.18, the space H^{κ,κ^-} is exactly the space of distributions T in $\mathcal{D}_0(\partial X)$ with $\mathcal{P}_x T$ belonging to $\ell^2(X_1)$, which by definition is equal to H_0^ω. By (5.2), for T in that space, we have $\|T\|^2 = 2q^{\kappa,\kappa^-}(T,T)$. □

In the next sections, our goal will be to get a kind of generalization of Corollary 5.23. More precisely, we will prove that, when a non-negative dual kernel (K, K^-) is Γ-invariant, the Hilbert space H^{K,K^-} contains the completion of H_0^ω with respect to a non-negative symmetric bilinear form Φ_w as in Section 3. We will also show that all forms Φ_w can be obtained in this way. These results will rely on a formula which we will establish in the next section.

6. An additive formula for dual kernels

Given a k-dual kernel (K, K^-), the purpose of this section is to construct a symmetric function $w : X_k \to \mathbb{R}$ such that the symmetric

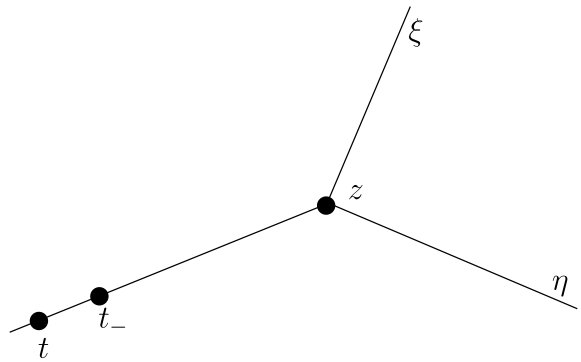

FIGURE 3. The points in the sum $S_z^{\ell,m}(\xi,\eta)$

bilinear forms associated to (K, K^-) may be defined by means of a formula which is the same as the one in Proposition 2.22. Unfortunately, this requires a lot of computations. At the first reading, it might be more comfortable to skip this section, admit Proposition 6.20 and go directly to Section 7.

6.1. **The first geodesic backtracking.** We start with some technical results. Let $k \geq 2$ and (K, K^-) be a k-dual kernel. We will prove that certain sums defined by using the dual prekernels $(K^j)_{j\geq k-1}$ obtained from (K, K^-) by successive orthogonal extensions are equal. These sums will play a key role in certain algebraic constructions.

For $\xi \neq \eta$ in ∂X and z in $(\xi\eta)$, let us denote by $z = x_0, x_1, x_2, \ldots$ and $z = y_0, y_1, y_2, \ldots$ the geodesic rays $[z\xi)$ and $[z\eta)$. If $\ell \in \mathbb{Z}$ and $m \geq 1$ are such that $2\ell + m \geq k$, we will define $S_z^{\ell,m}(\xi,\eta)$ as follows (see Figure 3).

If m is even, $m = 2n$, $n \geq 1$, we set

$$S_z^{\ell,m}(\xi,\eta) = \sum_{\substack{t\in S^n(z)\\ x_1,y_1\notin[zt]}} K_t^{2\ell+m}(x_\ell,y_\ell) - K_{t_-t}^{2\ell+m-1}(x_\ell,y_\ell)$$

(where for t in $S^n(z)$, t_- is the neighbour of t on $[tz]$).

If m is odd, $m = 2n-1$, $n \geq 1$, we set

$$S_z^{\ell,m}(\xi,\eta) = \sum_{\substack{t\in S^n(z)\\ x_1,y_1\notin[zt]}} K_{t_-t}^{2\ell+m}(x_\ell,y_\ell) - K_{t_-}^{2\ell+m-1}(x_\ell,y_\ell).$$

The following result tells us that, these sums are invariant under a backtracking from the geodesic $(\xi\eta)$.

Lemma 6.1. *For $\xi \neq \eta$ in ∂X, z in $(\xi\eta)$, $\ell \in \mathbb{Z}$ and $m \geq 1$ with $2\ell + m \geq k + 1$, we have*

$$S_z^{\ell,m}(\xi,\eta) = S_z^{\ell-1,m+1}(\xi,\eta).$$

The proof of this Lemma directly follows from the definition of the successive orthogonal extensions.

Proof. If m is even, $m = 2n$, $n \geq 1$, we have

$$S_z^{\ell,m}(\xi,\eta) = \sum_{\substack{t \in S^n(z) \\ x_1, y_1 \notin [zt]}} K_t^{2\ell+m}(x_\ell, y_\ell) - K_{t_-t}^{2\ell+m-1}(x_\ell, y_\ell).$$

Now, for t in $S^n(z)$ with $x_1, y_1 \notin [zt]$, we get

$$\begin{aligned} &K_t^{2\ell+m}(x_\ell, y_\ell) - K_{t_-t}^{2\ell+m-1}(x_\ell, y_\ell) \\ &\qquad = \sum_{\substack{t' \sim t \\ t' \neq t_-}} K_{tt'}^{2\ell+m-1}(x_{\ell-1}, y_{\ell-1}) - (d(t)-1) K_t^{2\ell+m-2}(x_{\ell-1}, y_{\ell-1}), \end{aligned}$$

hence, by replacing t with t' in the sum,

$$\begin{aligned} S_z^{\ell,m}(\xi,\eta) &= \sum_{\substack{t \in S^{n+1}(z) \\ x_1, y_1 \notin [zt]}} K_{t_-t}^{2\ell+m-1}(x_{\ell-1}, y_{\ell-1}) - K_{t_-}^{2\ell+m-2}(x_{\ell-1}, y_{\ell-1}) \\ &= S_z^{\ell-1,m+1}(\xi,\eta), \end{aligned}$$

since $m + 1 = 2(n+1) - 1$.

Now, if m is odd, $m = 2n - 1$, $n \geq 1$, we have

$$\begin{aligned} S_z^{\ell,m}(\xi,\eta) &= \sum_{\substack{t \in S^n(z) \\ x_1, y_1 \notin [zt]}} K_{t_-t}^{2\ell+m}(x_\ell, y_\ell) - K_{t_-}^{2\ell+m-1}(x_\ell, y_\ell) \\ &= \sum_{\substack{t \in S^n(z) \\ x_1, y_1 \notin [zt]}} K_t^{2\ell+m-1}(x_{\ell-1}, y_{\ell-1}) - K_{t_-t}^{2\ell+m-2}(x_{\ell-1}, y_{\ell-1}) \end{aligned}$$

□

In case $m = 1$, Lemma 6.1 gives

Corollary 6.2. *Let $\xi \neq \eta$ be in ∂X and z be in $(\xi\eta)$.*

For any $\ell \geq k-1$, we have $S_z^{\ell,1}(\xi,\eta) = 0$.

For any $\frac{k-1}{2} \leq \ell < k-1$, the sum $S_z^{\ell,1}(\xi,\eta)$ does not depend on x_i, y_i, $i > k - 1 - \ell$.

In the same spirit, Lemma 6.1 will allow us to prove that some other sums depend on less points than what would appear at a first glance. Recall that $k \geq 2$ and that (K, K^-) is a k-dual kernel.

Lemma 6.3. *Let $j \geq k-1$ and $(x_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line in X with origin ξ and endpoint η. Then, the quantity*

$$\sum_{h=1}^{j} K^{2k-3}_{x_{h-1}x_h}(\xi,\eta) - \sum_{h=1}^{j-1} K^{2k-2}_{x_h}(\xi,\eta)$$

only depends on $x_0,\dots,x_j$.

Proof. We will establish a more general statement, namely that, for any $\frac{k}{2} \leq \ell \leq k-1$, the quantity

$$A(\ell) = \sum_{h=k-\ell}^{j+\ell+1-k} K^{2\ell-1}_{x_{h-1}x_h}(\xi,\eta) - \sum_{h=k-\ell}^{j+\ell-k} K^{2\ell}_{x_h}(\xi,\eta)$$

only depends on $x_0,\dots,x_j$. This will be proved by induction on ℓ. For $\ell = k-1$, this is the desired result.

If k is even and $\ell = \frac{k}{2}$, we have

$$A(\ell) = \sum_{h=\ell}^{j+1-\ell} K^{-}_{x_{h-1}x_h}(x_{h-\ell},x_{h+\ell-1}) - \sum_{h=\ell}^{j-\ell} K_{x_h}(x_{h-\ell},x_{h+\ell})$$

and the right hand-side only depends on $x_0,\dots,x_j$.

If k is odd and $\ell = \frac{k+1}{2}$, we have

$$A(\ell) = \sum_{h=\ell-1}^{j+2-\ell} K_{x_{h-1}x_h}(\xi,\eta) - \sum_{h=\ell-1}^{j+1-\ell} K^{+}_{x_h}(\xi,\eta).$$

Now, for any $\ell-1 \leq h \leq j+1-\ell$, we have

$$\begin{aligned} K^{+}_{x_h}(\xi,\eta) = K_{x_{h-1}x_h}(\xi,\eta) + K_{x_hx_{h+1}}(\xi,\eta) \\ + \sum_{\substack{y\sim x_h \\ y\neq x_{h-1},x_{h+1}}} K_{x_hy}(\xi,\eta) - (d(x_h)-1)K^{-}_{x_h}(\xi,\eta). \end{aligned}$$

Thus, we get

$$\begin{aligned} A(\ell) = \sum_{h=\ell-1}^{j+1-\ell} (d(x_h)-1)K^{-}_{x_h}(x_{h-\ell+1},x_{h+\ell-1}) \\ - \sum_{h=\ell-1}^{j+1-\ell} \sum_{\substack{y\sim x_h \\ y\neq x_{h-1},x_{h+1}}} K_{x_hy}(x_{h-\ell+1},x_{h+\ell-1}) \\ - \sum_{h=\ell}^{j+1-\ell} K_{x_{h-1}x_h}(x_{h-\ell},x_{h+\ell-1}) \end{aligned} \tag{6.1}$$

and again the right hand-side only depends on $x_0, \dots, x_j$.

Assume now the result is true for some $\frac{k}{2} \leq \ell \leq k-2$ and let us prove that it still holds for $\ell+1$. To do this we will express $A(\ell+1)$ by means of $A(\ell)$. Indeed, we have

$$A(\ell+1) = \sum_{h=k-\ell-1}^{j+\ell+2-k} K^{2\ell+1}_{x_{h-1}x_h}(\xi,\eta) - \sum_{h=k-\ell-1}^{j+\ell+1-k} K^{2\ell+2}_{x_h}(\xi,\eta).$$

For any $k-\ell-1 \leq h \leq j+\ell+1-k$, we have

$$\begin{aligned} K^{2\ell+2}_{x_h}(\xi,\eta) = K^{2\ell+1}_{x_{h-1}x_h}(\xi,\eta) + K^{2\ell+1}_{x_h x_{h+1}}(\xi,\eta) \\ + \sum_{\substack{y\sim x_h \\ y\neq x_{h-1},x_{h+1}}} K^{2\ell+1}_{x_h y}(\xi,\eta) - (d(x_h)-1)K^{2\ell}_{x_h}(\xi,\eta). \end{aligned}$$

By putting $(d(x_h)-2)$-times the term $K^{2\ell}_{x_h}(\xi,\eta)$ under the sum, the latter quantity is equal to

$$K^{2\ell+1}_{x_{h-1}x_h}(\xi,\eta) + K^{2\ell+1}_{x_h x_{h+1}}(\xi,\eta) + S^{\ell,1}_{x_h}(\xi,\eta) - K^{2\ell}_{x_h}(\xi,\eta).$$

Thus, we get

$$\begin{aligned} A(\ell+1) = \sum_{h=k-\ell-1}^{j+\ell+1-k} K^{2\ell}_{x_h}(\xi,\eta) \\ - \sum_{h=k-\ell}^{j+\ell+1-k} K^{2\ell+1}_{x_{h-1}x_h}(\xi,\eta) - \sum_{h=k-\ell-1}^{j+\ell+1-k} S^{\ell,1}_{x_h}(\xi,\eta). \end{aligned}$$

Now, for any $k-\ell \leq h \leq j+\ell+1-k$, we have

$$K^{2\ell+1}_{x_{h-1}x_h}(\xi,\eta) = K^{2\ell}_{x_{h-1}}(\xi,\eta) + K^{2\ell}_{x_h}(\xi,\eta) - K^{2\ell-1}_{x_{h-1}x_h}(\xi,\eta).$$

We get

$$A(\ell+1) = A(\ell) - \sum_{h=k-\ell-1}^{j+\ell+1-k} S^{\ell,1}_{x_h}(\xi,\eta). \tag{6.2}$$

By the induction assumption, $A(\ell)$ only depends on $x_0, \dots, x_j$. By Corollary 6.2 , for any $k-\ell-1 \leq h \leq j+\ell+1-k$, $S^{\ell,1}_{x_h}(\xi,\eta)$ only depends on the points of $(\xi\eta)$ which are at distance $\leq k-\ell-1$ of x_h. As all these points belong to the segment $[x_0 x_j]$, the results follows. □

6.2. **Lifting of the forms q_{xy}^{2k-3} and q_x^{2k-2}.** Let still $k \geq 2$ and let (K, K^-) be a k-dual kernel. For any even $h \geq k-1$, $h = 2\ell$, $\ell \geq 1$ (resp. for any odd $h \geq k-1$, $h = 2\ell+1$, $\ell \geq 0$), for any x in X (resp. for any $x \sim y$ in X), we have an associated symmetric bilinear form $q_x^{K^h} = q_x^h$ (resp. $q_{xy}^{K^h} = q_{xy}^h$) on $V_0^\ell(x)$ (resp. $V_0^\ell(xy)$). We will now build bilinear forms on $V^\ell(x)$ (resp. $V^\ell(xy)$) whose restriction to $V_0^\ell(x)$ (resp. $V_0^\ell(xy)$) are equal to q_x^h (resp. q_{xy}^h). We will start with the cases where $\ell = k-1$.

To construct such forms, we will use

Lemma 6.4. *Let A be a finite set. Let V be the space of real-valued functions on A and*

$$V_0 = \{f \in V | \sum_{a \in A} f(a) = 0\}.$$

If q is a symmetric bilinear form on V, set, for a in A, $u_q(a) = q(\mathbf{1}_a, \mathbf{1}_a)$.

Let q_0 be a symmetric bilinear form on V_0. Then the map $q \mapsto u_q$ induces an affine isomorphism from the space of symmetric bilinear forms q on V with $q_{|V_0} = q_0$ onto V.

Any such form q will be called a lifting of q_0.

As in Lemma 6.4, lifting bilinear forms on $V_0^\ell(x)$ and $V_0^\ell(xy)$ to $V^\ell(x)$ and $V^\ell(xy)$ will require choices. These choices will be achieved by chosing what we will call a (K, K^-)-compatible function:

Definition 6.5. Let u be a function on X_{k-1}. Then u is said to be (K, K^-)-compatible if, for any x, y in X with $d(x, y) = k-1$, one has

$$u(x, y) + u(y, x) = \sum_{h=1}^{k-1} K_{z_{h-1} z_h}^{2k-3}(\xi, \eta) - \sum_{h=1}^{k-2} K_{z_h}^{2k-2}(\xi, \eta), \tag{6.3}$$

where $(z_h)_{h \in \mathbb{Z}}$ is any parametrized geodesic line with $z_0 = x$ and $z_{k-1} = y$ and ξ and η are its endpoints.

Remark 6.6. By Lemma 6.3, the right hand-side of (6.3) only depends on x and y. In particular, compatible functions u always exist and if (K, K^-) is Γ-invariant, one can chose u to be so.

To a (K, K^-)-compatible function, we associate its weight function:

Definition 6.7. If u is a (K, K^-)-compatible function, we define its weight function w on X_k by, for any x, y in X with $d(x, y) = k$,

$$w(x, y) = u(x, z_{k-1}) + u(y, z_1) + \sum_{h=1}^{k-1} K_{z_h}^{2k-2}(\xi, \eta) - \sum_{h=1}^{k} K_{z_{h-1} z_h}^{2k-3}(\xi, \eta), \tag{6.4}$$

where $(z_h)_{h\in\mathbb{Z}}$ is any parametrized geodesic line with $z_0 = x$ and $z_k = y$ and ξ and η are its endpoints.

Remark 6.8. Again, it follows from Lemma 6.3 that $w(x, y)$ only depends on the choice of x and y. Note that the weight function is symmetric.

Example 6.9. Recall the Busemann kernel from Subsection 5.5. A function u on X_1 is compatible with the Busemann kernel if and only if one has, for any $x \sim y$ in X,

$$u(xy) + u(yx) = 1.$$

In this case, the associated weight function w on X_2 is given by, for any x, y in X with $d(x, y) = 2$,

$$w(xy) = u(xz) + u(yz),$$

where z is the middle-point of the segment $[xy]$.

We are now ready to state our result on the lifting of the forms q_x^{2k-2} and q_{xy}^{2k-3}. The formulae which appear in the construction of these liftings are related to the ones in Proposition 2.22.

Proposition 6.10. *Let $k \geq 2$, (K, K^-) be a k-dual kernel, u be a (K, K^-)-compatible function and w be its weight function.*

Then there exists a unique family $(\hat{q}_x^{2k-2})_{x\in X}$ such that, for any x in X, $\hat{q}_x^{2k-2}$ is a symmetric bilinear form on $V^{k-1}(x)$ with $(\hat{q}_x^{2k-2})_{|V_0^{k-1}(x)} = q_x^{2k-2}$ and, for any z in $S^{k-1}(x)$,

$$\hat{q}_x^{2k-2}(\mathbf{1}_z, \mathbf{1}_z) = u(x, z).$$

In the same way, there exists a unique family $(\hat{q}_{xy}^{2k-3})_{x\in X}$ such that, for any $x \sim y$ in X, $\hat{q}_{xy}^{2k-3}$ is a symmetric bilinear form on $V^{k-2}(xy)$ with $(\hat{q}_{xy}^{2k-3})_{|V_0^{k-2}(xy)} = q_{xy}^{2k-3}$ and, for any z in $S^{k-2}(xy) \cap S^{k-1}(x)$,

$$\hat{q}_{xy}^{2k-3}(\mathbf{1}_z, \mathbf{1}_z) = u(x, z).$$

If $x_0, \dots, x_{2k-2}$ is a geodesic path in X, one has

$$\hat{q}_{x_{k-1}}^{2k-2}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2k-2}}) = -\frac{1}{2}\sum_{h=0}^{k-2} w(x_h, x_{h+k}) \tag{6.5}$$

$$\text{and } \hat{q}_{x_{k-2}x_{k-1}}^{2k-3}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2k-3}}) = -\frac{1}{2}\sum_{h=0}^{k-3} w(x_h, x_{h+k}). \tag{6.6}$$

Proof. The existence and uniqueness of the liftings is a direct translation of Lemma 6.4. We postpone the proof of (6.5) and (6.6) until next subsection. ☐

6.3. **The bias function** v**.** Let still $k \geq 2$ and (K, K^-) be a k-dual kernel. In order to prove formulae (6.5) and (6.6) as well as to study liftings of the forms q_x^{2j}, and q_{xy}^{2j-1}, $j \geq k-1$, we will need to define a last function associated to the k-dual kernel (K, K^-). This definition will rely on the

Lemma 6.11. *Let $x \sim y$ be neighbouring points in X and ξ in ∂X be such that $y \notin [x\xi)$. Pick η in ∂X with $x \notin [y\eta)$. Then, the quantity*

$$K_x^{2k-2}(\xi,\eta) - K_{xy}^{2k-3}(\xi,\eta)$$

does not depend on η.

Proof. Let $x_0 = y, x_1 = x, x_2, \dots$ be the geodesic ray $[y\xi)$ and $y_0 = x, y_1 = y, y_2, \dots$ be the geodesic ray $[x\eta)$. We will prove by induction on ℓ that, for any $\frac{k}{2} \leq \ell \leq k-1$, the quantity

$$\begin{aligned} B(\ell) &= K^{2\ell}_{x_{k-\ell}}(\xi,\eta) - K^{2\ell-1}_{x_{k-\ell}x_{k-\ell-1}}(\xi,\eta) \\ &= K^{2\ell}_{x_{k-\ell}}(x_k, y_{2\ell-k+1}) - K^{2\ell-1}_{x_{k-\ell}x_{k-\ell-1}}(x_{k-1}, y_{2\ell-k+1}) \end{aligned}$$

does not depend on η. For $\ell = k-1$, this is the desired result.

If k is even and $\ell = \frac{k}{2}$, we have

$$B(\ell) = K_{x_\ell}(x_k, y) - K^-_{x_\ell x_{\ell-1}}(x_{k-1}, y).$$

If k is odd and $\ell = \frac{k+1}{2}$, we have

$$B(\ell) = K^+_{x_{\ell-1}}(x_k, y_2) - K_{x_{\ell-1}x_{\ell-2}}(x_{k-1}, y_2),$$

which is equal to

$$K_{x_{\ell-1}x_\ell}(x_k, y) + \sum_{\substack{z \sim x_{\ell-1} \\ z \neq x_{\ell-2}, x_{\ell-2}}} K_{x_{\ell-1}z}(x_{k-1}, y) - (d(x_{\ell-1}) - 1) K^-_{x_{\ell-1}}(x_{k-1}, y).$$

Assume now the result is true for some $\frac{k}{2} \leq \ell \leq k-2$ and let us prove that it holds for $\ell + 1$. We have

$$B(\ell+1) = K^{2\ell+2}_{x_{k-\ell-1}}(\xi,\eta) - K^{2\ell+1}_{x_{k-\ell-1}x_{k-\ell-2}}(\xi,\eta),$$

which we can write as

$$\begin{aligned} B(\ell+1) &= K^{2\ell+1}_{x_{k-\ell-1}x_{k-\ell}}(\xi,\eta) \\ &\quad + \sum_{\substack{z \sim x_{k-\ell-1} \\ z \neq x_{k-\ell}, x_{k-\ell-2}}} K^{2\ell+1}_{x_{k-\ell-1}z}(\xi,\eta) - (d(x_{k-\ell-1}) - 1) K^{2\ell}_{x_{k-\ell-1}}(\xi,\eta). \end{aligned}$$

By putting $(d(x_{k-\ell-1}) - 2)$-times the expression $K^{2\ell}_{x_{k-\ell-1}}(\xi,\eta)$ under the sum sign, we get

$$B(\ell+1) = K^{2\ell+1}_{x_{k-\ell-1}x_{k-\ell}}(\xi,\eta) - K^{2\ell}_{x_{k-\ell-1}}(\xi,\eta) + S^{\ell,1}_{x_{k-\ell-1}}(\xi,\eta)$$

(where $S_z^{\ell,m}(\xi,\eta)$ has the same meaning as in Subsection 6.1). Hence

$$B(\ell+1) = B(\ell) + S^{\ell,1}_{x_{k-\ell-1}}(\xi,\eta).$$

By Corollary 6.2 , $S^{\ell,1}_{x_{k-\ell-1}}(\xi,\eta)$ only depends on the points of $(\xi\eta)$ which are at distance $\leq k-1-\ell$ of $x_{k-\ell-1}$. As all these points belong to $[y\xi)$, the results follows. $\square$

From Lemma 6.11, we can associate to the dual kernel (K,K^-) its bias function v on X_k which will play an important role in the sequel.

Definition 6.12. We define the bias function v of (K,K^-) as follows. If x,y are in X and $d(x,y)=k$, we set

$$v(x,y) = K^{2k-2}_{x_-}(y,z) - K^{2k-3}_{xx_-}(y_-,z), \tag{6.7}$$

where x_- and y_- are the neighbours of x and y on $[xy]$ and z is any point in X with $d(z,x)=k-2$ and $[zx]\cap[xy]=\{x\}$. By Lemma 6.11, the function v does not depend on the choice of z. Note that, by the relation $K^{2k-1}_{xx_-} = K^{2k-2}_x + K^{2k-2}_{x_-} - K^{2k-3}_{xx_-}$, we also have

$$v(x,y) = K^{2k-1}_{xx_-}(y,t) - K^{2k-2}_{x}(y_-,t), \tag{6.8}$$

for any t in X with $d(t,x)=k-1$ and $[tx]\cap[xy]=\{x\}$.

Example 6.13. The bias function v of the Busemann kernel is the constant function with value 1 on X_2.

The function v is related to the functions u and w by a cohomological equation:

Lemma 6.14. *Let u be a (K,K^-)-compatible function and w be the associated weight function. Let $(x_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line of X. We have*

$$u(x_0,x_{k-1}) + v(x_0,x_k) = u(x_1,x_k) + w(x_0,x_k).$$

Proof. By the definitions in (6.3) and (6.4), we have

$$\begin{aligned} w(x_0,x_k) &= u(x_0,x_{k-1}) + u(x_k,x_1) \\ &\quad + K^{2k-2}_{x_1}(x_{2-k},x_k) - K^{2k-3}_{x_0x_1}(x_{2-k},x_{k-1}) - (u(x_1,x_k)+u(x_k,x_1)). \end{aligned}$$

By (6.7), we have

$$K^{2k-2}_{x_1}(x_{2-k},x_k) - K^{2k-3}_{x_0x_1}(x_{2-k},x_{k-1}) = v(x_0,x_k)$$

and we are done. $\square$

Using these relations, we are no ready to give the

End of the proof of Proposition 6.10. We need to prove (6.5) and (6.6). Let $(x_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line.

First, we prove (6.5). On one hand, we have, by elementary properties of quadratic forms,

$$2\hat{q}^{2k-2}_{x_{k-1}}(\mathbf{1}_{x_0},\mathbf{1}_{x_{2k-2}}) = u(x_{k-1},x_0) + u(x_{k-1},x_{2k-2}) - K^{2k-2}_{x_{k-1}}(x_0,x_{2k-2}).$$

On the other hand, by Lemma 6.14, we have

$$\sum_{h=0}^{k-2} w(x_h,x_{h+k}) = u(x_0,x_{k-1}) - u(x_{k-1},x_{2k-2}) + \sum_{h=0}^{k-2} v(x_h,x_{h+k})$$

and, by (6.7),

$$\sum_{h=0}^{k-2} v(x_h,x_{h+k}) \\ = \sum_{h=1}^{k-1} K^{2k-2}_{x_h}(x_{h+1-k},x_{h+k-1}) - K^{2k-3}_{x_{h-1}x_h}(x_{h+1-k},x_{h+k-2}).$$

By (6.3), this gives

$$\sum_{h=0}^{k-2} v(x_h,x_{h+k}) = K^{2k-2}_{x_{k-1}}(x_0,x_{2k-2}) - u(x_0,x_{k-1}) - u(x_{k-1},x_0),$$

hence

$$\sum_{h=0}^{k-2} w(x_h,x_{h+k}) = K^{2k-2}_{x_{k-1}}(x_0,x_{2k-2}) - u(x_{k-1},x_0) - u(x_{k-1},x_{2k-2})$$

that is, (6.5) holds.

In the same way, let us prove (6.6). Again, we have, on one hand, by standard properties of quadratic forms,

$$2\hat{q}^{2k-3}_{x_{k-2}x_{k-1}}(\mathbf{1}_{x_0},\mathbf{1}_{x_{2k-3}}) \\ = u(x_{k-1},x_0) + u(x_{k-2},x_{2k-3}) - K^{2k-3}_{x_{k-2}x_{k-1}}(x_0,x_{2k-3})$$

and, on the other hand, still by Lemma 6.14,

$$\sum_{h=0}^{k-3} w(x_h,x_{h+k}) = u(x_0,x_{k-1}) - u(x_{k-2},x_{2k-3}) + \sum_{h=0}^{k-3} v(x_h,x_{h+k}).$$

By (6.7),

$$\sum_{h=0}^{k-3} v(x_h, x_{h+k})$$
$$= \sum_{h=1}^{k-2} K^{2k-2}_{x_h}(x_{h+1-k}, x_{h+k-1}) - K^{2k-3}_{x_{h-1}x_h}(x_{h+1-k}, x_{h+k-2}),$$

hence, by (6.3),

$$\sum_{h=0}^{k-3} v(x_h, x_{h+k}) = K^{2k-3}_{x_{k-2}x_{k-1}}(x_0, x_{2k-3}) - u(x_0, x_{k-1}) - u(x_{k-1}, x_0),$$

and (6.6) follows. □

6.4. **Lifting of the forms q^{2j-1}_{xy} and q^{2j}_x.** Recall that $k \geq 2$ and (K, K^-) is a k-dual kernel. We let v be the bias function of (K, K^-) as in Definition 6.12. More generally, for any $j \geq k$, we let $v_j : X_j \to \mathbb{R}$ be the bias function of the j-dual kernel (K^j, K^{j-1}).

Proposition 6.15. *Let $k \geq 2$, (K, K^-) be a k-dual kernel, u be a (K, K^-)-compatible function and w be its weight function. Fix $j \geq k-1$.*

Then there exists a unique family $(\hat{q}^{2j}_x)_{x\in X}$ such that, for any x in X, $\hat{q}^{2j}_x$ is a symmetric bilinear form on $V^j(x)$ with $(\hat{q}^{2j}_x)_{|V^j_0(x)} = q^{2j}_x$ and, for any z in $S^j(x)$,

$$\hat{q}^{2j}_x(\mathbf{1}_z, \mathbf{1}_z) = u(x, z_{k-1}) + \sum_{h=k}^{j} v_h(x, z_h),$$

where $z_0 = x, z_1, \ldots, z_j = z$ is the geodesic path from x to z. In the same way, there exists a unique family $(\hat{q}^{2j-1}_{xy})_{x\in X}$ such that, for any $x \sim y$ in X, $\hat{q}^{2j-1}_{xy}$ is a symmetric bilinear form on $V^{j-1}(xy)$ with $(\hat{q}^{2j-1}_{xy})_{|V^{j-1}_0(xy)} = q^{2j-1}_{xy}$ and, for any z in $S^{j-1}(xy) \cap S^j(x)$,

$$\hat{q}^{2j-1}_{xy}(\mathbf{1}_z, \mathbf{1}_z) = u(x, z_{k-1}) + \sum_{h=k}^{j} v_h(x, z_h),$$

where, as above, $z_0 = x, z_1, \ldots, z_j = z$ is the geodesic path from x to z.

If $x_0, \dots, x_{2j}$ is a geodesic path in X, one has

$$\hat{q}^{2j}_{x_j}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2j}}) = -\frac{1}{2}\sum_{h=0}^{k-2} w(x_{h+j+1-k}, x_{h+j+1}) \tag{6.9}$$

$$\text{and } \hat{q}^{2j-1}_{x_{j-1}x_j}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2j-1}}) = -\frac{1}{2}\sum_{h=0}^{k-3} w(x_{h+j+1-k}, x_{h+j+1}). \tag{6.10}$$

The proof of (6.9) and (6.10) relies on additional properties of the bias function v.

Lemma 6.16. *For any $x \sim y$ in X and any z, t in $S^{k-1}(xy)$ with $y \notin [xz]$ and $x \notin [yt]$, we have*

$$K^{2k-1}_{xy}(z,t) - K^{2k-3}_{xy}(z_-,t_-) = v(x,t) + v(y,z),$$

where z_- and t_- are the neighbours of z and t on $[zt]$.

Proof. By (6.7), we have

$$v(x,t) = K^{2k-2}_y(z_-,t) - K^{2k-3}_{xy}(z_-,t_-)$$
$$\text{and } v(y,z) = K^{2k-2}_x(z,t_-) - K^{2k-3}_{xy}(z_-,t_-).$$

The result now follows from the relation

$$K^{2k-1}_{xy}(z,t) = K^{2k-2}_x(z,t_-) + K^{2k-2}_y(z_-,t) - K^{2k-3}_{xy}(z_-,t_-).$$

□

Lemma 6.17. *For any x in X and any z, t in $S^k(x)$ with $x \in [zt]$, we have*

$$K^{2k}_x(z,t) - K^{2k-2}_x(z_-,t_-) = v(x,z) + v(x,t),$$

where z_- and t_- are the neighbours of z and t on $[zt]$.

Proof. Let z_1 and t_1 be the neighbours of x on $[xz]$ and $[xt]$. By (6.8), we have

$$v(x,z) = K^{2k-1}_{xz_1}(z,t_-) - K^{2k-2}_x(z_-,t_-)$$
$$\text{and } v(x,t) = K^{2k-1}_{xt_1}(z_-,t) - K^{2k-2}_x(z_-,t_-).$$

Now we have

$$\begin{aligned} &K^{2k}_x(z,t) - K^{2k-2}_x(z_-,t_-) \\ &\quad = v(x,z) + v(x,t) + \sum_{\substack{y\sim x\\ y\neq z_1,t_1}} K^{2k-1}_{xy}(z_-,t_-) - K^{2k-2}_x(z_-,t_-) \\ &\qquad\qquad = v(x,z) + v(x,t) + S^{k-1,1}_x(z,t), \end{aligned}$$

where by $S_x^{k-1,1}(z,t)$ we mean the same as $S_x^{k-1,1}(\xi,\eta)$ for some ξ,η in ∂X with $[xz] \subset [x\xi)$ and $[xt] \subset [x\eta)$ (see Subsection 6.1 for the definition of $S_z^{\ell,m}(\xi,\eta)$). By Corollary 6.2, we have $S_x^{k-1,1}(z,t) = 0$ and we are done. □

Proof of Proposition 6.15. Again, existence of the liftings is a direct consequence of Lemma 6.4.

We will now prove formulae (6.9) and (6.10). If $j = k-1$, they hold by Proposition 6.10. We can therefore assume that $j \geq k$.

We start by proving (6.9). We claim that, for any x in X and z,t in $S^j(x)$ with $x \in [z,t]$, if z_- and t_- are the neighbours of z and t in $S^{j-1}(x)$, we have

$$\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t) = \hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-}). \tag{6.11}$$

This implies (6.9) by induction on j. By definition, one has

$$\begin{aligned} \hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z) &= \hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{z_-}) + v_j(x,z) \\ \hat{q}_x^{2j}(\mathbf{1}_t, \mathbf{1}_t) &= \hat{q}_x^{2j-2}(\mathbf{1}_{t_-}, \mathbf{1}_{t_-}) + v_j(x,t), \end{aligned}$$

hence, by elementary properties of quadratic forms,

$$\begin{aligned} 2\hat{q}_x^{2j}&(\mathbf{1}_z, \mathbf{1}_t) - 2\hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-}) \\ &= v_j(x,z) + v_j(x,t) - K_x^{2j}(z,t) + K_x^{2j-2}(z_-,t_-). \end{aligned}$$

By Lemma 6.17, applied to the j-dual kernel (K^j, K^{j-1}), the latter is zero and (6.11) follows.

In the same way, let us prove (6.10). For any $x \sim y$ in X and z,t in $S^{j-1}(xy)$ with $z \in S^{j-1}(x)$ and $t \in S^{j-1}(y)$, we now claim that we have

$$\hat{q}_{xy}^{2j-1}(\mathbf{1}_z, \mathbf{1}_t) = \hat{q}_x^{2j-3}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-}). \tag{6.12}$$

Again this implies (6.10) by induction on j. By definition, one has

$$\begin{aligned} \hat{q}_{xy}^{2j-1}(\mathbf{1}_z, \mathbf{1}_z) &= \hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{z_-}) + v_j(y,z) \\ \hat{q}_x^{2j}(\mathbf{1}_t, \mathbf{1}_t) &= \hat{q}_x^{2j-2}(\mathbf{1}_{t_-}, \mathbf{1}_{t_-}) + v_j(x,t), \end{aligned}$$

hence, by elementary properties of quadratic forms,

$$\begin{aligned} 2\hat{q}_{xy}^{2j-1}&(\mathbf{1}_z, \mathbf{1}_t) - 2\hat{q}_{xy}^{2j-3}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-}) \\ &= v_j(x,z) + v_j(x,t) - K_{xy}^{2j-1}(z,t) + K_{xy}^{2j-3}(z_-,t_-). \end{aligned}$$

By Lemma 6.16, applied to the j-dual kernel (K^j, K^{j-1}), the latter is zero and (6.12) follows. □

6.5. **The second geodesic backtracking.** Our goal now is to obtain a formula for $\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)$, even when x is not on $[z,t]$. This will require to prove additional cancellation properties of certain sums defined by using dual kernels.

Let still $k \geq 2$ and (K, K^-) be a k-dual kernel. Let $x \sim y$ be in X and ξ, η be in ∂X. For $\ell \in \mathbb{Z}$ and $m \geq -1$ with $2\ell + m \geq k$, we define $T_{xy}^{\ell,m}(\xi,\eta)$ as follows.

If m is even, $m = 2n$, $n \geq 0$, we set

$$T_{xy}^{\ell,m}(\xi,\eta) = \sum_{\substack{z \in S^{n+1}(y) \\ x \in [yz]}} K_z^{2\ell+m}(\xi,\eta) - K_{z_-z}^{2\ell+m-1}(\xi,\eta)$$

(where for z in $S^{n+1}(y)$, z_- is the neighbour of z on $[yz]$).

If m is odd, $m = 2n-1$, $n \geq 0$, we set

$$T_{xy}^{\ell,m}(\xi,\eta) = \sum_{\substack{z \in S^{n+1}(y) \\ x \in [yz]}} K_{z_-z}^{2\ell+m}(\xi,\eta) - K_{z_-}^{2\ell+m-1}(\xi,\eta).$$

By backtracking from y, we get

Lemma 6.18. *For ξ, η in ∂X, $x \sim y$ in X, $\ell \in \mathbb{Z}$ and $m \geq -1$ with $2\ell + m \geq k+1$, we have*

$$T_{xy}^{\ell,m}(\xi,\eta) = T_{xy}^{\ell-1,m+1}(\xi,\eta).$$

Proof. The proof is the same as the one of Lemma 6.1. □

From this we deduce a property of invariance of certain values of (K, K^-) by backtracking.

Lemma 6.19. *Let x be in X and $\xi \neq \eta$ be in ∂X with $x \notin (\xi\eta)$. We set $i = d(x, (\xi\eta)) = \omega_x(\xi,\eta) \geq 1$. Let z be the element of $(\xi\eta)$ that is closest to x and y be the neighbour of x on $[xz]$ (see Figure 4). Then we have*

$$K_x^{2(i+k)-2}(\xi,\eta) = K_{xy}^{2(i+k)-3}(\xi,\eta) = K_z^{2k-2}(\xi,\eta).$$

Proof. It suffices to show that we have

$$K_x^{2(i+k)-2}(\xi,\eta) = K_{xy}^{2(i+k)-3}(\xi,\eta) = K_y^{2(i+k)-4}(\xi,\eta).$$

Now, by the recursive definition of kernels,

$$\begin{aligned} K_x^{2(i+k)-2}(\xi,\eta) &= K_{xy}^{2(i+k)-3}(\xi,\eta) + T_{xy}^{i+k-2,1}(\xi,\eta) \\ &= K_y^{2(i+k)-4}(\xi,\eta) + T_{xy}^{i+k-2,0}(\xi,\eta) + T_{xy}^{i+k-2,1}(\xi,\eta) \end{aligned}$$

By Lemma 6.18, we have

$$T_{xy}^{i+k-2,0}(\xi,\eta) = T_{xy}^{-i,2i+k-2}(\xi,\eta) \text{ and } T_{xy}^{i+k-2,1}(\xi,\eta) = T_{xy}^{-i-1,2i+k}(\xi,\eta).$$

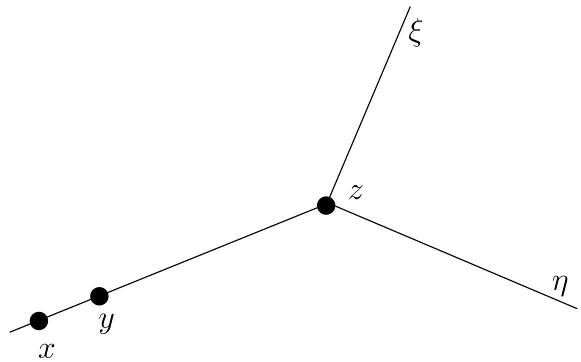

FIGURE 4. The points in Lemma 6.19

If k is even, $k = 2\ell$, $\ell \geq 1$, we have

$$T_{xy}^{-i,2i+k-2}(\xi,\eta) = \sum_{\substack{t \in S^{i+\ell}(y) \\ x \in [yt]}} K_t(\xi,\eta) - K^-_{t_-t}(\xi,\eta)$$

and, for any t in $S^{i+\ell}(y)$ with $x \in [yt]$, we have $\omega_t(\xi,\eta) = 2i + \ell - 1 \geq \ell + 1$, hence

$$K_t(\xi,\eta) = K^-_{t_-t}(\xi,\eta) = 0.$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we have

$$T_{xy}^{-i,2i+k-2}(\xi,\eta) = \sum_{\substack{t \in S^{i+\ell}(y) \\ x \in [yt]}} K_{tt_-}(\xi,\eta) - K^-_{t_-}(\xi,\eta)$$

and, for any t in $S^{i+\ell}(y)$ with $x \in [yt]$, we have $\omega_t(\xi,\eta) = 2i + \ell - 1 \geq \ell + 1$, hence

$$K_{tt_-}(\xi,\eta) = K^-_{t_-}(\xi,\eta) = 0.$$

In both cases, we get $T_{xy}^{i+k-2,0}(\xi,\eta) = 0$ and a fortiori $T_{xy}^{i+k-2,1}(\xi,\eta) = 0$ and the result follows. $\square$

6.6. **The additive formula.** We will show that for large enough j, $\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)$ is given by the same formula as in Proposition 2.22.

Proposition 6.20. *Let still $k \geq 2$ and (K, K^-) be a k-dual kernel. We chose a (K, K^-)-compatible function on X_{k-1} and we let w denote the associated weight function. For x in X and $\xi \neq \eta$ in ∂X, let $z_0 = x, z_1, \ldots$ be the geodesic ray $[x\xi)$ and $t_0 = x, t_1, \ldots$ be the geodesic ray $[x\eta)$. Set $i = \omega_x(\xi,\eta)$. Then, for every $j \geq i + k - 1$, we have*

$$\hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = \frac{1}{2}\sum_{h=0}^{i-1}(w(z_h, z_{h+k}) + w(t_h, t_{h+k})) - \frac{1}{2}\sum_{h=1}^{k-1} w(z_{i+h}, t_{i+k-h}).$$

If $i \geq 1$, we have $\hat{q}^{2j-1}_{xz_1}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = \hat{q}^{2j}_{x}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j})$.

To compute $\hat{q}^{2j}_x(\mathbf{1}_z, \mathbf{1}_z)$, we will need

Lemma 6.21. *Let $(x_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line. For any $j \geq k$, we have*

$$v_{j+1}(x_0, x_{j+1}) = v_j(x_1, x_{j+1}).$$

Proof. By (6.7), we have

$$v_{j+1}(x_0, x_{j+1}) = K^{2j}_{x_1}(x_{1-j}, x_{j+1}) - K^{2j-1}_{x_0x_1}(x_{1-j}, x_j),$$

whereas, by (6.8),

$$v_j(x_1, x_{j+1}) = K^{2j-1}_{x_1x_2}(x_{2-j}, x_{j+1}) - K^{2j-2}_{x_1}(x_{2-j}, x_j).$$

We get, by the recursive definition of kernels,

$$v_{j+1}(x_0, x_{j+1}) - v_j(x_1, x_{j+1}) = S^{j-1,1}_{x_1}(\xi, \eta),$$

where ξ and η are the endpoints of $(x_h)_{h\in\mathbb{Z}}$ and $S^{j-1,1}_{x_1}(\xi, \eta)$ is as in Subsection 6.1. As $j-1 \geq k-1$, by Corollary 6.2, we have $S^{j-1,1}_{x_1}(\xi, \eta) = 0$ and the result follows. $\square$

Corollary 6.22. *Let $(x_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line. For any $i \geq 0$ and $j \geq i + k - 1$, we have*

$$\hat{q}^{2j}_{x_0}(\mathbf{1}_{x_j}, \mathbf{1}_{x_j}) = \sum_{h=0}^{i-1} w(x_h, x_{h+k}) + \hat{q}^{2(j-i)}_{x_i}(\mathbf{1}_{x_j}, \mathbf{1}_{x_j}).$$

Proof. By the the definition in Proposition 6.15, we have

$$\hat{q}^{2j}_{x_0}(\mathbf{1}_{x_j}, \mathbf{1}_{x_j}) = u(x_0, x_{k-1}) + \sum_{h=k}^{j} v_h(x_0, x_h).$$

By Lemma 6.21, this may be written as

$$\hat{q}^{2j}_{x_0}(\mathbf{1}_{x_j}, \mathbf{1}_{x_j}) = u(x_0, x_{k-1}) + \sum_{h=0}^{j-k} v(x_h, x_{h+k}).$$

We get

$$\begin{aligned}&\hat{q}^{2j}_{x_0}(\mathbf{1}_{x_j}, \mathbf{1}_{x_j})\\ &\quad = u(x_0, x_{k-1}) + \sum_{h=0}^{i-1} v(x_h, x_{h+k}) - u(x_i, x_{i+k-1}) + \hat{q}^{2(j-i)}_{x_i}(\mathbf{1}_{x_j}, \mathbf{1}_{x_j})\end{aligned}$$

and the result follows from Lemma 6.14. $\square$

Proof of Proposition 6.20. Again by elementary properties of quadratic forms, we have

$$2\hat{q}_x^{2j}(\mathbf{1}_{z_j},\mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j},\mathbf{1}_{z_j}) + \hat{q}_x^{2j}(\mathbf{1}_{t_j},\mathbf{1}_{t_j}) - K_x^{2j}(z_j,t_j).$$

By Lemma 6.19, applied to the $(j-i+1)$-dual kernel (K^{j-i+1},K^{j-i}), we have

$$K_x^{2j}(z_j,t_j) = K_{z_i}^{2(j-i)}(z_j,t_j)$$

(note that $z_i = t_i$). By Corollary 6.22, we get

$$2\hat{q}_x^{2j}(\mathbf{1}_{z_j},\mathbf{1}_{t_j}) = 2\hat{q}_x^{2(j-i)}(\mathbf{1}_{z_j},\mathbf{1}_{t_j}) + \sum_{h=0}^{i-1} w(z_h,z_{h+k}) + \sum_{h=0}^{i-1} w(t_h,t_{h+k}).$$

Now, by Proposition 6.15, we have

$$2\hat{q}_x^{2(j-i)}(\mathbf{1}_{z_j},\mathbf{1}_{t_j}) = -\sum_{h=1}^{k-1} w(z_{i+h},t_{i+k-h})$$

and the formula for $\hat{q}_x^{2j}(\mathbf{1}_{z_j},\mathbf{1}_{t_j})$ follows.

If $i \geq 1$, we have $z_1 = t_1$ and, by the definition in Proposition 6.15,

$$\hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{z_j},\mathbf{1}_{z_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j},\mathbf{1}_{z_j})$$
$$\text{and } \hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{t_j},\mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{t_j},\mathbf{1}_{t_j}),$$

as well as, by Lemma 6.19, applied to the $(j-i+1)$-dual kernel (K^{j-i+1},K^{j-i}),

$$K_{xz_1}^{2j-1}(z_j,t_j) = K_x^{2j}(z_j,t_j).$$

We get $\hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{z_j},\mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j},\mathbf{1}_{t_j})$. □

7. Dual kernels and additive kernels

In this section, we use Proposition 6.20 to draw a link between the language of Section 3 and the language of dual kernels. We will prove that, given a Γ-invariant non-negative dual kernel (K,K^-), the associated space of distributions L^{K,K^-} always contains the space H_0^ω and that the symmetric bilinear form q^{K,K^-}, when restricted to H_0^ω, is equal to Φ_w, where w is a Γ-invariant weight function of (K,K^-). Conversely, we will prove that, if for a given symmetric Γ-invariant function w on X_k, $k \geq 2$, the bilinear form Φ_w is non-negative on H_0^ω, then there exists a Γ-invariant non-negative k-dual kernel (K,K^-) which admits w as a weight function.

7.1. **A dense subspace of H_0^ω.** First, we will need to gather additional information on the Hilbert space H_0^ω from Section 3. Recall that, if ν is a Borel probability measure ∂X, we write $\mathfrak{M}^\infty(\nu)$ for the space of signed Borel measures on ∂X which are absolutely continuous with respect to ν with a bounded density. The purpose of this Subsection is to construct a Borel probability measure ν such that $\mathfrak{M}^\infty(\nu)$ is a dense subspace of H^ω. We start with a general criterion for density.

Proposition 7.1. *Let x be in X and ν be a fully supported Borel probability measure on ∂X. Assume that one has*

$$\sup_{\substack{y\in X\\ y\neq x}} \frac{1}{\nu(U_{xy})^2} \sum_{\substack{z\in X\\ y\in[xz]}} \nu(U_{xz})^2 < \infty,$$

then ν is atom-free, ω is ν-integrable and $\mathfrak{M}^\infty(\nu)$ is dense in H^ω.

Remark 7.2. For $x \neq y$ in X, the quantity $\frac{1}{\nu(U_{xy})^2} \sum_{\substack{z\in X\\ y\in[xz]}} \nu(U_{xz})^2$ may be seen as a local version of the norm of H^ω.

Proof. The proof relies on a straightforward construction of approximations of the elements of H^ω by elements of $\mathfrak{M}^\infty(\nu)$. We will use again the language of Subsection 3.1.

Let, for any $y \neq x$ in X, y_- be the neighbour of y on $[x, y]$. First note that, since by definition one has $\mathcal{P}_x T(y_-, y) = \nu(U_{xy})$, the assumption implies that the function $\mathcal{P}_x\nu$ belongs to $\ell^2(X_1)$ hence, by Proposition 3.7, that ω is ν-integrable. In particular, by the same result, we have $\mathfrak{M}^\infty(\nu) \subset H^\omega$.

Now, if T is a distribution, for any $\ell \geq 1$, we define a smooth function on ∂X by setting, for any ξ in ∂X,

$$\varphi_\ell^T(\xi) = \frac{\langle T, \mathbf{1}_{U_{xy}}\rangle}{\nu(U_{xy})},$$

where y is the unique element of $S^\ell(x)$ with $y \in [x\xi)$. This makes sense since $\nu(U_{xy}) \neq 0$ as ν has full support. We define $\pi_\ell(T)$ as the distribution $\varphi_\ell^T \nu$ which belongs to $\mathcal{M}^\infty(\nu)$.

We now use again the notation of Section 3.1. If T belongs to H^ω, we claim that the assumption implies that $\pi_\ell(T) \xrightarrow[\ell\to\infty]{} T$ in H^ω. Indeed, by construction, for any $y \neq x$ with $d(x, y) \leq \ell$, we have $\langle \pi_\ell(T), \mathbf{1}_{U_{xy}}\rangle =$

$\langle T, \mathbf{1}_{U_{xy}}\rangle$, hence

$$\|\mathcal{P}_x(\pi_\ell(T) - T)\|_2^2 = 2 \sum_{\substack{y \in X \\ d(x,y) \geq \ell}} \langle \pi_\ell(T) - T, \mathbf{1}_{U_{xy}}\rangle^2$$
$$\leq 4 \sum_{\substack{y \in X \\ d(x,y) \geq \ell}} (\langle \pi_\ell(T), \mathbf{1}_{U_{xy}}\rangle^2 + \langle T, \mathbf{1}_{U_{xy}}\rangle^2).$$

Now, on one hand, as T belongs to H^ω, one has

$$\sum_{\substack{y \in X \\ d(x,y) \geq \ell}} \langle T, \mathbf{1}_{U_{xy}}\rangle^2 \xrightarrow[l \to \infty]{} 0.$$

On the other hand, we have

$$\sum_{d(x,y) \geq \ell} \langle \pi_\ell(T), \mathbf{1}_{U_{xy}}\rangle^2 = \sum_{\substack{y \in X \\ d(x,y) = \ell}} \sum_{\substack{z \in X \\ y \in [xz]}} \langle \pi_\ell(T), \mathbf{1}_{U_{xz}}\rangle^2$$
$$= \sum_{\substack{y \in X \\ d(x,y) = \ell}} \frac{\langle T, \mathbf{1}_{U_{xy}}\rangle^2}{\nu(U_{xy})^2} \sum_{\substack{z \in X \\ y \in [xz]}} \nu(U_{xz})^2.$$

By assumption, the latter goes to 0 as $\ell \to \infty$ and we are done. $\square$

Corollary 7.3. *There exists a fully supported atom-free Borel probability measure ν on ∂X such that ω is ν-integrable and $\mathfrak{M}^\infty(\nu)$ is dense in H^ω.*

Recall that, for x in X, $d(x) \geq 3$ is the number of neighbours of x.

Proof. For example, one can fix x in X and define the associate probability measure ν_x as the unique Borel probability measure such that, for any $y \neq x$ in X, if $x_0 = x, x_1, \ldots, x_\ell = y$ is the geodesic path from x to y, one has

$$\nu_x(U_{xy}) = \frac{1}{d(x)} \frac{1}{d(x_1) - 1} \cdots \frac{1}{d(x_{\ell-1}) - 1}.$$

Let us check that the criterion in Proposition 7.1 holds. By construction, for any $y \neq x$, and $z \sim y$ with $z \notin [xy]$, we have $\nu_x(U_{xz}) = \frac{1}{d(y)-1}\nu_x(U_{xy})$, hence

$$\sum_{\substack{z \sim y \\ z \notin [xy]}} \nu_x(U_{xz})^2 = \frac{1}{d(y) - 1}\nu_x(U_{xy})^2 \leq \frac{1}{2}\nu_x(U_{xy})^2.$$

By induction, we get, for $\ell \geq 0$, $\sum_{\substack{z\in S^\ell(y)\\ y\in[xz]}} \nu_x(U_{xz})^2 \leq \frac{1}{2^\ell}\nu_x(U_{xy})^2$, hence $\sum_{\substack{z\in X\\ y\in[xz]}} \nu(U_{xz})^2 \leq 2\nu(U_{xy})^2$ and the result follows by Proposition 7.1. $\square$

We can use the existence of ν to get a proof that the bilinear form Φ_w from Subsection 3.2 determines w up to cohomology.

Corollary 7.4. *Let w be a Γ-invariant symmetric function on X_k. Assume that Φ_w is zero on H_0^ω. Then the normalized smooth function on $\Gamma\backslash\mathscr{S}$ associated to w is cohomologuous to* 0.

The proof uses an elementary fact from measure theory.

Lemma 7.5. *Let (X,ν) be a probability space and Ω be a symmetric function in $\mathrm{L}^1(X\times X,\nu\otimes\nu)$. The following are equivalent.*
(i) *For every ρ in $\mathrm{L}^\infty(X,\nu)$ with $\int_X \rho \mathrm{d}\nu = 0$, we have*

$$\int_{X\times X} \Omega(x,y)\rho(x)\rho(y)\mathrm{d}\nu(x)\mathrm{d}\nu(y) = 0.$$

(ii) *There exists a function F in $\mathrm{L}^1(X,\nu)$ such that for $\nu\otimes\nu$-almost every (x,y) in $X\times X$, one has $\Omega(x,y) = F(x)+F(y)$.*

The function F is then uniquely determined by Ω.

Proof of Corollary 7.4. Let Ω be an additive kernel associated to w and, as in Corollary 7.3, let ν be a fully supported atom-free Borel probability measure on ∂X such that ω is ν-integrable.

Fix x in X. By Proposition 3.11, we have

$$\int_{\partial X\times\partial X} \Omega_x(\xi,\eta)\mathrm{d}\rho(\xi)\mathrm{d}\rho(\eta) = 0$$

for every ρ in $\mathfrak{M}_0^\infty(\nu)$. By Lemma 7.5, there exists a function F_x in $\mathrm{L}^1(\partial X,\nu)$ such that, for $\nu\otimes\nu$-almost every (ξ,η) in $\partial X\times\partial X$, one has $\Omega_x(\xi,\eta) = F_x(\xi)+F_x(\eta)$. As for every η in ∂X, the function $\xi\mapsto\Omega_x(\xi,\eta)$ is smooth on $\partial X\smallsetminus\{\eta\}$, the function F_x is smooth. As F_x is uniquely determined by Ω, the function $(x,\xi)\mapsto F_x(\xi)$ on $X\times\partial X$ is Γ-invariant. For every x,y in X and $\xi\neq\eta$ in ∂X, we have

$$\Omega_x(\xi,\eta)-\Omega_y(\xi,\eta) = (F_x(\xi)-F_y(\xi))+(F_x(\eta)-F_y(\eta)),$$

that is, Ω is an additive kernel associated to the trivial cohomology class. The conclusion follows by Lemma 2.19. $\square$

7.2. **From dual kernels to additive kernels.** Now, we will show how one can use Proposition 6.20 in order to associate an additive kernel to a dual kernel.

Theorem 7.6. *Let $k \geq 2$, (K, K^-) be a non-negative Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then the symmetric bilinear form Φ_w is non-negative on H_0^ω. More precisely, one has $H_0^\omega \subset L^{K,K^-}$ and the restriction of q^{K,K^-} to H_0^ω is equal to Φ_w.*

See Subsections 3.2 and 3.3 for the definition and properties of Φ_w. See Proposition 5.18 for the definition of the spaces L^{K,K^-} and H^{K,K^-} and the bilinear form q^{K,K^-} associated to the dual kernel (K, K^-). See Definitions 6.5 and 6.7 for the notion of a (K, K^-)-compatible function and its weight function.

Knowing Theorem 7.6, we can prove that the Hilbert space associated to (K, K^-) is large:

Corollary 7.7. *Let $k \geq 2$ and (K, K^-) be a Γ-invariant non-negative k-dual kernel. Then, for every $\ell \geq \frac{k}{2}$ and any x in X, the linear map $N_x^{\ell,*}$ maps H^{K,K^-} onto $V_0^\ell(x)/\ker q_x^{2\ell}$.*

If V is a vector space and Φ is a non-negative symmetric bilinear form on V, a linear functional φ on V is said to be bounded with respect to Φ if one cand find $C \geq 0$ with $\varphi(x)^2 \leq C\Phi(x, x)$ for any x in V. In other words, one has $\ker \Phi \subset \ker \varphi$ and φ is bounded with respect to the Euclidean structure associated to Φ on $V/\ker \Phi$.

Corollary 7.8. *Let $k \geq 2$, (K, K^-) be a Euclidean Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then, for every φ in $\overline{\mathcal{D}}(\partial X)$, the associated linear functional on H_0^ω is bounded with respect to Φ_w.*

For non necessarily non-negative dual kernels we also get

Corollary 7.9. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then, for every θ in H_0^ω, one has*

$$q_x^{2j}(N_x^{j,*}\theta, N_x^{j,*}\theta) \xrightarrow[j\to\infty]{} \Phi_w(\theta, \theta).$$

We now start the proof of Theorem 7.6 and its Corollaries. We will also estalish weaker versions of these results for non necessary non-negative dual kernels. This will be possible thanks to the easy

Lemma 7.10. *Let $k \geq 2$. Any Γ-invariant k-dual kernel may be written as the difference of two non-negative Γ-invariant k-dual kernels.*

Proof. Indeed, in the finite-dimensional space of all Γ-invariant k-dual kernels, the set of non-negative ones is a convex cone with non-empty interior by Proposition 5.14. $\square$

To dominate certain error terms, we shall use

Lemma 7.11. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then, there exits $C \geq 0$ such that, for any $j \geq k-1$, x in X, z, t in $S^j(x)$, one has*

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\right| \leq C(1+i+j),$$

where $i = |[xz] \cap [xt]|$ and $\hat{q}_x^{2j}$ is the symmetric bilinear form on $V^j(x)$ associated to the choices of (K, K^-) and u as in Subsection 6.4.

Proof. Thanks to Lemma 7.10, we may and will assume that (K, K^-) is non-negative.

Note that, as $\Gamma \backslash X$ is finite, the functions u and w are bounded. In particular, by Proposition 6.20, there exists $C_1 \geq 0$ such that, if $j \geq k-1$, x in X, z, t in $S^j(x)$, are such that $|[xz] \cap [xt]| \leq j+1-k$, one has

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\right| \leq C_2(1+j).$$

In the same way, by applying Corollary 6.22 to $i = j+1-k$, there exists $C_2 \geq 0$ such that, for any $j \geq k-1$, x in X, z in $S^j(x)$, one has

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z)\right| \leq C_1(1+j).$$

Now, let $j \geq k-1$, x in X, z, t in $S^j(x)$, be such that

$$i = |[xz] \cap [xt]| \geq j+1-k.$$

We set $\ell = i+k-1 \geq j$. We have

$$2\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t) = \hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z) + \hat{q}_x^{2j}(\mathbf{1}_t, \mathbf{1}_t) - K_x^{2j}(z, t)$$

and it only remains to get a bound for $K_x^{2j}(z, t)$. But, as the kernel (K, K^-) is non-negative, by Proposition 5.16, we have

$$0 \leq K_x^{2j}(z, t) \leq K_x^{2\ell}(z, t).$$

Now, again,

$$K_x^{2\ell}(z, t) = \hat{q}_x^{2\ell}(\mathbf{1}_z, \mathbf{1}_z) + \hat{q}_x^{2\ell}(\mathbf{1}_t, \mathbf{1}_t) - 2\hat{q}_x^{2\ell}(\mathbf{1}_z, \mathbf{1}_t),$$

hence

$$K_x^{2\ell}(z, t) \leq 2(C_1 + C_2)(1+\ell) = 2(C_1 + C_2)(k+i)$$

and

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\right| \leq C_1(1+j) + (C_1 + C_2)(k+i),$$

which should be proved. $\square$

Now, we will use the formulae in Propositions 3.11 and 6.20 to prove that q^{K,K^-} is equal to Φ_w on certain spaces:

Lemma 7.12. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. We also let ν be a Borel probability measure on ∂X such that ω is ν-integrable. Then, for every ρ in $\mathfrak{M}_0^\infty(\nu)$, one has*

$$q_x^{2j}(N_x^{j,*}\rho, N_x^{j,*}\rho) \xrightarrow[j\to\infty]{} \Phi_w(\rho,\rho).$$

See Subsection 3.1 for the definition of the space $\mathfrak{M}_0^\infty(\nu)$.

Proof. The proof is a direct consequence of the fact that the formulae appearing in Proposition 2.22 and Proposition 6.20 are the same.

More precisely, let us fix x in X. For any ρ in $\mathfrak{M}_0^\infty(\nu)$, we have, for $j \geq k-1$,

$$q_x^{2j}(N_x^{j,*}\rho, N_x^{j,*}\rho) = \sum_{z,t\in S^j(x)} \hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\rho(U_{xz})\rho(U_{xt}), \tag{7.1}$$

where, as above, $U_{xz} \subset \partial X$ is the set of ξ in ∂X with $z \in [x\xi)$ and, as in Subsection 5.4, $N_x^{j,*}$ is the linear operator that sends a distribution T on ∂X to the function $z \mapsto \langle T, \mathbf{1}_{U_{xz}}\rangle$ on $S^j(x)$. We define a smooth function on $\partial X \times \partial X$ by setting

$$\Omega_x^j(\xi,\eta) = \sum_{z,t\in S^j(x)} \hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\mathbf{1}_{U_{xz}}(\xi)\mathbf{1}_{U_{xt}}(\eta), \quad \xi,\eta \in \partial X. \tag{7.2}$$

We let Ω be as in Proposition 2.22, so that, by Proposition 3.11, for any ρ in $\mathfrak{M}_0^\infty(\nu)$, one has

$$\Phi_w(\rho,\rho) = \int_{\partial^2 X} \Omega_x(\xi,\eta)\mathrm{d}\rho(\xi)\mathrm{d}\rho(\eta).$$

Then, we claim that we have

$$\Omega_x^j \xrightarrow[j\to\infty]{} \Omega_x \text{ in } \mathrm{L}^1(\partial X \times \partial X, \nu\otimes\nu), \tag{7.3}$$

which, by (7.1), implies the Lemma.

Indeed, we split the sum in the right hand-side of (7.2) depending wether $|[xz]\cap[xt]| \leq j+1-k$ or $|[xz]\cap[xt]| \geq j+2-k$ and we get, by Proposition 6.20, for (ξ,η) in ∂X,

$$\begin{aligned}\Omega_x^j(\xi,\eta) = {}& \Omega_x(\xi,\eta)\mathbf{1}_{\omega_x(\xi,\eta)\leq j+1-k} \\ & + \sum_{\substack{z,t\in S^j(x)\\ |[xz]\cap[xt]|\geq j+2-k}} \hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\mathbf{1}_{U_{xz}}(\xi)\mathbf{1}_{U_{xt}}(\eta)\end{aligned} \tag{7.4}$$

Let us prove that the terms with $|[xz]\cap[xt]|\geq j+2-k$ in the right hand-side of (7.4) will play a negligible role. Let C be as in Lemma 7.11, so that, for any z,t in $S^j(x)$ with $i=|[xz]\cap[xt]|\geq j+2-k$, we have

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\right|\leq C(2i+k-1).$$

We get

$$\int_{\omega_x(\xi,\eta)\geq j+2-k}\left|\Omega_x^j(\xi,\eta)\right|\mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta)\\ \leq C\int_{\omega_x(\xi,\eta)\geq j+2-k}|2\omega_x(\xi,\eta)+k-1|\,\mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta),$$

hence, by the Dominated Convergence Theorem, as ω is ν-integrable,

$$\int_{\omega_x(\xi,\eta)\geq j+2-k}\left|\Omega_x^j(\xi,\eta)\right|\mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta)\xrightarrow[j\to\infty]{}0. \tag{7.5}$$

In the same way, by Lemma 2.20, Ω_x is $\nu\otimes\nu$-integrable and

$$\int_{\omega_x(\xi,\eta)\geq j+2-k}|\Omega_x(\xi,\eta)|\,\mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta)\xrightarrow[j\to\infty]{}0. \tag{7.6}$$

Now (7.3) follows from (7.4), (7.5) and (7.6). □

Proof of Theorem 7.6. We will get the result from Lemma 7.12 by an approximation argument. To this aim, we chose an atom-free Borel probability measure ν on ∂X such that ω is ν-integrable and $\mathfrak{M}^\infty(\nu)$ is dense in H^ω: such a measure exists by Corollary 7.3.

Fix x in X and let us note that, for any $\ell\geq 1$, the linear operator $N_x^{\ell,*}$ is bounded on H^ω. Indeed, for any y in $S^\ell(x)$ and T in $\mathcal{D}^*(\partial X)$, we have $N_x^{\ell,*}T(y)=\mathcal{P}_xT(y_-,y)$ where y_- is the neighbour of y on $[xy]$ and $\mathcal{P}_x$ is as in Subsection 3.1. In particular, for $\ell\geq\frac{k}{2}$, the bilinear form $(N_x^{\ell,*})^\star q_x^{2\ell}$ is bounded on H_0^ω.

Fix ρ in H_0^ω. By assumption, there exists a sequence (ρ_n) in $\mathfrak{M}^\infty(\nu)$ which converges to ρ in H_0^ω. By Lemma 7.12, for $\ell\geq\frac{k}{2}$ and any n, we have

$$q_x^{2\ell}(N_x^{\ell,*}\rho_n,N_x^{\ell,*}\rho_n)\leq\Phi_w(\rho_n,\rho_n),$$

hence, as $(N_x^{\ell,*})^\star q_x^{2\ell}$ is bounded on H_0^ω, by going to the limit, we get

$$q_x^{2\ell}(N_x^{\ell,*}\rho,N_x^{\ell,*}\rho)\leq\Phi_w(\rho,\rho).$$

As a consequence we have $H_0^\omega\subset L^{K,K^-}$ and, as q^{K,K^-} is non-negative, it is a bounded bilinear symmetric form on H_0^ω. Now, as again by Lemma 7.12, we have $q^{K,K^-}(\rho,\rho)=\Phi_w(\rho,\rho)$ for any ρ in $\mathfrak{M}^\infty(\nu)$, we also have $q^{K,K^-}(\theta,\theta)=\Phi_w(\theta,\theta)$ for any θ in H_0^ω. □

Proof of Corollary 7.7. Again, we use Corollary 7.3 to chose a fully supported atom-free Borel probability measure ν on ∂X such that ω is ν-integrable. By Lemma 7.12, we have $\mathfrak{M}_0^\infty(\nu) \subset H^{K,K^-}$. Now, as ν has full support, for any $\ell \geq 0$, the linear operator $N_x^{\ell,*}$ maps H^{K,K^-} onto $V_0^\ell(x)$. The result follows. $\square$

Proof of Corollary 7.8. Let φ be in $\mathcal{D}(\partial X)$. We have to show that the linear functional $\theta \mapsto \langle \theta, \varphi \rangle$ is bounded on H_0^ω with respect to the positive symmetric bilinear form Φ_w. By Theorem 7.6, it suffices to show that it is bounded on H^{K,K^-}. Indeed, fix x in X. By Lemma 4.16, we have $\varphi \in N_x^\ell V^\ell(x)$ for some $\ell \geq \frac{k}{2}$. As the the dual kernel (K, K^-) is Euclidean, for any $\ell \geq \frac{k}{2}$, the form $q_x^{2\ell}$ is positive definite on $V_0^\ell(x)$, hence there exists $C \geq 0$ such that, for any T in $\mathcal{D}_0(\partial X)$, one has $\langle T, \varphi \rangle^2 \leq C q_x^{2\ell}(N_x^{\ell,*}T, N_x^{\ell,*}T)$. The conclusion follows since, for T in H^{K,K^-}, one has $q_x^{2\ell}(N_x^{\ell,*}T, N_x^{\ell,*}T) \leq q^{K,K^-}(T,T)$ by construction. $\square$

Proof of Corollary 7.9. This is a direct consequence of Theorem 7.6 and Lemma 7.10. $\square$

7.3. **The image dual kernel.** We will now aim at proving a converse statement to Theorem 7.6. To do this, given a Γ-invariant function w on X_k, with Φ_w non-negative, we need to prove that Φ_w may be built by use of a dual kernel. In this subsection, we will define our candidate for being this dual kernel. This dual kernel will be constructed by taking Euclidean images of Φ_w (see Definition A.2 for the definition of the Euclidean image of a non-negative bilinear form under a surjective linear map, whose properties are discussed all along Appendix A).

Let us do this precisely. We need some more notation. Recall that, for any $\ell \geq 0$, for any x in X, we have a natural linear operator $N_x^\ell : V^\ell(x) \to \mathcal{D}(\partial X)$. In the same way, for $x \sim y$, and any $\ell \geq 0$, we define $N_{xy}^\ell : V^\ell(xy) \to \mathcal{D}(\partial X)$ as the linear operator such that for any z in $S^\ell(xy) \cap S^{\ell+1}(x)$, $N_{xy}^\ell(\mathbf{1}_z) = \mathbf{1}_{U_{xz}}$. One also let N_{xy}^ℓ denote the induced operator $\overline{V}^\ell(xy) \to \overline{\mathcal{D}}(\partial X)$. One has the compatibility relations

$$N_x^{\ell+1} I_{xy}^\ell = N_{xy}^\ell \text{ and } N_{xy}^\ell J_{xy}^\ell = N_x^\ell, \quad \ell \geq 0, \quad x \sim y \in X. \tag{7.7}$$

We denote the adjoint operators of N_x^ℓ and N_{xy}^ℓ in the usual way.

Lemma 7.13. *For $\ell \geq 1$ and x in X, we have $N_x^{\ell,*}(H_0^\omega) = V_0^\ell(x)$. For $\ell \geq 0$ and $x \sim y$ in X, we have $N_{xy}^{\ell,*}(H_0^\omega) = V_0^\ell(xy)$.*

Proof. As in Corollary 7.3, let ν be a fully-supported Borel probability measure on ∂X with $\mathfrak{M}_0^\infty(\nu) \subset H_0^\omega$. For $\ell \geq 1$, x in X and f in $V_0^\ell(x)$, we set φ to be the smooth function on ∂X defined by

$$\varphi = \sum_{y\in S^\ell(x)} \frac{1}{\nu(U_{xy})} f(y)\mathbf{1}_{U_{xy}}.$$

By construction, the distribution $\rho = \varphi\nu$ belongs to H_0^ω and $N_x^{\ell,*}(\rho) = f$. The proof in the odd case is analoguous. $\square$

Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume that Φ_w is non-negative on H_0^ω. In this case, thanks to Lemma 7.13, we can associate to w a family of dual prekernels as follows. Let j be an integer, $j \geq 1$.

If j is even, $j = 2\ell$, $\ell \geq 1$, for any x in X, set $q_x^j = (N_x^{\ell,*})_\star\Phi_w$ and define K_x^j as the associated function on $S^\ell(x)^2$ as in Lemma 5.1.

If j is odd $j = 2\ell+1$, $\ell \geq 0$, for any $x \sim y$ in X, set $q_{xy}^j = (N_{xy}^{\ell,*})_\star\Phi_w$ and define K_{xy}^j as the associated function on $S^\ell(xy)^2$ as in Lemma 5.1.

Then the relations (7.7) together with Lemma A.4 imply that, for any $j \geq 2$, (K^j, K^{j-1}) is an exact j-dual kernel.

Definition 7.14. Let k and w be as above. For any $j \geq 1$ the j-dual kernel K^j is called the image j-dual prekernel of w and, if $j \geq 2$, the j-dual kernel (K^j, K^{j-1}) is called the image j-dual kernel of w.

We can relate these kernels to the formalism developed in Section 5.

Lemma 7.15. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume that Φ_w is non-negative on H_0^ω. Let K^j, $j \geq 1$ be as above. Then, for any $j \geq k+1$, the j-dual kernel (K^j, K^{j-1}) is the orthogonal extension of the $(j-1)$-dual kernel (K^{j-1}, K^{j-2}).*

The proof of Lemma 7.15 will rely on the following abstract characterization of orthogonal extensions:

Lemma 7.16. *Let X be a finite-dimensional real vector space, $d \geq 2$ be an integer and $X_1, \dots, X_d$ be subspaces of X. We assume that there exists a subspace X_0 of X such that, for any $1 \leq i \neq j \leq d$, $X_i \cap X_j = X_0$ and $X/X_0 = \bigoplus_i X_i/X_0$. Let $p_0, p_1, \dots, p_d$ be positive definite symmetric bilinear forms on $X_0, X_1, \dots, X_d$ such that, for any $1 \leq i \leq d$, $p_{i|X_0} = p_0$. For $1 \leq i \leq d$, let $W_i \subset X^*$ be the orthogonal subspace of $\sum_{j\neq i} X_j$, that is, the kernel of the natural surjective map $X_i^* \to (\sum_{j\neq i} X_j)^*$. Let q be a positive definite symmetric bilinear form on X such that $p_{|X_i} = p_i$, $1 \leq i \leq d$. Then q is the orthogonal extension of $p_1, \dots, p_d$ if and only if, for any $1 \leq i \neq j \leq d$, the spaces W_i and W_j are orthogonal with respect to the dual form of q.*

Proof. Let p be the orthogonal extension of $p_1, \dots, p_d$ and, for $1 \leq i \leq d$, let Y_i be the orthogonal complement of X_0 in X_i with respect to p_i. We set $Y_0 = X_0$. By definition, we have $X = \bigoplus_{0 \leq i \leq d} Y_i$ and this decomposition is orthogonal with respect to p. Let W be the dual space of X and $T : X \to W$ be the linear isomorphism associated to p (that is, for any x, y in X, we have $p(x, y) = \langle Tx, y \rangle$). The dual form p^* of p is given by $p^* = (T^{-1})^\star p$. Let W_0 be the orthogonal complement of $\bigoplus_{1 \leq i \leq d} W_i$ with respect to p^*. We have $W = \bigoplus_{0 \leq i \leq d} W_i$ and, by construction, this decomposition is the image of the one of X by T, that is, $TY_i = W_i$, $0 \leq i \leq d$. In particular, the W_i, $0 \leq i \leq d$, are p^*-orthogonal to each other.

Let $A : X \to X$ be the endomorphism such that, for any x, y in X, $q(x, y) = p(Ax, y)$. To conclude, we need to prove that A is the identity map. One easily shows that the dual form q^* of q satisfies $q^*(v, w) = p^*(Bv, w)$, v, w in W, where $B = TA^{-1}T^{-1}$. Fix $1 \leq i \leq d$. Saying that W_i is q^*-orthogonal to all the W_j, $j \neq i$, amounts to saying that we have $BW_i \subset W_i \oplus W_0$. Pulling back this property by T, we get $Y_i \subset AX_i$. Now, we will use the other assumption on q, namely that its restriction to X_i is p_i. Indeed, fix y_i in Y_i. We have just seen that we can write $y_i = Ax_i$ for some x_i in X_i. For any z_i in X_i, we have $p(y_i, z_i) = p(Ax_i, z_i) = q(x_i, z_i) = p(x_i, z_i)$, hence as p_i is non-degenerate, $x_i = y_i$. Thus, we get $Ay_i = y_i$ for any y_i in Y_i. As A is p-symmetric and $X_0 = Y_0$ is the p-orthogonal complement of $\bigoplus_{1 \leq i \leq d} Y_i$, we get $AX_0 \subset X_0$. Since the restriction of q to X_0 is equal to the restriction of p, A is the identity map and $q = p$ as required. $\square$

Now, we will split the proof of Lemma 7.15 into several cases. First we will assume that Φ_w is coercive, that is, it is positive definite and defines the topology of H_0^ω. Note that, as $\overline{\mathcal{D}}(\partial X)$ may be seen as a (dense) subspace of the topological dual space of H_0^ω, the restriction of the dual bilinear form of Φ_w to $\overline{\mathcal{D}}(\partial X)$ defines a positive definite symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. We denote it by p.

We will now use again the language of Section 4 and study the quadratic fields obtained by pulling back p to our usual finite-dimensional spaces of functions. Let $j \geq 1$.

If j is even, $j = 2\ell$, $\ell \geq 1$, for any x in X, set $p_x^j = (N_x^\ell)^\star p$.

If j is odd $j = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, set $p_{xy}^j = (N_{xy}^\ell)^\star p$.

Then the relations (7.7) imply that p^j is a j-quadratic field. As p is positive definite, this field is Euclidean. The field p^j and the dual prekernel K^j are in duality as in Subsection 5.1.

Proof of Lemma 7.15 in case Φ_w is coercive and j is odd. We set $j = 2\ell+1$, $\ell \geq 1$. Let us fix $x \sim y$ in X and consider the positive symmetric bilinear form p^j_{xy} on $\overline{V}^\ell(xy)$. Its dual form on $V^\ell_0(xy)$ is q^j_{xy}. In order to apply Lemma 7.16, we need to show that the spaces $W_x = \ker J^{\ell,*}_{yx}$ and $W_y = \ker J^{\ell,*}_{xy}$ are orthogonal with respect to q^j_{xy}. Note that we have $W_x \oplus W_y = \ker M^{\ell-1,*}_{xy}$. Also note that, on $\mathcal{D}^*_0(\partial X)$, we have $M^{\ell-1,*}_{xy} N^{\ell,*}_{xy} = N^{\ell-1,*}_{xy}$ and that the space $\ker N^{\ell-1,*}_{xy}$ may be written as the direct sum of the spaces D_x and D_y of distributions defined by

$$
\begin{aligned}
D_x &= \{T \in \mathcal{D}_0(\partial X) | N^{\ell-1,*}_{xy} T = 0 \text{ and } \mathbf{1}_{U_{xy}} T = 0\} \\
D_y &= \{T \in \mathcal{D}_0(\partial X) | N^{\ell-1,*}_{xy} T = 0 \text{ and } \mathbf{1}_{U_{yx}} T = 0\}.
\end{aligned}
$$

Clearly, the map $N^{\ell,*}_{xy}$ sends D_x onto W_x and D_y onto W_y.

Consider the closed subspace $L = H^\omega_0 \cap \ker N^{\ell-1,*}_{xy}$ in H^ω_0. We set $L_x = L \cap D_x$ and $L_y = L \cap D_y$ and we have again a decomposition $L = L_x \oplus L_y$, which is now a decomposition of L as a direct sum of two orthogonal subspaces in H^ω_0. Here comes the crucial phenomenon in the proof: we claim that this decomposition is still orthogonal for the bilinear form Φ_w. Indeed, let ρ_x be in L_x and ρ_y be in L_y. We must prove that $\Phi_w(\rho_x, \rho_y) = 0$. By the definition of Φ_w in Subsection 3.2, we have

$$
\Phi_w(\rho_x, \rho_y) = \frac{1}{2} \sum_{(a,b) \in X_k} w(a,b) \mathcal{P}\rho_x(a, a_1) \mathcal{P}\rho_y(b_1, b), \tag{7.8}
$$

where, as usual, for $a \neq b$ in X, a_1 is the neighbour of a on $[ab]$ and b_1 the one of b.

We claim that for any (a, b) in X_k, we have $\mathcal{P}\rho_x(a, a_1)\mathcal{P}\rho_y(b_1, b) = 0$. Indeed, by construction, for any $s \sim t$ in X, if $\mathcal{P}\rho_x(s, t) \neq 0$, then $d(s, x)$ and $d(t, x)$ are $\geq \ell - 1$ and y is not in $[xs]$ nor in $[xt]$. In the same way, if $\mathcal{P}\rho_y(s, t) \neq 0$, then $d(s, y)$ and $d(t, y)$ are $\geq \ell - 1$ and x is not in $[ys]$ nor in $[yt]$. Therefore, for $a \neq b$ in X and a_1, b_1 as above, if $\mathcal{P}\rho_x(a, a_1)\mathcal{P}\rho_y(b_1, b) \neq 0$, we must have $d(a, x) \geq \ell$ and $d(b, y) \geq \ell$ and $[xy] \subset [ab]$, hence $d(a, b) \geq 2\ell + 1 = j > k$. By (7.8), we get $\Phi_w(\rho_x, \rho_y) = 0$.

By Lemma A.7, this implies that the spaces W_x and W_y are q^j_{xy}-orthogonal. By Lemma 7.16, p^j_{xy} is the orthogonal extension of p^{j-1}_x and p^{j-1}_y. $\square$

The proof in the even case follows the same lines.

Proof of Lemma 7.15 in case Φ_w is coercive and j is even. In this case, we set $j = 2\ell$, $\ell \geq 2$. We fix x in X and we consider the positive symmetric bilinear form p_x^j on $\overline{V}^\ell(x)$, with its dual form q_x^j on $V_0^\ell(x)$. We set, for any $y \sim x$, $W_y = \bigcap_{\substack{z\sim x\\ z\neq y}} \ker I_{xz}^{\ell-1,*}$. We get $\ker M_x^{\ell-1,*} = \bigoplus_{y\sim x} W_y$ and we need to show that this decomposition is q_x^j-orthogonal. Again, there is a related decomposition of $\mathcal{D}_0^*(\partial X)$: we may write $\ker N_{xy}^{\ell-1,*}$ as the direct sum of the spaces D_y, $y \sim x$ which are defined as

$$D_y = \{T \in \mathcal{D}_0(\partial X) | N_x^{\ell-1,*}T = 0 \text{ and } \forall z \sim x, z \neq y \quad \mathbf{1}_{U_{xz}}T = 0\}.$$

The map $N_x^{\ell,*}$ sends D_y onto W_y.

In H_0^ω, we define the closed subspaces $L = H_0^\omega \cap \ker N_x^{\ell-1,*}$ and, for $y \sim x$, $L_y = L \cap D_y$, so that we have the orthogonal decomposition $L = \bigoplus_{y\sim x} L_y$. Again, this decomposition is still orthogonal for the bilinear form Φ_w. Indeed, if y and z are two different neighbours of x and ρ_y and ρ_z are in L_y and L_z, we have, as in (7.8),

$$\Phi_w(\rho_y, \rho_z) = \frac{1}{2} \sum_{(a,b)\in X_k} w(a,b)\mathcal{P}\rho_y(a,a_1)\mathcal{P}\rho_z(b_1,b). \tag{7.9}$$

Again, we claim that $\mathcal{P}\rho_y(a,a_1)\mathcal{P}\rho_z(b_1,b) = 0$ for any (a,b) in X_k. Indeed, in this case, for any $s \sim t$ in X, if $\mathcal{P}\rho_y(s,t) \neq 0$, then $d(s,x)$ and $d(t,x)$ are $\geq \ell - 1$ and y is in $[xs]$ and in $[xt]$; if $\mathcal{P}\rho_z(s,t) \neq 0$, then $d(s,x)$ and $d(t,x)$ are $\geq \ell - 1$ and z is in $[xs]$ and in $[xt]$. Therefore, for $a \neq b$ in X, if $\mathcal{P}\rho_y(a,a_1)\mathcal{P}\rho_z(b_1,b) \neq 0$, we must have $d(a,x) \geq \ell$ and $d(b,x) \geq \ell$ and $x \in [ab]$, hence $d(a,b) \geq 2\ell = j > k$. By (7.9), we get $\Phi_w(\rho_y,\rho_z) = 0$. The conclusion follows as in the odd case. $\square$

Proof of Lemma 7.15 in the general case. We will use the approximation result from Proposition A.8 in the appendix to be brought back to the coercive case. Let us be more precise.

By definition, the scalar product of H_0^ω is the bilinear form associated to the constant function with value 2 on X_1. By Lemma 3.15, it is also the bilinear form associated to the constant function with value 2 on X_k. Therefore, as Φ_w is non-negative, for any $\varepsilon > 0$, the bilinear form Φ_{w_ε} associated to $w_\varepsilon = w + \varepsilon$ is coercive. Let, for any $\varepsilon > 0$ and $j \geq 1$, K_ε^j be the image j-dual prekernel of w_ε. Then, by Proposition A.8, for any $j \geq 1$, we have

$$K_\varepsilon^j \xrightarrow[\varepsilon\to 0]{} K^j$$

(where the convergence takes place in the finite-dimensional vector space $\mathcal{K}_j$ of Γ-invariant j-dual prekernels). Fix $j \geq k+1$. For $\varepsilon > 0$, as w_ε is Euclidean, $(K_\varepsilon^j, K_\varepsilon^{j-1})$ is the orthogonal extension of the $(j-1)$-dual kernel $(K_\varepsilon^{j-1}, K_\varepsilon^{j-2})$. As the relations defining the orthogonal extension are linear, they are continuous, hence (K^j, K^{j-1}) is the orthogonal extension of the $(j-1)$-dual kernel (K^{j-1}, K^{j-2}). □

7.4. **From additive kernels to dual kernels.** We are now ready to prove that w may be recovered from its image k-dual kernel.

Theorem 7.17. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume that Φ_w is non-negative on H_0^ω and let (K, K^-) be its image k-dual kernel. Then, there exists a Γ-invariant (K, K^-)-compatible function u on X_{k-1} with w as its weight function.*

Remark 7.18. Given $k \geq 2$ and a Γ-invariant function w on X_k, there is no direct way of deciding whether the bilinear form Φ_w is non-negative on the infinite dimensional vector space H_0^ω. Theorem 7.6 and Theorem 7.17 say that Φ_w is non-negative if and only if we may write w as the weight function of some Γ-invariant non-negative k-dual kernel. Checking whether a k-dual kernel is non-negative is not of the same order of difficulty, since it only requires to check whether bilinear forms on finite-dimensional vector spaces are non-negative.

From this, we draw results on the structure of the symmetric bilinear forms Φ_w. See Definitions 5.12 and 5.13 for the notion of an exact kernel.

Corollary 7.19. *Let $k \geq 2$, w be a symmetric Γ-invariant function on X_k and (K, K^-) be a Γ-invariant exact k-dual kernel. Assume that Φ_w is non-negative. Then (K, K^-) is the image k-dual kernel of Φ_w if and only if w is a weight function of (K, K^-) and H_0^ω has dense range in H^{K,K^-}.*

If H is a vector space and Φ is a non-negative symmetric bilinear form on H, the space of all bounded linear functionals of H with respect to Φ may be seen as the topological dual space of $H/\ker\Phi$ and comes with a natural Hilbert space structure.

Corollary 7.20. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k such that Φ_w is non-negative on H_0^ω. Let $U_w \subset \overline{\mathcal{D}}(\partial X)$ be the space of φ in $\overline{\mathcal{D}}(\partial X)$ such that the linear functional $T \mapsto \langle T, \varphi\rangle$ is bounded with respect to Φ_w on H_0^ω. Then U_w is dense in the space of all linear functionals on H_0^ω which are bounded with respect to Φ_w.*

We also have a statement in case Φ_w is not necessarily non-negative.

Corollary 7.21. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Then there exists a Γ-invariant k-dual kernel (K, K^-) such that, for any θ in H_0^ω, one has*

$$q_x^{2j}(N_x^{j,*}\theta, N_x^{j,*}\theta) \xrightarrow[j\to\infty]{} \Phi_w(\theta,\theta).$$

In particular, w is a weight function of (K, K^-).

Let us now prove this results. In the coercive case, Theorem 7.17 will follow from the easy

Lemma 7.22. *Let H be a Hilbert space with scalar product p and (K_ℓ) be decreasing sequence of closed subspaces of H with $\bigcap_\ell K_\ell = \{0\}$. For any ℓ, let π_ℓ be the quotient map $H \to H/K_\ell$. Then, for any x, y in H, we have*

$$(\pi_\ell)_\star p(\pi_\ell x, \pi_\ell y) \xrightarrow[\ell\to\infty]{} p(x, y).$$

Proof. For any ℓ, let H_ℓ be the orthogonal complement of K_ℓ and θ_ℓ be the orthogonal projection onto H_ℓ. Then $(\pi_\ell)_\star p(\pi_\ell x, \pi_\ell y) = p(\theta_\ell x, \theta_\ell y)$. As $\bigcup_\ell H_\ell$ is dense in H, $\theta_\ell x$ and $\theta_\ell y$ converge to x and y and the result follows. □

Proof of Theorem 7.17 in case Φ_w is coercive. In that case, it follows from Lemma 7.15 and Lemma 7.22 that, for any θ in H_0^ω, we have $\Phi_w(\theta, \theta) = q^{K,K^-}(\theta, \theta)$.

Now, we use the theory in Section 6: we chose a (K, K^-)-compatible function u' as in Definition 6.5 which is Γ-invariant (this is possible as noticed in Remark 6.6) and we let w' denote the associated weight function. By Theorem 7.6, the form q^{K,K^-} is equal $\Phi_{w'}$ on H_0^ω. Thus, we get $\Phi_w = \Phi_{w'}$ on H_0^ω. By Corollary 7.4, this tells us that the normalized smooth functions associated to w and w' are cohomologuous. Equivalently, by Lemma 3.13, there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that, for any (x, y) in X_k, one has $w(x, y) = w'(x, y) + v(x, y_1) - v(x_1, y)$, where as usual x_1 and y_1 are the neighbours of x and y on $[xy]$. We set $u(x, y) = u'(x, y) + v(x, y)$ for any (x, y) in X_{k-1}. As v is skew-symmetric, by Definition 6.5, u is still a (K, K^-)-compatible function and by Definition 6.7, its weight function is w. □

The proof in the general case will rely on the same approximation argument as the proof of Lemma 7.15. We will also need

Lemma 7.23. *Let $k \geq 2$ and (K, K^-) be a Γ-invariant k-dual kernel. Then the map $u \mapsto w$ is an affine isomorphism between the space of Γ-invariant (K, K^-)-compatible functions and the space of Γ-invariant weight functions.*

Proof. We need to prove that this map is injective. Let u and u' be Γ-invariant (K, K^-)-compatible functions on X_{k-1} with the same associated weight function. Then, as both u and u' are (K, K^-)-compatible, the function $u'' = u' - u$ is skew-symmetric. As u and u' have the same weight function, for any (x, y) in X_k, we have $u''(x, y_1) + u''(y, x_1) = 0$ (where x_1 and y_1 are the neighbours of x and y on $[xy]$), hence $u''(x, y_1) = u''(x_1, y)$. In particular, the smooth function $s = (x_h)_{h\in\mathbb{Z}} \mapsto u''(x_0, x_{k-1})$ is invariant under the shift map of Section 2. By Proposition 2.3, it is constant. Hence u'' is constant and, as it is skew-symmetric, it is zero, which should be proved. □

Proof of Theorem 7.17 in the general case. As in the proof of Lemma 7.15, we set $w_\varepsilon = w + \varepsilon$ for $\varepsilon > 0$, so that the bilinear form Φ_{w_ε} is coercive and, by Proposition A.8, we have $(K_\varepsilon, K_\varepsilon^-) \xrightarrow[\varepsilon\to 0]{} (K, K^-)$, where $(K_\varepsilon, K_\varepsilon^-)$ is the image k-dual kernel of w_ε. By the coercive case, for $\varepsilon > 0$, there exists a Γ-invariant $(K_\varepsilon, K_\varepsilon^-)$-compatible function u_ε on X_{k-1} such that w_ε is the associated weight function.

We claim that u_ε has a limit u as $\varepsilon \to 0$ which is a (K, K^-)-compatible function and the associated weight function is w, which finishes the proof. Indeed, from Lemma 7.23, we know that the map $(K, K^-, u) \mapsto (K, K^-, w)$ is a linear isomorphism from the space of Γ-invariant triples (K, K^-, u) where (K, K^-) is a k-dual kernel and u is a (K, K^-)-compatible function onto the space of Γ-invariant triples where w is a weight function. As all the involved spaces are finite-dimensional this linear isomorphism is a homeomorphism and the claim follows. □

Proof of Corollary 7.19. This is a direct consequence of Theorem 7.6, Theorem 7.17 and Lemma B.7. □

Proof of Corollary 7.20. Let (K, K^-) be the image k-dual kernel of Φ_w. Then, by Theorem 7.6 and Theorem 7.17, the restriction of q^{K,K^-} to H_0^ω is Φ_w and, by Corollary 7.19, H_0^ω has dense range in H^{K,K^-}. Thus, we must show that U_w has dense range in the topological dual space of H^{K,K^-}. As, by Proposition 5.18, H^{K,K^-} is complete with respect to q^{K,K^-}, this amounts to proving that the orthogonal space of U_w in H^{K,K^-} is 0. In other words, if $U_w^\perp$ is the space of those T in L^{K,K^-} such that $\langle T, \varphi\rangle = 0$ for any φ in U_w, we must show that we have $U_w^\perp \subset \ker q^{K,K^-}$.

We fix x in X and, for $\ell \geq 1$, we let, as in Subsection 5.4, N_x^ℓ denote the natural linear operator $V^\ell(x) \to \mathcal{D}(\partial X)$. We also set U_ℓ to be the orthogonal space of $\ker((N_x^{\ell,*})_\star \Phi_w)$ in $\overline{V}^\ell(x)$ (where as usual, we have identfied $\overline{V}^\ell(x)$ with the dual space of $V_0^\ell(x)$). Now, one easily checks that one has $U_w = \bigcup_{\ell\geq 1} N_x^\ell U_\ell$ so that a distribution T belongs to U_w if and only if, for any $\ell \geq 1$, $N_x^{\ell,*} T$ belongs to $\ker((N_x^{\ell,*})_\star \Phi_w)$ and we are done since (K, K^-) is the image k-dual kernel of Φ_w. □

Proof of Corollary 7.21. The set of Γ-invariant symmetric functions w' on X_k with $\Phi_{w'}$ coercive is non-empty, as it contains the constant positive functions. Since it is an open convex cone in the finite-dimensional vector space of symmetric Γ-invariant functions on X_k, any such function w may be written as a difference $w' - w''$, where $\Phi_{w'}$ and $\Phi_{w''}$ are coercive. By Theorem 7.17, we can find non-negative Γ-invariant k-dual kernels $(K', (K')^-)$ and $(K'', (K'')^-)$ which admit w' and w'' as weight functions. Therefore, w is a weight function of the dual kernel $(K, K^-) = (K' - K'', (K')^- - (K'')^-)$. The convergence follows from Theorem 7.6 (or from Corollary 7.9). Note that, as soon as this convergence takes place, w must be a weight function of (K, K^-) by Corollary 7.4. □

8. The weight map

Our aim now will be to give a characterization of those dual kernels which are the image kernels of a Γ-invariant function w with non-negative associated bilinear form on H_0^ω, or equivalently of those exact dual kernels (K, K^-) such that H_0^ω has dense range in H^{K,K^-}. This will require us to go back to the language of Section 6 and to study more carefully the map that sends a dual kernel to its weight functions.

More precisely, for $k \geq 2$, let as above $\mathcal{K}_k$ denote the real vector space of Γ-invariant k-dual kernels (which is finite-dimensional since $\Gamma\backslash X$ is finite). We also let $\mathcal{W}_k$ denote the quotient space of the space of symmetric Γ-invariant real valued functions on X_k by the space of functions of the form $(x, y) \mapsto u(x, x_1) + u(y, y_1)$, where u is a skew-symmetric Γ-invariant function on X_{k-1} and x_1 and y_1 are the neighbours of x and y on $[xy]$. By Lemma 3.13, the space $\mathcal{W}_k$ may be seen as a space of cohomology classes of smooth functions on $\Gamma\backslash\mathscr{S}$. By Definitions 6.5 and 6.7, if (K, K^-) is a Γ-invariant k-dual kernel, the set of its Γ-invariant weight functions is an equivalence class in $\mathcal{W}_k$. Thus, we have a well-defined linear map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ which we call the weight map. We will now prove that it is surjective and describe its null space.

8.1. **Surjectivity of the weight map.**

For Γ-invariant k-dual kernels, surjectivity of the weight map follows from Corollary 7.21. In this Subsection, we give a direct proof of this phenomenon by exhibiting an explicit section of the weight map. This construction will be used again in Section 11.

We need new notation. Let $k \geq 1$ and w be a symmetric function on X_k. For any x, y in X with $j = d(x,y) \geq k$, we set

$$\sum_{[xy]} w = \sum_{h=0}^{j-k} w(z_h, z_{h+k}),$$

where $x = z_0, z_1, \ldots, z_k = y$ is the geodesic path from x to y. If $d(x,y) < k$, we set $\sum_{[xy]} w = 0$.

We easily get

Lemma 8.1. *Let $k \geq 1$, w be a symmetric function on X_k, x, y be in X with $d(x,y) \geq k+1$ and x_1, y_1 be the neighbours of x and y on $[xy]$. We have*

$$\sum_{[xy]} w + \sum_{[x_1 y_1]} w = \sum_{[x y_1]} w + \sum_{[x_1 y]} w.$$

Now, let $k \geq 2$ and let still w be a symmetric function on X_k. For $j \geq k-1$, we define a j-dual prekernel $K^{w,j}$ by setting, if j is even, $j = 2\ell$, $\ell \geq 1$, for any x in X and z, t in $S^\ell(x)$,

$$K_x^{w,j}(z,t) = \sum_{[zt]} w$$

and in the same way, if j is odd, $j = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X and z, t in $S^\ell(x)$,

$$K_{xy}^{w,j}(z,t) = \sum_{[zt]} w$$

For $j = k$, we simply write K^w for $K^{w,k}$. Note that $K^{w,k-1} = 0$.

An elementary computation gives

Proposition 8.2. *Let $k \geq 2$ and w be a symmetric function on X_k. Then, for any $j \geq k$, the $(j+1)$-dual kernel $(K^{w,j+1}, K^{w,j})$ is the orthogonal extension of $(K^{w,j}, K^{w,j-1})$.*

See Definition 5.7 for the meaning of the orthogonal extension of a dual kernel.

Proof. Let temporarily $(L, K^{w,j})$ denote the orthogonal extension of $(K^{w,j}, K^{w,j-1})$. We have to prove that $L = K^{w,j+1}$.

Assume j is even, $j = 2\ell$, $\ell \geq 1$. We fix $x \sim y$ in X and $z \neq t$ in $S^\ell(xy)$. Let z_1 and t_1 be the neighbours of z and t on $[zt]$. If $[xy] \subset [zt]$

and, for example, $d(x,z)=\ell=d(y,t)$, we have

$$\begin{aligned}L_{xy}(z,t)&=K_x^{w,j}(z,t_1)+K_y^{w,j}(z_1,t)-K_{xy}^{w,j-1}(z_1t_1)\\&=\sum_{[zt_1]}w+\sum_{[z_1t]}w-\sum_{[z_1,t_1]}w\\&=\sum_{[zt]}w=K_{xy}^{w,j+1}(z,t),\end{aligned}$$

where we have applied Lemma 8.1 to the segment $[zt]$, which was possible since $d(z,t)=j+1\geq k+1$. Now, if $[xy]\not\subset[zt]$ and for example $d(z,x)=d(t,x)=\ell$, we have

$$K_y^{w,j}(z_1,t_1)=\sum_{[z_1t_1]}w=K_{xy}^{w,j-1}(z_1,t_1),$$

hence

$$L_{xy}(z,t)=K_x^{w,j}(z,t_1)=\sum_{[zt]}w=K_{xy}^{w,j+1}(z,t)$$

and we are done.

Assume j is odd, $j=2\ell+1$, $\ell\geq 1$. We fix x in X and $y\neq z$ in $S^{\ell+1}(x)$ and we let as above y_1,z_1 be the neighbours of y and z on $[yz]$. If x belongs to $[yz]$, we let a be the neighbour of x on $[xy]$ and b be its neighbour of $[xz]$. We have

$$\begin{aligned}L_x(y,z)=K_{xa}^{w,j}(y,z_1)&+K_{xb}^{w,j}(y,z_1)\\&+\sum_{\substack{c\sim x\\ c\notin\{a,b\}}}K_{xc}^{w,j}(y_1,z_1)-(d(x)-1)K_x^{w,j-1}(y_1,z_1),\end{aligned}$$

hence, as $d(y,z)=j+1\geq k+1$, by Lemma 8.1,

$$L_x(y,z)=\sum_{[yz_1]}w+\sum_{[y_1z]}w-\sum_{[y_1,z_1]}w=\sum_{[y,z]}w=K_x^{w,j+1}(y,z).$$

Now, if $x\notin[yz]$ and a is the neighbour of x with $d(y,a)=d(z,a)=\ell$, we have, for any $b\sim x$, $b\neq a$,

$$K_{xb}^{w,j}(y_1,z_1)=\sum_{[y_1z_1]}w=K_x^{w,j-1}(y_1,z_1),$$

hence

$$L_x(y,z)=K_{xa}^{w,j}(y,z)=\sum_{[yz]}w=K_x^{w,j+1}(y,z),$$

which should be proved. □

Corollary 8.3. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Then a function u on X_{k-1} is $(K^w, 0)$-compatible if and only if it is skew-symmetric. In particular, if $u = 0$, the associated weight function is w.*

Proof. By Definition 6.5 and Proposition 8.2, u is a $(K^w, 0)$-compatible function if and only if, for any x, y in X with $d(x, y) = k - 1$ and any parametrized geodesic line $(z_h)_{h\in\mathbb{Z}}$ with $z_0 = x$ and $z_{k-1} = y$,

$$u(x, y) + u(y, x) = \sum_{h=1}^{k-1} \sum_{i=h+1-k}^{h-2} w(z_i, z_{i+k}) - \sum_{h=1}^{k-2} \sum_{i=h+1-k}^{h-1} w(z_i, z_{i+k}).$$

In the right hand-side of the latter, the pairs (h, i) with $1 \leq h \leq k - 2$ and $h + 1 - k \leq i \leq h - 2$ appear twice. Thus, we get

$$u(x, y) + u(y, x) = \sum_{i=0}^{k-3} w(z_i, z_{i+k}) - \sum_{h=1}^{k-2} w(z_{h-1}, z_{h+k-1}) = 0,$$

that is, u is skew-symmetric.

Now, we let u be 0, so that, by Definition 6.7, the associated weight function w' must verify that, for any x, y in X with $d(x, y) = k$, if $(z_h)_{h\in\mathbb{Z}}$ is a parametrized geodesic line with $z_0 = x$ and $z_k = y$,

$$w'(x, y) = \sum_{h=1}^{k-1} \sum_{i=h+1-k}^{h-1} w(z_i, z_{i+k}) - \sum_{h=1}^{k} \sum_{i=h+1-k}^{h-2} w(z_i, z_{i+k}).$$

Again, in the right hand-side of the latter, the pairs (h, i) with $1 \leq h \leq k - 1$ and $h + 1 - k \leq i \leq h - 2$ appear twice and we get

$$w'(x, y) = \sum_{h=1}^{k-1} w(z_{h-1}, z_{h+k-1}) - \sum_{i=1}^{k-2} w(z_i, z_{i+k}) = w(x, y),$$

which should be proved. □

For Γ-invariant k-dual kernels, we retrieve Corollary 7.21.

Corollary 8.4. *For any $k \geq 2$, the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is surjective.*

8.2. **Pseudokernels.** Now that we have proved that the weight map is surjective, we will study its null space. This will be done by introducing a new vector space $\mathcal{L}_{k-1}$, together with an injective linear map $\mathcal{L}_{k-1} \hookrightarrow \mathcal{K}_k$ from $\mathcal{L}_{k-1}$ to the space of Γ-invariant k-dual kernels. The range of $\mathcal{L}_{k-1}$ under this map will exactly be the null space of the weight map. The proof of this result will be the objective of the next subsections.

We start by defining a new notion. Again, we have to split the definition according to the parity of k.

Definition 8.5. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$. A k-pseudokernel is a family $(L_{xy})_{(x,y)\in X_1}$ where for any (x, y) in X_1, L_{xy} is a symmetric function on $S^\ell(xy) \times S^\ell(xy)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(xy)$ associated to L_{xy} by Lemma 5.1 is denoted by r^L_{xy}.

Remark 8.6. Note that in the odd case, although the set on which L_{xy} is defined is symmetric in x and y, the function L_{xy} is not necessarily equal to L_{yx}.

Definition 8.7. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. A k-pseudokernel is a family $(L_{xy})_{(x,y)\in X_1}$ where for any (x, y) in X_1, L_{xy} is a symmetric function on $S^\ell(x) \times S^\ell(x)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(x)$ associated to L_{xy} by Lemma 5.1 is denoted by r^L_{xy}.

Remark 8.8. Note that in the even case, although the set on which L_{xy} is defined only depends on x, the function L_{xy} a priori also depends on the choice of a neighbour y of x.

As for dual kernels, when this will be convenient, we will sometimes think to the L_{xy} as being locally constant functions on $\partial X \times \partial X$.

For any $k \geq 1$, we define $\mathcal{L}_k$ as the vector space of Γ-invariant k-pseudokernels. Let us build a linear map from $\mathcal{L}_k$ to $\mathcal{K}_{k+1}$.

Definition 8.9. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$, and $L = (L_{xy})_{(x,y)\in X_1}$ be a k-pseudokernel. We define the $(k + 1)$-dual kernel (K, K^-) associated to L by the formulae

$$K_x = \sum_{y\sim x} L_{xy}, \quad x \in X,$$
$$K^-_{xy} = L_{xy} + L_{yx}, \quad x \sim y \in X.$$

Equivalently, the bilinear forms associated to (K, K^-) verify

$$q^K_x = \sum_{y\sim x} (I^{\ell,*}_{xy})^\star r^L_{xy}, \quad x \in X,$$
$$q^{K^-}_{xy} = r^L_{xy} + r^L_{yx}, \quad x \sim y \in X.$$

Definition 8.10. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$, and $L = (L_{xy})_{(x,y)\in X_1}$ be a k-pseudokernel. We define the $(k + 1)$-dual kernel (K, K^-) associated to L by the formulae

$$K_{xy} = L_{xy} + L_{yx}, \quad x \sim y \in X,$$
$$K^-_x = \frac{1}{d(x) - 1} \sum_{y\sim x} L_{xy}, \quad x \in X.$$

Equivalently, the bilinear forms associated to (K, K^-) verify

$$\begin{aligned} q_x^K &= (J_{xy}^{\ell,*})^\star r_{xy}^L + (J_{yx}^{\ell,*})^\star r_{yx}^L, \quad x \sim y \in X, \\ q_x^{K^-} &= \frac{1}{d(x)-1} \sum_{y \sim x} r_{xy}^L, \quad x \in X. \end{aligned}$$

This construction defines an injective linear map $\mathcal{L}_k \hookrightarrow \mathcal{K}_{k+1}$.

Proposition 8.11. *Let $k \geq 1$ and L be a k-pseudokernel. If the associated $(k+1)$-dual kernel is 0, then L is zero.*

The proof relies on a general property of symmetric bilinear forms which are built through surjective maps.

Lemma 8.12. *Let $W_0, W_1, \dots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. Assume $q_1, \dots q_d$ to be symmetric bilinear forms on $W_1, \dots, W_d$ and set $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$. Then we have $q = 0$ if and only if there exists symmetric bilinear forms $p_1, \dots, p_d$ on W_0 with $q_i = \varpi_i^\star p_i$, $1 \leq i \leq d$, and $p_1 + \cdots + p_d = 0$.

Proof. If $p_1, \dots, p_d$ exist, then clearly $q = 0$. Let us prove the converse statement.

Assume $q = 0$ and let us fix $1 \leq i \leq d$. Let us show that there exists a symmetric bilinear form p_i on W_0 with $q_i = \varpi_i^\star p_i$. In other words, we claim that, if w_i and w_i' are in W_i and $\varpi_i(w_i) = \varpi_i(w_i')$, then $q_i(w_i, w_i) = q_i(w_i', w_i')$. Indeed, for any $j \neq i$, pick w_j in W_j with $\varpi_j(w_j) = \varpi_i(w_i) = \varpi_i(w_i')$ and let w and w' be the unique elements of W such that

$$\begin{aligned} &\pi_i(w) = w_i \text{ and } \pi_i(w') = w_i'; \\ &\pi_j(w) = w_j \text{ and } \pi_j(w') = w_j, \quad j \neq i. \end{aligned}$$

As $q(w, w) = q(w', w') = 0$, we have

$$q_i(w_i, w_i) = -\sum_{j \neq i} q(w_j, w_j) = q_i(w_i', w_i'),$$

which should be proved.

Now, for any $1 \leq i \leq d$, we have built a symmetric bilinear form p_i on W_0 with $q_i = \varpi_i^\star p_i$. In particular, we have $0 = q = \pi_0^\star(p_1 + \cdots + p_d)$, hence $p_1 + \cdots + p_d = 0$. □

We shall also need the easy

Lemma 8.13. *Let A be a finite set with at least 3 elements and u be a real-valued function on A. Assume that, for any real-valued function f on A with $\sum_{a\in A} f(a) = 0$, we have $\sum_{a\in A} u(a)f(a)^2 = 0$. Then $u = 0$.*

Proof. Pick $a \neq b$ in A. By applying the assumption to $f = \mathbf{1}_a - \mathbf{1}_b$, we get $u(a) + u(b) = 0$. Now, chose c in A with $c \neq a$ and $c \neq b$, which is possible since A has at least three elements. We get $u(a) = -u(b) = u(c) = -u(a)$, hence $u(a) = 0$, which should be proved. □

Proof of Proposition 8.11. We prove the statement by induction on $k \geq 1$.

If $k = 1$, the data of a 1-pseudokernel L is equivalent to the data of the function $u : (x, y) \mapsto L_{xy}(x, y)$ on X_1. Now, saying that the 2-dual kernel associated to L is zero implies that, for any x in X, the quadratic form

$$f \mapsto \sum_{y\sim x} u(x, y)f(y)^2$$

is zero on $V_0^1(x)$. By Lemma 8.13, we get $u = 0$ as required.

Assume now $k \geq 2$ and the statement is proved for $k - 1$ and let us prove that it is still true for k. Let L be a k-pseudokernel such that the associated $(k + 1)$-dual kernel is 0.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have

$$(J_{xy}^{\ell,*})^\star r_{xy}^L + (J_{yx}^{\ell,*})^\star r_{yx}^L = 0.$$

Hence, by Lemma 8.12, there exists a family $(s_{xy})_{(x,y)\in X_1}$ where, for any (x, y) in X_1, s_{xy} is a symmetric bilinear form on $V_0^{\ell-1}(xy)$ with $r_{xy}^L = (I_{xy}^{\ell-1,*})^\star s_{xy}$ and $s_{xy} + s_{yx} = 0$. Equivalently, there exists a $(k - 1)$-pseudokernel M such that $L_{xy} = M_{xy}$ and $M_{xy} + M_{yx} = 0$, $(x, y) \in X_1$. As the $(k + 1)$-dual kernel associated to L is zero, we also get $\sum_{y\sim x} M_{xy} = 0$, $x \in X$, hence the k-dual kernel associated to M is zero. By induction, we get $M = 0$ and therefore $L = 0$.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any x in X, we have

$$\sum_{y\sim x} (I_{xy}^{\ell,*})^\star r_{xy}^L = 0.$$

Hence, by Lemma 8.12, there exists a family $(s_{xy})_{(x,y)\in X_1}$ where for any (x, y) in X_1, s_{xy} is a symmetric bilinear form on $V_0^\ell(x)$ with $r_{xy}^L = (J_{xy}^{\ell,*})^\star s_{xy}$ and we have $\sum_{y\sim x} s_{xy} = 0$, $x \in X$. Equivalently, there exists a $(k - 1)$-pseudokernel M such that $L_{xy} = M_{xy}$, $(x, y) \in X_1$, and $\sum_{y\sim x} M_{xy} = 0$, $x \in X$. As the $(k + 1)$-dual kernel associated to L is zero, we also get $M_{xy} + M_{yx} = 0$, $x \sim y \in X$, hence the k-dual kernel associated to M is zero. By induction, we get $M = 0$ and therefore $L = 0$. □

8.3. **Orthogonal extension of pseudokernels.** For $k \geq 1$, we have embedded $\mathcal{L}_k$ as a subspace of $\mathcal{K}_{k+1}$. Now, orthogonal extension defines an injective linear map $\mathcal{K}_{k+1} \hookrightarrow \mathcal{K}_{k+2}$. We will show how the restriction of orthogonal extension of dual kernels to pseudokernels may be obtained as an intrinsinc linear map from $\mathcal{L}_k$ to $\mathcal{L}_{k+1}$.

Definition 8.14. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$. If L is a k-pseudokernel, we define its orthogonal extension L^+ as the $(k+1)$-pseudokernel such that

$$L^+_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} L_{xz}, \quad (x, y) \in X_1.$$

Equivalently, the symmetric bilinear forms associated to L^+ are related to the ones associated to L by the formula

$$r^+_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} (I^{\ell,*}_{xz})^\star r_{xz}, \quad (x, y) \in X_1.$$

Definition 8.15. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. If L is a k-pseudokernel, we define its orthogonal extension L^+ as the $(k+1)$-pseudokernel such that

$$L^+_{xy} = L_{yx}, \quad (x, y) \in X_1.$$

Equivalently, the symmetric bilinear forms associated to L^+ are related to the ones associated to L by the formula

$$r^+_{xy} = (J^{\ell,*}_{yx})^\star r_{yx}, \quad (x, y) \in X_1.$$

The reader should beware the order of the variables!

Remark 8.16. The orthogonal extension map $L \mapsto L^+$ is injective. This is obvious in the even case. In the odd case, if $k = 2\ell + 1$, $\ell \geq 0$, and L is a k-pseudokernel, for $x \sim y$ in X, one has

$$\sum_{\substack{z \sim x \\ z \neq y}} L^+_{xz} = (d(x) - 1) L_{xy} + (d(x) - 2) L^+_{xy}$$

and injectivity follows.

These definitions are justified by the

Proposition 8.17. *Let $k \geq 1$ and L be a k-pseudokernel, with associated $(k+1)$-dual kernel (K, K^-). Then the $(k+2)$-dual kernel associated to the orthogonal extension L^+ of L is the orthogonal extension (K^+, K) of K.*

In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_k & \xrightarrow{+} & \mathcal{L}_{k+1} \\ \downarrow & & \downarrow \\ \mathcal{K}_{k+1} & \xrightarrow{+} & \mathcal{K}_{k+2}, \end{array}$$

where the horizontal arrows are orthogonal extensions.

Proof. The proof follows directly from the definitions. Let (H, H^-) be the $(k+2)$-dual kernel associated to L^+.

If k is odd, we have, for any $x \sim y$ in X,

$$\begin{aligned} H_{xy} = L^+_{xy} + L^+_{yx} &= \sum_{\substack{z\sim x\\ z\neq y}} L_{xz} + \sum_{\substack{t\sim y\\ t\neq x}} L_{yt} \\ &= \sum_{z\sim x} L_{xz} + \sum_{t\sim y} L_{yt} - (L_{xy} + L_{yx}) = K_x + K_y - K^-_{xy} = K^+_{xy} \end{aligned}$$

and also, for any x in X,

$$H^-_x = \frac{1}{d(x)-1} \sum_{y\sim x} L^+_{xy} = \frac{1}{d(x)-1} \sum_{y\sim x} \sum_{\substack{z\sim x\\ z\neq y}} L_{xz} = \sum_{z\sim x} L_{xz} = K_x,$$

which should be proved.

If k is even, we have, for any x in X,

$$\begin{aligned} K^+_x = \sum_{y\sim x} K_{xy} - (d(x)-1)K^-_x &= \sum_{y\sim x} (L_{xy} + L_{yx}) - \sum_{y\sim x} L_{xy} \\ &= \sum_{y\sim x} L_{yx} = \sum_{y\sim x} L^+_{xy} = H_x \end{aligned}$$

and also, for any $x \sim y$ in X,

$$K_{xy} = L_{xy} + L_{yx} = L^+_{yx} + L^+_{xy} = H^-_{xy},$$

and the result follows. $\square$

8.4. **Large extensions of pseudokernels.** Recall that our goal is to prove that the null space of the weight map is exactly the space of pseudokernels. To do this, we will apply to pseudokernels the formalism of Section 6 and prove that the weight functions of pseudokernels are coboundaries. This requires us to associate to a k-pseudokernel L a certain function on X_k. As for dual kernels, the definition of this function will use large orthogonal extensions of L. We start with describing those extensions.

The following result is an analogue for pseudokernels of Lemma 5.9 for dual kernels.

Lemma 8.18. *Let $k \geq 1$ and L be a k-pseudokernel. The orthogonal extensions of L may be defined by the following formulae. Fix $h \geq 0$ and $x \sim y$ in X. If k is odd, we have*

$$L_{xy}^{k+2h} = \sum_{\substack{z \in S^{h+1}(x) \\ x \notin [yz]}} L_{z_- z}.$$

If k is even, we have

$$L_{xy}^{k+2h} = \sum_{\substack{z \in S^{h+1}(y) \\ y \notin [xz]}} L_{zz_-}.$$

Proof. Assume for example that k is odd and let us prove the result by induction on $h \geq 0$. For $h = 0$, there is nothing to prove. If the result holds for $h \geq 0$, then, by Definition 8.14, for $x \sim y$ in X, we have

$$L_{xy}^{k+2h+1} = \sum_{\substack{z \sim x \\ z \neq y}} L_{xz}^{k+2h} = \sum_{\substack{z \sim x \\ z \neq y}} \sum_{\substack{t \in S^{h+1}(x) \\ x \notin [zt]}} L_{t_- t} = \sum_{\substack{t \in S^{h+2}(y) \\ y \notin [xt]}} L_{t_- t}.$$

The result follows since, by Definition 8.15, one has $L_{xy}^{k+2h+2} = L_{yx}^{k+2h+1}$. □

This directly gives, by using again Definition 8.15 in the even case,

Corollary 8.19. *Let $k \geq 1$ and L be a k-pseudokernel. For $x \sim y$ in X, we have*

$$L_{xy}^{2k-1} = \sum_{\substack{z \in S^{\ell+1}(x) \\ x \notin [yz]}} L_{z_- z}, \quad k = 2\ell + 1, \quad \ell \geq 0.$$

$$L_{xy}^{2k-1} = \sum_{\substack{z \in S^{\ell}(x) \\ x \notin [yz]}} L_{zz_-}, \quad k = 2\ell, \quad \ell \geq 1.$$

In particular, we get

Corollary 8.20. *Let $k \geq 1$ and L be a k-pseudokernel. For $x \sim y$ in X and ξ, η in U_{yx}, we have $L_{xy}^{2k-1}(\xi, \eta) = 0$.*

Proof. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, by Corollary 8.19, we have

$$L_{xy}^{2k-1}(\xi, \eta) = \sum_{\substack{z \in S^{\ell+1}(x) \\ x \notin [yz]}} L_{z_- z}(\xi, \eta).$$

By assumption, for z as above, the geodesic rays $[z\xi)$ and $[z\eta)$ both meet the sphere $S^\ell(zz_-)$ at x, hence $L_{z_-z}(\xi,\eta)=0$.

In the same way, if k is even, $k=2\ell$, $\ell\geq 1$, Corollary 8.19, gives

$$L^{2k-1}_{xy}(\xi,\eta)=\sum_{\substack{z\in S^\ell(x)\\ x\notin[yz]}} L_{zz_-}(\xi,\eta).$$

Now, for such a z, the geodesic rays $[z\xi)$ and $[z\eta)$ both meet the sphere $S^\ell(z)$ at x, hence $L_{zz_-}(\xi,\eta)=0$. □

By applying Proposition 8.17, from Lemma 8.19, we get

Corollary 8.21. *Let $k\geq 1$ and L be a k-pseudokernel. The associated dual prekernels may be defined by the following formulae. Fix $h\geq 0$ and $x\sim y$ in X. If k is odd, we have*

$$K^{k+2h+1}_x=\sum_{z\in S^{h+1}(x)} L_{z_-z} \text{ and } K^{k+2h}_{xy}=\sum_{z\in S^h(xy)} L_{z_-z}.$$

If k is even, we have

$$K^{k+2h+1}_{xy}=\sum_{z\in S^h(xy)} L_{zz_-} \text{ and } K^{k+2h}_x=\sum_{z\in S^h(x)} L_{zz_-},$$

where in the last equation, we assume $h\geq 1$.

Proof. For example, let us proof the first formula. Definition 8.9, Proposition 8.17 and Lemma 8.18 give

$$K^{k+2h+1}_x=\sum_{y\sim x} L^{k+2h}_{xy}=\sum_{y\sim x}\sum_{\substack{z\in S^{h+1}(x)\\ x\notin[yz]}} L_{z_-z}.$$

The formula follows as the sphere $S^{h+1}(x)$ may be written as the disjoint union

$$S^{h+1}(x)=\bigsqcup_{y\sim x}\{z\in S^{h+1}(x)|x\notin[yz]\}.$$

□

8.5. **Weight functions of pseudokernels.** We will now prove that, for $k\geq 2$, the weight map is zero on the image of $\mathcal{L}_{k-1}$ in $\mathcal{K}_k$. This will be achieved by giving an explicit formula for the compatible functions and weight functions of pseudokernels. As mentioned above, this requires the definition of a new function associated to a pseudokernel whose existence is warranted by

Lemma 8.22. *Let $k \geq 1$ and L be a k-pseudokernel. Let (x, y) be in X_k and $(z_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line with $z_0 = x$ and $z_k = y$. Then, the quantity*

$$L^{2k-1}_{z_0z_1}(z_{1-k}, z_k) = L^{2k-1}_{xz_1}(z_{1-k}, y)$$

only depends on x and y.

Proof. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, by Corollary 8.19, we have

$$L^{2k-1}_{z_0z_1}(z_{1-k}, z_k) = \sum_{\substack{t\in S^{\ell+1}(z_0)\\ z_0\notin[z_1t]}} L_{t_-t}(z_{1-k}, z_k).$$

For t as above, the segment $[z_{1-k}t]$ meets the sphere $S^\ell(tt_-)$ at z_0, hence $L_{t_-t}(z_{1-k}, z_k)$ does not depend on the choice of the points $(z_h)_{h<0}$.

In the same way, if k is even, $k = 2\ell$, $\ell \geq 1$, Corollary 8.19, gives

$$L^{2k-1}_{z_0z_1}(z_{1-k}, z_k) = \sum_{\substack{t\in S^{\ell}(z_0)\\ z_0\notin[z_1t]}} L_{tt_-}(z_{1-k}, z_k).$$

Now, for such a z, the segment $[z_{1-k}t]$ meets the sphere $S^\ell(t)$ at z_0, hence $L_{t_-t}(z_{1-k}, z_k)$ again does not depend on the choice of the points $(z_h)_{h<0}$. □

We can define a natural function associated to a pseudokernel.

Definition 8.23. Let $k \geq 1$ and L be a k-pseudokernel. We define its pseudoweight v as the unique function on X_k such that, for any (x, y) in X_k, one has

$$v(x, y) = L^{2k-1}_{z_0z_1}(z_{1-k}, z_k) = L^{2k-1}_{xz_1}(z_{1-k}, y),$$

where $(z_h)_{h\in\mathbb{Z}}$ is any parametrized geodesic line with $z_0 = x$ and $z_k = y$.

Note that if L is Γ-invariant, so is v.

We now get a formula for weight functions of pseudokernels.

Proposition 8.24. *Let $k \geq 1$, L be a k-pseudokernel, v be the pseudoweight of L and (K, K^-) be the $(k+1)$-dual kernel associated to L. Then a function u on X_k is (K, K^-)-compatible if and only if, for any (x, y) in X_k, one has*

$$u(x, y) + u(y, x) = v(x, y) + v(y, x).$$

If u is such a function, the associated weight function w on X_{k+1} verifies, for any (x, y) in X_{k+1},

$$w(x, y) = u(x, y_1) + u(y, x_1) - v(x, y_1) - v(y, x_1)$$

(where as usual x_1 and y_1 are the neighbours of x and y on $[xy]$).

This directly implies

Corollary 8.25. *Assume L is Γ-invariant. Then $W_k(K, K^-) = 0$.*

Proof of Proposition 8.24. The proof relies on straightforward but tedious computations.

First, we establish the formula for u. Let (x, y) be in X_k and $(z_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line with $z_0 = x$ and $z_k = y$. Denote by ξ and η the endpoints of $(z_h)_{h\in\mathbb{Z}}$. By Definition 6.5, we have

$$u(x,y) + u(y,x) = \sum_{h=1}^{k} K^{2k-1}_{z_{h-1}z_h}(\xi,\eta) - \sum_{h=1}^{k-1} K^{2k}_{z_h}(\xi,\eta).$$

By Proposition 8.17, we have,

$$K^{2k}_a = \sum_{b\sim a} L^{2k-1}_{ab}, \quad a \in X,$$
$$\text{and } K^{2k-1}_{ab} = L^{2k-1}_{ab} + L^{2k-1}_{ba}, \quad a \sim b \in X.$$

Therefore, we get

$$u(x,y) + u(y,x) = \sum_{h=1}^{k} (L^{2k-1}_{z_{h-1}z_h}(\xi,\eta) + L^{2k-1}_{z_h z_{h-1}}(\xi,\eta))$$
$$- \sum_{h=1}^{k-1} (L^{2k-1}_{z_h z_{h-1}}(\xi,\eta) + L^{2k-1}_{z_h z_{h+1}}(\xi,\eta)) - \sum_{h=1}^{k-1} \sum_{\substack{w\sim z_h \\ w\neq z_{h-1},z_{h+1}}} L^{2k-1}_{z_h w}(\xi,\eta).$$

By Corollary 8.20, in the right hand-side of the latter, the third sum is zero, so that this equation gives

$$u(x,y) + u(y,x) = L^{2k-1}_{z_0 z_1}(\xi,\eta) + L^{2k-1}_{z_k z_{k-1}}(\xi,\eta) = v(x,y) + v(y,x),$$

and we are done.

Now, we prove the formula for w. Let (x, y) be in X_{k+1} and $(z_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line with $z_0 = x$ and $z_{k+1} = y$. Still denote by ξ and η its endpoints. By Definition 6.7, we have

$$w(x,y) = u(z_0, z_k) + u(z_{k+1}, z_1) + \sum_{h=1}^{k} K^{2k}_{z_h}(\xi,\eta) - \sum_{h=1}^{k+1} K^{2k-1}_{z_{h-1}z_h}(\xi,\eta).$$

Again, by Proposition 8.17 and Corollary 8.20, this gives

$$w(x,y) = u(z_0, z_k) + u(z_{k+1}, z_1)$$
$$+ \sum_{h=1}^{k} (L^{2k-1}_{z_h z_{h-1}}(\xi,\eta) + L^{2k-1}_{z_h z_{h+1}}(\xi,\eta)) - \sum_{h=1}^{k+1} (L^{2k-1}_{z_{h-1}z_h}(\xi,\eta) + L^{2k-1}_{z_h z_{h-1}}(\xi,\eta)).$$

We get

$$\begin{aligned} w(x,y) &= u(z_0,z_k) + u(z_{k+1},z_1) - L^{2k-1}_{z_0z_1}(\xi,\eta) - L^{2k-1}_{z_{k+1}z_k}(\xi,\eta) \\ &= u(x,y_1) + u(y,x_1) - v(x,y_1) - v(y,x_1) \end{aligned}$$

as required. □

8.6. **Weight functions of orthogonal extensions.** Thanks to Proposition 8.24 and Corollary 8.25, we know that the weight map is 0 on pseudokernels. It remains to show the converse statement, that if a k-dual kernel admits a weight function which is a coboundary, then this dual kernel is the one associated to some $(k-1)$-pseudokernel. Here, by a coboundary, we mean in the sense of cohomology following from Lemma 3.13.

Our strategy will be to start by a weaker statement, namely that if a k-dual kernel admits a weight function (and hence has all its weight functions) of the form $(x,y) \mapsto v(x,y_1) + v(y,x_1)$ for some non-necessarily skew-symmetric function v on X_{k-1}, then it must be the sum of a pseudokernel with an orthogonal extension. As a preliminary, the purpose of this subsection is to prove that the weight functions of orthogonal extensions are actually of this form.

Proposition 8.26. *Let $k \geq 2$ and (K,K^-) be a k-dual kernel with orthogonal extension (K^+,K). Let u be a (K,K^-)-compatible function and w be the associated weight function for (K,K^-). Then a function u^+ on X_k is (K^+,K)-compatible if and only if one has, for any (x,y) in X_k,*

$$u^+(x,y) + u^+(y,x) = w(x,y) + u(x_1,y) + u(y_1,x).$$

In that case, the associated weight function w^+ for (K^+,K) is defined by, for any (x,y) in X_{k+1},

$$w^+(x,y) = u^+(x,y_1) + u^+(y,x_1) - u(x_1,y_1) - u(y_1,x_1).$$

Corollary 8.27. *Let $k \geq 2$ and (K,K^-) be a k-dual kernel with orthogonal extension (K^+,K). Let w be a weight function for (K,K^-) and w^+ be a weight function for (K^+,K). Then there exists a skew-symmetric function v on X_k such that, for any (x,y) in X_{k+1}, one has*

$$w^+(x,y) = \frac{1}{2}(w(x,y_1) + w(x_1,y)) + v(x,y_1) - v(x_1,y).$$

If (K,K^-), w and w^+ are Γ-invariant, one can chose v to be so.

In both statements, for $x \neq y$ in X, we have as usual denoted by x_1 and y_1 the neighbours of x and y on $[xy]$.

Note that, in Corollary 8.27, the functions w and w^+ are related in the same way as the functions w and w' in Lemma 3.13.

Again, the proofs will follow from the definitions by straightforward computations.

Proof of Proposition 8.26. By Definition 6.5, saying that the function u^+ is (K^+, K)-compatible amounts to saying that, for any (x, y) in X_k, if $(z_h)_{h\in\mathbb{Z}}$ is a parametrized geodesic line with $z_0 = x$ and $z_k = y$, we have

$$u^+(x, y) + u^+(y, x) = \sum_{h=1}^{k} K^{2k-1}_{z_{h-1}z_h}(\xi, \eta) - \sum_{h=1}^{k-1} K^{2k}_{z_h}(\xi, \eta), \tag{8.1}$$

where ξ and η are the endpoints of $(z_h)_{h\in\mathbb{Z}}$. Now, for any $1 \le h \le k-1$, we have

$$\begin{aligned} K^{2k}_{z_h}(\xi, \eta) = K^{2k-1}_{z_{h-1}z_h}(\xi, \eta) &+ K^{2k-1}_{z_h z_{h+1}}(\xi, \eta) \\ &+ \sum_{\substack{t\sim z_h \\ t\notin\{z_{h-1}, z_{h+1}\}}} K^{2k-1}_{z_h t}(\xi, \eta) - (d(z_h) - 1)K^{2k-2}_{z_h}(\xi, \eta). \end{aligned}$$

This can be written as

$$\begin{aligned} K^{2k}_{z_h}(\xi, \eta) = K^{2k-1}_{z_{h-1}z_h}(\xi, \eta) &+ K^{2k-1}_{z_h z_{h+1}}(\xi, \eta) \\ &+ S^{k-1,1}_{z_h}(\xi, \eta) - K^{2k-2}_{z_h}(\xi, \eta), \end{aligned}$$

where $S^{k-1,1}_{z_h}$ is as in Subsection 6.1. By Corollary 6.2, we have $S^{k-1,1}_{z_h} = 0$ and hence (8.1) gives

$$u^+(x, y) + u^+(y, x) = \sum_{h=1}^{k-1} K^{2k-2}_{z_h}(\xi, \eta) - \sum_{h=2}^{k-1} K^{2k-1}_{z_{h-1}z_h}(\xi, \eta).$$

As $K^{2k-1}_{z_{h-1}z_h} = K^{2k-2}_{z_{h-1}} + K^{2k-2}_{z_h} - K^{2k-3}_{z_{h-1}z_h}$, $2 \le h \le k-1$, we get

$$u^+(x, y) + u^+(y, x) = \sum_{h=2}^{k-1} K^{2k-3}_{z_{h-1}z_h}(\xi, \eta) - \sum_{h=2}^{k-2} K^{2k-2}_{z_h}(\xi, \eta).$$

Now we use Definitions 6.5 and 6.7 for (K, K^-), which give

$$\begin{aligned} u^+(x, y) + u^+(y, x) = (u(x, y_1) + u(y_1, x)) &+ (u(x_1, y) + u(y, x_1)) \\ &+ (w(x, y) - u(x, y_1) - u(y, x_1)), \end{aligned}$$

that is

$$u^+(x, y) + u^+(y, x) = w(x, y) + u(y_1, x) + u(x_1, y),$$

which should be proved.

For weight functions, the same computation yields, for any (x,y) in X_{k+1}, any parametrized geodesic line $(z_h)_{h\in\mathbb{Z}}$ with endpoints ξ and η and $z_0 = x$ and $z_{k+1} = y$,

$$\begin{aligned} w^+(x,y) &= u^+(x,y_1) + u^+(y,x_1) + \sum_{h=2}^{k-1} K_{z_h}^{2k-2}(\xi,\eta) - \sum_{h=2}^{k} K_{z_{h-1}z_h}^{2k-3}(\xi,\eta) \\ &= u^+(x,y_1) + u^+(y,x_1) - u(x_1,y_1) - u(y_1,x_1). \end{aligned}$$

□

Proof of Corollary 8.27. Let u (resp. u^+) be the (K,K^-)-compatible (resp. (K^+,K)-compatible) function with w (resp. w^+) as its weight function. Fix (x,y) in X_{k+1}, let x_1 and y_1 be their neighbours on $[xy]$ and x_2 and y_2 be the neighbours of x_1 and y_1 on $[x_1y_1]$. Proposition 8.26 gives

$$\begin{aligned} w(x,y_1) &= u^+(x,y_1) + u^+(y_1,x) - u(x_1,y_1) - u(y_2,x) \\ w(x_1,y) &= u^+(x_1,y) + u^+(y,x_1) - u(x_2,y) - u(y_1,x_1) \\ w^+(x,y) &= u^+(x,y_1) + u^+(y,x_1) - u(x_1,y_1) - u(y_1,x_1), \end{aligned}$$

hence

$$w^+(x,y) - \frac{1}{2}(w(x,y_1) + w(x_1,y)) = v(x,y_1) + v(y,x_1),$$

where, for any (a,b) in X_k,

$$v(a,b) = \frac{1}{2}(u^+(a,b) - u^+(b,a) + u(b_1,a) - u(a_1,b)).$$

□

8.7. **Split weight functions.** Our goal is still to prove that if a dual kernel admits a weight function which is a coboundary, then it is a pseudokernel. As mentionned above, we will first prove a weaker version of this statement which will play a key rule in the final proof. We start by introducing a new notion.

Let $k \geq 2$ and w be a symmetric function on X_k. We shall say that w is split if there exists a function v on X_{k-1} such that for any (x,y) in X_k, one has $w(x,y) = v(x,y_1) + v(y,x_1)$ (note that we don't require v to have any symmetry property).

By Definition 6.7, if a dual kernel admits a split weight function, all its weight functions are split. Proposition 8.24 and Corollary 8.27 tell us that the weight functions of a pseudokernel and those of an orthogonal extension are split. We have a converse statement:

Proposition 8.28. *Let $k \geq 3$ and (K, K^-) be a k-dual kernel. Then the weight functions of (K, K^-) are split if and only if there exists a $(k-1)$-pseudokernel L, with associate k-dual kernel (J, J^-), and a $(k-1)$-dual kernel (H, H^-), with orthogonal extension (H^+, H), such that $K = H^+ + J$ and $K^- = H + J^-$.*

Let $k \geq 2$ and (K, K^-) be a k-dual kernel. We begin the proof of this fact by introducing a new function on X_k associated to (K, K^-).

If k is even, $k = 2\ell$, $\ell \geq 1$, we define the preweight of (K, K^-) as the function w_- on X_k such that, for any (x, y) in X_k, $w_-(x, y) = K_a(x, y)$, where a is the middle point of $[xy]$, that is, a is the unique point of $[xy]$ with $d(x, a) = \ell = d(y, a)$.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we define the preweight of (K, K^-) as the function w_- on X_k such that, for any (x, y) in X_k, $w_-(x, y) = K_{ab}(x, y)$, where $[ab]$ is the middle edge of $[xy]$, that is, a and b are the unique points of $[xy]$ with $d(x, a) = \ell = d(y, b)$.

Remark 8.29. Note that the preweight actually does not depend on K^-.

Lemma 8.30. *Let $k \geq 2$, (K, K^-) be a k-dual kernel, w_- be the preweight of (K, K^-) and w be a weight function of (K, K^-). Then the function $w - w_-$ is split.*

Proof. The proof of this fact follows from a careful rereading of the proof of Lemma 6.3.

More precisely, it follows from this proof (in case $j = k$) that, for any (x, y) in X_k and any $\frac{k}{2} \leq \ell \leq k - 1$, the number

$$w_\ell(x, y) = \sum_{h=k-\ell}^{\ell} K_{z_h}^{2\ell}(\xi, \eta) - \sum_{h=k-\ell}^{\ell+1} K_{z_{h-1} z_h}^{2\ell-1}(\xi, \eta)$$

does not depend on the choice of a parametrized geodesic line $(z_h)_{h \in \mathbb{Z}}$ with $z_0 = x$ and $z_k = y$ and endpoints ξ and η.

By Definition 6.7, the function $w - w_{k-1}$ is split. We claim that, for any $\frac{k}{2} \leq \ell \leq k-2$, the function $w_{\ell+1} - w_\ell$ is split. To prove this we will use again the notation $S_z^{\ell,m}(\xi, \eta)$ which was introduced in Subsection 6.1. For any (x, y) in X_{k-1}, we set

$$v_\ell(x, y) = S_{z_{k-\ell-1}}^{\ell,1}(\xi, \eta) + \frac{1}{2} \sum_{h=k-\ell}^{\ell} S_{z_h}^{\ell,1}(\xi, \eta),$$

where $x = z_0, \ldots, z_{k-1} = y$ is the geodesic path from x to y and $(\xi\eta)$ is any geodesic line with $[xy] \subset (\xi\eta)$. It follows from Corollary 6.2 that

$v_\ell(x,y)$ does not depend on the choice of $(\xi\eta)$. Now, Equation (6.2) in the proof of Lemma 6.3 gives, for any (x,y) in X_k,

$$w_{\ell+1}(x,y) = w_\ell(x,y) + v_\ell(x,y_1) + v_\ell(y,x_1)$$

and $w_{\ell+1} - w_\ell$ is indeed split.

To conclude, it remains to compute w_ℓ for the lowest possible value of ℓ.

If k is even and $\ell = \frac{k}{2}$, we have, for any (x,y) in X_k, if z is the middle point of $[xy]$ and a and b are the neighbours of z respectively on $[xz]$ and on $[yz]$,

$$\begin{aligned} w_\ell(x,y) &= K_z(x,y) - K^-_{az}(x,y_1) - K^-_{bz}(y,x_1) \\ &= w_-(x,y) - K_{az}(x,y_1) - K_{bz}(y,x_1), \end{aligned}$$

hence $w_\ell - w_-$ is split.

If k is odd and $\ell = \frac{k+1}{2}$, fix (x,y) in X_k and let $[zt]$ be the middle edge of $[xy]$ (with $d(x,z) = \ell - 1 = d(y,t)$). Equation (6.1) in the proof of Lemma 6.3 reads as

$$\begin{aligned} w_\ell(x,y) = w_-(x,y) - (d(z)-1)K^-_z(x,y_1) - (d(t)-1)K^-_t(x_1,y) \\ + \sum_{\substack{a\sim z \\ a\notin[xy]}} K_{az}(x_1,y_1) + \sum_{\substack{b\sim t \\ b\notin[xy]}} K_{bt}(x_1,y_1). \end{aligned}$$

In particular, $w_\ell - w_-$ is split and the lemma follows. □

We have an abstract criterion for a bilinear form to split as a sum.

Lemma 8.31. *Let $W_0, W_1, \dots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i,j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. For any $1 \leq i \leq d$, we set $X_i = \bigcap_{j\neq i} \ker \pi_j \subset W$.

Let q be a symmetric bilinear form on W. Then there exists symmetric bilinear forms $q_1, \dots q_d$ on $W_1, \dots, W_d$ with $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$ if and only if, for any $1 \leq i \neq j \leq d$, the spaces X_i and X_j are q-orthogonal, that is, $q(X_i, X_j) = 0$.

Proof. Clearly, if $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$ for some symmetric bilinear forms $q_1, \dots q_d$ on $W_1, \dots, W_d$, then for any $i \neq j$, the spaces X_i and X_j are q-orthogonal.

Conversely, assume this is the case and let us build $q_1, \dots q_d$. We chose a subspace X_0 of W such that the restriction of π_0 to X_0 is an isomorphism onto W_0. We then have $W = X_0 \oplus X_1 \oplus \cdots \oplus X_d$ and, by assumption, if $w = x_0 + \cdots + x_d$ is in W, with $x_i \in X_i$, $0 \leq i \leq d$, one has

$$q(w,w) = \sum_{i=0}^{d} q(x_i, x_i) + 2\sum_{i=1}^{d} q(x_0, x_i).$$

Now, for any $1 \leq i \leq d$, the restriction of π_i to $X_0 + X_i$ is an isomorphism onto W_i. Therefore, there exists a unique symmetric bilinear form q_i on W_i such that, for x_0 in X_0 and x_i in X_i, one has

$$\pi_i^\star q_i(x_0 + x_i, x_0 + x_i) = \frac{1}{d} q_0(x_0, x_0) + q_i(x_i, x_i) + 2q_i(x_0, x_i).$$

By construction, one has $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$. □

Proof of Proposition 8.28. Let (K, K^-) be a k-dual kernel which admits a split weight function. By Lemma 8.30, the preweight w_- of (K, K^-) is split. We will show that this amounts to saying that one can apply Lemma 8.31 to the bilinear forms associated to K. Fix a function v on X_{k-1} with $w_-(x,y) = v(x, y_1) + v(y, x_1)$, $(x,y) \in X_k$.

First assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$. Fix x, y in X. We claim that the subspaces

$$\ker J_{xy}^{\ell,*} \text{ and } \ker J_{yx}^{\ell,*}$$

of $V_0^\ell(xy)$ are q_{xy}^K-orthogonal. Indeed pick f_x in $\ker J_{yx}^{\ell,*}$ and f_y in $\ker J_{xy}^{\ell,*}$. We have

$$\begin{aligned}
f_x(b) &= 0 \quad b \in S^\ell(y) \cap S^{\ell+1}(x) \\
f_y(a) &= 0 \quad a \in S^\ell(x) \cap S^{\ell+1}(y) \\
\sum_{\substack{a \sim a_1 \\ a \notin [xa_1]}} f_x(a) &= 0 \quad a_1 \in S^{\ell-1}(x) \cap S^\ell(y) \\
\sum_{\substack{b \sim b_1 \\ b \notin [yb_1]}} f_y(b) &= 0 \quad b_1 \in S^{\ell-1}(y) \cap S^\ell(x).
\end{aligned}$$

Therefore, by Lemma 5.1,

$$\begin{aligned}
q_{xy}^K(f_x, f_y) &= -\frac{1}{2} \sum_{\substack{a \in S^\ell(x) \cap S^{\ell+1}(y) \\ b \in S^\ell(y) \cap S^{\ell+1}(x)}} w_-(a,b) f_x(a) f_y(b) \\
&= -\frac{1}{2} \sum_{\substack{a_1 \in S^{\ell-1}(x) \cap S^\ell(y) \\ b_1 \in S^{\ell-1}(y) \cap S^\ell(x)}} \sum_{\substack{a \sim a_1 \\ a \notin [xa_1]}} \sum_{\substack{b \sim b_1 \\ b \notin [yb_1]}} (v(a, b_1) + v(b, a_1)) f_x(a) f_y(b) = 0.
\end{aligned}$$

By Lemma 8.31, we can find a family $(s_{xy})_{(x,y) \in X_1}$, where, for any (x,y), s_{xy} is a symmetric bilinear form on $V_0^\ell(x)$ and $q_{xy}^K = (J_{xy}^{\ell,*})^\star s_{xy} +$

$(J^{\ell,*}_{yx})^\star s_{yx}$. In other words, there exists a $(k-1)$-pseudokernel M with $K_{xy} = M_{xy} + M_{yx}$ for any $x \sim y$ in X. This is not over since for the moment there is no relation between K^- and M. To correct this, we set

$$\begin{aligned} H_x &= \sum_{y\sim x} M_{xy} - (d(x)-1)K^-_x \quad x \in X, \\ L_{xy} &= M_{xy} - H_x \quad (x,y) \in X_1 \end{aligned}$$

and we set $H^- = 0$ and we consider $(H, H^-) = (H, 0)$ as a $(k-1)$-dual kernel and L as a $(k-1)$-pseudokernel. Let (H^+, H) be the orthogonal extension of $(H, 0)$ and (J, J^-) be the k-dual kernel associated to L. By construction, we have, for $x \sim y$ in X,

$$K_{xy} = M_{xy} + M_{yx} = L_{xy} + L_{yx} + H_x + H_y = J_{xy} + H^+_{xy}$$

and, for x in X,

$$\begin{aligned} K^-_x &= \frac{1}{d(x)-1}\left(\sum_{y\sim x} M_{xy} - H_x\right) \\ &= \frac{1}{d(x)-1}\left(\sum_{y\sim x}(L_{xy} + H_x) - H_x\right) = \frac{1}{d(x)-1}\sum_{y\sim x} L_{xy} + H_x \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = J^-_x + H_x, \end{aligned}$$

which should be proved.

Now assume k is even, $k = 2\ell$, $\ell \geq 2$, and let us proceed in the same way. We fix x in X and we set, for $y \sim x$,

$$W_y = \bigcap_{\substack{z\sim x \\ z\neq y}} \ker I^{\ell-1,*}_{xz} \subset V_0(x).$$

We claim that these spaces are q^K_x-orthogonal to each other. Indeed, for $y \sim x$ and f_y in W_y, we have

$$\begin{aligned} f_y(b) &= 0 \quad b \in S^\ell(x) \cap S^{\ell+1}(y) \\ \sum_{\substack{a\sim a_1 \\ a\notin[xa_1]}} f_y(a) &= 0 \quad a_1 \in S^{\ell-1}(x) \cap S^{\ell-2}(y). \end{aligned}$$

Now pick $y \neq z$ among the neighbours of x and chose f_y in W_y and f_z in W_z. Again by Lemma 5.1, we have

$$q_x^K(f_y, f_z) = -\frac{1}{2} \sum_{\substack{a \in S^\ell(x) \cap S^{\ell-1}(y) \\ b \in S^\ell(x) \cap S^{\ell-1}(z)}} w_-(a,b) f_y(a) f_z(b)$$

$$= -\frac{1}{2} \sum_{\substack{a_1 \in S^{\ell-1}(x) \cap S^{\ell-2}(y) \\ b_1 \in S^{\ell-1}(x) \cap S^{\ell-2}(z)}} \sum_{\substack{a \sim a_1 \\ a \notin [xa_1]}} \sum_{\substack{b \sim b_1 \\ b \notin [xb_1]}} (v(a,b_1) + v(b,a_1)) f_y(a) f_z(b) = 0.$$

By Lemma 8.31, we can now find a $(k-1)$-pseudokernel M with $K_x = \sum_{y \sim x} M_{xy}$. We set

$$\begin{aligned} H_{xy} &= M_{xy} + M_{yx} - K^-_{xy} \quad x \sim y \in X \\ L_{xy} &= M_{xy} - H_{xy} \quad x \in X. \end{aligned}$$

Again, we let (H^+, H) denote the orthogonal extension of the $(k-1)$-dual kernel $(H, 0)$ and (J, J^-) the k-dual kernel associated to the $(k-1)$-pseudokernel L and we have, for any x in X,

$$K_x = \sum_{y \sim x} M_{xy} = \sum_{y \sim x} L_{xy} + \sum_{y \sim x} H_{xy} = J_x + H_x^+$$

and, for $x \sim y$,

$$K^-_{xy} = M_{xy} + M_{yx} - H_{xy} = L_{xy} + L_{yx} + H_{xy} = J^-_{xy} + H_{xy},$$

which should be proved. □

8.8. **The null space of the weight map.** We are now ready to conclude:

Theorem 8.32. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Then the following are equivalent.*
(i) *There exists a skew-symmetric function v on X_{k-1} such that the function $(x,y) \mapsto v(x,y_1) + v(y,x_1)$ on X_k is a weight function of (K, K^-) (where as usual, for (x,y) in X_k, x_1 and y_1 are the neighbours of x and y on $[xy]$).*
(ii) *There exists a $(k-1)$-pseudokernel L such that (K, K^-) is the k-dual kernel associated to L.*
In case (K, K^-) is Γ-invariant, the function v in (i) *may be chosen to be Γ-invariant.*

In other words, for Γ-invariant kernels, we have

Corollary 8.33. *For any $k \geq 2$, the null space of the weight map $\mathcal{K}_k \to \mathcal{W}_k$ is the space $\mathcal{L}_{k-1}$ of $(k-1)$-pseudokernels.*

We will need the elementary

Lemma 8.34. *Let $k \geq 2$ and φ be a function on X_k such that, for any (x, y) in X_{k+1}, one has $\varphi(x, y_1) + \varphi(y, x_1) = 0$. Then, there exists a skew-symmetric function ψ on X_{k-1} such that, for any (x, y) in X_{k-1}, one has $\varphi(x, y) = \psi(x_1, y)$.*

Proof. Fix (x, y) in X_{k-1}. Let z be a neighbour of y that is not on $[xy]$. If t is a neighbour of x that is not on $[xy]$, we have $\varphi(t, y) = -\varphi(z, x)$, hence this value does not depend on t. We define it as $\psi(x, y)$. By construction, ψ is skew-symmetric and we are done. $\square$

Proof of Theorem 8.32. (ii)⇒(i) is Proposition 8.24.

We prove *(i)⇒(ii)* by induction on $k \geq 2$.

First assume $k = 2$. Pick a 2-dual kernel (K, K^-) which satisfies the assumptions. In this case, a function u on X_1 is (K, K^-)-compatible if and only if, for any $x \sim y$ in X, one has

$$u(x, y) + u(y, x) = K^-_{xy}(x, y).$$

Now, by assumption, there exists such a function u as well as a skew-symmetric function v on X_1 such that, for any x in X any $y \neq z$ in $S^1(x)$, one has

$$\begin{aligned} v(y, x) + v(z, x) &= u(y, x) + u(z, x) + K_x(y, z) - K^-_{xy}(x, y) - K^-_{xz}(x, z) \\ &= -u(x, y) - u(x, z) + K_x(y, z). \end{aligned}$$

We define a 1-pseudokernel by setting, for any (x, y) in X_1, $L_{xy}(x, y) = v(y, x) + u(x, y)$. The relations above directly imply that (K, K^-) is the 2-dual kernel associated to L.

Assume now $k \geq 3$ and the result is true for $k - 1$. Again we chose a k-dual kernel (K, K^-) which satisfies the assumptions of the Theorem, that is, there exists a skew-symmetric function v on X_{k-1} such that the functon $(x, y) \mapsto v(x, y_1) + v(y, x_1)$ on X_k is a weight function of (K, K^-). In particular, this weight function is split, hence, by Proposition 8.28, there exists a $(k-1)$-pseudokernel L and a $(k-1)$-dual kernel (H, H^-) such that (K, K^-) is the sum of the k-dual kernel associated with L and of the orthogonal extension (H^+, H) of (H, H^-). To conclude, it suffices to prove that (H, H^-) is the $(k-1)$-dual kernel associated to some $(k-2)$-pseudokernel. We will get this by applying the induction hypothesis to (H, H^-). To this aim, we chose a weight function w on X_{k-1} for (H, H^-). By Proposition 8.24, Corollary 8.27 and the assumption, there exists a skew-symmetric function v' on X_{k-1} such that, for any (x, y) in X_k, one has

$$w(x, y_1) + w(y, x_1) = v'(x, y_1) + v'(y, x_1)$$

(recall that weight functions are symmetric). By Lemma 8.34, there exists a skew-symmetric function v'' on X_{k-2} such that, for any (x, y) in X_{k-1}, one has

$$w(x, y) = v'(x, y) + v''(x_1, y).$$

As w is symmetric and v' is skew-symmetric, we have

$$w(x, y) = \frac{1}{2}(v''(x_1, y) + v''(y_1, x)).$$

Now, the induction assumption tells us that (H, H^-) is the $(k-1)$-dual kernel associated to some $(k-2)$-pseudokernel. By Propositon 8.17, (H^+, H) is the k-dual kernel associated to some $(k-1)$-pseudokernel. Therefore, (K, K^-) also is of this form, which should be proved. □

9. Image dual kernels

For $k \geq 2$, we have introduced in Definition 7.14 the notion of the image dual kernel of a Γ-invariant function w on X_k such that the symmetric bilinear form Φ_w on H_0^ω is non-negative. We will simply say that a k-dual kernel (K, K^-) is an image dual kernel if one can find such a w with (K, K^-) being the image dual kernel of w. Note that, in view of Lemma A.4, this implies in particular that (K, K^-) is not only non-negative but exact in the sense of Definitions 5.12 and 5.13.

In the present Section, we will use the characterization of the null space of the weight map obtained above to give a geometric criterion for an exact dual kernel to be an image dual kernel.

9.1. **Non-negative pseudokernels.** We have the following natural

Definition 9.1. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$, and L be a k-pseudokernel. We say that L is non-negative if, for any $x \sim y$ in X, the symmetric bilinear form r_{xy}^L associated to L on $V_0^\ell(xy)$ is non-negative.

Definition 9.2. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$, and L be a k-pseudokernel. We say that L is non-negative if, for any $x \sim y$ in X, the symmetric bilinear form r_{xy}^L associated to L on $V_0^\ell(x)$ is non-negative.

Recall that we write $\mathcal{K}_k$, $k \geq 2$, for the space of Γ-invariant k-dual kernels and $\mathcal{L}_k$, $k \geq 1$, for the space of Γ-invariant k-pseudokernels. As above, we let $\mathcal{K}_k^+ \subset \mathcal{K}_k$ stand for the set of non-negative Γ-invariant k-dual kernels. In the same way, we let $\mathcal{L}_k^+ \subset \mathcal{L}_k$ stand for the set of non-negative Γ-invariant k-pseudokernels. The sets $\mathcal{K}_k^+$ and $\mathcal{L}_k^+$ are closed convex cones (see Proposition 5.14 in the former case). As in

Section 8, for $k \geq 2$, we identify $\mathcal{L}_{k-1}$ with a subspace of $\mathcal{K}_k$. Note that there is no obvious relation between $\mathcal{K}_k^+$ and $\mathcal{L}_{k-1}^+$. The main result of this Section is

Theorem 9.3. *Let $k \geq 2$ and (K, K^-) be in $\mathcal{K}_k^+$. Then $(K, K^-) + \mathcal{L}_{k-1}$ contains a unique image kernel (H, H^-) and the $(k-1)$-pseudokernel L with $(H, H^-) = (K, K^-) + L$ is non-negative. In particular, the following are equivalent:*
(i) *(K, K^-) is an image kernel.*
(ii) *we have $((K, K^-) + \mathcal{L}_{k-1}^+) \cap \mathcal{K}_k^+ = (K, K^-)$.*
(iii) *we have $((K, K^-) + \mathcal{L}_{k-1}) \cap \mathcal{K}_k^+ \subset (K, K^-) - \mathcal{L}_{k-1}^+$.*

Let us explain the underlying ideas in this result. Given (K, K^-) as above, we pick a weight function w for (K, K^-). Theorem 7.6 tells us that the pre-Hilbert space of distributions associated to (K, K^-) contains H_0^ω and that the restriction of q^{K,K^-} to H_0^ω is equal to Φ_w. In particular, Φ_w is non-negative. We let (H, H^-) be its image dual kernel. Theorem 7.17 tells us that w is a weight function for (H, H^-). Therefore, Theorem 8.32 tells us that (H, H^-) belongs to $(K, K^-) + \mathcal{L}_{k-1}$. Now, we would like to prove that (H, H^-) actually belongs to $(K, K^-) + \mathcal{L}_{k-1}^+$. This is the main difficulty of the proof. Indeed, from the construction, it is clear that, for any $j \geq k-1$, the dual prekernel $H^j - K^j$ is non-negative. But saying that the pseudokernel $(H - K, H^- - K^-)$ is non-negative (as a pseudokernel) is an a priori stronger property. Therefore, the proof of this result will require us to compare several notions of non-negativity.

9.2. **Weakly non-negative pseudokernels.** We introduce a new notion of non-negativity for pseudokernels which will play a central role in the proof of Theorem 9.3.

Let L be a k-pseudokernel. In Definitions 8.14 and 8.15, we have introduced the orthogonal extension of L. If L is non-negative, its orthogonal extension L^+ is a non-negative $(k+1)$-pseudokernel, so that all its successive orthogonal extensions L^j, $j \geq k$, are non-negative.

As in Definitions 5.10 and 5.11, we shall say that a dual prekernel is non-negative if the associated bilinear forms are non-negative. Let (K, K^-) be the $(k+1)$-dual kernel associated to L and let K^j, $j \geq k$, be the dual prekernels obtained from (K, K^-) by successive orthogonal extensions. From Definitions 8.9 and 8.10, it is clear that if L is non-negative, all the K^j, $j \geq k$, are non-negative dual prekernels.

We shall need a weaker notion of non-negativity for pseudokernels.

Definition 9.4. Let $k \geq 1$ and L be a k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). We say that L is weakly non-negative if the dual prekernels K^j, $j \geq k$, are non-negative.

The previous discussion directly gives

Lemma 9.5. *Let $k \geq 1$. Any non-negative k-pseudokernel is also weakly non-negative.*

The converse of this statement is not true. Nevertheless, for Γ-invariant pseudokernels, we have a criterion for being weakly non-negative which involves only finitely many kernels.

Proposition 9.6. *Let $k \geq 1$ and L be a Γ-invariant k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). Then L is weakly non-negative if and only if L^{2k-1} is non-negative and the dual prekernels K^j, $k \leq j \leq 2k-3$, are non-negative.*

This technical result is the main ingredient of the proof of Theorem 9.3. One of the directions of the equivalence is easier to prove and actually holds without assuming the kernel to be Γ-invariant. Indeed, it will follow from the following general formula.

Lemma 9.7. *Let $k \geq 2$ and L be a k-pseudokernel. For any x in X, we have*

$$K_x^{2k-2} = \frac{1}{d(x)-1} \sum_{y \sim x} L_{yx}^{2k-1}.$$

For any $j \geq 0$ and $x \sim y$ in X, we have

$$K_{xy}^{2(j+k)-1} = \sum_{z \in S^j(xy)} L_{z_- z}^{2k-1} \text{ and } K_x^{2(j+k)} = \sum_{z \in S^{j+1}(x)} L_{z_- z}^{2k-1}.$$

In this statement, for $j \geq 1$ and z in $S^j(xy)$ we have denoted by z_- the neighbour of z in $S^{j-1}(xy)$. For $j = 0$, we write $x_- = y$ and $y_- = x$.

Proof. As $k \geq 2$, we have $2k-2 \geq k$ and Proposition 8.17 says that K^{2k-2} is the $(2k-2)$-predual kernel associated to L^{2k-2}. By Definitions 8.10 and 8.15, we get

$$K_x^{2k-2} = \frac{1}{d(x)-1} \sum_{y \sim x} L_{xy}^{2k-2} = \frac{1}{d(x)-1} \sum_{y \sim x} L_{yx}^{2k-1}.$$

The other formulae follow from Proposition 8.17 and Corollary 8.21. $\square$

This gives a first direction in Proposition 9.6.

Corollary 9.8. *Let $k \geq 1$ and L be a k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). If L^{2k-1} is non-negative, then, for every $j \geq \max(k, 2k-2)$, the j-dual prekernel K^j is non-negative.*

Proof. If $k = 1$, there is nothing to prove since $2k - 1 = 1$. If $k \geq 2$, the result directly follows from Lemma 9.7. $\square$

9.3. **Negative edges.** To finish the proof of Proposition 9.6, we will show that for Γ-invariant pseudokernels, the converse to Corollary 9.8 holds. In this Subsection, we begin by showing that, if L is a weakly non-negative k-pseudokernel, for most of the edges (x, y), L^{2k-1}_{xy} must represent a non-negative symmetric bilinear form. This fact will rely on the following abstract

Lemma 9.9. *Let $d \geq 1$ be an integer and $V_1, \dots, V_d$ be real vector spaces. Set $V = V_1 \oplus \cdots \oplus V_d$. For $1 \leq i \leq d$, we let φ_i be a non-zero linear functional on V_i and q_i be a symmetric bilinear form on V_i. We set φ to be the linear functional $\varphi_1 + \cdots + \varphi_d$ and q to be the symmetric bilinear form $q_1 + \cdots + q_d$ on V. Assume that q is non-negative on the hyperplane $\ker\varphi$ of V. Then there exists at most one $1 \leq i \leq d$ such that q_i admits negative vectors. In that case the maximal negative subspaces of q_i have dimension one.*

Proof. Assume there exists $1 \leq i \neq j \leq d$ and v_i in V_i and v_j in V_j with $q_i(v_i, v_i) < 0$ as well as $q_j(v_j, v_j) < 0$. Then q is negative definite on the 2-plane $W = \mathbb{R}v_i \oplus \mathbb{R}v_j \subset V$. As $W \cap \ker\varphi$ is non zero, q can not be non-negative on $\ker\varphi$. Now, let i be such that q_i admits negative vectors. For the same reason as above, any negative subspace of V_i must have zero intersection with $\ker\varphi_i$, hence, it must be a line. $\square$

From Lemma 9.9, we will deduce a geometric property of a certain set of exceptional edges associated to a weakly negative pseudokernel. Let $N \subset X_1$ be a set of oriented edges of X. We will say that N meets the spheres at most once if, for any x in X and $h \geq 1$, we have

$$|\{z \in S^h(x) | (z_-, z) \in N\}| \leq 1.$$

Let $k \geq 1$ and L be a k-pseudokernel. We define the set N_L of negative edges of L as the set of those (x, y) in X_1 such that the symmetric bilinear form associated to L^{2k-1}_{xy} on $V_0^{k-1}(xy)$ has negative vectors.

Lemma 9.10. *Let $k \geq 1$ and L be a k-pseudokernel. Assume L is weakly non-negative. Then the set N_L of negative edges of L meets the spheres at most once.*

Proof. For $\ell \geq 0$ and $x \sim y$ in X, we define the set $S_+^\ell(xy)$ as

$$S_+^\ell(xy) = \{x\} \cup \{z \in S^\ell(y) | x \notin [yz]\}.$$

This is the boundary of a rooted subtree in X. We also define $W^\ell(xy)$ as the vector space of all real-valued functions on $S_+^\ell(xy)$ and as usual, we let $\overline{W}^\ell(xy)$ denote the quotient of $W^\ell(xy)$ by the space of constant functions and $W_0^\ell(xy)$ denote the space of those f in $W^\ell(xy)$ with $\sum_{z \in S_+^\ell(xy)} f(z) = 0$, which we identify in the usual way with the dual space of $\overline{W}^\ell(xy)$.

Let $k \geq 1$ and L be a k-pseudokernel. Pick $x \sim y$ in X. By Lemma 8.22, we may see L_{xy}^{2k-1} as a symmetric function on $S_+^{k-1}(xy) \times S_+^{k-1}(xy)$ which is zero on the diagonal. Thanks to Lemma 5.1, we associate to it a symmetric bilinear form r_{xy}^{+L} on the space $W_0^{k-1}(xy)$. Note that, saying that (x, y) belongs to N_L is the same as saying that r_{xy}^{+L} admits negative vectors.

Now, let x, y be in X with $x \neq y$ and let still y_- be the neighbour of y on $[xy]$. Set $h = d(x, y) \geq 1$ and pick $\ell \geq 0$. Then, we have a natural injective linear map $H_{xy}^\ell : W^\ell(y_-y) \to V^{h+\ell}(x)$ defined as follows. For z in $S^{h+\ell}(x)$ and f in $W^\ell(y_-y)$, we set

$$\begin{aligned} H_{xy}^\ell f(z) =& f(z) \quad \text{if } y \in [xz] \\ H_{xy}^\ell f(z) =& f(y_-) \quad \text{else.} \end{aligned}$$

Note that saying that y belongs to $[xz]$ amounts to saying that z belongs to $S_+^\ell(y_-y)$ or that y_- does not belong to $[yz]$. One still let H_{xy}^ℓ denote the induced map $\overline{W}^\ell(y_-y) \hookrightarrow \overline{V}^{h+\ell}(x)$.

Let still $h \geq 1$ and $\ell \geq 0$. For x in X, one easily checks that, as $S^{h+\ell}(x)$ may be written as the disjoint union

$$S^{h+\ell}(x) = \bigsqcup_{y \in S^h(x)} \{z \in S^\ell(y) | y_- \notin [yz]\},$$

the space $\overline{V}^{h+\ell}(x)$ is spanned by the spaces $H_{xy}^\ell \overline{W}^\ell(y_-y)$, $y \in S^h(x)$, and that the kernel of the surjective map

$$\bigoplus_{y \in S^h(x)} H_{xy}^\ell : \bigoplus_{y \in S^h(x)} \overline{W}^\ell(y_-y) \to \overline{V}^{h+\ell}(x)$$

is the line spanned by the vector $\bigoplus_{y \in S^h(x)} \mathbf{1}_{y_-}$. By duality, this tells us that the adjoint map

$$\bigoplus_{y \in S^h(x)} H_{xy}^{\ell,*} : V_0^{h+\ell}(x) \to \bigoplus_{y \in S^h(x)} W_0^\ell(y_-y)$$

is injective and that its range is the set of vectors $\bigoplus_{y\in S^h(x)} f_y$ in $\bigoplus_{y\in S^h(x)} W_0^\ell(y_-y)$ with $\sum_{y\in S^h(x)} f_y(y_-)=0$. Therefore, we are in the same situation as in Lemma 9.9. We will now introduce symmetric bilinear forms in order to precisely apply this Lemma.

Let $k\geq 1$ and L be a weakly nonnegative k-pseudokernel and still let K^j, $j\geq k$ denote the associated dual prekernels. By Lemma 8.22 and Lemma 9.7, for any $h\geq 1$ and x in X, we have

$$q_x^{2k+2h-2}=\sum_{y\in S^h(x)}(H_{xy}^{k-1,*})^\star r_{y_-y}^{+L},$$

where $q_x^{2k+2h-2}$ is the symmetric bilinear form associated to $K_x^{2k+2h-2}$ on $V_0^{k+h-1}(x)$. By assumption, this symmetric bilinear form is nonnegative. By Lemma 9.9, at most one of the symmetric bilinear forms $r_{y_-y}^{+L}$, $y\in S^h(x)$, admits negative vectors, which should be proved. □

9.4. **A mixing argument.** In this Subsection, we will strenghten Lemma 9.10 by showing that, if L is a weakly negative Γ-invariant pseudokernel, the set N_L of its negative edges must be empty. This will require us to study a certain linear operator acting on the space of Γ-invariant functions on edges.

Thus, let F_1 be the vector space of all Γ-invariant functions on the set of edges X_1 of X. As $\Gamma\backslash X$ is finite, F_1 has finite dimension. We define an endomorphism of F_1 by setting, for φ in F_1 and $x\sim y$ in X,

$$T\varphi(x,y)=\frac{1}{d(y)-1}\sum_{\substack{z\sim y\\ z\neq x}}\varphi(y,z).$$

Note that $T\mathbf{1}=\mathbf{1}$.

We will start by describing the adjoint operator of T. To this aim, we will need to define a convenient scalar product on F_1. In order to deal with the case where Γ stabilizes an edge of X, we will use the following elementary combinatorial result:

Lemma 9.11. *Let A be a set and G be a group acting on A such that, for any a in A, its stabilizer G_a in G is finite. Then, for any non-negative G-invariant function φ on A^2, we have*

$$\sum_{(a,b)\in G\backslash A^2}\frac{1}{|G_a\cap G_b|}\varphi(a,b)=\sum_{a\in G\backslash A}\frac{1}{|G_a|}\overline{\varphi}(a),$$

where, for any a in A, $\overline{\varphi}(a)=\sum_{b\in A}\varphi(a,b)$.

Recall that, with the notation of the Lemma, if φ is a G-invariant function on A, $\sum_{a\in G\backslash A}\varphi(a)$ means the sum of φ on a system of representatives in A of the elements of $G\backslash A$ (when this makes sense). See [2] for related volume formulae in quotients of trees by discrete subgroups.

For any x in X, we still denote by Γ_x the stabilizer of x in Γ, which is a finite subgroup of Γ. We define a scalar product on F_1 by setting, for any φ, ψ in F_1,

$$\begin{aligned}\langle\varphi,\psi\rangle &= \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi(x,y)\psi(x,y)\\ &= \sum_{x\in\Gamma\backslash X}\frac{1}{|\Gamma_x|}\sum_{y\sim x}\varphi(x,y)\psi(x,y),\end{aligned}$$

where the latter equality follows from Lemma 9.11.

Lemma 9.12. *The adjoint operator of T with respect to the scalar product on F_1 is the operator $T^\dagger$ such that, for any ψ in E and $x\sim y$ in X,*

$$T^\dagger\psi(x,y) = \frac{1}{d(x)-1}\sum_{\substack{z\sim x\\ z\neq y}}\psi(z,x).$$

Note that again $T^\dagger\mathbf{1}=\mathbf{1}$.

Proof. Let temporarily S stand for the operator defined in the statement of the Lemma. For any φ,ψ in F_1, we have, by Lemma 9.11,

$$\begin{aligned}\langle T\varphi,\psi\rangle &= \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\frac{1}{d(y)-1}\sum_{\substack{z\sim y\\ z\neq x}}\varphi(y,z)\psi(x,y)\\ &= \sum_{y\in\Gamma\backslash X}\frac{1}{|\Gamma_y|}\frac{1}{d(y)-1}\sum_{\substack{x,z\sim y\\ x\neq z}}\varphi(y,z)\psi(x,y).\end{aligned}$$

The same argument shows that the latter quantity also equals $\langle\varphi,S\psi\rangle$, which should be proved. $\square$

We are now ready to prove

Lemma 9.13. *Let $N\subset X_1$ be a set of oriented edges of X which meets the spheres at most once. If N is Γ-invariant, it is empty.*

The intuition of the Lemma is that the graph $\Gamma\backslash X$ is a discrete analogue of a compact negatively curved Riemannian manifold. The geodesic flow of such a manifold has mixing properties which imply equidistribution properties of large spheres. Thus, the images in $\Gamma\backslash X$

of large spheres in X should satisfy an equidistribution property. We now make this precise.

Proof. As every x in X has at least three neighbours, for any $n \geq 1$, we have, for $x \sim y$ in X,

$$T^n \mathbf{1}_N(x,y) \leq 2^{-n} |\{z \in S^n(y) | (z_-, z) \in N\}| \leq 2^{-n},$$

hence $T^n \mathbf{1}_N \xrightarrow[n\to\infty]{} 0$ in F_1. Now, as $T^\dagger \mathbf{1} = \mathbf{1}$ by Lemma 9.12, we get $\langle \mathbf{1}, \mathbf{1}_N \rangle = 0$, that is by definition,

$$\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \mathbf{1}_N(x,y) = 0$$

and $N = \emptyset$. □

As announced, we can now conclude the

Proof of Proposition 9.6. Let L be Γ-invariant k-pseudokernel with associated dual prekernels K^j, $j \geq k$.

Assume the dual prekernels K^j, $k \leq j \leq 2k-3$ are non-negative. If the $(2k-1)$-pseudokernel L^{2k-1} is non-negative, by Corollary 9.8, the dual prekernels K^j, $j \geq 2k-2$ are non-negative, hence L is weakly non-negative.

Conversely, assume L to be weakly non-negative and let N_L be its set of negative edges. By Lemma 9.10, N_L meets the spheres at most once. Now, as L is Γ-invariant, N_L is a Γ-invariant subset of X_1. Thus, by Lemma 9.13, N_L is empty, which should be proved. □

9.5. **A geometric criterion for image kernels.** We will now use Proposition 9.6 to prove Theorem 9.3. A key argument in the proof will be

Lemma 9.14. *Let $k \geq 2$ be even, L be a $(k-1)$-pseudokernel and (K, K^-) be a non-negative k-dual kernel. Assume that the k-pseudokernel L^+ is non-negative and that the k-dual kernel $(K, K^-) + L$ is exact. Then L is non-negative.*

Proof. Set $\ell = \frac{k}{2}$. As usual, for x in X denote by q_x^K the symmetric bilinear form associated with K_x on $V_0^\ell(x)$ and, for $x \sim y$ in X, denote by $q_{xy}^{K^-}$ and r_{xy}^L the symmetric bilinear forms associated to K_{xy}^- and L_{xy} on $V_0^{\ell-1}(xy)$. Note that the symmetric bilinear form $r_{xy}^{L^+}$ associated to L_{xy}^+ on $V_0^\ell(x)$ is defined by

$$r_{xy}^{L^+} = \sum_{\substack{z\sim x\\ z\neq y}} (I_{xz}^{\ell-1,*})^\star r_{xz}^L.$$

As $(K, K^-) + L$ is exact, we have

$$(I_{xy}^{\ell-1,*})_\star(q_x^K + \sum_{z\sim x}(I_{xz}^{\ell-1,*})^\star r_{xz}^L) = q_{xy}^{K^-} + r_{xy}^L + r_{yx}^L,$$

which gives, by Lemma A.6,

$$(I_{xy}^{\ell-1,*})_\star(q_x^K + r_{xy}^{L^+}) = q_{xy}^{K^-} + r_{yx}^L,$$

As L^+ is non-negative, by Lemma A.5, we get

$$(I_{xy}^{\ell-1,*})_\star q_x^K + (I_{xy}^{\ell-1,*})_\star r_{xy}^{L^+} \leq q_{xy}^{K^-} + r_{yx}^L,$$

hence

$$r_{yx}^L \geq (I_{xy}^{\ell-1,*})_\star q_x^K - q_{xy}^{K^-}.$$

As (K, K^-) is non-negative, we have $(I_{xy}^{\ell-1,*})_\star q_x^K \geq q_{xy}^{K^-}$ and therefore r_{yx}^L is non-negative, which should be proved. □

Note that, if k is even and if L is a k-pseudokernel, L is non-negative if and only if L^+ is. Therefore, by an easy induction argument which relies on Propositon 5.16 and Proposition 8.17, we get

Corollary 9.15. *Let $k \geq 2$, L be a $(k-1)$-pseudokernel and (K, K^-) be a non-negative k-dual kernel. Assume that the j-pseudokernel L^j is non-negative for some $j \geq k-1$ and that the k-dual kernel $(K, K^-)+L$ is exact. Then L is non-negative.*

Together with Proposition 9.6, this gives

Corollary 9.16. *Let $k \geq 2$, L be a weakly non-negative Γ-invariant $(k-1)$-pseudokernel and (K, K^-) be a non-negative Γ-invariant k-dual kernel. Assume that the k-dual kernel $(K, K^-) + L$ is exact. Then the $(k-1)$-pseudokernel L is non-negative.*

We are now ready to conclude the

Proof of Theorem 9.3. Let (K, K^-) be as in the setting a Γ-invariant non-negative k-dual kernel. As in Proposition 5.18, we let L^{K,K^-} be the space of distributions associated to (K, K^-) equipped with its natural non-negative symmetric bilinear form q^{K,K^-}. We chose a Γ-invariant weight function w for (K, K^-). By Theorem 7.6, we have $H_0^\omega \subset L^{K,K^-}$ and the restriction of q^{K,K^-} to H_0^ω is the bilinear form Φ_w from Section 3. We let (H, H^-) be the image k-dual kernel of Φ_w, as in Definition 7.14. By Theorem 7.17, w is a weight function of (H, H^-). Therefore, by Corollary 8.33, there exists L in $\mathcal{L}_{k-1}$ with $(H, H^-) = (K, K^-) + L$.

We claim that (H, H^-) is the unique image kernel in $(K, K^-) + \mathcal{L}_{k-1}$ indeed, let w' be a symmetric Γ-invariant function on X_k such that $\Phi_{w'}$ is non-negative on H_0^ω. If the image dual kernel of w' belongs to

$(H, H^-) + \mathcal{L}_{k-1}$, by Corollary 8.33 and Lemma 3.14, we have $\Phi_w = \Phi_{w'}$, hence the image dual kernel of $\Phi_{w'}$ is (H, H^-).

We will now show that the $(k-1)$-pseudokernel L is non-negative.

First, we show that it is weakly non-negative, as in Definition 9.4. As usual, for $j \geq k-1$, we set H^j and K^j, to be the j-dual prekernels associated to (H, H^-) and (K, K^-) by successive orthogonal extensions. We claim that $H^j - K^j$ is a non-negative dual prekernel. Indeed, assume that j is even, $j = 2\ell$, $\ell \geq 1$. Fix x in X and let, as is Subsection 5.4, N_x^ℓ be the natural linear operator $V^\ell(x) \hookrightarrow \mathcal{D}(\partial X)$. By definition, we have $q^{K,K^-} \geq (N_x^{\ell,*})^\star q_x^{K^j}$ on L^{K,K^-} and, by Lemma 7.15, $q_x^{H^j} = (N_x^{\ell,*})_\star \Phi_w$. Thus, we get $q_x^{H^j} \geq q_x^{K^j}$ as required. The proof is analoguous in the odd case.

Now, as (H, H^-) is exact (see Lemma A.4), and (K, K^-) is non-negative, by Corollary 9.16, L is a non-negative $(k-1)$-pseudokernel, that is, L belongs to the cone $\mathcal{L}_k^+$. This finishes the proof of the first part of the Proposition. The second part follows easily. $\square$

9.6. **The harmonic kernel.** As an example of the use of Theorem 9.3, we will now apply it to show that the harmonic kernel (χ, χ^-) from Subsection 5.5 is an image kernel. Recall that this 2-dual kernel is defined by

$$\chi_x(y, z) = 2\frac{d(x) - 1}{d(x)}, \quad x \in X, \quad y \neq z \in S^1(x),$$
$$\chi_{xy}(x, y) = 1, \quad x \sim y \in X.$$

By Proposition 5.21, the harmonic kernel is Euclidean.

Proposition 9.17. *The harmonic kernel is an image kernel.*

The proof will use the following elementary extension of Lemma 5.20, which follows from a straight forward computation using Lemma A.10.

Lemma 9.18. *Let A be a finite set with at least two elements, V_0 be the vector space of functions with zero sum on A and u be a positive function on A. We set q to be the symmetric bilinear form*

$$(f, g) \mapsto \sum_{a \in A} u(a) f(a) g(a)$$

on V_0. Then, for every a in A, if e_a is the evaluation linear functional $f \mapsto f(a)$ on V_0, one has

$$(e_a)_\star q = u(a) \left(1 - \frac{1}{u(a)S}\right)^{-1},$$

where $S = \sum_{b \in A} \frac{1}{u(b)}$.

Proof of Proposition 9.17. By Theorem 9.3, we must show that, if L is a Γ-invariant non-negative 1-pseudokernel such that the 2-dual kernel $(\chi, \chi^-) + L$ is non-negative, then $L = 0$. Now, for any $x \sim y$ in X, the space $V_0^0(xy)$ is a line spanned by the vector $\mathbf{1}_y - \mathbf{1}_x$ and the linear operator $I_{xy}^{0,*}$ sends a function f in $V_0^1(x)$ to $f(y)(\mathbf{1}_y - \mathbf{1}_x)$. Therefore, by Lemma 5.20, Definition 8.9 and Lemma 9.18, we must show that, if u is a Γ-invariant non-negative function on X_1 such that, for any $x \sim y$ in X, one has

$$\begin{aligned}(9.1)\quad u(x,y) &+ \frac{d(x)-1}{d(x)} \\ &\geq (1 + u(x,y) + u(y,x)) \left(1 - \frac{1}{\left(u(x,y) + \frac{d(x)-1}{d(x)}\right) S(x)} \right),\end{aligned}$$

where

$$S(x) = \sum_{z \sim x} \frac{1}{u(x,z) + \frac{d(x)-1}{d(x)}},$$

then necessarily $u = 0$.

Let us prove this claim. We let u and S be as above. From (9.1), we get, for (x, y) in X_1,

(9.2)

$$1 + u(x,y) + u(y,x) \geq \left(u(x,y) + \frac{d(x)-1}{d(x)} \right) \left(u(y,x) + \frac{1}{d(x)} \right) S(x).$$

By setting

$$S(x,y) = \sum_{\substack{z \sim x \\ z \neq y}} \frac{1}{u(x,z) + \frac{d(x)-1}{d(x)}},$$

we have

$$\left(u(x,y) + \frac{d(x)-1}{d(x)} \right) S(x) = 1 + \left(u(x,y) + \frac{d(x)-1}{d(x)} \right) S(x,y)$$

and (9.2) becomes

$$1 - \frac{1}{d(x)} + u(xy) \geq \left(u(x,y) + \frac{d(x)-1}{d(x)} \right) \left(u(y,x) + \frac{1}{d(x)} \right) S(x,y),$$

or, equivalently, as $u(x,y) + \frac{d(x)-1}{d(x)} > 0$,

$$(9.3)\qquad \frac{1}{S(x,y)} \geq u(y,x) + \frac{1}{d(x)}.$$

Set $m = \max_{(x,y)\in X_1} u(x,y)$. For $x \sim y$ in X_1, we have

$$S(x,y) \geq \frac{d(x)-1}{m + \frac{d(x)-1}{d(x)}},$$

hence, from (9.3),

$$m + \frac{1}{d(x)} \leq \frac{m + \frac{d(x)-1}{d(x)}}{d(x)-1} = \frac{m}{d(x)-1} + \frac{1}{d(x)}.$$

As $d(x) \geq 3$, we get $m = 0$, which should be proved. $\square$

We have just proved that the harmonic kernel is an image kernel or, equivalently by Corollary 7.19, that the space H_0^ω is dense in the Hilbert space of distributions H^{χ,χ^-} associated to (χ, χ^-). In Proposition 10.13 below, we will show that these two spaces are actually equal.

10. Admissible kernels

We have described the image kernels. These are the dual kernels which are the image kernels associated to a non-negative bilinear form Φ_w, where w is a symmetric Γ-invariant function on X_k. We will now focus on the case where Φ_w is coercive, that is where Φ_w defines on H_0^ω the same topology as the standard scalar product.

We will need to use again part of the language that was introduced in Section 4. Recall in particular that a k-Euclidean field is a k-quadratic field whose associated bilinear forms are positive definite (see Definition 4.18). To such a field, we have associated a k-dual kernel in Section 5, where such dual kernels are called Euclidean dual kernels (see Definition 5.12 and Definition 5.13). The data of a Euclidean field or of the associated Euclidean dual kernel are equivalent.

Definition 10.1. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field, with associated Euclidean k-dual kernel (K, K^-). We shall say that p and (K, K^-) are admissible if there exists a symmetric Γ-invariant function w on X_k such that Φ_w is coercive and (K, K^-) is the image dual kernel of w.

The purpose of this section is to give a criterion for a Γ-invariant Euclidean field to be admissible which only involves finite-dimensional spaces.

10.1. Convolution operators.

In this subsection, we relate the fact that a Euclidean field is admissible with the fact that a certain convolution operator is bounded in $\ell^2(X_1)$. This will require us tu use again the language of Section 4.

Recall that X_* stands for the space of pairs (x,y) in X^2 with $x \neq y$ and X_1 for the pairs (x,y) in X^2 with $x \sim y$. If φ is a function on X_*, we will associate to φ an operator P_φ acting on skew-symmetric functions on X_1 as follows. Given a finitely supported skew-symmetric function ψ on X_1, we set, for (x,y) in X_1,

$$P_\varphi\psi(x,y) = \sum_{\substack{(a,b)\in X_1 \\ y,b\in[xa]}} \varphi(x,a)\psi(b,a) - \sum_{\substack{(a,b)\in X_1 \\ x,b\in[ya]}} \varphi(y,a)\psi(b,a) - \frac{1}{2}(\varphi(x,y)+\varphi(y,x))\psi(x,y), \tag{10.1}$$

which by construction is a skew-symmetric function on X_1. Note that, if φ is symmetric, for any $x \neq y$ in X, if x_1 and y_1 are the neighbours of x and y on $[xy]$, we have

$$P_\varphi(\mathbf{1}_{yy_1} - \mathbf{1}_{y_1y})(x,x_1) = \varphi(x,y). \tag{10.2}$$

The operator P_φ was defined in order to warrant this latter property. Note also that if φ is Γ-invariant, the operator P_φ commutes with the action of Γ. In this case, we call P_φ the convolution operator of φ.

We let $\ell^2_-(X_1)$ denote the Hilbert space of skew-symmetric square-summable functions on X_1. By (standard) abuse of language, we shall say that P_φ is bounded on $\ell^2_-(X_1)$ if there exists a constant $C > 0$ such that, for every finitely supported skew-symmetric function ψ on X_1, one has $\|P_\varphi\psi\|_2 \leq C\,\|\psi\|_2$.

Now, let $k \geq 2$ and p be a k-Euclidean field. In Section 4 (see in particular Subsection 4.6), we have associated to p a symmetric function φ_p^∞. This function describes the scalar product obtained from p on $\overline{\mathcal{D}}(\partial X)$ by successive orthogonal extensions. Here comes a criterion for p to be admissible.

Proposition 10.2. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then p is admissible if and only if the convolution operator $P_{\varphi_p^\infty}$ is bounded in $\ell^2_-(X_1)$.*

Proof. First assume that p is admissible. Then, by definition, there exists a Γ-invariant symmetric function w on X_k such that the Euclidean dual kernel (K, K^-) associated to p (see Subsection 5.1) is the image kernel of Φ_w (see Definition 7.14). Then, let Θ be the self-adjoint operator of H_0^ω which represents Φ_w. As Φ_w is coercive, Θ is invertible. For θ in $\mathcal{D}(\partial X)$, let θ^* be the element of H_0^ω which represents the bounded linear functional $T \mapsto T(\theta)$ on H_0^ω . By Theorem 7.6 and Theorem 7.17, saying that (K, K^-) is the image kernel of Φ_w amounts to saying that, for any θ_1, θ_2 in $\mathcal{D}(\partial X)$, one has

$$p^\infty(\theta_1, \theta_2) = \langle \Theta^{-1}\theta_1^*, \theta_2^* \rangle.$$

Now let $\mathcal{P}$ be the linear map defined in Subsection 3.1: by Lemma 3.5, $\mathcal{P}$ is an isomorphism from H_0^ω onto a closed subspace of $\ell^2_-(X_1)$. Let Π be the orthogonal projection from $\ell^2_-(X_1)$ onto this subspace. By construction of the map $\mathcal{P}$, for any $x \sim y$ in X, for any T in H_0^ω, we have

$$T(\mathbf{1}_{U_{xy}}) = \mathcal{P}T(x,y) = \langle \mathcal{P}T, \mathbf{1}_{(x,y)} \rangle = \frac{1}{2} \langle \mathcal{P}T, \mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)} \rangle, \tag{10.3}$$

hence $\mathcal{P}(\mathbf{1}^*_{U_{xy}}) = \Pi(\frac{1}{2}(\mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}))$. Let Υ be the bounded operator of $\ell^2_-(X_1)$ such that, for ψ in $\ell^2_-(X_1)$, we have $\Upsilon\psi = \mathcal{P}(\Theta^{-1}T)$ where T is the distribution T in H_0^ω with $\mathcal{P}(T) = \Pi\psi$. By construction, for any $a \sim b$ and $x \sim y$ in X with $b, y \in [ax]$, we have

$$\begin{aligned} \langle \Upsilon(\mathbf{1}_{(a,b)} - \mathbf{1}_{(b,a)}), (\mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}) \rangle &= 4p^\infty(\mathbf{1}_{U_{ab}}, \mathbf{1}_{U_{xy}}) \\ &= -4\varphi_p^\infty(a,x) = -2\langle P_{\varphi_p^\infty}(\mathbf{1}_{(a,b)} - \mathbf{1}_{(b,a)}), (\mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}) \rangle, \end{aligned}$$

where we have used (10.2). By linearity, we get $2P_{\varphi_p^\infty}\psi = -\Upsilon\psi$ for any skew-symmetric finitely supported function ψ on X_1 and hence $P_{\varphi_p^\infty}$ is bounded.

Let us keep the same notation and prove the converse statement. We now assume that $P_{\varphi_p^\infty}$ is a bounded endomorphism of $\ell^2_-(X_1)$. Note in particular that (10.2) implies that $P_{\varphi_p^\infty}$ is self-adjoint. Besides, by the description of the space $\mathcal{P}H_0^\omega$ in Lemma 3.5 and still by (10.2), a direct computation shows that the range of $P_{\varphi_p^\infty}$ is contained in $\mathcal{P}H_0^\omega$. Therefore, there exists a bounded self-adjoint operator Ξ of H_0^ω such that, for any ψ in $\ell^2_-(X_1)$, one has $P_{\varphi_p^\infty}\psi = \mathcal{P}(\Xi T)$ where T is the distribution T in H_0^ω with $\mathcal{P}(T) = \Pi\psi$. Pick θ in $\mathcal{D}(\partial X)$. By (10.3) above, as the $\mathbf{1}_{U_{xy}}$, $x \sim y \in X$, span $\mathcal{D}(\partial X)$ as a vector space, we can find a finitely supported skew-symmetric function ψ on X_1 with $\Pi\psi = \mathcal{P}\theta^*$. Now, (10.2) gives

$$p^\infty(\theta, \theta) = -\langle P_{\varphi_p^\infty}\psi, \psi \rangle = -\langle \Xi\theta^*, \theta^* \rangle.$$

In particular, there exists $C > 0$ such that

$$p^\infty(\theta, \theta) \leq C \left\| \theta^* \right\|^2 \tag{10.4}$$

for any θ in $\mathcal{D}(\partial X)$.

Let (K, K^-) be the Euclidean k-dual kernel associated to p as in Subsection 5.1 and H^{K,K^-} be the Hilbert space of distributions associated to (K, K^-) as in Subsection 5.4. We claim that the latter inequality implies the inclusion $H^{K,K^-} \subset H_0^\omega$ as spaces of distributions. Indeed, by Corollary 5.19, the space H^{K,K^-} is exactly the topological dual space of the space $\overline{\mathcal{D}}(\partial X)$, equipped with the scalar product p^∞. Hence, if T is a distribution in H^{K,K^-}, we can find $C' > 0$ with $T(\theta)^2 \leq C' p^\infty(\theta, \theta)$, $\theta \in \mathcal{D}(\partial X)$. From (10.4), we get $T(\theta)^2 \leq CC' \|\theta^*\|^2$, hence, for any skew-symmetric finitely supported function ψ on X_1, $\langle \mathcal{P}T, \psi \rangle^2 \leq CC' \|\Pi\psi\|^2 \leq CC' \|\psi\|^2$. Therefore, $\mathcal{P}T$ belongs to $\ell^2(X_1)$, that is, T belongs to H_0^ω as claimed.

By Theorem 7.6, we know that we have $H_0^\omega \subset H^{K,K^-}$ and that the inclusion map is bounded. We just proved that this inclusion map is surjective, so that by the open mapping theorem it is an isomophism of Banach spaces. Therefore, still by Theorem 7.6, if w is a weight function of (K, K^-), the bilinear form Φ_w is coercive. Finally, we note that, by Lemma B.7, as the dual kernel (K, K^-) is exact, the bilinear forms associated to (K, K^-) are the images of the scalar product of H^{K,K^-} by the natural surjective maps (see Definition 5.12 and Definition 5.13 for the notion of an exact kernel). Now we just proved that H^{K,K^-} was equal to H_0^ω and that the scalar product was a coercive bilinear form Φ_w, so that by definition, the Euclidean field is admissible. □

10.2. **Quadratic pseudofields.** We will now look for a condition to ensure that the convolution operator associated to a quadratic type function obtained by successive orthogonal extensions is bounded. This condition will use a recursive formula for such quadratic type functions. To state this formula, we will need to use a new vector space which can be seen as a concrete version of the dual space of the space $\mathcal{L}_k$ of Γ-invariant k-pseudokernels.

Recall that, for any $\ell \geq 0$ and any x in X (resp. any $x \sim y$ in X), the space $\overline{V}^\ell(x)$ (resp. $\overline{V}^\ell(xy)$) is the quotient space of the space of functions on $S^\ell(x)$ (resp. $S^\ell(xy)$) by the line of constant functions.

Fix $k \geq 1$. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, a k-quadratic pseudofield is a family $(s_{xy})_{(x,y) \in X_1}$ such that, for any (x, y) in X_1, s_{xy} is a symmetric bilinear form on $\overline{V}^\ell(xy)$. If k is even, $k = 2\ell$, $\ell \geq 1$, a k-quadratic pseudofield is a family $(s_{xy})_{(x,y) \in X_1}$ such that, for any (x, y) in X_1, s_{xy} is a symmetric bilinear form on $\overline{V}^\ell(x)$. The space of all Γ-invariant k-quadratic pseudofields is denoted by $\mathcal{M}_k$. Let us identify $\mathcal{M}_k$ with the dual space of $\mathcal{L}_k$.

Let $s = (s_{xy})_{(x,y) \in X_1}$ be in $\mathcal{M}_k$ and L be in $\mathcal{L}_k$, that is s is a Γ-invariant k-quadratic pseudofield and L is a Γ-invariant k-pseudokernel. For any (x, y) in X_1, L defines a symmetric bilinear form r^L_{xy} on the dual space of the space where s_{xy} is defined. By making use of the quadratic duality from Appendix C, we get a well-defined real number $\langle r^L_{xy}, s_{xy}\rangle$ which comes from the duality between these spaces. Now, to define a duality between $\mathcal{L}_k$ and $\mathcal{M}_k$, we need to average these numbers over $\Gamma\backslash X_1$. As in Subsection 9.4, we just have to be careful to deal with the case where Γ fixes some edges in X.

Recall that, for any x in X, we denote by Γ_x the stabilizer of x in Γ, which is a finite subgroup of Γ. If s is in $\mathcal{M}_k$ and L is in $\mathcal{L}_k$, we set

$$\langle L, s\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle r^L_{xy}, s_{xy}\rangle. \tag{10.5}$$

By Lemma 9.11, we can also write

$$\langle L, s\rangle = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \sum_{y\sim x} \langle r^L_{xy}, s_{xy}\rangle. \tag{10.6}$$

From now on, we shall use this duality to identify $\mathcal{M}_k$ with the dual space of $\mathcal{L}_k$.

As an example of the use of Formulae 10.5 and 10.6, we will compute the adjoint operator of the orthogonal extension of Γ-invariant pseudokernels which is a linear map $\mathcal{L}_k \to \mathcal{L}_{k+1}$.

Let $s = (s_{xy})_{(x,y)\in X_1}$ be a $(k+1)$-quadratic pseudofield. We define the reduction s^- of s which will be a k-quadratic pseudofield. If k is odd, $k = 2\ell+1$, $\ell \geq 0$, we let s^- be the k-quadratic pseudofield defined by

$$s^-_{xy} = (I^\ell_{xy})^\star \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}, \quad (x,y) \in X_1.$$

If k is even, $k = 2\ell$, $\ell \geq 1$, we let s^- be the k-quadratic pseudofield defined by

$$s^-_{xy} = (J^\ell_{xy})^\star s_{yx}, \quad (x,y) \in X_1.$$

As announced, we get

Lemma 10.3. *The reduction operator $s \mapsto s^-, \mathcal{M}_{k+1} \to \mathcal{M}_k$ is the adjoint operator of the orthogonal extension operator $L \mapsto L^+, \mathcal{L}_k \to \mathcal{L}_{k+1}$.*

The proof is closely related to the one of Lemma 9.12.

Proof. Let s be a Γ-invariant $(k+1)$-quadratic pseudofield and L be a Γ-invariant k-pseudokernel, with associated bilinear forms $(r_{xy})_{(x,y)\in X_1}$.

First assume k is odd, $k = 2\ell+1$, $\ell \geq 0$. By the duality formula (10.6), $\langle L^+, s\rangle$ is the sum over $\Gamma\backslash X$ of the Γ-invariant function on X

$$x \mapsto \frac{1}{|\Gamma_x|} \sum_{y\sim x} \langle r^+_{xy}, s_{xy}\rangle.$$

By definition, we have, for any (x,y) in X_1, $r^+_{xy}=\sum_{\substack{z\sim x\\ z\neq y}}(I^{\ell,*}_{xz})^\star r_{xz}$, Thus, for x in X, we have

$$\sum_{y\sim x}\langle r^+_{xy},s_{xy}\rangle=\sum_{\substack{y,z\sim x\\ y\neq z}}\langle (I^{\ell,*}_{xz})^\star r_{xz},s_{xy}\rangle$$
$$=\sum_{\substack{y,z\sim x\\ y\neq z}}\langle r_{xz},(I^{\ell}_{xz})^\star s_{xy}\rangle=\sum_{z\sim x}\langle r_{xz},s^-_{xz}\rangle,$$

where the second equality comes from Lemma C.2. Therefore, again by the duality formula (10.6), $\langle L^+,s\rangle=\langle L,s^-\rangle$, which should be proved.

Assume now k is even, $k=2\ell$, $\ell\geq 1$, so that we now have, for (x,y) in X_1, $r^+_{xy}=(J^{\ell,*}_{yx})^\star r_{xy}$. By the duality formula (10.5) and again by Lemma C.2, $\langle L^+,s\rangle$ is the sum over $\Gamma\backslash X_1$ of the Γ-invariant function on X_1

$$(x,y)\mapsto\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle (J^{\ell,*}_{yx})^\star r_{xy},s_{xy}\rangle=\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle r_{yx},(J^{\ell}_{yx})^\star s_{xy}\rangle,$$

which is also equal to $\langle L,s^-\rangle$. □

10.3. **Quadratic transfer operators.** Recall that our aim is to give a recursive formula for quadratic type functions obtained by successive orthogonal extensions. This formula will involve the powers of a linear operator acting on Γ-invariant quadratic pseudofields that we will now define. We call these operators the quadratic transfer operators as they are analoguous to the transfer operators of hyperbolic dynamics studied for example in [30].

Let $k\geq 2$ and p be a k-Euclidean quadratic field (which we do not assume to be Γ-invariant for the moment).

Let x,y be in X with $x\sim y$. With p (and its orthogonal extensions) come Euclidean structures on the spaces $\overline{V}^\ell(x)$ and $\overline{V}^\ell(xy)$, for any $\ell\geq 0$. In particular, the injective linear operators $I^\ell_{xy}:\overline{V}^\ell(xy)\to\overline{V}^{\ell+1}(x)$ and $J^\ell_{xy}:\overline{V}^\ell(x)\to\overline{V}^\ell(xy)$ admit adjoint operators with respect to these Euclidean structures. We denote these adjoint operators as

$$I^{\ell,\dagger p}_{xy}:\overline{V}^{\ell+1}(x)\to\overline{V}^\ell(xy)$$
$$\text{and } J^{\ell,\dagger p}_{xy}:\overline{V}^{\ell}(xy)\to\overline{V}^\ell(x).$$

These are surjective operators which heavily depend on p.

Let us now define the quadratic transfer operator T_p. Again, we need to split the definition according to the parity of k.

Definition 10.4. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$ and p be a k-Euclidean field. If $s = (s_{xy})_{(x,y)\in X_1}$ is a $(k-1)$-quadratic pseudofield, we set, for any (x,y) in X_1,

$$(T_p s)_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} (I^{\ell-1,\dagger p}_{xz} I^{\ell-1}_{xy})^\star s_{zx}.$$

Definition 10.5. (k odd) Let $k \geq 3$ be an odd integer, $k = 2\ell+1$, $\ell \geq 1$ and p be a k-Euclidean field. If $s = (s_{xy})_{(x,y)\in X_1}$ is a $(k-1)$-quadratic pseudofield, we set, for any (x,y) in X_1,

$$(T_p s)_{xy} = (J^{\ell,\dagger p}_{yx} J^{\ell}_{xy})^\star \sum_{\substack{z\sim y\\ z\neq x}} s_{yz}.$$

We will show later that, when p is Γ-invariant, it is is admissible if and only if the spectral radius of T_p on $\mathcal{M}_{k-1}$ is < 1. As a first step towards this result, let us study the behaviour of T_p under orthogonal extensions.

Lemma 10.6. *Let $k \geq 2$, p be a k-Euclidean field with orthogonal extension p^+ and s be a k-quadratic pseudofield. Then, if k is even, $k = 2\ell$, for any (x,y) in X_1, we have*

$$(T_{p^+} s)_{xy} = (I^{\ell-1,\dagger p}_{xy})^\star s^-_{yx}.$$

If k is odd, $k = 2\ell+1$, for any (x,y) in X_1, we have

$$(T_{p^+} s)_{xy} = (J^{\ell,\dagger p}_{xy})^\star \sum_{\substack{z\sim x\\ z\neq y}} s^-_{xz}.$$

In both cases, this gives in particular $(T_{p^+} s)^- = T_p(s^-)$.

Corollary 10.7. *Assume p to be Γ-invariant. Then the spectrum of T_{p^+} in $\mathcal{M}_k$ is the union of $\{0\}$ and the spectrum of T_p in $\mathcal{M}_{k-1}$.*

Proof. By Remark 8.16, the orthogonal extension operator is injective on pseudokernels. Therefore, by Lemma 10.3, the reduction map $\varpi_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is surjective. Now, Lemma 10.6 implies that T_p^+ is 0 on the null space of ϖ_k and that the endomorphism induced by T_{p^+} on $\mathcal{M}_k / \ker \varpi_k \simeq \mathcal{M}_{k-1}$ is conjugated to T_p. The result follows. $\square$

The proof of Lemma 10.6 uses

Lemma 10.8. *Let $k \geq 2$ and p be a k-Euclidean field. For any $\ell \geq \frac{k}{2}$, for any $x \sim y$ in X, we have*

$$J^{\ell,\dagger p}_{yx} J^{\ell}_{xy} = I^{\ell-1}_{yx} I^{\ell-1,\dagger p}_{xy}.$$

For any $\ell \geq \frac{k-1}{2}$, for any x in X and any y, z in $S^1(x)$ with $y \neq z$, we have

$$I_{xz}^{\ell,\dagger p} I_{xy}^{\ell} = J_{xz}^{\ell} J_{xy}^{\ell,\dagger p}.$$

Proof. In the first case, this is a direct consequence of the fact that, under the assumptions, the scalar product of the space $\overline{V}^{\ell}(xy)$ is obtained from the scalar products on the subspaces $J_{xy}^{\ell}\overline{V}^{\ell}(x)$ and $J_{yx}^{\ell}\overline{V}^{\ell}(y)$ by orthogonal extension.

In the same way, in the second case, this follows from the fact that the scalar product of the space $\overline{V}^{\ell+1}(x)$ is obtained from the scalar products on the subspaces $I_{xy}^{\ell}\overline{V}^{\ell}(xy)$, $y \sim x$, through orthogonal extension. □

Proof of Lemma 10.6. Assume k is even, $k = 2\ell$. For $x \sim y$ in X, by Lemma 10.8, we have $J_{yx}^{\ell,\dagger p} J_{xy}^{\ell} = I_{yx}^{\ell-1} I_{xy}^{\ell-1,\dagger p}$. Plugging this relation in the definition of T_{p^+}, we get

$$\begin{aligned}(T_{p^+}s)_{xy} = (I_{yx}^{\ell-1} I_{xy}^{\ell-1,\dagger p})^{\star} \sum_{\substack{z\sim y\\ z\neq x}} s_{yz} = (I_{xy}^{\ell-1,\dagger p})^{\star} (I_{yx}^{\ell-1})^{\star} \sum_{\substack{z\sim y\\ z\neq x}} s_{yz} \\ = (I_{xy}^{\ell-1,\dagger p})^{\star} s_{yx}^{-}.\end{aligned}$$

This gives

$$(T_{p^+}s)_{xy}^{-} = (I_{xy}^{\ell-1})^{\star} \sum_{\substack{z\sim x\\ z\neq y}} (T_{p^+}s)_{xz} = (I_{xy}^{\ell-1})^{\star} \sum_{\substack{z\sim x\\ z\neq y}} (I_{xz}^{\ell-1,\dagger p})^{\star} s_{zx}^{-} = (T_p s^{-})_{xy}.$$

Now, assume k is odd, $k = 2\ell + 1$. For x in X, again by Lemma 10.8, we have $I_{xz}^{\ell,\dagger p} I_{xy}^{\ell} = J_{xz}^{\ell} J_{xy}^{\ell,\dagger p}$. Now, the definition of T_{p^+} gives

$$(T_{p^+}s)_{xy} = (J_{xy}^{\ell,\dagger p})^{\star} \sum_{\substack{z\sim x\\ z\neq y}} (J_{xz}^{\ell})^{\star} s_{zx} = (J_{xy}^{\ell,\dagger p})^{\star} \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}^{-}.$$

Thus, we get

$$(T_{p^+}s)_{xy}^{-} = (J_{xy}^{\ell})^{\star} (T_{p^+}s)_{yx} = (J_{xy}^{\ell})^{\star} (J_{yx}^{\ell,\dagger p})^{\star} \sum_{\substack{z\sim y\\ z\neq x}} s_{yz}^{-} = (T_p s^{-})_{xy}.$$

□

10.4. **Computing quadratic type functions.** We will now give a formula for the quadratic type function associated to a Euclidean field. Thanks to this formula, we will be able to relate the question whether the associated convolution operator is bounded to the domination of the spectral radius of the quadratic transfer operator.

Proposition 10.9. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Assume that the associated quadratic transfer operator has spectral radius < 1 on the finite-dimensional vector space $\mathcal{M}_{k-1}$ of Γ-invariant $(k-1)$-quadratic pseudofields. Then p is admissible.*

As we already said, the converse is also true, but we will prove it only later.

We now give our formula for the quadratic type function. To state it, we need to introduce a new notation in order to avoid some possible confusions. For $x \neq z$ in X and $\ell = d(x,z)$, we let $\mathbf{1}_z^x$ denote the characteristic function of $\{z\}$, viewed as an element of the space $\overline{V}^\ell(x)$. In the same way, for $x \sim y$ and z in X, if $\ell = \min(d(x,z), d(x,y))$, we let $\mathbf{1}_z^{xy}$ denote the characteristic function of $\{z\}$, viewed as an element of the space $\overline{V}^\ell(xy)$.

Lemma 10.10. *Let $k \geq 2$ and p be a k-Euclidean field. Let a, b be in X with $j = d(a,b) \geq k$ and let $c_0 = a, c_1, \ldots, c_j = b$ be the geodesic path from a to b.*

If k is even, $k = 2\ell$, $\ell \geq 1$, we have

(10.7)
$$\varphi_p^\infty(a,b) = -p_{c_\ell}(\mathbf{1}_a^{c_\ell}, I^{\ell-1}_{c_\ell c_{\ell+1}} I^{\ell-1,\dagger p}_{c_{\ell+1} c_\ell} I^{\ell-1}_{c_{\ell+1} c_{\ell+2}} \cdots I^{\ell-1}_{c_{j-\ell-1} c_{j-\ell}} I^{\ell-1,\dagger p}_{c_{j-\ell} c_{j-\ell-1}} \mathbf{1}_b^{c_{j-\ell}}).$$

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, we have

$$\begin{aligned}(10.8)\quad \varphi_p^\infty(a,b) = -p_{c_\ell c_{\ell+1}}(&\mathbf{1}_a^{c_\ell c_{\ell+1}}, \\ &J^\ell_{c_{\ell+1} c_\ell} J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+2} c_{\ell+1}} \cdots J^\ell_{c_{j-\ell} c_{j-\ell-1}} J^{\ell,\dagger p}_{c_{j-\ell-1} c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1} c_{j-\ell}}).\end{aligned}$$

Given p a k-Euclidean field, for a, b in X with $d(a,b) = k$, we directly know how to compute the scalar product between $\mathbf{1}_a$ and $\mathbf{1}_b$ with respect to p. The formulas in the lemma say that, when the distance is greater than k, in order to compute the scalar product between $\mathbf{1}_a$ and $\mathbf{1}_b$ with respect to large orthogonal extensions of the Euclidean field p, we can use the operators of Subsection 4.2 and their Euclidean adjoint operators with respect to p to let the vector $\mathbf{1}_b$ travel along the segment $[ab]$ until it is close enough to $\mathbf{1}_a$.

Proof. We fix j and we prove this result by descending induction on k with $2 \leq k \leq j$. For $k = j$, the result is the very definition of $\varphi_p^\infty(a,b) = \varphi_p(a,b)$. Now, assume $k \leq j-1$ and the result is true for $k+1$. By definition, we have $\varphi_p^\infty(a,b) = \varphi_{p^+}^\infty(a,b)$, so that we can apply the induction assumption to compute this number.

If k is even, $k = 2\ell$, the induction assumption tells us that $-\varphi_p^\infty(a,b)$ is the $p^+_{c_\ell c_{\ell+1}}$-scalar product of the vectors $\mathbf{1}_a^{c_\ell c_{\ell+1}}$ and

$$J^\ell_{c_{\ell+1}c_\ell} J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+2}c_{\ell+1}} \cdots J^\ell_{c_{j-\ell}c_{j-\ell-1}} J^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}}$$

in the space $\overline{V}^\ell(c_\ell c_{\ell+1})$. Now, we have

$$\mathbf{1}_a^{c_\ell c_{\ell+1}} = J^\ell_{c_\ell c_{\ell+1}} \mathbf{1}_a^{c_\ell} \text{ and } \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}} = J^\ell_{c_{j-\ell}c_{j-\ell-1}} \mathbf{1}_b^{c_{j-\ell}}.$$

Therefore, by the definition of the adjoint operators, $-\varphi_p^\infty(a,b)$ is the p_{c_ℓ}-scalar product of the vectors $\mathbf{1}_a^{c_\ell}$ and

$$J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+1}c_\ell} J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+2}c_{\ell+1}} \cdots J^\ell_{c_{j-\ell}c_{j-\ell-1}} J^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell}} J^\ell_{c_{j-\ell}c_{j-\ell-1}} \mathbf{1}_b^{c_{j-\ell}}$$

in the space $\overline{V}^\ell(c_\ell)$. By Lemma 10.8, for $\ell \leq h \leq j-\ell-1$, we have

$$J^{\ell,\dagger p}_{c_h c_{h+1}} J^\ell_{c_{h+1}c_h} = I^{\ell-1}_{c_h c_{h+1}} I^{\ell-1,\dagger p}_{c_{h+1}c_h},$$

and (10.7) follows.

In the same way, if k is odd, $k = 2\ell+1$, the induction assumption tells us that $-\varphi_p^\infty(a,b)$ is the $p^+_{c_\ell}$-scalar product of the vectors $\mathbf{1}_a^{c_{\ell+1}}$ and

$$I^\ell_{c_{\ell+1}c_{\ell+2}} I^{\ell,\dagger p}_{c_{\ell+2}c_{\ell+1}} I^\ell_{c_{\ell+2}c_{\ell+3}} \cdots I^\ell_{c_{j-\ell-2}c_{j-\ell-1}} I^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell-2}} \mathbf{1}_b^{c_{j-\ell-1}}$$

in the space $\overline{V}^{\ell+1}(c_{\ell+1})$. As we have

$$\mathbf{1}_a^{c_{\ell+1}} = I^\ell_{c_{\ell+1}c_\ell} \mathbf{1}_a^{c_\ell c_{\ell+1}} \text{ and } \mathbf{1}_b^{c_{j-\ell-1}} = I^\ell_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}},$$

by the definition of the adjoint operators, $-\varphi_p^\infty(a,b)$ is the $p_{c_\ell c_{\ell+1}}$-scalar product of the vectors $\mathbf{1}_a^{c_\ell c_{\ell+1}}$ and

$$I^{\ell,\dagger p}_{c_{\ell+1}c_\ell} I^\ell_{c_{\ell+1}c_{\ell+2}} I^{\ell,\dagger p}_{c_{\ell+2}c_{\ell+1}} I^\ell_{c_{\ell+2}c_{\ell+3}} \cdots I^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell-2}} I^\ell_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}}$$

in the space $\overline{V}^\ell(c_\ell c_{\ell+1})$. By Lemma 10.8, for $\ell \leq h \leq j-\ell-2$, we have

$$I^{\ell,\dagger p}_{c_{h+1}c_h} I^\ell_{c_{h+1}c_{h+2}} = J^\ell_{c_{h+1}c_h} J^{\ell,\dagger p}_{c_{h+1}c_{h+2}}$$

and (10.8) follows. □

The main idea of the proof of Proposition 10.9 is to use quadratic transfer operators in order to give a simpler form of (10.7) and (10.8).

Proof of Proposition 10.9. Assume k is even, $k = 2\ell$, $\ell \geq 1$ and let us define a Γ-invariant $(k-1)$-quadratic pseudofield s. For any $x \sim y$ in X, we set s_{xy} to be the symmetric bilinear form on $\overline{V}^{\ell-1}(xy)$ such that, for f in $\overline{V}^{\ell-1}(xy)$, one has

$$s_{xy}(f,f) = \sum_{\substack{a \in S^\ell(x) \\ y \notin [xa]}} p_y(I^{\ell-1}_{xy} f, \mathbf{1}_a)^2 = \sum_{\substack{a \in S^\ell(x) \\ y \notin [xa]}} p^-_{xy}(f, I^{\ell-1,\dagger p}_{xy} \mathbf{1}_a)^2.$$

By using Definition 10.4, where the quadratic transfer operator is constructed, and Lemma 10.10, we get, for any b in $S^\ell(y)$ with $x \notin [yb]$, for any $n \geq 0$,

$$(T_p^n s)_{xy}(I_{yx}^{\ell-1,\dagger p}\mathbf{1}_b, I_{yx}^{\ell-1,\dagger p}\mathbf{1}_b) = \sum_{\substack{a \in S^{k+n+1}(b) \\ [xy]\subset[ab]}} \varphi_p^\infty(a,b)^2.$$

Therefore, if T_p has spectral radius < 1, we can find $\rho < 1$ such that

$$\sum_{(a,b)\in\Gamma\backslash X_*} \rho^{-d(a,b)}\varphi_p^\infty(a,b)^2 < \infty$$

(recall that $\Gamma\backslash X$ is finite). By Haagerup inequality (see Proposition D.3), the convolution operator $P_{\varphi_p^\infty}$ is bounded in $\ell^2_-(X_1)$. By Proposition 10.2, the Euclidean field p is admissible.

Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$. Now, we define a Γ-invariant $(k-1)$-quadratic pseudofield s by setting, for every $x \sim y$ in X and every f in $\overline{V}^\ell(x)$,

$$s_{xy}(f,f) = \sum_{\substack{b\in S^\ell(y)\\ x\notin[yb]}} p_{xy}(J_{xy}^\ell f, \mathbf{1}_b)^2 = \sum_{\substack{b\in S^\ell(y)\\ x\notin[yb]}} p_x^-(f, J_{xy}^{\ell,\dagger p}\mathbf{1}_b)^2.$$

Now, using Definition 10.4 and Lemma 10.10 yields, for any a in $S^{\ell+1}(x)$ with $y \notin [xa]$, for any $n \geq 0$, if z is the neighbour of x on $[ax]$,

$$(T_p^n s)_{xy}(J_{xz}^{\ell,\dagger p}\mathbf{1}_a, J_{xz}^{\ell,\dagger p}\mathbf{1}_a) = \sum_{\substack{b \in S^{k+n+1}(a) \\ [xy]\subset[ab]}} \varphi_p^\infty(a,b)^2.$$

As above, if T_p has spectral radius < 1, we can find $\rho < 1$ such that

$$\sum_{(a,b)\in\Gamma\backslash X_*} \rho^{-d(a,b)}\varphi_p^\infty(a,b)^2 < \infty$$

and Proposition D.3 and Proposition 10.2 give the conclusion. □

10.5. **The harmonic field.** As an example and for further use, we will apply the previous constructions and results to the harmonic kernel from Subsections 5.5 and 9.6.

We start by giving a more explicit definition of the adjoint operators in case $k = 2$. We keep using the notation of Lemma 10.10.

Lemma 10.11. *Let p be a 2-Euclidean quadratic field. For any $x \sim y$ in X and any f in $\overline{V}^1(x)$, we have*

$$I_{xy}^{0,\dagger p} f = \frac{p_x(f, \mathbf{1}_y^x)}{p_{xy}^-(\mathbf{1}_y^{xy}, \mathbf{1}_y^{xy})}\mathbf{1}_y^{xy}.$$

Proof. Indeed, in case $k = 2$, $\overline{V}^0(xy)$ is a line which is spanned by $\mathbf{1}_y^{xy}$, I_{xy}^0 sends $\mathbf{1}_y^{xy}$ to $\mathbf{1}_y^x$ and we have by definition $p_x(\mathbf{1}_y^x, \mathbf{1}_y^x) = p_{xy}^-(\mathbf{1}_y^{xy}, \mathbf{1}_y^{xy})$. □

Corollary 10.12. *Let p be a 2-Euclidean field. Then the associated quadratic type function φ_p^∞ on X_* may be computed as follows. Fix $x \neq y$ in X.*

If $d(x,y) = 1$, one has $\varphi_p^\infty(x,y) = \varphi_{p^-}(x,y) = p_{xy}^-(\mathbf{1}_x^{xy}, \mathbf{1}_x^{xy})$.

If $d(x,y) = 2$, one has $\varphi_p^\infty(x,y) = \varphi_p(x,y) = -p_z(\mathbf{1}_x^z, \mathbf{1}_y^z)$, where z is the middle point of $[xy]$.

In general, if $j = d(x,y) \geq 2$ and $x = z_0, z_1, \ldots, z_j = y$ is the geodesic path from x to y, one has

$$\varphi_p^\infty(x,y) = \frac{\prod_{h=1}^{j-1} \varphi_p(z_{h-1}, z_{h+1})}{\prod_{h=1}^{j-2} \varphi_{p^-}(z_h, z_{h+1})}.$$

These formulae are closely related to the ones appearing in the work of Młotkowski [28].

Proof. For $j = 1, 2$ this is the definition of φ_p^∞. Now, for $j \geq 3$, note that we have, by Lemma 10.11,

$$\begin{aligned} I^{0,\dagger p}_{z_{j-1}z_{j-2}} \mathbf{1}_y^{z_{j-1}} &= \frac{p_{z_{j-1}}(\mathbf{1}_y^{z_{j-1}}, \mathbf{1}_{z_{j-2}}^{z_{j-1}})}{p^-_{z_{j-1}z_{j-2}}(\mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}}, \mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}})} \mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}} \\ &= \frac{\varphi_p(z_{j-2}, y)}{\varphi_{p^-}(z_{j-2}, z_{j-1})} \mathbf{1}_{z_{j-1}}^{z_{j-1}z_{j-2}}, \end{aligned}$$

where, in the second equality, we have used the relation $\mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}} + \mathbf{1}_{z_{j-1}}^{z_{j-1}z_{j-2}} = 0$. We obtain

$$I^0_{z_{j-2}z_{j-1}} I^{0,\dagger p}_{z_{j-1}z_{j-2}} \mathbf{1}_y^{z_{j-1}} = \frac{\varphi_p(z_{j-2}, y)}{\varphi_{p^-}(z_{j-2}, z_{j-1})} \mathbf{1}_{z_{j-1}}^{z_{j-2}}.$$

By Lemma 10.10, this gives

$$\varphi_p^\infty(x,y) = \varphi_p^\infty(x, z_{j-1}) \frac{\varphi_p(z_{j-2}, y)}{\varphi_{p^-}(z_{j-2}, z_{j-1})},$$

whence the result. □

Now, we go back to the harmonic kernel (χ, χ^-) of Subsection 5.5. We let π be the associated 2-Euclidean field, which we call the harmonic field.

Proposition 10.13. *The harmonic field is admissible. The associated quadratic transfer operator T_π on the space $\mathcal{M}_1$ of Γ-invariant 1-quadratic pseudofields has spectral radius $\leq \frac{1}{2}$.*

Proof. We will apply Proposition 10.9. To this aim, we need to say more on the quadratic transfer operator T_π. Let s be a 1-quadratic pseudofield and, for (x, y) in X_1, set $u(xy) = s_{xy}(\mathbf{1}_x^{xy}, \mathbf{1}_x^{xy}) = s_{xy}(\mathbf{1}_y^{xy}, \mathbf{1}_y^{xy})$. Let p be a 2-Euclidean field. By Lemma 10.11, for x in X and y, z neighbours of x, we have

$$I_{xz}^{0,\dagger p} I_{xy}^0 \mathbf{1}_y^{xy} = \frac{p_x(\mathbf{1}_y^x, \mathbf{1}_z^x)}{p_{xz}^-(\mathbf{1}_z^{xz}, \mathbf{1}_z^{xz})} \mathbf{1}_z^{xz}.$$

Therefore, by Definition 10.4, we can identify T_p with the operator that sends a function u on X_1 to the function

$$(x, y) \mapsto \sum_{\substack{z \sim x \\ z \neq y}} \frac{p_x(\mathbf{1}_y^x, \mathbf{1}_z^x)^2}{p_{xz}^-(\mathbf{1}_z^{xz}, \mathbf{1}_z^{xz})^2} u(zx).$$

Now, for the harmonic field π, by Lemma 5.20, we have

$$\pi_x(\mathbf{1}_y^x, \mathbf{1}_y^x) = 1 \text{ and } \pi_x(\mathbf{1}_y^x, \mathbf{1}_z^x) = -\frac{1}{d(x) - 1} \text{ if } y \neq z.$$

Thus T_π may be seen as the operator that sends a function u on X_1 to the function

$$(x, y) \mapsto \frac{1}{(d(x) - 1)^2} \sum_{\substack{z \sim x \\ z \neq y}} u(zx).$$

With respect to the uniform norm on functions on X_1, this operator has norm $\leq \sup_{x \in X} \frac{1}{d(x)-1} \leq \frac{1}{2}$, hence it has spectral radius $\leq \frac{1}{2}$. By Proposition 10.9, the Euclidean field π is admissible. $\square$

10.6. **Tangent dual kernels.** We now aim at establishing the converse of Theorem 10.9, namely that if a Γ-invariant Euclidean kernel is admissible, its quadratic transfer operator has spectral radius < 1. Our argument is differential geometric and requires us to compute the tangent space of the space of Euclidean kernels, viewed as a submanifold of the vector space of dual kernels. This is the purpose of this subsection.

Fix $k \geq 2$. We denote by $\mathcal{P}_k$ the set of all Γ-invariant k-Euclidean fields. The space $\mathcal{P}_k$ is an open subset of the finite-dimensional vector space of all Γ-invariant k-quadratic fields. In particular, it comes with a natural manifold structure. As explained in Subsection 5.1, there is a natural injective map $\mathcal{P}_k \hookrightarrow \mathcal{K}_k$ where, as in Section 8, $\mathcal{K}_k$ stands for the space of Γ-invariant k-dual kernels. It will turn out that this map is an immersion and that we can describe the tangent spaces of its range. To do this we again need to define a family of linear operators.

Let p be a k-Euclidean field. Then, for $\ell \geq 0$ and $x \sim y$ in X, p defines a Euclidean structure on the spaces $V_0^\ell(x)$ and $V_0^\ell(xy)$ that is dual to the Euclidean structure on the spaces $\overline{V}^{\ell}(x)$ and $\overline{V}^{\ell}(xy)$. Now, we have linear surjective operators $I_{xy}^{\ell,*} : V_0^{\ell+1}(x) \to V_0^{\ell}(xy)$ and $J_{xy}^{\ell,*} : V_0^{\ell}(xy) \to V_0^{\ell}(x)$. We define the operators

$$I_{xy}^{\ell,*\dagger p} : V_0^{\ell}(xy) \to V_0^{\ell+1}(x)$$
$$\text{and } J_{xy}^{\ell,*\dagger p} : V_0^{\ell}(x) \to V_0^{\ell}(xy)$$

as being the adjoints of these operators with respect to the Euclidean structure p. They are injective operators which can also be seen as the adjoint operators, with respect to the duality, of the above introduced operators $I_{xy}^{\ell,\dagger p}$ and $J_{xy}^{\ell,\dagger p}$.

Proposition 10.14. *Let $k \geq 2$. The natural map $\mathcal{P}_k \hookrightarrow \mathcal{K}_k$ is an immersion. Fix p in $\mathcal{P}_k$ and let $\mathrm{T}_p\mathcal{P}_k$ denote the tangent space of $\mathcal{P}_k$, viewed as a subspace of $\mathcal{K}_k$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, then $\mathrm{T}_p\mathcal{P}_k$ is the space of all Γ-invariant k-dual kernels whose associated bilinear forms $(q_x)_{x\in X}$ and $(q_{xy}^-)_{x\sim y\in X}$ satisfy the relations

$$(I_{xy}^{\ell-1,*\dagger p})^\star q_x = q_{xy}^- = (I_{yx}^{\ell-1,*\dagger p})^\star q_y, \quad x \sim y \in X.$$

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, then $\mathrm{T}_p\mathcal{P}_k$ is the space of all Γ-invariant k-dual kernels whose associated bilinear forms $(q_{xy})_{x\sim y\in X}$ and $(q_x^-)_{x\in X}$ satisfy the relations

$$(J_{xy}^{\ell,*\dagger p})^\star q_{xy} = q_x^- = (J_{xz}^{\ell,*\dagger p})^\star q_{xz}, \quad x \in X, \quad y, z \sim x.$$

Proof. As $\mathcal{P}_k$ is an open subspace of the vector space of all k-quadratic fields, its tangent space may be identified with this vector space. Now, fix p in $\mathcal{P}$. The Euclidean structures associated with p on the spaces $\overline{V}^{\ell}(x)$ and $\overline{V}^{\ell}(xy)$, $\ell \geq 0$, $x \sim y$ in X, give rise to isomorphisms between these spaces and the spaces $V_0^{\ell}(x)$ and $V_0^{\ell}(xy)$. These isomorphisms conjugate the linear maps I_{xy}^{ℓ} and J_{xy}^{ℓ} with the above defined linear maps $I_{xy}^{\ell,*\dagger p}$ and $J_{xy}^{\ell,*\dagger p}$. The conclusion follows from these facts and standard considerations on the tangent space of the space of scalar products on a finite-dimensional vector space. □

10.7. **The adjoint quadratic transfer operator.** Our goal is still to prove that a Euclidean field p is admissible if and only if the associated quadratic transfer operator T_p has spectral radius < 1. We will actually need to use the adjoint operator of the quadratic transfer operator which we will now describe.

Recall that we have identified the dual space of the space $\mathcal{M}_k$ of k-quadratic pseudofields with the space $\mathcal{L}_k$ of k-pseudokernels.

Lemma 10.15. *Let $k \geq 2$ and p be a k-Euclidean field. Define a linear operator T_p^* on the space of $(k-1)$-pseudokernels in the following way. Let L be a $(k-1)$-pseudokernel with associated bilinear forms $(r_{xy})_{(x,y)\in X_1}$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, then $T_p^* L$ is the $(k-1)$-pseudokernel with associated bilinear forms defined by, for (x, y) in X_1,

$$(T_p^* r)_{xy} = \sum_{\substack{z\sim y\\ z\neq x}} (I_{yz}^{\ell-1,*} I_{yx}^{\ell-1,*\dagger p})^\star r_{yz} = (I_{yx}^{\ell-1,*\dagger p})^\star \sum_{\substack{z\sim y\\ z\neq x}} (I_{yz}^{\ell-1,*})^\star r_{yz}.$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, then $T_p^* L$ is the $(k-1)$-pseudokernel with associated bilinear forms defined by, for (x, y) in X_1,

$$(T_p^* r)_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} (J_{zx}^{\ell,*} J_{xz}^{\ell,*\dagger p})^\star r_{zx}.$$

Then if p is Γ-invariant, the operator $T_p^* : \mathcal{L}_{k-1} \to \mathcal{L}_{k-1}$ is the adjoint operator of the quadratic transfer operator $T_p : \mathcal{M}_{k-1} \to \mathcal{M}_{k-1}$.

Proof. The proof is closely related to the one of Lemmas 9.12 and 10.3. We recall the argument. We keep the notation of Subsection 10.2. In particular $\langle ., . \rangle$ is the duality between $\mathcal{L}_{k-1}$ and $\mathcal{M}_{k-1}$. We pick r in $\mathcal{L}_{k-1}$ and s in $\mathcal{M}_{k-1}$ and we need to show that we have $\langle T_p^* r, s\rangle = \langle r, T_p s\rangle$.

If k is even, $k = 2\ell$, $\ell \geq 1$, by Definition 10.4 and Lemma 9.11, $\langle r, T_p s\rangle$ is the sum on $\Gamma\backslash X$ of the Γ-invariant function on X

$$x \mapsto \frac{1}{|\Gamma_x|} \sum_{y\sim x} \langle r_{xy}, (T_p s)_{xy}\rangle = \frac{1}{|\Gamma_x|} \sum_{\substack{y,z\sim x\\ y\neq z}} \langle r_{xy}, (I_{xz}^{\ell-1,\dagger p} I_{xy}^{\ell-1})^\star s_{zx}\rangle.$$

Now, for x in X and $y, z \sim x$ with $y \neq z$, by Lemma C.2, we have

$$\begin{aligned}\langle r_{xy}, (I_{xz}^{\ell-1,\dagger p} I_{xy}^{\ell-1})^\star s_{zx}\rangle &= \langle (I_{xy}^{\ell-1,*})^\star r_{xy}, (I_{xz}^{\ell-1,\dagger p})^\star s_{zx}\rangle \\ &= \langle (I_{xy}^{\ell-1,*} I_{xz}^{\ell-1,*\dagger p})^\star r_{xy}, s_{zx}\rangle.\end{aligned}$$

Hence $\langle r, T_p s\rangle$ is the sum on $\Gamma\backslash X$ of the Γ-invariant function on X

$$x \mapsto \frac{1}{|\Gamma_x|} \sum_{z\sim x} \langle (T_p^* r)_{zx}, s_{zx}\rangle,$$

which, still by Lemma 9.11, is equal to $\langle T_p^* r, s\rangle$.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, by Definition 10.5 and Lemma 9.11, $\langle r, T_p s\rangle$ is the sum on $\Gamma\backslash X$ of the Γ-invariant function on X

$$y \mapsto \frac{1}{|\Gamma_y|} \sum_{x\sim y} \langle r_{xy}, (T_p s)_{xy}\rangle = \frac{1}{|\Gamma_y|} \sum_{\substack{x,z\sim y\\ x\neq z}} \langle r_{xy}, (J_{yx}^{\ell,\dagger p} J_{xy}^{\ell})^\star s_{yz}\rangle.$$

As above, for y in X and $x, z \sim y$, $x \neq z$, by Lemma C.2, we have

$$\langle r_{xy}, (J_{yx}^{\ell,\dagger p} J_{xy}^{\ell})^\star s_{yz}\rangle = \langle (J_{xy}^{\ell,*} J_{yx}^{\ell,*\dagger p})^\star r_{xy}, s_{yz}\rangle.$$

Hence $\langle r, T_p s\rangle$ is the sum on $\Gamma\backslash X$ of the Γ-invariant function on X

$$y \mapsto \frac{1}{|\Gamma_y|} \sum_{z\sim y} \langle (T_p^* r)_{yz}, s_{yz}\rangle,$$

which, again by Lemma 9.11, is equal to $\langle T_p^* r, s\rangle$. □

We summarize the computation of the tangent space of $\mathcal{P}_k$ and the definition of the adjoint quadratic transfer operator.

Proposition 10.16. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean kernel. Then, as subspaces of $\mathcal{L}_{k-1}$, we have*

$$\mathcal{L}_{k-1} \cap \mathrm{T}_p\mathcal{P}_k = \ker(T_p^* - 1).$$

Proof. We pick L in $\mathcal{L}_{k-1}$ which we view as a family $(r_{xy})_{(x,y)\in X_1}$ of symmetric bilinear forms.

Asume k is even, $k = 2\ell$, $\ell \geq 1$. Then by Definition 8.9 and Proposition 10.14, saying that r is tangent to $\mathcal{P}_k$ at p is saying that, for any (x, y) in X_1, one has

$$(I_{xy}^{\ell-1,*\dagger p})^\star \sum_{z\sim x} (I_{xz}^{\ell-1,*})^\star r_{xz} = r_{xy} + r_{yx}.$$

As, by construction, $I_{xy}^{\ell-1,*} I_{xy}^{\ell-1,*\dagger p}$ is the identity operator of $V_0^{\ell-1}(xy)$, this is equivalent to saying that

$$(I_{xy}^{\ell-1,*\dagger p})^\star \sum_{\substack{z\sim x\\ z\neq y}} (I_{xz}^{\ell-1,*})^\star r_{xz} = r_{yx},$$

which, by Lemma 10.15, reads as $(T_p^* r)_{yx} = r_{yx}$.

Now, if k is odd, $k = 2\ell+1$, $\ell \geq 1$, by Definition 8.10 and Proposition 10.14, saying that r is tangent to $\mathcal{P}_k$ at p is saying that, for any (x, y) in X_1, one has

$$(J_{xy}^{\ell,*\dagger p})^\star ((J_{xy}^{\ell,*})^\star r_{xy} + (J_{yx}^{\ell,*})^\star r_{yx}) = \frac{1}{d(x) - 1} \sum_{z\sim x} r_{xz}.$$

As above, by construction, $J_{xy}^{\ell,*} J_{xy}^{\ell,*\dagger p}$ is the identity operator of $V_0^{\ell}(x)$, so that this is equivalent to saying that

$$(J_{yx}^{\ell,*} J_{xy}^{\ell,*\dagger p})^\star r_{yx} = \frac{1}{d(x) - 1} \sum_{\substack{z\sim x\\ z\neq y}} r_{xz} - \frac{d(x) - 2}{d(x) - 1} r_{xy}. \tag{10.9}$$

For (x, y) in X_1, we sum (10.9) applied to the pairs (x, z) with $z \sim x$, $z \neq y$. We get

$$\sum_{\substack{z\sim x\\ z\neq y}} (J^{\ell,*}_{zx} J^{\ell,*\dagger p}_{xz})^\star r_{zx} = \frac{1}{d(x)-1} \sum_{\substack{z\sim x\\ z\neq y}} \sum_{\substack{t\sim x\\ t\neq z}} r_{xt} - \frac{d(x)-2}{d(x)-1} \sum_{\substack{z\sim x\\ z\neq y}} r_{xz} = r_{xy}.$$

Thus, if r is in $\mathrm{T}_p\mathcal{P}_k$, we have $T^*_p r = r$. Conversely, if $T^*_p r = r$, the same computation shows that (10.9) holds and hence that r is in the tangent space $\mathrm{T}_p\mathcal{P}_k$. □

10.8. **The weight map as a diffeomorphism.** We are ready to state and prove

Theorem 10.17. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean kernel. Then p is admissible if and only if the quadratic transfer operator T_p has spectral radius < 1 on the space $\mathcal{M}_{k-1}$ of $(k-1)$-quadratic pseudofields.*

In the course of the proof, we shall use a classical generalization of the Perron-Frobenius Theorem. Recall that, if V is a finite-dimensional vector space, a closed convex cone $\mathcal{C} \subset V$ is said to be proper if it does not contain any vector line.

Lemma 10.18. *Let V be a finite-dimensional real vector space and T be an endomorphism of V which preserves a proper closed convex cone $\mathcal{C}$ of V with nonempty interior, that is, $T\mathcal{C} \subset \mathcal{C}$. Then the spectral radius of T is an eigenvalue of T associated to an eigenvector in $\mathcal{C}$.*

Proof. If T is nilpotent, there is nothing to prove. Else, let $\rho > 0$ be the spectral radius of T. By replacing T with $\rho^{-1}T$, we can assume $\rho = 1$. Let V' be the subspace of V whose complexification is the sum of all eigenspaces of T associated with eigenvalues of modulus 1 of T. Then T preserves V' and the closure in $\mathrm{GL}(V')$ of the sub-semigroup spanned by the restriction T' of T to V' is a compact subgroup K of $\mathrm{GL}(V')$.

Fix any norm on V. It follows from the Jordan reduction of T that there exists a proper subspace W of V with $T^{-1}W = W$ and, for any v in $V \smallsetminus W$, any limit point in V of $\frac{1}{\|T^n v\|} T^n v$ belongs to V'.

Now, as $\mathcal{C}$ has non-empty interior, we can pick such a v in $\mathcal{C}$. Therefore, the closed convex cone $\mathcal{C}' = \mathcal{C} \cap V'$ is non-zero. Pick v' in $\mathcal{C}'$. Then $v'' = \int_K kv' \mathrm{d}k$, the average of kv' with respect to the Haar measure of K, is K-invariant. To conclude, it suffices to prove that $v'' \neq 0$. But, by the Hahn-Banach Theorem, $\mathcal{C}'$ being a proper closed convex cone in V', there exists a linear functional φ on V' which is positive on $\mathcal{C}' \smallsetminus \{0\}$. As, for any k in K, kv' belongs to $\mathcal{C}'$, we have $\varphi(kv') > 0$, hence $\varphi(v'') > 0$ and $v'' \neq 0$. The result follows. □

In our case, the quadratic transfer operators preserve a natural convex cone. We say that a quadratic pseudofield is non-negative if all the associated symmetric bilinear forms are non-negative. From the fact that T_p is defined by taking sums of pull-back maps between vector spaces, we directly get

Lemma 10.19. *Let $k \geq 2$, p be a k-Euclidean field and s be a non-negative $(k-1)$-quadratic pseudofield. Then $T_p s$ is non-negative.*

For $k \geq 1$, let $\mathcal{M}_k^+ \subset \mathcal{M}_k$ be the cone of Γ-invariant non-negative k pseudofields. This is a proper closed convex cone in $\mathcal{M}_k$ with non-empty interior.

Proof of Theorem 10.17. Proposition 10.9 says that if T_p has spectral radius < 1, then p is admissible. Let us prove the converse statement. We will use the results of Section 7 to show that the weight map is a local diffeomorphism from the space of admissible kernels to the space of cohomology classes of functions and then conclude by using the study of the weight map from Section 8 and Proposition 10.16. Let us do this precisely.

As in Section 8, we let $\mathcal{W}_k$ stand for the space of cohomology classes of Γ-invariant symmetric functions on X_k and $W_k : \mathcal{K}_k \to \mathcal{W}_k$ for the weight map. We also let $\iota_k : \mathcal{P}_k \hookrightarrow \mathcal{K}_k$ denote the natural injection.

Let us introduce maps that are related to the Hilbert space H_0^ω. We let $\mathcal{Q}^\infty(H_0^\omega)$ denote the space of continuous quadratic forms on the Hilbert space H_0^ω and $\mathcal{Q}^\infty_{++}(H_0^\omega) \subset \mathcal{Q}^\infty(H_0^\omega)$ denote the open subset of coercive positive quadratic forms. The image map $\Pi_k : \mathcal{Q}^\infty_{++}(H_0^\omega) \to \mathcal{P}_k$ is well-defined. Finally, we have a linear map $F_k : W_k \to \mathcal{Q}^\infty(H_0^\omega)$ that sends the cohomology class of a function w to the quadratic form Φ_w. We let $\mathcal{U}_k = F_k^{-1}\mathcal{Q}^\infty_{++}(H_0^\omega)$ be the open set of cohomology classes of those w such that Φ_w is coercive. To summarize, we have maps:

$$W_k\iota_k : \mathcal{P}_k \to \mathcal{W}_k \text{ and } \Pi_k F_k : \mathcal{U}_k \to \mathcal{P}_k \tag{10.10}$$

and we want to describe the set $\mathcal{P}_k^{\mathrm{ad}} = \Pi_k F_k(\mathcal{U}_k)$ of admissible Γ-invariant k-Euclidean kernels. This situation is pictured in Figure 5.

Here comes the key observation of the proof, that is, Theorem 7.17 says that

$$W_k\iota_k\Pi_k F_k w = w \tag{10.11}$$

for any w in $\mathcal{U}_k$. Now, let us notice that all the maps involved in (10.10) are smooth. Indeed, W_k and F_k are linear maps defined on finite-dimensional vector spaces and ι_k is smooth by Proposition 10.14, whereas Π_k is smooth by Proposition A.16. Therefore, (10.11) gives,

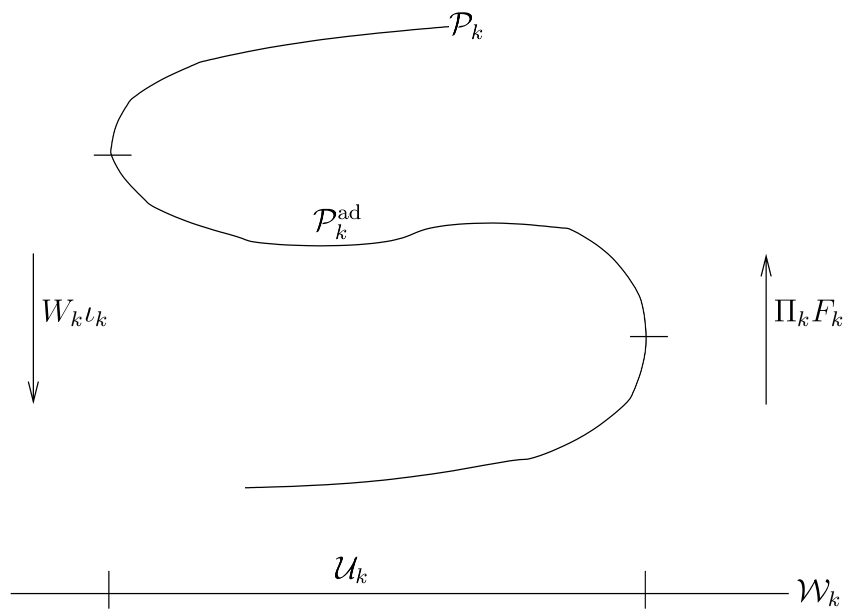

FIGURE 5. The objects in the proof of Theorem 10.17

by the chain-rule,

$$\mathrm{d}_{\Pi_k F_k w}(W_k\iota_k)\mathrm{d}_w(\Pi_k F_k) = \mathrm{Id}_{\mathcal{W}_k} \tag{10.12}$$

for w in $\mathcal{U}_k$. Set $p = \Pi_k F_k w$. As the map W_k is linear, we have $\mathrm{d}_p(W_k\iota_k) = W_k\mathrm{d}_p\iota_k$ and (10.12) implies that $W_k(\mathrm{T}_p\mathcal{P}_k) = \mathcal{W}_k$, that is W_k maps $\mathrm{T}_p\mathcal{P}_k$ onto $\mathcal{W}_k$.

Let us show that, for p in $\mathcal{P}_k^{\mathrm{ad}}$, W_k actually induces a linear isomorphism from $\mathrm{T}_p\mathcal{P}_k$ onto $\mathcal{W}_k$. As we have just shown this map to be surjective, this amounts to proving that both $\mathcal{P}_k$ and $\mathcal{W}_k$ have the same dimension. To do this, let π be the harmonic field, as in Subsection 10.5, which is a Γ-invariant 2-Euclidean field. We write π^k for the $(k-2)$-th orthogonal extension of π: this is a Γ-invariant k-Euclidean field. Proposition 10.13 says that π, and hence π^k, is admissible, and also that the quadratic transfer operator T_π has spectral radius $\leq \frac{1}{2}$ on $\mathcal{M}_1$. Then, it follows from Corollary 10.7, that T_{π^k} has spectral radius $\leq \frac{1}{2}$ on $\mathcal{M}_{k-1}$. By duality, the adjoint quadratic transfer operator $T^*_{\pi^k}$ has spectral radius $\leq \frac{1}{2}$ on $\mathcal{L}_{k-1}$. By Proposition 10.16, we have therefore $\mathcal{L}_{k-1} \cap \mathrm{T}_{\pi^k}\mathcal{P}_k = \{0\}$. By Corollary 8.33, $\mathcal{L}_{k-1}$ is exactly the null space of W_k, so that we have just shown that W_k is injective on $\mathrm{T}_{\pi^k}\mathcal{P}_k$, and hence that $\mathcal{P}_k$ and $\mathcal{W}_k$ have the same dimension.

Now we know that for p in $\mathcal{P}_k^{\mathrm{ad}}$, W_k is injective on $\mathrm{T}_p\mathcal{P}_k$, that is, still by Corollary 8.33, $\mathcal{L}_{k-1} \cap \mathrm{T}_p\mathcal{P}_k = \{0\}$. By Proposition 10.16, 1 is not an eigenvalue of the quadratic transfer operator T_p. By Lemma 10.19, the operator T_p preserves the cone $\mathcal{M}_{k-1}^+ \subset \mathcal{M}_{k-1}$ of Γ-invariant non-negative $(k-1)$-quadratic pseudofields. Hence, by Lemma 10.18, since 1 is not an eigenvalue of T_p, the spectral radius of T_p is $\neq 1$. As $\mathcal{Q}_{++}^\infty(H_0^\omega)$ is convex and F_k is linear, the open set $\mathcal{U}_k = F_k^{-1}\mathcal{Q}_{++}^\infty(H_0^\omega) \subset \mathcal{W}_k$ is convex and hence $\mathcal{P}_k^{\mathrm{ad}} = \Pi_k F_k \mathcal{U}_k$ is connected, since Π_k is continuous on $\mathcal{Q}_{++}^\infty(H_0^\omega)$ by Proposition A.16. As T_p depends continuously on p on $\mathcal{P}_k$, so does the spectral radius of T_p. Now, for π^k the $(k-2)$-th orthogonal extension of the harmonic kernel, we have shown above that T_{π^k} has spectral radius < 1. Therefore, for any p in $\mathcal{P}_k^{\mathrm{ad}}$, T_p has spectral radius < 1, which should be proved. □

As a Corollary of the proof, we get

Corollary 10.20. *Let $k \geq 2$. Then the space $\mathcal{W}_k$ has the same dimension as the space of Γ-invariant k-quadratic fields. The set $\mathcal{P}_k^{\mathrm{ad}}$ of admissible Γ-invariant k-Euclidean fields is open in $\mathcal{P}_k$. The weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ induces a smooth diffeomorphism from $\mathcal{P}_k^{\mathrm{ad}}$ onto its image.*

Remark 10.21. The fact that $\mathcal{W}_k$ and the space of Γ-invariant k-quadratic fields have the same dimension also follows from the duality between these spaces established in Proposition 11.2 below.

11. The admissible Riemannian metric

In this Section, for $k \geq 2$, we will define a natural Riemannian metric on the space $\mathcal{P}_k^{\mathrm{ad}}$ of admissible Γ-invariant k-Euclidean fields. The orthogonal extension embedding $\mathcal{P}_k^{\mathrm{ad}} \hookrightarrow \mathcal{P}_{k+1}^{\mathrm{ad}}$ will be proved to a Riemannian immersion. This metric may be seen as an analogue of the natural Riemannian metric on the space of positive definite symmetric bilinear forms of a finite-dimensional vector space (see [21]).

11.1. **Invariant quadratic type functions.**

The construction of this Riemannian metric will rely on certain duality properties on the space of Γ-invariant functions on X_k, $k \geq 2$. To introduce these properties, we go back to the point of view of quadratic type functions from Subsection 4.1 and say a little more about Γ-invariant ones.

First, we have a natural surjectivity result:

Lemma 11.1. *Let $k \geq 2$. The reduction map $\varphi \mapsto \varphi^-$ maps Γ-invariant quadratic type functions on X_k onto Γ-invariant quadratic type functions on X_{k-1}.*

To prove this, let us introduce some notation which extends the objects of Subsection 9.4. As usual, for $x \neq y$ in X, we let x_1 and y_1 denote the neighbours of x and y on $[xy]$.

Fix $k \geq 1$. We let F_k denote the finite-dimensional vector space of Γ-invariant functions on X_k. We define $S_k : F_k \to F_k$ as being the natural symmetry operator, $S_k v(x,y) = v(y,x)$, $v \in F_k$, $(x,y) \in X_k$, and we set $F_k^+ \subset F_k$ and $F_k^- \subset F_k$ to be respectively the space of symmetric and skew-symmetric functions. We also let L_k and R_k be the left and right augmentation operators $F_k \to F_{k+1}$ defined by, for v in F_k and (x,y) in X_{k+1},

$$L_k v(x,y) = v(x_1, y) \text{ and } R_k v(x,y) = v(x, y_1).$$

Note that one has $S_{k+1}L_k = R_k S_k$ and $L_{k+1}R_k = R_{k+1}L_k$. Lastly, we equip F_k with the S_k-invariant scalar product $\langle .,. \rangle$ defined by, for u, v in F_k,

$$\langle u, v \rangle = \sum_{(x,y) \in \Gamma \backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} u(x,y) v(x,y). \tag{11.1}$$

A direct computation using as usual Lemma 9.11 shows that, with respect to this scalar product, the adjoint maps of L_k and R_k are the reduction operators $L_k^\dagger$ and $R_k^\dagger$, defined by, for v in F_{k+1} and (x,y) in X_k,

$$L_k^\dagger v(x,y) = \sum_{\substack{z \sim x \\ z \neq x_1}} v(z,y) \text{ and } R_k^\dagger v(x,y) = \sum_{\substack{t \sim y \\ t \neq y_1}} v(x,t).$$

In particular, by Definition 4.2, a function φ in F_{k+1} has quadratic type if and only if it is symmetric and the function $L_k^\dagger \varphi$ in F_k is symmetric. By Lemma 4.3, one then has $L_k^\dagger \varphi = R_k^\dagger \varphi = \varphi^-$.

Let $\mathcal{F}_k$, $k \geq 2$, denote the finite-dimensional vector space of Γ-invariant k-quadratic fieds.

Proof of Lemma 11.1. Recall the reduction map $p \mapsto p^-$ of Subsection 4.2. For $k \geq 3$, we denote by $\rho_k : \mathcal{F}_k \mapsto \mathcal{F}_{k-1}$ the reduction map of Γ-invariant k-quadratic fields. This is a linear map. If $k \geq 3$, by Proposition 4.21, for any Γ-invariant $(k-1)$-Euclidean field p, one has $(p^+)^- = p$. Hence the space $\rho_k(\mathcal{F}_k)$ contains the open subset of Euclidean fields in $\mathcal{F}_{k-1}$ (which is nonempty by Proposition 5.21). We get $\rho_k(\mathcal{F}_k) = \mathcal{F}_{k-1}$. The conclusion follows, by the identification of quadratic fields with quadratic type functions in Proposition 4.11 and Lemma 4.13.

It remains to prove the case where $k = 2$. Recall that $F_2^+ \subset F_2$ is the space of symmetric functions. We claim that $L_1^\dagger$ maps F_2^+ onto

F_1. This amounts to proving that the adjoint map of the restriction of $L_1^\dagger$ to F_2^+ is injective. As the orthogonal projection of F_2 onto F_2^+ is $\frac{1}{2}(1+S_2)$, this adjoint map is $\frac{1}{2}(1+S_2)L_1$. Let now u be in F_1 with $\frac{1}{2}(1+S_2)L_1 u = 0$. For any x in X and $y \neq z$ in $S^1(x)$, we have $u(x,y)+u(x,z)=0$. Pick a third neighbour t of x (which exists by assumption). We have $u(x,y) = -u(x,z) = u(x,t) = -u(x,y)$, hence $u=0$ as required. Thus $L_1^\dagger(F_2^+) = F_1$. In particular, if φ is a symmetric function on X_1, we can find a symmetric function ψ on X_2 with $L_1^\dagger\psi = \varphi$. By definition, ψ has quadratic type and $\psi^- = \varphi$. □

Now, we can show that, among symmetric functions, the orthogonal complement of Γ-invariant quadratic type functions on X_k is the space of Γ-invariant functions that are coboundaries.

Proposition 11.2. *Let $k \geq 1$ and w be a symmetric Γ-invariant function on X_k. Then one has $\langle w, \varphi\rangle = 0$ for every quadratic type function φ on X_k if and only if there exists a skew-symmetric Γ-invariant function v on X_{k-1} such that, for any (x,y) in X_k, $w(x,y) = v(x,y_1) - v(x_1,y)$.*

Note that we have exceptionnally denoted by X_0 the diagonal in $X\times X$. Skew-symmetric functions on X_0 are zero! The Proposition says that the space of Γ-invariant quadratic type functions on X_k may be seen as the dual space to the space of cohomology classes of symmetric Γ-invariant functions on X_k.

Remark 11.3. In particular, Proposition 11.2 implies that the dimension of the space of Γ-invariant quadratic type functions on X_k grows exponentially with k. Since by Proposition 4.11, the set $\mathcal{P}_k$ of Γ-invariant Euclidean fields may be seen as an open subset in the latter space, as k grows, we are building more and more Γ-invariant scalar products on $\overline{\mathcal{D}}(\partial X)$, hence more and more unitary representations of Γ.

The proof of Proposition 11.2 relies on a classical phenomenon in duality that we state in the framework of Euclidean spaces.

Lemma 11.4. *Let V and W be Euclidean spaces and $T: V \to W$ be a linear map and $X \subset W$ be a subspace. Then the orthogonal complement of $T^{-1}X$ is given by*

$$(T^{-1}X)^\perp = T^\dagger(X^\perp),$$

where $T^\dagger : W \to V$ is the Euclidean adjoint operator of T.

Proof. If $Y \subset W$ is a subspace, we claim that $(T^\dagger Y)^\perp = T^{-1}(Y^\perp)$. Indeed, for v in V, we have

$$v \in (T^\dagger Y)^\perp \Leftrightarrow (\forall y \in Y \quad \langle v, T^\dagger y\rangle = 0) \Leftrightarrow (\forall y \in Y \quad \langle Tv, y\rangle = 0) \\ \Leftrightarrow Tv \in Y^\perp.$$

The resut follows by taking $Y = X^\perp$. □

Proof of Proposition 11.2. If $k = 1$, a quadratic type function on X_1 is simply a symmetric function. Thus, by assumption, we have $\langle w, w\rangle = 0$, hence $w = 0$ and we are done.

Assume $k \geq 2$. Recall that F_k^+ and F_k^- are the spaces of symmetric and skew-symmetric functions in F_k. Thus, by Definition 4.2, the space of Γ-invariant quadratic type functions on X_k is

$$F_k^+ \cap (L_{k-1}^\dagger)^{-1} F_{k-1}^+.$$

As F_k^- is the orthogonal complement of F_k^+ in F_k, Lemma 11.4 implies that w belongs to the space

$$F_k^- + L_{k-1}(F_{k-1}^-),$$

that is, we may write $w = u + L_{k-1}v$ where u and v are skew-symmetric functions on X_k and X_{k-1}. Now, w being symmetric, we get

$$w = S_k w = S_k u + S_k L_{k-1} v = -u + R_{k-1} S_{k-1} v = -u - R_{k-1} v,$$

hence

$$w = \frac{1}{2} w + \frac{1}{2} S_k w = \frac{1}{2} L_{k-1} v - \frac{1}{2} R_{k-1} v$$

and the result follows. □

11.2. **The weight formula.** Recall that our goal is to construct a natural Riemannian metric on the space $\mathcal{P}_k^{\mathrm{ad}}$ of admissible Γ-invariant k-quadratic fields. One of the main features of this Riemannian metric is that it can be defined by two natural formulae. We will first prove that these two definitions are equivalent.

Theorem 11.5. *Let $k \geq 2$. Let p be a Γ-invariant k-quadratic field and $\varphi_p : X_k \to \mathbb{R}$ be the associated quadratic type function. Let also (K, K^-) be a Γ-invariant k-dual kernel and $w : X_k \to \mathbb{R}$ be a Γ-invariant weight function of (K, K^-). If k is even, we have*

(11.2)
$$\sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle p_x, q_x^K \rangle - \frac{1}{2} \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}^-, q_{xy}^{K^-} \rangle = \frac{1}{2} \langle \varphi_p, w \rangle.$$

If k is odd, we have

(11.3)
$$\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy},q^K_{xy}\rangle-\sum_{x\in\Gamma\backslash X}\frac{d(x)-1}{|\Gamma_x|}\langle p^-_x,q^{K^-}_x\rangle=\frac{1}{2}\langle\varphi_p,w\rangle.$$

The reader should compare (11.2) and (11.3) with the formulae in Lemma 5.9.

As usual, for any $\ell\geq 1$, if K is a 2ℓ-dual prekernel, for x in X, we have denoted by q^K_x the symmetric bilinear form associated with K_x on $V^\ell_0(x)$. If K is a $(2\ell+1)$-dual prekernel, for $x\sim y$ in X, we have denoted by q^K_{xy} the symmetric bilinear form associated with K_{xy} on $V^\ell_0(xy)$. See Section 4 for the notions of a quadratic field and the associated quadratic type function. See Definition 6.7 for the notion of a weight function of a dual kernel. As in Subsection 11.1, we have denoted by $\langle .,.\rangle$ the natural scalar product on the space of Γ-invariant functions on X_k which has been defined by Equation (11.1). As in Appendix C, we have also denoted by $\langle .,.\rangle$ the natural duality between the space of symmetric bilinear forms on a vector space and on its dual space.

Definition 11.6. The bilinear pairing defined between dual kernels and quadratic fields in Theorem 11.5 will be called the weight pairing. We denote it by $(p,K,K^-)\mapsto[p,(K,K^-)]$.

From the elementary properties of Γ-invariant quadratic type functions, we get a nice compatibility property of the weight pairing with orthogonal extensions. Recall the notion of the reduction of a quadratic field from Subsection 4.2.

Corollary 11.7. *Let $k\geq 2$. Let p be a Γ-invariant $(k+1)$-quadratic field with reduction p^- and (K,K^-) be a Γ-invariant k-dual kernel with orthogonal extension (K^+,K). We have*

$$[p,(K^+,K)]=[p^-,(K,K^-)].$$

Proof. We keep the notation from Subsection 11.1. Let $w:X_k\to\mathbb{R}$ be a Γ-invariant weight function for (K,K^-). Then, by Corollary 8.27, the function $\frac{1}{2}(R_kw+L_kw)$ is a weight function for (K^+,K). Now, note that, by Lemma 4.13, the quadratic type functions associated to p^- and p satisfy the relation $\varphi_{p^-}=(\varphi_p)^-=R^\dagger_k\varphi_p=L^\dagger_k\varphi_p$. Thus, we get

$$[p, (K^+, K)] = \frac{1}{4}\langle \varphi_p, R_k w + L_k w\rangle = \frac{1}{4}\langle R_k^\dagger \varphi_p + L_k^\dagger \varphi_p, w\rangle$$
$$= \frac{1}{2}\langle \varphi_{p^-}, w\rangle = [p^-, (K, K^-)],$$

which should be proved. □

We now give the

Proof of Theorem 11.5. We will use the study of the weight map in Section 8 to deduce the general case of Proposition 11.5 from particular ones.

First, we assume that (K, K^-) is the k-dual kernel associated with a $(k-1)$-pseudokernel L. In that case, we will prove that both hand-sides of (11.2) and (11.3) are zero.

Indeed, on one hand, Theorem 8.32 tells us that w is a coboundary, that is, there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that, for any (x, y) in X_1, one has $w(x, y) = v(x, y_1) - v(x_1, y)$ where x_1 and y_1 are the neighbours of x and y on $[xy]$. Thus, by Proposition 11.2, we have $\langle \varphi_p, w\rangle = 0$.

On the other hand, assume first that k is even, $k = 2\ell$. For $x \sim y$ in X, we let as usual r_{xy}^L be the symmetric bilinear form associated with L_{xy} on $V_0^{\ell-1}(xy)$. By construction, we have $q_x^K = \sum_{y\sim x} (I_{xy}^{\ell-1,*})^\star r_{xy}^L$, hence, again using Lemma 9.11,

$$\sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\langle p_x, q_x^K\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle p_x, (I_{xy}^{\ell-1,*})^\star r_{xy}^L\rangle.$$

By Lemma C.2, for $x \sim y$ in X, we have

$$\langle p_x, (I_{xy}^{\ell-1,*})^\star r_{xy}^L\rangle = \langle (I_{xy}^{\ell-1})^\star p_x, r_{xy}^L\rangle = \langle p_{xy}^-, r_{xy}^L\rangle,$$

hence

$$\sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\langle p_x, q_x^K\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle p_{xy}^-, r_{xy}^L\rangle.$$

Now, we also have, for $x \sim y$ in X, $q_{xy}^{K^-} = r_{xy}^L + r_{yx}^L$ and thus

$$\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle p_{xy}^-, q_{xy}^{K^-}\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle p_{xy}^-, r_{xy}^L\rangle$$

and the left hand-side of (11.2) is zero.

In the same way, if k is odd, $k = 2\ell + 1$, by Lemma C.2 and Lemma 9.11, we have

$$\frac{1}{2} \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}, q_{xy}^K \rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_x^-, r_{xy}^L \rangle$$
$$= \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \sum_{y\sim x} \langle p_x^-, r_{xy}^L \rangle = \sum_{x\in\Gamma\backslash X} \frac{d(x)-1}{|\Gamma_x|} \langle p_x^-, q_x^{K^-} \rangle,$$

where as usual, for $x \sim y$ in X, r_{xy}^L stand for the symmetric bilinear form associated with L_{xy} on $V_0^\ell(x)$.

This finishes the case where (K, K^-) is the dual kernel associated to a pseudokernel. In the general case, now, we will use again the dual kernel $(K^w, 0)$ from Subsection 8.1. Recall from Corollary 8.3 that w is a weight function of $(K^w, 0)$. Therefore, by Theorem 8.32, the dual kernel $(K, K^-) - (K^w, 0)$ is associated to a certain pseudokernel. As we have just shown that (11.2) and (11.3) are true for pseudokernels, it suffices to show that they are true when $(K, K^-) = (K^w, 0)$.

In that case, assume first that k is even, $k = 2\ell$. Then, for any x in X and y, z in $S^\ell(x)$, we have

$$K_x(y,z) = w(y,z) \quad x \in [yz]$$
$$K_x(y,z) = 0 \quad \text{else.}$$

Therefore, by Lemma C.5,

$$\langle p_x, q_x^K \rangle = -\frac{1}{2} \sum_{\substack{(y,z)\in S^\ell(x)\times S^\ell(x)\\ x\in[yz]}} w(y,z) p_x(\mathbf{1}_y, \mathbf{1}_z)$$
$$= \frac{1}{2} \sum_{\substack{(y,z)\in S^\ell(x)\times S^\ell(x)\\ x\in[yz]}} w(y,z) \varphi_p(y,z).$$

Now, (11.2) follows from Lemma 9.11.

In the same way, if k is odd, $k = 2\ell + 1$, for any $x \sim y$ in X and z, t in $S^\ell(x)$, we have

$$K_{xy}(z,t) = w(z,t) \quad [xy] \subset [yz]$$
$$K_{xy}(z,t) = 0 \quad \text{else.}$$

Therefore, by Lemma C.5,

$$\langle p_{xy}, q_{xy}^K \rangle = \frac{1}{2} \sum_{\substack{(z,t)\in S^\ell(xy)\times S^\ell(xy)\\ [xy]\subset[zt]}} w(y,z) \varphi_p(z,t)$$

and (11.3) again follows from Lemma 9.11. □

11.3. **The weight tensor.** Given $k \geq 2$, we will use the weight pairing to define a natural smooth section g of the vector bundle $\mathcal{Q}(\mathrm{T}\mathcal{P}_k)$ of symmetric bilinear forms on the tangent space of $\mathcal{P}_k$. It turns out that $\mathcal{P}_k^{\mathrm{ad}}$ is precisely a connected component of the set of p such that g_p is positive.

We now give the precise definition of g. Recall that in Subsection 5.1, we have defined an embedding from the space of k-Euclidean fields into the space of k-dual kernels. In case of Γ-invariant Euclidean fields, the spaces are finite-dimensional and we have studied this embedding from the point of view of differential geometry. In particular, we have shown in Proposition 10.14 that it is a smooth map. As in Subsection 10.8, let us now denote this map by $\iota_k : \mathcal{P}_k \hookrightarrow \mathcal{K}_k$. As above, we also denote by $\mathcal{F}_k$ the space of Γ-invariant k-quadratic fields.

Let p in $\mathcal{P}_k$ be a Γ-invariant k-Euclidean field. If q and r are Γ-invariant k-quadratic fields (which we view as tangent vectors to $\mathcal{P}_k$), we set

$$g_p(q, r) = -[q, \mathrm{d}_p\iota_k(r)],$$

where $[.,.]$ is the weight pairing. We call g the weight tensor on $\mathcal{P}_k$. The reason why we don't mention the dependance on k in our notation for the weight tensor will become clear in the next subsections.

Lemma 11.8. *Let $k \geq 2$. For any p in $\mathcal{P}_k$, g_p is a symmetric bilinear form on $\mathcal{F}_k$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, let q, r be in $\mathcal{F}_k$. For x in X, let A_x and B_x be the p_x-symmetric endomorphisms of $\overline{V}^{\ell}(x)$ which represent q_x and r_x with respect to p_x. For $x \sim y$ in X, let A^-_{xy} and B^-_{xy} be the p^-_{xy}-symmetric endomorphisms of $\overline{V}^{\ell-1}(xy)$ which represent q^-_{xy} and r^-_{xy} with respect to p^-_{xy}. One has

$$g_p(q, r) = \sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_x|} \operatorname{tr}(A_x B_x) - \frac{1}{2} \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \operatorname{tr}(A^-_{xy} B^-_{xy}).$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, let q, r be in $\mathcal{F}_k$. For $x \sim y$ in X, let A_{xy} and B_{xy} be the p_{xy}-symmetric endomorphisms of $\overline{V}^{\ell}(xy)$ which represent q_{xy} and r_{xy} with respect to p_{xy}. For x in X, let A^-_x and B^-_x be the p^-_x-symmetric endomorphisms of $\overline{V}^{\ell}(x)$ which represent q^-_x and r^-_x with respect to p^-_x. One has

$$g_p(q, r) = \frac{1}{2} \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \operatorname{tr}(A_{xy} B_{xy}) - \sum_{x \in \Gamma\backslash X} \frac{d(x) - 1}{|\Gamma_x|} \operatorname{tr}(A^-_x B^-_x).$$

Proof. The formulae are a direct consequence of the definition of the tensor g, Theorem 11.5 and Lemma C.4. Symmetry follows. $\square$

Remark 11.9. Note that, with the notation of Lemma 11.8, for $x \sim y$ in X, one has, if k is even, $I_{xy}^{\ell-1,\dagger p} A_x I_{xy}^{\ell-1} = A_{xy}^- = I_{yx}^{\ell-1,\dagger p} A_y I_{yx}^{\ell-1}$ and, if k is odd, $A_x^- = J_{xy}^{\ell,\dagger p} A_{xy} J_{xy}^{\ell}$.

11.4. **Derivative of the orthogonal extension.** We will prove that the weight tensor is natural in the sense that, for $k \geq 2$, the weight tensor of $\mathcal{P}_k$ is the pull-back of the one of $\mathcal{P}_{k+1}$ by the orthogonal extension map. This will require us to first prove that this map is smooth.

Let $\eta_k : \mathcal{P}_k \to \mathcal{P}_{k+1}$ denote the orthogonal extension map $p \mapsto p^+$.

Proposition 11.10. *Let $k \geq 2$. The orthogonal extension map η_k is smooth. Let p be in $\mathcal{P}_k$ and q in $\mathcal{F}_k$. We have the following formulae for describing the Γ-invariant $(k+1)$-quadratic field $\mathrm{d}_p\eta_k(q)$. If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we have*

$$\mathrm{d}_p\eta_k(q)_{xy} = (J_{xy}^{\ell,\dagger p})^\star q_x + (J_{yx}^{\ell,\dagger p})^\star q_y - (M_{xy}^{\ell-1,\dagger p})^\star q_{xy}^-.$$

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, for x in X, we have

$$\mathrm{d}_p\eta_k(q)_x = \sum_{y\sim x} (I_{xy}^{\ell,\dagger p})^\star q_{xy} - (d(x)-1)(M_x^{\ell,\dagger p})^\star q_x^-.$$

The linear maps M_x^ℓ, $x \in X$, $\ell \geq 1$, and M_{xy}^ℓ, $x \sim y \in X$, $\ell \geq 0$, have been defined in Subsection 4.2. See Lemma 4.4 for their main properties. We have denoted their adjoint linear maps with respect to the Euclidean structures associated to p in the usual way.

Proposition 11.10 immediately follows from the following abstract result. As usual, for a vector space V, we denote by $\mathcal{Q}(V)$ the space of symmetric bilinear forms on V and by $\mathcal{Q}_{++}(V) \subset \mathcal{Q}(V)$ the set of positive definite forms.

Lemma 11.11. *Let X be a finite-dimensional vector space, $d \geq 2$ an integer and $X_0, X_1, \ldots, X_d$ be subspaces of X. Assume that, for any $1 \leq i \neq j \leq d$, one has $X_i \cap X_j = X_0$ and that $X/X_0 = \bigoplus_{i=1}^d X_i/X_0$. Set $\mathcal{F}$ to be the space of all $q = (q_0, q_1, \ldots, q_d)$ in $\mathcal{Q}(X_0) \times \mathcal{Q}(X_1) \times \ldots \times \mathcal{Q}(X_d)$ with $(q_i)_{|V_i} = q_0$, $1 \leq i \leq d$, and $\mathcal{P} \subset \mathcal{F}$ to be the set of those p in $\mathcal{F}$ such that each of the p_i, $0 \leq i \leq d$, is positive definite. Then the orthogonal extension map $\eta : \mathcal{P} \to \mathcal{Q}_{++}(X)$ is smooth. If p is in $\mathcal{Q}$ and q is in $\mathcal{F}$, one has*

$$\mathrm{d}_p\eta(q) = P_1^\star q_1 + \cdots + P_d^\star q_d - (d-1)P_0^\star q_0,$$

where, for $0 \leq i \leq d$, P_i is the $\eta(p)$-orthogonal projection $X \to X_i$.

Proof. As in Subsection C.2, for a vector space V, we let

$$\delta_V : \mathcal{Q}_{++}(V) \to \mathcal{Q}_{++}(V^*)$$

denote the natural smooth diffeomorphism between scalar products on V and on its dual space V^*.

For $0 \leq i \leq d$, we let $\pi_i : X^* \to X_i^*$ be the restriction map. Set $\mathcal{K} = \mathcal{Q}(X_0^*) \times \mathcal{Q}(X_1^*) \times \ldots \times \mathcal{Q}(X_d^*)$ and let $\sigma : \mathcal{K} \to \mathcal{Q}(X^*)$ be the linear map

$$(r_0, r_1, \ldots, r_d) \mapsto \pi_1^\star r_1 + \cdots + \pi_d^\star r_d - (d-1)\pi_0^\star r_0.$$

We also set $\mathcal{D} : \mathcal{P} \to \mathcal{K}$ to be the product map

$$(p_0, p_1, \ldots, p_d) \mapsto (\delta_{X_0}(p_0), \delta_{X_1}(p_1), \ldots, \delta_{X_d}(p_d)).$$

By Lemma 5.3, we may write η as the product map $\eta = \delta_X^{-1}\sigma\mathcal{D}$. The result now follows from the chain-rule and Lemma C.3. □

For $k \geq 3$, we still denote by $\rho_k : \mathcal{F}_k \mapsto \mathcal{F}_{k-1}$ the reduction map of Γ-invariant k-quadratic fields. This is a linear map.

Corollary 11.12. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. We have $\rho_{k+1}\mathrm{d}_p\eta_k = \mathrm{Id}_{\mathcal{F}_k}$. The map η_k is a closed immersion.*

Proof. By Proposition 4.21, we have $(p^+)^- = p$ for any p in $\mathcal{P}_k$, hence, by differentiating this identity, $\rho_{k+1}\mathrm{d}_p\eta_k = \mathrm{Id}_{\mathcal{F}_k}$ (which can also be checked directly by using the formulae in Proposition 11.10). In particular, $\mathrm{d}_p\eta_k$ is injective. That it has closed range follows from the characterization of orthogonal extensions in Lemma 7.16. □

11.5. **Naturality of the weight tensor.** We can now examine the behaviour of the weight tensor under orthogonal extension.

Proposition 11.13. *Let $k \geq 2$ and p in $\mathcal{P}_k$ be a Γ-invariant k-Euclidean field. Chose Γ-invariant quadratic fields q in $\mathcal{F}_k$ and r in $\mathcal{F}_{k+1}$. One has*

$$g_{p^+}(\mathrm{d}_p\eta_k(q), r) = g_p(q, \rho_{k+1}r).$$

Proof. By definition, we have

$$g_{p^+}(\mathrm{d}_p\eta_k(q), r) = -[r, \mathrm{d}_{p^+}\iota_{k+1}\mathrm{d}_p\eta_k(r)] = -[r, \mathrm{d}_p(\iota_{k+1}\eta_k)(q)].$$

Now, Proposition 5.2 tells us that the dual kernel associated to the orthogonal extension of p is the orthogonal extension of the dual kernel associated to p. By differentiating this property at p, we get that the $(k+1)$-dual kernel $\mathrm{d}_p(\iota_{k+1}\eta_k)(q)$ is the orthogonal extension of the k-dual kernel $\mathrm{d}_p(\iota_k)(q)$. By Corollary 11.7, this gives

$$g_{p^+}(\mathrm{d}_p\eta_k(q), r) = -[\rho_{k+1}r, \mathrm{d}_p(\iota_k)(r)] = g_p(q, \rho_{k+1}r),$$

which should be proved. □

In case $\rho_{k+1} r = 0$, Proposition 11.13 gives

Corollary 11.14. *Let $k \geq 2$, p be in $\mathcal{P}_k$, q be in $\mathcal{F}_k$ and r be in $\mathcal{F}_{k+1}$ with $\rho_{k+1}(r) = r^- = 0$. One has $g_{p^+}(\mathrm{d}_p \eta_k(q), r) = 0$.*

In case r belongs to $\mathrm{d}_p \eta_k(\mathcal{F}_k)$, Proposition 11.13 gives

Corollary 11.15. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. One has $(\mathrm{d}_p \eta_k)^\star g_{p^+} = g_p$.*

In other words, the pull-back of the weight tensor of $\mathcal{P}_{k+1}$ by the orthogonal extension map is the weight tensor of $\mathcal{P}_k$.

Proof. By Corollary 11.12, for q in $\mathcal{F}_k$, one has $\rho_{k+1} \mathrm{d}_p \eta_k(q) = q$, and the result follows by by Proposition 11.13. □

We will later use the following consequence of these results:

Corollary 11.16. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. Assume that g_p is positive on $\mathcal{F}_k$. Then g_{p^+} is positive on $\mathcal{F}_{k+1}$.*

Proof. By Corollary 11.12, we have $\mathcal{F}_{k+1} = \ker \rho_{k+1} \oplus \mathrm{d}_p \eta_k(\mathcal{F}_k)$. By Corollary 11.14, these two subspaces are g_{p^+}-orthogonal to each other. Now, by the assumption and Corollary 11.15, g_{p^+} is positive on the space $\mathrm{d}_p \eta_k(\mathcal{F}_k)$, whereas by Lemma 11.8, it is positive on $\ker \rho_{k+1}$. The result follows. □

11.6. **Positivity and admissibility.** We can use the previous results to give a new criterion for a Euclidean field to be admissible.

Theorem 11.17. *Let $k \geq 2$. The set $\mathcal{P}_k^{\mathrm{ad}} \subset \mathcal{P}_k$ of admissible Γ-invariant k-Euclidean fields is a connected component of the set of p in $\mathcal{P}_k$ such that the symmetric bilinear form g_p on $\mathcal{F}_k$ is positive.*

See Definition 10.1 for the notion of an admissible kernel. It may be true that $\mathcal{P}_k^{\mathrm{ad}}$ is actually equal to the set of p in $\mathcal{P}_k$ such that g_p is positive.

Theorem 11.17 implies in particular that g induces on $\mathcal{P}_k^{\mathrm{ad}}$ the structure of a Riemannian manifold.

We start the proof with a general positivity result.

Lemma 11.18. *Let A be a finite set with n elements, $n \geq 3$, V be the space of real-valued functions on A and $\overline{V}$ be its quotient by the line of constant functions. For f in V set*

$$p(f, f) = \sum_{a \in A} f(a)^2 - \frac{1}{n-1} \sum_{\substack{(a,b) \in A^2 \\ a \neq b}} f(a) f(b)$$

and view p as a scalar product on $\overline{V}$. Then, for any non-zero p-symmetric endomorphism S of $\overline{V}$, we have

$$\mathrm{tr}(S^2) > \frac{1}{2}\sum_{a\in A} p(S\mathbf{1}_a, \mathbf{1}_a)^2.$$

Proof. Equip as usual V with the standard scalar product q defined by, for f in V, $q(f,f) = \sum_{a\in A} f(a)^2$ and let P be the q-orthogonal projection on $V_0 = \{f \in V | \sum_a f(a) = 0\}$ which is the q-orthogonal complement of constant functions. A direct computation shows that, for f in V, one has $p(f,f) = \frac{n}{n-1} q(Pf, f)$. In particular, p-symmetric endomorphisms of $\overline{V}$ may be identified with q-symmetric endomorphisms S of V such that $S\mathbf{1} = 0$. For any such S, set $\Phi(S) = \mathrm{tr}(S^2) - \frac{1}{2}\sum_{a\in A} p(S\mathbf{1}_a, \mathbf{1}_a)^2$. One has

$$\Phi(S) = \sum_{(a,b)\in A^2} q(S\mathbf{1}_a, \mathbf{1}_b)^2 - \frac{n^2}{2(n-1)^2}\sum_{a\in A} q(S\mathbf{1}_a, \mathbf{1}_a)^2. \tag{11.4}$$

If $n \geq 4$, we have $\frac{n^2}{2(n-1)^2} < 1$ and the result follows. It remains to deal with the case where $n = 3$. Then, denote by a, b, c the three elements of A and set $u = q(S\mathbf{1}_b, \mathbf{1}_c)$, $v = q(S\mathbf{1}_c, \mathbf{1}_a)$ and $w = q(S\mathbf{1}_a, \mathbf{1}_b)$. As $S\mathbf{1} = 0$, we get from (11.4),

$$\begin{aligned}\Phi(S) &= 2(u^2 + v^2 + w^2) - \frac{1}{8}((u+v)^2 + (v+w)^2 + (w+u)^2)\\ &= \frac{7}{4}(u^2 + v^2 + w^2) - \frac{1}{4}(uv + vw + wu)\end{aligned}$$

and a direct computation shows that this quadratic form on $\mathbb{R}^3$ is positive definite. □

To prove Theorem 11.17, we will again use the harmonic field which was studied in Subsections 5.5, 9.6 and 10.5. Recall from Proposition 10.13 that π is an admissible 2-Euclidean field.

Corollary 11.19. *The symmetric bilinear form g_π is positive on $\mathcal{F}_2$.*

Proof. By construction, for x in X and f in $\overline{V}^1(x)$, we have

$$\pi_x(f,f) = \sum_{y\sim x} f(y)^2 - \frac{1}{n(x)-1}\sum_{\substack{y,z\sim x\\ y\neq z}} f(y)f(z),$$

and we can therefore aim at applying Lemma 11.19 to the set $S^1(x)$ and the bilinear form π_x.

Fix q in $\mathcal{F}_k$ and, as in Lemma 11.8, for x in X, let us denote by A_x the endomorphism of $\overline{V}^1(x)$ which represents q_x with respect to p_x. By Lemma 9.11, Lemma 10.11, Lemma 11.8 and Remark 11.9, we have

$$g_\pi(q,q) = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\left(\operatorname{tr}(A_x^2) - \frac{1}{2}\sum_{y\sim x}\pi_x(A_x\mathbf{1}_y,\mathbf{1}_y)^2\right),$$

which, by Lemma 11.19, is positive as soon as $q \neq 0$. $\square$

Next, we will use the weight formula to characterize those p such that g_p is non-degenerate on $\mathcal{F}_k$.

Lemma 11.20. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. Then the symmetric bilinear form g_p is non-degenerate on $\mathcal{F}_k$ if and only if the quadratic transfer operator of p does not admit 1 as an eigenvalue.*

See Definitions 10.4 and 10.5 for the description of the quadratic transfer operator.

Proof. Assume that the quadratic transfer operator of p admits 1 as an eigenvalue. By Proposition 10.16, there exists a non-zero q in $\mathcal{F}_k$ such that $\mathrm{d}_p\iota_k(q)$ is a pseudokernel. Then, by Corollary 8.33, Proposition 11.2 and Theorem 11.5, for any r in $\mathcal{F}_k$, one has $g_p(q,r) = 0$, hence g_p is degenerate.

Conversely, assume q is a non-zero element in $\mathcal{F}_k$ and $g_p(q,r) = 0$ for any r in $\mathcal{F}_k$. Chose a Γ-invariant weight function w for the Γ-invariant k-dual kernel $\mathrm{d}_p\iota_k(q)$. By Proposition 11.2 and Theorem 11.5, for any r in $\mathcal{F}_k$, one has $\langle w,\varphi_r\rangle = 0$. By Proposition 4.11, for any Γ-invariant quadratic type function φ on X_k, one has $\langle w,\varphi\rangle = 0$, hence, by Proposition 11.2, there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that $w(x,y) = v(x,y_1) - v(x_1,y)$ for (x,y) in X_k. By Theorem 8.32, the k-dual kernel $\mathrm{d}_p\iota_k(q)$ is a pseudokernel. Again by Proposition 10.16, the quadratic transfer operator of p admits 1 as an eigenvalue. $\square$

Proof of Theorem 11.17. Note that, being the image of a convex set by a continuous map, the set $\mathcal{P}_k^{\mathrm{ad}}$ of admissible kernels is connected.

By Corollary 11.16 and Corollary 11.19, the symmetric form g_{π^k} is positive on $\mathcal{F}_k$, where π^k denotes the $(k-2)$-th orthogonal extension of the harmonic kernel. By Proposition 10.13, π^k belongs to $\mathcal{P}_k^{\mathrm{ad}}$ and, by Theorem 10.17 and Lemma 11.20, for any p in $\mathcal{P}_k^{\mathrm{ad}}$, the symmetric form g_p is non-degenerate. Therefore, as $\mathcal{P}_k^{\mathrm{ad}}$ is connected, g_p is positive for any p in $\mathcal{P}_k^{\mathrm{ad}}$.

Let $\mathcal{P}'_k$ be the connected component of g_{π^k} in the set of those p in $\mathcal{P}_k$ such that g_p is positive. We have just shown the inclusion $\mathcal{P}_k^{\mathrm{ad}} \subset \mathcal{P}'_k$.

Conversely, by Corollary 10.7 and Proposition 10.13, the quadratic transfer operator T_{π^k} of the Euclidean field π^k has spectral radius < 1. By Lemma 11.20, the spectral radius of T_p is $\neq 1$ for any p in $\mathcal{P}'_k$. As the spectral radius is continuous and $\mathcal{P}'_k$ is connnected by assumption, the spectral radius of T_p is < 1 for any p in $\mathcal{P}'_k$, hence p is admissible by Theorem 10.17. Thus we get $\mathcal{P}'_k \subset \mathcal{P}^{\mathrm{ad}}_k$ as required. □

Appendix A. Euclidean images

In this appendix, we study the notion of the Euclidean image of a non-negative symmetric bilinear form by a surjective map.

A.1. **Definition and first properties.** We start by defining the Euclidean image of a non-negative symmetric bilinear form under a surjective linear map. This relies on the

Lemma A.1. *Let V and W be real vector spaces and let q be a non-negative symmetric bilinear form form on V and $\pi : V \to W$ be a surjective linear map. For any w in W, we set*

$$\Phi(w) = \inf_{\substack{v \in V \\ \pi(v) = w}} q(v, v).$$

Then Φ is a non-negative quadratic form on W.

Definition A.2. Let the notation be as above. The polar form of Φ is called the Euclidean image of q by π and denoted by $\pi_\star q$.

Remark A.3. Assume V is a Hilbert space with scalar product q and π is bounded. Let X be the orthogonal complement of $\ker \pi$ in V. Then π induces a linear isomorphism from X onto W and $\pi_\star q$ is the image by this linear isomorphism of the restriction of q to X.

Proof of Lemma A.1. We will proceed to several reductions in order to be brought back to the case in Remark A.3.

First, we will reduce the proof to the case where Φ is non-zero on non-zero vectors. Let W_0 be the set of w in W such that $\Phi(w) = 0$ and let us show that Φ is W_0-invariant, that is, for any w in W and w_0 in W_0, we have $\Phi(w + w_0) = \Phi(w)$. Indeed, for such w and w_0, for any $\varepsilon > 0$, we can find v and v_0 in V with $\pi(v) = w$, $\pi(v_0) = w_0$, $q(v, v) \leq \Phi(w) + \varepsilon$ and $q(v_0, v_0) \leq \varepsilon$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} q(v + v_0, v + v_0) &= q(v, v) + q(v_0, v_0) + 2q(v, v_0) \\ &\leq \Phi(v) + 2\varepsilon + 2\sqrt{\varepsilon(q(v_0, v_0) + \varepsilon)}. \end{aligned}$$

As ε is arbitrary, this gives $\Phi(w + w_0) \leq \Phi(w)$. By symmetry, we get $\Phi(w + w_0) = \Phi(w)$. In particular, if w is also in W_0, we have $\Phi(w + w_0) = 0$ and W_0 is a subspace of W. For w in W, we have

$$\inf_{\substack{v \in V \\ \pi(v) \in w + W_0}} q(v, v) = \inf_{w_0 \in W_0} \Phi(w + w_0) = \Phi(w).$$

Thus, by replacing W with the quotient space W/W_0, we can assume that we have $W_0 = 0$.

Now, we can also assume that q is positive definite. Indeed, if $V_0 \subset V$ is the null space of q, we have $\pi(V_0) \subset W_0$, hence $\pi(V_0) = 0$. Therefore, we can replace V with the quotient space V/V_0 and assume that q is a scalar product. We equip V with the associated topology.

Let $U \subset V$ be the null space of π. We claim that U is closed with respect to this topology. Indeed, if (v_n) is a sequence in U that converges to v in V, we have, by definition of the topology, $q(v - v_n, v - v_n) = \|v - v_n\|^2 \xrightarrow[n\to\infty]{} 0$, hence $\Phi(\pi(v)) = 0$ and $\pi(v) = 0$.

Let H be the completion of V with respect to the positive definite bilinear form q and let X be orthogonal complement of the closure of U in H. Then, as U is closed in V, the orthogonal projection $H \to X$ induces an embedding of $\theta : W \simeq V/U \hookrightarrow X$ and, for w in W, we have $\Phi(w) = q(\theta w, \theta w)$. The result follows. $\square$

Let us give some elementary properties of Euclidean images. This operation behaves well under composition of surjective maps.

Lemma A.4. *Let V, W, X be real vector spaces and $\pi : V \to W$ and $\theta : W \to X$ be surjective linear maps. If q is a non-negative symmetric bilinear form on V, we have*

$$\theta_\star \pi_\star q = (\theta\pi)_\star q.$$

Also, it satisfies a concavity property.

Lemma A.5. *Let V, W be real vector spaces and $\pi : V \to W$ be a surjective linear map. If p and q are non-negative symmetric bilinear forms on V, we have*

$$\pi_\star(p + q) \geq \pi_\star p + \pi_\star q.$$

We have an invariance under certain translations:

Lemma A.6. *Let V, W be real vector spaces and $\pi : V \to W$ be a surjective linear map. Let p be a non-negative symmetric bilinear form on V and q be a symmetric bilinear form on W. Then $p + \pi^\star q$ is non-negative if and only if $\pi_* p + q$ is non-negative and we then have*

$$\pi_\star(p + \pi^\star q) = \pi_\star p + q.$$

Orthogonal splittings are preserved.

Lemma A.7. *Let $V_1, \dots, V_d, W_1, \dots, W_d$ be real vector spaces and $\pi_i : V_i \to W_i$ be surjective linear maps. Set $V = \bigoplus_{i=1}^d V_i$ and $W = \bigoplus_{i=1}^d W_i$ and write π for the sum map $V \to W$. Then if q is a non-negative symmetric bilinear form on V such that the $V_1, \dots, V_d$ are q-orthogonal to each other, the $W_1, \dots, W_d$ are $\pi_* q$-orthogonal to each other.*

A.2. **Approximation of Euclidean images.** In the course of the article, we have used the following approximation property of Euclidean images.

Proposition A.8. *Let H be Hilbert space with scalar product p, W be a finite-dimensional real vector space and $\pi : H \to W$ be a continuous surjective linear map. We assume that q is a continuous non-negative symmetric bilinear form on H. Then we have the following convergence of bilinear forms on W:*

$$\pi_\star(\varepsilon p + q) \xrightarrow[\varepsilon \to 0]{} \pi_\star q.$$

Remark A.9. Even in finite-dimensional vector spaces, the map $q \mapsto \pi_* q$ has bad continuity properties at the undefinite bilinear forms. For example, if $V = \mathbb{R}^2$ and, for any $n \geq 1$, q_n is the polar form of the quadratic form

$$(x, y) \mapsto (x + (1/n)y)^2,$$

then q_n has a non-zero limit, whereas, for any non-zero linear functional φ of V, one has $\varphi_\star q_n \xrightarrow[n \to \infty]{} 0$. This explains why, in Proposition A.8, we have made some additional assumptions to get a limit.

We will prove Proposition A.8 in several steps. The main idea is to reduce it to the case where π is a linear functional. We start by studying this situation.

If V is a vector space, φ is a non-zero linear functional and q is a non-negative symmetric bilinear form, we shall identify the bilinear form $\varphi_\star q$ and the real number

$$\varphi_\star q(1, 1) = \inf_{\substack{v \in V \\ \langle \varphi, v \rangle = 1}} q(v, v).$$

This number is easy to compute:

Lemma A.10. *Let V be a real vector space, equipped with a non-negative symmetric bilinear form q, $W \subset V$ be the null space of q and φ be a non-zero linear functional of V.*

If φ is not zero on W, then $\varphi_\star q = 0$.

If φ is zero on W and φ is not continuous with respect to the topology induced by q on V/W, then again $\varphi_\star q = 0$.

Finally, if φ is zero on W and continuous with respect to the topology on V/W, then

$$\varphi_\star q = \frac{1}{\|\varphi\|^2},$$

where $\|\varphi\|$ stands for the norm of φ as a bounded linear functional of the normed space V/W.

Proof. If $\varphi_{|W} \neq 0$, we can find w in W with $\langle \varphi, w\rangle = 1$ and hence $\varphi_\star q = q(w,w) = 0$ and we are done.

Else, we can assume $W = 0$ and q is a scalar product. If φ is not continuous with respect to the topology induced by q, there exists a sequence (u_n) in V with $q(u_n, u_n) = 1$ and $\langle \varphi, u_n\rangle \xrightarrow[n\to\infty]{} \infty$. We set $v_n = \frac{1}{\langle \varphi, u_n\rangle} u_n$ and we have $\langle \varphi, v_n\rangle = 1$ and $q(v_n, v_n) \xrightarrow[n\to\infty]{} 0$, hence $\varphi_\star q = 0$.

Finally, if φ is continuous, we can assume V to be complete with respect to q. Now, let u be the unique vector of V such that $\langle \varphi, v\rangle = q(u, v)$, $v \in V$, so that $\|\varphi\| = \|u\| = q(u,u)^{\frac{1}{2}}$. Any vector v in V with $\langle \varphi, v\rangle = 1$ may be written as $v = \frac{1}{q(u,u)} u + w$ with $q(u,w) = 0$. In particular, we then have $q(v,v) = \frac{1}{q(u,u)} + q(w,w) \geq \frac{1}{q(u,u)}$ and the result follows. □

The data of the numbers $\varphi_\star q$ allows to recover p.

Lemma A.11. *Let V be a real-vector space and q be a non-negative symmetric bilinear form on V. For any $v \neq 0$ on V, we have*

$$q(v,v) = \sup_{\substack{\varphi \in V^* \\ \langle \varphi, v\rangle = 1}} \varphi_* q.$$

Proof. The statement is a direct consequence of Cauchy-Schwarz inequality. Let us be more precise.

By construction, we have $q(v,v) \geq \sup_{\substack{\varphi \in V^* \\ \langle \varphi, v\rangle = 1}} \varphi_* q$ and we only need to prove the reverse inequality.

We fix v in V with $q(v,v) \neq 0$ (if $q(v,v) = 0$, the statement is evident). We consider the linear functional

$$\varphi : w \mapsto \frac{q(v,w)}{q(v,v)}$$

so that $\varphi(v) = 1$. Now, Cauchy-Schwarz inequality gives, for any w in V,

$$\varphi(w) \leq \frac{q(w,w)^{\frac{1}{2}}}{q(v,v)^{\frac{1}{2}}},$$

hence, if $\varphi(w) = 1$, $q(w,w) \geq q(v,v)$. Thus, by definition, we get $\varphi_\star q = q(v,v)$ and the result follows. □

Recall that, if V is a finite-dimensional real vector space, we denote by $\mathcal{Q}(V)$ the space of symmetric bilinear forms on V and by $\mathcal{Q}_+(V) \subset \mathcal{Q}(V)$ the set of non-negative ones. The set $\mathcal{Q}_+(V)$ comes with its natural topology as a closed subset of a finite-dimensional vector space. We have an evident semicontinuity property.

Lemma A.12. *Let V be a finite-dimensional vector space. For any $\varphi \neq 0$ in V^*, the function $q \mapsto \varphi_\star q$ is upper semicontinuous on $\mathcal{Q}_+(V)$, that is, we have, for q in $\mathcal{Q}_+(V)$,*

$$\varphi_\star q = \limsup_{p \to q} \varphi_\star p.$$

Proof. Indeed, for any v in V, the function $q \mapsto q(v,v)$ is continuous on $\mathcal{Q}(V)$, hence the function $q \mapsto \varphi_\star q$ is the infimum of a family of continuous functions. □

From this, we can deduce a continuity property:

Lemma A.13. *Let V be a finite-dimensional vector space and q be in $\mathcal{Q}_+(V)$. Assume (q_n) is a sequence in $\mathcal{Q}_+(V)$ such that $q_n \xrightarrow[n\to\infty]{} q$ with $q_n \geq q$, $n \geq 0$. Then, for any $\varphi \neq 0$ in V^*, we have $\varphi_\star q_n \xrightarrow[n\to\infty]{} \varphi_\star q$.*

Proof. This is a consequence of semicontinuity and concavity. Indeed, on one hand, by Lemma A.12, we have

$$\limsup \varphi_\star q_n \leq \varphi_\star q. \tag{A.1}$$

On the other hand, for any n, we set $p_n = q_n - q$, so that by assumption, the bilinear form p_n is non-negative and $p_n \xrightarrow[n\to\infty]{} 0$. In particular, again by Lemma A.12, we have $\varphi_\star p_n \xrightarrow[n\to\infty]{} 0$. Now, by Lemma A.5, for any n, we have

$$\varphi_\star q_n \geq \varphi_\star q + \varphi_\star p_n,$$

hence

$$\liminf \varphi_\star q_n \geq \varphi_\star q. \tag{A.2}$$

The result follows from (A.1) and (A.2). □

Next, we give a formula for $\varphi_\star q$.

Lemma A.14. *Let H be Hilbert space with scalar product p, u be a non-zero vector of H, T be a bounded non-negative self-adjoint operator of H and ν be the spectral measure of u with respect to T. Then if φ is the linear functional $v \mapsto p(u,v)$ and q is the bilinear form $(v,w) \mapsto p(Tv,w)$, we have*

$$\varphi_\star q = \left(\int_0^\infty \frac{\mathrm{d}\nu(t)}{t} \right)^{-1}.$$

In particular, we have $\varphi_\star q = 0$ if and only if $\int_0^\infty \frac{\mathrm{d}\nu(t)}{t} = \infty$.

Proof. Recall that ν is a compactly supported positive Radon measure on $[0,\infty)$. By the Spectral Theorem, we only need to prove the formula when $H = \mathrm{L}^2([0,\infty),\nu)$, u is the constant function $\mathbf{1}$ and T is the operator $f(t) \mapsto tf(t)$.

Note that if $\nu(0) > 0$, we have $q(\mathbf{1}_0, \mathbf{1}_0) = 0$ and $\varphi(\mathbf{1}_0) > 0$, hence $\varphi_* q = 0$ and the result holds. Therefore, we will now assume that we have $\nu(0) = 0$.

In this case, by definition, we have

$$(\varphi_\star q)^{-1} = \sup_{f \in H \smallsetminus \{0\}} \frac{(\int_0^\infty f \mathrm{d}\nu)^2}{\int_0^\infty t f(t)^2 \mathrm{d}\nu(t)}.$$

We let μ be the Radon measure with $\mathrm{d}\mu(t) = t\mathrm{d}\nu(t)$. The supremum above is finite if and only if the function $t \mapsto \frac{1}{t}$ belongs to $\mathrm{L}^2([0,\infty),\mu)$, that is, the function $t \mapsto \frac{1}{t}$ belongs to $\mathrm{L}^1([0,\infty),\mu)$. When this holds, the supremum is equal to $\int_0^\infty t^{-2}\mathrm{d}\mu(t) = \int_0^\infty t^{-1}\mathrm{d}\nu(t)$. □

From this formula, we can deduce a first case of Proposition A.8.

Corollary A.15. *Let H be Hilbert space with scalar product p and φ be a non-zero continuous linear functional on H. Then, if q is a continuous non-negative symmetric bilinear form on H, we have:*

$$\varphi_\star(\varepsilon p + q) \xrightarrow[\varepsilon \to 0]{} \varphi_\star q.$$

Proof. Let u be the vector in H which represents φ and T be the bounded self-adjoint operator on H which represents q. If ν is the spectral measure of u with respect to T, by Lemma A.14, for any $\varepsilon \geq 0$, we have

$$\varphi_\star(\varepsilon p + q) = \left(\int_0^\infty \frac{\mathrm{d}\nu(t)}{t + \varepsilon} \right)^{-1}.$$

The conclusion follows from the Monotone Convergence Theorem. □

We are now ready to conclude.

Proof of Proposition A.8. For any $\varepsilon > 0$, we set $r_\varepsilon = \pi_\star(\varepsilon p + q)$. As the family $(r_\varepsilon)_{\varepsilon>0}$ is an non-decreasing family of non-negative forms, it has a limit r as $\varepsilon \to 0$. We need to prove that $r = \pi_* q$. On one hand, by Lemma A.4 and Corollary A.15, for any non-zero linear functional φ on W, we have

$$\varphi_\star r_\varepsilon = (\varphi\pi)_\star(\varepsilon p + q) \xrightarrow[\varepsilon\to 0]{} (\varphi\pi)_\star q = \varphi_\star \pi_\star q.$$

On the other hand, as, for any $\varepsilon > 0$, $r_\varepsilon \geq r$, by Lemma A.13, we have

$$\varphi_\star r_\varepsilon \xrightarrow[\varepsilon\to 0]{} \varphi_\star r.$$

We get $\varphi_\star \pi_\star q = \varphi_\star r$ for any φ, hence $\pi_\star q = r$ by Lemma A.11. $\square$

A.3. **Derivative of the image map.** For a finite-dimensional vector space V, the map $q \mapsto \pi_* q$ is smooth on the space $\mathcal{Q}_{++}(V)$ of positive definite bilinear forms on V. We have even used an infinite-dimensional version of this result that we will now prove.

Fix a Hilbert space H with scalar product p. We recall that a continuous symmetric bilinear form q on H is said to be coercive if there exists $\varepsilon > 0$ with $q(v, v) \geq \varepsilon p(v, v)$, $v \in H$, or equivalently if q is positive definite and defines the same topology as p on H. We denote by $\mathcal{Q}^\infty_{++}(H)$ the space of coercive continuous bilinear symmetric forms on H, which is an open subset for the norm topology of the space $\mathcal{Q}^\infty(H)$ of continuous bilinear symmetric forms.

Proposition A.16. *Let H be a Hilbert space, W be a finite-dimensional vector space and $\pi : H \to W$ be a continuous surjective linear map. We denote by $\pi^{\dagger p} : W \to H$ the adjoint operator of π, that is, the linear map with $(\pi_\star p)(\pi(v), w) = p(v, \pi^{\dagger p}(w))$, $v \in H$, $w \in W$. Then the map $q \mapsto \pi_* q$, $\mathcal{Q}^\infty_{++}(H) \to \mathcal{Q}_{++}(W)$ is smooth. Its derivative at p is the linear map $q \mapsto (\pi^{\dagger p})^\star q$, $\mathcal{Q}^\infty(H) \to \mathcal{Q}(W)$.*

Proof. Let q be in $\mathcal{Q}^\infty_{++}(H)$. We will compute the adjoint $\pi^{\dagger q}$ of π with respect to the scalar product q on H. Let T be the unique self-adjoint bounded operator on H such that $q(v_1, v_2) = p(Tv_1, v_2)$ for v_1, v_2 in H. As q is coercive, T is invertible. Let $L \subset H$ be the kernel of π. The space $\pi^{\dagger p}(W)$ is the orthogonal complement $L^{\perp p}$ of L with respect to p. By construction, the orthogonal complement $L^{\perp q}$ of L with respect to q is $T^{-1}\pi^{\dagger p}(W)$. In particular, the endomorphism $\pi T^{-1}\pi^{\dagger p}$ of W is injective, hence bijective. We denote by U its inverse. We claim that we have $\pi^{\dagger q} = T^{-1}\pi^{\dagger p}U$. Indeed, on one hand, the range of the linear operator $T^{-1}\pi^{\dagger p}U$ is $T^{-1}\pi^{\dagger p}(W) = L^{\perp q}$ and on the other hand we have $\pi T^{-1}\pi^{\dagger p}U = \mathrm{Id}_W$. Therefore, we have $\pi^{\dagger q} = T^{-1}\pi^{\dagger p}U$ and, for w_1, w_2

in W,

$$\pi_* q(w_1, w_2) = \pi^{\dagger q}(w_1, w_2) = q(T^{-1}\pi^{\dagger p} U w_1, T^{-1}\pi^{\dagger p} U w_2) \\ = p(\pi^{\dagger p} U w_1, T^{-1}\pi^{\dagger p} U w_2). \tag{A.3}$$

As the inverse map is smooth on the space of invertible operators, $\pi_* q$ depends smoothly on q. Let us now compute its derivative $\mathrm{d}_p(\pi_*)$ at p. For this, we will derivate the quantity in (A.3) for T close to the identity operator.

We let $\mathcal{B}(H)$ denote the space of bounded operators on H. The derivative of the inverse map at the identity operator is $\Theta \mapsto -\Theta, \mathcal{B}(H) \to \mathcal{B}(H)$. Hence, by the chain rule, the derivative of the map $T \mapsto (\pi T^{-1} \pi^{\dagger p})^{-1}$ at the identity operator is the map $\Theta \mapsto \pi \Theta \pi^{\dagger p}, \mathcal{B}(H) \to \mathrm{End}(W)$. Let q be in $\mathcal{Q}^\infty(H)$ and Θ be the self-adjoint operator associated to q. We get, from (A.3), for w_1, w_2 in W,

$$\begin{aligned} &\mathrm{d}_p(\pi_*)(q) \\ &= p(\pi^{\dagger p}\pi\Theta\pi^{\dagger p} w_1, \pi^{\dagger p} w_2) - p(\pi^{\dagger p} w_1, \Theta \pi^{\dagger p} w_2) + p(\pi^{\dagger p} w_1, \pi^{\dagger p}\pi\Theta\pi^{\dagger p} w_2). \end{aligned}$$

Now we claim that the three numbers in the right hand-side of the latter are actually equal to each other. Indeed, by construction the operator $\pi^{\dagger p}\pi$ is the p-orthogonal projection on $\pi^\dagger(W)$, so that

$$\begin{aligned} p(\pi^{\dagger p}\pi\Theta\pi^{\dagger p} w_1, \pi^{\dagger p} w_2) = p(\Theta\pi^{\dagger p} w_1, \pi^{\dagger p} w_2) &= p(\pi^{\dagger p} w_1, \Theta\pi^{\dagger p} w_2) \\ &= p(\pi^{\dagger p} w_1, \pi^{\dagger p}\pi\Theta\pi^{\dagger p} w_2) \end{aligned}$$

and

$$\mathrm{d}_p(\pi_*)(q) = p(\pi^{\dagger p} w_1, \Theta\pi^{\dagger p} w_2) = q(\pi^{\dagger p} w_1, \pi^{\dagger p} w_2)$$

as required. □

Appendix B. Euclidean projective limits

The purpose of this appendix is to define and study the notion of a Euclidean projective limit that has been used throughout the article.

B.1. **Non-negative projective systems.** We first define non-negative projective systems and the associated Euclidean projective limits.

Definition B.1. A non-negative projective system is a sequence $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ where, for any integer $\ell \geq 0$, X_ℓ is a finite-dimensional real vector space, q_ℓ is a non-negative symmetric bilinear form on X_ℓ and $\pi_\ell : X_{\ell+1} \to X_\ell$ is a surjective map such that $(\pi_\ell)^\star q_\ell \leq q_{\ell+1}$ (that is, the bilinear symmetric form $q_{\ell+1} - (\pi_\ell)^\star q_\ell$ is non-negative).

Remark B.2. For any $\ell \geq 0$, let $Y_\ell \subset X_\ell$ be the null space of q_ℓ. Then we have $\pi_\ell Y_{\ell+1} \subset Y_\ell$. Set $\overline{X}_\ell = X_\ell / Y_\ell$ and let $\overline{\pi}_\ell : \overline{X}_{\ell+1} \to \overline{X}_\ell$ be the natural map and $\overline{q}_\ell$ be the bilinear form induced by q_ℓ on $\overline{X}_\ell$. Then the sequence $(\overline{X}_\ell, \overline{q}_\ell, \overline{\pi}_\ell)_{\ell \geq 0}$ is again a non-negative projective system and the forms $(\overline{q}_\ell)_{\ell \geq 0}$ are all positive definite. We shall use this construction repeatedly.

Recall that the algebraic projective limit X of the projective system (X_ℓ, π_ℓ) is defined as

$$X = \varprojlim_{\ell \geq 0} X_\ell = \left\{ x = (x_\ell)_{\ell \geq 0} \in \prod_{\ell \geq 0} X_\ell \,\middle|\, \forall \ell \geq 0 \quad x_\ell = \pi_\ell(x_{\ell+1}) \right\}.$$

To any non-negative projective system we can associate a natural Hilbert space.

Lemma B.3. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a non-negative projective system and let X be the algebraic projective limit of the projective system (X_ℓ, π_ℓ). We let $L \subset X$ be the set of those $x = (x_\ell)_{\ell \geq 0}$ in X such that*

$$\sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell) < \infty.$$

Then L is a vector subspace of X and there exists a unique non-negative symmetric bilinear form q on L with

$$q(x, x) = \sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell), \quad x \in L.$$

The space $H = L / \ker q$, equipped with the positive definite bilinear form induced by q is complete.

Definition B.4. If $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ is a non-negative projective system the Hilbert space H from Lemma B.3 is called the Euclidean projective limit of the system.

Remark B.5. Note that in general, there is no reason for H not to be reduced to 0. We will address this question in the next subsections.

Proof of Lemma B.3. By Minkowski inequality (that is, the triangle inequality for non-negative quadratic forms), for any x, y in L, we have, for $\ell \geq 0$,

$$q_\ell(x_\ell + y_\ell, x_\ell + y_\ell) \leq (q_\ell(x_\ell)^{\frac{1}{2}} + q_\ell(y_\ell)^{\frac{1}{2}})^2,$$

hence $x + y$ belongs to L. Now, we set

$$q(x, y) = \lim_{\ell \to \infty} \frac{1}{2}(q_\ell(x + y, x + y) - q_\ell(x, x) - q_\ell(y, y))$$

One checks that q is a symmetric bilinear form and that, for any x in L,

$$q(x,x) = \sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell).$$

In particular, q is non-negative.

It remains to prove that the space $H = L/\ker q$ is complete for the bilinear form induced by q, which we still denote by q. First, we assume that, for any $\ell \geq 0$, q_ℓ is positive definite. In this case, we have $\ker q = \{0\}$ and $H = L$. We need to prove that any absolutely convergent series in H is convergent. Let us pick a sequence (x_n) in H and assume it is absolutely convergent, that is,

$$\sum_n q(x_n, x_n)^{\frac{1}{2}} < \infty.$$

Set, for any n, $x_n = (x_{\ell,n})_{\ell \geq 0}$. We have, for $\ell \geq 0$,

$$\sum_n q_\ell(x_{\ell,n}, x_{\ell,n})^{\frac{1}{2}} < \infty.$$

Hence the series $\sum_n x_{\ell,n}$ converges in the finite-dimensional Hilbert space (X_ℓ, q_ℓ) towards an element x_ℓ. By uniqueness of the limit, we have $\pi_\ell(x_{\ell+1}) = x_\ell$. Therefore, the element $x = (x_\ell)_{\ell \geq 0}$ in $\prod_{\ell \geq 0} X_\ell$ actually belongs to the algebraic projective limit X. Now, for any ℓ, we have

$$q_\ell(x_\ell, x_\ell) \leq \left(\sum_n q_\ell(x_{\ell,n}, x_{\ell,n})^{\frac{1}{2}} \right)^2 \leq \left(\sum_n q(x_n, x_n)^{\frac{1}{2}} \right)^2,$$

hence x belongs to H. In the same way, one checks that $\sum_n x_n = x$ in H.

In the general case, we let $(\overline{X}_\ell, \overline{q}_\ell, \overline{\pi}_\ell)_{\ell \geq 0}$ be as in Remark B.2 above, that is $\overline{X}_\ell$ is the quotient of X_ℓ by the null space of q_ℓ, $\overline{q}_\ell$ is the induced symmetric bilinear form on $\overline{X}_\ell$ and $\overline{\pi}_\ell : \overline{X}_{\ell+1} \to \overline{X}_\ell$ is the natural map. Then the algebraic projective limit $\overline{X}$ of the projective system $(\overline{X}_\ell, \overline{q}_\ell)_{\ell \geq 0}$ is exactly the quotient of X by the space

$$\ker q = \{x = (x_\ell)_{\ell \geq 0} \in X | \forall \ell \geq 0 \quad x_\ell \in \ker q_\ell\}.$$

We are brought back to the case where all the q_ℓ are positive definite. □

B.2. **Straight systems.** We now would like to have conditions for the Euclidean projective limit to be large. To this aim, we introduce a new notion.

Definition B.6. Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a non-negative projective system. We shall say that the system is straight if, for every $\ell \geq 0$, we have $(\pi_\ell)_\star q_{\ell+1} = q_\ell$, that is, q_ℓ is the Euclidean image of $q_{\ell+1}$.

Straight systems have good Euclidean projective limits.

Lemma B.7. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a straight non-negative projective system and let H be its Euclidean projective limit. Then, for any $\ell \geq 0$, the natural map $\rho_\ell : H \to X_\ell / \ker q_\ell$ is onto and one has $(\rho_\ell)_\star q = q_\ell$.*

A subspace $L \subset H$ is dense in H if and only if, for any $\ell \geq 0$, $\rho_\ell(L) = X_\ell / \ker q_\ell$ and $(\rho_\ell)_\star q_{|L} = q_\ell$.

Proof. As noticed in Remark B.2, we can assume that, for any $\ell \geq 0$, the symmetric bilinear form q_ℓ is positive definite. Set $W_0 = X_0$ and $p_0 = q_0$ and, for $\ell \geq 1$, set $W_\ell = \ker \pi_{\ell-1}$ and let p_ℓ be the restriction of q_ℓ to W_ℓ. The system being straight, we have an isomorphism $X_\ell \to W_0 \oplus \cdots W_\ell$ which sends q_ℓ to $p_0 + \cdots + p_\ell$ and which identifies π_ℓ with the natural map $W_0 \oplus \cdots W_{\ell+1} \to W_0 \oplus \cdots W_\ell$. Now we see that H may be defined as Hilbertian direct sum of the Euclidean spaces $(W_\ell, p_\ell)_{\ell \geq 0}$. In particular, the first part of the lemma follows easily.

Let now L be a closed subspace of H such that, for any $\ell \geq 0$, $\rho_\ell(L) = X_\ell$ and $(\rho_\ell)_\star q_{|L} = q_\ell$ and let us prove that $L = H$. Indeed, we can identify ρ_ℓ with the orthogonal projection $H \to X_\ell = W_0 \oplus \cdots W_\ell$. Now, as $(\rho_\ell)_\star q_{|L} = q_\ell$, there exists a closed subspace Y_ℓ of L such that ρ_ℓ is an isometry from Y_ℓ onto X_ℓ. But as ρ_ℓ is an orthogonal projection of H, this implies that $Y_\ell = X_\ell$, hence $X_\ell \subset L$. As this is true, for any ℓ, we get $L = H$ as required. $\square$

B.3. **Straightenable systems.** We shall now see how can build a straight system from one that is not.

For $k \geq \ell$, let us write $\pi_{k,\ell}$ for the natural product map

$$\pi_{k,\ell} = \pi_{k-1} \cdots \pi_\ell : X_k \to X_\ell.$$

If the system is not straight, we can try to straighten it. There is a natural formula for doing so.

Lemma B.8. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a non-negative projective system. Assume that, for any ℓ and any x in X_ℓ, one has*

$$\Phi_\ell(x) = \sup_{k \geq \ell} (\pi_{k,\ell})_\star q_k(x, x) = \sup_{k \geq \ell} \inf_{\substack{y \in X_k \\ \pi_{k,\ell}(y) = x}} q_k(y, y) < \infty.$$

Then, for any ℓ, Φ_ℓ is a quadratic form on X_ℓ. Let p_ℓ be its polar form. The family $(X_\ell, p_\ell, \pi_\ell)_{\ell \geq 0}$ is a straight non-negative projective system.

Definition B.9. Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a non-negative projective system. We say that it is straightenable if, for any ℓ and any x in X_ℓ, we have

$$\sup_{k\geq \ell}(\pi_{k,\ell})_\star q_k(x,x) < \infty.$$

In this case, the straight non-negative projective system $(X_\ell, p_\ell, \pi_\ell)_{\ell\geq 0}$ from Lemma B.8 is called the straightening of $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$.

Proof of Lemma B.8. Fix $\ell \geq 0$. For any $k \geq \ell$, let Φ_ℓ^k denote the quadratic form $x \mapsto (\pi_{k,\ell})_\star q_k(x,x)$ on X_ℓ. If $k > \ell$, since $q_k \geq (\pi_{k-1})^\star q_{k-1}$, we have, for any x in X_ℓ

$$\Phi_\ell^k(x) \geq \inf_{\substack{y\in X_k \\ \pi_{k,\ell}(y)=x}} q_{k-1}(\pi_k(y), \pi_k(y)) = \Phi_\ell^{k-1}(x),$$

where the latter inequality holds because of the surjectivity of π_{k-1}. In particular, we have

$$\Phi_\ell^k(x) \xrightarrow[k\to\infty]{} \Phi_\ell(x)$$

and Φ_ℓ is a quadratic form. As in the statement, we let p_ℓ denote its polar form. Since $\Phi_\ell \geq \Phi_\ell^\ell$ and Φ_ℓ^ℓ is the quadratic form associated to q_ℓ, Φ_ℓ is non-negative. Finally, for any x in $X_{\ell+1}$ and any $k \geq \ell + 1$, we have

$$\{y \in X_k | \pi_{k,\ell+1}(y) = x\} \subset \{y \in X_k | \pi_{k,\ell}(y) = \pi_\ell(y)\},$$

hence $\Phi_{\ell+1} \geq (\pi_\ell)^\star \Phi_\ell$ and the family $(X_\ell, p_\ell, \pi_\ell)_{\ell\geq 0}$ is a non-negative projective system. It remains to prove that this system is straight, which is the main difficulty of the proof.

To this aim, we need to introduce more notation. For any ℓ, let $W_\ell \subset X_\ell$ be the null space of Φ_ℓ. As Φ_ℓ is the non-decreasing limit of the Φ_ℓ^k, $k \geq \ell$, there exists a smallest $k \geq \ell$ such that W_ℓ is the null space of Φ_ℓ^k. We denote it by $j(\ell)$. We also set $V_\ell = X_\ell/W_\ell$.

Now, fix $\ell \geq 0$ and x in X_ℓ. We claim that there exists y in $X_{\ell+1}$ such that $\pi_\ell(y) = x$ and, for any $k \geq \ell + 1$, $\Phi_{\ell+1}^k(y) \leq \Phi_\ell(y)$, which finishes the proof

Indeed, by Lemma A.4, for $k \geq \ell+1$, we have $(\pi_\ell)_\star \Phi_{\ell+1}^k = \Phi_\ell^k \leq \Phi_\ell$. As $X_{\ell+1}$ is finite-dimensional, this implies that the set

$$A_k = \{y \in X_{\ell+1} | \pi_\ell(y) = x \text{ and } \Phi_{\ell+1}^k(y) \leq \Phi_\ell(x)\}$$

is not empty. Note that one has $A_{k+1} \subset A_k$. We let B_k be the image of A_k in $V_{\ell+1}$. Then, if $k \geq j(\ell + 1)$, B_k is a compact subset of V_k. As this sequence is non-increasing, we have $\bigcap_{k\geq j(\ell+1)} B_k \neq \emptyset$ and we are done. $\square$

The straightened system and the original one have the same Euclidean projective limit:

Lemma B.10. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a straightenable non-negative projective system and let $(X_\ell, p_\ell, \pi_\ell)_{\ell\geq 0}$ be its straightening. For any $x = (x_\ell)_{\ell\geq 0}$ in the algebraic projective limit, we have*

$$\sup_{\ell\geq 0} p_\ell(x_\ell, x_\ell) = \sup_{\ell\geq 0} q_\ell(x_\ell, x_\ell).$$

In particular, both systems have the same Euclidean projective limit.

Proof. On one-hand, we have, for any $\ell \geq 0$, $q_\ell \leq p_\ell$, hence

$$\sup_{\ell\geq 0} q_\ell(x_\ell, x_\ell) \leq \sup_{\ell\geq 0} p_\ell(x_\ell, x_\ell).$$

On the other hand, for any $k \geq \ell \geq 0$, we have $\pi_{k,\ell}(x_k) = x_\ell$, hence

$$p_\ell(x_\ell, x_\ell) = \sup_{k\geq \ell}(\pi_{k,\ell})_\star q_k(x, x) = \sup_{k\geq\ell} \inf_{\substack{y\in X_k \\ \pi_{k,\ell}(y)=x_\ell}} q_k(y, y) \leq \sup_{k\geq\ell} q_k(x_k, x_k).$$

□

The notion of a straightenable system allows us to characterize the case where the Euclidean projective limit is large enough.

Proposition B.11. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a non-negative projective system with Euclidean projective limit H. The following are equivalent:*
(i) *The system is straightenable.*
(ii) *For any $\ell \geq 0$, the natural map $H \to X_\ell/\ker q_\ell$ is onto.*
(iii) *There exists a Hilbert space K and a family $(\theta_\ell)_{\ell\geq 0}$ where, for any $\ell \geq 0$, θ_ℓ is a surjective continuous linear map $K \to X_\ell/\ker q_\ell$ with $\theta_\ell = \pi_\ell\theta_{\ell+1}$ and $q_\ell(\theta_\ell(v), \theta_\ell(v)) \leq \|v\|^2$ for any v in K.*

Proof. (i)⇒(ii) This follows from Lemmas B.7, B.8 and B.10.

(ii)⇒(iii) This is evident by taking $H = K$.

(iii)⇒(i) Let $\ell \geq 0$ and v be in K. For any $k \geq \ell$, we have $\pi_{k,\ell}(\theta_k(v)) = \theta_\ell(v)$. Hence,

$$\sup_{k\geq\ell} \inf_{\substack{y\in X_k \\ \pi_{k,\ell}(y)=\theta_\ell(v)}} q_k(y, y) \leq \|v\|^2.$$

As the maps θ_ℓ are surjective, the system is straightenable. □

Appendix C. Quadratic duality

We recall here some basic facts about the duality between the spaces of symmetric bilinear forms on a vector space and on its dual space.

C.1. **Definition and elementary properties.** Let V be a finite-dimensional vector space. As usual, we let V^* denote its dual space and $\mathcal{Q}(V)$ denote the space of symmetric bilinear forms on V.

For φ, ψ in V^*, we let $\varphi\psi \in \mathcal{Q}(V)$ denote the bilinear form

$$(v, w) \mapsto \frac{1}{2}(\varphi(v)\psi(w) + \psi(v)\varphi(w))$$

on V. If $\varphi = \psi$, we write φ^2 for $\varphi\varphi$. In the same way, for v, w in V, we let vw denote the bilinear form

$$(\varphi, \psi) \mapsto \frac{1}{2}(\varphi(v)\psi(w) + \psi(v)\varphi(w))$$

on V^* and, when $v = w$, we write v^2 for vv. In the formalism of multilinear algebra, the map $\varphi \mapsto \varphi^2$ (resp. $v \mapsto v^2$) defines an isomorphism between the spaces $\mathrm{S}^2 V^*$ (resp. $\mathrm{S}^2 V$) and $\mathcal{Q}(V)$ (resp. $\mathcal{Q}(V^*)$).

Any p in $\mathcal{Q}(V)$ defines a linear map $\theta_p : V \to V^*$ such that, for v, w in V, one has $p(v, w) = \langle \theta_p v, w \rangle$. The fact that p is symmetric translates into saying that θ_p is equal to its adjoint operator, that is, $\theta_p = \theta_p^*$ (when V is identified with the dual space of V^*!)

For any p in $\mathcal{Q}(V)$ and q in $\mathcal{Q}(V^*)$, we set

$$\langle p, q \rangle = \mathrm{tr}(\theta_p \theta_q) = \mathrm{tr}(\theta_q \theta_p).$$

Lemma C.1. *Let V be a finite-dimensional vector space. For any p in $\mathcal{Q}(V)$ and v in V, we have $\langle p, v^2 \rangle = p(v, v)$. In particular, the pairing $\langle ., . \rangle$ between $\mathcal{Q}(V)$ and $\mathcal{Q}(V^*)$ is non-degenerate. For any φ in V^* and v in V, one has $\langle \varphi^2, v^2 \rangle = \varphi(v)^2$ and this property uniquely determines the pairing $\langle ., . \rangle$.*

The pairing $\langle ., . \rangle$ is called the quadratic duality in this paper.

Proof. For v in V and $q = v^2$, the linear map $\theta_q : V^* \to V$ reads as $\varphi \mapsto \varphi(v)v$. Thus, for p in $\mathcal{Q}(V)$, $\theta_p \theta_q$ is the endomorphism $w \mapsto p(v, w)v$ of V whose trace is $p(v, v)$. Non-degeneracy follows. The uniqueness property comes from the fact that the φ^2, $\varphi \in V^*$, span $\mathcal{Q}(V)$ as a vector space. $\square$

The quadratic duality behaves well under linear maps.

Lemma C.2. *Let V and W be finite-dimensional real vector spaces and $T : V \to W$ be a linear map with adjoint linear map $T^* : W^* \to V^*$. If p is a symmetric bilinear form on W and q is a symmetric bilinear form on V^*, we have*

$$\langle T^\star p, q \rangle = \langle p, (T^*)^\star q \rangle.$$

Proof. Indeed, one has $\theta_{T^\star p} = T^*\theta_p T$ and $\theta_{(T^*)^\star q} = T\theta_q T^*$, hence

$$\langle T^\star p, q\rangle = \operatorname{tr}(T^*\theta_p T\theta_q) = \operatorname{tr}(\theta_p T\theta_q T^*) = \langle p, (T^*)^\star q\rangle.$$

□

C.2. **A Euclidean formula.** Let V be a finite-dimensional vector space. We will see how the formalism of the quadratic duality allows to describe the classical $\mathrm{GL}(V)$-invariant Riemannian metric on the set $\mathcal{Q}_{++}(V)$ of positive definite symmetric bilinear forms on V.

If p is in $\mathcal{Q}_{++}(V)$, the map $\theta_p : V \to V^*$ is a linear isomorphism. We define the dual form $\delta_V(p)$ of p as the positive symmetric bilinear form $(\theta_p^{-1})^\star p$ on V^*. The map $\delta_V : \mathcal{Q}_{++}(V) \to \mathcal{Q}_{++}(V^*)$ is a smooth diffeomorphism and we can compute its derivative.

Lemma C.3. *Let V be a finite-dimensional vector space, p be a scalar product on V and q be in $\mathcal{Q}(V)$. We have*

$$\mathrm{d}_p\delta_V(q) = -(\theta_p^{-1})^\star q.$$

For p, q as in the setting, there exists a unique p-symmetric endomorphism A of V with $q(v, w) = p(Av, w)$ for v, w in V. We say that A is the endomorphism which represents q with respect to p.

Proof. Let first q be in $\mathcal{Q}_{++}(V)$ and A be the p-symmetric endomorphism which represents q with respect to p. For any φ, ψ in V^*, let $v = \theta_p^{-1}\varphi$ and $w = \theta_p^{-1}\psi$ be the vectors such that $\varphi(u) = p(u, v)$ and $\psi(u) = p(u, w)$ for u in V. We have, by definition, $\theta_q = \theta_p A$, hence

$$\delta_V(q)(\varphi, \psi) = q(A^{-1}v, A^{-1}w) = p(v, A^{-1}w).$$

Therefore, for q in $\mathcal{Q}(V)$, $\mathrm{d}_p\delta_V(q)(\varphi, \psi) = -p(v, Aw)$, which we may write as $\mathrm{d}_p\delta_V(q) = -(\theta_p^{-1})^\star q$. □

The derivative of δ_V allows to define the natural Riemannian metric of $\mathcal{Q}_{++}(V)$ (see [21, Chapter VI]).

Lemma C.4. *Let V be a finite-dimensional vector space, p be a scalar product on V and q, r be in $\mathcal{Q}(V)$. We have*

$$-\langle q, \mathrm{d}_p\delta_V(r)\rangle = \operatorname{tr}(AB),$$

where A and B are the p-symmetric endomorphisms of V which represent q and r with respect to p.

Proof. We have $\theta_q = \theta_p A$ and $\theta_r = \theta_p B$. By Lemma C.3, we get $\theta_{\mathrm{d}_p\delta_V(r)} = -B\theta_p^{-1}$, hence, by definition, $\langle q, \mathrm{d}_p\delta_V(r)\rangle = -\operatorname{tr}(\theta_p AB\theta_p^{-1}) = -\operatorname{tr}(AB)$. □

C.3. **A formula for finite sets.** We give a formula for the quadratic duality which was used in the proof of the weight formula in Subsection 11.2.

Let A be a finite set and V be the space of real-valued functions on A. We identify V with its dual space through the bilinear form $(f, g) \mapsto \sum_{a \in A} f(a)g(a)$. As usual, we set $\overline{V}$ to be the quotient space of V by the line of constant functions and we identify the dual space of V with the space $V_0 = \{f \in V | \sum_a f(a) = 0\}$.

If p and q are symmetric bilinear forms on $\overline{V}$ and V_0, we set, for a, b in A,

$$\begin{aligned} \varphi_p(a, b) &= -p(\mathbf{1}_a, \mathbf{1}_b) \\ K_q(a, b) &= q(\mathbf{1}_a - \mathbf{1}_b, \mathbf{1}_a - \mathbf{1}_b), \end{aligned}$$

where by abuse of notation, we write $\mathbf{1}_a, \mathbf{1}_b$ for their images in $\overline{V}$.

Lemma C.5. *Let p and q be symmetric bilinear forms on $\overline{V}$ and V_0. We have*

$$\langle p, q \rangle = \frac{1}{2} \sum_{(a,b) \in A^2} \varphi_p(a, b) K_q(a, b).$$

Proof. Let $T : V \to \overline{V}$ be the natural quotient map, so that T^* is the inclusion $V_0 \hookrightarrow V$. We define a symmetric bilinear form $\hat{q}$ on V by setting, for f, g in V,

$$\hat{q}(f, g) = -\frac{1}{2} \sum_{(a,b) \in A^2} K_q(a, b) f(a) g(b).$$

Then the restriction of $\hat{q}$ to V_0 is q, that is, $(T^*)^\star \hat{q} = q$. Thus, by Lemma C.2, we have

$$\langle p, q \rangle = \langle p, (T^*)^\star \hat{q} \rangle = \langle T^\star p, \hat{q} \rangle.$$

Through the identification between V and its dual space, the basis $(\mathbf{1}_a)_{a \in A}$ is equal to its dual basis, so that, by definition, we have

$$\langle T^\star p, q \rangle = \sum_{(a,b) \in A^2} p(T\mathbf{1}_a, T\mathbf{1}_b) \hat{q}(\mathbf{1}_a, \mathbf{1}_b)$$

and the result follows. □

Appendix D. Haagerup inequality

In the course of the article, we have used Haagerup inequality from [18] to ensure that certain convolution operators were bounded. In this appendix, we show precisely how to adapt the original statement and proof by Haagerup in order to have them fitting in our framework. This adaptation could also be seen as following from [19] and [23].

We keep the notation of the article. In particular, X is a tree and Γ is a discrete group of automorphisms of X such that $\Gamma\backslash X$ is finite.

D.1. **Norms of convolutors.** The original Haagerup inequality dominates the norm of the convolution operator on $\ell^2(\Gamma)$ associated to a function f on $\ell^2(\Gamma)$ by a weighted ℓ^2-norm of f. As, in our case, the action of Γ on X is not necessarily transitive, we use convolution operators by Γ-invariant functions on X_*. We shall first define precisely which kind of norms we will use on the space of such functions.

For φ a Γ-invariant function on X^2, we set

$$(\|\varphi\|_2^\Gamma)^2 = \sum_{(x,y)\in\Gamma\backslash X^2} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi(x,y)^2.$$

Lemma D.1. *There exists $C > 0$ such that, for any Γ invariant function on φ on X_*, we have*

$$\frac{1}{C}\|\varphi\|_2^\Gamma \leq \sup_{x\in X}\|\varphi(x,.)\|_2 \leq C\|\varphi\|_2^\Gamma,$$

where, for x in X, $\|\varphi(x,.)\|_2$ is the norm of the function $y \mapsto \varphi(x,y)$ in $\ell^2(X)$ if this function belongs to this space and ∞ else.

Proof. Lemma 9.11 gives

$$(\|\varphi\|_2^\Gamma)^2 = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\|\varphi(x,.)\|_2^2.$$

The conclusion follows from the fact that $\Gamma\backslash X$ is finite. □

D.2. **Bounded convolution operators.** We recall that X_* stands for the set of (x,y) in X^2 with $x \neq y$ and that, for $k \geq 1$, X_k stands for the set of (x,y) with $d(x,y) = k$. As above, we write $\ell^2_-(X_1)$ for the space of square-integrable skew-symmetric functions on X_1. For a Γ-invariant function φ on X_*, we defined in (10.1) the associated convolution operator P_φ by the following formula: if ψ is a finitely supported skew-symmetric function ψ on X_1, for (x,y) in X_1,

$$P_\varphi\psi(x,y) = \sum_{\substack{(a,b)\in X_1\\ y,b\in[xa]}} \varphi(x,a)\psi(b,a) - \sum_{\substack{(a,b)\in X_1\\ x,b\in[ya]}} \varphi(y,a)\psi(b,a) - \frac{1}{2}(\varphi(x,y)+\varphi(y,x))\psi(x,y).$$

Haagerup inequality states as

Proposition D.2. *Let φ be a Γ-invariant function φ on X_* such that*

$$\sum_{(x,y)\in\Gamma\backslash X_*} \varphi(x,y)^2 d(x,y)^\alpha < \infty$$

for some $\alpha > 2$. Then the convolution operator P_φ is bounded in $\ell^2_-(X_1)$.

We will prove this by following the same lines as in [18]. First, we translate [18, Lemma 1.3] which is the key observation of the proof. We fix a point o in X that will play the role of an origin and, for any integer $k \geq 1$, we set

$$Y_k = \{(x,y) \in X_1 | \max(d(o,x), d(o,y)) = k\}.$$

We get

Lemma D.3. *There exists $C > 0$ such that the following holds. Let $j,k,\ell \geq 1$ and φ be a Γ-invariant function on X_*, with support on X_j and ψ be a skew-symmetric function on X_1 with support on Y_k. If $j \leq k+\ell$, $k \leq \ell + j$ and $\ell \leq j+k$, we have*

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2 \leq C \|\varphi\|_2^\Gamma \|\psi\|_2 .$$

In all other cases, we have $P_\varphi\psi = 0$ on Y_ℓ.

Proof. If $j = 1$, P_φ is just the multiplication operator by the function $(x,y) \mapsto \frac{1}{2}(\varphi(x,y) + \varphi(y,x))$ on $\ell^2_-(X_1)$. The required inequality follows since all the norms on the finite-dimensional space of Γ-invariant functions on X_1 are equivalent.

Therefore, we assume $j \geq 2$. In that case, for any function ψ in $\ell^2_-(X_1)$ and any $x \sim y$ in X, we have

$$P_\varphi\psi(x,y) = \sum_{\substack{a\in S^j(x)\\ y\in[xa]}} \varphi(x,a)\psi(a_1,a) - \sum_{\substack{b\in S^j(y)\\ x\in[yb]}} \varphi(y,b)\psi(b_1,b),$$

where, for a, b as above a_1, b_1 are their neighbour which are closest to $[xy]$. For x in X with $d(x,o) = x \neq o$, we let x_- denote its neighbour on $[ox]$. We must dominate the quantity

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 = 2 \sum_{d(x,o)=\ell} (P_\varphi\psi(x_-,x))^2.$$

Now, recall that φ has support in X_j and ψ has support in Y_k. Hence if, for some x with $d(x,o) = \ell$, we have $P_\psi\varphi(x_-,x) \neq 0$, then three possibilities can occur:

– *(i)* there exists a in X with $d(x,a) = j$, $d(a,o) = k$, $x_- \in [ax]$ and $a \notin [ox]$. In that case, we have, by the triangle identity, $j \leq k+\ell$, $k \leq \ell + j$, and $\ell \leq j+k$. Let y be the point such that $[ao] \cap [xo] = [yo]$

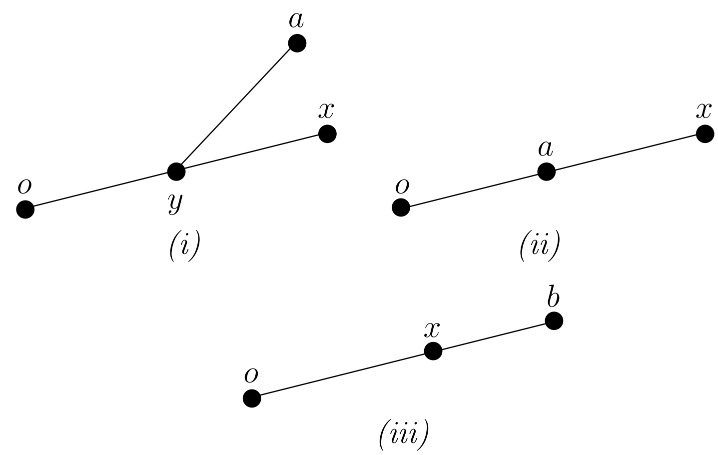

FIGURE 6. The three cases in the proof of Lemma D.3

and $i = d(y, o)$. As x_- belongs to $[ax]$, we have $y \neq x$ hence $i \leq \ell - 1$. Besides, we have

$$j = d(a, x) = d(a, y) + d(y, x) = (k - i) + (\ell - i) = k + \ell - 2i,$$

so that $2i = k + \ell - j$ and this number must be even. Also, as $i \leq \ell - 1$, we have $k \leq j + \ell - 2$ and, as $a \notin [ox]$, we have $i \leq k - 1$, hence $\ell \leq j + k - 2$.
– *(ii)* there exists a in $[ox_-]$ with $d(x, a) = j$ and $d(a, o) = k - 1$. In that case, we have $\ell = d(x, o) = j + k - 1$.
– *(iii)* there exists b in X with $d(x, b) = j - 1$, $d(b, o) = k$ and $x_- \notin [xb]$. In that case, as x_- is not in $[xb]$, we have $x \in [ob]$ hence $k = \ell + j - 1$. The three cases are pictured on Figure 6.

Note in particular that the inequalities over j, k, ℓ imply that no two of those three cases can happen simultaneanously.

We now prove the inequality in case *(i)*, that is, we assume that we have $j \leq k + \ell$, $k \leq \ell + j - 2$, $\ell \leq j + k - 2$ and that $j + k + \ell$ is even and we set $i = \frac{1}{2}(k + \ell - j)$. The reasoning above, gives
(D.1)

$$\|(P_\varphi \psi)\mathbf{1}_{Y_\ell}\|_2^2 = 2 \sum_{d(y,o)=i} \sum_{\substack{d(x,y)=\ell-i \\ y \in [xo]}} \left(\sum_{\substack{d(a,y)=k-i \\ [ao] \cap [xo] = [yo]}} \varphi(x, a)\psi(a_-, a) \right)^2 .$$

We define a new Γ-invariant function φ' on X_*. For any (x,y) in X_*, if $d(x,y)=\ell-i$, we set

$$\varphi'(x,y)=\left(\sum_{\substack{d(x,z)=j\\ y\in[xz]}}\psi(x,z)^2\right)^{\frac{1}{2}}.$$

Else, we set $\varphi'(x,y)=0$. By (D.1) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 &\leq 2\sum_{d(y,o)=i}\sum_{\substack{d(a,o)=k\\ y\in[ao]}}\psi(a_-,a)^2\sum_{\substack{d(x,o)=\ell\\ y\in[xo]}}\varphi'(x,y)^2\\ &\leq \sup_{y\in X}\|\varphi'(.,y)\|_2^2\,\|\psi\|_2^2\,.\end{aligned}$$

Now, let C be as in Lemma D.1, as φ' is Γ-invariant, we have

$$\sup_{y\in X}\|\varphi'(.,y)\|_2\leq C^2\sup_{x\in X}\|\varphi'(x,.)\|_2$$

and by the definition of φ',

$$\sup_{x\in X}\|\varphi'(x,.)\|_2=\sup_{x\in X}\|\varphi(x,.)\|_2\leq C\,\|\varphi\|_2^\Gamma\,.$$

The result follows.

In case *(ii)*, we have $\ell=j+k-1$ and we can write

$$\begin{aligned}\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 &= 2\sum_{d(y,o)=k}\sum_{\substack{d(x,y)=j-1\\ y\in[xo]}}\varphi(x,y_-)^2\psi(y,y_-)^2\\ &\leq \sup_{y\in X}\|\varphi(.,y)\|_2^2\,\|\psi\|_2^2\,,\end{aligned}$$

and we conclude again by Lemma D.1.

Finally, in case *(iii)*, we have $k=\ell+j-1$ and

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2=2\sum_{d(x,o)=\ell}\left(\sum_{\substack{d(b,o)=k\\ x\in[bo]}}\varphi(x_-,b)\psi(b_-,b)\right)^2.$$

As in case *(i)*, we define a Γ-invariant function φ' on X_1 by setting, for any (u,v) in X_1,

$$\varphi'(u,v) = \left(\sum_{\substack{d(u,w)=j \\ y\in[xz]}} \psi(u,w)^2 \right)^{\frac{1}{2}},$$

so that Cauchy-Schwarz inequality gives

$$\|(P_\varphi \psi)\mathbf{1}_{Y_\ell}\|_2 \leq \sup_{(u,v)\in X_1} \varphi'(u,v) \, \|\psi\|_2 \, ,$$

and we conclude as above. □

From Lemma D.3, we easily deduce Proposition D.2 as in [18, Lemma 1.4, Lemma 1.5].

Proof of Proposition D.2. Pick a Γ-invariant function φ on X_*. Let us for the moment fix $j \geq 1$ and set $\varphi_j = \varphi \mathbf{1}_{X_j}$. Let ψ be a finitely suported skew-symmetric function on X_1. For $k \geq 1$, we write $\psi_k = \psi \mathbf{1}_{Y_k}$, so that $\psi = \sum_{k\geq 1} \psi_k$. By Lemma D.3, we can find $C > 0$ such that, for any such ψ, for $\ell \geq 1$, one has

$$\left\|P_{\varphi_j}\psi \mathbf{1}_{Y_\ell}\right\|_2 \leq \sum_{k\geq 1} \left\|(P_{\varphi_j}\psi_k)\mathbf{1}_{Y_\ell}\right\|_2 \leq C \, \|\varphi_j\|_2^\Gamma \sum_{k=|j-\ell|}^{j+\ell} \|\psi_k\|_2 \, .$$

Indeed, for any $k \notin [|j-\ell|, j+\ell]$, Lemma D.3 says that $(P_{\varphi_j}\psi_k)\mathbf{1}_{Y_\ell} = 0$. We can dominate the norm of $P_{\varphi_j}\psi$ by

$$\begin{aligned}
\left\|P_{\varphi_j}\psi\right\|_2^2 &= \sum_{\ell\geq 1} \left\|P_{\varphi_j}\psi \mathbf{1}_{Y_\ell}\right\|_2^2 \\
&\leq C^2 (\|\varphi_j\|_2^\Gamma)^2 \sum_{\ell\geq 1} \left(\sum_{k=|j-\ell|}^{j+\ell} \|\psi_k\|_2 \right)^2 \\
&\leq C^2 (\|\varphi_j\|_2^\Gamma)^2 \sum_{\ell\geq 1} (2\min(j,\ell)+1) \sum_{k=|j-\ell|}^{j+\ell} \|\psi_k\|_2^2 \\
&\leq C^2 3j (\|\varphi_j\|_2^\Gamma)^2 \sum_{k\geq 1} (2\min(j,k)+1) \, \|\psi_k\|_2^2 \\
&\leq C^2 9j^2 (\|\varphi_j\|_2^\Gamma)^2 \, \|\psi\|_2^2 \, ,
\end{aligned}$$

where we have used Cauchy-Schwarz inequality.

Now, we have

$$\|P_\varphi \psi\|_2 \leq \sum_{j\geq 1} \left\|P_{\varphi_j}\psi\right\|_2 \leq 3C\,\|\psi\|_2 \sum_{j\geq 1} j\,\|\varphi_j\|_2^\Gamma .$$

Fix $\alpha > 2$ and set $C' = \sum_{j\geq 1} j^{1-\alpha}$. For any sequence $(x_j)_{j\geq 1}$ of non-negative real numbers, Cauchy-Schwarz inequality gives $(\sum_{j\geq 1} jx_j)^2 \leq C' \sum_{j\geq 1} j^\alpha x_j^2$. Thus, we get

$$\|P_\varphi \psi\|_2^2 \leq 9C^2 C'\,\|\psi\|_2^2 \sum_{j\geq 1} j^\alpha (\|\varphi_j\|_2^\Gamma)^2$$

and we are done. □

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