Additive representations of tree lattices

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CHAPTER 1

Introduction

1.1. Motivations

This book grew out of a will to understand spectral gap problems. I was in particular fascinated by two conjectures.

The first one is about dense subgroups of a compact simple Lie group G. Given a dense subgroup Γ of G, it is believed that Γ has no almost invariant vectors for the left action ρ of G in the space $\mathrm{L}^2_0(G)$ of square integrable functions on G with zero integral with respect to the Haar measure. This is known to hold in many cases due to the work of Bourgain and Gamburd [**11, 12**] and Benoist and de Saxcé [**9**]. This amounts to saying that, if $g_1, \ldots, g_r$ are elements of G which span a dense subgroup, the self-adjoint operator $Q = \frac{1}{2r}\sum_{i=1}^r (\rho(g_i) + \rho(g_i)^{-1})$ has norm < 1 in the space $\mathrm{L}^2_0(G)$.

The second one is related to a construction of Bourgain [**10**], which we extended with Benoist [**8**]. A general question which follows from this construction is the following: given a simple Lie group G and a unitary irreducible representation ρ of G in a Hilbert space H, is there a bound for the essential spectral radii of the operators Q as above, when $g_1, \ldots, g_r$ span a dense subgroup of G?

Motivated by these problems, in order to have a better understanding of the obstructions to spectral gap, I tried to build examples of unitary operators $U_1, \ldots, U_r$ of a Hilbert space H such that the operator $Q = \frac{1}{2r}\sum_{i=1}^r (U_i + U_i^{-1})$ would have spectral radius < 1, but close to 1. In other words, I wanted to build unitary representations of a free group Γ with r generators, which would have a small spectral gap.

For $r = 1$, one knows in some sense all unitary representations of $\mathbb{Z}$. Indeed, it follows from the spectral theorem for unitary operators [**44**] that the data of a cyclic representation with a marked cyclic unitary vector amounts to the data of Borel probability measure on the circle $\mathbb{R}/\mathbb{Z}$.

In the case of a non Abelian free group Γ, one could try to build unitary representations by building Γ-invariant non-negative symmetric bilinear forms on the space of finitely supported functions on Γ. Somehow, this space is too big for providing examples which could be

handleable. With the idea that Γ should be thought of as a negatively curved object, to find how to build unitary representations, one can look at the theory of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ [**34**] or of the group of automorphisms of a regular tree [**21**]. In both cases, there, a particular role is played by the Steinberg representation for $\mathrm{SL}_2(\mathbb{R})$ or the special representation for the group of automorphisms of a regular tree. We recall the construction of these representations.

1.2. Steinberg representation of $\mathrm{SL}_2(\mathbb{R})$

One can define the Steinberg representation of $\mathrm{SL}_2(\mathbb{R})$ in the following way. Let H_0 be the space of all distributions T in the Sobolev space $H = \mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$ such that $\langle T, \mathbf{1}\rangle = 0$, where $\mathbf{1}$ is the constant function with value 1. The group $\mathrm{SL}_2(\mathbb{R})$ acts on H and H_0 in a natural way. Let us construct an invariant scalar product for this action. For $\xi \neq \eta$ in $\mathbb{P}^1_{\mathbb{R}}$, and p in the hyperbolic plane $\mathbb{H}$, let $(\xi|\eta)_p$ denote the Gromov product of ξ and η viewed from p. Equivalently, if $\xi = \mathbb{R}v$ and $\eta = \mathbb{R}w$ for some non-zero vectors v and w in $\mathbb{R}^2$, we set

$$(\xi|\eta)_p = -\frac{1}{4}\log\left(\frac{\|v \wedge w\|_p}{\|v\|_p \|w\|_p}\right),$$

where $\|.\|_p$ stands for the Euclidean norms associated to p on $\mathbb{R}^2$ and $\wedge^2\mathbb{R}^2$. Then the symmetric bilinear form defined formally by

$$\Phi_p : (\varphi, \theta) \mapsto \int_{\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}} (\xi|\eta)_p \mathrm{d}\varphi(\xi)\mathrm{d}\theta(\eta)$$

is bounded on $\mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$. It is positive definite and defines the usual topology of $\mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$. Now the restriction of Φ_p to H_0 does not depend on p. This follows from the following additive property of the Gromov product

$$(\xi|\eta)_p - (\xi|\eta)_q = \frac{1}{2}(b_\xi(p,q) + b_\eta(p,q)) \quad p, q \in \mathbb{H} \quad \xi \neq \eta \in \mathbb{P}^1_{\mathbb{R}}, \tag{1.1}$$

where $b : \mathbb{H} \times \mathbb{H} \times \mathbb{P}^1_{\mathbb{R}} \to \mathbb{R}$ is the Busemann cocycle which can be defined in this instance by

$$b_\xi(p,q) = \frac{1}{2}\log\left(\frac{\|v\|_p}{\|v\|_q}\right),$$

for p, q in $\mathbb{H}$ and ξ the vector line spanned by the non-zero vector v in $\mathbb{R}^2$.

This representation is irreducible if one considers distributions with coefficients in $\mathbb{R}$; if one considers distributions with coefficients in $\mathbb{C}$,

it is the sum of two irreducible components. In this case, these irreducible components are discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ which are complex conjugate to each other (see [**34**]).

Moreover, the very definition of the space of this representation as a hyperplane in $H = \mathrm{H}^{-\frac{1}{2}}(\mathbb{P}^1_{\mathbb{R}})$ says that the representation H_0 appears in an exact sequence

$$0 \to H_0 \to H \to \mathbb{C} \to 0,$$

where H is a representation of $\mathrm{SL}_2(\mathbb{R})$ by bounded automorphisms which is not uniformly bounded. This defines a natural non trivial 1-cohomology class of H_0.

Bargmann's classification of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ is given in [**34**]. One can check that they all can be constructed from the representation in H_0 by abstract algebraic operations, such as taking tensor products (as in the work of Repka [**41**]) or exponentials defined by the natural 1-cohomology class (this latter procedure is described in Section 2.11 of the book of Bekka, de la Harpe and Valette [**6**]).

To summarize, the additive property (1.1) allows to define an invariant symmetric bilinear form on certain function spaces on the boundary of $\mathbb{H}$. This bilinear form is positive definite and most of the representation theory of $\mathrm{SL}_2(\mathbb{R})$ can be recovered from the unitary representation defined by this data.

1.3. Special representations of automorphisms of trees

We will now recall an analogue construction for trees. Let $q \geq 2$ be an integer and X be a regular tree with valence $q+1$, that is, every vertex x in X has exactly $q+1$ neighbours. Later, X will be the tree associated to a free group. The group G of automorphisms of X comes with a natural locally compact topology. The study of the unitary irreducible representations of G was initiated by Cartier [**17**] and a full classification of them was given by Olshanski [**38**]. This classification is described in detail in the book of Figà-Talamanca and Nebbia [**21**], and it shares some deep analogy with the representation theory of $\mathrm{SL}_2(\mathbb{R})$.

In particular, let ∂X be the boundary of X, $\partial^2 X$ be the set of pairs of different points in ∂X and $\omega : X \times \partial^2 X \to \mathbb{Z}$ be the Gromov product. Then, there is a certain Hilbert space H^ω of distributions on ∂X whose scalar product is formally defined as the symmetric bilinear form

$$(\varphi, \theta) \mapsto \int_{\partial X \times \partial X} \omega_x(\xi, \eta) \mathrm{d}\varphi(\xi) \mathrm{d}\theta(\eta),$$

where x is a fixed vertex of X. Again, if we set H_0^ω to be the space of those T in H^ω with $\langle T, \mathbf{1}\rangle = 0$, the restriction of this scalar product to H_0^ω does not depend on x, which is due to the relation

(1.2)
$$\omega_x(\xi,\eta) - \omega_y(\xi,\eta) = \frac{1}{2}(b_\xi(x,y) + b_\eta(x,y)) \quad x,y \in X \quad \xi \neq \eta \in \partial X,$$

where $b : X \times X \times \partial X \to \mathbb{Z}$ again is the Busemann cocycle. The construction of the Hilbert spaces H^ω and H_0^ω is recalled in Section 3.1.

1.4. Additive kernels

Let $g_1, \dots, g_r$ be elements of $\mathrm{SL}_2(\mathbb{R})$ which span a dense subgroup. We denote by Γ be the abstract free group spanned by $g_1, \dots, g_r$ and by X the Cayley graph of Γ with respect to this system of generators, which is a regular tree with valence $2r$. The pull-back of the Steinberg representation to Γ can, be defined directly by looking at functions on the tree X.

Indeed, boundary theory gives a Γ-equivariant measurable map $F : \partial X \to \mathbb{P}^1_{\mathbb{R}}$ which is defined ν_x-almost everywhere. It follows from the construction of this map by means of probability theory (see [**7**, Sec. 9, 13]) that the $\nu_x \otimes \nu_x$-almost everywhere defined function

$$\Omega_x : (\xi,\eta) \mapsto (F(\xi)|F(\eta))_p$$

is $\nu_x \otimes \nu_x$-integrable. Thus, it defines a symmetric bilinear form Ψ_x on the space $\mathfrak{M}^\infty(\partial X, \nu_x)$ of Borel signed measures on ∂X which are absolutely continuous with respect to ν_x with a bounded Radon derivative. As the bilinear form Φ_p of Section 1.2 is non-negative, so is Ψ_x. Now, from (1.1), one can get a relation of the form

(1.3)
$$\Omega_x(\xi,\eta) - \Omega_y(\xi,\eta) = \frac{1}{2}(B_\xi(x,y) + B_\eta(x,y)) \quad x,y \in X \quad \xi \neq \eta \in \partial X,$$

which implies again that the restriction of Ψ_x to the space $\mathfrak{M}_0^\infty(\partial X, \nu_x)$ of signed measures ρ with $\rho(\mathbf{1}) = 0$ is Γ-invariant. The completion of this space with respect to this Γ-invariant non-negative bilinear form defines a representation of Γ which is a subrepresentation of $\rho \circ \theta$. Thus, at least part of the representation ρ may be seen as being obtained from the data of the function Ω_x and the additive relation (1.1).

Functions Ω which are Γ-invariant and satisfy a relation as (1.3) will be called additive kernels in this article. They are rather common in negatively curved geometry (see Ledrappier's survey paper [**35**], whose results are adapted to our framework in Chapter 2). In this book, we will be interested in the case where, as in the above construction, the

bilinear forms defined by such functions are non-negative (and hence define unitary representations of Γ). We will develop a general theory of these functions Ω when they are smooth, that is, when for x in X, Ω_x is locally constant on $\partial^2 X$. The language we will use only involves the action of Γ on X. Therefore, from now on, we will not assume that Γ is a free group acting on the Cayley graph associated with a free system of generators, but rather that Γ is a discrete group of automorphisms of a tree X and that Γ only has finitely many orbits in X (which, by [**3**], implies Γ to be virtually free).

It turns out that, when Ω is a Γ-invariant additive kernel which defines non-negative bilinear forms, the precise definition of the Hilbert space associated with such an additive kernel is dual to the construction of a Γ-invariant non-negative bilinear form on the space $\overline{\mathcal{D}}(\partial X) = \mathcal{D}(\partial X)/\mathbb{R}\mathbf{1}$, which is the quotient of the space $\mathcal{D}(\partial X)$ of locally constant functions on ∂X by the line of constant functions. This space will be central in our study. The data of a Γ-invariant non-negative bilinear form on $\overline{\mathcal{D}}(\partial X)$ is actually equivalent to the data of a unitary representation of Γ on a Hilbert space H, together with a harmonic 1-cocycle, whose values span a dense subspace of H. This point of view is detailed in Appendix F.

The bilinear forms on $\overline{\mathcal{D}}(\partial X)$ which are dual to non-negative bilinear forms associated to Γ-invariant additive kernels will be obtained by an inductive process called orthogonal extension, starting from a finite-dimensional data which will be called a Euclidean field. The first chapters of the book are devoted to the precise description of the above mentioned duality phenomenon as well as to the construction of orthogonal extension.

For example, as a byproduct of this structure theory, we will show in Theorem 15.1 that if $\Omega : X \times \partial^2 X$ is a Γ-invariant locally constant additive kernel, and if the associated bilinear forms are non-negative, then, there exists a Γ-invariant locally constant function $F : X \times \partial X \to \mathbb{R}$ such that, for any x in X and $\xi \neq \eta$ in ∂X, one has

$$\Omega_x(\xi, \eta) + F(x, \xi) + F(x, \eta) \geq 0.$$

1.5. The spectral problem for unitary representaions of tree lattices

We assume that the tree X is regular, with valence $q + 1$, for some integer $q \geq 2$. Once we have defined orthogonal extension, we will study the spectral theory of the action of Γ on the completion of $\overline{\mathcal{D}}(\partial X)$ with respect to a scalar product obtained by orthogonal extension.

Let us precise what kind of spectral theory we have in mind. Given a vector space V equipped with an action of Γ, we let $\mathcal{F}(X,V)^\Gamma$ denote the space of all maps $X \to V$ which are Γ-equivariant. This space comes with a natural linear operator Q associated to the geometry of X. For f in $\mathcal{F}(X,V)^\Gamma$ and x in X, we write

$$Qf(x) = \frac{1}{q+1} \sum_{y \sim x} f(y).$$

Assume V is equipped with a Hilbert space structure and the action of Γ on V is unitary. Then, we can equip $\mathcal{F}(X,V)^\Gamma$ with the natural scalar product defined by, for f, g in $\mathcal{F}(X,V)^\Gamma$,

$$\langle f, g \rangle = \sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|} \langle f(x), g(x) \rangle$$

(where, for x in X, Γ_x is the stabilizer of x in Γ). One easily checks that the operator Q has norm ≤ 1 and that it is self-adjoint with respect to this scalar product. We are interested in understanding the spectral invariants of this self-adjoint operator.

Example 1.1. Assume Γ is the free group generated by a finite set S with $|S| = r \geq 2$ and that X is the natural tree associated with this data (so that $q = 2r-1$). Let x be the vertex of X whose neighbours are the sx and $s^{-1}x$ for s in S. Then, the map $f \mapsto f(x)$ is an isometry from $\mathcal{F}(X,V)^\Gamma$ onto V which conjugates the operator Q and the operator $v \mapsto \frac{1}{2r} \sum_{s \in S} (sv + s^{-1}v)$ on V.

Example 1.2. Let V be $\ell^2(\Gamma)$, equipped with the left regular representation. In this case, elements of $\mathcal{F}(X,V)^\Gamma$ can be thought of as Γ-invariant functions f on $\Gamma \times X$. The map $f \mapsto f(e,x)$ is an isometry from $\mathcal{F}(X,V)^\Gamma$ onto $\ell^2(X)$. Then, we can see Q as the natural Markov operator on X defined by

$$Qg(x) = \frac{1}{q+1} \sum_{y \sim x} g(y), \quad g \in \ell^2(X), \quad x \in X.$$

The spectrum of this operator was computed by Kesten [**32**]: this is the interval $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$. We give a proof of this fact in Corollary 22.10.

According to our knowledge, very few examples are known, where the spectral invariants of Q can be explicitely computed. In this book, we provide a wide family of unitary representations where such computations can be realized. This will be achieved in Theorem 31.1, where we establish a Plancherel formula. The elements of this formula will be built later. It yields

COROLLARY 1.3. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H^p be the associated unitary representation of Γ. The spectrum of the operator Q in $\mathcal{F}(X, H^p)^\Gamma$ is $\Sigma_p \cup [-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$. The associated spectral measures are absolutely continuous on the interval $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$.*

The set Σ_p is a finite subset of $(-1, -\frac{2\sqrt{q}}{q+1}) \cup (\frac{2\sqrt{q}}{q+1}, 1)$ which can be empty in certain cases. It is defined precisely in (26.1) by means of the spectral values of the simple transfer operator S_p, which is a linear endomorphism of a finite-dimensional vector space.

1.6. The Ihara trace formula

The constructions leading to the spectral theorem for scalar products obtained by orthogonal extension on $\overline{\mathcal{D}}(\partial X)$ rely on analogies with the spectral theory of the operator Q associated with the trivial representation V of Γ. Then, the space $\mathcal{F}(X, V)^\Gamma$ is simply the space $\mathcal{H}_0$ of Γ-invariant functions on X. In that case, the spectrum of the operator Q is related to the spectrum of an other operator, acting on the space $\mathcal{H}_1$ of Γ-invariant functions on X_1: in our study the role of the latter operator will be played by the simple transfer operator mentioned above. The relations between those two operators on $\mathcal{H}_0$ and $\mathcal{H}_1$ are at the core of the Ihara trace formula [**28**]. To motivate the reader, we will now briefly recall the proof of this formula.

We equip the space $\mathcal{H}_1$ with the operator T defined by

$$Tu(x, y) = \sum_{\substack{z \sim y \\ z \neq x}} u(y, z), \quad u \in \mathcal{H}_1, \quad x \sim y \in X.$$

The original reason for studying this operator was that, for $n \geq 1$, the number $\frac{1}{n} \operatorname{tr}(T^n)$ can be interpretated as a number of closed loops of length n in the quotient graph $\Gamma \backslash X$ when this makes sense.

To relate the spectrum of Q, acting on $\mathcal{H}_0$, to the one of T, acting on $\mathcal{H}_1$, we start by equipping these spaces with the natural scalar products defined by, for a, b in $\mathcal{H}_0$ and u, v in $\mathcal{H}_1$,

$$\langle a, b \rangle = \sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|} a(x) b(x)$$

$$\text{and } \langle u, v \rangle = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} u(x, y) v(x, y).$$

There are two natural embeddings $L : \mathcal{H}_0 \to \mathcal{H}_1$ and $R : \mathcal{H}_0 \to \mathcal{H}_1$ which may be defined by

$$La(x, y) = a(x) \text{ and } Ra(x, y) = a(y), \quad a \in \mathcal{H}_0, \quad x \sim y \in X.$$

With respect to the natural Euclidean structures, the adjoint operators $L^\dagger$ and $R^\dagger$ may be defined by

$$L^\dagger u(x) = \sum_{y\sim x} u(x,y) \text{ and } R^\dagger u(x) = \sum_{y\sim x} u(y,x), \quad u \in \mathcal{H}_1, \quad x \in X.$$

Let $\mathcal{E}$ be the orthogonal complement in $\mathcal{H}_1$ of the space $L\mathcal{H}_0 + R\mathcal{H}_0$. It will turn out that both $\mathcal{E}$ and $\mathcal{E}^\perp$ are T-invariant.

Indeed, the space $\mathcal{E}$ is the space of all functions u on X_1 wich satisfy, for any x in X,

$$\sum_{y\sim x} u(x,y) = \sum_{y\sim x} u(y,x) = 0.$$

In particular, for such a function, we have $Tu(x,y) = -u(y,x)$ for any $x \sim y$ in X. Thus, $\mathcal{E}$ is stable under T and the spectrum of T in $\mathcal{E}$ is (at most) $\{-1,1\}$. The dimensions of the associated eigenspaces are combinatorial invariants of the action of Γ on X which may be related to the cardinalities of the sets $\Gamma\backslash X$ and $\Gamma\backslash X_1$ when Γ is torsion free.

Now, we analyse the action of T on $\mathcal{E}^\perp = L\mathcal{H}_0 + R\mathcal{H}_0$. For a, b in $\mathcal{H}_0$, set $B\begin{pmatrix} a \\ b \end{pmatrix} = La + Rb$. A direct computation shows the relation

$$TB\begin{pmatrix} a \\ b \end{pmatrix} = B\begin{pmatrix} 0 & -1 \\ q & (q+1)Q \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Note that, for t in $\mathbb{R}$ with $(q+1)^2t^2 \neq 4q$, the eigenvalues of the matrix $\begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix}$ are the two roots of the equation $q + u^2 = (q+1)tu$. For $t = \frac{2\sqrt{q}}{q+1}$ (resp. $t = -\frac{2\sqrt{q}}{q+1}$), the matrix is not diagonalizable and its eigenvalue is $\sqrt{q}$ (resp. $-\sqrt{q}$). Thus, the spectrum of the operator $\begin{pmatrix} 0 & -1 \\ q & (q+1)Q \end{pmatrix}$, acting on $\mathcal{H}_0^2$, is the set of all roots of the equation $q + u^2 = (q+1)tu$, where t runs among the eigenvalues of Q, acting on $\mathcal{H}_0$.

To conclude, we need to analyse the null space of the linear map B. Say that a function u on X is constant on neighbours if, for every x, y, z in X with $x \sim y$ and $x \sim z$, we have $u(y) = u(z)$. If u is such a function, let Iu be the function on X whose value on x in X is the value of u on the neighbours of x: again, Iu is constant on neighbours. Write $\mathcal{H}_{-1}$ for the space of Γ-invariant functions that are constant on neighbours. Then, the null space of B is the space

$$\left\{ \begin{pmatrix} u \\ -Iu \end{pmatrix} \middle| \, u \in \mathcal{H}_{-1} \right\} \subset \mathcal{H}_0^2.$$

Let us describe $\mathcal{H}_{-1}$ more precisely. There are two cases, depending on whether the action of Γ on X is bipartite or not: we say that the action of Γ on X is bipartite if, for some (equivalently any) x in X, all the integral numbers $d(x, \gamma x)$, $\gamma \in \Gamma$, are even. If the action of Γ on X is not bipartite, the space $\mathcal{H}_{-1}$ is reduced to constant functions. If the action of Γ on X is bipartite, the space $\mathcal{H}_{-1}$ has dimension 2. Depending on the case, the spectrum of Q in $\mathcal{H}_{-1}$ is either $\{1\}$ or $\{-1, 1\}$. Besides, an application of the maximum principle shows that all the eigenvalues of Q in $\mathcal{H}_0/\mathcal{H}_{-1}$ are contained in $(-1, 1)$.

Summarizing this discussion, we can relate the spectra of Q and T as follows.

THEOREM 1.4 (Ihara [**28**]). *The eigenspace of T associated to 1 (resp. -1) is the space of all skew symmetric (resp. symmetric) functions u on X_1 with $\sum_{y\sim x} u(x, y) = 0$ for any x in X. The operator T admits q as an eigenvalue with multiplicity 1. The associated eigenline is the space of constant functions. If the action of Γ on X is bipartite, the operator T admits $-q$ as an eigenvalue with multiplicity 1. The other eigenvalues of T are the roots of the equation $q + u^2 = (q + 1)tu$ where t runs among the eigenvalues of P in $(-1, 1)$. For each such t and u, the dimension of the eigenspace of T associated to u is equal to the dimension of the eigenspace of P associated to t. If $(q+1)^2t^2 \neq 4q$, then u is a simple eigenvalue of T. If $(q + 1)^2t^2 = 4q$, the generalized eigenspace associated to u has dimension twice the dimension of the eigenspace.*

In a finite-dimensional vector space, an eigenvalue of a linear operator is said to be simple if the associated eigenspace is equal to the associated generalized eigenspace; in other words, the dimension of the eigenspace is equal to the multiplicity of the eigenvalue as a root of the characteristic polynomial.

The spaces $\mathcal{H}_{-1}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ will play a role in this article. More generally, we will handle a sequence of finite-dimensional spaces $(\mathcal{H}_k)_{k\geq -1}$. For $k \geq -1$, the elements of $\mathcal{H}_k$ will be called Γ-invariant k-pseudofunctions. The space $\mathcal{H}_k$ will be equipped with an automorphism $H \mapsto H^\vee$ and an embedding $H \mapsto H^>$ into $\mathcal{H}_{k+1}$. The embeddings L and R will be respectively written as $H \mapsto H^>$ and $H \mapsto H^{\vee>\vee}$.

The construction of the objects that appear in the Plancherel formula of Theorem 31.1 will use the action of the matrix $\begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix}$ on the space $\mathcal{H}_k^2$, for t in $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}] \cup \Sigma_p$.

1.7. Spectral obstructions

We still let $q \geq 2$ be an integer and X be a homogeneous tree of degree $q+1$, equipped with a proper, cofinite action of a discrete group Γ. As said before, we will develop a spectral theory of Γ-invariant scalar products on the space $\overline{\mathcal{D}}(\partial X)$ which are obtained by orthogonal extension of Euclidean fields. This theory will involve the space $\mathcal{H}_\infty = \mathcal{F}(X_1, \overline{\mathcal{D}}(\partial X))^\Gamma$ of Γ-equivariant map from the space X_1 of edges of X to $\overline{\mathcal{D}}(\partial X)$. Later in the book, the space $\mathcal{H}_\infty$ will be called the space of Γ-invariant ∞-pseudofunctions. It will be written as a non decreasing union $\mathcal{H}_\infty = \bigcup_{k\geq -1} \mathcal{H}_k$ of finite-dimensional subspaces. For $k \geq -1$, the space $\mathcal{H}_k$ will be called the space of Γ-invariant k-pseudofunctions.

The spectral analysis of the operator Q will rely on a more refined analysis of the algebra $\mathcal{A}$ of operators of $\mathcal{H}_\infty$ which is spanned by the operators R and S defined as follows: for $x \sim y$ in X and H in $\mathcal{H}_\infty$, one has

$$(RH)_{xy} = \sum_{\substack{z\sim x \\ z\neq y}} H_{xz} \text{ and } (SH)_{xy} = H_{yx}.$$

Our initial question of understanding spectral gap properties of unitary representations of Γ defined by an additive kernel Ω has evolved into understanding such properties for completions of the space $\overline{\mathcal{D}}(\partial X)$ with respect to a Γ-invariant non-negative symmetric bilinear form. Any such bilinear form p defines in a natural way a symmetric bilinear form (still denoted by p) on $\mathcal{H}_\infty$, for which the operators R and S are symmetric: given H, J in $\mathcal{H}_\infty$, one sets

$$p(f,g) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p(H_{xy}, J_{xy}).$$

This yields an inclusion

$$\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \hookrightarrow \mathcal{Q}(\mathcal{H}_\infty)^{R,S}, \tag{1.4}$$

where the former stands for the space of all Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ and the latter for the space of all symmetric bilinear forms on $\mathcal{H}_\infty$ for which the operators R and S are symmetric. Denoting by $\mathcal{Q}_+(\overline{\mathcal{D}}(\partial X))^\Gamma \subset \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and $\mathcal{Q}_+(\mathcal{H}_\infty)^{R,S} \subset \mathcal{Q}(\mathcal{H}_\infty)^{R,S}$ the cones of non-negative forms, one has $\mathcal{Q}_+(\overline{\mathcal{D}}(\partial X))^\Gamma = \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \cap \mathcal{Q}_+(\mathcal{H}_\infty)^{R,S}$.

Part of the tools we have developed for understanding the spectral theory of (orthogonal extensions of) Euclidean fields can be used to describe the cone $\mathcal{Q}_+(\mathcal{H}_\infty)^{R,S}$. A natural question is then to understand the difference between the spaces $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and $\mathcal{Q}(\mathcal{H}_\infty)^{R,S}$. The last

chapters of the book are devoted to a partial result in the description of this difference.

More precisely, by restriction of (1.4), for $k \geq -1$, we get a natural map

$$\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^{\Gamma} \to \mathcal{Q}(\mathcal{AH}_k)^{R,S} \tag{1.5}$$

where $\mathcal{AH}_k$ is the (infinite-dimensional) subspace of $\mathcal{H}_\infty$ spanned by the images of $\mathcal{H}_k$ under the elements of the algebra $\mathcal{A}$ and $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ is the space of all symmetric bilinear forms on $\mathcal{AH}_k$ for which the operators R and S are symmetric. Theorem 42.16 will yield

COROLLARY 1.5. *For $k \geq 0$, the image of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^{\Gamma}$ in the space $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ has finite codimension.*

1.8. Related works

The group $\mathrm{SL}_2(\mathbb{R})$ or the group of automorphisms of a regular tree are type I groups. In other words, the space of all their irreducible unitary representations may be parametrized by a standard Borel space. Such a parametrization can not be obtained for discrete groups with infinite conjugacy classes. These notions and facts are explained in [**5**, Chap. 6, 7]. In other words, there is no hope of getting a classification of the irreducible unitary representations of Γ. Nevertheless one can try and describe special examples of such representations.

As far as I know, the study of unitary representations of discrete groups acting on negatively curved spaces can be traced back to the work of of Figà-Talamanca and Picardello [**22**] who proved that the restriction to a free group of a spherical irreducible representation of the group of automorphisms of its tree stays irreducible. For Lie groups, Cowling and Steger [**19**] determined under which condition the restriction to a lattice of a unitary irreducible representation of a semisimple Lie group remains irreducible.

The construction, from the geometry of its tree, of unitary representations of a free group that are not necessarily representations of the full group of automorphisms was initiated by Kuhn and Steger [**33**]. This work was recently pursued by Iozzi, Kuhn and Steger [**29**]. The

A main development of the field was the proof of the irreducibility of the quasi-regular representation associated with the Patterson-Sullivan measure of the fundamental group of a compact negatively curved manifold by Bader and Muchnik [**2**]. This result was extended to groups of isometries of CAT(-1)-spaces by Boyer [**13**] and then to a wider class of quasi-regular representations associated to Gibbs measures by Boyer and Garncarek [**14**]. In this latter work, there appears a strong

relation between the unitary representation theory of the fundamental group and the thermodynamic formalism of the geodesic flow. This connection also exists in the present paper.

We notice that the representations that are studied by these authors may be seen as analogues or deformations of the principal series representations of $\mathrm{SL}_2(\mathbb{R})$ or of the group of isometries of a regular tree. In particular, the Hilbert spaces on which they are defined are easily constructed: they are the L^2 spaces associated with a certain quasi-invariant measure on the boundary of the group. Our point of view is different in as much as the representations that we build are analogues of the special representations mentioned in Sections 1.2 and 1.3. In particular, the definition of the associated Hilbert spaces is more intricate. One could say that we study the additive representation theory of Γ, whereas the above mentioned authors study the multiplicative representation theory of Γ. Both theories are related through the exponentiation process associated to a certain 1-cohomology class in the additive representations (see again [**6**, Sec. 2.11]). The precise study of this relation will also be the subject of a later work, as will be the use of this exponentiation process to build generalized complementary series.

1.9. Miscellaneous notation

When speaking of a function, we shall always mean a function with values in $\mathbb{R}$. Except when expressively specified, all vector spaces considered in this paper and in particular all Hilbert spaces will be defined over $\mathbb{R}$.

If V is a vector space, we shall denote its algebraic dual space by V^*. If W is another vector space and $\theta : V \to W$ is a linear map, we write $\theta^* : W^* \to V^*$ for the adjoint linear map. If V (resp. W) is equipped with a scalar product p (resp. q), we write $\theta^{\dagger pq} : W \to V$ for the adjoint linear map of T with respect to these Euclidean structures. When the choices of p and q are clear from the context, we simply write $\theta^\dagger$ for $\theta^{\dagger pq}$. The null space[1] of θ is denoted by $\ker\theta$.

The space of all symmetric bilinear forms on V is denoted by $\mathcal{Q}(V)$. The space of non-negative (resp. positive definite) forms is denoted by $\mathcal{Q}_+(V)$ (resp. $\mathcal{Q}_{++}(V)$). If q is a symmetric bilinear form on W, then

[1]There is a problem with terminology here. In linear algebra, the kernel of a linear map is the space where it cancels. In functional analysis, the kernel of a bilinear form on a space of functions is a function with two variables. To avoid confusions, we will only use the word kernel with the latter meaning and speak of the null space of a linear map.

$\theta^\star q$ stands for the pull-back of q under θ, that is, the bilinear form $(v,w) \mapsto q(Tv,Tw)$ on V. Thus, θ^* and $\theta^\star$ are both pull-back maps, but they don't act on the same spaces. Unfortunately, at some point, we will need to write $(\theta^*)^\star$.

If U is a totally discontinuous locally compact topological space, we say that a function φ on U is smooth if it is locally constant. The space of all compactly supported smooth functions on U will be denoted by $\mathcal{D}(U)$. A distribution on U is a linear functional on this space. The space of all distributions on U is therefore the algebraic dual space of $\mathcal{D}(U)$. We denote it by $\mathcal{D}^*(U)$. This notion of a distribution and its use in the representation theory of groups acting on totally discontinuous spaces can be traced back to [**16**].

If φ is in $\mathcal{D}(U)$ and T is in $\mathcal{D}^*(U)$, we write $\langle T, \varphi\rangle$ for the evaluation of T on φ. We write φT for the distribution $\psi \mapsto \langle T, \varphi\psi\rangle$.

If U is compact, we set $\overline{\mathcal{D}}(U) = \mathcal{D}(U)/\mathbb{R}$ to be the quotient of the space of smooth functions by the line of constant functions. Its dual space can be identified with the space of distributions which kill the function $\mathbf{1}$. It is denoted by $\mathcal{D}_0^*(U)$.

If V is another totally discontinuous locally compact topological space, for φ in $\mathcal{D}(U)$ and ψ in $\mathcal{D}(V)$, we write $\varphi \otimes \psi$ for the function $(u,v) \mapsto \varphi(u)\psi(v)$ on $U \times V$, which belongs to $\mathcal{D}(U \times V)$. This identifies $\mathcal{D}(U \times V)$ with the algebraic tensor product $\mathcal{D}(U) \otimes \mathcal{D}(V)$. In particular, if ρ is a distribution in $\mathcal{D}^*(U)$ and θ is a distribution in $\mathcal{D}^*(V)$, we define a distribution $\rho \otimes \theta$ in $\mathcal{D}^*(U \times V)$ by setting

$$\langle \rho \otimes \theta, \varphi \otimes \psi\rangle = \langle \rho, \varphi\rangle\langle \theta, \psi\rangle, \quad \varphi \in \mathcal{D}(U), \quad \psi \in \mathcal{D}(V).$$

The indicator function of a subset V in a set U is denoted by $\mathbf{1}_V^U$ or more simply by $\mathbf{1}_V$ when there is no ambiguity.

Let Γ be a group acting on a set X and $S \subset X$ be a system of representatives of the elements of $\Gamma\backslash X$. If φ is a Γ-invariant function on X which is summable on S, we write $\sum_{x \in \Gamma\backslash X} \varphi(x)$ for $\sum_{x \in S} \varphi(x)$.

If E is a set, we let $E^{\mathbb{N}}$ be the set of sequences of elements of E.

If V is a vector space, we write $V^{(\mathbb{N})} \subset V^{\mathbb{N}}$ for the set of finitely supported sequences, that is, sequences $(x_i)_{i \geq 0}$ such that $x_i = 0$ for all large enough i. For v in V and $i \geq 0$, we let $v\mathbf{1}_i$ be the sequence in $V^{(\mathbb{N})}$ all of whose coefficients are 0, except the i-th one which is equal to v.

We will often write elements u of V^2 as column matrices $u = \begin{pmatrix} v \\ w \end{pmatrix}$, where v and w are in V. In this case, we write $u^* = \begin{pmatrix} v & w \end{pmatrix}$ for the associate line matrix.

Let V be a real vector space. We write $V_{\mathbb{C}}$ for the complexification of V, that is, $V_{\mathbb{C}} = \mathbb{C} \otimes V$. One has a natural embedding $V \subset V_{\mathbb{C}}$ and $V_{\mathbb{C}}$ may be written as $V_{\mathbb{C}} = V \oplus iV$. We then write $v \mapsto \overline{v}$ for the natural complex conjugation in $V_{\mathbb{C}}$, so that, for v in V, one has $\overline{v} = v$ and $\overline{iv} = -iv$

Any object defined in terms of real linear algebra on real vector spaces defines an analoguous complexified object on complexified spaces: we usually denote the two objects by the same letters. For example, if $T : V \to V$ is a real linear map, we still write $T : V_{\mathbb{C}} \to V_{\mathbb{C}}$ for the complex linear map defined by $T(v+iw) = Tv+iTw$ for v, w in V. In particular, we adopt the following convention: if p is a symmetric bilinear form on V, then p also stands for the complex bilinear form on $V_{\mathbb{C}}$ whose restriction to V is p. Thus, the natural Hermitian form on $V_{\mathbb{C}}$ associated to p is defined as $\tilde{p} : (v, w) \mapsto p(\overline{v}, w), V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. Then, p is positive definite as a real bilinear form on V if and only if $\tilde{p}$ is positive definite as a Hermitian form on $V_{\mathbb{C}}$.

The space of polynomial functions $\mathbb{R} \to V$ is denoted by $V[t]$.

Part 1

Quadratic fields and dual kernels

CHAPTER 2

Tree lattices and additive kernels

2.1. Trees

In all the book, the letter X will stand for a locally finite tree. We start by giving a precise definition of the version of this notion that we will use.

We let X be countable set equipped with a symmetric relation $\sim$ such that for any x in X, the set of neighbours of x, that is, the set $S^1(x)$ of y in X with $x \sim y$, is finite (X is locally finite) and does not contain x. We let $d(x)$ denote its number of elements. To avoid technicalities, we assume that $d(x) \geq 3$ for any x in X. Starting from Chapter 20, d will be assumed to be constant.

We assume that $(X, \sim)$ is connected, that is, for every x, y in X there exists a sequence $z_0 = x, z_1, \ldots, z_n = y$ of elements of X such that $z_{h-1} \sim z_h$ for $1 \leq h \leq n$ and $z_{h-1} \neq z_{h+1}$ for $1 \leq h \leq n-1$. Such a sequence will be called a geodesic path from x to y. The integer n is called the length of the path.

We assume that $(X, \sim)$ is simply connected, that is, for every x, y in X there exists a unique geodesic path $z_0 = x, z_1, \ldots, z_n = y$ from x to y. The set $\{z_0, \ldots, z_n\}$ is called the geodesic segment between x and y and is denoted by $[xy]$. The length of this path is called the distance between x and y and denoted by $d(x, y)$. For any $n \geq 0$, we write $S^n(x)$ for the sphere with radius n and center x for this distance. When no confusion is possible, the element of the segment $[xy]$ which is at distance h from x is denoted by x_h.

A sequence $(x_h)_{h \geq 0}$ of elements of X is called a geodesic ray if, for all $H \geq 0$, the sequence $(x_h)_{0 \leq h \leq H}$ is a geodesic path. The element x_0 is called the origin of the geodesic ray.

Two geodesic rays $(x_h)_{h \geq 0}$ and $(y_h)_{h \geq 0}$ are said to be equivalent if there exists a relative integer $k \in \mathbb{Z}$ with $x_{h+k} = y_h$ for any large enough h. This is an equivalence relation among geodesic rays and the set of equivalence classes is called the boundary of X and is denoted by ∂X. For any x in X and ξ in ∂X, there exists a unique geodesic ray $(x_h)_{h \geq 0}$ with origin x in the equivalence class defined by ξ. Note that the parametrization of the set $\{x_h | h \geq 0\}$ which makes it into a

geodesic ray is unique. By abuse of notations, we shall identify the geodesic ray $(x_h)_{h\geq 0}$ and the set $\{x_h | h \geq 0\}$ and denote both of them by $[x\xi)$. The elements ξ is called the endpoint of the ray $[x\xi)$.

Fix x in X. The set of geodesic rays with origin x embeds naturally as a subset of the product set $\prod_{h\geq 0} S^h(x)$, which is closed for the product topology of the discrete topologies on the spheres. We equip this set with the induced topology which is compact. The image of this topology on ∂X does not depend on x. We shall henceforward equip ∂X with this topology. It is compact and totally discontinuous.

Let $\partial^2 X$ denote the set of pairs of different points in ∂X.

A sequence $(x_h)_{h\in\mathbb{Z}}$ of elements of X is called a parametrized geodesic line if, for all $H \geq 0$, the sequence $(x_h)_{|h|\leq H}$ is a geodesic path. Let $\mathscr{S}$ be the set of all parametrized geodesic lines of X.

Let $s = (x_h)_{h\in\mathbb{Z}}$ be in $\mathscr{S}$. The point x_0 is called the base point of s and denoted by $\pi(s)$. The sequence $(x_{h+1})_{h\in\mathbb{Z}}$ (resp. $(x_{-h})_{h\in\mathbb{Z}}$) is again a parametrized geodesic line. It is denoted by Ts (resp. ιs). The maps $T : \mathscr{S} \to \mathscr{S}$ and $\iota : \mathscr{S} \to \mathscr{S}$ are called the time shift and the time reversal.

If $s = (x_h)_{h\in\mathbb{Z}}$ is in $\mathscr{S}$, the endpoints ξ and η of the geodesic rays $(x_{-h})_{h\geq 0}$ and $(x_h)_{h\geq 0}$ are different. They are respectively called the origin and endpoint of s and denoted by s_- and s_+.

Conversely, given $\xi \neq \eta$ in $\partial^2 X$, there exists a parametrized geodesic line $(x_h)_{h\in\mathbb{Z}}$ with origin ξ and endpoint η. The set $\{x_h | h \in \mathbb{Z}\}$ only depends on ξ and η and is denoted by $(\xi\eta)$. It is called the geodesic line between ξ and η. The parametrizations of this geodesic line are unique up to time shift and time reversal.

The map $\mathscr{S} \to X \times \partial^2 X, s \mapsto (\pi(s), s_-, s_+)$ is injective and its range is a closed subset of $X \times \partial^2 X$ (where X is equipped with the discrete topology). We equip $\mathscr{S}$ with the topology induced by this injection. This makes $\mathscr{S}$ a locally compact totally discontinuous space and the maps T and ι homeomorphisms of $\mathscr{S}$.

An automorphism of X is a map $g : X \to X$ such that, for any x, y in X, one has $gx \sim gy$ if and only if $x \sim y$. A group Γ of automorphisms of X is said to be discrete if it acts properly on X. It is said to be cofinite if the quotient $\Gamma\backslash X$ is finite. A cofinite lattice of X is a discrete cofinite group of automorphisms of X. In the sequel we fix a cofinite lattice Γ.

Below are two examples of such a tree lattice that the reader may keep in mind along the book.

EXAMPLE 2.1. Let A be a finite set with at least three elements and set $d = |A|$ to be the cardinality of A. We assume that X is d-regular

(that is, $d(x) = d$ for every x in X). We fix a map $w : X_1 \to A$ from the set $X_1 = \{(x, y) \in X^2 | x \sim y\}$ of edges of X towards A which is symmetric (that is, $w(x, y) = w(y, x)$ for $x \sim y$ in X) and such that, for every x in X, the map $y \mapsto w(x, y)$ is a bijection from the set $S^1(x)$ of neighbours of x onto A. We then let Γ be the group of automorphisms of X which preserve the map w. Then Γ is a cofinite lattice, which as an abstract group is the free product of d copies of $\mathbb{Z}/2\mathbb{Z}$.

EXAMPLE 2.2. Let A be a finite set with at least two elements. Set $d = 2|A|$ and let X be a d-regular tree. We now fix a map $w : X_1 \to A \times \{-1, 1\}$ which is skew-symmetric (in the sense that, for every $x \sim y$ in X, if $w(x, y) = (a, \epsilon)$, then $w(y, x) = (a, -\epsilon)$) and again such that, for every x in X, the map $y \mapsto w(x, y)$ is a bijection from the set $S^1(x)$ of neighbours of x onto $A \times \{-1, 1\}$. We then let Γ be the group of automorphisms of X which preserve the map w. Then Γ is a cofinite lattice, which as an abstract group is the free product of d copies of $\mathbb{Z}$: this is the classical construction of the tree of a free group.

More generally, trees appear naturallly as universal covers of finite graphs and tree lattices as their fundamental groups. We refer the reader to [**3**] for more on tree lattices.

2.2. Dynamical properties

The action of Γ on $\mathscr{S}$ is proper and the space $\Gamma\backslash\mathscr{S}$ is compact. Since the action of Γ on $\mathscr{S}$ commutes with the maps T and ι, the latter induce homeomorphisms of the compact space $\Gamma\backslash\mathscr{S}$. By abuse of notation, we still denote these maps by T and ι. The map T may be seen as an analogue of the geodesic flow for the quotient of X by Γ. In this Section, we prove that this geodesic flow is topologically transitive. This property will be used in the sequel to prove uniqueness of the solutions of certain functional equations.

PROPOSITION 2.3. *The map T admits dense orbits on $\Gamma\backslash\mathscr{S}$. Equivalently, the group Γ admits dense orbits on $\partial^2 X$.*

This rather standard result will follow from classical arguments of hyperbolic dynamical systems and hyperbolic geometry as in [**18**]. Most of the steps of the proof could be deduced from general properties of hyperbolic groups as in [**23**]. As these properties are much easier to prove in our particular case, we include a sketch of their proofs here.

We start with an easy consequence of the fact that $\Gamma\backslash X$ is finite.

LEMMA 2.4. *Let x, y, z be in X with $x \sim y$. Then there exists g in Γ with $x \notin [y(gz)]$.*

PROOF. Indeed, the set $\{t \in X | x \notin [yt]\}$ is unbounded whereas, as $\Gamma\backslash X$ is finite, one has $\sup_{t\in X} \inf_{g\in\Gamma} d(t, gz) < \infty$. □

An automorphism g of X will be called hyperbolic if there exists a geodesic line $(\xi\eta)$ such that $g(\xi\eta) = (\xi\eta)$ and the restriction of g to $(\xi\eta)$ is a non-trivial translation. More precisely, there exists $k \neq 0$ such that, if $(x_h)_{h\in\mathbb{Z}}$ is a parametrization of $(\xi\eta)$ with origin ξ and endpoint η, one has $gx_h = x_{h+k}$, $h \in \mathbb{Z}$. Up to reversing the roles of ξ and η, one can assume $k > 0$. In that case, for every $\zeta \neq \xi$ in ∂X, one has $g^n\zeta \xrightarrow[n\to\infty]{} \eta$. In particular, the fixed points ξ and η of g and the positive integer k are uniquely determined by g. They are respectively called the repulsive fixed point, the attractive fixed point and the translation length of g. The geodesic line $(\xi\eta)$ is called the axis of g.

Let us give an easy criterion for an automorphism to be hyperbolic. This is a version of the closing Lemma from hyperbolic dynamics (see [**26**]).

LEMMA 2.5. *Let x be in X and g be an automorphism of X. Assume $gx \neq x$. Let y be the neighbour of x on $[x(gx)]$. Then, if gy does not belong to the segment $[x(gx)]$, g is hyperbolic with translation length $k = d(x, gx)$ and x belongs to the axis of g.*

PROOF. Let $x_0 = x, x_1 = y, \ldots, x_k = gx$ be the parametrization of the segment $[x(gx)]$. For any h in $\mathbb{Z} \smallsetminus [0, k]$, if $h = \ell k + m$, $\ell \in \mathbb{Z}$, $0 \leq m \leq k-1$, we set $x_h = g^k x_m$. Then one easily checks that $(x_h)_{h\in\mathbb{Z}}$ is a parametrized geodesic line and that $gx_h = x_{h+k}$, $h \in \mathbb{Z}$. Thus, g is hyperbolic and we are done. □

From this, we deduce that Γ contains hyperbolic elements.

LEMMA 2.6. *The group Γ contains hyperbolic elements. More precisely, the set of attractive fixed points of hyperbolic elements of Γ is dense in ∂X.*

PROOF. Fix $x \neq y$ in X and let us build a hyperbolic element γ of Γ whose attractive fixed point ξ is such that $y \in [x\xi)$.

By Lemma 2.4, we can find an element g in Γ with $[xy] \subset [x(gx)]$. We let z be the neighbour of x on $[x(gx)]$. Then, if gz does not belong to $[x(gx)]$, by Lemma 2.5, g is hyperbolic and x belongs to the axis of g. In particular, as $[xy] \subset [x(gx)]$, we can set $\gamma = g$.

If not, again by Lemma 2.4, we can find h in Γ with $hx \neq x$ and $[x(gx)] \cap [x(hx)] = \{x\}$. We let t be the neighbour of x on $[x(hx)]$.

Assume $ht \notin [x(hx)]$. Then, still by by Lemma 2.5, h is hyperbolic and its attractive fixed point η satisfies $[x\eta) \cap [x(gx)] = \{x\}$. As t is the neighbour of x on $[x\eta)$, gt is the neighbour of gx on $[(gx)(g\eta))$.

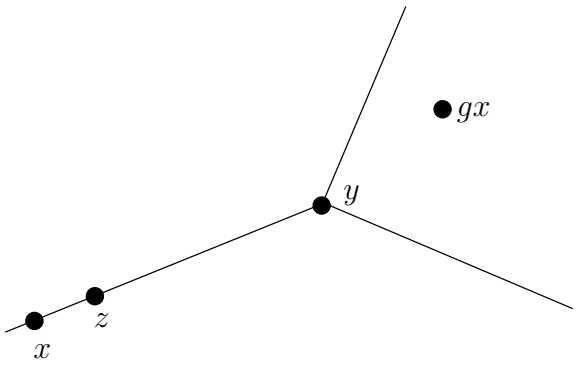

FIGURE 1. Proof of Lemma 2.6

Since $t \neq z$ and gz is the neighbour of gx on $[x(gx)]$, we have $[xy] \subset [x(gx)] \subset [x(g\eta))$ and we can set $\gamma = ghg^{-1}$.

Finally, if ht belongs to $[x(hx)]$, we claim that gh^{-1} is hyperbolic. Indeed, by construction the geodesic segment $[(hx)(gx)]$ is equal to the union $[x(gx)] \cup [x(hx)]$. Now, the neighbour of hx on this segment is ht and the one of gx is gz. Since by assumption, $z \neq t$, we have $gt = (gh^{-1})ht \neq gz$, hence gt does not belong to $[(hx)(gx)]$. Thus, again by Lemma 2.5, gh^{-1} is hyperbolic and its attractive fixed point ζ satisfies $[xy] \subset [(hx)(gx)] \subset [(hx)\zeta)$, so that we can set $\gamma = gh^{-1}$. □

We recall the definition of the Busemann cocycle: this is a a first example of a smooth boundary cocycle, a notion which will play a key role in this book.

Let x and y be in X and ξ be in ∂X. The set $[x\xi) \cap [y\xi)$ is a geodesic ray. The number $d(x,z) - d(y,z)$ does not depend on z when z varies in $[x\xi) \cap [y\xi)$. We denote it by $b_\xi(x,y)$. The map $b : \partial X \times X \times X \to \mathbb{R}$ is smooth and is invariant under all automorphisms of X. It satisfies the cocycle relation:

$$b_\xi(x,z) = b_\xi(x,y) + b_\xi(y,z), \quad x,y,z \in X, \quad \xi \in \partial X.$$

For ξ in ∂X, we let Γ_ξ be the stabilizer of ξ in Γ. Fix x in X. By the cocycle property, the map $g \mapsto b_\xi(x, gx)$ is a homomorphism from Γ_ξ to $\mathbb{Z}$ which does not depend on x. We denote this homomorphism by χ_ξ.

Set $U_\xi = \partial X \smallsetminus \{\xi\}$. We fix x in X. We define a ultrametric distance on U_ξ by setting, for $\eta \neq \zeta$ in U_ξ, $D_x^\xi(\eta,\zeta) = e^{b_\xi(x,z)}$ where z in X is such that $(\xi\eta) \cap (\xi\zeta) = (\xi z]$. This is a proper distance, meaning that the associated balls are compact. It defines the locally compact topology of U_ξ, viewed as a subset of ∂X.

For x, y in X, one has $D_x^\xi = e^{b_\xi(x,y)} D_y^\xi$ and for g in Γ_ξ and η, ζ in U_ξ, one has $D_x^\xi(g\eta, g\zeta) = e^{\chi_\xi(g)} D_x^\xi(\eta, \zeta)$. Thus, a fixed point argument gives:

LEMMA 2.7. *Let ξ be in ∂X and g be in Γ with $\chi_\xi(g) < 0$. Then g is hyperbolic and its repulsive fixed point is ξ.*

Let Γ_ξ^0 be the kernel of χ_ξ in Γ_ξ.

LEMMA 2.8. *Let ξ be in ∂X. The action of the group Γ_ξ^0 on the locally compact space $U_\xi = \partial X \smallsetminus \{\xi\}$ is proper.*

PROOF. We need to prove that for any compact subset K of U_ξ, the set of g in Γ_ξ^0 with $gK \cap K \neq \emptyset$ is finite.

For x in X, define $K_{\xi x}$ as the set of those η in U_ξ such that x belongs to $(\xi\eta)$. These sets are the balls of the above introduced distances on U_ξ. In particular $K_{\xi x}$ is a compact subset of U_ξ and every compact subset of U_ξ is contained in $K_{\xi x}$ for some x. Thus, to check that the action is proper, we can assume that K above is of the form $K_{\xi x}$.

Now, for $x \neq y$ in X with $b_\xi(x, y) = 0$, we have $K_{\xi x} \cap K_{\xi y} = \emptyset$. Therefore, we get

$$\{g \in \Gamma_\xi^0 | gK_{\xi x} \cap K_{\xi x} \neq \emptyset\} = \{g \in \Gamma_\xi^0 | gx = x\},$$

and the latter is finite by assumption. □

The group Γ_ξ can not be too large:

LEMMA 2.9. *Let ξ be in ∂X. If $\chi_\xi \neq 0$ on Γ_ξ, then Γ_ξ fixes a point in U_ξ. In particular, we always have $\Gamma_\xi \neq \Gamma$.*

PROOF. Assume $\chi_\xi = 0$. Then, for every x in X, we have

$$\Gamma_\xi x \subset \{y \in X | b_\xi(x, y) = 0\},$$

hence $\Gamma_\xi \neq \Gamma$ by Lemma 2.4.

Assume $\chi_\xi \neq 0$, that is, $\chi_\xi(\Gamma_\xi)$ is a non trivial subgroup of $\mathbb{Z}$. Let g be in Γ_ξ such that $\chi_\xi(g) < 0$ and $\chi_\xi(\Gamma_\xi) = \chi_\xi(g)\mathbb{Z}$. By Lemma 2.7, the automorphism g is hyperbolic with repulsive fixed point ξ. Let $\eta \in U_\xi$ be its attractive fixed point. We claim that η is fixed by Γ_ξ. Indeed, as by Lemma 2.8, the action of Γ_ξ^0 on U_ξ is proper, there exists a neighborhood V of η in U_ξ such that $V \cap \Gamma_\xi^0 \eta = \{\eta\}$. As η is the attractive fixed point of g, for every ζ in U_ξ, there exists $n \geq 0$ with $g^n \zeta \in V$. Since Γ_ξ^0 is normal in Γ_ξ and $g\eta = \eta$, we have $g\Gamma_\xi^0 \eta = \Gamma_\xi^0 \eta$. Thus, we get $\Gamma_\xi^0 \eta = \{\eta\}$. By assumption, we have $\Gamma_\xi = g^{\mathbb{Z}} \Gamma_\xi^0$ and therefore η is a fixed point of Γ_ξ. In particular, for x in $(\xi\eta)$, we have $\Gamma_\xi x \subset (\xi\eta)$, hence again $\Gamma_\xi \neq \Gamma$ by Lemma 2.4. □

The action of Γ on ∂X is minimal.

LEMMA 2.10. *Let ξ be in ∂X. Then $\Gamma\xi$ is dense in ∂X.*

PROOF. Let g be a hyperbolic element of Γ with attractive fixed point ζ and repulsive fixed point η. If $\xi \neq \eta$, we have $g^n\xi \xrightarrow[n\to\infty]{} \zeta$. If $\xi = \eta$, by Lemma 2.9, we have $\Gamma_\xi \neq \Gamma$, that is, we can find h in Γ with $h\xi \neq \xi$. Then, $g^n h\xi \xrightarrow[n\to\infty]{} \zeta$. In both cases, ζ belongs to the closure of $\Gamma\xi$, hence $\Gamma\xi$ is dense by Lemma 2.6. □

We can now finish the proof of Proposition 2.3. This relies on the classical shadowing argument from hyperbolic dynamics (see [**26**]).

PROOF OF PROPOSITION 2.3. First, note that the two statements in the proposition are equivalent. Indeed, for s in $\mathscr{S}$, saying that the image of s in $\Gamma\backslash\mathscr{S}$ has dense orbit under T is saying that s has dense orbit under the $(\Gamma \times \mathbb{Z})$-action on $\mathscr{S}$ defined by

$$(\gamma, n) \cdot s = \gamma(T^n s) = T^n(\gamma s), \quad \gamma \in \Gamma, \quad n \in \mathbb{Z}, \quad s \in \mathscr{S}.$$

Now, the surjective map $\mathscr{S} \to \partial^2 X, s \mapsto (s_-, s_+)$ identifies $\partial^2 X$ with the quotient of $\mathscr{S}$ by the $T^{\mathbb{Z}}$-action, so that saying that s has dense $(\Gamma \times \mathbb{Z})$-orbit in $\mathscr{S}$ is saying that (s_-, s_+) has dense Γ-orbit in $\partial^2 X$.

We will now show that the action of Γ on $\partial^2 X$ admits dense orbits. As $\partial^2 X$ is a locally compact topological space with a countable basis, it suffices to prove that, for every non empty open subsets U and V of $\partial^2 X$, there exists γ in Γ with $\gamma U \cap V \neq \emptyset$ (existence of a dense set of points with dense orbits then follows from a Baire category argument).

One can assume that U (resp. V) is of the form $U^- \times U^+$ (resp. $V^- \times V^+$) where U^-, U^+, V^- and V^+ are non empty open subsets of ∂X and $U^- \cap U^+ = V^- \cap V^+ = \emptyset$. We fix a hyperbolic element g in Γ (which exists by Lemma 2.6). Let g^- be the repulsive fixed point of g and g^+ be its attractive fixed point. By Lemma 2.10, there exists γ in Γ with $\gamma g^+ \in V^+$. Thus, up to replacing V by its image by γ^{-1}, we can assume that g^+ belongs to V^+. In the same way, we can assume that g^- belongs to U^-. Now, we fix ξ in U^+ and η in V^-. We can find an integer $n \geq 0$ such that one has $g^n\xi \in V^+$ and $g^{-n}\eta \in U^-$. In particular, we have $(g^{-n}\eta, \xi) \in U^- \times U^+$ whereas $(\eta, g^n\xi) \in V^- \times V^+$, hence $g^n U \cap V \neq \emptyset$ and the result follows. □

2.3. Boundary cocycles

We now introduce smooth boundary cocycles: they are generalizations of the Busemann cocycle. Recall that, given a locally compact

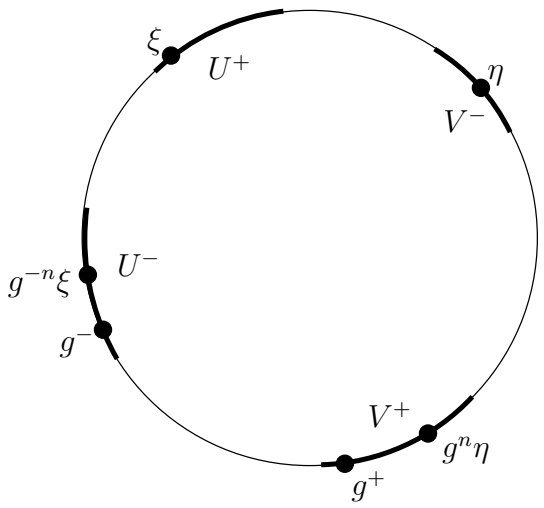

FIGURE 2. Proof of Proposition 2.3

topological space U, we let $\mathcal{D}(U)$ denote the space of smooth functions with compact support on U. We will freely identify functions in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ with Γ-invariant smooth functions on $\mathscr{S}$.

Two smooth functions f and g in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ are said to be cohomologuous if there exists some smooth function h such that $f - g = h \circ T - h$. Note that in particular, f is cohomologuous with $f \circ T$. By Proposition 2.3, the smooth function h such that $f - g = h \circ T - h$ is unique up to the addition of a constant. A smooth function f is said to be even if f is cohomologuous to $f \circ \iota$.

By a smooth boundary cocycle, we shall mean a set-theoretic cocycle $X \times X \to \mathcal{D}(\partial X)$. More precisely, such a cocycle B is a map $\partial X \times X \times X \to \mathbb{R}, (\xi, x, y) \mapsto B_\xi(x, y)$ such that, for any ξ in ∂X and x, y, z in X,

$$B_\xi(x, z) = B_\xi(x, y) + B_\xi(y, z)$$

and that, for any x, y in X, the function $\xi \mapsto B_\xi(x, y)$ is locally constant on ∂X. The group of automorphisms of X acts in a natural way on the space of smooth boundary cocycles and in the sequel, we shall only consider Γ-invariant smooth boundary cocyles, that is, we require that for any γ in Γ, ξ in ∂X and x, y in X,

$$B_{\gamma\xi}(\gamma x, \gamma y) = B_\xi(x, y).$$

Two Γ-invariant smooth boundary cocycles B and C are said to be cohomologuous if there exists a Γ-invariant smooth function F on $X \times$

∂X such that, for any ξ in ∂X and x, y in X,

$$B_\xi(x,y) - C_\xi(x,y) = F(x,\xi) - F(y,\xi).$$

EXAMPLE 2.11. The Busemann cocycle is a smooth boundary cocycle which is invariant under all automorphisms of X.

There is a general philosophy, coming from [**35**], that under some regularity assumptions, there is a bijection between cohomology classes of functions on $\Gamma\backslash\mathscr{S}$ and cohomology classes of Γ-invariant boundary cocycles on X. We shall make it explicit in the case of smooth objects (see [**15**] for the case of Hölder continuous objects).

To begin with, let us give an alternate definition of a smooth boundary cocycle. This is a kind of generalization of the construction of the Busemann cocycle.

LEMMA 2.12. *For any smooth function f on $X \times \partial X$, there exists a unique smooth boundary cocycle B such that, for any x in X and ξ in ∂X, if x_1 is the unique neighbour of x on $[x\xi)$, we have*

$$B_\xi(x, x_1) = f(x,\xi).$$

PROOF. Let x and y be in X, ξ be in ∂X and z be a point in $[x\xi) \cap [y\xi)$. We denote by $x_0 = x, x_1, \ldots, x_n = z$ the geodesic path from x to z and by $y_0 = y, y_1, \ldots, y_p = z$ the geodesic path from y to z. The number

$$\sum_{h=0}^{n-1} f(x_h,\xi) - \sum_{h=0}^{p-1} f(y_h,\xi)$$

does not depend on z. We denote it by $B_\xi(x,y)$. One easily checks that B is then the unique smooth boundary cocycle satisfying the requirements of the lemma. □

Let us now focus on Γ-invariant objects. Consider f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ and a Γ-invariant smooth boundary cocycle B. We shall say that f is a potential for B if f is cohomologuous to the smooth function $s \mapsto B_{s_+}(\pi(s), \pi(Ts))$.

PROPOSITION 2.13. *The map which sends a Γ-invariant smooth boundary cocycle to the set of its potentials induces a bijection between the set of cohomology classes of Γ-invariant smooth boundary cocycles and the set of cohomology classes of smooth functions on $\Gamma\backslash\mathscr{S}$.*

Before proving Proposition 2.13, let us give a lemma which will allow us to get surjectivity of the involved map between cohomology classes.

There exists a natural surjective map $\mathscr{S} \to X \times \partial X$, namely the map $s \mapsto (\pi(s), s_+)$. For f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$, if f factors through a smooth function on $X \times \partial X$, then by Lemma 2.12 we can associate to it a Γ-invariant smooth boundary cocycle. To extend this to any f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$, let us describe more precisely the fibers of the map $\mathscr{S} \to X \times \partial X$.

For s in $\mathscr{S}$, define M_s as the set of those t in $\mathscr{S}$ such that $\pi(s) = \pi(t)$ and $s_+ = t_+$. This is a compact subset of $\mathscr{S}$ and we have $TM_s \subset M_{Ts}$. We say that a function f in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ is M-invariant if for any s in $\mathscr{S}$, for any t in M_s, we have $f(s) = f(t)$.

From the dynamical point of view, M_s plays the role of a local stable leaf for s. Thus, we get

LEMMA 2.14. *Let f be in $\mathcal{D}(\Gamma\backslash\mathscr{S})$. There exists $k \geq 0$ such that $f \circ T^k$ is M-invariant.*

PROOF. Heuristically, M_s being a piece of the stable leaf of s with respect to the transformation T, if t belongs to M_s, the points $T^k\Gamma s$ and $T^k\Gamma t$ must get closer and closer in $\Gamma\backslash\mathscr{S}$. The conclusion follows since f is locally constant. Let us make this argument more precise.

For k in $\mathbb{N}$ and s in $\mathscr{S}$, set $M_s^k = T^k M_{T^{-k}s}$. One has $\bigcap_{k\geq 0} M_s^k = \{s\}$. We let D be the diagonal in $(\Gamma\backslash\mathscr{S})^2$ and $D^k \subset (\Gamma\backslash\mathscr{S})^2$ be the image of the set $\{(s,t) \in \mathscr{S} | t \in M_s^k\}$ under the natural map $\mathscr{S}^2 \to (\Gamma\backslash\mathscr{S})^2$.

We claim that we have $\bigcap_{k\geq 0} D^k = D$ in $\Gamma\backslash\mathscr{S}$. Indeed, let s and t be in $\mathscr{S}$ and assume that, for every $k \geq 0$, one has $(\Gamma s, \Gamma t) \in D^k$. We need to prove that s belongs to Γt. By assumption, for every $k \geq 0$, there exists γ_k in Γ with $\gamma_k t \in M_s^k$. We can assume that $\gamma_0 = e$ the identity element. Then, let $x = \pi(s) = \pi(t)$ and $\xi = s_+ = t_+$ be the common base point and the common endpoint of s and t. For any $k \geq 0$, as $\gamma_k t$ belongs to $M_s^k \subset M_s$, we also have

$$\gamma_k x = \pi(\gamma_k t) = \pi(s) = x \text{ and } \gamma_k \xi = (\gamma_k t)_+ = s_+ = \xi.$$

Thus, γ_k belongs to the group Γ_ξ^0 defined in Section 2.2. Let $\eta = s_-$ and $\zeta = t_-$ be the origins of s and t. By Lemma 2.8, the action of Γ_ξ^0 on $U_\xi = \partial X \smallsetminus \{\xi\}$ is proper, hence $\Gamma_\xi^0\zeta$ is a closed subset of U_ξ. As, for every $k \geq 0$, $\gamma_k t$ belongs to M_s^k, we have $\gamma_k\zeta \xrightarrow[k\to\infty]{} \eta$. Therefore, there exists $k \geq 0$ with $\gamma_k\zeta = \eta$. As $\gamma_k x = x$ and $\gamma_k\xi = \xi$, we get $\gamma_k t = s$ and we are done.

Now, let (U_i) be an open cover of $\Gamma\backslash\mathscr{S}$ such that f is constant on each of the U_i. The set $U = \bigcup_i U_i \times U_i$ is open in $(\Gamma\backslash\mathscr{S})^2$ and contains D. By compactness, there exists $k \geq 0$ with $D^k \subset U$.

Let s be in $\mathscr{S}$ and t be in M_s, and let $\overline{s}$ and $\overline{t}$ be their images in $\Gamma\backslash\mathscr{S}$. By definition we have $(T^k\overline{s}, T^k\overline{t}) \in D^k$, hence $(T^k\overline{s}, T^k\overline{t}) \in U$. Now, from the definition of U, we get $f(T^k\overline{s}) = f(T^k\overline{t})$, which should be proved. □

To prove injectivity of the map between cohomology classes, we will need

LEMMA 2.15. *Let f be an M-invariant function in $\mathcal{D}(\Gamma\backslash\mathscr{S})$. If f is cohomologuous to 0, then any function h in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ such that $f = h - h\circ T$ is M-invariant.*

PROOF. Let us first construct an M-invariant function h with $f = h - h\circ T$. Let h_1 be in $\mathcal{D}(\Gamma\backslash\mathscr{S})$ with $f = h_1 - h_1\circ T$. By Lemma 2.14, there exists $k \geq 0$ such that $h_2 = h_1 \circ T^k$ is M-invariant. We have

$$f - h_2 + h_2 \circ T = f - f \circ T^k = \sum_{j=0}^{k-1} f \circ T^j - \sum_{j=0}^{k-1} f \circ T^{j+1}$$

and we can set $h = h_2 + \sum_{j=0}^{k-1} f \circ T^j$.

Now, by Proposition 2.3, if h' is any other smooth function with $f = h' - h'\circ T$, then $h - h'$ is constant, hence h' also is M-invariant. □

PROOF OF PROPOSITION 2.13. By Lemmas 2.12 and 2.14, every smooth function on $\Gamma\backslash\mathscr{S}$ is the potential of some Γ-invariant boundary cocycle. To conclude the proof it only remains to prove that such a cocycle B is cohomologuous to 0 if and only if its potentials are cohomologuous to 0.

Assume first that B is cohomologuous to 0. Then there exists a smooth Γ-invariant function h on $X \times \partial X$ such that, for any x, y in X and ξ in ∂X, we have

$$B_\xi(x, y) = h(x, \xi) - h(y, \xi).$$

For s in $\mathscr{S}$, we get

$$B_{s_+}(\pi(s), \pi(Ts)) = h(\pi(s), s_+) - h(\pi(Ts), s_+),$$

hence the potentials of B are cohomologuous to 0.

Conversely, assume that the potentials of B are cohomologuous to 0. By Lemma 2.15, there exists a Γ-invariant smooth function h on $X \times \partial X$ such that, for any x in X and ξ in ∂X,

$$B_\xi(x, x_1) = h(x, \xi) - h(x_1, \xi),$$

where x_1 is the unique neighbour of x on $[x\xi)$. By the uniqueness part in Lemma 2.12, we get that B is a coboundary. □

2.4. Additive kernels

Still following the main lines of [**35**], one can associate to a cohomology class on $\Gamma\backslash\mathscr{S}$ a family of smooth functions on $\partial^2 X$. In case the cohomology class is even, the associated functions on $\partial^2 X$ are symmetric. This fact will be crucial later when using these functions to define symmetric bilinear forms.

PROPOSITION 2.16. *Let B be a Γ-invariant smooth boundary cocycle. Assume that the potentials of B are even. Then, there exists a smooth Γ-invariant function Ω on $X \times \partial^2 X$ such that, for any x in X, the function $(\xi, \eta) \mapsto \Omega_x(\xi, \eta)$ is symmetric and that, for any x, y in X and (ξ, η) in $\partial^2 X$, one has*

$$\Omega_x(\xi, \eta) - \Omega_y(\xi, \eta) = \frac{1}{2}(B_\xi(x, y) + B_\eta(x, y)). \tag{2.1}$$

The function Ω is unique up to a constant.

DEFINITION 2.17. Such a map Ω will be called an additive kernel associated to B. More generally, we will speak of the additive kernels associated to cocycles which are cohomologuous to B as the additive kernels associated to the cohomology class of B.

EXAMPLE 2.18. When $B = b$, the Busemann cocycle, let ω be the Gromov product, that is, for any x in X and (ξ, η) in ∂X, $\omega_x(\xi, \eta)$ is the distance from x to the geodesic line $(\xi\eta)$. Then ω satisifies the conclusions of the Proposition.

PROOF. Let us first define $\Omega_x(\xi, \eta)$ when x belongs to the geodesic line $(\xi\eta)$. We let f be the smooth function $s \mapsto B_{s_+}(\pi(s), \pi(Ts))$ on $\Gamma\backslash\mathscr{S}$. By assumption, the functions f and $f \circ \iota$ are cohomologuous. Hence, the functions f and $f \circ \iota T$ are cohomologuous. We chose a smooth function h such that

$$f - f \circ \iota T = h - h \circ T.$$

We claim that h is then invariant under ι, that is, $h \circ \iota = h$. Indeed, we have

$$\begin{aligned} h \circ \iota - h \circ \iota T &= (h - h \circ T^{-1}) \circ \iota = (h \circ T - h) \circ T^{-1}\iota \\ &= (f \circ \iota T - f) \circ T^{-1}\iota = f - f \circ \iota T = h - h \circ T. \end{aligned}$$

By Proposition 2.3, $h - h \circ \iota$ is a constant function. Let c be its value. As ι^2 is the identity map, we have $h = h \circ \iota + c = h + 2c$, hence $c = 0$, and h is ι-invariant.

If (ξ, η) is in $\partial^2 X$ and if x belongs to the geodesic line $(\xi\eta)$, we set

$$\Omega_x(\xi, \eta) = \frac{1}{2} h(s),$$

where s is the unique parametrized geodesic line such that $s_- = \xi$, $s_+ = \eta$ and $\pi(s) = x$. As h is ι-invariant, we have $\Omega_x(\xi, \eta) = \Omega_x(\eta, \xi)$.

Let us check that (2.1) holds on $(\xi\eta)$. We let y be the unique neighbour of x on $[x\eta)$. By definition we have, on one hand,

$$\Omega_y(\xi, \eta) = \frac{1}{2} h(Ts)$$

and, on the other hand,

$$B_\eta(x, y) = f(s) \text{ and } B_\xi(x, y) = -B_\xi(y, x) = -f(\iota T s).$$

Hence (2.1) holds for any two neighbouring points x and y on $(\xi\eta)$. By the cocycle identity, it holds for any two points.

Now, if x is any element in X, we set

$$\Omega_x(\xi, \eta) = \Omega_y(\xi, \eta) + \frac{1}{2}(B_\xi(x, y) + B_\eta(x, y)),$$

where y is on the geodesic line $(\xi\eta)$. As (2.1) holds on $(\xi\eta)$, this does not depend on y. One easily checks that (2.1) holds everywhere.

Uniqueness follows from the fact that Γ has a dense orbit on $\partial^2 X$ (see Proposition 2.3). □

The additive kernels determine the cocycle.

LEMMA 2.19. *Let B be an even smooth Γ-invariant boundary cocycle. Assume the cohomology class of B admits 0 as an additive kernel. Then B is cohomologuous to 0.*

PROOF. Indeed, up to replacing B by a cohomologuous cocycle, one can assume that, for any x, y in X and $\xi \neq \eta$ in ∂X, one has $B_\xi(x, y) + B_\eta(x, y) = 0$. Let ξ, η, ζ be three different points in ∂X, which exist due to our assumptions on X. We get $B_\xi(x, y) = -B_\eta(x, y) = B_\zeta(x, y) = -B_\xi(x, y)$, hence $B_\xi(x, y) = 0$ and we are done. □

Let still ω be the Gromov product, as in Example 2.18. To prove that certain formulas make sense, we shall use

LEMMA 2.20. *Let Ω be an additive kernel associated to a Γ-invariant smooth even boundary cocycle B. Then there exists $C \geq 0$ such that, for any x in X and $\xi \neq \eta$ in ∂X, one has*

$$|\Omega_x(\xi, \eta)| \leq C(1 + \omega_x(\xi, \eta)).$$

PROOF. The Γ-invariant function $h : s \mapsto \Omega_{\pi(s)}(s_-, s_+)$ on $\mathscr{S}$ is smooth. As $\Gamma\backslash\mathscr{S}$ is compact, it is bounded. In the same way, the smooth Γ-invariant function $f : s \mapsto B_{s_+}(\pi(s), \pi(Ts))$ on $\mathscr{S}$ is bounded. We chose $C \geq 0$ with $C \geq \max(\|f\|_\infty, \|h\|_\infty)$ and the result follows by the defining properties of Ω. □

2.5. Normalized smooth functions

We will give a formula for the function Ω in a particular case that will play an important role in the book.

For any $k \geq 1$, we denote by X_k the set

$$X_k = \{(x, y) \in X^2 | d(x, y) = k\}.$$

Note that, for $k = 1$, the set X_1 is the set of oriented edges of X.

Given a Γ-invariant symmetric function w on X_k, we can define an even smooth function f on $\Gamma\backslash\mathscr{S}$ by setting, for $s = (x_h)_{h\in\mathbb{Z}}$ in $\mathscr{S}$,

$$f(s) = w(x_0, x_k).$$

We then say that f is a normalized even function.

LEMMA 2.21. *Any smooth even function on $\Gamma\backslash\mathscr{S}$ is cohomologuous to a normalized even function.*

PROOF. By Lemma 2.14, we can assume that f is M-invariant. Again, by Lemma 2.14, applied to the function $f \circ \iota$, there exists $k \geq 0$ such that, for any $s = (x_h)_{h\in\mathbb{Z}}$ and $t = (y_h)_{h\in\mathbb{Z}}$ in $\mathscr{S}$, if $x_h = y_h$ for any $h \geq k$, then $f(s) = f(t)$. In other words, there exists a Γ-invariant function v on X_k, such that, for any $s = (x_h)_{h\in\mathbb{Z}}$, $f(s) = v(x_0, x_k)$. This gives also, $f(\iota T^k s) = v(x_k, x_0)$, hence

$$\frac{1}{2}(f(s) + f(\iota T^k s)) = \frac{1}{2}(v(x_0, x_k) + v(x_k, x_0)).$$

Now, f being even, it is cohomologuous to $\frac{1}{2}(f + f \circ \iota T^k)$ and we are done. □

For normalized functions, we have an explicit formula for the associated additive kernel.

PROPOSITION 2.22. *Let w be a Γ-invariant symmetric function on X_k for some $k \geq 1$. Let f be the associated normalized smooth even function on $\Gamma\backslash\mathscr{S}$ and B be the smooth Γ-invariant boundary cocycle defined as in Lemma 2.12. Then, let us give a formula for the function Ω from Proposition 2.16. For x in X and (ξ, η) in $\partial^2 X$, we let $(y_i)_{i\geq 0}$ and $(z_i)_{i\geq 0}$ denote the geodesic rays $[x\xi)$ and $[x\eta)$. Let $j = \omega_x(\xi, \eta)$ be*

the distance from x to the geodesic line $(\xi\eta)$ so that $y_0 = z_0, \dots, y_j = z_j$ but $y_{j+1} \neq z_{j+1}$. Then one can set

$$\Omega_x(\xi,\eta) = \frac{1}{2}\sum_{h=0}^{j-1}(w(y_h, y_{h+k}) + w(z_h, z_{h+k})) - \frac{1}{2}\sum_{h=1}^{k-1} w(y_{j+h}, z_{j+k-h}).$$

The proof is straightforward.

DEFINITION 2.23. If w is a Γ-invariant symmetric function on X_k for some $k \geq 1$, we say that the additive kernels Ω associated to the cohomology class of the normalized function

$$s = (x_h)_{h\in\mathbb{Z}} \mapsto w(x_0, x_k)$$

on $\mathscr{S}$ are the additive kernels associated to w.

CHAPTER 3

Normalized kernels and bilinear forms

In this chapter, we will associate to every bounded symmetric function w on X_k, $k \geq 1$, a bounded symmetric bilinear form on a certain Hilbert space H_0^ω. The definition of this bilinear form will then be related to the formula in Proposition 2.22.

Recall that, for $k \geq 1$, X_k stands for the set of pairs (a, b) in X^2 with $d(a, b) = k$. We equip the countable set X_k with its counting measure and, for $1 \leq p < \infty$, we let $\ell^p(X_k)$ denote the associated space of p-integrable functions, equipped with its natural norm. In other words, a function $\theta : X_k \to \mathbb{R}$ is in $\ell^p(X_k)$ if and only if one has $\sum_{\substack{a,b \in X \\ d(a,b)=k}} |\theta(a, b)|^p < \infty$ and the latter number is then $\|\theta\|_p^p$.

3.1. The Hilbert spaces H^ω and H_0^ω

We start by building the space H_0^ω. In case X is homogeneous (that is, for any x in X, its number of neighbourgs $d(x)$ is independent of x), the space H_0^ω is the skew-symmetric special representation of the group of automorphisms of X as built in [**38**] and described in [**21**].

We fix x in X which will play the role of an origin and we associate to it a smooth function χ_x on $X_1 \times \partial X$ as follows: for any $a \sim b$ in X and ξ in ∂X, we set

$$\begin{aligned} &\chi_x(a, b, \xi) = 1 && \text{if } [ab] \subset [x\xi) \text{ and } [xa] \cap [b\xi) = \emptyset. \\ &\chi_x(a, b, \xi) = -1 && \text{if } [ab] \subset [x\xi) \text{ and } [xb] \cap [a\xi) = \emptyset. \\ &\chi_x(a, b, \xi) = 0 && \text{else.} \end{aligned}$$

In other words, $\chi_x(., ., \xi)$ may be seen as an oriented indicator function of the geodesic ray $[x\xi)$. In particular, note that $\chi_x(a, b, \xi)$ is skew-symmetric in (a, b).

Let us precisely describe how this map depends on x. From a direct computation, we get

LEMMA 3.1. *For any x, y and $a \sim b$ in X, the function $\chi_x(a,b,.) - \chi_y(a,b,.)$ is constant on ∂X. Its value $\delta_{xy}(a,b)$ is given by*

$$\begin{aligned} \delta_{xy}(a,b) &= 1 && \text{if } [ab] \subset [xy] \text{ and } [xa] \cap [by] = \emptyset. \\ \delta_{xy}(a,b) &= -1 && \text{if } [ab] \subset [xy] \text{ and } [xb] \cap [ay] = \emptyset. \\ \delta_{xy}(a,b) &= 0 && \text{else.} \end{aligned}$$

In particular, δ_{xy} is a finitely supported function on X_1.

Now, here comes the key observation which will allow us to relate the scalar product on a certain subspace of $\ell^2(X_1)$ to integral formulas on the boundary. Recall that ω is the Gromov product of X, that is, for any x in X and $\xi \neq \eta$ in ∂X, $\omega_x(\xi,\eta)$ is the distance from x to the geodesic line $(\xi\eta)$ (see Example 2.18). The proof of the following is immediate.

LEMMA 3.2. *Fix x in X and $\xi \neq \eta$ in ∂X. Let y in X be such that $[x\xi) \cap [x\eta) = [xy]$. Then, for $a \sim b$ in X, we have*

$$\begin{aligned} \chi_x(a,b,\xi)\chi_x(a,b,\eta) &= 1 && \text{if } [ab] \subset [xy] \\ &= 0 && \text{else.} \end{aligned}$$

In particular, the function $(a,b) \mapsto \chi_x(a,b,\xi)\chi_x(a,b,\eta)$ is finitely supported on X_1 and we have

$$\sum_{a \sim b \in X} \chi_x(a,b,\xi)\chi_x(a,b,\eta) = 2\omega_x(\xi,\eta).$$

Recall that, if U is a totally discontinuous compact topological space, the space of distributions on U is denoted by $\mathcal{D}^*(U)$: this is the dual space to the space $\mathcal{D}(U)$ of smooth functions on U. Also, $\mathcal{D}_0^*(U)$ denotes the space of distributions T on U such that $\langle T, \mathbf{1}\rangle = 0$, which we view as the dual space to $\overline{\mathcal{D}}(U) = \mathcal{D}(U)/\mathbb{R}$.

As $\chi_x(a,b,\xi)$ depends smoothly on ξ, we can use it to define a linear map from distributions to functions on X_1. Fix x in X. If T is in $\mathcal{D}(\partial X)$, we let $\mathcal{P}_x T$ be the skew-symmetric function on X_1 such that, for any $a \sim b$ in X,

$$\mathcal{P}_x T(a,b) = \langle T, \chi_x(a,b,.)\rangle.$$

From Lemma 3.1 we immediately get

LEMMA 3.3. *For x, y in X, T in $\mathcal{D}^*(\partial X)$ and $a \sim b$ in X, we have*

$$\mathcal{P}_x T(a,b) - \mathcal{P}_y T(a,b) = \delta_{xy}(a,b)\langle T, \mathbf{1}\rangle \text{ and } \langle T, \mathbf{1}\rangle = \sum_{z \sim x} \mathcal{P}_x T(x,z).$$

In particular $\mathcal{P}_x T - \mathcal{P}_y T$ is a finitely supported function on X_1 and, if T is in $\mathcal{D}_0^*(\partial X)$, the function $\mathcal{P}_x T = \mathcal{P}T$ does not depend on x.

REMARK 3.4. Since the map $\mathcal{P}$ on $\mathcal{D}_0^*(\partial X)$ does not depend on the choice of x, it commutes with the natural action of the group of automorphisms of X. This fact will be instrumental in our next constructions.

We can describe the spaces $\mathcal{P}_x\mathcal{D}^*(\partial X)$ and $\mathcal{P}\mathcal{D}_0^*(\partial X)$.

LEMMA 3.5. *For any x in X, the map $\mathcal{P}_x$ establishes a linear isomorphism between the space $\mathcal{D}^*(\partial X)$ and the space of skew-symmetric functions θ on X_1 such that, for any $a \neq x$ in X, one has $\sum_{b\sim a}\theta(a,b) = 0$.*

The map $\mathcal{P}$ establishes a linear isomorphism between the space $\mathcal{D}_0^*(\partial X)$ and the space of skew-symmetric functions θ on X_1 such that, for any a in X, one has $\sum_{b\sim a}\theta(a,b) = 0$.

In the proof, we shall need the following notation which will also be used later in the book. If $x \neq y$ are in X, we let U_{xy} be the closed open subset in ∂X defined by

$$U_{xy} = \{\xi \in \partial X | [xy] \cap [y\xi) = \{y\}\}.$$

In the language of hyperbolic geometry, the set U_{xy} is the shadow of y viewed from x (see for example [**45**]). By definition, for any x in X, the open sets U_{xy}, $y \in X \smallsetminus \{x\}$, generate the topology of ∂X. From this, we get

LEMMA 3.6. *Fix x in X. Let φ be in $\mathcal{D}(\partial X)$. There exists $\ell \geq 1$ and a function f on $S^\ell(x)$ such that $\varphi = \sum_{y\in S^\ell(x)} f(y)\mathbf{1}_{U_{xy}}$.*

PROOF. By definition, for every ξ in ∂X, there exists $y \neq x$ such that $\xi \in U_{xy}$ and f is constant on U_{xy}. By compactness, there exists finitely many $y_1, \ldots, y_n$ in $X \smallsetminus \{x\}$ such that, for $1 \leq i \leq n$, f is constant on U_{xy_i} and these open subset cover ∂X. The result follows by taking $\ell = \max_{1\leq i\leq n} d(x, y_i)$. □

PROOF OF LEMMA 3.5. Note that the second part of the Lemma follows from the first and the formula for $\langle T, \mathbf{1}\rangle$, $T \in \mathcal{D}^*(\partial X)$, from Lemma 3.3.

Now, let us prove the first part. Let θ be a skew-symmetric function on X_1 such that, for any $a \neq x$ in X, one has $\sum_{b\sim a}\theta(a,b) = 0$ and let us build T in $\mathcal{D}^*(\partial X)$ with $\mathcal{P}_x T = \theta$. For y in X, $y \neq x$, we let y_- be the unique neighbour of y on $[xy]$.

Pick φ and chose, as in Lemma 3.6, some $\ell \geq 1$ and a function f on $S^\ell(x)$ such that $\varphi = \sum_{y\in S^\ell(x)} f(y)\mathbf{1}_{U_{xy}}$. We claim that the number $u_\ell = \sum_{y\in S^\ell(x)} f(y)\theta(y_-, y)$ does not depend on ℓ. Indeed, for any y in

$S^\ell(x)$, we have

$$\theta(y_-, y) = \sum_{\substack{z\sim y \\ z\neq y_-}} \theta(y, z),$$

hence $u_\ell = u_{\ell+1}$. As this number clearly depends linearly on φ, we can define a distribution T by setting $\langle T, \varphi\rangle = u_\ell$ for ℓ large enough.

Let us check that we have $\theta = \mathcal{P}_x T$. To this aim, we pick (a, b) in X_1. With no loss of generality, we can assume that we have $d(x, b) = \ell \geq 1$ and $a \in [xb]$. Then, by construction, we have $\chi_x(a, b, .) = \mathbf{1}_{U_{xb}}$, hence $\mathcal{P}_x T(a, b) = \langle T, \mathbf{1}_{U_{xb}}\rangle = \theta(a, b)$ and the description of $\mathcal{P}_x\mathcal{D}^*(\partial X)$ follows. □

We can now use the map $\mathcal{P}$ to define a remarkable Hilbert space. We let H^ω denote the space of distributions ρ in $\mathcal{D}^*(\partial X)$ such that, for some x in X, the function $\mathcal{P}_x\rho$ belongs to $\ell^2(X_1)$. By Lemma 3.3, this condition does not depend on x. We equip it with the norm induced by this embedding: the restriction of the norm to $H_0^\omega = \mathcal{D}_0^*(\partial X) \cap H^\omega$ is independent of x. As, by Lemma 3.5, $\mathcal{P}\mathcal{D}^*(\partial X) \cap \ell^2(X_1)$ is a closed subspace of $\ell^2(X_1)$, the space H_0^ω is a Hilbert space.

Let ν be a Borel probability measure on ∂X. For any $1 \leq p \leq \infty$, we let $\mathfrak{M}^p(\nu) \subset \mathcal{D}(\partial X)$ denote the space of signed Borel measures on ∂X whose density is p-integrable with respect to ν. We write $\mathfrak{M}_0^p(\nu) = \mathfrak{M}^p(\nu) \cap \mathcal{D}_0^*(\partial X)$ for the set of those ρ in $\mathfrak{M}^p(\nu)$ with $\rho(\mathbf{1}) = 0$.

Assume ν is atom-free, so that for any x in X, ω_x is defined $\nu\otimes\nu$-almost everywhere. For $p \geq 1$, we say that ω is ν-p-integrable if ω_x is $\nu\otimes\nu$-p-integrable: this condition does not depend on the choice of x since, for any x, y in X, one has $|\omega_x - \omega_y| \leq d(x, y)$.

These properties are closely related to the structure of the space H^ω by the following Proposition which is a rather straightforward consequence of Lemma 3.2:

Proposition 3.7. *Let ν be an atom-free Borel probability measure on ∂X and x be in X. Then ω is ν-integrable if and only if ν belongs to H^ω. In this case, we have*

$$\|\mathcal{P}_x\nu\|_2^2 = 2\int_{\partial^2 X} \omega_x \mathrm{d}(\nu\otimes\nu).$$

In the same way, for $1 \leq p < \infty$ and $1 < p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, if ω is ν-p-integrable then $\mathfrak{M}^{p'}(\nu)$ is contained in H^ω. In this case, for every ρ in $\mathfrak{M}^{p'}(\nu)$, we have

$$\|\mathcal{P}_x\rho\|_2^2 = 2\int_{\partial^2 X} \omega_x \mathrm{d}(\rho\otimes\rho).$$

PROOF. For y in X, $y \neq x$, let y_- denote the neighbour of y on $[xy]$. For any symmetric function θ in $\ell^1(X_1)$, we have

$$\sum_{a\sim b\in X} \theta(a,b) = 2 \sum_{y\in X\smallsetminus\{x\}} \theta(y_-,y). \tag{3.1}$$

For any $k \geq 0$ the function $\omega_x^k = \min(\omega_x, k)$ is smooth on $\partial X \times \partial X$. Let ρ be any element of $\mathcal{D}^*(\partial X)$. By Lemma 3.2, we have

$$\sum_{\substack{y\in X\smallsetminus\{x\}\\ d(x,y)\leq k}} \mathcal{P}_x\rho(y_-,y)^2 = \rho\otimes\rho(\omega_x^k), \tag{3.2}$$

where $\rho\otimes\rho$ is the tensor square distribution of ρ when we use the natural identification $\mathcal{D}(\partial X\times\partial X) \simeq \mathcal{D}(\partial X)\otimes\mathcal{D}(\partial X)$. By (3.1), the left hand-side of (3.2) is increasing, with finite limit if and only if ρ belongs to H^ω. If $\rho = \nu$ is a Borel probability measure, by the Monotone Convergence Theorem, the right hand-side is increasing with finite limit if and only if ω is ν-integrable. The first part of the proposition follows by taking the limit as $k \to \infty$ in (3.2).

Assume now ω is ν-p-integrable and assume the ρ in (3.2) belongs to $\mathfrak{M}^{p'}(\nu)$. Then the right hand-side of (3.2) converges to $\int_{\partial^2 X} \omega_x \mathrm{d}(\rho\otimes\rho)$. Hence the left hand-side has a finite limit, that is, $\mathcal{P}_x\rho$ belongs to $\ell^2(X_1)$. The computation of the norm follows by taking limits in (3.2). □

3.2. Bilinear forms on H^ω

We will now define symmetric bilinear forms on H^ω for which an analogue of Proposition 3.7 will be true, where ω will be replaced by an additive kernel as in Proposition 2.22.

Fix $k \geq 1$. If θ is a function on X_1, we define $\theta_{2,k}$ as the function on X_k such that, for any a, b in X with $d(a,b) = k$, one has $\theta_{2,k}(a,b) = \theta(a,a_1)\theta(b_1,b)$, where a_1 is the neighbour of a on $[ab]$ and b_1 the neighbour of b on $[ab]$. Note that if θ is skew-symmetric, then $\theta_{2,k}$ is symmetric.

If $k = 1$, we have $\theta_{2,1} = \theta^2$. In general, we easily get

LEMMA 3.8. *For any θ in $\ell^2(X_1)$, the function $\theta_{2,k}$ belongs to $\ell^1(X_k)$ and we have $\|\theta_{2,k}\|_1 \leq (D-1)^{k-1}\|\theta\|_2^2$, where $D = \sup_{x\in X} d(x)$.*

Let w be a bounded symmetric function on X_k. We associate to w the symmetric bilinear form Φ_w on $\ell^2(X_1)$ such that, for any θ in

$\ell^2(X_1)$,

$$\Phi_w(\theta,\theta) = \frac{1}{2}\sum_{(a,b)\in X_k} w(a,b)\theta_{2,k}(a,b). \tag{3.3}$$

By Lemma 3.8 above, this bilinear form is well-defined and bounded.

For any x in X and $\xi \neq \eta$ in ∂X, we also set, as in Proposition 2.22,

$$\Omega_x^w(\xi,\eta) = \frac{1}{2}\sum_{h=0}^{j-1}(w(y_h,y_{h+k}) + w(z_h,z_{h+k})) - \frac{1}{2}\sum_{h=1}^{k-1} w(y_{j+h},z_{j+k-h}),$$

where $(y_h)_{h\geq 0}$ and $(z_h)_{h\geq 0}$ are the geodesic rays $[x\xi)$ and $[x\eta)$ and $j = \omega_x(\xi,\eta)$. Note that one has $|\Omega_x^w| \leq (\omega_x + (k-1))\,\|w\|_\infty$, so that integrability properties for ω imply the same for Ω^w.

We now have an analogue of Proposition 3.7 for the bilinear form Φ_w.

Proposition 3.9. *Fix $k \geq 1$. Let w be a symmetric bounded function on X_k and ν be an atom-free Borel probability on ∂X such that ω is ν-integrable. Then, for any x in X, we have*

$$\Phi_w(\mathcal{P}_x\nu,\mathcal{P}_x\nu) = \int_{\partial^2 X} \Omega_x^w \mathrm{d}(\nu\otimes\nu).$$

In the same way, for $1 \leq p < \infty$ and $1 < p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, if ω is ν-p-integrable then, for any x in X and ρ,θ in $\mathfrak{M}^{p'}(\nu)$, we have

$$\Phi_w(\mathcal{P}_x\rho,\mathcal{P}_x\theta) = \int_{\partial^2 X} \Omega_x^w \mathrm{d}(\rho\otimes\theta).$$

As for Proposition 3.7, the proof of Proposition 3.9 relies on an elementary computation, which is a generalization of Lemma 3.2:

Lemma 3.10. *Let $k \geq 1$. Fix x in X and $\xi \neq \eta$ in ∂X. Let $(y_h)_{h\geq 0}$ and $(z_h)_{h\geq 0}$ be the geodesic rays $[x\xi)$ and $[x\eta)$ and $j = \omega_x(\xi,\eta)$. Then, if a,b are in X with $d(a,b) = k$ and a_1 and b_1 are the neighbours of a and b on $[ab]$, we have*

$$\chi_x(a,a_1,\xi)\chi_x(b_1,b,\eta) = \sum_{h=0}^{j-1}(\mathbf{1}_{(b,a)=(y_h,y_{h+k})} + \mathbf{1}_{(a,b)=(z_h,z_{h+k})}) - \sum_{h=1}^{k-1}\mathbf{1}_{(a,b)=(y_{j+h},z_{j+k-h})}.$$

The above formula has to be understood as an equality of numbers, where the left hand side is either -1, 0 or 1.

PROOF OF PROPOSITION 3.9. Again, this is an easy consequence of Lemma 3.10. Let us be more precise.

We start by defining a truncated version of Ω_x^w. Fix $\ell \geq 0$ and pick ξ, η in ∂X. We let again $(y_h)_{h\geq 0}$ and $(z_h)_{h\geq 0}$ be the geodesic rays $[x\xi)$ and $[x\eta)$. Now, if $\xi \neq \eta$ and $\omega_x(\xi,\eta) < \ell$, we set

$$\Omega_x^{w,\ell}(\xi,\eta) = \frac{1}{2}\sum_{\substack{0\leq h\leq j-1\\ h+k\leq \ell}} (w(y_h, y_{h+k}) + w(z_h, z_{h+k})) \\ - \frac{1}{2}\sum_{\substack{1\leq h\leq k-1\\ j+h\leq \ell\\ j+k-h\leq \ell}} w(y_{j+h}, z_{j+k-h}).$$

Else, we just set

$$\Omega_x^{w,\ell}(\xi,\eta) = \frac{1}{2}\sum_{h=0}^{\ell-k} w(y_h, y_{h+k}) + \frac{1}{2}\sum_{h=0}^{\ell-k} w(z_h, z_{h+k}).$$

Then, $\Omega_x^{w,\ell}$ is a smooth function on $\partial X \times \partial X$ and, by Lemma 3.10, for any ρ in $\mathcal{D}^*(\partial X)$, we have

$$\frac{1}{2}\sum_{\substack{(a,b)\in X_k\\ d(x,a)\leq \ell\\ d(x,b)\leq \ell}} (\mathcal{P}_x\rho)_{2,k}(a,b) = (\rho\otimes\rho)(\Omega_x^{w,\ell}). \tag{3.4}$$

Now, on one hand, by definition, if ρ is in H^ω, the left hand-side of (3.4) goes to $\Phi_w(\mathcal{P}_x\rho, \mathcal{P}_x\rho)$ as $\ell \to \infty$.

On the other hand, assume that ω is ν-integrable so that, by Proposition 3.7, ν belongs to H^ω. We have $\left|\Omega_x^{w,\ell}\right| \leq (\omega_x + (k-1))\left\|w\right\|_\infty$ and for any $\xi \neq \eta$ in ∂X, $\Omega_x^{w,\ell}(\xi,\eta) \xrightarrow[\ell\to\infty]{} \Omega_x^w(\xi,\eta)$. Hence, by the Dominated Convergence Theorem, for $\rho = \nu$, the right hand-side of (3.4) goes to $2\int_{\partial^2 X} \Omega_x^w \mathrm{d}(\nu\otimes\nu)$ as $\ell \to \infty$ and the first part of the Proposition follows.

The second part is proved in the same way. □

3.3. Bilinear forms on H_0^ω

We will now focus on the case where w is Γ-invariant and prove that the restriction of Φ_w to H_0^ω only depends on the cohomology class of the normalized smooth function associated to w.

PROPOSITION 3.11. *Let f be a smooth even Γ-invariant function on $\mathscr{S}$, $k \geq 1$ and w be a Γ-invariant symmetric function on X_k such that the normalized smooth function associated to w is cohomologuous*

to f. Then the symmetric bilinear form $(\rho,\theta)\mapsto\Phi_w(\mathcal{P}\rho,\mathcal{P}\theta)$ on H_0^ω is Γ-invariant. This bilinear form only depends on the cohomology class of f; in particular it does not depend on the choices of k and w as soon as f is cohomologuous to the normalized smooth function associated to w.

Let Ω be an additive kernel associated to the cohomology class of f. Then, for $1\leq p<\infty$ and $1<p'\leq\infty$ with $\frac{1}{p}+\frac{1}{p'}=1$, if ν is an atom-free Borel probability measure on ∂X such that ω is ν-p-integrable, for any x in X and ρ,θ in $\mathfrak{M}_0^{p'}(\nu)$, we have

$$\Phi_w(\mathcal{P}\rho,\mathcal{P}\theta)=\int_{\partial^2 X}\Omega_x\mathrm{d}(\rho\otimes\theta).$$

Remark 3.12. Let ν be as above. It easy to check that the formula in Proposition 3.11 defines a bilinear form on $\mathfrak{M}_0^{p'}(\nu)$ which does not depend on the choice of x nor of the one of Ω.

Indeed, pick a smooth boundary B cocycle as in (2.1). For any x,y in X, we have, for ρ in $\mathfrak{M}_0^{p'}(\nu)$,

$$\int_{\partial X\times\partial X}(\Omega_x(\xi,\eta)-\Omega_y(\xi,\eta))\mathrm{d}\rho(\xi)\mathrm{d}\rho(\eta)$$
$$=\int_{\partial X}B_\xi(x,y)\mathrm{d}\rho(\xi)\int_{\partial X}\mathrm{d}\rho(\eta)=0$$

since $\rho(\mathbf{1})=0$. In the same way, by the uniqueness statement in Proposition 2.16, if f is cohomologuous to 0, Ω is of the form

$$(x,\xi,\eta)\mapsto F(x,\xi)+F(y,\eta)$$

for some smooth function F on $X\times\partial X$ and $\int_{\partial X\times\partial X}\Omega_x\mathrm{d}(\rho\otimes\rho)=0$ by the same argument.

Therefore, one way of proving the independence statement in Proposition 3.11 would be to exhibit a Borel probability measure ν on ∂X such that $\mathfrak{M}^\infty(\nu)$ is dense in H^ω, as will be done later in Corollary 8.3. Here, we will chose an other more direct approach.

For normalized smooth even functions, we have a criterion for cohomology:

Lemma 3.13. *Let $k\geq k'\geq 1$ and $w:X_k\to\mathbb{R}$ and $w':X_{k'}\to\mathbb{R}$ be Γ-invariant symmetric functions. Then the smooth normalized functions associated to w and w' are cohomologuous if and only if there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that, for*

any x, y in X with $d(x,y) = k$, one has

$$w(x,y) = \frac{1}{k-k'+1} \sum_{h=0}^{k-k'} w'(x_h, x_{h+k'}) + v(x, y_{k-1}) - v(x_1, y), \tag{3.5}$$

where $x_0 = x, x_1, \dots, x_k = y$ is the geodesic path from x to y.

When $k = 1$, by convention, a skew-symmetric function on X_0 is 0.

By abuse of language, when w and w' are as above, we shall say that they are cohomologuous and, when w is cohomologuous to 0, we shall say that w is a coboundary.

Proof. First, one easily checks that the normalized smooth function associated to the function defined by the right hand-side of (3.5) is cohomologuous to the normalized smooth function defined by w'. This gives the "if" part of the statement and reduces the proof of the "only if"part to the case where $k' = k$.

In other words, to conclude, we need to show that, if for some Γ-invariant symmetric function w on X_k, the associated normalized smooth function is cohomologuous to 0, then there exists a skew-symmetric function v on X_{k-1} such that, for any (x,y) in X_k, one has $w(x,y) = v(x,y_1) - v(x_1,y)$, where x_1 and y_1 are the neighbours of x and y on $[xy]$. To do this, we will use the language of Section 2.3.

Indeed, by assumption, there exists a Γ-invariant smooth function h on $\mathscr{S}$ such that, for any $s = (x_h)_{h\in\mathbb{Z}}$, one has $w(x_0,x_k) = h(s) - h(Ts)$. Now, by Lemma 2.15, the function h is M-invariant, that is, it does not depend on the coordinates $(x_h)_{h<0}$. In the same way, for any such s, one has $w(x_0, x_{-k}) = h(\iota s) - h(T\iota s)$, hence

$$w(x_k, x_0) = h(\iota T^k s) - h(T\iota T^k s) = h(\iota T^k s) - h(\iota T^{k-1} s),$$

so that, again by Lemma 2.15, the function $h \circ \iota T^{k-1}$ is M-invariant, that is, h does not depend on the coordinates $(x_h)_{h\geq k}$. In other words, there exists a Γ-invariant function v on X_{k-1} such that, for any s as above, one has $h(s) = v(x_0, x_{k-1})$. We get, for any (x,y) in X_k, $w(x,y) = v(x,y_1) - v(x_1,y)$, where x_1 and y_1 are the neighbours of x and y on $[xy]$ and it only remains to prove that one can chose v to be skew-symmetric.

Let still x, y, x_1, y_1 be as above. We have

$$v(x,y_1) - v(x_1,y) = w(x,y) = w(y,x) = v(y,x_1) - v(y_1,x),$$

hence $v(x,y_1) + v(y_1,x) = v(x_1,y) + v(y,x_1)$. In other words, the Γ-invariant smooth function on $\mathscr{S}$,

$$s = (x_h)_{h\in\mathbb{Z}} \mapsto v(x_0, x_{k-1}) + v(x_{k-1}, x_0)$$

is T-invariant. By Proposition 2.3, this function is constant, that is, there exists c such that, for any (x, y) in X_{k-1}, one has $v(x, y) + v(y, x) = c$. The result follows by replacing v with $v - \frac{c}{2}$. □

By using Lemma 3.13, we can split the proof of independence in Proposition 3.11 into two cases.

LEMMA 3.14. *Let $k \geq 1$ and w be a bounded symmetric function on X_k. Assume that there exists a bounded skew-symmetric function v on X_{k-1} such that, for any (x, y) in X_k,*

$$w(x, y) = v(x, y_1) - v(x_1, y),$$

where x_1 and y_1 are the neighbours of x and y on $[xy]$. Then the bilinear form Φ_w is zero on the space $\mathcal{P}H_0^\omega$.

PROOF. By Lemma 3.5, we have to prove that, if θ is a skew-symmetric function in $\ell^2(X_1)$ such that, for any x in X, one has $\sum_{y\sim x} \theta(x, y) = 0$, then $\Phi_w(\theta, \theta) = 0$.

Indeed, if $k = 1$, then $v = 0$ and there is nothing to prove. If $k \geq 2$, we have

$$\Phi_w(\theta, \theta) = \sum_{x\in X} \sum_{x_1\sim x} \sum_{\substack{d(y_1,x)=k-1 \\ x_1\in[xy_1]}} \sum_{\substack{y\sim y_1 \\ y\notin[xy_1]}} \theta(x, x_1)\theta(y_1, y)v(x, y_1).$$

Now, if x and y_1 are as under the sum, we have

$$\sum_{\substack{y\sim y_1 \\ y\notin[xy_1]}} \theta(y_1, y) = \theta(y, y_2),$$

where y_2 is the neighbour of y_1 on $[xy_1]$. Thus, we get

$$\Phi_w(\theta, \theta) = -\sum_{(a,b)\in X_{k-1}} \theta_{2,k-1}(a, b)v(a, b),$$

where $\theta_{2,k-1}$ is the same as in Section 3.2. As $\theta_{2,k-1}$ is symmetric and v is skew-symmetric, the latter sum is zero and we are done. □

LEMMA 3.15. *Let $k \geq k' \geq 1$ and w and w' be bounded symmetric functions on X_k and $X_{k'}$. Assume that, for any (x, y) in X_k,*

$$w(x, y) = \frac{1}{k - k' + 1} \sum_{h=0}^{k-k'} w'(x_h, x_{h+k'}),$$

where $x_0 = x, x_1, \ldots, x_k = y$ is the geodesic path from x to y. Then the bilinear forms Φ_w and $\Phi_{w'}$ are equal to each other on the space $\mathcal{P}H_0^\omega$.

PROOF. Again, it suffices to prove that, if θ is a skew-symetric function in $\ell^2(X_1)$ with $\sum_{y\sim x}\theta(x,y) = 0$ for any x in X, one has $\Phi_w(\theta,\theta) = \Phi_{w'}(\theta,\theta)$. For such a θ, we have

$$2\Phi_w(\theta,\theta) \tag{3.6}$$
$$= \frac{1}{k-k'+1}\sum_{h=0}^{k-k'}\sum_{(x,y)\in X_{k'}}\sum_{\substack{d(a,y)=h+k'\\ x\in[ay]}}\sum_{\substack{d(b,x)=k-h\\ y\in[xb]}} w'(x,y)\theta(a,a_-)\theta(b_-,b),$$

where, if a and b are as under the sum, a_- and b_- are the neighbours of a and b in $[ay]$ and $[xb]$. Now, for any x, y in X with $x\sim y$, an easy induction argument shows that, for $h\geq 0$,

$$\sum_{\substack{d(a,y)=h+1\\ x\in[ay]}}\theta(a,a_-) = \theta(x,y),$$

hence, for (x,y) in $X_{k'}$ and $0\leq h\leq k-k'$,

$$\sum_{\substack{d(a,y)=h+k'\\ x\in[ay]}}\theta(a,a_-)\sum_{\substack{d(b,x)=k-h\\ y\in[xb]}}\theta(b_-,b) = \theta_{2,k'}(x,y).$$

By (3.6), we get $\Phi_w(\theta,\theta) = \Phi'_w(\theta,\theta)$ as required. □

PROOF OF PROPOSITION 3.11 . The fact that the definition of the bilinear form is independent on w follows from Lemmas 3.13, 3.14 and 3.15.

As the linear map $\mathcal{P} : H_0^\omega \to \ell^2(X_1)$ and the quadratic maps $\theta\mapsto\theta_{2,k}$, $k\geq 0$, commute with the action of the group of automorphisms of X, the bilinear form Φ_w is Γ-invariant on $\mathcal{P}H_0^\omega$ as soon as w is Γ-invariant.

Finally, the integral formula follows from the one in Proposition 3.9 and from Remark 3.12. □

In the next chapters, we will study those Γ-invariant functions w such that the associated symmetric bilinear form Φ_w is non-negative on H_0^ω. This will require us to introduce first several notions related to non-negative bilinear forms on vector spaces associated with X.

CHAPTER 4

Bilinear forms on smooth functions

In this chapter, we build scalar products on spaces of smooth functions. The topological dual spaces of these spaces with respect to these scalar products will later turn out to be defined by additive kernels.

4.1. Quadratic type functions

Recall that $\overline{\mathcal{D}}(\partial X) = \mathcal{D}(\partial X)/\mathbb{R}$ is the quotient space of $\mathcal{D}(\partial X)$ by the space of constant functions on ∂X. We will give algebraic constructions of symmetric bilinear forms on the space $\overline{\mathcal{D}}(\partial X)$.

Recall that, for any $k \geq 1$, we let X_k stand for the set of (x, y) in X^2 with $d(x, y) = k$. We set

$$X_* = \{(x, y) \in X^2 | x \neq y\}.$$

A function $\varphi : X_* \to \mathbb{R}$ is said to be of quadratic type if it is symmetric and if, for every (x, y) in X_*, we have

$$\varphi(x, y) = \sum_{\substack{z \sim y \\ z \notin [xy]}} \varphi(x, z).$$

Recall also that, for (x, y) in X_*, we let U_{xy} be the closed open subset of ∂X

$$U_{xy} = \{\xi \in \partial X | [xy] \cap [y\xi) = \{y\}\}.$$

Let p be a symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. We associate to p a symmetric function φ_p on X_* by setting, for any (x, y) in X_*,

$$\varphi_p(x, y) = -p(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{yx}}).$$

We get the following characterization of quadratic type functions, which will be used throughout the book:

PROPOSITION 4.1. *The map $p \mapsto \varphi_p$ is a linear isomorphism between the space of symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ and the space of quadratic type functions on X_*.*

The proof of this result will follow from a truncated version of it which we will now give. We first define quadratic type functions on X_k, $k \geq 1$.

DEFINITION 4.2. Let $k \geq 1$. If $k = 1$, a function $\varphi : X_1 \to \mathbb{R}$ is said to be of quadratic type if it is symmetric. If $k \geq 2$, a function $\varphi : X_k \to \mathbb{R}$ is said to be of quadratic type if it is symmetric and if the function

$$\varphi^- : (x, y) \mapsto \sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi(z, y), X_{k-1} \to \mathbb{R} \tag{4.1}$$

is symmetric. The function φ^- is called the reduction of φ.

The reduction of a quadratic type function is of quadratic type.

LEMMA 4.3. *Let $k \geq 2$ and φ be a quadratic type function on X_k. Then φ^- is of quadratic type and, for (x, y) in X_k, we have*

$$\sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi(z, y) = \varphi^-(x, y) = \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(x, t).$$

PROOF. We first prove the formula. As both φ and φ^- are symmetric, by (4.1), we have, for (x, y) in X_k,

$$\varphi^-(x, y) = \varphi^-(y, x) = \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(t, x) = \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(x, t).$$

Now, about the first statement, if $k = 2$, there is nothing to prove. If $k \geq 3$, for $(x, y) \in X_{k-1}$, we have

$$\sum_{\substack{z \sim x \\ z \notin [xy]}} \varphi^-(z, y) = \sum_{\substack{z \sim x \\ z \notin [xy]}} \sum_{\substack{t \sim y \\ t \notin [xy]}} \varphi(z, t),$$

which is clearly symmetric in (x, y). Thus, φ^- is of quadratic type. □

In particular, any quadratic type function φ on X_k defines in a natural way a quadratic type function on all the X_h, $1 \leq h \leq k$. By abuse of notation, this function will be sometimes again denoted by φ.

4.2. Quadratic fields

We will now give an interpretation of this notion in terms of symmetric bilinear forms on certain spaces.

If $\ell \geq 0$ is an integer, recall that, for any x in X, we denote by $S^\ell(x)$, the sphere with center x and radius ℓ in X, that is, the set of y in X with $d(x, y) = \ell$. We let $V^\ell(x)$ denote the space of real-valued functions on $S^\ell(x)$ and $\overline{V}^\ell(x) = V^\ell(x)/\mathbb{R}$ denote its quotient by the line of constant functions.

If x and y are neighboring elements of X (that is, $x \sim y$), we let $S^\ell(xy)$ denote the set

$$S^\ell(xy) = \{z \in S^\ell(x) | y \notin [xz]\} \cup \{z \in S^\ell(y) | x \notin [yz]\}.$$

We let $V^\ell(xy)$ denote the space of real-valued functions on $S^\ell(xy)$ and $\overline{V}^\ell(xy) = V^\ell(xy)/\mathbb{R}$ denote its quotient by the line of constant functions.

For any $\ell \geq 0$ and any x, y in X with $x \sim y$, we define linear maps

$$I^\ell_{xy} : V^\ell(xy) \to V^{\ell+1}(x)$$
$$J^\ell_{xy} : V^\ell(x) \to V^\ell(xy)$$

as follows.

If f is in $V^\ell(xy)$, then $I^\ell_{xy} f$ is the function on $S^{\ell+1}(x)$ such that, for any z in $S^{\ell+1}(x)$, one has

$I^\ell_{xy} f(z) = f(z)$ if y is on $[xz]$ (and hence $d(y,z) = \ell$).

$I^\ell_{xy} f(z) = f(w)$ if y is not on $[xz]$ and w is the neighbor of z on $[xz]$.

If f is in $V^\ell(x)$, then $J^\ell_{xy} f$ is the function on $S^\ell(xy)$ such that, for any z in $S^\ell(xy)$, one has

$J^\ell_{xy} f(z) = f(z)$ if y is not on $[xz]$ (and hence $d(x,z) = \ell$).

$J^\ell_{xy} f(z) = f(w)$ if y is on $[xz]$ and w is the neighbor of z on $[xz]$.

These maps are injections and they send constant functions to constant functions. In particular, they induce linear injections $\overline{V}^\ell(xy) \to \overline{V}^{\ell+1}(x)$ and $\overline{V}^\ell(x) \to \overline{V}^\ell(xy)$ which we still denote by I^ℓ_{xy} and J^ℓ_{xy}.

Finally, for any $\ell \geq 0$ and x in X, we let $M^\ell_x : V^\ell(x) \to V^{\ell+1}(x)$ be the map that sends a function f in $V^\ell(x)$ towards the function $M^\ell_x f$ such that, for any z in $S^{\ell+1}(x)$,

$$M^\ell_x f(z) = f(w) \text{ where } w \text{ is the neighbor of } z \text{ on } [xz].$$

In the same way, if x, y are in X and $x \sim y$, we let $M^\ell_{xy} : V^\ell(xy) \to V^{\ell+1}(xy)$ be the map that sends a function f in $V^\ell(xy)$ towards the function $M^\ell_x f$ such that, for any z in $S^{\ell+1}(xy)$,

$$M^\ell_{xy} f(z) = f(w) \text{ where } w \text{ is the neighbor of } z \text{ on } [xz].$$

Again, we still denote by M^ℓ_x and by M^ℓ_{xy} the associated injections $\overline{V}^\ell(x) \to \overline{V}^{\ell+1}(x)$ and $\overline{V}^\ell(xy) \to \overline{V}^{\ell+1}(xy)$.

An immediate computation gives

LEMMA 4.4. *For any $\ell \geq 0$ and x, y in X with $x \sim y$, we have*

$$I_{xy}^{\ell} J_{xy}^{\ell} = M_x^{\ell}$$
$$J_{xy}^{\ell+1} I_{xy}^{\ell} = M_{xy}^{\ell}.$$

Let us describe how these maps allow to split the spaces into smaller ones.

PROPOSITION 4.5. *For any $\ell \geq 1$ and x in X, the space $\overline{V}^{\ell}(x)$ is spanned by the subspaces*

$$I_{xy}^{\ell-1} \overline{V}^{\ell-1}(xy), \quad y \sim x.$$

For y, z in X with $y \sim x$, $z \sim x$ and $y \neq z$, we have

$$I_{xy}^{\ell-1} \overline{V}^{\ell-1}(xy) \cap I_{xz}^{\ell-1} \overline{V}^{\ell-1}(xz) = M_x^{\ell-1} \overline{V}^{\ell-1}(x).$$

If $\ell \geq 2$, we have

$$\overline{V}^{\ell}(x) / M_x^{\ell-1} \overline{V}^{\ell-1}(x) = \bigoplus_{y \sim x} I_{xy}^{\ell-1} \overline{V}^{\ell-1}(xy) / M_x^{\ell-1} \overline{V}^{\ell-1}(x).$$

PROPOSITION 4.6. *For any $\ell \geq 1$ and x, y in X with $x \sim y$, the space $\overline{V}^{\ell}(xy)$ is spanned by the subspaces*

$$J_{xy}^{\ell} \overline{V}^{\ell}(x) \text{ and } J_{yx}^{\ell} \overline{V}^{\ell}(y).$$

The intersection of these two subspaces is

$$J_{xy}^{\ell} \overline{V}^{\ell}(x) \cap J_{yx}^{\ell} \overline{V}^{\ell}(y) = M_{xy}^{\ell-1} \overline{V}^{\ell-1}(xy).$$

The proofs are immediate.

For $k \geq 1$, we will now define the notion of a k-quadratic field. The definition depends on the parity of k.

DEFINITION 4.7. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. A k-quadratic field is a family $(p_x)_{x \in X}$ where, for any x in X, p_x is a symmetric bilinear form on $\overline{V}^{\ell}(x)$, such that, for any x, y in X with $x \sim y$, we have

$$(I_{xy}^{\ell-1})^{\star} p_x = (I_{yx}^{\ell-1})^{\star} p_y.$$

This bilinear form on $\overline{V}^{\ell-1}(xy)$ is denoted by p_{xy}^{-}.

DEFINITION 4.8. (k odd) Let k be an odd integer, $k = 2\ell+1$, $\ell \geq 0$. A k-quadratic field is a family $(p_{xy})_{x \sim y \in X}$ where, for any x, y in X with $x \sim y$, $p_{xy} = p_{yx}$ is a symmetric bilinear form on $\overline{V}^{\ell}(xy)$, such that, for any x in X, the bilinear forms

$$(J_{xy}^{\ell})^{\star} p_{xy}, \quad y \sim x,$$

are all equal to each other. This bilinear form on $\overline{V}^\ell(x)$ is denoted by p_x^-.

From the combinatorial properties of the spaces of functions on spheres, we have

PROPOSITION 4.9. *Let $k \geq 2$ and let p be a k-quadratic field. Then p^- is a $(k-1)$-quadratic field.*

We call p^- the reduction of p.

PROOF. First assume k is even, $k = 2\ell$, $\ell \geq 1$. For x in X and y with $y \sim x$, we need to prove that the bilinear form

$$(J_{xy}^{\ell-1})^\star p_{xy}^-$$

does not depend on y. By definition, we have

$$p_{xy}^- = (I_{xy}^{\ell-1})^\star p_x,$$

hence

$$(J_{xy}^{\ell-1})^\star p_{xy}^- = (I_{xy}^{\ell-1} J_{xy}^{\ell-1})^\star p_x.$$

Now, by Lemma 4.4,

$$I_{xy}^{\ell-1} J_{xy}^{\ell-1} = M_x^{\ell-1}$$

and the result follows.

Assume now k is odd, $k = 2\ell + 1$, $\ell \geq 1$. For x, y in X with $x \sim y$, we need to prove that the bilinear forms

$$(I_{xy}^{\ell-1})^\star p_x^- \text{ and } (I_{yx}^{\ell-1})^\star p_y^-$$

are equal to each other. Again, by definition, we have

$$p_x^- = (J_{xy}^\ell)^\star p_{xy}$$

hence

$$(I_{xy}^{\ell-1})^\star p_x^- = (J_{xy}^\ell I_{xy}^{\ell-1})^\star p_{xy}.$$

Still by Lemma 4.4, we have

$$J_{xy}^\ell I_{xy}^{\ell-1} = M_{xy}^{\ell-1}$$

and the result follows. □

REMARK 4.10. If $k = 1$ the compatibility condition in the definition of a 1-quadratic field is empty. In particular, such a field is simply the data of the symmetric function $(x, y) \mapsto p_{xy}(\mathbf{1}_x, \mathbf{1}_y)$ on X_1. We shall extend this correspondance to higher k.

4.3. Fields and quadratic type functions

Let $k \geq 1$ and p be a k-quadratic field. We will associate to p a symmetric function φ_p on X_k as follows. Pick x, y in X and let $z_0 = x, z_1, \ldots, z_k = y$ be the geodesic path from x to y. If k is even, $k = 2\ell$, $\ell \geq 1$, we set

$$\varphi_p(x, y) = -p_{z_\ell}(\mathbf{1}_x, \mathbf{1}_y).$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, we set

$$\varphi_p(x, y) = -p_{z_\ell z_{\ell+1}}(\mathbf{1}_x, \mathbf{1}_y).$$

PROPOSITION 4.11. *Fix $k \geq 1$. The map $p \mapsto \varphi_p$ is a linear isomorphism between the space of k-quadratic field and the one of quadratic type functions on X_k.*

REMARK 4.12. From Definition 4.7 and Definition 4.8, we see that the interpretation of a quadratic type function as a field of quadratic forms depends on the parity of k. When k is even, one has to take the point of view of vertices, whereas when k is odd, one should take the point of view of edges. From now on, we will often need to split the proofs according to the parity of k.

We will prove Proposition 4.11 in several steps.

LEMMA 4.13. *For any $k \geq 1$, if p is a k-quadratic field, then the function φ_p is of quadratic type. If $k \geq 2$, one has $\varphi_{p^-} = (\varphi_p)^-$.*

PROOF. If $k = 1$, this has already been noticed in Remark 4.10.

Assume $k \geq 2$. Recall that, by Proposition 4.9, p^- is a $(k-1)$-quadratic field. Let us prove that, for any (x, y) in X_{k-1}, we have

$$\varphi_{p^-}(x, y) = \sum_{\substack{z \sim x \\ z \notin [x,y]}} \varphi_p(z, y). \tag{4.2}$$

Let $z_0 = x, z_1, \ldots, z_{k-1} = y$ be the geodesic path from x to y.

If k is even, $k = 2\ell$, $\ell \geq 1$, we have

$$-\varphi_{p^-}(x, y) = p^-_{z_{\ell-1} z_\ell}(\mathbf{1}_x, \mathbf{1}_y) = p_{z_{\ell-1}}(I^{\ell-1}_{z_{\ell-1} z_\ell} \mathbf{1}_x, I^{\ell-1}_{z_{\ell-1} z_\ell} \mathbf{1}_y).$$

Now, by definition, we have

$$I^{\ell-1}_{z_{\ell-1} z_\ell} \mathbf{1}_y = \mathbf{1}_y \text{ and } I^{\ell-1}_{z_{\ell-1} z_\ell} \mathbf{1}_x = \sum_{\substack{z \sim x \\ z \neq z_1}} \mathbf{1}_z$$

and the result follows.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we have

$$-\varphi_{p^-}(x, y) = p^-_{z_\ell}(\mathbf{1}_x, \mathbf{1}_y) = p_{z_\ell z_{\ell-1}}(J^{\ell}_{z_\ell z_{\ell-1}} \mathbf{1}_x, J^{\ell}_{z_\ell z_{\ell-1}} \mathbf{1}_y).$$

Again, by definition, we have

$$J^{\ell}_{z_\ell z_{\ell-1}}\mathbf{1}_y = \mathbf{1}_y \text{ and } J^{\ell}_{z_\ell z_{\ell-1}}\mathbf{1}_x = \sum_{\substack{z\sim x\\ z\neq z_1}} \mathbf{1}_z$$

and we are done.

In particular, as φ_{p^-} is symmetric, φ_p is of quadratic type. By comparing (4.1) with (4.2), we get $\varphi_{p^-} = (\varphi_p)^-$. □

We will now prove that the map $p \mapsto \varphi_p$ is injective. Recall that, for any quadratic type function φ on X_k, we still denote by φ its natural extension to $\bigcup_{1\leq\ell\leq k} X_\ell$. We get the following easy formula for recovering p from φ_p.

LEMMA 4.14. *Let $k \geq 1$ and p be a k-quadratic field.*

If k is even, $k = 2\ell$, $\ell \geq 1$, for any x in X and $z \neq t$ in $S^\ell(x)$, we have

$$p_x(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_p(z,t). \tag{4.3}$$

If k is odd, $k = 2\ell+1$, $\ell \geq 0$, for any x, y in X with $x \sim y$ and any $z \neq t$ in $S^\ell(xy)$, we have

$$p_{xy}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_p(z,t). \tag{4.4}$$

PROOF. We prove this by induction on k. If $k = 1$, this is obvious.

Assume $k \geq 2$ and the result is true for $k-1$.

Assume k is even, $k = 2\ell$, $\ell \geq 1$ and pick x in X and $z \neq t$ in $S^\ell(x)$. If x belongs to $[zt]$, (4.3) follows from the definition of φ_p. Else, there exists a neighbour y of x such that z and t belong to $S^{\ell-1}(y)$. We then get

$$\mathbf{1}_z = I^{\ell-1}_{xy}(\mathbf{1}_z) \text{ and } \mathbf{1}_t = I^{\ell-1}_{xy}(\mathbf{1}_t),$$

hence

$$p_x(\mathbf{1}_z, \mathbf{1}_t) = p^-_{xy}(\mathbf{1}_z, \mathbf{1}_t).$$

Now, by the induction assumption, the latter is equal to $\varphi_p(z,t)$ and we are done.

In the same way, if k is odd, $k = 2\ell+1$, $\ell \geq 1$, we pick x, y in X with $x \sim y$ and $z \neq t$ in $S^\ell(xy)$. If $[xy] \subset [zt]$, again, we have (4.4) by definition. Else, up to exchanging the roles of x and y, we can assume $z, t \in S^\ell(x)$, hence

$$\mathbf{1}_z = J^{\ell}_{xy}(\mathbf{1}_z) \text{ and } \mathbf{1}_t = J^{\ell}_{xy}(\mathbf{1}_t)$$

and

$$p_{xy}(\mathbf{1}_z, \mathbf{1}_t) = p^-_x(\mathbf{1}_z, \mathbf{1}_t).$$

Again, the result now follows from the induction assumption. □

Surjectivity will follow from the following elementary

LEMMA 4.15. *Let A be a finite set. Let V be the space of real-valued functions on A and $\overline{V} = V/\mathbb{R}$ be its quotient by the space of constant functions. Set $A_2 = \{(a,b) \in A^2 | a \neq b\}$. If p is a symmetric bilinear form on $\overline{V}$, let φ_p be the function on A_2 defined by*

$$\varphi_p(a,b) = -p(\mathbf{1}_a, \mathbf{1}_b), \quad a \neq b.$$

Then the map $p \mapsto \varphi_p$ is a linear isomorphism between the space of symmetric bilinear form on $\overline{V}$ and the space of symmetric real-valued functions on A_2.

We are now ready to give the full

PROOF OF PROPOSITION 4.11. Let $k \geq 1$. By Lemma 4.13, the map $p \mapsto \varphi_p$ sends k-quadratic fields to quadratic type functions on X_k. By Lemmas 4.14 and 4.15, this map is injective. It remains to prove that it is surjective. Fix φ a quadratic type function on X_k and let us construct p such that $\varphi = \varphi_p$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any x in X, by Lemma 4.15, there exists a unique symmetric bilinear form p_x on $\overline{V}^\ell(x)$ such that

$$p_x(\mathbf{1}_z, \mathbf{1}_w) = -\varphi(z,w), \quad z \neq w \in S^\ell(x).$$

Let us show that the family $p = (p_x)_{x \in X}$ is a k-quadratic field. We claim that, for any $x \sim y$ in X, for any $z \neq t$, in $S^{\ell-1}(xy)$ we have

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_t) = -\varphi(z,t),$$

which, by Lemma 4.15, implies that $(I_{xy}^{\ell-1})^\star p_x = (I_{yx}^{\ell-1})^\star p_y$. Indeed, if z and t are in $S^{\ell-1}(y)$, we have

$$I_{xy}^{\ell-1}(\mathbf{1}_z) = \mathbf{1}_z \text{ and } I_{xy}^{\ell-1}(\mathbf{1}_t) = \mathbf{1}_t,$$

hence by definition

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_t) = -\varphi(z,t).$$

If z is in $S^{\ell-1}(x)$ and t is in $S^{\ell-1}(y)$, we have

$$I_{xy}^{\ell-1}(\mathbf{1}_z) = \sum_{\substack{z' \sim z \\ z' \notin [xz]}} \mathbf{1}_{z'} \text{ and } I_{xy}^{\ell-1}(\mathbf{1}_t) = \mathbf{1}_t,$$

hence

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_t) = -\sum_{\substack{z' \sim z \\ z' \notin [xz]}} \varphi(z',t) = -\varphi(z,t).$$

Finally, if z and t are in $S^{\ell-1}(x)$, we have

$$I_{xy}^{\ell-1}(\mathbf{1}_z) = \sum_{\substack{z'\sim z \\ z'\notin[xz]}} \mathbf{1}_{z'} \text{ and } I_{xy}^{\ell-1}(\mathbf{1}_t) = \sum_{\substack{t'\sim t \\ t'\notin[xz]}} \mathbf{1}_{t'},$$

and again

$$p_x(I_{xy}^{\ell-1}\mathbf{1}_z, I_{xy}^{\ell-1}\mathbf{1}_z) = -\sum_{\substack{z'\sim z \\ z'\notin[xz]}} \sum_{\substack{t'\sim t \\ t'\notin[xz]}} \varphi(z',t')$$
$$= -\sum_{\substack{z'\sim z \\ z'\notin[xz]}} \varphi(z',t) = -\varphi(z,t).$$

If k is odd, $k = 2\ell$, $\ell \geq 0$, still by Lemma 4.15, for any $x \sim y$ in X, there exists a unique symmetric bilinear form p_{xy} on $\overline{V}^{\ell}(xy)$ such that

$$p_{xy}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi(z,t), \quad z \neq t \in S^{\ell}(xy).$$

We also show that the family $p = (p_{xy})_{x\sim y\in X}$ is a k-quadratic field. This will now follow from the fact that, for any $x \sim y$ in X, for any $z \neq w$, in $S^{\ell}(x)$,

$$p_{xy}(J_{xy}^{\ell}\mathbf{1}_z, J_{xy}^{\ell}\mathbf{1}_t) = -\varphi(z,t),$$

which we prove as above. □

4.4. Fields and bilinear forms on smooth functions

We will now give the proof of Proposition 4.1. To this aim, let us introduce a new set of linear operators. For x in X and $\ell \geq 0$, we let

$$N_x^{\ell} : V^{\ell}(x) \to \mathcal{D}(\partial X)$$

be the linear operator such that, for any f in $V^{\ell}(x)$, y in $S^{\ell}(x)$ and ξ in U_{xy}, one has

$$N_x^{\ell} f(\xi) = f(y).$$

Again, one still denotes by N_x^{ℓ} the induced operator $\overline{V}^{\ell}(x) \to \overline{\mathcal{D}}(\partial X)$.

We will use the easy

Lemma 4.16. *For any x in X, one has*

$$\mathcal{D}(\partial X) = \bigcup_{\ell\geq 0} N_x^{\ell} V^{\ell}(x)$$

and, for any $\ell \geq 0$,

$$N_x^{\ell+1} M_x^{\ell} = N_x^{\ell}.$$

Proof. The first part is a rewriting of Lemma 3.6. The second part follows from a straightforward computation. □

PROOF OF PROPOSITION 4.1. Let p be a symmetric bilinear form on the space $\overline{\mathcal{D}}(\partial X)$. Then, since for any $x \neq y$ in X, the closed open set U_{yx} is the disjoint union of the U_{yz}, $z \sim x$, $z \notin [xy]$, we have

$$\varphi_p(x,y) = \sum_{\substack{z\sim x \\ z\notin[xy]}} \varphi_p(z,y),$$

hence φ_p is of quadratic type.

If φ_p is 0, we claim that p is 0. Indeed, as the characteristic functions of the closed open subsets U_{xy}, $x \sim y \in X$, span $\mathcal{D}(\partial X)$, it suffices to check that for any $x \sim y$ and $z \sim w$ in X, we have $p(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{zw}}) = 0$. As $\mathbf{1}_{U_{xy}} + \mathbf{1}_{U_{yx}} = \mathbf{1}$ and $\mathbf{1}$ is in the null space of p, we can assume that $y \neq w$ and x and z belong to $[yw]$. We then have $U_{xy} = U_{wy}$ and $U_{zw} = U_{yw}$ hence

$$p(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{zw}}) = -\varphi_p(y,w) = 0,$$

and we are done.

Finally, if φ is a quadratic type function on X_*, for any $k \geq 1$, let φ_k be the restriction of φ to X_k which is a quadratic type function on X_k. By Proposition 4.11, there exists a unique k-quadratic field p^k such that $\varphi_{p^k} = \varphi_k$. By Lemma 4.13, for $k \geq 2$, one has $(p^k)^- = p^{k-1}$. Fix x in X. We get, for any $\ell \geq 1$,

$$(M_x^\ell)^\star p^{2(\ell+1)} = p^{2\ell}.$$

By Lemma 4.16, this tells us that there exists a unique symmetric bilinear form p on $\overline{\mathcal{D}}(\partial X)$ such that, for any $\ell \geq 1$,

$$(N_x^\ell)^\star p = p^{2\ell}.$$

One verifies that $\varphi = \varphi_p$. □

4.5. Orthogonal extension of Euclidean fields

We just saw how a globally defined quadratic type function on X_* gives rise to quadratic type functions on X_k for any $k \geq 1$. We will now introduce a reverse operation in the Euclidean case. It will rely on the following lemma, which we will apply to the decompositions of spaces of functions on spheres from Proposition 4.5 and Proposition 4.6.

LEMMA 4.17. *Let X be a finite-dimensional real vector space, $d \geq 2$ be an integer and $X_1, \dots, X_d$ be subspaces of X. We assume that there exists a subspace X_0 of X such that, for any $1 \leq i \neq j \leq d$, $X_i \cap X_j = X_0$ and $X/X_0 = \bigoplus_i X_i/X_0$. Let $p_0, p_1, \dots, p_d$ be positive definite symmetric bilinear forms on $X_0, X_1, \dots, X_d$ such that, for any $1 \leq i \leq d$, $p_{i|X_0} = p_0$. For $1 \leq i \leq d$, let $Y_i \subset X_i$ be the orthogonal complement of X_0 in X_i with respect to p_i. Then, there exists a unique*

positive definite symmetric bilinear form p on X such that, for any $1 \leq i \leq d$, $p_{|X_i} = p_i$ and, for any $1 \leq i \neq j \leq d$, the spaces Y_i and Y_j are orthogonal with respect to p. The form p is called the orthogonal extension of $p_1, \ldots, p_d$ to X.

DEFINITION 4.18. Let $k \geq 1$. If p is a k-quadratic field, we shall say that p is a k-Euclidean field if the associated symmetric bilinear forms are positive definite.

If $k \geq 2$, we will build an orthogonal extension of these fields which is a $(k+1)$-Euclidean field.

DEFINITION 4.19. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. If p is a k-Euclidean field, for any $x \sim y$ in X, we let p^+_{xy} denote the orthogonal extension of p_x and p_y to $\overline{V}^\ell(xy)$, where $\overline{V}^\ell(x)$ and $\overline{V}^\ell(y)$ are identified to subspaces of $\overline{V}^\ell(xy)$ through the maps J^ℓ_{xy} and J^ℓ_{yx}. The family $p^+ = (p^+_{xy})_{x \sim y \in X}$ is called the orthogonal extension of p.

DEFINITION 4.20. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 1$. If p is a k-Euclidean field, for any x in X, we let p^+_x denote the orthogonal extension of $(p_{xy})_{y \sim x}$ to $\overline{V}^{\ell+1}(x)$, where the spaces $\overline{V}^\ell(xy)$, $y \sim x$, are identified to subspaces of $\overline{V}^{\ell+1}(x)$ through the maps I^ℓ_{xy}, $y \sim x$. The family $p^+ = (p^+_x)_{x \in X}$ is called the orthogonal extension of p.

The orthogonal extension is again a quadratic field. More precisely, we have the following result, whose proof directly follows from the definitions:

PROPOSITION 4.21. *Let $k \geq 2$ and p be a k-Euclidean field. Then its orthogonal extension p^+ is a $(k+1)$-Euclidean field and $(p^+)^- = p$.*

Recall that the reduction p^- of p was introduced in Definition 4.7 and Definition 4.8.

4.6. The Hilbert space of a Euclidean field

Let p be a k-Euclidean field with $k \geq 2$. The successive orthogonal extensions of p allow to define a quadratic type function φ^∞_p on X_*, or equivalently, by Proposition 4.1, a symmetric bilinear form p^∞ on $\overline{\mathcal{D}}(\partial X)$, which clearly turns out to be positive definite. We let H^p be the completion of $\overline{\mathcal{D}}(\partial X)$ with respect to p^∞ and we call it the Hilbert space of p.

If p is Γ-invariant, so is φ^∞_p, hence p^∞ is Γ-invariant and H^p comes with a natural action of Γ which makes it a unitary representation.

In the next chapter we will study the topological dual space of H^p.

CHAPTER 5

Hilbert spaces of distributions

5.1. Dual kernels

We will now introduce dual notions to the ones studied above. We start with a dual statement to Lemma 4.15.

LEMMA 5.1. *Let A be a finite set. Let V be the space of real-valued functions on A and*

$$V_0 = \{f \in V | \sum_{a \in A} f(a) = 0\}.$$

If q is a symmetric bilinear form on V_0, let K^q be the function on $A \times A$ defined by

$$K^q(a,b) = q(\mathbf{1}_a - \mathbf{1}_b, \mathbf{1}_a - \mathbf{1}_b), \quad a, b \in A.$$

Then the map $q \mapsto K^q$ is a linear isomorphism between the space of symmetric bilinear forms on V_0 and the space of symmetric real-valued functions on $A \times A$ which are zero on the diagonal. If K is such a function and q is the associated bilinear form, for any f, g in V_0, we have

$$q(f,g) = -\frac{1}{2} \sum_{(a,b) \in A^2} K(a,b) f(a) f(b).$$

In the sequel, if A is a finite set and V is the space of real-valued functions on A, we always identify the space V with its dual space through the positive definite bilinear form on V

$$(f,g) \mapsto \sum_{a \in A} f(a) g(a).$$

Let V_0 be the space of functions in V with zero sum,

$$V_0 = \{f \in V | \sum_{a \in A} f(a) = 0\}.$$

The space V_0 may now be seen as the dual space of $\overline{V} = V/\mathbb{R}$.

In particular, for any $\ell \geq 0$, for any x in X, let $V_0^\ell(x)$ denote the set of real-valued functions on $S^\ell(x)$ with zero sum, and, for any $x \sim y$ in X, let $V_0^\ell(xy)$ denote the set of real-valued functions on $S^\ell(xy)$ with

zero sum. We regard these spaces as the dual spaces of $\overline{V}^{\ell}(x)$ and $\overline{V}^{\ell}(xy)$.

Recall that if V is a finite-dimensional real vector space, to any non-degenerate symmetric bilinear form p on V, we can associate its dual bilinear form q on the dual space V^* of V. The form q is defined as the image of p by the linear isomorphism from V to V^* associated with the form p.

Let p be a k-Euclidean field for some $k \geq 1$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any x in X, we let q_x be the dual symmetric bilinear form to p_x on $V_0^{\ell}(x)$. If z and w are in $S^{\ell}(x)$, we set

$$K_x^p(z, w) = q_x(\mathbf{1}_z - \mathbf{1}_w, \mathbf{1}_z - \mathbf{1}_w).$$

If ξ, η are in ∂X, we write

$$K_x^p(\xi, \eta) = K_x^p(z, w),$$

where z (resp. w) is the intersection point of the geodesic ray $[x\xi)$ (resp. $[x\eta)$) with $S^{\ell}(x)$.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we let q_{xy} be the dual symmetric bilinear form to p_{xy} on $V_0^{\ell}(xy)$. If z and w are in $S^{\ell}(xy)$, we set

$$K_{xy}^p(z, w) = q_{xy}(\mathbf{1}_z - \mathbf{1}_w, \mathbf{1}_z - \mathbf{1}_w).$$

If ξ, η are in ∂X, we write

$$K_{xy}^p(\xi, \eta) = K_{xy}^p(z, w),$$

where z (resp. w) is the intersection point of the geodesic ray $[x\xi)$ (resp. $[x\eta)$) with $S^{\ell}(xy)$.

By Lemma 5.1, the Euclidean field p is completely determined by the data of K^p. We have a nice way of computing K^{p^+} from K^p and K^{p^-}. Recall that, for x in X, $d(x)$ is the number of neighbours of x.

PROPOSITION 5.2. *Let p be a k-Euclidean field for some $k \geq 2$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have, as functions on $\partial X \times \partial X$,

$$K_{xy}^{p^+} = K_x^p + K_y^p - K_{xy}^{p^-}.$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any x in X, we have, as functions on $\partial X \times \partial X$,

$$K_x^{p^+} = \sum_{y \sim x} K_{xy}^p - (d(x) - 1) K_x^{p^-}.$$

PROOF. The proof is a direct translation of Lemma 5.3 below. □

LEMMA 5.3. *Let $X, X_0, \dots, X_d$ and $p_0, \dots, p_d$ be as in Lemma 4.17. Let p be the orthogonal extension of $p_1, \dots, p_d$ to X. Equip the dual spaces $X^*, X_0^*, X_1^*, \dots, X_d^*$ of $X, X_0, X_1, \dots, X_d$ with the bilinear forms $q, q_0, q_1 \dots, q_d$ which are dual to $p, p_0, p_1, \dots, p_d$. Then, for every φ, ψ in X^*, we have*

$$q(\varphi, \psi) = q_1(\varphi_{|X_1}, \psi_{|X_1}) + \cdots + q_d(\varphi_{|X_d}, \psi_{|X_d}) - (d-1)q_0(\varphi_{|X_0}, \psi_{|X_0}).$$

PROOF. For $1 \leq i \leq d$, let $Y_i \subset X_i$ be the orthogonal complement of X_0 in X_i. Set u_i to be the vector in X_i which represents $\varphi_{|X_i}$ with respect to p_i, that is, such that $\varphi(x_i) = p_i(u_i, x_i)$ for x_i in X_i. Write $u_i = v_i + w_i$, with v_i in X_0 and w_i in Y_i. By definition, we have $q_i(\varphi_{|X_i}, \varphi_{|X_i}) = p_i(u_i, u_i) = p_i(v_i, v_i) + p_i(w_i, w_i)$.

We claim that $v_1, \dots, v_d$ are equal to each other. Indeed, for $1 \leq i \leq d$, we have, for any x_0 in X_0, $p_0(v_i, x_0) = p_i(v_i, x_0) = p_i(u_i, x_0) = \varphi(x_0)$, which does not depend on i, hence v_i does not depend on i since p_0 is positive definite.

Set $v = v_1 = \cdots = v_d$ and $u = v + w_1 + \cdots + w_d$. We claim that the vector u represents the linear functional φ on X with respect to p. Indeed, for x in X, write $x = x_0 + x_1 + \cdots + x_d$, with x_0 in X_0 and x_i in Y_i, $1 \leq i \leq d$. We have

$$\begin{aligned}\varphi(x) &= \varphi(x_0) + \varphi(x_1) + \cdots + \varphi(x_d)\\ &= p_0(v, x_0) + p_1(u_1, x_1) + \cdots + p_d(u_d, x_d)\\ &= p_0(v, x_0) + p_1(w_1, x_1) + \cdots + p_d(w_d, x_d) = p(u, x),\end{aligned}$$

where the latter equality follows from the definition of p in Lemma 4.17. We get, still by this definition,

$$\begin{aligned}q(\varphi, \varphi) = p(u, u) &= p_0(v, v) + p_1(w_1, w_1) + \cdots + p_d(w_d, w_d)\\ &= p_1(u_1, u_1) + \cdots + p_d(u_d, u_d) - (d-1)p_0(v, v),\end{aligned}$$

and the result follows. □

We will now axiomatize the relations which appear in Proposition 5.2.

DEFINITION 5.4. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. A k-dual prekernel is a family $(K_x)_{x \in X}$ where, for any x in X, K_x is a symmetric function on $S^\ell(x) \times S^\ell(x)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(x)$ associated to K_x by Lemma 5.1 is denoted by q_x^K.

DEFINITION 5.5. (k odd) Let k be an odd integer, $k = 2\ell+1$, $\ell \geq 0$. A k-dual prekernel is a family $(K_{xy})_{x \sim y \in X}$ where, for any $x \sim y$ in X, $K_{xy} = K_{yx}$ is a symmetric function on $S^\ell(xy) \times S^\ell(xy)$ which is zero

on the diagonal. The symmetric bilinear form on $V_0^\ell(xy)$ associated to K_{xy} by Lemma 5.1 is denoted by q_{xy}^K.

As above, depending on the context, we may also consider dual prekernels as families of locally constant functions on $\partial X \times \partial X$.

DEFINITION 5.6. Let $k \geq 2$ be an integer. Then a k-dual kernel is a pair (K, K^-) where K is a k-dual prekernel and K^- is a $(k-1)$-dual prekernel.

Dual kernels admit orthogonal extensions which behave as in Proposition 5.2.

DEFINITION 5.7. Let $k \geq 2$ be an integer and (K, K^-) be a k-dual kernel.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, set

$$K_{xy}^+ = K_x + K_y - K_{xy}^-.$$

Then K^+ is a $(k+1)$-dual prekernel.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any x in X, set

$$K_x^+ = \sum_{y \sim x} K_{xy} - (d(x) - 1)K_x^-.$$

Then K^+ is a $(k+1)$-dual prekernel.

In both cases, the $(k+1)$-dual kernel (K^+, K) is called the orthogonal extension of the k-dual kernel (K, K^-). More generally, for any $j \geq k$, we denote by (K^j, K^{j-1}) the $(j-k)$-th orthogonal extension of (K, K^-).

REMARK 5.8. The orthogonal extension map $(K, K^-) \mapsto (K^+, K)$ is a linear embedding from the vector space of k-dual kernels into the vector space of $(k+1)$-dual kernels.

5.2. Large extensions of dual kernels

As an example of the use of these notions, let us give formulas for the K^j, $j \geq k+1$. For $h \geq 0$ and x in X, we set $B^h(x) = \bigcup_{0 \leq \ell \leq h} S^\ell(x)$ to be the ball with center x and radius h in X. In the same way, for $x \sim y$ in X, we set $B^h(x) = \bigcup_{0 \leq \ell \leq h} S^\ell(xy)$. Successive orthogonal extensions are defined by summing the kernels on points and edges in these sets.

LEMMA 5.9. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. The orthogonal extensions of (K, K^-) may be defined by the following formulas.*

Fix $h \geq 1$ and $x \sim y$ in X. If k is even, we have

$$K_x^{k+2h} = \sum_{z \in B^h(x)} K_z - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^-$$

$$\text{and } K_{xy}^{k+2h-1} = \sum_{z \in B^{h-1}(xy)} K_z - \frac{1}{2} \sum_{\substack{z,t \in B^{h-1}(xy) \\ z \sim t}} K_{zt}^-.$$

If k is odd, we have

$$K_x^{k+2h-1} = \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt} - \sum_{z \in B^{h-1}(x)} (d(z)-1) K_z^-$$

$$\text{and } K_{xy}^{k+2h} = \frac{1}{2} \sum_{\substack{z,t \in B^h(xy) \\ z \sim t}} K_{zt} - \sum_{z \in B^{h-1}(xy)} (d(z)-1) K_z^-.$$

PROOF. We fix $j \geq 3$ and we prove the formula for K^j when (K, K^-) is a k-dual kernel by descending induction on k with $2 \leq k \leq j-1$.

If $k = j-1$, the formula is the same as in Definition 5.7.

Now, assume that $k \leq j-2$ and that the formula holds for $k+1$. We will prove it for k. We need to split the discussion according to the parities of j and k. Assume j is even, $j = 2m$, $m \geq 2$.

If k is, $k = 2\ell$, $\ell \geq 1$, we set $h = m - \ell \geq 1$. By the induction assumption, applied to the $(k+1)$-dual kernel (K^+, K), we have

$$K_x^j = \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^+ - \sum_{z \in B^{h-1}(x)} (d(z)-1) K_z.$$

By Definition 5.7, we get

$$K_x^j = \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} (K_z + K_t - K_{zt}^-) - \sum_{z \in B^{h-1}(x)} (d(z)-1) K_z,$$

which equals

$$\sum_{z \in B^h(x)} |S^1(z) \cap B^h(x)| K_z - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^- - \sum_{z \in B^{h-1}(x)} (d(z)-1) K_z,$$

where $|.|$ is the cardinality of finite sets. For z in $B^{h-1}(x)$, we have $S^1(z) \subset B^h(x)$, hence $|S^1(z) \cap B^h(x)| = d(z)$. For z in $S^h(x)$, we have

$|S^1(z) \cap B^h(x)| = 1$. Thus,

$$K_x^j = \sum_{z \in B^h(x)} K_z - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}^-,$$

which should be proved.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we set $h = m - \ell - 1 \geq 1$. The induction assumption and Definition 5.7 now give

$$\begin{aligned} K_x^j &= \sum_{z \in B^h(x)} K_z^+ - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt} \\ &= \sum_{z \in B^h(x)} \left(\sum_{t \sim z} K_{zt} - (d(z) - 1) K_z^- \right) - \frac{1}{2} \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt}. \end{aligned}$$

Note that

$$\sum_{z \in B^h(x)} \sum_{t \sim z} K_{zt} = \sum_{\substack{z,t \in B^h(x) \\ z \sim t}} K_{zt} + \sum_{z \in S^{h+1}(x)} K_{zz_-},$$

where, for z in $S^{h+1}(x)$, z_- is the neighbour of z on $[xz]$. Thus, we get

$$K_x^j = \frac{1}{2} \sum_{\substack{z,t \in B^{h+1}(x) \\ z \sim t}} K_{zt} - \sum_{z \in B^h(x)} (d(z) - 1) K_z^-,$$

as required.

The proofs in case j is odd are analogous. □

5.3. Non-negative dual kernels

We will introduce a non-negativity property that is satisfied by the dual kernels of the form (K^p, K^{p^-}), where p is a Euclidean field. When this property holds, we can associate a Hilbert space to a dual kernel.

We first start by introducing a natural notion for prekernels.

Definition 5.10. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and K be a k-dual prekernel. We say that K is non-negative if, for any x in X, the bilinear form q_x^K is non-negative.

Definition 5.11. (k odd) Let $k \geq 1$ be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$, and K be a k-dual prekernel. We say that K is non-negative if, for any $x \sim y$ in X, the bilinear form q_{xy}^K is non-negative.

Let us now define a related notion for dual kernels. We need some more notation. For $x \sim y$ in X and $\ell \geq 0$, the adjoint maps of the maps I_{xy}^{ℓ} and J_{xy}^{ℓ} will be denoted by $I_{xy}^{\ell,*}$ and $J_{xy}^{\ell,*}$. In other words, for any $x \sim y$ in X, we have linear maps

$$I_{xy}^{\ell,*} : V^{\ell+1}(x) \to V^{\ell}(xy)$$
$$J_{xy}^{\ell,*} : V^{\ell}(xy) \to V^{\ell}(x)$$

defined as follows.

If f is in $V^{\ell+1}(x)$, then $I_{xy}^{\ell,*} f$ is the function on $S^{\ell}(xy)$ such that, for any z in $S^{\ell}(xy)$, one has

$$I_{xy}^{\ell,*} f(z) = f(z) \text{ if } y \text{ is on } [xz].$$
$$I_{xy}^{\ell,*} f(z) = \sum_{\substack{w \sim z \\ w \notin [xz]}} f(w) \text{ if } y \text{ is not on } [xz].$$

If f is in $V^{\ell}(xy)$, then $J_{xy}^{\ell,*} f$ is the function on $S^{\ell}(x)$ such that, for any z in $S^{\ell}(x)$, one has

$$J_{xy}^{\ell,*} f(z) = f(z) \text{ if } y \text{ is not on } [xz].$$
$$J_{xy}^{\ell,*} f(z) = \sum_{\substack{w \sim z \\ w \notin [xz]}} f(w) \text{ if } y \text{ is on } [xz].$$

Let V and W be real vector spaces and let $\pi : V \to W$ be a surjective linear map. If q is a non-negative symmetric bilinear form on V, we define in Appendix A the Euclidean image $\pi_\star q$ of q: this is a non-negative symmetric bilinear form on W. For any w in W, we have

$$\pi_\star q(w, w) = \inf_{\substack{v \in V \\ \pi(v) = w}} q(v, v).$$

Definition 5.12. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and (K, K^-) be a k-dual kernel.

We say that (K, K^-) is non-negative if the dual prekernels K and K^- are non-negative and if, for any $x \sim y$ in X, we have

$$q_x^K \geq (I_{xy}^{\ell-1,*})^\star q_{xy}^{K^-}.$$

We say that (K, K^-) is exact if it is non-negative and, for any $x \sim y$ in X, we have

$$(I_{xy}^{\ell-1,*})_\star q_x^K = q_{xy}^{K^-}.$$

We say that (K, K^-) is Euclidean if it is exact and, for any x in X, the bilinear form q_x^K is positive definite.

DEFINITION 5.13. (k odd) Let $k \geq 2$ be an odd integer, $k = 2\ell + 1$, $\ell \geq 1$, and (K, K^-) be a k-dual kernel.

We say that (K, K^-) is non-negative if the dual prekernels K and K^- are non-negative and if, for any $x \sim y$ in X, we have

$$q^K_{xy} \geq (J^{\ell,*}_{xy})^\star q^{K^-}_x.$$

We say that (K, K^-) is exact if it is non-negative and, for any $x \sim y$ in X, we have

$$(J^{\ell,*}_{xy})_\star q^K_{xy} = q^{K^-}_x.$$

We say that (K, K^-) is Euclidean if it is exact and, for any $x \sim y$ in X, the bilinear form q^K_{xy} is positive definite.

As $\Gamma\backslash X$ is finite, the vector space of Γ-invariant k-dual kernels has finite dimension. We denote it by $\mathcal{K}_k$ and we set $\mathcal{K}^+_k \subset \mathcal{K}_k$ to be the set of Γ-invariant non-negative k-dual kernels. Elementary properties of Euclidean images give

PROPOSITION 5.14. *Let $k \geq 2$ and (H, H^-) and (K, K^-) be non-negative k-dual kernels. Then $(H + K, H^- + K^-)$ is non-negative.*

The set of non-negative k-dual kernels is a convex cone in the vector space of all k-dual kernels.

The set $\mathcal{K}^+_k$ of Γ-invariant non-negative k-dual kernels is a closed convex cone with non-empty interior inside the vector space $\mathcal{K}_k$ of Γ-invariant k-dual kernels.

For x in X, we let Γ_x be the stabilizer of x in Γ, which by assumption is a finite subgroup.

PROOF. The fact that, for (H, H^-) and (K, K^-) as above, the dual kernel $(H + K, H^- + K^-)$ is non-negative follows from Lemma A.5. As the set of non-negative k-dual kernels is clearly stable by multiplication by non-negative real numbers, it is a convex cone.

The set $\mathcal{K}^+_k$ is closed in $\mathcal{K}_k$ as being defined by a set of closed inequalities. It remains to prove that it has non-empty interior.

We let $S \subset X$ be a finite set of representatives for the Γ-action on vertices of X, that is, $X = \Gamma S$ and, for every x in S, $\Gamma x \cap S = \{x\}$. In the same way, we let $\overline{X}_1 = \{\{x, y\} | (x, y) \in X_1\}$ be the set of non oriented edges of X and $T \subset \overline{X}_1$ be a finite set of representatives for the Γ-action on $\overline{X}_1$. Now, we first define Γ-invariant dual prekernels as follows.

Fix $\ell \geq 1$. For x in S, we chose a Γ_x-invariant positive definite symmetric bilinear form $p^{2\ell}_x$ on $V^\ell_0(x)$. For x in X, we set $p^{2\ell}_x = (g^{-1})^\star p^{2\ell}_{gx}$ where g in Γ is such that gx is in S. We let $H^{2\ell}_x$ be the associated function on $S^\ell(x) \times S^\ell(x)$ as in Lemma 5.1.

Fix $\ell \geq 0$. For $\{x, y\}$ in T, we set $\Gamma_{xy} = \{g \in \Gamma | g\{x, y\} = \{x, y\}\}$ and we chose a Γ_{xy}-invariant positive definite symmetric bilinear form $p_{xy}^{2\ell+1}$ on $V_0^\ell(xy)$. For $x \sim y$ in X, we set $p_{xy}^{2\ell+1} = (g^{-1})^\star p_{(gx)(gy)}^{2\ell+1}$ where g in Γ is such that $\{gx, gy\}$ is in T. We let $H_{xy}^{2\ell+1}$ be the associated function on $S^\ell(xy) \times S^\ell(xy)$ as in Lemma 5.1.

Now, let k be even, $k = 2\ell$, $\ell \geq 1$. For $x \sim y$ in X, we set $K_x = H_x^k + \sum_{z \sim x} H_{xz}^{k-1}$ and $K_{xy}^- = H_{xy}^{k-1}$. Then (K, K^-) is a Γ-invariant k-dual kernel which clearly lies in the interior of $\mathcal{K}_k^+$.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we set $K_{xy} = H_{xy}^k + H_x^{k-1} + H_y^{k-1}$ and $K_x^- = H_x^{k-1}$. Again, (K, K^-) is an interior point of $\mathcal{K}_k^+$. □

Euclidean kernels are in one-to-one correspondance with Euclidean fields.

PROPOSITION 5.15. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Then (K, K^-) is Euclidean if and only if there exists a k-Euclidean field p such that $(K, K^-) = (K^p, K^{p^-})$.*

The notions we have defined behave well with respect to orthogonal extension.

PROPOSITION 5.16. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. If (K, K^-) is non-negative (resp. exact, resp. Euclidean), so is the $(k+1)$-dual kernel (K^+, K).*

PROOF. This is a direct consequence of the abstract lemma below, which we apply to the structures that are dual to the ones in Proposition 4.5 and Proposition 4.6. □

LEMMA 5.17. *Let $W_0, W_1, \ldots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces, and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. Assume $q_0, q_1, \ldots q_d$ to be non-negative symmetric bilinear forms on $W_0, W_1, \ldots, W_d$ with $q_i \geq \varpi_i^\star q_0$, $1 \leq i \leq d$, and set

$$q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d - (d-1)\pi_0^\star q_0.$$

Then,
(i) *the symmetric bilinear form q is non-negative and $q \geq \pi_i^\star q_i$ for $1 \leq i \leq d$.*
(ii) *if we have $(\varpi_i)_\star q_i = q_0$ for $1 \leq i \leq d$, then we also have $(\pi_i)_\star q = q_i$ for $1 \leq i \leq d$.*

(iii) *if the forms $q_1, \dots q_d$ are positive definite, the form q is positive definite.*

PROOF. *(i)* Pick $1 \leq i \leq d$. We have

$$q = \pi_i^\star q_i + \sum_{j \neq i} (\pi_j^\star q_j - \pi_0^\star q_0) = \pi_i^\star q_i + \sum_{j \neq i} \pi_j^\star (q_j - \varpi_j^\star q_0),$$

hence $q \geq \pi_i^\star q_i$. In particular, q is non-negative.

(ii) Still fix $1 \leq i \leq d$ and let w_i be a vector in W_i. We set $w_0 = \varpi_i(w_i)$. For $j \neq i$, as $(\varpi_j)_\star q_j = q_0$, we can find w_j in W_j with $\varpi_j(w_j) = w_0$ and $q_j(w_j, w_j) = q_0(w_0, w_0)$. Now the vector $w = (w_1, \dots, w_d)$ belongs to W and by construction, we have $\pi(w) = w_i$ and $q(w, w) = q_i(w_i, w_i)$.

(iii) Let $w = (w_1, \dots, w_d)$ be in W with $q(w, w) = 0$. By *(i)*, for $1 \leq i \leq d$, we have $q_i(w_i, w_i) \leq q(w, w)$, hence $w_i = 0$. We get $w = 0$. □

5.4. The Hilbert space of a non-negative dual kernel

Recall that, by definition, the space $\mathcal{D}^*(\partial X)$ of distributions on X is the dual space of the space $\mathcal{D}(\partial X)$ of smooth functions on X and that $\mathcal{D}_0^*(\partial X)$ is the set of distributions T with $\langle T, \mathbf{1} \rangle = 0$, which we freely identify with the dual space of $\overline{\mathcal{D}}(\partial X) = \mathcal{D}(\partial X)/\mathbb{R}$.

Recall also that we have defined natural linear operators, for x in X and $\ell \geq 0$,

$$N_x^\ell : V^\ell(x) \to \mathcal{D}(\partial X).$$

Again, we let

$$N_x^{\ell,*} : \mathcal{D}^*(\partial X) \to V^\ell(x)$$

denote the adjoint operator of N_x^ℓ.

To a non-negative kernel, we will associate a natural Hilbert space of distributions by using the results in Appendix B.

Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Recall that $(K^j)_{j \geq k}$ denote the successive predual kernels obtained from (K, K^-) by orthogonal extension. As above, for any even $j \geq k-1$, $j = 2\ell$, $\ell \geq 1$, for any x in X, we associate to K_x^j a symmetric bilinear form $q_x^{K^j}$ on $V_0^\ell(x)$. When there is no ambiguity, we shall write q_x^j for $q_x^{K^j}$. In the same way, for any odd $j \geq k-1$, $j = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we let $q_{xy}^{K^j}$ or q_{xy}^j denote the symmetric bilinear form associated to K_{xy}^j on $V_0^\ell(xy)$.

PROPOSITION 5.18. *Let $k \geq 2$ and (K, K^-) be a non-negative k-dual kernel. Fix x in X and let L^{K,K^-} denote the set of distributions*

θ in $\mathcal{D}_0(\partial X)$ such that

$$\sup_{\ell \geq \frac{k}{2}} q_x^{2\ell}(N_x^{\ell,*}\theta, N_x^{\ell,*}\theta) < \infty.$$

Then L^{K,K^-} is a vector subspace of $\mathcal{D}_0(\partial X)$ and the map

$$\theta \mapsto \sup_{\ell \geq \frac{k}{2}} q_x^{2\ell}(N_x^{\ell,*}\theta, N_x^{\ell,*}\theta)$$

is a non-negative quadratic form on L^{K,K^-}. Let q^{K,K^-} be its polar form. Both L^{K,K^-} and q^{K,K^-} do not depend on the choice of x. The space $H^{K,K^-} = L^{K,K^-}/\ker q^{K,K^-}$, equipped with the positive definite bilinear form induced by q^{K,K^-} is complete.

From the definition of the Hilbert space associated to a Euclidean field, we get

Corollary 5.19. *If (K, K^-) is Euclidean and p is the k-Euclidean field such that $(K, K^-) = (K^p, K^{p^-})$, then $L^{K,K^-} = H^{K,K^-}$ is exactly the space of distributions which are bounded linear functional for the scalar product p^∞ on $\overline{\mathcal{D}}(\partial X)$. In particular, H^{K,K^-} may be seen as the topological dual space of the Hilbert space H^p associated to p.*

The space H^{K,K^-}, equipped with its natural scalar product, will be called the Hilbert space associated with the dual kernel (K, K^-).

Proof of Proposition 5.18. Let us check that the definition of the objects is independent on x. To this aim, let $x \sim y$ be in X. For any $\ell \geq 0$, the linear operator $I_{xy}^\ell J_{yx}^\ell$ embeds $V^\ell(y)$ as a subspace in $V^{\ell+1}(x)$. One easily checks that one has $N_x^{\ell+1} I_{xy}^\ell J_{yx}^\ell = N_y^\ell$. Hence, if $2\ell \geq k$, we get

$$(N_y^{\ell,*})^\star q_y^{2\ell} = (N_x^{\ell+1,*})^\star (I_{xy}^{\ell,*})^\star (J_{yx}^{\ell,*})^\star q_y^{2\ell}.$$

By Proposition 5.16, we have

$$(J_{yx}^{\ell,*})^\star q_y^{2\ell} \leq q_{xy}^{2\ell+1} \text{ and } (I_{xy}^{\ell,*})^\star q_{xy}^{2\ell+1} \leq q_x^{2\ell+2}.$$

Thus, we have

$$(N_y^{\ell,*})^\star q_y^{2\ell} \leq (N_x^{\ell+1,*})^\star q_x^{2\ell+2}.$$

In particular, for any θ in $\mathcal{D}_0(\partial X)$, we have

$$\sup_{\ell \geq 1} q_x^{2\ell}(N_x^{\ell,*}\theta, N_x^{\ell,*}\theta) = \sup_{\ell \geq 1} q_y^{2\ell}(N_y^{\ell,*}\theta, N_y^{\ell,*}\theta).$$

By connectedness of X, the latter equality holds for any x, y in X, hence the constructions in the Proposition do not depend on the choice of x.

The rest of the proof directly follows from Lemma 4.16, Proposition 5.16 and Lemma B.3: indeed, the family $(V_0^\ell(x), M_x^{\ell+1,*}, q_x^{2\ell})_{\ell \geq \frac{k}{2}}$ is a

non-negative projective system in the sense of Definition B.1, whose algebraic projective limit may be identified with $\mathcal{D}_0(\partial X)$ (see Appendix B for more details). □

Note that for the moment, we don't know whether L^{K,K^-} is not reduced to 0. We will later prove that, when (K, K^-) is Γ-invariant, L^{K,K^-} contains the space H_0^ω from Chapter 3. We will first show how this phenomenon appears on a particular example.

5.5. Examples of dual kernels

In this Section, we give two examples of non-negative dual kernels. Their constructions are based on the elementary

LEMMA 5.20. *Let A be a finite set and V be the space of real-valued functions on A and V_0 be the subspace of functions f with $\sum_{a\in A} f(a) = 0$. We let q be the scalar product $(f, g) \mapsto \sum_{a\in A} f(a)g(a)$ on V_0 and, for a in A, we let e_a denote the evaluation linear functional $f \mapsto f(a)$. Then if q^* is the scalar product dual to q on the dual space of V_0, for $a \neq b$ in A, we have $q^*(e_a, e_a) = \frac{n-1}{n}$ and $q^*(e_a, e_b) = -\frac{1}{n}$, where $n = |A|$ is the cardinality of A.*

Let us define a 2-dual kernel (χ, χ^-). For any $x \sim y$ in X, we set $\chi^-_{xy}(x, y) = 1$. For any x in X and any neighbours $y \neq z$ of x, we set $\chi_x(y, z) = 2\frac{d(x)-1}{d(x)}$. We call (χ, χ^-) the harmonic kernel.

PROPOSITION 5.21. *The harmonic kernel is a Euclidean kernel.*

PROOF. Let $x \sim y$ be in X and q_x and q^-_{xy} be the symmetric bilinear forms associated with χ and χ^- on the spaces $V_0^1(x)$ and $V_0^0(xy)$. By construction (see Lemma 5.1), one has $q_x(f, f) = \frac{d(x)-1}{d(x)} \sum_{z\sim x} f(z)^2$ for any f in $V_0^1(x)$ and $q^-_{xy}(\mathbf{1}_x - \mathbf{1}_y, \mathbf{1}_x - \mathbf{1}_y) = 1$. We must check that we have $q^-_{xy} = (I^{0,*}_{xy})_\star q_x$. Now, for any f in $V_0^1(x)$, we have $I^{0,*}_{xy} f = f(y)(\mathbf{1}_y - \mathbf{1}_x)$, so that, by Lemma 5.20 and Lemma A.10, we have

$$(I^{0,*}_{xy})_\star q_x(\mathbf{1}_y - \mathbf{1}_x, \mathbf{1}_y - \mathbf{1}_x) = \frac{1}{p_x(\mathbf{1}_y, \mathbf{1}_y)} = 1,$$

where p_x is the scalar product dual to q_x on $\overline{V}^1(x)$. □

We shall pursue the study of the harmonic kernel in Sections 10.6 and 11.6. In particular, we will prove in Proposition 11.21 that the Hilbert space of distributions H^{χ,χ^-} associated to (χ, χ^-) in Proposition 5.18 is exactly the Hilbert space H_0^ω that has been studied in Chapter 3.

By changing slightly the construction, we can build another dual kernel, that is not exact any more, but for which the computations are easier. We define the Busemann kernel $(\kappa, \kappa^-) = (\kappa^2, \kappa^1)$ as follows. For any $x \sim y$ in X, we set $\kappa^1_{xy}(x, y) = 1$. For any x in X and any neighbours $y \neq z$ of x, we set $\kappa^2_x(y, z) = 2$. We denote by $(\kappa^k)_{k\geq 1}$ the dual prekernels obtained from (κ^2, κ^1) by successive orthogonal extensions as in Definition 5.7. For the Busemann kernel, all the objects that have been introduced in Chapter 7 can be computed explicitely.

PROPOSITION 5.22. *The Busemann kernel (κ^2, κ^1) is a non-negative 2-dual kernel. Let $k \geq 1$. If k is even $k = 2\ell$, $\ell \geq 1$, for any x in X and z, t in $S^\ell(x)$, one has $\kappa^k_x(z, t) = d(z, t)$. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X and z, t in $S^\ell(xy)$, one has $\kappa^k_{xy}(z, t) = d(z, t)$.*

PROOF. Let still (χ, χ^-) be the harmonic kernel. For any x in X, we have $\kappa_x = \frac{d(x)}{d(x)-1}\chi_x$ and, for $x \sim y$ in X, we have $\kappa^-_{xy} = \chi^-_{xy}$. As, for any x, $\frac{d(x)}{d(x)-1} \geq 1$ and as, by Proposition 5.21, the harmonic kernel is Euclidean, the k-dual kernel $(\kappa - \chi, \kappa^- - \chi^-) = (\kappa - \chi, 0)$ is non-negative, hence (κ, κ^-) is non-negative by Proposition 5.14.

Let us compute κ^k, $k \geq 3$. Assume k is even, $k = 2\ell$, $\ell \geq 2$. Fix x in X. For z, t in $S^\ell(x)$, Lemma 5.9 and the definition of (κ, κ^-) give

$$\kappa^k_x(z, t) = 2|B^{\ell-1}(x) \cap [zt]| \\ - \frac{1}{2}|\{(u, v) \in B^{\ell-1}(x) | u \sim v \text{ and } [uv] \subset [zt]\}|.$$

If $z = t$, all these numbers are 0. Else, one has $d(z, t) \geq 2$ and

$$|B^{\ell-1}(x) \cap [zt]| = d(z, t) - 1$$
$$|\{(u, v) \in B^{\ell-1}(x) | u \sim v \text{ and } [uv] \subset [zt]\}| = 2(d(z, t) - 2)$$

and the result follows. The proof in the odd case is analogous. □

COROLLARY 5.23. *The Hilbert space of distributions*

$$H^{\kappa,\kappa^-} \subset \mathcal{D}_0(\partial X)$$

associated with the Busemann kernel is exactly the space H^ω_0.

PROOF. For x, y, z in X, let t be such that $[xy] \cap [xz] = [xt]$. We set $\omega_x(y, z) = d(x, t)$. This extends the definition of the Gromov product (see Example 2.18). If $d(x, y) = d(x, z) = \ell$, we get

$$d(y, z) = 2\ell - 2\omega_x(y, z). \tag{5.1}$$

Let x be in X, $\ell \geq 1$ and $q^{2\ell}_x$ be the symmetric bilinear form associated to $\kappa^{2\ell}_x$ on $V^\ell_0(x)$. By Lemma 5.1, Proposition 5.22 and (5.1), for

f in $V_0^\ell(x)$, we have

$$q_x^{2\ell}(f,f) = \sum_{y,z\in S^\ell(x)} (\omega_x(y,z)-\ell)f(y)f(z)$$

$$= \sum_{y,z\in S^\ell(x)} \omega_x(y,z)f(y)f(z) = \sum_{k=1}^{\ell} \sum_{y,z\in S^\ell(x)} \mathbf{1}_{\omega_x(y,z)\geq k} f(y)f(z).$$

Fix T in $\mathcal{D}_0(\partial X)$. We get

$$q_x^{2\ell}(N_x^{\ell,*}T, N_x^{\ell,*}T) = \sum_{\substack{t\in X \\ 1\leq d(x,t)\leq \ell}} \sum_{\substack{y,z\in S^\ell(x) \\ t\in[xy]\cap[xz]}} \langle T, \mathbf{1}_{U_{xy}}\rangle \langle T, \mathbf{1}_{U_{xz}}\rangle \tag{5.2}$$

$$= \sum_{\substack{t\in X \\ 1\leq d(x,t)\leq \ell}} \langle T, \mathbf{1}_{U_{xt}}\rangle^2 = \frac{1}{2} \sum_{\substack{(u,v)\in X_1 \\ d(x,u)\leq \ell \\ d(x,v)\leq \ell}} \mathcal{P}_x T(u,v)^2,$$

where $\mathcal{P}_x$ is as in Section 3.1. By Proposition 5.18, the space H^{κ,κ^-} is exactly the space of distributions T in $\mathcal{D}_0(\partial X)$ with $\mathcal{P}_x T$ belonging to $\ell^2(X_1)$, which by definition is equal to H_0^ω. By (5.2), for T in that space, we have $\|T\|^2 = 2q^{\kappa,\kappa^-}(T,T)$. □

In the next chapters, our goal will be to get a kind of generalization of Corollary 5.23. More precisely, we will prove that, when a non-negative dual kernel (K, K^-) is Γ-invariant, the Hilbert space H^{K,K^-} contains the completion of H_0^ω with respect to a non-negative symmetric bilinear form Φ_w as in Chapter 3. We will also show that all forms Φ_w can be obtained in this way.

CHAPTER 6

Pseudokernels

The description of the bilinear form Φ_w associated with a non-negative dual kernel (K, K^-) will rely on a formula for computing the scalar product of the Hilbert space H^{K,K^-}. To establish this purely algebraic formula, we will study a new object, whose introduction will help us in many computations.

6.1. The definition of a pseudokernel

We start by defining this new notion. Again, we have to split the definition according to the parity of k.

DEFINITION 6.1. (k odd) Let k be an odd integer, $k = 2\ell+1$, $\ell \geq 0$. A k-pseudokernel is a family $(L_{xy})_{(x,y)\in X_1}$ where for any (x, y) in X_1, L_{xy} is a symmetric function on $S^\ell(xy) \times S^\ell(xy)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(xy)$ associated to L_{xy} by Lemma 5.1 is denoted by r^L_{xy}.

REMARK 6.2. Note that in the odd case, although the set on which L_{xy} is defined is symmetric in x and y, the function L_{xy} is not necessarily equal to L_{yx}.

DEFINITION 6.3. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$. A k-pseudokernel is a family $(L_{xy})_{(x,y)\in X_1}$ where for any (x, y) in X_1, L_{xy} is a symmetric function on $S^\ell(x) \times S^\ell(x)$ which is zero on the diagonal. The symmetric bilinear form on $V_0^\ell(x)$ associated to L_{xy} by Lemma 5.1 is denoted by r^L_{xy}.

REMARK 6.4. Note that in the even case, although the set on which L_{xy} is defined only depends on x, the function L_{xy} a priori also depends on the choice of a neighbour y of x.

As for dual kernels, when this will be convenient, we will sometimes think to the L_{xy} as being locally constant functions on $\partial X \times \partial X$.

For any $k \geq 1$, we define $\mathcal{L}_k$ as the vector space of Γ-invariant k-pseudokernels.

6.2. Operations on pseudokernels

We introduce natural algebraic operations on pseudokernels which will play a fundamental role in all our computations.

For $x \sim y$, the operators I_{xy}^{ℓ}, $\ell \geq 0$, and J_{xy}^{ℓ}, $\ell \geq 1$, are defined in Section 4.2.

DEFINITION 6.5. Let $k \geq 1$ and L be a k-pseudokernel.

If k is even, for any $x \sim y$ in X, we set $L_{xy}^{\vee} = \sum_{z \sim x, z \neq y} L_{xz}$. We call $L^{\vee}$ the reversal of L. The map $L \mapsto L^{\vee}$ is a linear automorphisms on the space of k-pseudokernels.

If k is odd, for any $x \sim y$ in X, we set $L_{xy}^{\vee} = L_{yx}$. We call $L^{\vee}$ the inversion of L. The map $L \mapsto L^{\vee}$ is an involution of the space of k-pseudokernels.

DEFINITION 6.6. We define the direct extension of pseudokernels as follows. Let $k \geq 1$ and L be a k-pseudokernel. We let $L^{>}$ be the $(k+1)$-pseudokernel with $L_{xy}^{>} = L_{xy}$, $x \sim y$ in X. In other words, the symmetric bilinear forms $(r_{xy}^{L})_{x \sim y \in X}$ and $(r_{xy}^{L^{>}})_{x \sim y \in X}$ are related in the following way.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we have $r_{xy}^{L^{>}} = (I_{xy}^{\ell,*})^{\star} r_{xy}^{L}$. If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have $r_{xy}^{L^{>}} = (J_{xy}^{\ell,*})^{\star} r_{xy}^{L}$.

Direct extension is a linear embedding from the space of k-pseudokernels into the space of $(k+1)$-pseudokernels.

Finally, we define orthogonal extension of pseudokernels by combining reversal, inversion and direct extension. The relevance of this notion will become clear later.

DEFINITION 6.7. Let $k \geq 1$ and L be a k-pseudokernel. We define the orthogonal extension L^{+} of L by setting $L^{+} = L^{>\vee}$.

Here comes a fundamental commutation property of those maps.

LEMMA 6.8. *Let $k \geq 1$ and L be a k-pseudokernel. We have $L^{\vee>>} = L^{>>\vee}$.*

PROOF. Assume that k is odd, $k = 2\ell + 1$, $\ell \geq 0$. From Lemma 4.4, we get, for $x \sim y$ in X, $J_{xy}^{\ell+1} I_{xy}^{\ell} = M_{xy}^{\ell}$, where $M_{xy}^{\ell} = M_{yx}^{\ell}$ is the natural embedding $\overline{V}^{\ell}(xy) \hookrightarrow \overline{V}^{\ell+1}(xy)$. Therefore, if r_{xy}^{L} is the symmetric bilinear form associated to L_{xy} on $V_0^{\ell}(xy)$, the symmetric bilinear form associated with $L^{>>}$ on $V_0^{\ell+1}(xy)$ is $(M_{xy}^{\ell,*})^{\star} r_{xy}^{L}$ and the conclusion follows.

In the same way, if k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, the symmetric bilinear form associated with $L^{>>}$ on $V_0^{\ell+1}(x)$ is $(M_x^{\ell,*})^\star r_{xy}^L$. □

Conversely, we have

PROPOSITION 6.9. *Let $k \geq 1$ and L, M be k-pseudokernels. Assume that we have $L^> = M^{>\vee}$.*

If $k = 1$, then $L = M = 0$.

If $k \geq 2$, then there exists a $(k-1)$-pseudokernel N with $L = N^{\vee>}$ and $M = N^>$.

The proof relies on a general property of symmetric bilinear forms which are built through surjective maps.

LEMMA 6.10. *Let $W_0, W_1, \ldots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. Assume $q_1, \ldots q_d$ to be symmetric bilinear forms on $W_1, \ldots, W_d$ and set $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$. Then we have $q = 0$ if and only if there exists symmetric bilinear forms $p_1, \ldots, p_d$ on W_0 with $q_i = \varpi_i^\star p_i$, $1 \leq i \leq d$, and $p_1 + \cdots + p_d = 0$.

PROOF. If $p_1, \ldots, p_d$ exist, then clearly $q = 0$. Let us prove the converse statement.

Assume $q = 0$ and let us fix $1 \leq i \leq d$. Let us show that there exists a symmetric bilinear form p_i on W_0 with $q_i = \varpi_i^\star p_i$. In other words, we claim that, if w_i and w_i' are in W_i and $\varpi_i(w_i) = \varpi_i(w_i')$, then $q_i(w_i, w_i) = q_i(w_i', w_i')$. Indeed, for any $j \neq i$, pick w_j in W_j with $\varpi_j(w_j) = \varpi_i(w_i) = \varpi_i(w_i')$ and let w and w' be the unique elements of W such that

$$\pi_i(w) = w_i \text{ and } \pi_i(w') = w_i';$$
$$\pi_j(w) = w_j \text{ and } \pi_j(w') = w_j, \quad j \neq i.$$

As $q(w, w) = q(w', w') = 0$, we have

$$q_i(w_i, w_i) = -\sum_{j \neq i} q(w_j, w_j) = q_i(w_i', w_i'),$$

which should be proved.

Now, for any $1 \leq i \leq d$, we have built a symmetric bilinear form p_i on W_0 with $q_i = \varpi_i^\star p_i$. In particular, we have $0 = q = \pi_0^\star(p_1 + \cdots + p_d)$, hence $p_1 + \cdots + p_d = 0$. □

PROOF OF PROPOSITION 6.9. As usual, we denote by r^L_{xy} and r^M_{xy}, $x \sim y$ in X, the symmetric bilinear forms associated with L and M.

Assume $k = 1$. Then there exists functions u and v on X_1 such that, for $x \sim y$ in X and f in $V^0_0(xy)$, one has $r^L_{xy}(f, f) = u(x, y)f(y)^2$ and $r^M_{xy}(f, f) = v(x, y)f(y)^2$. The equation $L^> = M^{>\vee}$ reads as

$$u(x, y)f(y)^2 = \sum_{\substack{z\sim x\\ z\neq y}} v(x, z)f(z)^2, \quad f \in V^1_0(x).$$

By taking $f = \mathbf{1}_y - \mathbf{1}_z$ for $z \neq y$, we get $u(x, y) = v(x, z)$. As $d(x) \geq 3$, this implies that $u(x, .)$ and $v(x, .)$ are constant on $S^1(x)$ with the same value. Would this value be non-zero, we would get

$$f(y)^2 = \sum_{\substack{z\sim x\\ z\neq y}} f(z)^2, \quad f \in V^1_0(x).$$

The quadratic form on the right hand-side is positive definite on $V^1_0(x)$, whereas the one on the left hand-side has rank 1. Both can not be equal since $\dim V^1_0(x) = d(x) - 1 \geq 2$. We get $u = v = 0$ as required.

Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$. Then, for $x \sim y$ in X, we have

$$(I^{\ell,*}_{xy})^\star r^L_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} (I^{\ell,*}_{xz})^\star r^M_{xz}.$$

Proposition 4.5 and Lemma 6.10 imply that there exists families

$$(s_{xy})_{x\sim y\in X} \text{ and } (t_{xy})_{x\sim y\in X}$$

where, for $x \sim y$ in X, s_{xy} and t_{xy} are symmetric bilinear forms on $V^\ell_0(x)$ and

$$\begin{aligned} r^L_{xy} &= (J^{\ell,*}_{xy})^\star s_{xy} \\ r^M_{xy} &= (J^{\ell,*}_{xy})^\star t_{xy} \\ s_{xy} &= \sum_{\substack{z\sim x\\ z\neq y}} t_{xz}. \end{aligned}$$

The conclusion directly follows, by taking N to be the $(k-1)$-pseudokernel associated with $(t_{xy})_{x\sim y\in X}$.

The proof in the even case is analogous. □

6.3. The canonical pseudokernel

We now introduce a natural embedding of the space of k-dual kernels into the one of k-pseudokernels.

DEFINITION 6.11. (k even) Let $k \geq 2$ be an even integer and (K, K^-) be a k-dual kernel. The canonical k-pseudokernel L of (K, K^-) is defined by

$$L_{xy} = K_x - K^-_{xy} \quad x \sim y \in X.$$

DEFINITION 6.12. (k odd) Let $k \geq 3$ be an odd integer and (K, K^-) be a k-dual kernel. The canonical k-pseudokernel L of (K, K^-) is defined by

$$L_{xy} = K_{xy} - K^-_x \quad x \sim y \in X.$$

Dual kernels are defined in Section 5.1.

REMARK 6.13. Let us state this definition in an other way. For $k \geq 1$, any k-dual prekernel K may be considered as a k-pseudokernel. Indeed, if k is odd, K may be seen as the k-pseudokernel L with $L_{xy} = K_{xy} = L_{yx}$, $x \sim y \in X$. If k is even, K may be seen as the k-pseudokernel L with $L_{xy} = K_x$, $x \sim y \in X$. Now, if $k \geq 2$ and (K, K^-) is a k-dual kernel, the definitions say that the canonical k-pseudokernel L of (K, K^-) is $L = K - K^{->}$.

We let $C_k : \mathcal{K}_k \to \mathcal{L}_k$ denote the linear map that sends a Γ-invariant k-dual kernel to its canonical k-pseudokernel. We call C_k the canonical map. We can relate the canonical map to the previously introduced operations.

LEMMA 6.14. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with canonical k-pseudokernel L and orthogonal extension (K^+, K). The canonical $(k+1)$-pseudokernel of (K^+, K) is the orthogonal extension L^+ of L.*

Note that this explains why we called $L^+ = L^{>\vee}$ the orthogonal extension of L (see Definition 6.7).

PROOF. Let M be the canonical $(k+1)$-pseudokernel of (K^+, K). Fix $x \sim y$ in X. If k is even, we have

$$M_{xy} = K^+_{xy} - K_x = (K_x + K_y - K^-_{xy}) - K_x = K_y - K^-_{xy} = L_{yx}.$$

If k is odd, we have

$$M_{xy} = K^+_x - K_{xy} = \sum_{z \sim x} K_{xz} - (d(x) - 1)K^-_x - K_{xy}$$
$$= \sum_{\substack{z \sim x \\ z \neq y}} (K_{xz} - K_x) = \sum_{\substack{z \sim x \\ z \neq y}} L_{xz}.$$

□

The canonical map is injective.

LEMMA 6.15. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. If the canonical k-pseudokernel of (K, K^-) is 0, then $(K, K^-) = 0$.*

PROOF. We use the convention established in Remark 6.13, that is, we consider dual prekernels as pseudokernels. We set $L = K^-$ if k is odd and, if k is even, we let L be the $(k-1)$-pseudokernel defined by $L_{xy} = (d(x) - 1)K^-_{xy}$, $x \sim y \in X$. In both cases, as by assumption, we have $K = K^{->}$, we get $K^{->\vee} = L^>$.

We prove the result by induction on $k \geq 2$.

If $k = 2$, by Proposition 6.9, since $K^{->\vee} = L^>$, we have $K^- = 0$, hence $K = 0$.

Assume now $k \geq 3$ and the result holds for $k-1$. Still by Proposition 6.9, as $K^{->\vee} = L^>$, there exists a $(k-2)$-pseudokernel H with $K^- = H^>$ and $L = H^{\vee>}$. If k is odd, we have $K^- = L$, hence $H^> = H^{\vee>}$ and therefore $H = H^\vee$, which says that H is a $(k-2)$-dual prekernel. Similarly, if k is even, for $x \sim y$ in X, we have $L_{xy} = (d(x) - 1)K^-_{xy}$, hence $H^{\vee>}_{xy} = (d(x) - 1)H^>_{xy}$ and $H^\vee_{xy} = (d(x) - 1)H_{xy}$, which also says that H is a $(k-2)$-dual prekernel. In both cases, the $(k-1)$-dual kernel (K^-, H) precisely satisfies $K^- = H^>$, that is, it has zero canonical $(k-1)$-pseudokernel. By induction, we get $(K^-, H) = 0$ and, as $K = K^{->}$, $K = 0$, which should be proved. □

6.4. The $(k+1)$-dual kernel associated with a k-pseudokernel

We define a linear map from $\mathcal{L}_k$ to $\mathcal{K}_{k+1}$ which will later play an important role in Chapter 9.

DEFINITION 6.16. (k odd) Let k be an odd integer, $k = 2\ell+1$, $\ell \geq 0$, and $L = (L_{xy})_{(x,y)\in X_1}$ be a k-pseudokernel. We define the $(k+1)$-dual kernel (K, K^-) associated to L by the formulas

$$K_x = \sum_{y\sim x} L_{xy}, \qquad x \in X,$$
$$K^-_{xy} = L_{xy} + L_{yx}, \qquad x \sim y \in X.$$

Equivalently, the bilinear forms associated to (K, K^-) verify

$$q^K_x = \sum_{y\sim x} (I^{\ell,*}_{xy})^\star r^L_{xy}, \qquad x \in X,$$
$$q^{K^-}_{xy} = r^L_{xy} + r^L_{yx}, \qquad x \sim y \in X.$$

DEFINITION 6.17. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$, and $L = (L_{xy})_{(x,y)\in X_1}$ be a k-pseudokernel. We define the $(k+1)$-dual

kernel (K, K^-) associated to L by the formulas

$$\begin{aligned} K_{xy} &= L_{xy} + L_{yx}, && x \sim y \in X, \\ K^-_x &= \frac{1}{d(x)-1} \sum_{y \sim x} L_{xy}, && x \in X. \end{aligned}$$

Equivalently, the bilinear forms associated to (K, K^-) verify

$$\begin{aligned} q^K_x &= (J^{\ell,*}_{xy})^\star r^L_{xy} + (J^{\ell,*}_{yx})^\star r^L_{yx}, && x \sim y \in X, \\ q^{K^-}_x &= \frac{1}{d(x)-1} \sum_{y \sim x} r^L_{xy}, && x \in X. \end{aligned}$$

This construction defines an injective linear map $\mathcal{L}_k \hookrightarrow \mathcal{K}_{k+1}$.

PROPOSITION 6.18. *Let $k \geq 1$ and L be a k-pseudokernel. If the associated $(k+1)$-dual kernel is 0, then L is zero.*

PROOF. By definition, with the language of Section 6.2, we have

$$L + L^\vee = 0 \text{ and } L^> + L^{>\vee} = 0. \tag{6.1}$$

We will prove by induction on $k \geq 1$ that these equations imply that $L = 0$.

If $k = 1$, by Proposition 6.9, the second equation in (6.1) already implies $L = 0$.

Assume now $k \geq 2$ and the statement is proved for $k-1$ and let us prove that it is still true for k. By Proposition 6.9, the second equation in (6.1) implies that there exists a $(k-1)$-pseudokernel M with $L = M^{\vee>} = -M^>$ and hence $M + M^\vee = 0$. Now, the first equation reads as $M^> + M^{>\vee} = 0$. By the induction hypothesis, we obtain $M = 0$, hence $L = 0$ as required. □

In case (K, K^-) is the k-dual kernel associated to a $(k-1)$-pseudokernel, the canonical k-pseudokernel of (K, K^-) may be described.

LEMMA 6.19. *Let $k \geq 2$, M be a $(k-1)$-pseudokernel and (K, K^-) be the k-dual kernel associated to $M^\vee$. Let L be the canonical k-pseudokernel of (K, K^-). We have $L = M^{\vee>\vee} - M^>$.*

PROOF. Fix $x \sim y$ in X. If k is even, we have

$$L_{xy} = K_x - K^-_{xy} = \sum_{z \sim x} M_{zx} - (M_{xy} + M_{yx}) = \sum_{\substack{z \sim x \\ z \neq y}} M_{zx} - M_{xy}.$$

If k is odd, we have

$$\begin{aligned} L_{xy} = K_{xy} - K_x^- &= M_{xy}^\vee + M_{yx}^\vee - \frac{1}{d(x)-1}\sum_{z\sim x} M_{xz}^\vee \\ &= M_{xy}^\vee + M_{yx}^\vee - \frac{1}{d(x)-1}\sum_{\substack{z,t\sim x\\ z\neq t}} M_{xt} \\ &= M_{xy}^\vee + M_{yx}^\vee - \sum_{t\sim x} M_{xt} = M_{yx}^\vee - M_{xy}. \end{aligned}$$

□

From Lemma 6.19 and Lemma 6.15, we deduce

COROLLARY 6.20. *Let $k \geq 2$, (K, K^-) be a k-dual kernel, L be its canonical k-pseudokernel and M be a $(k-1)$-pseudokernel. Then $L = M^{\vee>\vee} - M^>$ if and only if (K, K^-) is the k-dual kernel associated to $M^\vee$.*

Finally, the following result also explains why the map $L \mapsto L^{>\vee} = L^+$ was called the orthogonal extension of pseudokernels.

PROPOSITION 6.21. *Let $k \geq 1$ and L be a k-pseudokernel, with associated $(k+1)$-dual kernel (K, K^-). Then the $(k+2)$-dual kernel associated to the orthogonal extension L^+ of L is the orthogonal extension (K^+, K) of K.*

In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_k & \xrightarrow{+} & \mathcal{L}_{k+1} \\ \downarrow & & \downarrow \\ \mathcal{K}_{k+1} & \xrightarrow{+} & \mathcal{K}_{k+2}, \end{array}$$

where the horizontal arrows are orthogonal extensions.

PROOF. The proof follows directly from the definitions. Let us write (H, H^-) for the $(k+2)$-dual kernel associated to L^+.

If k is odd, we have, for any $x \sim y$ in X,

$$\begin{aligned} H_{xy} = L_{xy}^+ + L_{yx}^+ &= \sum_{\substack{z\sim x\\ z\neq y}} L_{xz} + \sum_{\substack{t\sim y\\ t\neq x}} L_{yt} \\ &= \sum_{z\sim x} L_{xz} + \sum_{t\sim y} L_{yt} - (L_{xy} + L_{yx}) = K_x + K_y - K_{xy}^- = K_{xy}^+ \end{aligned}$$

and also, for any x in X,

$$H_x^- = \frac{1}{d(x)-1}\sum_{y\sim x} L_{xy}^+ = \frac{1}{d(x)-1}\sum_{y\sim x}\sum_{\substack{z\sim x\\ z\neq y}} L_{xz} = \sum_{z\sim x} L_{xz} = K_x,$$

which should be proved.

If k is even, we have, for any x in X,

$$\begin{aligned} K_x^+ = \sum_{y\sim x} K_{xy} - (d(x)-1)K_x^- &= \sum_{y\sim x}(L_{xy}+L_{yx}) - \sum_{y\sim x} L_{xy} \\ &= \sum_{y\sim x} L_{yx} = \sum_{y\sim x} L_{xy}^+ = H_x \end{aligned}$$

and also, for any $x\sim y$ in X,

$$K_{xy} = L_{xy} + L_{yx} = L_{yx}^+ + L_{xy}^+ = H_{xy}^-,$$

and the result follows. □

6.5. Large extensions of pseudokernels

To conclude this chapter, we give formulas for pseudokernels which are analogues to the formulas of Section 5.2 for dual kernels. These formulas and the algebraic formalism of pseudokernels will be instrumental in the next chapter.

For $j \geq k \geq 1$ and L a k-pseudokernel, we denote by L^{j-k} the j-pseudokernel obtained from L through $(j-k)$ orthogonal extensions, that is, $L^j = L^{+^{j-k}}$.

LEMMA 6.22. *Let $k \geq 1$ and L be a k-pseudokernel. The orthogonal extensions of L may be defined by the following formulas. Fix $h \geq 0$ and $x\sim y$ in X. If k is odd, we have*

$$L_{xy}^{k+2h} = \sum_{\substack{z\in S^{h+1}(x)\\ x\notin[yz]}} L_{z_z}.$$

If k is even, we have

$$L_{xy}^{k+2h} = \sum_{\substack{z\in S^{h+1}(y)\\ y\notin[xz]}} L_{zz_}.$$

PROOF. Assume for example that k is odd and let us prove the result by induction on $h \geq 0$. For $h = 0$, there is nothing to prove. If

the result holds for $h \geq 0$, then, by Definition 6.7, for $x \sim y$ in X, we have

$$L_{xy}^{k+2h+1} = \sum_{\substack{z \sim x \\ z \neq y}} L_{xz}^{k+2h} = \sum_{\substack{z \sim x \\ z \neq y}} \sum_{\substack{t \in S^{h+1}(x) \\ x \notin [zt]}} L_{t_t} = \sum_{\substack{t \in S^{h+2}(y) \\ y \notin [xt]}} L_{t_t}.$$

The result follows since, by Definition 6.7, one has $L_{xy}^{k+2h+2} = L_{yx}^{k+2h+1}$. □

This directly gives, by using again Definition 6.7 in the even case,

COROLLARY 6.23. *Let $k \geq 1$ and L be a k-pseudokernel. For $x \sim y$ in X, we have*

$$L_{xy}^{2k-1} = \sum_{\substack{z \in S^{\ell+1}(x) \\ x \notin [yz]}} L_{z_z}, \qquad k = 2\ell + 1, \ell \geq 0.$$

$$L_{xy}^{2k-1} = \sum_{\substack{z \in S^{\ell}(x) \\ x \notin [yz]}} L_{zz_}, \qquad k = 2\ell, \ell \geq 1.$$

In particular, we get

COROLLARY 6.24. *Let $k \geq 1$ and L be a k-pseudokernel. For $x \sim y$ in X and ξ, η in U_{yx}, we have $L_{xy}^{2k-1}(\xi, \eta) = 0$.*

PROOF. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, by Corollary 6.23, we have

$$L_{xy}^{2k-1}(\xi, \eta) = \sum_{\substack{z \in S^{\ell+1}(x) \\ x \notin [yz]}} L_{z_z}(\xi, \eta).$$

By assumption, for z as above, the geodesic rays $[z\xi)$ and $[z\eta)$ both meet the sphere $S^{\ell}(zz_)$ at x, hence $L_{z_z}(\xi, \eta) = 0$.

In the same way, if k is even, $k = 2\ell$, $\ell \geq 1$, Corollary 6.23, gives

$$L_{xy}^{2k-1}(\xi, \eta) = \sum_{\substack{z \in S^{\ell}(x) \\ x \notin [yz]}} L_{zz_}(\xi, \eta).$$

Now, for such a z, the geodesic rays $[z\xi)$ and $[z\eta)$ both meet the sphere $S^{\ell}(z)$ at x, hence $L_{zz_}(\xi, \eta) = 0$. □

By applying Proposition 6.21, from Lemma 6.23, we get

COROLLARY 6.25. *Let $k \geq 1$ and L be a k-pseudokernel. The associated dual prekernels may be defined by the following formulas. Fix $h \geq 0$ and $x \sim y$ in X. If k is odd, we have*

$$K_x^{k+2h+1} = \sum_{z \in S^{h+1}(x)} L_{z_z} \text{ and } K_{xy}^{k+2h} = \sum_{z \in S^{h}(xy)} L_{z_z}.$$

If k is even, we have

$$K_{xy}^{k+2h+1} = \sum_{z\in S^h(xy)} L_{zz_-} \text{ and } K_x^{k+2h} = \sum_{z\in S^h(x)} L_{zz_-},$$

where in the last equation, we assume $h \geq 1$.

PROOF. For example, let us proof the first formula. Definition 6.16, Proposition 6.21 and Lemma 6.22 give

$$K_x^{k+2h+1} = \sum_{y\sim x} L_{xy}^{k+2h} = \sum_{y\sim x} \sum_{\substack{z\in S^{h+1}(x)\\ x\notin[yz]}} L_{z_-z}.$$

The formula follows as the sphere $S^{h+1}(x)$ may be written as the disjoint union

$$S^{h+1}(x) = \bigsqcup_{y\sim x} \{z \in S^{h+1}(x) | x \notin [yz]\}.$$

□

The last result of this section will be the starting point of the next chapter.

COROLLARY 6.26. *Fix $k \geq 2$ and a k-dual kernel (K, K^-). Let $x_0, x_1, \ldots, x_{2k-2}$ be a geodesic path in X. Then, the quantity*

$$K_{x_{k-1}}^{2k-2}(x_0, x_{2k-2}) - K_{x_{k-2}x_{k-1}}^{2k-3}(x_0, x_{2k-3})$$

only depends $x_{k-2}, \ldots, x_{2k-2}$.

PROOF. Let L be the canonical k-pseudokernel of (K, K^-) (see Definition 6.11 and Definition 6.12). By Lemma 6.14, we have

$$K_{x_{k-1}}^{2k-2}(x_0, x_{2k-2}) - K_{x_{k-2}x_{k-1}}^{2k-3}(x_0, x_{2k-3}) = L_{x_{k-1}x_{k-2}}^{2k-2}(x_0, x_{2k-2}).$$

We fix a parametrized geodesic line $(x_h)_{h\in\mathbb{Z}}$ containing $x_0, x_1, \ldots, x_{2k-2}$. We denote its origin by ξ and its endpoint by η.

If k is even, $k = 2\ell$, $\ell \geq 1$, since $2k-2 = k+2(\ell-1)$, by Lemma 6.22, we have

$$L_{x_{k-1}x_{k-2}}^{2k-2}(\xi, \eta) = \sum_{\substack{z\in S^\ell(x_{k-2})\\ x_{k-2}\notin[x_{k-1}z]}} L_{zz_-}(\xi, \eta).$$

For z as above and $h \leq k-2$, we have $d(x_h, z) = \ell + k - 2 - h$, hence $L_{zz_-}(\xi, \eta)$ does not depend on the $(x_h)_{h<k-2}$.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 0$, since $2k - 2 = k + 1 + 2(\ell - 1)$, still by Lemma 6.22, we have

$$L^{2k-2}_{x_{k-1}x_{k-2}}(\xi, \eta) = \sum_{\substack{t \in S^{\ell}(x_{k-2}) \\ x_{k-2} \notin [x_{k-1}t]}} L^{+}_{t_{-}t}(\xi, \eta) = \sum_{\substack{z \in S^{\ell+1}(x_{k-2}) \\ x_{k-2} \notin [x_{k-1}z]}} L_{zz_{-}}(\xi, \eta),$$

where the last equality follows from Definition 6.5 and Definition 6.7. Again, for z as above and $h \leq k - 2$, we have $d(x_h, z) = \ell + k - 2 - h$, and $L_{zz_{-}}(\xi, \eta)$ does not depend on the $(x_h)_{h<k-2}$. □

Part 2

Dual kernels and weights

CHAPTER 7

An additive formula for dual kernels

As announced, given a k-dual kernel (K, K^-), we will construct in this chapter a symmetric function $w : X_k \to \mathbb{R}$ such that the symmetric bilinear forms associated to (K, K^-) may be defined by means of a formula which is the same as the one in Proposition 2.22.

7.1. Geodesic sums

We begin by examining the consequences of Corollary 6.26. For convenience, we shift the parameter k as regards the convention in Corollary 6.26.

PROPOSITION 7.1. *Let $k \geq 1$. Assume that φ is a symmetric function on X_{2k} and ψ is a symmetric function on X_{2k-1}. The following are equivalent:*
(i) *For any geodesic path $x_0, x_1, \ldots, x_{2k}$ in X, the quantity*

$$\varphi(x_0, x_{2k}) - \psi(x_0, x_{2k-1})$$

only depends $x_{k-1}, \ldots, x_{2k}$.
(ii) *For any $j \geq k$ and any geodesic path $x_0, x_1, \ldots, x_{j+2k-2}$ in X, the quantity*

$$\sum_{h=0}^{j-2} \varphi(x_h, x_{h+2k}) - \sum_{h=0}^{j-1} \psi(x_h, x_{h+2k-1}) \tag{7.1}$$

only depends on $x_{k-1}, \ldots, x_{j+k-1}$.
(iii) *The above holds for $j = k$ and $j = k + 1$.*
(iv) *There exists a function u on X_k and a symmetric function w on X_{k+1} such that, for any geodesic path $x_0, x_1, \ldots, x_0, \ldots, x_{2k}$ in X, one has*

$$\varphi(x_0, x_{2k}) = \sum_{h=0}^{k-1} w(x_h, x_{h+k+1}) + u(x_k, x_0) + u(x_k, x_{2k}) \text{ and} \tag{7.2}$$

$$\psi(x_0, x_{2k-1}) = \sum_{h=0}^{k-2} w(x_h, x_{h+k+1}) + u(x_k, x_0) + u(x_{k-1} x_{2k-1}). \tag{7.3}$$

PROOF. *(i)⇒(ii).* Assume *(i)* holds, take $j \geq k$ and let

$$x_0, \dots, x_{j+2k-2}$$

be a geodesic path. By symmetry, it suffices to prove that (7.1) only depends on $x_{k-1}, \dots, x_{j+2k-2}$. Now, we have

$$\sum_{h=0}^{j-2} \varphi(x_h, x_{h+2k}) - \sum_{h=0}^{j-1} \psi(x_h, x_{h+2k-1}) \\ = \sum_{h=0}^{j-2} (\varphi(x_h, x_{h+2k}) - \psi(x_h, x_{h+2k-1})) + \psi(x_{j-1}, x_{j+2k-2}).$$

The last term only depends on the elements $x_{j-1}, \dots, x_{j+2k-2}$, hence on the elements $x_{k-1}, \dots, x_{j+2k-2}$ as $j \geq k$. Besides, for $0 \leq h \leq j-2$, the assumption says that $\varphi(x_h, x_{h+2k}) - \psi(x_h, x_{h+2k-1})$ only depends on $x_{h+k-1}, \dots, x_{h+2k}$, hence also on $x_{k-1}, \dots, x_{j+2k-2}$ as required.

(ii)⇒(iii) is trivial.

(iii)⇒(i). Formally, it is not necessary to prove this implication, but the idea for showing it will be used later. Take a geodesic path $x_0, \dots, x_{3k-1}$. Then, the quantity $\varphi(x_0, x_{2k}) - \psi(x_0, x_{2k-1})$ may be written as the difference of (7.1) for $j = k+1$ applied to the path $x_0, \dots, x_{3k-1}$, with (7.1) for $j = k$ applied to the path $x_1, \dots, x_{3k-1}$. The conclusion follows.

(iv)⇒(i) is direct since, assuming *(iv)* to hold, we obtain, for any geodesic path $x_0, x_1, \dots, x_{2k}$,

$$\varphi(x_0, x_{2k}) - \psi(x_0, x_{2k-1}) = w(x_{k-1}, x_{2k}) + u(x_k x_{2k}) - u(x_{k-1}, x_{2k-1}).$$

It remains to show *(iii)⇒(iv)* which we now do. This is the most delicate part of the proof. Assuming *(iii)* to hold, we choose a function u on X_k such that, for any geodesic path $x_0, \dots, x_{3k-2}$, we have

(7.4)
$$u(x_{k-1}, x_{2k-1}) + u(x_{2k-1}, x_{k-1}) = \sum_{h=0}^{k-1} \psi(x_h, x_{h+2k-1}) - \sum_{h=0}^{k-2} \varphi(x_h, x_{h+2k}).$$

This makes sense thanks to (7.1) for $j = k$. Now, due to (7.1) for $j = k+1$, there exists a symmetric function w on X_k such that, for any geodesic path $x_0, \dots, x_{3k-1}$, we have

(7.5) $w(x_{k-1}, x_{2k}) =$

$$u(x_{k-1}, x_{2k-1}) + u(x_{2k}, x_k) + \sum_{h=0}^{k-1} \varphi(x_h, x_{h+2k}) - \sum_{h=0}^{k} \psi(x_h, x_{h+2k-1}).$$

We also define a function v on X_k by setting, for a geodesic path $x_0, \dots, x_{k+1}$,

$$v(x_0, x_{k+1}) = w(x_0, x_{k+1}) + u(x_1, x_{k+1}) - u(x_0, x_k). \tag{7.6}$$

Using the definitions of u and w above and reasoning as in the proof of *(iii)⇒(i)* yields, for a geodesic path $x_0, x_1, \dots, x_{2k}$,

$$\begin{aligned}\varphi(x_0, x_{2k}) - \psi(x_0, x_{2k-1}) &= (w(x_{k-1}, x_{2k}) - u(x_{k-1}, x_{2k-1}) - u(x_{2k}, x_k)) \\ &\quad + (u(x_k, x_{2k}) + u(x_{2k}, x_k)) = v(x_{k-1}, x_{2k}).\end{aligned} \tag{7.7}$$

We will now show that (7.2) and (7.3) hold. Indeed, let $(x_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line in X. Thanks to the definition of v in (7.6), we have

$$\sum_{h=0}^{k-1} w(x_h, x_{h+k+1}) + u(x_k, x_0) + u(x_k, x_{2k}) = \sum_{h=0}^{k-1} v(x_h, x_{h+k+1}) + u(x_0, x_k) + u(x_k, x_0).$$

Using the definition of u in (7.4) together with (7.7), we obtain

$$\begin{aligned}&\sum_{h=0}^{k-1} w(x_h, x_{h+k+1}) + u(x_k, x_0) + u(x_k, x_{2k}) = \\ &\sum_{h=0}^{k-1} (\varphi(x_{h-k+1}, x_{h+k+1}) - \psi(x_{h-k+1}, x_{h+k})) + \sum_{h=0}^{k-1} \psi(x_{h-k+1}, x_{h+k}) \\ &\qquad - \sum_{h=0}^{k-2} \varphi(x_{h-k+1}, x_{h+k+1}) = \varphi(x_0, x_{2k})\end{aligned}$$

and (7.2) follows. In the same way, we have

$$\sum_{h=0}^{k-2} w(x_h, x_{h+k+1}) + u(x_k, x_0) + u(x_{k-1}, x_{2k-1}) = \sum_{h=0}^{k-2} v(x_h, x_{h+k+1}) + u(x_0, x_k) + u(x_k, x_0).$$

Reasoning as above, we now get

$$\sum_{h=0}^{k-2} w(x_h, x_{h+k+1}) + u(x_k, x_0) + u(x_{k-1}, x_{2k-1}) =$$
$$\sum_{h=0}^{k-2} (\varphi(x_{h-k+1}, x_{h+k+1}) - \psi(x_{h-k+1}, x_{h+k})) + \sum_{h=0}^{k-1} \psi(x_{h-k+1}, x_{h+k})$$
$$- \sum_{h=0}^{k-2} \varphi(x_{h-k+1}, x_{h+k+1}) = \psi(x_0, x_{2k-1}),$$

that is, we have proved (7.3). □

7.2. Functions associated with a dual kernel

Let $k \geq 2$ and (K, K^-) be a k-dual kernel. We will now use Corollary 6.26 in conjunction with Proposition 7.1 (applied to $k-1$) to give formulas for the orthogonal extensions of (K, K^-).

We begin by giving names to the objects appearing in Proposition 7.1 and its proof. The next two definitions are inspired by (7.4) and (7.5).

Definition 7.2. Let u be a function on X_{k-1}. Then u is said to be (K, K^-)-compatible if, for any x, y in X with $d(x, y) = k-1$, one has

$$u(x, y) + u(y, x) = \sum_{h=1}^{k-1} K^{2k-3}_{z_{h-1} z_h}(\xi, \eta) - \sum_{h=1}^{k-2} K^{2k-2}_{z_h}(\xi, \eta), \tag{7.8}$$

where $(z_h)_{h \in \mathbb{Z}}$ is any parametrized geodesic line with $z_0 = x$ and $z_{k-1} = y$ and ξ and η are its endpoints.

Remark 7.3. By Corollary 6.26 and Proposition 7.1, the right hand-side of (7.8) only depends on x and y. In particular, compatible functions u always exist and if (K, K^-) is Γ-invariant, one can chose u to be so.

To a (K, K^-)-compatible function, we associate its weight function:

Definition 7.4. If u is a (K, K^-)-compatible function, we define its weight function w on X_k by, for any x, y in X with $d(x, y) = k$,

$$w(x, y) = u(x, z_{k-1}) + u(y, z_1) + \sum_{h=1}^{k-1} K^{2k-2}_{z_h}(\xi, \eta) - \sum_{h=1}^{k} K^{2k-3}_{z_{h-1} z_h}(\xi, \eta), \tag{7.9}$$

where $(z_h)_{h\in\mathbb{Z}}$ is any parametrized geodesic line with $z_0 = x$ and $z_k = y$ and ξ and η are its endpoints.

REMARK 7.5. Again, it follows from Corollary 6.26 and Proposition 7.1 that $w(x, y)$ only depends on the choice of x and y. Note that the weight function is symmetric.

EXAMPLE 7.6. Recall the Busemann kernel from Section 5.5. A function u on X_1 is compatible with the Busemann kernel if and only if one has, for any $x \sim y$ in X,

$$u(xy) + u(yx) = 1.$$

In this case, the associated weight function w on X_2 is given by, for any x, y in X with $d(x, y) = 2$,

$$w(xy) = u(xz) + u(yz),$$

where z is the middle-point of the segment $[xy]$.

We restate the conclusion of Proposition 7.1 for dual kernels.

COROLLARY 7.7. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Choose a (K, K^-)-compatible function u on X_{k-1} and let w be the associated weight function on X_k. For any geodesic path $x_0, x_1, \ldots, x_{2k-2}$ in X, we have*

(7.10)

$$K^{2k-2}_{x_{k-1}}(x_0, x_{2k-2}) = \sum_{h=0}^{k-2} w(x_h, x_{h+k}) + u(x_{k-1}, x_0) + u(x_{k-1}, x_{2k-2})$$

and

(7.11)

$$K^{2k-3}_{x_{k-2}x_{k-1}}(x_0, x_{2k-3}) = \sum_{h=0}^{k-3} w(x_h, x_{h+k}) + u(x_{k-1}, x_0) + u(x_{k-2}x_{2k-3}).$$

PROOF. This is a direct consequence of Corollary 6.26, Proposition 7.1 and its proof. □

In the course of the proof of Proposition 7.1, we have considered a certain function v. Since we will also need it later, we give it a name. The next definition makes sense thanks to Corollary 6.26

DEFINITION 7.8. Let $k \geq 2$ and (K, K^-) be a k-dual kernel. We define the bias function v of (K, K^-) as follows. If x, y are in X and $d(x, y) = k$, we set

$$v(x, y) = K^{2k-2}_{x_-}(y, z) - K^{2k-3}_{xx_-}(y_-, z), \tag{7.12}$$

where x_- and y_- are the neighbours of x and y on $[xy]$ and z is any point in X with $d(z,x) = k-2$ and $[zx] \cap [xy] = \{x\}$. By Corollary 6.26, the function v does not depend on the choice of z. Note that, by the relation $K^{2k-1}_{xx_-} = K^{2k-2}_x + K^{2k-2}_{x_-} - K^{2k-3}_{xx_-}$, we also have

$$v(x,y) = K^{2k-1}_{xx_-}(y,t) - K^{2k-2}_x(y_-,t), \tag{7.13}$$

for any t in X with $d(t,x) = k-1$ and $[tx] \cap [xy] = \{x\}$.

EXAMPLE 7.9. The bias function v of the Busemann kernel is the constant function with value 1 on X_2.

The function v is related to the functions u and w by a cohomological equation:

LEMMA 7.10. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel, u be a (K, K^-)-compatible function and w be the associated weight function. Let $x_0, x_1, \ldots, x_k$ be a geodesic path of X. We have*

$$u(x_0, x_{k-1}) + v(x_0, x_k) = u(x_1, x_k) + w(x_0, x_k).$$

PROOF. This follows from (7.6) and (7.7) in the proof of Proposition 7.1. □

7.3. Lifting of bilinear forms

Let still $k \geq 2$ and let (K, K^-) be a k-dual kernel. For any even $h \geq k-1$, $h = 2\ell$, $\ell \geq 1$ (resp. for any odd $h \geq k-1$, $h = 2\ell+1$, $\ell \geq 0$), for any x in X (resp. for any $x \sim y$ in X), we have an associated symmetric bilinear form $q_x^{K^h} = q_x^h$ (resp. $q_{xy}^{K^h} = q_{xy}^h$) on $V_0^\ell(x)$ (resp. $V_0^\ell(xy)$). We will now build bilinear forms on $V^\ell(x)$ (resp. $V^\ell(xy)$) whose restriction to $V_0^\ell(x)$ (resp. $V_0^\ell(xy)$) are equal to q_x^h (resp. q_{xy}^h).

To construct such forms, we will use

LEMMA 7.11. *Let A be a finite set. Let V be the space of real-valued functions on A and*

$$V_0 = \{f \in V | \sum_{a \in A} f(a) = 0\}.$$

If q is a symmetric bilinear form on V, set, for a in A, $u_q(a) = q(\mathbf{1}_a, \mathbf{1}_a)$.

Let q_0 be a symmetric bilinear form on V_0. Then the map $q \mapsto u_q$ induces an affine isomorphism from the space of symmetric bilinear forms q on V with $q_{|V_0} = q_0$ onto V.

Any such form q will be called a lifting of q_0.

As in Lemma 7.11, lifting bilinear forms on $V_0^\ell(x)$ and $V_0^\ell(xy)$ to $V^\ell(x)$ and $V^\ell(xy)$ will require choices. These choices will be achieved by chosing a (K, K^-)-compatible function.

We are now ready to state our result on the lifting of the bilinear forms associated with a dual kernel. The formulas which appear in the construction of these liftings are related to the ones in Proposition 2.22.

PROPOSITION 7.12. *Let $k \geq 2$, (K, K^-) be a k-dual kernel with bias function v, u be a (K, K^-)-compatible function and w be its weight function. Fix $j \geq k-1$.*

Then there exists a unique family $(\hat{q}_x^{2j})_{x\in X}$ such that, for any x in X, $\hat{q}_x^{2j}$ is a symmetric bilinear form on $V^j(x)$ with $(\hat{q}_x^{2j})_{|V_0^j(x)} = q_x^{2j}$ and, for any z in $S^j(x)$,

$$\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z) = u(x, z_{k-1}) + \sum_{h=0}^{j-k} v(z_h, z_{h+k}) = \sum_{h=0}^{j-k} w(z_h, z_{h+k}) + u(z_{j-k+1}, z_j),$$

where $z_0 = x, z_1, \ldots, z_j = z$ is the geodesic path from x to z. In the same way, there exists a unique family $(\hat{q}_{xy}^{2j-1})_{x\in X}$ such that, for any $x \sim y$ in X, $\hat{q}_{xy}^{2j-1}$ is a symmetric bilinear form on $V^{j-1}(xy)$ with $(\hat{q}_{xy}^{2j-1})_{|V_0^{j-1}(xy)} = q_{xy}^{2j-1}$ and, for any z in $S^{j-1}(xy) \cap S^j(x)$,

$$\hat{q}_{xy}^{2j-1}(\mathbf{1}_z, \mathbf{1}_z) = u(x, z_{k-1}) + \sum_{h=0}^{j-k} v(z_h, z_{h+k}) = \sum_{h=0}^{j-k} w(z_h, z_{h+k}) + u(z_{j-k+1}, z_j),$$

where, as above, $z_0 = x, z_1, \ldots, z_j = z$ is the geodesic path from x to z.

If $x_0, \ldots, x_{2j}$ is a geodesic path in X, one has

$$\hat{q}_{x_j}^{2j}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2j}}) = -\frac{1}{2} \sum_{h=0}^{k-2} w(x_{h+j+1-k}, x_{h+j+1}) \tag{7.14}$$

$$\text{and } \hat{q}_{x_{j-1}x_j}^{2j-1}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2j-1}}) = -\frac{1}{2} \sum_{h=0}^{k-3} w(x_{h+j+1-k}, x_{h+j+1}). \tag{7.15}$$

The proof of (7.14) and (7.15) relies on Corollary 7.7 and on additional properties of the bias function v.

LEMMA 7.13. *For any $x \sim y$ in X and any z, t in $S^{k-1}(xy)$ with $y \notin [xz]$ and $x \notin [yt]$, we have*

$$K_{xy}^{2k-1}(z,t) - K_{xy}^{2k-3}(z_-, t_-) = v(x,t) + v(y,z),$$

where z_- and t_- are the neighbours of z and t on $[zt]$.

PROOF. By (7.12), we have

$$\begin{aligned} v(x,t) &= K_y^{2k-2}(z_-, t) - K_{xy}^{2k-3}(z_-, t_-) \\ \text{and } v(y,z) &= K_x^{2k-2}(z, t_-) - K_{xy}^{2k-3}(z_-, t_-). \end{aligned}$$

The result now follows from the relation

$$K_{xy}^{2k-1}(z,t) = K_x^{2k-2}(z, t_-) + K_y^{2k-2}(z_-, t) - K_{xy}^{2k-3}(z_-, t_-).$$

□

LEMMA 7.14. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with bias function v. For any x in X and any z, t in $S^k(x)$ with $x \in [zt]$, we have*

$$K_x^{2k}(z,t) - K_x^{2k-2}(z_-, t_-) = v(x,z) + v(x,t),$$

where z_- and t_- are the neighbours of z and t on $[zt]$.

PROOF. Let z_1 and t_1 be the neighbours of x on $[xz]$ and $[xt]$. By (7.13), we have

$$\begin{aligned} v(x,z) &= K_{xz_1}^{2k-1}(z, t_-) - K_x^{2k-2}(z_-, t_-) \\ \text{and } v(x,t) &= K_{xt_1}^{2k-1}(z_-, t) - K_x^{2k-2}(z_-, t_-). \end{aligned}$$

From the definition of the orthogonal extension, we obtain

$$\begin{aligned} &K_x^{2k}(z,t) - K_x^{2k-2}(z_-, t_-) \\ &\qquad = v(x,z) + v(x,t) + \sum_{\substack{y \sim x \\ y \neq z_1, t_1}} K_{xy}^{2k-1}(z_-, t_-) - K_x^{2k-2}(z_-, t_-). \end{aligned}$$

Let L be the canonical k-pseudokernel of (K, K^-) (see Definition 6.11 and Definition 6.12). By Lemma 6.14, we get

$$\begin{aligned} K_x^{2k}(z,t) - K_x^{2k-2}(z_-, t_-) &= v(x,z) + v(x,t) + \sum_{\substack{y \sim x \\ y \neq z_1, t_1}} L_{xy}^{2k-1}(z_-, t_-) \\ &= v(x,z) + v(x,t), \end{aligned}$$

where the last equality follows from Corollary 6.24. □

We will need a formula for the bias function of an orthogonal extension.

LEMMA 7.15. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with bias function v. Then, the bias function of the orthogonal extension (K^+, K) of (K, K^-) is the function v^+ defined by, for (x, y) in X_{k+1}, $v^+(x, y) = v(x_1, y)$, where x_1 is the neighbour of x on $[x, y]$.*

PROOF. Let v^+ be the bias function of (K^+, K). Fix a parametrized geodesic line $(x_h)_{h\in\mathbb{Z}}$ in X. By (7.12), we have

$$v^+(x_0, x_{k+1}) = K^{2k}_{x_1}(x_{1-k}, x_{k+1}) - K^{2k-1}_{x_0x_1}(x_{1-k}, x_k),$$

whereas, by (7.13),

$$v(x_1, x_{k+1}) = K^{2k-1}_{x_1x_2}(x_{2-k}, x_{k+1}) - K^{2k-2}_{x_1}(x_{2-k}, x_k).$$

We get, by the recursive definition of kernels,

$$\begin{aligned} v^+(x_0, x_{k+1}) - v(x_1, x_{k+1}) &= \sum_{\substack{y\sim x_1 \\ y\notin\{x_0,x_2\}}} K^{2k-1}_{x_1y}(x_{2-k}, x_k) - K^{2k-2}_{x_1}(x_{2-k}, x_k) \\ &= \sum_{\substack{y\sim x_1 \\ y\notin\{x_0,x_2\}}} L^{2k-1}_{x_1y}(x_{2-k}, x_k), \end{aligned}$$

where L is the canonical k-pseudokernel of (K, K^-) (see Definition 6.11 and Definition 6.12) and we have used Lemma 6.14. Using Corollary 6.24, we obtain $v^+(x_0, x_{k+1}) = v(x_1, x_{k+1})$ as required. □

PROOF OF PROPOSITION 7.12. Existence of the liftings is a direct consequence of Lemma 7.11. Equality in the two formulas for chosing the lifting is a consequence of Lemma 7.10.

We will now prove formulas (7.14) and (7.15).

First, we assume $j = k-1$. In that case, if $x_0, \dots, x_{2k-2}$ is a geodesic path in X, by elementary properties of bilinear forms, we have

$$\begin{aligned} &2\hat{q}^{2k-2}_{x_{k-1}}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2k-2}}) = \\ &\hat{q}^{2k-2}_{x_{k-1}}(\mathbf{1}_{x_0}, \mathbf{1}_{x_0}) + \hat{q}^{2k-2}_{x_{k-1}}(\mathbf{1}_{x_{2k-2}}, \mathbf{1}_{x_{2k-2}}) - \hat{q}^{2k-2}_{x_{k-1}}(\mathbf{1}_{x_0} - \mathbf{1}_{x_{2k-2}}, \mathbf{1}_{x_0} - \mathbf{1}_{x_{2k-2}}) \\ &\qquad = u(x_{k-1}, x_0) + u(x_{k-1}, x_{2k-2}) - K^{2k-2}_{x_{k-1}}(x_0, x_{2k-2}) \end{aligned}$$

and (7.14) follows from (7.10) in Corollary 7.7. In the same way, we have

$$\begin{aligned} &2\hat{q}^{2k-3}_{x_{k-2}x_{k-1}}(\mathbf{1}_{x_0}, \mathbf{1}_{x_{2k-3}}) = \\ &\qquad u(x_{k-1}, x_0) + u(x_{k-2}, x_{2k-3}) - K^{2k-3}_{x_{k-2}x_{k-1}}(x_0, x_{2k-3}) \end{aligned}$$

and (7.15) now follows from (7.11) in Corollary 7.7.

From now on, we assume that $j \geq k$.

We start by proving (7.14). We claim that, for any x in X and z, t in $S^j(x)$ with $x \in [z,t]$, if z_- and t_- are the neighbours of z and t in $S^{j-1}(x)$, we have

$$\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t) = \hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-}). \tag{7.16}$$

This implies (7.14) by induction on j. Now, we show (7.16). By definition, one has

$$\begin{aligned}\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z) &= \hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{z_-}) + v(z_k, z)\\ \hat{q}_x^{2j}(\mathbf{1}_t, \mathbf{1}_t) &= \hat{q}_x^{2j-2}(\mathbf{1}_{t_-}, \mathbf{1}_{t_-}) + v(t_k, t),\end{aligned}$$

where z_k and t_k are the elements of the segments $[x,z]$ and $[x,t]$ at distance k from z and t. Hence, by elementary properties of quadratic forms,

$$\begin{aligned}2\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t) - 2\hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-})&\\ = v(z_k, z) + v(t_k, t) - K_x^{2j}(z,t) + K_x^{2j-2}(z_-, t_-).&\end{aligned}$$

By Lemma 7.14, applied to the j-dual kernel (K^j, K^{j-1}), and Lemma 7.15, the latter is zero and (7.16) follows.

In the same way, let us prove (7.15). For any $x \sim y$ in X and z, t in $S^{j-1}(xy)$ with $z \in S^{j-1}(x)$ and $t \in S^{j-1}(y)$, we now claim that we have

$$\hat{q}_{xy}^{2j-1}(\mathbf{1}_z, \mathbf{1}_t) = \hat{q}_x^{2j-3}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-}). \tag{7.17}$$

Again this implies (7.15) by induction on j. Tos show (7.17), notice that by definition, one has

$$\begin{aligned}\hat{q}_{xy}^{2j-1}(\mathbf{1}_z, \mathbf{1}_z) &= \hat{q}_x^{2j-2}(\mathbf{1}_{z_-}, \mathbf{1}_{z_-}) + v(z_k, z)\\ \hat{q}_x^{2j}(\mathbf{1}_t, \mathbf{1}_t) &= \hat{q}_x^{2j-2}(\mathbf{1}_{t_-}, \mathbf{1}_{t_-}) + v(t_k, t),\end{aligned}$$

hence, by elementary properties of quadratic forms,

$$\begin{aligned}2\hat{q}_{xy}^{2j-1}(\mathbf{1}_z, \mathbf{1}_t) - 2\hat{q}_{xy}^{2j-3}(\mathbf{1}_{z_-}, \mathbf{1}_{t_-})&\\ = v(z_k, z) + v(t_k, t) - K_{xy}^{2j-1}(z,t) + K_{xy}^{2j-3}(z_-, t_-).&\end{aligned}$$

By Lemma 7.13, applied to the j-dual kernel (K^j, K^{j-1}), and Lemma 7.15, the latter is zero and (7.17) follows. □

7.4. The additive formula

We will show that for large enough j, $\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)$ is given by the same formula as in Proposition 2.22.

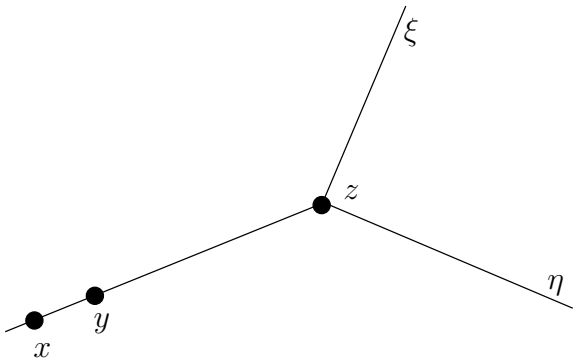

FIGURE 1. The points in Lemma 7.17

PROPOSITION 7.16. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. We chose a (K, K^-)-compatible function on X_{k-1} and we let w denote the associated weight function. For x in X and $\xi \neq \eta$ in ∂X, let $z_0 = x, z_1, \dots$ be the geodesic ray $[x\xi)$ and $t_0 = x, t_1, \dots$ be the geodesic ray $[x\eta)$. Set $i = \omega_x(\xi, \eta)$. Then, for every $j \geq i + k - 1$, we have*

$$\hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = \frac{1}{2}\sum_{h=0}^{i-1}(w(z_h, z_{h+k}) + w(t_h, t_{h+k})) - \frac{1}{2}\sum_{h=1}^{k-1} w(z_{i+h}, t_{i+k-h}).$$

If $i \geq 1$, we have $\hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j})$.

The main ingredient of the proof is the following property of invariance of certain values of (K, K^-) by backtracking on geodesics.

LEMMA 7.17. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Take x in X and $\xi \neq \eta$ in ∂X with $x \notin (\xi\eta)$. We set $i = d(x, (\xi\eta)) = \omega_x(\xi, \eta) \geq 1$. Let z be the element of $(\xi\eta)$ that is closest to x and y be the neighbour of x on $[xz]$ (see Figure 1). Then we have*

$$K_x^{2(i+k)-2}(\xi, \eta) = K_{xy}^{2(i+k)-3}(\xi, \eta) = K_z^{2k-2}(\xi, \eta).$$

PROOF. It suffices to show that we have

$$K_x^{2(i+k)-2}(\xi, \eta) = K_{xy}^{2(i+k)-3}(\xi, \eta) = K_y^{2(i+k)-4}(\xi, \eta).$$

Let L be the canonical pseudokernel of (K, K^-) (see Definition 6.11 and Definition 6.12). By Lemma 6.14, we have

$$\begin{aligned} &K_x^{2(i+k)-2}(\xi,\eta) - K_y^{2(i+k)-4}(\xi,\eta) = \\ &(K_x^{2(i+k)-2}(\xi,\eta) - K_{xy}^{2(i+k)-3}(\xi,\eta)) + (K_{xy}^{2(i+k)-3}(\xi,\eta) - K_y^{2(i+k)-4}(\xi,\eta)) \\ &\qquad = L_{xy}^{2(i+k)-2}(\xi,\eta) + L_{yx}^{2(i+k)-3}(\xi,\eta) \\ &\qquad\qquad = L_{yx}^{2(i+k)-1}(\xi,\eta) + L_{yx}^{2(i-1+k)-1}(\xi,\eta), \end{aligned}$$

which makes sense as $i \geq 1$. By Corollary 6.24 the latter is zero and we are done. □

Proof of Proposition 7.16. By elementary properties of quadratic forms, we have

$$2\hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{z_j}) + \hat{q}_x^{2j}(\mathbf{1}_{t_j}, \mathbf{1}_{t_j}) - K_x^{2j}(z_j, t_j).$$

By Lemma 7.17, applied to the $(j-i+1)$-dual kernel (K^{j-i+1}, K^{j-i}), we have

$$K_x^{2j}(z_j, t_j) = K_{z_i}^{2(j-i)}(z_j, t_j)$$

(note that $z_i = t_i$). By Proposition 7.12, we get

$$2\hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = 2\hat{q}_x^{2(j-i)}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) + \sum_{h=0}^{i-1} w(z_h, z_{h+k}) + \sum_{h=0}^{i-1} w(t_h, t_{h+k})$$

as well as

$$2\hat{q}_x^{2(j-i)}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = -\sum_{h=1}^{k-1} w(z_{i+h}, t_{i+k-h})$$

and the formula for $\hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j})$ follows.

If $i \geq 1$, we have $z_1 = t_1$ and, by the definition in Proposition 7.12,

$$\hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{z_j}, \mathbf{1}_{z_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{z_j})$$
$$\text{and } \hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{t_j}, \mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{t_j}, \mathbf{1}_{t_j}),$$

as well as, by Lemma 7.17, applied to the $(j-i+1)$-dual kernel (K^{j-i+1}, K^{j-i}),

$$K_{xz_1}^{2j-1}(z_j, t_j) = K_x^{2j}(z_j, t_j).$$

We get $\hat{q}_{xz_1}^{2j-1}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j}) = \hat{q}_x^{2j}(\mathbf{1}_{z_j}, \mathbf{1}_{t_j})$. □

CHAPTER 8

Dual kernels and additive kernels

In this chapter, we use Proposition 7.16 to draw a link between the language of Chapter 3 and the language of dual kernels. We will prove that, given a Γ-invariant non-negative dual kernel (K, K^-), the associated space of distributions L^{K,K^-} always contains the space H_0^ω and that the symmetric bilinear form q^{K,K^-}, when restricted to H_0^ω, is equal to Φ_w, where w is a Γ-invariant weight function of (K, K^-). Conversely, we will prove that, if for a given symmetric Γ-invariant function w on X_k, $k \geq 2$, the bilinear form Φ_w is non-negative on H_0^ω, then there exists a Γ-invariant non-negative k-dual kernel (K, K^-) which admits w as a weight function.

8.1. A dense subspace of H_0^ω

First, we will need to gather additional information on the Hilbert space H_0^ω from Chapter 3. Recall that, if ν is a Borel probability measure ∂X, we write $\mathfrak{M}^\infty(\nu)$ for the space of signed Borel measures on ∂X which are absolutely continuous with respect to ν with a bounded density. The purpose of this Section is to construct a Borel probability measure ν such that $\mathfrak{M}^\infty(\nu)$ is a dense subspace of H^ω. We start with a general criterion for density.

PROPOSITION 8.1. *Let x be in X and ν be a fully supported Borel probability measure on ∂X. Assume that one has*

$$\sup_{\substack{y\in X\\ y\neq x}} \frac{1}{\nu(U_{xy})^2} \sum_{\substack{z\in X\\ y\in[xz]}} \nu(U_{xz})^2 < \infty,$$

then ν is atom-free, ω is ν-integrable and $\mathfrak{M}^\infty(\nu)$ is dense in H^ω.

REMARK 8.2. For $x \neq y$ in X, the quantity $\frac{1}{\nu(U_{xy})^2} \sum_{\substack{z\in X\\ y\in[xz]}} \nu(U_{xz})^2$ may be seen as a local version of the norm of H^ω.

PROOF. The proof relies on a straightforward construction of approximations of the elements of H^ω by elements of $\mathfrak{M}^\infty(\nu)$. We will use again the language of Section 3.1.

Let, for any $y \neq x$ in X, y_- be the neighbour of y on $[x, y]$. First note that, since by definition one has $\mathcal{P}_x T(y_-, y) = \nu(U_{xy})$, the assumption implies that the function $\mathcal{P}_x \nu$ belongs to $\ell^2(X_1)$ hence, by Proposition 3.7, that ω is ν-integrable. In particular, by the same result, we have $\mathfrak{M}^\infty(\nu) \subset H^\omega$.

Now, if T is a distribution, for any $\ell \geq 1$, we define a smooth function on ∂X by setting, for any ξ in ∂X,

$$\varphi_\ell^T(\xi) = \frac{\langle T, \mathbf{1}_{U_{xy}} \rangle}{\nu(U_{xy})},$$

where y is the unique element of $S^\ell(x)$ with $y \in [x\xi)$. This makes sense since $\nu(U_{xy}) \neq 0$ as ν has full support. We define $\pi_\ell(T)$ as the distribution $\varphi_\ell^T \nu$ which belongs to $\mathfrak{M}^\infty(\nu)$.

We now use again the notation of Chapter 3.1. If T belongs to H^ω, we claim that the assumption implies that $\pi_\ell(T) \xrightarrow[\ell \to \infty]{} T$ in H^ω. Indeed, by construction, for any $y \neq x$ with $d(x, y) \leq \ell$, we have $\langle \pi_\ell(T), \mathbf{1}_{U_{xy}} \rangle = \langle T, \mathbf{1}_{U_{xy}} \rangle$, hence

$$\begin{aligned} \|\mathcal{P}_x(\pi_\ell(T) - T)\|_2^2 &= 2 \sum_{\substack{y \in X \\ d(x,y) \geq \ell}} \langle \pi_\ell(T) - T, \mathbf{1}_{U_{xy}} \rangle^2 \\ &\leq 4 \sum_{\substack{y \in X \\ d(x,y) \geq \ell}} (\langle \pi_\ell(T), \mathbf{1}_{U_{xy}} \rangle^2 + \langle T, \mathbf{1}_{U_{xy}} \rangle^2). \end{aligned}$$

Now, on one hand, as T belongs to H^ω, one has

$$\sum_{\substack{y \in X \\ d(x,y) \geq \ell}} \langle T, \mathbf{1}_{U_{xy}} \rangle^2 \xrightarrow[l \to \infty]{} 0.$$

On the other hand, we have

$$\begin{aligned} \sum_{d(x,y) \geq \ell} \langle \pi_\ell(T), \mathbf{1}_{U_{xy}} \rangle^2 &= \sum_{\substack{y \in X \\ d(x,y) = \ell}} \sum_{\substack{z \in X \\ y \in [xz]}} \langle \pi_\ell(T), \mathbf{1}_{U_{xz}} \rangle^2 \\ &= \sum_{\substack{y \in X \\ d(x,y) = \ell}} \frac{\langle T, \mathbf{1}_{U_{xy}} \rangle^2}{\nu(U_{xy})^2} \sum_{\substack{z \in X \\ y \in [xz]}} \nu(U_{xz})^2. \end{aligned}$$

By assumption, the latter goes to 0 as $\ell \to \infty$ and we are done. □

Corollary 8.3. *There exists a fully supported atom-free Borel probability measure ν on ∂X such that ω is ν-integrable and $\mathfrak{M}^\infty(\nu)$ is dense in H^ω.*

Recall that, for x in X, $d(x) \geq 3$ is the number of neighbours of x.

PROOF. For example, one can fix x in X and define the associate probability measure ν_x as the unique Borel probability measure such that, for any $y \neq x$ in X, if $x_0 = x, x_1, \dots, x_\ell = y$ is the geodesic path from x to y, one has

$$\nu_x(U_{xy}) = \frac{1}{d(x)} \frac{1}{d(x_1) - 1} \cdots \frac{1}{d(x_{\ell-1}) - 1}.$$

Let us check that the criterion in Proposition 8.1 holds. By construction, for any $y \neq x$, and $z \sim y$ with $z \notin [xy]$, we have $\nu_x(U_{xz}) = \frac{1}{d(y)-1}\nu_x(U_{xy})$, hence

$$\sum_{\substack{z \sim y \\ z \notin [xy]}} \nu_x(U_{xz})^2 = \frac{1}{d(y) - 1} \nu_x(U_{xy})^2 \leq \frac{1}{2} \nu_x(U_{xy})^2.$$

By induction, we get, for $\ell \geq 0$, $\sum_{\substack{z \in S^\ell(y) \\ y \in [xz]}} \nu_x(U_{xz})^2 \leq \frac{1}{2^\ell} \nu_x(U_{xy})^2$, hence $\sum_{\substack{z \in X \\ y \in [xz]}} \nu(U_{xz})^2 \leq 2\nu(U_{xy})^2$ and the result follows by Proposition 8.1. □

We can use the existence of ν to get a proof that the bilinear form Φ_w from Section 3.2 determines w up to cohomology.

COROLLARY 8.4. *Let w be a Γ-invariant symmetric function on X_k. Assume that Φ_w is zero on H_0^ω. Then the normalized smooth function on $\Gamma \backslash \mathscr{S}$ associated to w is cohomologuous to 0.*

The proof uses an elementary fact from measure theory.

LEMMA 8.5. *Let (X, ν) be a probability space and Ω be a symmetric function in $\mathrm{L}^1(X \times X, \nu \otimes \nu)$. The following are equivalent.*
(i) *For every ρ in $\mathrm{L}^\infty(X, \nu)$ with $\int_X \rho \mathrm{d}\nu = 0$, we have*

$$\int_{X \times X} \Omega(x, y) \rho(x) \rho(y) \mathrm{d}\nu(x) \mathrm{d}\nu(y) = 0.$$

(ii) *There exists a function F in $\mathrm{L}^1(X, \nu)$ such that for $\nu \otimes \nu$-almost every (x, y) in $X \times X$, one has $\Omega(x, y) = F(x) + F(y)$.*

The function F is then uniquely determined by Ω.

PROOF OF COROLLARY 8.4. Let Ω be an additive kernel associated to w and, as in Corollary 8.3, let ν be a fully supported atom-free Borel probability measure on ∂X such that ω is ν-integrable.

Fix x in X. By Proposition 3.11, we have

$$\int_{\partial X \times \partial X} \Omega_x(\xi, \eta) \mathrm{d}\rho(\xi) \mathrm{d}\rho(\eta) = 0$$

for every ρ in $\mathfrak{M}_0^\infty(\nu)$. By Lemma 8.5, there exists a function F_x in $\mathrm{L}^1(\partial X, \nu)$ such that, for $\nu \otimes \nu$-almost every (ξ, η) in $\partial X \times \partial X$, one has $\Omega_x(\xi, \eta) = F_x(\xi) + F_x(\eta)$. As for every η in ∂X, the function $\xi \mapsto \Omega_x(\xi, \eta)$ is smooth on $\partial X \smallsetminus \{\eta\}$, the function F_x is smooth. As F_x is uniquely determined by Ω, the function $(x, \xi) \mapsto F_x(\xi)$ on $X \times \partial X$ is Γ-invariant. For every x, y in X and $\xi \neq \eta$ in ∂X, we have

$$\Omega_x(\xi, \eta) - \Omega_y(\xi, \eta) = (F_x(\xi) - F_y(\xi)) + (F_x(\eta) - F_y(\eta)),$$

that is, Ω is an additive kernel associated to the trivial cohomology class. The conclusion follows by Lemma 2.19. □

8.2. From dual kernels to additive kernels

Now, we will show how one can use Proposition 7.16 in order to associate an additive kernel to a dual kernel.

THEOREM 8.6. *Let $k \geq 2$, (K, K^-) be a non-negative Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then the symmetric bilinear form Φ_w is non-negative on H_0^ω. More precisely, one has $H_0^\omega \subset L^{K,K^-}$ and the restriction of q^{K,K^-} to H_0^ω is equal to Φ_w.*

See Sections 3.2 and 3.3 for the definition and properties of Φ_w. See Proposition 5.18 for the definition of the spaces L^{K,K^-} and H^{K,K^-} and the bilinear form q^{K,K^-} associated to the dual kernel (K, K^-). See Definitions 7.2 and 7.4 for the notion of a (K, K^-)-compatible function and its weight function.

Knowing Theorem 8.6, we can prove that the Hilbert space associated to (K, K^-) is large:

COROLLARY 8.7. *Let $k \geq 2$ and (K, K^-) be a Γ-invariant non-negative k-dual kernel. Then, for every $\ell \geq \frac{k}{2}$ and any x in X, the linear map $N_x^{\ell,*}$ maps H^{K,K^-} onto $V_0^\ell(x)/\ker q_x^{2\ell}$.*

If V is a vector space and Φ is a non-negative symmetric bilinear form on V, a linear functional φ on V is said to be bounded with respect to Φ if one cand find $C \geq 0$ with $\varphi(x)^2 \leq C\Phi(x, x)$ for any x in V. In other words, one has $\ker \Phi \subset \ker \varphi$ and φ is bounded with respect to the Euclidean structure associated to Φ on $V/\ker \Phi$.

COROLLARY 8.8. *Let $k \geq 2$, (K, K^-) be a Euclidean Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then, for every φ in $\overline{\mathcal{D}}(\partial X)$, the associated linear functional on H_0^ω is bounded with respect to Φ_w.*

For non necessarily non-negative dual kernels we also get

COROLLARY 8.9. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then, for every θ in H_0^ω, one has*

$$q_x^{2j}(N_x^{j,*}\theta, N_x^{j,*}\theta) \xrightarrow[j\to\infty]{} \Phi_w(\theta, \theta).$$

We now start the proof of Theorem 8.6 and its Corollaries. We will also estalish weaker versions of these results for non necessary non-negative dual kernels. This will be possible thanks to the easy

LEMMA 8.10. *Let $k \geq 2$. Any Γ-invariant k-dual kernel may be written as the difference of two non-negative Γ-invariant k-dual kernels.*

PROOF. Indeed, in the finite-dimensional space of all Γ-invariant k-dual kernels, the set of non-negative ones is a convex cone with non-empty interior by Proposition 5.14. □

To dominate certain error terms, we shall use

LEMMA 8.11. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, u a Γ-invariant (K, K^-)-compatible function on X_{k-1} and w its weight function. Then, there exits $C \geq 0$ such that, for any $j \geq k-1$, x in X, z, t in $S^j(x)$, one has*

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\right| \leq C(1 + i + j),$$

where $i = |[xz] \cap [xt]|$ and $\hat{q}_x^{2j}$ is the symmetric bilinear form on $V^j(x)$ associated to the choices of (K, K^-) and u as in Section 7.3.

PROOF. Thanks to Lemma 8.10, we may and will assume that (K, K^-) is non-negative.

Note that, as $\Gamma\backslash X$ is finite, the functions u and w are bounded. In particular, by Proposition 7.16, there exists $C_1 \geq 0$ such that, if $j \geq k-1$, x in X, z, t in $S^j(x)$, are such that $|[xz] \cap [xt]| \leq j + 1 - k$, one has

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t)\right| \leq C_1(1 + j).$$

In the same way, by Proposition 7.12, there exists $C_2 \geq 0$ such that, for any $j \geq k-1$, x in X, z in $S^j(x)$, one has

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z)\right| \leq C_2(1 + j).$$

Now, let $j \geq k-1$, x in X, z, t in $S^j(x)$, be such that

$$i = |[xz] \cap [xt]| \geq j + 1 - k.$$

We set $\ell = i + k - 1 \geq j$. We have

$$2\hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_t) = \hat{q}_x^{2j}(\mathbf{1}_z, \mathbf{1}_z) + \hat{q}_x^{2j}(\mathbf{1}_t, \mathbf{1}_t) - K_x^{2j}(z, t)$$

and it only remains to get a bound for $K_x^{2j}(z,t)$. But, as the kernel (K,K^-) is non-negative, by Proposition 5.16, we have

$$0 \leq K_x^{2j}(z,t) \leq K_x^{2\ell}(z,t).$$

Now, again,

$$K_x^{2\ell}(z,t) = \hat{q}_x^{2\ell}(\mathbf{1}_z,\mathbf{1}_z) + \hat{q}_x^{2\ell}(\mathbf{1}_t,\mathbf{1}_t) - 2\hat{q}_x^{2\ell}(\mathbf{1}_z,\mathbf{1}_t),$$

hence

$$K_x^{2\ell}(z,t) \leq 2(C_1+C_2)(1+\ell) = 2(C_1+C_2)(k+i)$$

and

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\right| \leq C_1(1+j) + (C_1+C_2)(k+i),$$

which should be proved. □

Now, we will use the formulas in Propositions 3.11 and 7.16 to prove that q^{K,K^-} is equal to Φ_w on certain spaces:

LEMMA 8.12. *Let $k \geq 2$, (K,K^-) be a Γ-invariant k-dual kernel, u a Γ-invariant (K,K^-)-compatible function on X_{k-1} and w its weight function. We also let ν be a Borel probability measure on ∂X such that ω is ν-integrable. Then, for every ρ in $\mathfrak{M}_0^\infty(\nu)$, one has*

$$q_x^{2j}(N_x^{j,*}\rho, N_x^{j,*}\rho) \xrightarrow[j\to\infty]{} \Phi_w(\rho,\rho).$$

See Section 3.1 for the definition of the space $\mathfrak{M}_0^\infty(\nu)$.

PROOF. The proof is a direct consequence of the fact that the formulas appearing in Proposition 2.22 and Proposition 7.16 are the same.

More precisely, let us fix x in X. For any ρ in $\mathfrak{M}_0^\infty(\nu)$, we have, for $j \geq k-1$,

$$q_x^{2j}(N_x^{j,*}\rho, N_x^{j,*}\rho) = \sum_{z,t\in S^j(x)} \hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\rho(U_{xz})\rho(U_{xt}), \tag{8.1}$$

where, as above, $U_{xz} \subset \partial X$ is the set of ξ in ∂X with $z \in [x\xi)$ and, as in Section 5.4, $N_x^{j,*}$ is the linear operator that sends a distribution T on ∂X to the function $z \mapsto \langle T, \mathbf{1}_{U_{xz}}\rangle$ on $S^j(x)$. We define a smooth function on $\partial X \times \partial X$ by setting

$$\Omega_x^j(\xi,\eta) = \sum_{z,t\in S^j(x)} \hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\mathbf{1}_{U_{xz}}(\xi)\mathbf{1}_{U_{xt}}(\eta), \quad \xi,\eta \in \partial X. \tag{8.2}$$

We let Ω be as in Proposition 2.22, so that, by Proposition 3.11, for any ρ in $\mathfrak{M}_0^\infty(\nu)$, one has

$$\Phi_w(\rho,\rho) = \int_{\partial^2 X} \Omega_x(\xi,\eta)\mathrm{d}\rho(\xi)\mathrm{d}\rho(\eta).$$

Then, we claim that we have

$$\Omega_x^j \xrightarrow[j\to\infty]{} \Omega_x \text{ in } \mathrm{L}^1(\partial X \times \partial X, \nu\otimes\nu), \tag{8.3}$$

which, by (8.1), implies the Lemma.

Indeed, we split the sum in the right hand-side of (8.2) depending wether $|[xz]\cap[xt]| \le j+1-k$ or $|[xz]\cap[xt]| \ge j+2-k$ and we get, by Proposition 7.16, for (ξ,η) in ∂X,

$$\begin{aligned}\Omega_x^j(\xi,\eta) = \Omega_x(\xi,\eta)\mathbf{1}_{\omega_x(\xi,\eta)\le j+1-k} \\ + \sum_{\substack{z,t\in S^j(x)\\ |[xz]\cap[xt]|\ge j+2-k}} \hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\mathbf{1}_{U_{xz}}(\xi)\mathbf{1}_{U_{xt}}(\eta)\end{aligned} \tag{8.4}$$

Let us prove that the terms with $|[xz]\cap[xt]| \ge j+2-k$ in the right hand-side of (8.4) will play a negligible role. Let C be as in Lemma 8.11, so that, for any z,t in $S^j(x)$ with $i = |[xz]\cap[xt]| \ge j+2-k$, we have

$$\left|\hat{q}_x^{2j}(\mathbf{1}_z,\mathbf{1}_t)\right| \le C(2i+k-1).$$

We get

$$\begin{aligned}\int_{\omega_x(\xi,\eta)\ge j+2-k} \left|\Omega_x^j(\xi,\eta)\right| \mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta) \\ \le C\int_{\omega_x(\xi,\eta)\ge j+2-k} |2\omega_x(\xi,\eta)+k-1|\,\mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta),\end{aligned}$$

hence, by the Dominated Convergence Theorem, as ω is ν-integrable,

$$\int_{\omega_x(\xi,\eta)\ge j+2-k} \left|\Omega_x^j(\xi,\eta)\right| \mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta) \xrightarrow[j\to\infty]{} 0. \tag{8.5}$$

In the same way, by Lemma 2.20, Ω_x is $\nu\otimes\nu$-integrable and

$$\int_{\omega_x(\xi,\eta)\ge j+2-k} |\Omega_x(\xi,\eta)|\,\mathrm{d}\nu(\xi)\mathrm{d}\nu(\eta) \xrightarrow[j\to\infty]{} 0. \tag{8.6}$$

Now (8.3) follows from (8.4), (8.5) and (8.6). □

Proof of Theorem 8.6. We will get the result from Lemma 8.12 by an approximation argument. To this aim, we chose an atom-free Borel probability measure ν on ∂X such that ω is ν-integrable and $\mathfrak{M}^\infty(\nu)$ is dense in H^ω: such a measure exists by Corollary 8.3.

Fix x in X and let us note that, for any $\ell \ge 1$, the linear operator $N_x^{\ell,*}$ is bounded on H^ω. Indeed, for any y in $S^\ell(x)$ and T in $\mathcal{D}^*(\partial X)$, we have $N_x^{\ell,*}T(y) = \mathcal{P}_x T(y_-,y)$ where y_- is the neighbour of y on $[xy]$ and $\mathcal{P}_x$ is as in Section 3.1. In particular, for $\ell \ge \frac{k}{2}$, the bilinear form $(N_x^{\ell,*})^\star q_x^{2\ell}$ is bounded on H_0^ω.

Fix ρ in H_0^ω. By assumption, there exists a sequence (ρ_n) in $\mathfrak{M}^\infty(\nu)$ which converges to ρ in H_0^ω. By Lemma 8.12, for $\ell \geq \frac{k}{2}$ and any n, we have

$$q_x^{2\ell}(N_x^{\ell,*}\rho_n, N_x^{\ell,*}\rho_n) \leq \Phi_w(\rho_n, \rho_n),$$

hence, as $(N_x^{\ell,*})^\star q_x^{2\ell}$ is bounded on H_0^ω, by going to the limit, we get

$$q_x^{2\ell}(N_x^{\ell,*}\rho, N_x^{\ell,*}\rho) \leq \Phi_w(\rho, \rho).$$

As a consequence we have $H_0^\omega \subset L^{K,K^-}$ and, as q^{K,K^-} is non-negative, it is a bounded bilinear symmetric form on H_0^ω. Now, as again by Lemma 8.12, we have $q^{K,K^-}(\rho,\rho) = \Phi_w(\rho,\rho)$ for any ρ in $\mathfrak{M}^\infty(\nu)$, we also have $q^{K,K^-}(\theta,\theta) = \Phi_w(\theta,\theta)$ for any θ in H_0^ω. □

Proof of Corollary 8.7. Again, we use Corollary 8.3 to chose a fully supported atom-free Borel probability measure ν on ∂X such that ω is ν-integrable. By Lemma 8.12, we have $\mathfrak{M}_0^\infty(\nu) \subset H^{K,K^-}$. Now, as ν has full support, for any $\ell \geq 0$, the linear operator $N_x^{\ell,*}$ maps H^{K,K^-} onto $V_0^\ell(x)$. The result follows. □

Proof of Corollary 8.8. Let φ be in $\mathcal{D}(\partial X)$. We have to show that the linear functional $\theta \mapsto \langle \theta, \varphi\rangle$ is bounded on H_0^ω with respect to the positive symmetric bilinear form Φ_w. By Theorem 8.6, it suffices to show that it is bounded on H^{K,K^-}. Indeed, fix x in X. By Lemma 4.16, we have $\varphi \in N_x^\ell V^\ell(x)$ for some $\ell \geq \frac{k}{2}$. As the the dual kernel (K, K^-) is Euclidean, for any $\ell \geq \frac{k}{2}$, the form $q_x^{2\ell}$ is positive definite on $V_0^\ell(x)$, hence there exists $C \geq 0$ such that, for any T in $\mathcal{D}_0(\partial X)$, one has $\langle T, \varphi\rangle^2 \leq C q_x^{2\ell}(N_x^{\ell,*}T, N_x^{\ell,*}T)$. The conclusion follows since, for T in H^{K,K^-}, one has $q_x^{2\ell}(N_x^{\ell,*}T, N_x^{\ell,*}T) \leq q^{K,K^-}(T,T)$ by construction. □

Proof of Corollary 8.9. This is a direct consequence of Theorem 8.6 and Lemma 8.10. □

8.3. The image dual kernel

We will now aim at proving a converse statement to Theorem 8.6. To do this, given a Γ-invariant function w on X_k, with Φ_w non-negative, we need to prove that Φ_w may be built by means of a dual kernel. In this section, we will define our candidate for being this dual kernel. This dual kernel will be constructed by taking Euclidean images of Φ_w (see Definition A.2 for the definition of the Euclidean image of a non-negative bilinear form under a surjective linear map, whose properties are discussed all along Appendix A).

Let us do this precisely. We need some more notation. Recall that, for any $\ell \geq 0$, for any x in X, we have a natural linear operator $N_x^\ell : V^\ell(x) \to \mathcal{D}(\partial X)$. In the same way, for $x \sim y$, and any $\ell \geq 0$,

we define $N^\ell_{xy} : V^\ell(xy) \to \mathcal{D}(\partial X)$ as the linear operator such that for any z in $S^\ell(xy) \cap S^{\ell+1}(x)$, $N^\ell_{xy}(\mathbf{1}_z) = \mathbf{1}_{U_{xz}}$. One also let N^ℓ_{xy} denote the induced operator $\overline{V}^\ell(xy) \to \overline{\mathcal{D}}(\partial X)$. One has the compatibility relations

$$N^{\ell+1}_x I^\ell_{xy} = N^\ell_{xy} \text{ and } N^\ell_{xy} J^\ell_{xy} = N^\ell_x, \quad \ell \geq 0, \quad x \sim y \in X. \tag{8.7}$$

We denote the adjoint operators of N^ℓ_x and N^ℓ_{xy} in the usual way.

LEMMA 8.13. *For $\ell \geq 1$ and x in X, we have $N^{\ell,*}_x(H^\omega_0) = V^\ell_0(x)$. For $\ell \geq 0$ and $x \sim y$ in X, we have $N^{\ell,*}_{xy}(H^\omega_0) = V^\ell_0(xy)$.*

PROOF. As in Corollary 8.3, let ν be a fully-supported Borel probability measure on ∂X with $\mathfrak{M}^\infty_0(\nu) \subset H^\omega_0$. For $\ell \geq 1$, x in X and f in $V^\ell_0(x)$, we set φ to be the smooth function on ∂X defined by

$$\varphi = \sum_{y \in S^\ell(x)} \frac{1}{\nu(U_{xy})} f(y) \mathbf{1}_{U_{xy}}.$$

By construction, the distribution $\rho = \varphi\nu$ belongs to H^ω_0 and $N^{\ell,*}_x(\rho) = f$. The proof in the odd case is analogous. □

Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume that Φ_w is non-negative on H^ω_0. In this case, thanks to Lemma 8.13, we can associate to w a family of dual prekernels as follows. Let j be an integer, $j \geq 1$.

If j is even, $j = 2\ell$, $\ell \geq 1$, for any x in X, set $q^j_x = (N^{\ell,*}_x)_\star \Phi_w$ and define K^j_x as the associated function on $S^\ell(x)^2$ as in Lemma 5.1.

If j is odd $j = 2\ell+1$, $\ell \geq 0$, for any $x \sim y$ in X, set $q^j_{xy} = (N^{\ell,*}_{xy})_\star \Phi_w$ and define K^j_{xy} as the associated function on $S^\ell(xy)^2$ as in Lemma 5.1.

Then the relations (8.7) together with Lemma A.4 imply that, for any $j \geq 2$, (K^j, K^{j-1}) is an exact j-dual kernel.

DEFINITION 8.14. Let k and w be as above. For any $j \geq 1$ the j-dual kernel K^j is called the image j-dual prekernel of w and, if $j \geq 2$, the j-dual kernel (K^j, K^{j-1}) is called the image j-dual kernel of w.

We can relate these kernels to the formalism developed in Chapter 5.

LEMMA 8.15. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume that Φ_w is non-negative on H^ω_0. Let K^j, $j \geq 1$ be as above. Then, for any $j \geq k+1$, the j-dual kernel (K^j, K^{j-1}) is the orthogonal extension of the $(j-1)$-dual kernel (K^{j-1}, K^{j-2}).*

The proof of Lemma 8.15 will rely on the following abstract characterization of orthogonal extensions:

LEMMA 8.16. *Let X be a finite-dimensional real vector space, $d \geq 2$ be an integer and $X_1, \ldots, X_d$ be subspaces of X. We assume that there exists a subspace X_0 of X such that, for any $1 \leq i \neq j \leq d$, $X_i \cap X_j = X_0$ and $X/X_0 = \bigoplus_i X_i/X_0$. Let $p_0, p_1, \ldots, p_d$ be positive definite symmetric bilinear forms on $X_0, X_1, \ldots, X_d$ such that, for any $1 \leq i \leq d$, $p_{i|X_0} = p_0$. For $1 \leq i \leq d$, let $W_i \subset X^*$ be the orthogonal subspace of $\sum_{j \neq i} X_j$, that is, the kernel of the natural surjective map $X_i^* \to (\sum_{j \neq i} X_j)^*$. Let q be a positive definite symmetric bilinear form on X such that $p_{|X_i} = p_i$, $1 \leq i \leq d$. Then q is the orthogonal extension of $p_1, \ldots, p_d$ if and only if, for any $1 \leq i \neq j \leq d$, the spaces W_i and W_j are orthogonal with respect to the dual form of q.*

PROOF. Let p be the orthogonal extension of $p_1, \ldots, p_d$ and, for $1 \leq i \leq d$, let Y_i be the orthogonal complement of X_0 in X_i with respect to p_i. We set $Y_0 = X_0$. By definition, we have $X = \bigoplus_{0 \leq i \leq d} Y_i$ and this decomposition is orthogonal with respect to p. Let W be the dual space of X and $T : X \to W$ be the linear isomorphism associated to p (that is, for any x, y in X, we have $p(x, y) = \langle Tx, y \rangle$). The dual form p^* of p is given by $p^* = (T^{-1})^\star p$. Let W_0 be the orthogonal complement of $\bigoplus_{1 \leq i \leq d} W_i$ with respect to p^*. We have $W = \bigoplus_{0 \leq i \leq d} W_i$ and, by construction, this decomposition is the image of the one of X by T, that is, $TY_i = W_i$, $0 \leq i \leq d$. In particular, the W_i, $0 \leq i \leq d$, are p^*-orthogonal to each other.

Let $A : X \to X$ be the endomorphism such that, for any x, y in X, $q(x, y) = p(Ax, y)$. To conclude, we need to prove that A is the identity map. One easily shows that the dual form q^* of q satisfies $q^*(v, w) = p^*(Bv, w)$, v, w in W, where $B = TA^{-1}T^{-1}$. Fix $1 \leq i \leq d$. Saying that W_i is q^*-orthogonal to all the W_j, $j \neq i$, amounts to saying that we have $BW_i \subset W_i \oplus W_0$. Pulling back this property by T, we get $Y_i \subset AX_i$. Now, we will use the other assumption on q, namely that its restriction to X_i is p_i. Indeed, fix y_i in Y_i. We have just seen that we can write $y_i = Ax_i$ for some x_i in X_i. For any z_i in X_i, we have $p(y_i, z_i) = p(Ax_i, z_i) = q(x_i, z_i) = p(x_i, z_i)$, hence as p_i is non-degenerate, $x_i = y_i$. Thus, we get $Ay_i = y_i$ for any y_i in Y_i. As A is p-symmetric and $X_0 = Y_0$ is the p-orthogonal complement of $\bigoplus_{1 \leq i \leq d} Y_i$, we get $AX_0 \subset X_0$. Since the restriction of q to X_0 is equal to the restriction of p, A is the identity map and $q = p$ as required. □

Now, we will split the proof of Lemma 8.15 into several cases. First we will assume that Φ_w is coercive, that is, it is positive definite and defines the topology of H_0^ω. Note that, as $\overline{\mathcal{D}}(\partial X)$ may be seen as a (dense) subspace of the topological dual space of H_0^ω, the restriction

of the dual bilinear form of Φ_w to $\overline{\mathcal{D}}(\partial X)$ defines a positive definite symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. We denote it by p.

We will now use again the language of Chapter 4 and study the quadratic fields obtained by pulling back p to our usual finite-dimensional spaces of functions. Let $j \geq 1$.

If j is even, $j = 2\ell$, $\ell \geq 1$, for any x in X, set $p^j_x = (N^\ell_x)^\star p$.

If j is odd $j = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, set $p^j_{xy} = (N^\ell_{xy})^\star p$.

Then the relations (8.7) imply that p^j is a j-quadratic field. As p is positive definite, this field is Euclidean. The field p^j and the dual prekernel K^j are in duality as in Section 5.1.

Proof of Lemma 8.15 when Φ_w is coercive and j odd. We set $j = 2\ell+1$, $\ell \geq 1$. Let us fix $x \sim y$ in X and consider the positive symmetric bilinear form p^j_{xy} on $\overline{V}^\ell(xy)$. Its dual form on $V^\ell_0(xy)$ is q^j_{xy}. In order to apply Lemma 8.16, we need to show that the spaces $W_x = \ker J^{\ell,*}_{yx}$ and $W_y = \ker J^{\ell,*}_{xy}$ are orthogonal with respect to q^j_{xy}. Note that we have $W_x \oplus W_y = \ker M^{\ell-1,*}_{xy}$. Also note that, on $\mathcal{D}^*_0(\partial X)$, we have $M^{\ell-1,*}_{xy} N^{\ell,*}_{xy} = N^{\ell-1,*}_{xy}$ and that the space $\ker N^{\ell-1,*}_{xy}$ may be written as the direct sum of the spaces D_x and D_y of distributions defined by

$$D_x = \{T \in \mathcal{D}_0(\partial X) | N^{\ell-1,*}_{xy} T = 0 \text{ and } \mathbf{1}_{U_{xy}} T = 0\}$$
$$D_y = \{T \in \mathcal{D}_0(\partial X) | N^{\ell-1,*}_{xy} T = 0 \text{ and } \mathbf{1}_{U_{yx}} T = 0\}.$$

Clearly, the map $N^{\ell,*}_{xy}$ sends D_x onto W_x and D_y onto W_y.

Consider the closed subspace $L = H^\omega_0 \cap \ker N^{\ell-1,*}_{xy}$ in H^ω_0. We set $L_x = L \cap D_x$ and $L_y = L \cap D_y$ and we have again a decomposition $L = L_x \oplus L_y$, which is now a decomposition of L as a direct sum of two orthogonal subspaces in H^ω_0. Here comes the crucial phenomenon in the proof: we claim that this decomposition is still orthogonal for the bilinear form Φ_w. Indeed, let ρ_x be in L_x and ρ_y be in L_y. We must prove that $\Phi_w(\rho_x, \rho_y) = 0$. By the definition of Φ_w in Section 3.2, we have

$$\Phi_w(\rho_x, \rho_y) = \frac{1}{2} \sum_{(a,b) \in X_k} w(a,b) \mathcal{P}\rho_x(a, a_1) \mathcal{P}\rho_y(b_1, b), \tag{8.8}$$

where, as usual, for $a \neq b$ in X, a_1 is the neighbour of a on $[ab]$ and b_1 the one of b.

We claim that for any (a, b) in X_k, we have $\mathcal{P}\rho_x(a, a_1)\mathcal{P}\rho_y(b_1, b) = 0$. Indeed, by construction, for any $s \sim t$ in X, if $\mathcal{P}\rho_x(s,t) \neq 0$, then $d(s, x)$ and $d(t, x)$ are $\geq \ell - 1$ and y is not in $[xs]$ nor in $[xt]$. In the same way, if $\mathcal{P}\rho_y(s,t) \neq 0$, then $d(s, y)$ and $d(t, y)$ are $\geq \ell - 1$ and x

is not in $[ys]$ nor in $[yt]$. Therefore, for $a \neq b$ in X and a_1, b_1 as above, if $\mathcal{P}\rho_x(a,a_1)\mathcal{P}\rho_y(b_1,b) \neq 0$, we must have $d(a,x) \geq \ell$ and $d(b,y) \geq \ell$ and $[xy] \subset [ab]$, hence $d(a,b) \geq 2\ell+1 = j > k$. By (8.8), we get $\Phi_w(\rho_x, \rho_y) = 0$.

By Lemma A.7, this implies that the spaces W_x and W_y are q^j_{xy}-orthogonal. By Lemma 8.16, p^j_{xy} is the orthogonal extension of p^{j-1}_x and p^{j-1}_y. □

The proof in the even case follows the same lines.

Proof of Lemma 8.15 when Φ_w is coercive and j even. In this case, we set $j = 2\ell$, $\ell \geq 2$. We fix x in X and we consider the positive symmetric bilinear form p^j_x on $\overline{V}^\ell(x)$, with its dual form q^j_x on $V^\ell_0(x)$. We set, for any $y \sim x$, $W_y = \bigcap_{\substack{z\sim x\\ z\neq y}} \ker I^{\ell-1,*}_{xz}$. We get $\ker M^{\ell-1,*}_x = \bigoplus_{y\sim x} W_y$ and we need to show that this decomposition is q^j_x-orthogonal. Again, there is a related decomposition of $\mathcal{D}^*_0(\partial X)$: we may write $\ker N^{\ell-1,*}_{xy}$ as the direct sum of the spaces D_y, $y \sim x$ which are defined as

$$D_y = \{T \in \mathcal{D}_0(\partial X) | N^{\ell-1,*}_x T = 0 \text{ and } \forall z \sim x, z \neq y \quad \mathbf{1}_{U_{xz}} T = 0\}.$$

The map $N^{\ell,*}_x$ sends D_y onto W_y.

In H^ω_0, we define the closed subspaces $L = H^\omega_0 \cap \ker N^{\ell-1,*}_x$ and, for $y \sim x$, $L_y = L \cap D_y$, so that we have the orthogonal decomposition $L = \bigoplus_{y\sim x} L_y$. Again, this decomposition is still orthogonal for the bilinear form Φ_w. Indeed, if y and z are two different neighbours of x and ρ_y and ρ_z are in L_y and L_z, we have, as in (8.8),

$$\Phi_w(\rho_y, \rho_z) = \frac{1}{2} \sum_{(a,b)\in X_k} w(a,b)\mathcal{P}\rho_y(a,a_1)\mathcal{P}\rho_z(b_1,b). \tag{8.9}$$

Again, we claim that $\mathcal{P}\rho_y(a,a_1)\mathcal{P}\rho_z(b_1,b) = 0$ for any (a,b) in X_k. Indeed, in this case, for any $s \sim t$ in X, if $\mathcal{P}\rho_y(s,t) \neq 0$, then $d(s,x)$ and $d(t,x)$ are $\geq \ell-1$ and y is in $[xs]$ and in $[xt]$; if $\mathcal{P}\rho_z(s,t) \neq 0$, then $d(s,x)$ and $d(t,x)$ are $\geq \ell-1$ and z is in $[xs]$ and in $[xt]$. Therefore, for $a \neq b$ in X, if $\mathcal{P}\rho_y(a,a_1)\mathcal{P}\rho_z(b_1,b) \neq 0$, we must have $d(a,x) \geq \ell$ and $d(b,x) \geq \ell$ and $x \in [ab]$, hence $d(a,b) \geq 2\ell = j > k$. By (8.9), we get $\Phi_w(\rho_y, \rho_z) = 0$. The conclusion follows as in the odd case. □

Proof of Lemma 8.15 in the general case. We will use the approximation result from Proposition A.8 in the appendix to deduce this from the coercive case. Let us be more precise.

By definition, the scalar product of H^ω_0 is the bilinear form associated to the constant function with value 2 on X_1. By Lemma 3.15, it is also the bilinear form associated to the constant function with value

2 on X_k. Therefore, as Φ_w is non-negative, for any $\varepsilon > 0$, the bilinear form Φ_{w_ε} associated to $w_\varepsilon = w + \varepsilon$ is coercive. Let, for any $\varepsilon > 0$ and $j \geq 1$, K_ε^j be the image j-dual prekernel of w_ε. Then, by Proposition A.8, for any $j \geq 1$, we have

$$K_\varepsilon^j \xrightarrow[\varepsilon \to 0]{} K^j$$

(where the convergence takes place in the finite-dimensional vector space $\mathcal{K}_j$ of Γ-invariant j-dual prekernels). Fix $j \geq k+1$. For $\varepsilon > 0$, as w_ε is Euclidean, $(K_\varepsilon^j, K_\varepsilon^{j-1})$ is the orthogonal extension of the $(j-1)$-dual kernel $(K_\varepsilon^{j-1}, K_\varepsilon^{j-2})$. As the relations defining the orthogonal extension are linear, they are continuous, hence (K^j, K^{j-1}) is the orthogonal extension of the $(j-1)$-dual kernel (K^{j-1}, K^{j-2}). □

8.4. From additive kernels to dual kernels

We are now ready to prove that w may be recovered from its image k-dual kernel.

THEOREM 8.17. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume that Φ_w is non-negative on H_0^ω and let (K, K^-) be its image k-dual kernel. Then, there exists a Γ-invariant (K, K^-)-compatible function u on X_{k-1} with w as its weight function.*

REMARK 8.18. Given $k \geq 2$ and a Γ-invariant function w on X_k, there is no direct way of deciding whether the bilinear form Φ_w is non-negative on the infinite dimensional vector space H_0^ω. Theorem 8.6 and Theorem 8.17 say that Φ_w is non-negative if and only if we may write w as the weight function of some Γ-invariant non-negative k-dual kernel. Checking whether a k-dual kernel is non-negative is not of the same order of difficulty, since it only requires to check whether bilinear forms on finite-dimensional vector spaces are non-negative.

From Theorem 8.17, we draw results on the structure of the symmetric bilinear forms Φ_w. See Definitions 5.12 and 5.13 for the notion of an exact kernel.

COROLLARY 8.19. *Let $k \geq 2$, w be a symmetric Γ-invariant function on X_k and (K, K^-) be a Γ-invariant exact k-dual kernel. Assume that Φ_w is non-negative. Then (K, K^-) is the image k-dual kernel of Φ_w if and only if w is a weight function of (K, K^-) and H_0^ω has dense range in H^{K,K^-}.*

If H is a vector space and Φ is a non-negative symmetric bilinear form on H, the space of all bounded linear functionals of H with respect to Φ may be seen as the topological dual space of $H/\ker\Phi$ and comes with a natural Hilbert space structure.

COROLLARY 8.20. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k such that Φ_w is non-negative on H_0^ω. Let $U_w \subset \overline{\mathcal{D}}(\partial X)$ be the space of φ in $\overline{\mathcal{D}}(\partial X)$ such that the linear functional $T \mapsto \langle T, \varphi \rangle$ is bounded with respect to Φ_w on H_0^ω. Then U_w is dense in the space of all linear functionals on H_0^ω which are bounded with respect to Φ_w.*

We also have a statement in case Φ_w is not necessarily non-negative.

COROLLARY 8.21. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Then there exists a Γ-invariant k-dual kernel (K, K^-) such that, for any θ in H_0^ω, one has*

$$q_x^{2j}(N_x^{j,*}\theta, N_x^{j,*}\theta) \xrightarrow[j\to\infty]{} \Phi_w(\theta, \theta).$$

In particular, w is a weight function of (K, K^-).

Let us now prove this results. In the coercive case, Theorem 8.17 will follow from the easy

LEMMA 8.22. *Let H be a Hilbert space with scalar product p and (K_ℓ) be decreasing sequence of closed subspaces of H with $\bigcap_\ell K_\ell = \{0\}$. For any ℓ, let π_ℓ be the quotient map $H \to H/K_\ell$. Then, for any x, y in H, we have*

$$(\pi_\ell)_\star p(\pi_\ell x, \pi_\ell y) \xrightarrow[\ell\to\infty]{} p(x, y).$$

PROOF. For any ℓ, let H_ℓ be the orthogonal complement of K_ℓ and θ_ℓ be the orthogonal projection onto H_ℓ. Then $(\pi_\ell)_\star p(\pi_\ell x, \pi_\ell y) = p(\theta_\ell x, \theta_\ell y)$. As $\bigcup_\ell H_\ell$ is dense in H, $\theta_\ell x$ and $\theta_\ell y$ converge to x and y and the result follows. □

PROOF OF THEOREM 8.17 WHEN Φ_w IS COERCIVE. In that case, it follows from Lemma 8.15 and Lemma 8.22 that, for any θ in H_0^ω, we have $\Phi_w(\theta, \theta) = q^{K,K^-}(\theta, \theta)$.

Now, we use the theory in Chapter 7: we chose a function u' which is (K, K^-)-compatible as in Definition 7.2 which is Γ-invariant (this is possible as noticed in Remark 7.3) and we let w' denote the associated weight function. By Theorem 8.6, the form q^{K,K^-} is equal $\Phi_{w'}$ on H_0^ω. Thus, we get $\Phi_w = \Phi_{w'}$ on H_0^ω. By Corollary 8.4, this tells us that the normalized smooth functions associated to w and w' are cohomologuous. Equivalently, by Lemma 3.13, there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that, for any (x, y) in X_k, one has $w(x, y) = w'(x, y) + v(x, y_1) - v(x_1, y)$, where as usual x_1 and y_1 are the neighbours of x and y on $[xy]$. We set $u(x, y) = u'(x, y) + v(x, y)$ for any (x, y) in X_{k-1}. As v is skew-symmetric, by Definition 7.2, u is still a (K, K^-)-compatible function and by Definition 7.4, its weight function is w. □

The proof in the general case will rely on the same approximation argument as the proof of Lemma 8.15. We will also need

LEMMA 8.23. *Let $k \geq 2$ and (K, K^-) be a Γ-invariant k-dual kernel. Then the map $u \mapsto w$ is an affine isomorphism between the space of Γ-invariant (K, K^-)-compatible functions and the space of Γ-invariant weight functions.*

PROOF. We need to prove that this map is injective. Let u and u' be Γ-invariant (K, K^-)-compatible functions on X_{k-1} with the same associated weight function. Then, as both u and u' are (K, K^-)-compatible, the function $u'' = u' - u$ is skew-symmetric. As u and u' have the same weight function, for any (x, y) in X_k, we have $u''(x, y_1) + u''(y, x_1) = 0$ (where x_1 and y_1 are the neighbours of x and y on $[xy]$), hence $u''(x, y_1) = u''(x_1, y)$. In particular, the smooth function $s = (x_h)_{h\in\mathbb{Z}} \mapsto u''(x_0, x_{k-1})$ is invariant under the shift map of Chapter 2. By Proposition 2.3, it is constant. Hence u'' is constant and, as it is skew-symmetric, it is zero, which should be proved. □

PROOF OF THEOREM 8.17 IN THE GENERAL CASE. As for proving Lemma 8.15, we set $w_\varepsilon = w + \varepsilon$ for $\varepsilon > 0$, so that the bilinear form Φ_{w_ε} is coercive and, by Proposition A.8, we have $(K_\varepsilon, K_\varepsilon^-) \xrightarrow[\varepsilon\to 0]{} (K, K^-)$, where $(K_\varepsilon, K_\varepsilon^-)$ is the image k-dual kernel of w_ε. By the coercive case, for $\varepsilon > 0$, there exists a Γ-invariant $(K_\varepsilon, K_\varepsilon^-)$-compatible function u_ε on X_{k-1} such that w_ε is the associated weight function.

We claim that u_ε has a limit u as $\varepsilon \to 0$ which is a (K, K^-)-compatible function and the associated weight function is w, which finishes the proof. Indeed, from Lemma 8.23, we know that the map $(K, K^-, u) \mapsto (K, K^-, w)$ is a linear isomorphism from the space of Γ-invariant triples (K, K^-, u) where (K, K^-) is a k-dual kernel and u is a (K, K^-)-compatible function onto the space of Γ-invariant triples (K, K^-, w), where w is a weight function. As all the involved spaces are finite-dimensional this linear isomorphism is a homeomorphism and the claim follows. □

PROOF OF COROLLARY 8.19. This is a direct consequence of Theorem 8.6, Theorem 8.17 and Lemma B.7. □

PROOF OF COROLLARY 8.20. Let (K, K^-) be the image k-dual kernel of Φ_w. Then, by Theorem 8.6 and Theorem 8.17, the restriction of q^{K,K^-} to H_0^ω is Φ_w and, by Corollary 8.19, H_0^ω has dense range in H^{K,K^-}. Thus, we must show that U_w has dense range in the topological dual space of H^{K,K^-}. As, by Proposition 5.18, H^{K,K^-} is complete with respect to q^{K,K^-}, this amounts to proving that the orthogonal space

of U_w in H^{K,K^-} is 0. In other words, if $U_w^\perp$ is the space of those T in L^{K,K^-} such that $\langle T, \varphi\rangle = 0$ for any φ in U_w, we must show that we have $U_w^\perp \subset \ker q^{K,K^-}$.

We fix x in X and, for $\ell \geq 1$, we let, as in Section 5.4, N_x^ℓ denote the natural linear operator $V^\ell(x) \to \mathcal{D}(\partial X)$. We also set U_ℓ to be the orthogonal space of $\ker((N_x^{\ell,*})_\star \Phi_w)$ in $\overline{V}^\ell(x)$ (where as usual, we have identfied $\overline{V}^\ell(x)$ with the dual space of $V_0^\ell(x)$). Now, one easily checks that one has $U_w = \bigcup_{\ell\geq 1} N_x^\ell U_\ell$ so that a distribution T belongs to U_w if and only if, for any $\ell \geq 1$, $N_x^{\ell,*}T$ belongs to $\ker((N_x^{\ell,*})_\star \Phi_w)$ and we are done since (K, K^-) is the image k-dual kernel of Φ_w. □

Proof of Corollary 8.21. The set of Γ-invariant symmetric functions w' on X_k with $\Phi_{w'}$ coercive is non-empty, as it contains the constant positive functions. Since it is an open convex cone in the finite-dimensional vector space of symmetric Γ-invariant functions on X_k, any such function w may be written as a difference $w' - w''$, where $\Phi_{w'}$ and $\Phi_{w''}$ are coercive. By Theorem 8.17, we can find non-negative Γ-invariant k-dual kernels $(K', (K')^-)$ and $(K'', (K'')^-)$ which admit w' and w'' as weight functions. Therefore, w is a weight function of the dual kernel $(K, K^-) = (K' - K'', (K')^- - (K'')^-)$. The convergence follows from Theorem 8.6 (or from Corollary 8.9). Note that, as soon as this convergence takes place, w must be a weight function of (K, K^-) by Corollary 8.4. □

CHAPTER 9

The weight map

Our aim now will be to give a characterization of those dual kernels which are the image kernels of a Γ-invariant function w with non-negative associated bilinear form on H_0^ω, or equivalently of those exact dual kernels (K, K^-) such that H_0^ω has dense range in H^{K,K^-}. This will require us to study more carefully the map that sends a dual kernel to its weight functions.

More precisely, for $k \geq 2$, let as above $\mathcal{K}_k$ denote the real vector space of Γ-invariant k-dual kernels (which is finite-dimensional since $\Gamma\backslash X$ is finite). We also let $\mathcal{W}_k$ denote the quotient space of the space of symmetric Γ-invariant real valued functions on X_k by the space of functions of the form $(x, y) \mapsto u(x, x_1) + u(y, y_1)$, where u is a skew-symmetric Γ-invariant function on X_{k-1} and x_1 and y_1 are the neighbours of x and y on $[xy]$. By Lemma 3.13, the space $\mathcal{W}_k$ may be seen as a space of cohomology classes of smooth functions on $\Gamma\backslash\mathscr{S}$. By Definitions 7.2 and 7.4, if (K, K^-) is a Γ-invariant k-dual kernel, the set of its Γ-invariant weight functions is an equivalence class in $\mathcal{W}_k$. Thus, we have a well-defined linear map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ which we call the weight map. We will now prove that it is surjective and describe its null space.

The main objective of this chapter is to show

THEOREM 9.1. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. Then the following are equivalent.*
(i) *There exists a skew-symmetric function v on X_{k-1} such that the function $(x, y) \mapsto v(x, y_1) + v(y, x_1)$ on X_k is a weight function of (K, K^-) (where as usual, for (x, y) in X_k, x_1 and y_1 are the neighbours of x and y on $[xy]$).*
(ii) *There exists a $(k-1)$-pseudokernel L such that (K, K^-) is the k-dual kernel associated to L.*
In case (K, K^-) is Γ-invariant, the function v in (i) *may be chosen to be Γ-invariant.*

See Definition 6.16 and Definition 6.17 for the k-dual kernel associated with a $(k-1)$-pseudokernel.

In other words, for Γ-invariant kernels, we have

COROLLARY 9.2. *For any $k \geq 2$, the null space of the weight map $\mathcal{K}_k \to \mathcal{W}_k$ is the space $\mathcal{L}_{k-1}$ of $(k-1)$-pseudokernels.*

9.1. Surjectivity of the weight map

For Γ-invariant k-dual kernels, surjectivity of the weight map follows from Corollary 8.21. In this Section, we give a direct proof of this phenomenon by exhibiting an explicit section of the weight map. This construction will be used again in Chapter 16.

We need new notation. Let $k \geq 1$ and w be a symmetric function on X_k. For any x, y in X with $j = d(x,y) \geq k$, we set

$$\sum_{[xy]} w = \sum_{h=0}^{j-k} w(z_h, z_{h+k}),$$

where $x = z_0, z_1, \ldots, z_k = y$ is the geodesic path from x to y. If $d(x,y) < k$, we set $\sum_{[xy]} w = 0$.

We easily get

LEMMA 9.3. *Let $k \geq 1$, w be a symmetric function on X_k, x, y be in X with $d(x,y) \geq k+1$ and x_1, y_1 be the neighbours of x and y on $[xy]$. We have*

$$\sum_{[xy]} w + \sum_{[x_1 y_1]} w = \sum_{[xy_1]} w + \sum_{[x_1 y]} w.$$

Now, let $k \geq 2$ and let still w be a symmetric function on X_k. For $j \geq k-1$, we define a j-dual prekernel $K^{w,j}$ by setting, if j is even, $j = 2\ell$, $\ell \geq 1$, for any x in X and z, t in $S^\ell(x)$,

$$K_x^{w,j}(z,t) = \sum_{[zt]} w$$

and in the same way, if j is odd, $j = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X and z, t in $S^\ell(x)$,

$$K_{xy}^{w,j}(z,t) = \sum_{[zt]} w$$

For $j = k$, we simply write K^w for $K^{w,k}$. Note that $K^{w,k-1} = 0$.

An elementary computation gives

PROPOSITION 9.4. *Let $k \geq 2$ and w be a symmetric function on X_k. Then, for any $j \geq k$, the $(j+1)$-dual kernel $(K^{w,j+1}, K^{w,j})$ is the orthogonal extension of $(K^{w,j}, K^{w,j-1})$.*

See Definition 5.7 for the meaning of the orthogonal extension of a dual kernel.

PROOF. Let temporarily $(H, K^{w,j})$ denote the orthogonal extension of $(K^{w,j}, K^{w,j-1})$. We have to prove that $H = K^{w,j+1}$.

Assume j is even, $j = 2\ell$, $\ell \geq 1$. We fix $x \sim y$ in X and $z \neq t$ in $S^\ell(xy)$. Let z_1 and t_1 be the neighbours of z and t on $[zt]$. If $[xy] \subset [zt]$ and, for example, $d(x,z) = \ell = d(y,t)$, we have

$$\begin{aligned} H_{xy}(z,t) &= K^{w,j}_x(z,t_1) + K^{w,j}_y(z_1,t) - K^{w,j-1}_{xy}(z_1t_1) \\ &= \sum_{[zt_1]} w + \sum_{[z_1t]} w - \sum_{[z_1,t_1]} w \\ &= \sum_{[zt]} w = K^{w,j+1}_{xy}(z,t), \end{aligned}$$

where we have applied Lemma 9.3 to the segment $[zt]$, which was possible since $d(z,t) = j+1 \geq k+1$. Now, if $[xy] \not\subset [zt]$ and for example $d(z,x) = d(t,x) = \ell$, we have

$$K^{w,j}_y(z_1,t_1) = \sum_{[z_1t_1]} w = K^{w,j-1}_{xy}(z_1,t_1),$$

hence

$$H_{xy}(z,t) = K^{w,j}_x(z,t_1) = \sum_{[zt]} w = K^{w,j+1}_{xy}(z,t)$$

and we are done.

Assume j is odd, $j = 2\ell+1$, $\ell \geq 1$. We fix x in X and $y \neq z$ in $S^{\ell+1}(x)$ and we let as above y_1, z_1 be the neighbours of y and z on $[yz]$. If x belongs to $[yz]$, we let a be the neighbour of x on $[xy]$ and b be its neighbour of $[xz]$. We have

$$\begin{aligned} H_x(y,z) = K^{w,j}_{xa}(y,z_1) + K^{w,j}_{xb}(y,z_1) \\ + \sum_{\substack{c\sim x \\ c\notin\{a,b\}}} K^{w,j}_{xc}(y_1,z_1) - (d(x)-1)K^{w,j-1}_x(y_1,z_1), \end{aligned}$$

hence, as $d(y,z) = j+1 \geq k+1$, by Lemma 9.3,

$$H_x(y,z) = \sum_{[yz_1]} w + \sum_{[y_1z]} w - \sum_{[y_1,z_1]} w = \sum_{[y,z]} w = K^{w,j+1}_x(y,z).$$

Now, if $x \notin [yz]$ and a is the neighbour of x with $d(y,a) = d(z,a) = \ell$, we have, for any $b \sim x$, $b \neq a$,

$$K^{w,j}_{xb}(y_1,z_1) = \sum_{[y_1z_1]} w = K^{w,j-1}_x(y_1,z_1),$$

hence

$$H_x(y,z) = K_{xa}^{w,j}(y,z) = \sum_{[yz]} w = K_x^{w,j+1}(y,z),$$

which should be proved. □

COROLLARY 9.5. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Then a function u on X_{k-1} is $(K^w, 0)$-compatible if and only if it is skew-symmetric. In particular, if $u = 0$, the associated weight function is w.*

PROOF. By Definition 7.2 and Proposition 9.4, u is a $(K^w, 0)$-compatible function if and only if, for any x, y in X with $d(x,y) = k-1$ and any parametrized geodesic line $(z_h)_{h\in\mathbb{Z}}$ with $z_0 = x$ and $z_{k-1} = y$,

$$u(x,y) + u(y,x) = \sum_{h=1}^{k-1} \sum_{i=h+1-k}^{h-2} w(z_i, z_{i+k}) - \sum_{h=1}^{k-2} \sum_{i=h+1-k}^{h-1} w(z_i, z_{i+k}).$$

In the right hand-side of the latter, the pairs (h,i) with $1 \leq h \leq k-2$ and $h+1-k \leq i \leq h-2$ appear twice. Thus, we get

$$u(x,y) + u(y,x) = \sum_{i=0}^{k-3} w(z_i, z_{i+k}) - \sum_{h=1}^{k-2} w(z_{h-1}, z_{h+k-1}) = 0,$$

that is, u is skew-symmetric.

Now, we let u be 0, so that, by Definition 7.4, the associated weight function w' must verify that, for any x, y in X with $d(x,y) = k$, if $(z_h)_{h\in\mathbb{Z}}$ is a parametrized geodesic line with $z_0 = x$ and $z_k = y$,

$$w'(x,y) = \sum_{h=1}^{k-1} \sum_{i=h+1-k}^{h-1} w(z_i, z_{i+k}) - \sum_{h=1}^{k} \sum_{i=h+1-k}^{h-2} w(z_i, z_{i+k}).$$

Again, in the right hand-side of the latter, the pairs (h,i) with $1 \leq h \leq k-1$ and $h+1-k \leq i \leq h-2$ appear twice and we get

$$w'(x,y) = \sum_{h=1}^{k-1} w(z_{h-1}, z_{h+k-1}) - \sum_{i=1}^{k-2} w(z_i, z_{i+k}) = w(x,y),$$

which should be proved. □

For Γ-invariant k-dual kernels, we retrieve Corollary 8.21.

COROLLARY 9.6. *For any $k \geq 2$, the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is surjective.*

9.2. Weight functions of pseudokernels

Now that we have proved that the weight map is surjective, we will study its null space: we will actually prove that, for $k \geq 2$, the null space of W_k is the image of the space $\mathcal{L}_{k-1}$ of Γ-invariant $(k-1)$-pseudokernels in $\mathcal{K}_k$.

In the present section, we start by showing that the weight map is zero on the image of $\mathcal{L}_{k-1}$ in $\mathcal{K}_k$. This will be achieved by giving an explicit formula for the compatible functions and weight functions of pseudokernels. This requires the definition of a new function associated to a pseudokernel whose existence is warranted by

LEMMA 9.7. *Let $k \geq 1$ and L be a k-pseudokernel. Let (x, y) be in X_k and $(z_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line with $z_0 = x$ and $z_k = y$. Then, the quantity*

$$L^{2k-1}_{z_0 z_1}(z_{1-k}, z_k) = L^{2k-1}_{x z_1}(z_{1-k}, y)$$

only depends on x and y.

PROOF. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, by Corollary 6.23, we have

$$L^{2k-1}_{z_0 z_1}(z_{1-k}, z_k) = \sum_{\substack{t\in S^{\ell+1}(z_0)\\ z_0\notin[z_1 t]}} L_{t_- t}(z_{1-k}, z_k).$$

For t as above, the segment $[z_{1-k}t]$ meets the sphere $S^\ell(tt_-)$ at z_0, hence $L_{t_- t}(z_{1-k}, z_k)$ does not depend on the choice of the points $(z_h)_{h<0}$.

In the same way, if k is even, $k = 2\ell$, $\ell \geq 1$, Corollary 6.23, gives

$$L^{2k-1}_{z_0 z_1}(z_{1-k}, z_k) = \sum_{\substack{t\in S^{\ell}(z_0)\\ z_0\notin[z_1 t]}} L_{tt_-}(z_{1-k}, z_k).$$

Now, for such a z, the segment $[z_{1-k}t]$ meets the sphere $S^\ell(t)$ at z_0, hence $L_{t_- t}(z_{1-k}, z_k)$ again does not depend on the choice of the points $(z_h)_{h<0}$. □

We can define a natural function associated to a pseudokernel.

DEFINITION 9.8. Let $k \geq 1$ and L be a k-pseudokernel. We define its pseudoweight v as the unique function on X_k such that, for any (x, y) in X_k, one has

$$v(x, y) = L^{2k-1}_{z_0 z_1}(z_{1-k}, z_k) = L^{2k-1}_{x z_1}(z_{1-k}, y),$$

where $(z_h)_{h\in\mathbb{Z}}$ is any parametrized geodesic line with $z_0 = x$ and $z_k = y$.

Note that if L is Γ-invariant, so is v. We get a formula for weight functions of pseudokernels.

PROPOSITION 9.9. *Let $k \geq 1$, L be a k-pseudokernel, v be the pseudoweight of L and (K, K^-) be the $(k+1)$-dual kernel associated to L. Then a function u on X_k is (K, K^-)-compatible if and only if, for any (x, y) in X_k, one has*

$$u(x, y) + u(y, x) = v(x, y) + v(y, x).$$

If u is such a function, the associated weight function w on X_{k+1} verifies, for any (x, y) in X_{k+1},

$$w(x, y) = u(x, y_1) + u(y, x_1) - v(x, y_1) - v(y, x_1)$$

(where as usual x_1 and y_1 are the neighbours of x and y on $[xy]$).

This directly implies

COROLLARY 9.10. *Assume L is Γ-invariant. Then $W_{k+1}(K, K^-) = 0$.*

PROOF OF PROPOSITION 9.9. The proof relies on straightforward but tedious computations.

First, we establish the formula for u. Let (x, y) be in X_k and $(z_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line with $z_0 = x$ and $z_k = y$. Denote by ξ and η the endpoints of $(z_h)_{h\in\mathbb{Z}}$. By Definition 7.2, we have

$$u(x, y) + u(y, x) = \sum_{h=1}^{k} K^{2k-1}_{z_{h-1}z_h}(\xi, \eta) - \sum_{h=1}^{k-1} K^{2k}_{z_h}(\xi, \eta).$$

By Proposition 6.21, we have,

$$K^{2k}_a = \sum_{b\sim a} L^{2k-1}_{ab}, \quad a \in X,$$
$$\text{and } K^{2k-1}_{ab} = L^{2k-1}_{ab} + L^{2k-1}_{ba}, \quad a \sim b \in X.$$

Therefore, we get

$$\begin{aligned} u(x, y) + u(y, x) = &\sum_{h=1}^{k} (L^{2k-1}_{z_{h-1}z_h}(\xi, \eta) + L^{2k-1}_{z_h z_{h-1}}(\xi, \eta)) \\ &- \sum_{h=1}^{k-1} (L^{2k-1}_{z_h z_{h-1}}(\xi, \eta) + L^{2k-1}_{z_h z_{h+1}}(\xi, \eta)) - \sum_{h=1}^{k-1} \sum_{\substack{w\sim z_h \\ w\neq z_{h-1}, z_{h+1}}} L^{2k-1}_{z_h w}(\xi, \eta). \end{aligned}$$

By Corollary 6.24, in the right hand-side of the latter, the third sum is zero, so that this equation gives

$$u(x, y) + u(y, x) = L^{2k-1}_{z_0 z_1}(\xi, \eta) + L^{2k-1}_{z_k z_{k-1}}(\xi, \eta) = v(x, y) + v(y, x),$$

and we are done.

Now, we prove the formula for w. Let (x, y) be in X_{k+1} and $(z_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line with $z_0 = x$ and $z_{k+1} = y$. Still denote by ξ and η its endpoints. By Definition 7.4, we have

$$w(x,y) = u(z_0,z_k) + u(z_{k+1},z_1) + \sum_{h=1}^{k} K^{2k}_{z_h}(\xi,\eta) - \sum_{h=1}^{k+1} K^{2k-1}_{z_{h-1}z_h}(\xi,\eta).$$

Again, by Proposition 6.21 and Corollary 6.24, this gives

$$\begin{aligned} w(x,y) &= u(z_0,z_k) + u(z_{k+1},z_1) \\ &+ \sum_{h=1}^{k}(L^{2k-1}_{z_hz_{h-1}}(\xi,\eta) + L^{2k-1}_{z_hz_{h+1}}(\xi,\eta)) - \sum_{h=1}^{k+1}(L^{2k-1}_{z_{h-1}z_h}(\xi,\eta) + L^{2k-1}_{z_hz_{h-1}}(\xi,\eta)). \end{aligned}$$

We get

$$\begin{aligned} w(x,y) &= u(z_0,z_k) + u(z_{k+1},z_1) - L^{2k-1}_{z_0z_1}(\xi,\eta) - L^{2k-1}_{z_{k+1}z_k}(\xi,\eta) \\ &= u(x,y_1) + u(y,x_1) - v(x,y_1) - v(y,x_1) \end{aligned}$$

as required. □

As a remark, we also relate the pseudoweight of a canonical pseudokernel with the weight of a dual kernel.

LEMMA 9.11. *Let $k \geq 2$, (K, K^-) be a k-dual kernel and L be its canonical k-pseudokernel. Then the bias function of (K, K^-) is the pseudoweight of L.*

See Definition 6.11 and Definition 6.12 for the canonical pseudokernel of a dual kernel. See Definition 7.8 for the introduction of the bias function of a dual kernel.

PROOF. By Lemma 6.14, the canonical $(2k-1)$-pseudokernel of the $(2k-1)$-dual kernel (K^{2k-1}, K^{2k-2}) is L^{2k-1}. The result now directly follows from the definitions. □

For $k \geq 1$ and v a function on X_k, the symmetrization of v is the function $(x, y) \mapsto \frac{1}{2}(v(x, y) + v(y, x))$.

COROLLARY 9.12. *Let $k \geq 2$, (K, K^-) be a k-dual kernel and L be its canonical k-pseudokernel. Let v be the bias function of (K, K^-) which is also the pseudoweight of L. Then the symmetrization of v is a weight function of (K, K^-).*

PROOF. It follows from Definition 7.2 that (K, K^-) admits a unique symmetric compatible function u. Let w be the associated weight function as in Definition 7.4. By Lemma 7.10, for (x, y) in X_k, we have

$$v(x,y) = u(x_1,y) - u(x,y_1) + w(x,y),$$

where x_1 and y_1 are the neighbours of x and y on $[xy]$. Summing with the symmetric equality, we get, u and w being symmetric,

$$v(x,y) + v(y,x) = 2w(x,y)$$

as required. □

9.3. Weight functions of orthogonal extensions

Thanks to Proposition 9.9 and Corollary 9.10, we know that the weight map is 0 on pseudokernels. It remains to show the converse statement, that if a k-dual kernel admits a weight function which is a coboundary, then this dual kernel is the one associated to some $(k-1)$-pseudokernel. Here, by a coboundary, we mean in the sense of cohomology following from Lemma 3.13.

Our strategy will be to start by a weaker statement, namely that if a k-dual kernel admits a weight function (and hence has all its weight functions) of the form $(x,y) \mapsto v(x,y_1) + v(y,x_1)$ for some non-necessarily skew-symmetric function v on X_{k-1}, then it must be the sum of a pseudokernel with an orthogonal extension. As a preliminary, the purpose of this section is to prove that the weight functions of orthogonal extensions are actually of this form.

PROPOSITION 9.13. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with orthogonal extension (K^+, K). Let u be a (K, K^-)-compatible function and w be the associated weight function for (K, K^-). Then a function u^+ on X_k is (K^+, K)-compatible if and only if one has, for any (x,y) in X_k,*

$$u^+(x,y) + u^+(y,x) = w(x,y) + u(x_1,y) + u(y_1,x).$$

In that case, the associated weight function w^+ for (K^+, K) is defined by, for any (x,y) in X_{k+1},

$$w^+(x,y) = u^+(x,y_1) + u^+(y,x_1) - u(x_1,y_1) - u(y_1,x_1).$$

COROLLARY 9.14. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with orthogonal extension (K^+, K). Let w be a weight function for (K, K^-) and w^+ be a weight function for (K^+, K). Then there exists a skew-symmetric function v on X_k such that, for any (x,y) in X_{k+1}, one has*

$$w^+(x,y) = \frac{1}{2}(w(x,y_1) + w(x_1,y)) + v(x,y_1) - v(x_1,y).$$

If (K, K^-), w and w^+ are Γ-invariant, one can chose v to be so.

In both statements, for $x \neq y$ in X, we have as usual denoted by x_1 and y_1 the neighbours of x and y on $[xy]$.

Note that, in Corollary 9.14, the functions w and w^+ are related in the same way as the functions w and w' in Lemma 3.13.

Again, the proofs will follow from the definitions by straightforward computations.

Proof of Proposition 9.13. By Definition 7.2, saying that the function u^+ is (K^+, K)-compatible amounts to saying that, for any (x, y) in X_k, if $(z_h)_{h\in\mathbb{Z}}$ is a parametrized geodesic line with $z_0 = x$ and $z_k = y$, we have

$$u^+(x,y) + u^+(y,x) = \sum_{h=1}^{k} K^{2k-1}_{z_{h-1}z_h}(\xi,\eta) - \sum_{h=1}^{k-1} K^{2k}_{z_h}(\xi,\eta), \tag{9.1}$$

where ξ and η are the endpoints of $(z_h)_{h\in\mathbb{Z}}$. Now, for any $1 \leq h \leq k-1$, we have

$$K^{2k}_{z_h}(\xi,\eta) = K^{2k-1}_{z_{h-1}z_h}(\xi,\eta) + K^{2k-1}_{z_hz_{h+1}}(\xi,\eta) + \sum_{\substack{t\sim z_h \\ t\notin\{z_{h-1},z_{h+1}\}}} K^{2k-1}_{z_ht}(\xi,\eta) - (d(z_h)-1)K^{2k-2}_{z_h}(\xi,\eta).$$

By Lemma 6.14, this can be written as

$$K^{2k}_{z_h}(\xi,\eta) = K^{2k-1}_{z_{h-1}z_h}(\xi,\eta) + K^{2k-1}_{z_hz_{h+1}}(\xi,\eta) - K^{2k-2}_{z_h}(\xi,\eta) + \sum_{\substack{t\sim z_h \\ t\notin\{z_{h-1},z_{h+1}\}}} L^{2k-1}_{z_ht}(\xi,\eta),$$

where L is the canonical pseudokernel of (K, K^-). By Corollary 6.24, the last sum is zero and hence (9.1) gives

$$u^+(x,y) + u^+(y,x) = \sum_{h=1}^{k-1} K^{2k-2}_{z_h}(\xi,\eta) - \sum_{h=2}^{k-1} K^{2k-1}_{z_{h-1}z_h}(\xi,\eta).$$

As $K^{2k-1}_{z_{h-1}z_h} = K^{2k-2}_{z_{h-1}} + K^{2k-2}_{z_h} - K^{2k-3}_{z_{h-1}z_h}$, $2 \leq h \leq k-1$, we get

$$u^+(x,y) + u^+(y,x) = \sum_{h=2}^{k-1} K^{2k-3}_{z_{h-1}z_h}(\xi,\eta) - \sum_{h=2}^{k-2} K^{2k-2}_{z_h}(\xi,\eta).$$

Now we use Definitions 7.2 and 7.4 for (K, K^-), which give

$$u^+(x,y) + u^+(y,x) = (u(x,y_1) + u(y_1,x)) + (u(x_1,y) + u(y,x_1)) + (w(x,y) - u(x,y_1) - u(y,x_1)),$$

that is

$$u^+(x,y) + u^+(y,x) = w(x,y) + u(y_1,x) + u(x_1,y),$$

which should be proved.

For weight functions, the same computation yields, for any (x,y) in X_{k+1}, any parametrized geodesic line $(z_h)_{h\in\mathbb{Z}}$ with endpoints ξ and η and $z_0 = x$ and $z_{k+1} = y$,

$$\begin{aligned} w^+(x,y) &= u^+(x,y_1) + u^+(y,x_1) + \sum_{h=2}^{k-1} K_{z_h}^{2k-2}(\xi,\eta) - \sum_{h=2}^{k} K_{z_{h-1}z_h}^{2k-3}(\xi,\eta) \\ &= u^+(x,y_1) + u^+(y,x_1) - u(x_1,y_1) - u(y_1,x_1). \end{aligned}$$

□

Proof of Corollary 9.14. Let u (resp. u^+) be the (K,K^-)-compatible (resp. (K^+,K)-compatible) function with w (resp. w^+) as its weight function. Fix (x,y) in X_{k+1}, let x_1 and y_1 be their neighbours on $[xy]$ and x_2 and y_2 be the neighbours of x_1 and y_1 on $[x_1y_1]$. Proposition 9.13 gives

$$\begin{aligned} w(x,y_1) &= u^+(x,y_1) + u^+(y_1,x) - u(x_1,y_1) - u(y_2,x) \\ w(x_1,y) &= u^+(x_1,y) + u^+(y,x_1) - u(x_2,y) - u(y_1,x_1) \\ w^+(x,y) &= u^+(x,y_1) + u^+(y,x_1) - u(x_1,y_1) - u(y_1,x_1), \end{aligned}$$

hence

$$w^+(x,y) - \frac{1}{2}(w(x,y_1) + w(x_1,y)) = v(x,y_1) + v(y,x_1),$$

where, for any (a,b) in X_k,

$$v(a,b) = \frac{1}{2}(u^+(a,b) - u^+(b,a) + u(b_1,a) - u(a_1,b)).$$

□

9.4. Split weight functions

Our goal is still to prove that if a dual kernel admits a weight function which is a coboundary, then it is a pseudokernel. As mentionned above, we will first prove a weaker version of this statement which will play a key rule in the final proof. We start by introducing a new notion.

Let $k \geq 2$ and w be a symmetric function on X_k. We shall say that w is split if there exists a function v on X_{k-1} such that for any (x,y) in X_k, one has $w(x,y) = v(x,y_1) + v(y,x_1)$ (note that we don't require v to have any symmetry property).

By Definition 7.4, if a dual kernel admits a split weight function, all its weight functions are split. Proposition 9.9 and Corollary 9.14 tell us

that the weight functions of a pseudokernel and those of an orthogonal extension are split. We have a converse statement:

PROPOSITION 9.15. *Let $k \geq 3$ and (K, K^-) be a k-dual kernel. Then the weight functions of (K, K^-) are split if and only if there exists a $(k-1)$-pseudokernel L, with associate k-dual kernel (J, J^-), and a $(k-1)$-dual kernel (H, H^-), with orthogonal extension (H^+, H), such that $K = H^+ + J$ and $K^- = H + J^-$.*

Let $k \geq 2$ and (K, K^-) be a k-dual kernel. We begin the proof of this fact by introducing a new function on X_k associated to (K, K^-).

If k is even, $k = 2\ell$, $\ell \geq 1$, we define the preweight of (K, K^-) as the function w_- on X_k such that, for any (x, y) in X_k, $w_-(x, y) = K_a(x, y)$, where a is the middle point of $[xy]$, that is, a is the unique point of $[xy]$ with $d(x, a) = \ell = d(y, a)$.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we define the preweight of (K, K^-) as the function w_- on X_k such that, for any (x, y) in X_k, $w_-(x, y) = K_{ab}(x, y)$, where $[ab]$ is the middle edge of $[xy]$, that is, a and b are the unique points of $[xy]$ with $d(x, a) = \ell = d(y, b)$.

REMARK 9.16. Note that the preweight actually does not depend on K^-.

LEMMA 9.17. *Let $k \geq 2$, (K, K^-) be a k-dual kernel, w_- be the preweight of (K, K^-) and w be a weight function of (K, K^-). Then the function $w - w_-$ is split.*

PROOF. By Proposition 9.4 and Corollary 9.5, the result is true if $K = 0$. Therefore, it suffices to prove that, if K^- is a $(k-1)$-predual kernel, the weight functions of the k-dual kernel $(0, K^-)$ are split.

If $k = 2$, a direct computation using Definition 7.2 and Definition 7.4 shows that a function u is $(0, K^-)$ compatible if and only if, for any $x \sim y$ in X, one has $u(x, y) + u(y, x) = K^-_{xy}(x, y)$. Then, the associated weigh function is defined by, for x in X and $y \neq z$ in $S^1(x)$, $w(y, z) = -u(x, y) - u(x, z)$. Thus, the function w is split.

If $k \geq 3$, we let L be the $(k-1)$-pseudokernel such that, for $x \sim y$ in X, $L_{xy} = K^-_{xy}$ if k is even and $L_{xy} = (d(x) - 1)K^-_x$ if k is odd. Then, by Definition 6.16 and Definition 6.17, the associated k-dual kernel (H, H^-) is given by, for $x \sim y$ in X, if k is even,

$$H_x = \sum_{y \sim x} K^-_{xy} \text{ and } H^-_{xy} = 2K^-_{xy};$$

if k is odd,

$$H_{xy} = (d(x) - 1)K^-_x + (d(y) - 1)K^-_y \text{ and } H^-_x = d(x)K^-_x.$$

Also, let (J, J^-) be the $(k-1)$-dual kernel such that $J^- = 0$ and, if k is even, $J = K^-$, wheras, if k is odd, for x in X, $J_x = (d(x)-1)K^-_x$. Then, the orthogonal extension (J^+, J) of (J, J^-) is defined by, for $x \sim y$ in X, if k is even,

$$J^+_x = \sum_{y\sim x} K^-_{xy} \text{ and } J_{xy} = K^-_{xy};$$

if k is odd,

$$J^+_{xy} = (d(x)-1)K^-_x + (d(y)-1)K^-_y \text{ and } J_x = (d(x)-1)K^-_x.$$

In both cases, we have $(H, H^-) - (J^+, J) = (0, K^-)$. Thus, the k-dual kernel $(0, K^-)$ is the sum of the k-dual kernel associated with a $(k-1)$-pseudokernel with the orthogonal extension of a $(k-1)$-dual kernel. By Proposition 9.9 and Proposition 9.13, the weight functions of $(0, K^-)$ are split. □

We have an abstract criterion for a bilinear form to split as a sum.

LEMMA 9.18. *Let $W_0, W_1, \dots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. For any $1 \leq i \leq d$, we set $X_i = \bigcap_{j\neq i} \ker \pi_j \subset W$.

Let q be a symmetric bilinear form on W. Then there exists symmetric bilinear forms $q_1, \dots q_d$ on $W_1, \dots, W_d$ with $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$ if and only if, for any $1 \leq i \neq j \leq d$, the spaces X_i and X_j are q-orthogonal, that is, $q(X_i, X_j) = 0$.

PROOF. Clearly, if $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$ for some symmetric bilinear forms $q_1, \dots q_d$ on $W_1, \dots, W_d$, then for any $i \neq j$, the spaces X_i and X_j are q-orthogonal.

Conversely, assume this is the case and let us build $q_1, \dots q_d$. We chose a subspace X_0 of W such that the restriction of π_0 to X_0 is an isomorphism onto W_0. We then have $W = X_0 \oplus X_1 \oplus \cdots \oplus X_d$ and, by assumption, if $w = x_0 + \cdots + x_d$ is in W, with $x_i \in X_i$, $0 \leq i \leq d$, one has

$$q(w, w) = \sum_{i=0}^{d} q(x_i, x_i) + 2\sum_{i=1}^{d} q(x_0, x_i).$$

Now, for any $1 \leq i \leq d$, the restriction of π_i to $X_0 + X_i$ is an isomorphism onto W_i. Therefore, there exists a unique symmetric bilinear

form q_i on W_i such that, for x_0 in X_0 and x_i in X_i, one has

$$\pi_i^\star q_i(x_0+x_i, x_0+x_i) = \frac{1}{d} q_0(x_0,x_0) + q_i(x_i,x_i) + 2q_i(x_0,x_i).$$

By construction, one has $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$. □

Proof of Proposition 9.15. Let (K, K^-) be a k-dual kernel which admits a split weight function. By Lemma 9.17, the preweight w_- of (K, K^-) is split. We will show that this amounts to saying that one can apply Lemma 9.18 to the bilinear forms associated to K. Fix a function v on X_{k-1} with $w_-(x,y) = v(x,y_1) + v(y,x_1)$, $(x,y) \in X_k$.

First assume k is odd, $k = 2\ell+1$, $\ell \geq 1$. Fix x, y in X. We claim that the subspaces

$$\ker J_{xy}^{\ell,*} \text{ and } \ker J_{yx}^{\ell,*}$$

of $V_0^\ell(xy)$ are q_{xy}^K-orthogonal. Indeed pick f_x in $\ker J_{yx}^{\ell,*}$ and f_y in $\ker J_{xy}^{\ell,*}$. We have

$$\begin{aligned}
f_x(b) &= 0 && b \in S^\ell(y) \cap S^{\ell+1}(x)\\
f_y(a) &= 0 && a \in S^\ell(x) \cap S^{\ell+1}(y)\\
\sum_{\substack{a\sim a_1\\ a\notin[xa_1]}} f_x(a) &= 0 && a_1 \in S^{\ell-1}(x) \cap S^\ell(y)\\
\sum_{\substack{b\sim b_1\\ b\notin[yb_1]}} f_y(b) &= 0 && b_1 \in S^{\ell-1}(y) \cap S^\ell(x).
\end{aligned}$$

Therefore, by Lemma 5.1,

$$\begin{aligned}
q_{xy}^K(f_x,f_y) &= -\frac{1}{2} \sum_{\substack{a\in S^\ell(x)\cap S^{\ell+1}(y)\\ b\in S^\ell(y)\cap S^{\ell+1}(x)}} w_-(a,b) f_x(a) f_y(b)\\
&= -\frac{1}{2} \sum_{\substack{a_1\in S^{\ell-1}(x)\cap S^\ell(y)\\ b_1\in S^{\ell-1}(y)\cap S^\ell(x)}} \sum_{\substack{a\sim a_1\\ a\notin[xa_1]}} \sum_{\substack{b\sim b_1\\ b\notin[yb_1]}} (v(a,b_1)+v(b,a_1)) f_x(a) f_y(b) = 0.
\end{aligned}$$

By Lemma 9.18, we can find a family $(s_{xy})_{(x,y)\in X_1}$, where, for any (x,y), s_{xy} is a symmetric bilinear form on $V_0^\ell(x)$ and $q_{xy}^K = (J_{xy}^{\ell,*})^\star s_{xy} + (J_{yx}^{\ell,*})^\star s_{yx}$. In other words, there exists a $(k-1)$-pseudokernel M with $K_{xy} = M_{xy} + M_{yx}$ for any $x \sim y$ in X. This is not over since for the moment there is no relation between K^- and M. To correct this, we

set

$$H_x = \sum_{y\sim x} M_{xy} - (d(x)-1)K_x^-, \qquad x \in X,$$
$$L_{xy} = M_{xy} - H_x, \qquad (x,y) \in X_1$$

and $H^- = 0$. We consider $(H, H^-) = (H, 0)$ as a $(k-1)$-dual kernel and L as a $(k-1)$-pseudokernel. Let (H^+, H) be the orthogonal extension of $(H, 0)$ and (J, J^-) be the k-dual kernel associated to L. By construction, we have, for $x \sim y$ in X,

$$K_{xy} = M_{xy} + M_{yx} = L_{xy} + L_{yx} + H_x + H_y = J_{xy} + H^+_{xy}$$

and, for x in X,

$$\begin{aligned} K_x^- &= \frac{1}{d(x)-1}\left(\sum_{y\sim x} M_{xy} - H_x\right) \\ &= \frac{1}{d(x)-1}\left(\sum_{y\sim x}(L_{xy}+H_x) - H_x\right) = \frac{1}{d(x)-1}\sum_{y\sim x} L_{xy} + H_x \\ &= J_x^- + H_x, \end{aligned}$$

which should be proved.

Now assume k is even, $k = 2\ell$, $\ell \geq 2$, and let us proceed in the same way. We fix x in X and we set, for $y \sim x$,

$$W_y = \bigcap_{\substack{z\sim x \\ z\neq y}} \ker I_{xz}^{\ell-1,*} \subset V_0(x).$$

We claim that these spaces are q_x^K-orthogonal to each other. Indeed, for $y \sim x$ and f_y in W_y, we have

$$\begin{aligned} f_y(b) &= 0 \quad b \in S^\ell(x) \cap S^{\ell+1}(y) \\ \sum_{\substack{a\sim a_1 \\ a\notin[xa_1]}} f_y(a) &= 0 \quad a_1 \in S^{\ell-1}(x) \cap S^{\ell-2}(y). \end{aligned}$$

Now pick $y \neq z$ among the neighbours of x and chose f_y in W_y and f_z in W_z. Again by Lemma 5.1, we have

$$\begin{aligned} q_x^K(f_y, f_z) &= -\frac{1}{2} \sum_{\substack{a\in S^\ell(x)\cap S^{\ell-1}(y) \\ b\in S^\ell(x)\cap S^{\ell-1}(z)}} w_-(a,b) f_y(a) f_z(b) \\ &= -\frac{1}{2} \sum_{\substack{a_1\in S^{\ell-1}(x)\cap S^{\ell-2}(y) \\ b_1\in S^{\ell-1}(x)\cap S^{\ell-2}(z)}} \sum_{\substack{a\sim a_1 \\ a\notin[xa_1]}} \sum_{\substack{b\sim b_1 \\ b\notin[xb_1]}} (v(a,b_1)+v(b,a_1)) f_y(a) f_z(b) = 0. \end{aligned}$$

By Lemma 9.18, we can now find a $(k-1)$-pseudokernel M with $K_x = \sum_{y\sim x} M_{xy}$. We set

$$\begin{aligned} H_{xy} &= M_{xy} + M_{yx} - K^-_{xy}, && x \sim y \in X, \\ L_{xy} &= M_{xy} - H_{xy}, && x \in X. \end{aligned}$$

Again, we let (H^+, H) denote the orthogonal extension of the $(k-1)$-dual kernel $(H, 0)$ and (J, J^-) the k-dual kernel associated to the $(k-1)$-pseudokernel L and we have, for any x in X,

$$K_x = \sum_{y\sim x} M_{xy} = \sum_{y\sim x} L_{xy} + \sum_{y\sim x} H_{xy} = J_x + H^+_x$$

and, for $x \sim y$,

$$K^-_{xy} = M_{xy} + M_{yx} - H_{xy} = L_{xy} + L_{yx} + H_{xy} = J^-_{xy} + H_{xy},$$

which should be proved. □

9.5. The null space of the weight map

We are now ready to conclude the proof of Theorem 9.1. We will need the elementary

LEMMA 9.19. *Let $k \geq 2$ and φ be a function on X_k such that, for any (x, y) in X_{k+1}, one has $\varphi(x, y_1)+\varphi(y, x_1) = 0$. Then, there exists a skew-symmetric function ψ on X_{k-1} such that, for any (x, y) in X_{k-1}, one has $\varphi(x, y) = \psi(x_1, y)$.*

PROOF. Fix (x, y) in X_{k-1}. Let z be a neighbour of y that is not on $[xy]$. If t is a neighbour of x that is not on $[xy]$, we have $\varphi(t, y) = -\varphi(z, x)$, hence this value does not depend on t. We define it as $\psi(x, y)$. By construction, ψ is skew-symmetric and we are done. □

PROOF OF THEOREM 9.1. *(ii)⇒(i)* is Proposition 9.9.

We prove *(i)⇒(ii)* by induction on $k \geq 2$.

First assume $k = 2$. Pick a 2-dual kernel (K, K^-) which satisfies the assumptions. In this case, a function u on X_1 is (K, K^-)-compatible if and only if, for any $x \sim y$ in X, one has

$$u(x, y) + u(y, x) = K^-_{xy}(x, y).$$

Now, by assumption, there exists such a function u as well as a skew-symmetric function v on X_1 such that, for any x in X any $y \neq z$ in $S^1(x)$, one has

$$\begin{aligned} v(y, x) + v(z, x) &= u(y, x) + u(z, x) + K_x(y, z) - K^-_{xy}(x, y) - K^-_{xz}(x, z) \\ &= -u(x, y) - u(x, z) + K_x(y, z). \end{aligned}$$

We define a 1-pseudokernel by setting, for any (x, y) in X_1, $L_{xy}(x, y) = v(y, x) + u(x, y)$. The relations above directly imply that (K, K^-) is the 2-dual kernel associated to L.

Assume now $k \geq 3$ and the result is true for $k - 1$. Again we chose a k-dual kernel (K, K^-) which satisfies the assumptions of the Theorem, that is, there exists a skew-symmetric function v on X_{k-1} such that the functon $(x, y) \mapsto v(x, y_1) + v(y, x_1)$ on X_k is a weight function of (K, K^-). In particular, this weight function is split, hence, by Proposition 9.15, there exists a $(k-1)$-pseudokernel L and a $(k-1)$-dual kernel (H, H^-) such that (K, K^-) is the sum of the k-dual kernel associated with L and of the orthogonal extension (H^+, H) of (H, H^-). To conclude, it suffices to prove that (H, H^-) is the $(k-1)$-dual kernel associated to some $(k-2)$-pseudokernel. We will get this by applying the induction hypothesis to (H, H^-). To this aim, we chose a weight function w on X_{k-1} for (H, H^-). By Proposition 9.9, Corollary 9.14 and the assumption, there exists a skew-symmetric function v' on X_{k-1} such that, for any (x, y) in X_k, one has

$$w(x, y_1) + w(y, x_1) = v'(x, y_1) + v'(y, x_1)$$

(recall that weight functions are symmetric). By Lemma 9.19, there exists a skew-symmetric function v'' on X_{k-2} such that, for any (x, y) in X_{k-1}, one has

$$w(x, y) = v'(x, y) + v''(x_1, y).$$

As w is symmetric and v' is skew-symmetric, we have

$$w(x, y) = \frac{1}{2}(v''(x_1, y) + v''(y_1, x)).$$

Now, the induction assumption tells us that (H, H^-) is the $(k-1)$-dual kernel associated to some $(k-2)$-pseudokernel. By Propositon 6.21, (H^+, H) is the k-dual kernel associated to some $(k-1)$-pseudokernel. Therefore, (K, K^-) also is of this form, which should be proved. $\square$

CHAPTER 10

Image dual kernels

For $k \geq 2$, we have introduced in Definition 8.14 the notion of the image dual kernel of a Γ-invariant function w on X_k such that the symmetric bilinear form Φ_w on H_0^ω is non-negative. We will simply say that a k-dual kernel (K, K^-) is an image dual kernel if one can find such a w with (K, K^-) being the image dual kernel of w. Note that, in view of Lemma A.4, this implies in particular that (K, K^-) is not only non-negative but exact in the sense of Definitions 5.12 and 5.13.

In the present Chapter, we will use the characterization of the null space of the weight map obtained above to give a geometric criterion for an exact dual kernel to be an image dual kernel.

10.1. Non-negative pseudokernels

We have the following natural

DEFINITION 10.1. (k odd) Let k be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$, and L be a k-pseudokernel. We say that L is non-negative if, for any $x \sim y$ in X, the symmetric bilinear form r_{xy}^L associated to L on $V_0^\ell(xy)$ is non-negative.

DEFINITION 10.2. (k even) Let k be an even integer, $k = 2\ell$, $\ell \geq 1$, and L be a k-pseudokernel. We say that L is non-negative if, for any $x \sim y$ in X, the symmetric bilinear form r_{xy}^L associated to L on $V_0^\ell(x)$ is non-negative.

Recall that we write $\mathcal{K}_k$, $k \geq 2$, for the space of Γ-invariant k-dual kernels and $\mathcal{L}_k$, $k \geq 1$, for the space of Γ-invariant k-pseudokernels. As above, we let $\mathcal{K}_k^+ \subset \mathcal{K}_k$ stand for the set of non-negative Γ-invariant k-dual kernels. In the same way, we let $\mathcal{L}_k^+ \subset \mathcal{L}_k$ stand for the set of non-negative Γ-invariant k-pseudokernels. The sets $\mathcal{K}_k^+$ and $\mathcal{L}_k^+$ are closed convex cones (see Proposition 5.14 in the former case). Using the construction of the k-dual kernel associated with a $(k-1)$-pseudokernel in Section 6.4, for $k \geq 2$, we may identify $\mathcal{L}_{k-1}$ with a subspace of $\mathcal{K}_k$. Note that there is no obvious relation between $\mathcal{K}_k^+$ and $\mathcal{L}_{k-1}^+$. The main result of this Chapter is

THEOREM 10.3. *Let $k \geq 2$ and (K, K^-) be in $\mathcal{K}_k^+$. Then $(K, K^-) + \mathcal{L}_{k-1}$ contains a unique image kernel (H, H^-) and the $(k-1)$-pseudokernel L with $(H, H^-) = (K, K^-) + L$ is non-negative. In particular, the following are equivalent:*
(i) *(K, K^-) is an image kernel.*
(ii) *we have $((K, K^-) + \mathcal{L}_{k-1}^+) \cap \mathcal{K}_k^+ = (K, K^-)$.*
(iii) *we have $((K, K^-) + \mathcal{L}_{k-1}) \cap \mathcal{K}_k^+ \subset (K, K^-) - \mathcal{L}_{k-1}^+$.*

Let us explain the underlying ideas in this result. Given (K, K^-) as above, we pick a weight function w for (K, K^-). Theorem 8.6 tells us that the pre-Hilbert space of distributions associated to (K, K^-) contains H_0^ω and that the restriction of q^{K,K^-} to H_0^ω is equal to Φ_w. In particular, Φ_w is non-negative. We let (H, H^-) be its image dual kernel. Theorem 8.17 tells us that w is a weight function for (H, H^-). Therefore, Theorem 9.1 tells us that (H, H^-) belongs to $(K, K^-) + \mathcal{L}_{k-1}$. Now, we would like to prove that (H, H^-) actually belongs to $(K, K^-) + \mathcal{L}_{k-1}^+$. This is the main difficulty of the proof. Indeed, from the construction, it is clear that, for any $j \geq k-1$, the dual prekernel $H^j - K^j$ is non-negative. But saying that the pseudokernel $(H - K, H^- - K^-)$ is non-negative (as a pseudokernel) is an a priori stronger property. Therefore, the proof of this result will require us to compare several notions of non-negativity.

10.2. Weakly non-negative pseudokernels

We introduce a new notion of non-negativity for pseudokernels which will play a central role in the proof of Theorem 10.3.

Let L be a k-pseudokernel. In Definition 6.7, we have introduced the orthogonal extension of L. If L is non-negative, its orthogonal extension L^+ is a non-negative $(k+1)$-pseudokernel, so that all its successive orthogonal extensions L^j, $j \geq k$, are non-negative.

As in Definitions 5.10 and 5.11, we shall say that a dual prekernel is non-negative if the associated bilinear forms are non-negative. Let (K, K^-) be the $(k+1)$-dual kernel associated to L and let K^j, $j \geq k$, be the dual prekernels obtained from (K, K^-) by successive orthogonal extensions. From Definitions 6.16 and 6.17, it is clear that if L is non-negative, all the K^j, $j \geq k$, are non-negative dual prekernels.

We shall need a weaker notion of non-negativity for pseudokernels.

DEFINITION 10.4. Let $k \geq 1$ and L be a k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). We say that L is weakly non-negative if the dual prekernels K^j, $j \geq k$, are non-negative.

The previous discussion directly gives

LEMMA 10.5. *Let $k \geq 1$. Any non-negative k-pseudokernel is also weakly non-negative.*

The converse of this statement is not true. Nevertheless, for Γ-invariant pseudokernels, we have a criterion for being weakly non-negative which involves only finitely many kernels.

PROPOSITION 10.6. *Let $k \geq 1$ and L be a Γ-invariant k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). Then L is weakly non-negative if and only if L^{2k-1} is non-negative and the dual prekernels K^j, $k \leq j \leq 2k-3$, are non-negative.*

This technical result is the main ingredient of the proof of Theorem 10.3. One of the directions of the equivalence is easier to prove and actually holds without assuming the kernel to be Γ-invariant. Indeed, it will follow from the following general formula.

LEMMA 10.7. *Let $k \geq 2$ and L be a k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). For any x in X, we have*

$$K_x^{2k-2} = \frac{1}{d(x)-1} \sum_{y \sim x} L_{yx}^{2k-1}.$$

For any $j \geq 0$ and $x \sim y$ in X, we have

$$K_{xy}^{2(j+k)-1} = \sum_{z \in S^j(xy)} L_{z_- z}^{2k-1} \text{ and } K_x^{2(j+k)} = \sum_{z \in S^{j+1}(x)} L_{z_- z}^{2k-1}.$$

In this statement, for $j \geq 1$ and z in $S^j(xy)$ we have denoted by z_- the neighbour of z in $S^{j-1}(xy)$. For $j = 0$, we write $x_- = y$ and $y_- = x$.

PROOF. As $k \geq 2$, we have $2k-2 \geq k$ and Proposition 6.21 says that K^{2k-2} is the $(2k-2)$-predual kernel associated to L^{2k-2}. By Definitions 6.17 and 6.7, we get

$$K_x^{2k-2} = \frac{1}{d(x)-1} \sum_{y \sim x} L_{xy}^{2k-2} = \frac{1}{d(x)-1} \sum_{y \sim x} L_{yx}^{2k-1}.$$

The other formulas follow from Proposition 6.21 and Corollary 6.25. □

This gives a first direction in Proposition 10.6.

COROLLARY 10.8. *Let $k \geq 1$ and L be a k-pseudokernel with associated $(k+1)$-dual kernel (K, K^-). If L^{2k-1} is non-negative, then, for every $j \geq \max(k, 2k-2)$, the j-dual prekernel K^j is non-negative.*

PROOF. If $k = 1$, there is nothing to prove since $2k-1 = 1$. If $k \geq 2$, the result directly follows from Lemma 10.7. □

10.3. Negative edges

To finish the proof of Proposition 10.6, we will show that for Γ-invariant pseudokernels, the converse to Corollary 10.8 holds. In this section, we begin by showing that, if L is a weakly non-negative k-pseudokernel, for most of the edges (x, y), L_{xy}^{2k-1} must represent a non-negative symmetric bilinear form. This fact will rely on the following abstract

LEMMA 10.9. *Let $d \geq 1$ be an integer and $V_1, \ldots, V_d$ be real vector spaces. Set $V = V_1 \oplus \cdots \oplus V_d$. For $1 \leq i \leq d$, we let φ_i be a non-zero linear functional on V_i and q_i be a symmetric bilinear form on V_i. We set φ to be the linear functional $\varphi_1 + \cdots + \varphi_d$ and q to be the symmetric bilinear form $q_1 + \cdots + q_d$ on V. Assume that q is non-negative on the hyperplane $\ker \varphi$ of V. Then there exists at most one $1 \leq i \leq d$ such that q_i admits negative vectors. In that case the maximal negative subspaces of q_i have dimension one.*

PROOF. Assume there exists $1 \leq i \neq j \leq d$ and v_i in V_i and v_j in V_j with $q_i(v_i, v_i) < 0$ as well as $q_j(v_j, v_j) < 0$. Then q is negative definite on the 2-plane $W = \mathbb{R}v_i \oplus \mathbb{R}v_j \subset V$. As $W \cap \ker \varphi$ is non zero, q can not be non-negative on $\ker \varphi$. Now, let i be such that q_i admits negative vectors. For the same reason as above, any negative subspace of V_i must have zero intersection with $\ker \varphi_i$, hence, it must be a line. □

From Lemma 10.9, we will deduce a geometric property of a certain set of exceptional edges associated to a weakly negative pseudokernel. Let $N \subset X_1$ be a set of oriented edges of X. We will say that N meets the spheres at most once if, for any x in X and $h \geq 1$, we have

$$|\{z \in S^h(x) | (z_-, z) \in N\}| \leq 1.$$

Let $k \geq 1$ and L be a k-pseudokernel. We define the set N_L of negative edges of L as the set of those (x, y) in X_1 such that the symmetric bilinear form associated to L_{xy}^{2k-1} on $V_0^{k-1}(xy)$ has negative vectors.

LEMMA 10.10. *Let $k \geq 1$ and L be a k-pseudokernel. Assume L is weakly non-negative. Then the set N_L of negative edges of L meets the spheres at most once.*

PROOF. For $\ell \geq 0$ and $x \sim y$ in X, we define the set $S_+^\ell(xy)$ as

$$S_+^\ell(xy) = \{x\} \cup \{z \in S^\ell(y) | x \notin [yz]\}.$$

This is the boundary of a rooted subtree in X. We also define $W^\ell(xy)$ as the vector space of all real-valued functions on $S_+^\ell(xy)$ and as usual,

we let $\overline{W}^\ell(xy)$ denote the quotient of $W^\ell(xy)$ by the space of constant functions and $W_0^\ell(xy)$ denote the space of those f in $W^\ell(xy)$ with $\sum_{z\in S_+^\ell(xy)} f(z) = 0$, which we identify in the usual way with the dual space of $\overline{W}^\ell(xy)$.

Let $k \geq 1$ and L be a k-pseudokernel. Pick $x \sim y$ in X. By Lemma 9.7, we may see L_{xy}^{2k-1} as a symmetric function on $S_+^{k-1}(xy) \times S_+^{k-1}(xy)$ which is zero on the diagonal. Thanks to Lemma 5.1, we associate to it a symmetric bilinear form r_{xy}^{+L} on the space $W_0^{k-1}(xy)$. Note that, saying that (x, y) belongs to N_L is the same as saying that r_{xy}^{+L} admits negative vectors.

Now, let x, y be in X with $x \neq y$ and let still y_- be the neighbour of y on $[xy]$. Set $h = d(x, y) \geq 1$ and pick $\ell \geq 0$. Then, we have a natural injective linear map $H_{xy}^\ell : W^\ell(y_-y) \to V^{h+\ell}(x)$ defined as follows. For z in $S^{h+\ell}(x)$ and f in $W^\ell(y_-y)$, we set

$$\begin{aligned} H_{xy}^\ell f(z) =& f(z) \quad \text{if } y \in [xz] \\ H_{xy}^\ell f(z) =& f(y_-) \quad \text{else.} \end{aligned}$$

Note that saying that y belongs to $[xz]$ amounts to saying that z belongs to $S_+^\ell(y_-y)$ or that y_- does not belong to $[yz]$. One still let H_{xy}^ℓ denote the induced map $\overline{W}^\ell(y_-y) \hookrightarrow \overline{V}^{h+\ell}(x)$.

Let still $h \geq 1$ and $\ell \geq 0$. For x in X, one easily checks that, as $S^{h+\ell}(x)$ may be written as the disjoint union

$$S^{h+\ell}(x) = \bigsqcup_{y\in S^h(x)} \{z \in S^\ell(y) | y_- \notin [yz]\},$$

the space $\overline{V}^{h+\ell}(x)$ is spanned by the spaces $H_{xy}^\ell \overline{W}^\ell(y_-y)$, $y \in S^h(x)$, and that the kernel of the surjective map

$$\bigoplus_{y\in S^h(x)} H_{xy}^\ell : \bigoplus_{y\in S^h(x)} \overline{W}^\ell(y_-y) \to \overline{V}^{h+\ell}(x)$$

is the line spanned by the vector $\bigoplus_{y\in S^h(x)} \mathbf{1}_{y_-}$. By duality, this tells us that the adjoint map

$$\bigoplus_{y\in S^h(x)} H_{xy}^{\ell,*} : V_0^{h+\ell}(x) \to \bigoplus_{y\in S^h(x)} W_0^\ell(y_-y)$$

is injective and that its range is the set of vectors $\bigoplus_{y\in S^h(x)} f_y$ in $\bigoplus_{y\in S^h(x)} W_0^\ell(y_-y)$ with $\sum_{y\in S^h(x)} f_y(y_-) = 0$. Therefore, we are in the same situation as in Lemma 10.9. We will now introduce symmetric bilinear forms in order to precisely apply this Lemma.

Let $k \geq 1$ and L be a weakly nonnegative k-pseudokernel and still let K^j, $j \geq k$ denote the associated dual prekernels. By Lemma 9.7 and Lemma 10.7, for any $h \geq 1$ and x in X, we have

$$q_x^{2k+2h-2} = \sum_{y \in S^h(x)} (H_{xy}^{k-1,*})^\star r_{y_- y}^{+L},$$

where $q_x^{2k+2h-2}$ is the symmetric bilinear form associated to $K_x^{2k+2h-2}$ on $V_0^{k+h-1}(x)$. By assumption, this symmetric bilinear form is nonnegative. By Lemma 10.9, at most one of the symmetric bilinear forms $r_{y_- y}^{+L}$, $y \in S^h(x)$, admits negative vectors, which should be proved. □

10.4. A mixing argument

In this section, we will strenghten Lemma 10.10 by showing that, if L is a weakly negative Γ-invariant pseudokernel, the set N_L of its negative edges must be empty. This will require us to study a certain linear operator acting on the space of Γ-invariant functions on edges.

Thus, let $\mathscr{V}_1$ be the vector space of all Γ-invariant functions on the set of edges X_1 of X. As $\Gamma\backslash X$ is finite, $\mathscr{V}_1$ has finite dimension. We define an endomorphism of $\mathscr{V}_1$ by setting, for φ in $\mathscr{V}_1$ and $x \sim y$ in X,

$$T\varphi(x,y) = \frac{1}{d(y)-1} \sum_{\substack{z \sim y \\ z \neq x}} \varphi(y,z).$$

Note that $T\mathbf{1} = \mathbf{1}$.

We will start by describing the adjoint operator of T. To this aim, we will need to define a convenient scalar product on $\mathscr{V}_1$. In order to deal with the case where Γ stabilizes an edge of X, we will use the following elementary combinatorial result:

LEMMA 10.11. *Let A be a set and G be a group acting on A such that, for any a in A, its stabilizer G_a in G is finite. Then, for any non-negative G-invariant function φ on A^2, we have*

$$\sum_{(a,b) \in G\backslash A^2} \frac{1}{|G_a \cap G_b|} \varphi(a,b) = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \overline{\varphi}(a),$$

where, for any a in A, $\overline{\varphi}(a) = \sum_{b \in A} \varphi(a,b)$.

Recall that, with the notation of the Lemma, if φ is a G-invariant function on A, $\sum_{a \in G\backslash A} \varphi(a)$ means the sum of φ on a system of representatives in A of the elements of $G\backslash A$ (when this makes sense). See [**3**] for related volume formulas in quotients of trees by discrete subgroups.

For any x in X, we still denote by Γ_x the stabilizer of x in Γ, which is a finite subgroup of Γ. We define a scalar product on $\mathscr{V}_1$ by setting, for any φ, ψ in $\mathscr{V}_1$,

$$\langle \varphi, \psi \rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \varphi(x,y)\psi(x,y)$$
$$= \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \sum_{y\sim x} \varphi(x,y)\psi(x,y),$$

where the latter equality follows from Lemma 10.11.

LEMMA 10.12. *The adjoint operator of T with respect to the scalar product on $\mathscr{V}_1$ is the operator $T^\dagger$ such that, for any ψ in E and $x \sim y$ in X,*

$$T^\dagger \psi(x,y) = \frac{1}{d(x)-1} \sum_{\substack{z\sim x \\ z\neq y}} \psi(z,x).$$

Note that again $T^\dagger \mathbf{1} = \mathbf{1}$.

PROOF. Let temporarily S stand for the operator defined in the statement of the Lemma. For any φ, ψ in $\mathscr{V}_1$, we have, by Lemma 10.11,

$$\langle T\varphi, \psi \rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \frac{1}{d(y)-1} \sum_{\substack{z\sim y \\ z\neq x}} \varphi(y,z)\psi(x,y)$$
$$= \sum_{y\in\Gamma\backslash X} \frac{1}{|\Gamma_y|} \frac{1}{d(y)-1} \sum_{\substack{x,z\sim y \\ x\neq z}} \varphi(y,z)\psi(x,y).$$

The same argument shows that the latter quantity also equals $\langle \varphi, S\psi \rangle$, which should be proved. □

We are now ready to prove

LEMMA 10.13. *Let $N \subset X_1$ be a set of oriented edges of X which meets the spheres at most once. If N is Γ-invariant, it is empty.*

The intuition of the Lemma is that the graph $\Gamma\backslash X$ is a discrete analogue of a compact negatively curved Riemannian manifold. The geodesic flow of such a manifold has mixing properties which imply equidistribution properties of large spheres. Thus, the images in $\Gamma\backslash X$ of large spheres in X should satisfy an equidistribution property. We now make this precise.

PROOF. As every x in X has at least three neighbours, for any $n \geq 1$, we have, for $x \sim y$ in X,

$$T^n \mathbf{1}_N(x,y) \leq 2^{-n} |\{z \in S^n(y) | (z_-, z) \in N\}| \leq 2^{-n},$$

hence $T^n \mathbf{1}_N \xrightarrow[n\to\infty]{} 0$ in $\mathscr{V}_1$. Now, as $T^\dagger \mathbf{1} = \mathbf{1}$ by Lemma 10.12, we get $\langle \mathbf{1}, \mathbf{1}_N \rangle = 0$, that is by definition,

$$\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \mathbf{1}_N(x,y) = 0$$

and $N = \emptyset$. □

As announced, we can now conclude the

PROOF OF PROPOSITION 10.6. Let L be Γ-invariant k-pseudokernel with associated dual prekernels K^j, $j \geq k$.

Assume the dual prekernels K^j, $k \leq j \leq 2k-3$ are non-negative. If the $(2k-1)$-pseudokernel L^{2k-1} is non-negative, by Corollary 10.8, the dual prekernels K^j, $j \geq 2k-2$ are non-negative, hence L is weakly non-negative.

Conversely, assume L to be weakly non-negative and let N_L be its set of negative edges. By Lemma 10.10, N_L meets the spheres at most once. Now, as L is Γ-invariant, N_L is a Γ-invariant subset of X_1. Thus, by Lemma 10.13, N_L is empty, which should be proved. □

10.5. A geometric criterion for image kernels

We will now use Proposition 10.6 to prove Theorem 10.3. A key argument in the proof will be

LEMMA 10.14. *Let $k \geq 2$ be even, L be a $(k-1)$-pseudokernel and (K, K^-) be a non-negative k-dual kernel. Assume that the k-pseudokernel L^+ is non-negative and that the k-dual kernel $(K, K^-) + L$ is exact. Then L is non-negative.*

PROOF. Set $\ell = \frac{k}{2}$. As usual, for x in X denote by q_x^K the symmetric bilinear form associated with K_x on $V_0^\ell(x)$ and, for $x \sim y$ in X, denote by $q_{xy}^{K^-}$ and r_{xy}^L the symmetric bilinear forms associated to K_{xy}^- and L_{xy} on $V_0^{\ell-1}(xy)$. Note that the symmetric bilinear form $r_{xy}^{L^+}$ associated to L_{xy}^+ on $V_0^\ell(x)$ is defined by

$$r_{xy}^{L^+} = \sum_{\substack{z\sim x \\ z\neq y}} (I_{xz}^{\ell-1,*})^\star r_{xz}^L.$$

As $(K, K^-) + L$ is exact, we have

$$(I_{xy}^{\ell-1,*})_\star(q_x^K + \sum_{z\sim x}(I_{xz}^{\ell-1,*})^\star r_{xz}^L) = q_{xy}^{K^-} + r_{xy}^L + r_{yx}^L,$$

which gives, by Lemma A.6,

$$(I_{xy}^{\ell-1,*})_\star(q_x^K + r_{xy}^{L^+}) = q_{xy}^{K^-} + r_{yx}^L,$$

As L^+ is non-negative, by Lemma A.5, we get

$$(I_{xy}^{\ell-1,*})_\star q_x^K + (I_{xy}^{\ell-1,*})_\star r_{xy}^{L^+} \leq q_{xy}^{K^-} + r_{yx}^L,$$

hence

$$r_{yx}^L \geq (I_{xy}^{\ell-1,*})_\star q_x^K - q_{xy}^{K^-}.$$

As (K, K^-) is non-negative, we have $(I_{xy}^{\ell-1,*})_\star q_x^K \geq q_{xy}^{K^-}$ and therefore r_{yx}^L is non-negative, which should be proved. □

Note that, if k is even and if L is a k-pseudokernel, L is non-negative if and only if L^+ is. Therefore, by an easy induction argument which relies on Propositon 5.16 and Proposition 6.21, we get

Corollary 10.15. *Let $k \geq 2$, L be a $(k-1)$-pseudokernel and (K, K^-) be a non-negative k-dual kernel. Assume that the j-pseudokernel L^j is non-negative for some $j \geq k-1$ and that the k-dual kernel $(K, K^-) + L$ is exact. Then L is non-negative.*

Together with Proposition 10.6, this gives

Corollary 10.16. *Let $k \geq 2$, L be a weakly non-negative Γ-invariant $(k-1)$-pseudokernel and (K, K^-) be a non-negative Γ-invariant k-dual kernel. Assume that the k-dual kernel $(K, K^-) + L$ is exact. Then the $(k-1)$-pseudokernel L is non-negative.*

We are now ready to conclude the

Proof of Theorem 10.3. Let (K, K^-) be as in the setting a Γ-invariant non-negative k-dual kernel. As in Proposition 5.18, we let L^{K,K^-} be the space of distributions associated to (K, K^-) equipped with its natural non-negative symmetric bilinear form q^{K,K^-}. We chose a Γ-invariant weight function w for (K, K^-). By Theorem 8.6, we have $H_0^\omega \subset L^{K,K^-}$ and the restriction of q^{K,K^-} to H_0^ω is the bilinear form Φ_w from Chapter 3. We let (H, H^-) be the image k-dual kernel of Φ_w, as in Definition 8.14. By Theorem 8.17, w is a weight function of (H, H^-). Therefore, by Corollary 9.2, there exists L in $\mathcal{L}_{k-1}$ with $(H, H^-) = (K, K^-) + L$.

We claim that (H, H^-) is the unique image kernel in $(K, K^-) + \mathcal{L}_{k-1}$ indeed, let w' be a symmetric Γ-invariant function on X_k such that

$\Phi_{w'}$ is non-negative on H_0^ω. If the image dual kernel of w' belongs to $(H, H^-) + \mathcal{L}_{k-1}$, by Corollary 9.2 and Lemma 3.14, we have $\Phi_w = \Phi_{w'}$, hence the image dual kernel of $\Phi_{w'}$ is (H, H^-).

We will now show that the $(k-1)$-pseudokernel L is non-negative.

First, we show that it is weakly non-negative, as in Definition 10.4. As usual, for $j \geq k-1$, we set H^j and K^j, to be the j-dual prekernels associated to (H, H^-) and (K, K^-) by successive orthogonal extensions. We claim that $H^j - K^j$ is a non-negative dual prekernel. Indeed, assume that j is even, $j = 2\ell$, $\ell \geq 1$. Fix x in X and let, as is Section 5.4, N_x^ℓ be the natural linear operator $V^\ell(x) \hookrightarrow \mathcal{D}(\partial X)$. By definition, we have $q^{K,K^-} \geq (N_x^{\ell,*})^\star q_x^{K^j}$ on L^{K,K^-} and, by Lemma 8.15, $q_x^{H^j} = (N_x^{\ell,*})_\star \Phi_w$. Thus, we get $q_x^{H^j} \geq q_x^{K^j}$ as required. The proof is analogous in the odd case.

Now, as (H, H^-) is exact (see Lemma A.4), and (K, K^-) is non-negative, by Corollary 10.16, L is a non-negative $(k-1)$-pseudokernel, that is, L belongs to the cone $\mathcal{L}_k^+$. This finishes the proof of the first part of the Proposition. The second part follows easily. □

10.6. The harmonic kernel

As an example of the use of Theorem 10.3, we will now apply it to show that the harmonic kernel (χ, χ^-) from Section 5.5 is an image kernel. Recall that this 2-dual kernel is defined by

$$\chi_x(y,z) = 2\frac{d(x)-1}{d(x)}, \quad x \in X, \quad y \neq z \in S^1(x),$$
$$\chi_{xy}(x,y) = 1, \quad x \sim y \in X.$$

By Proposition 5.21, the harmonic kernel is Euclidean.

Proposition 10.17. *The harmonic kernel is an image kernel.*

The proof will use the following elementary extension of Lemma 5.20, which follows from a straight forward computation using Lemma A.10.

Lemma 10.18. *Let A be a finite set with at least two elements, V_0 be the vector space of functions with zero sum on A and u be a positive function on A. We set q to be the symmetric bilinear form*

$$(f,g) \mapsto \sum_{a \in A} u(a) f(a) g(a)$$

on V_0. Then, for every a in A, if e_a is the evaluation linear functional $f \mapsto f(a)$ on V_0, one has

$$(e_a)_\star q = u(a)\left(1 - \frac{1}{u(a)S}\right)^{-1},$$

where $S = \sum_{b\in A} \frac{1}{u(b)}$.

Proof of Proposition 10.17. By Theorem 10.3, we must show that, if L is a Γ-invariant non-negative 1-pseudokernel such that the 2-dual kernel $(\chi, \chi^-) + L$ is non-negative, then $L = 0$. Now, for any $x \sim y$ in X, the space $V_0^0(xy)$ is a line spanned by the vector $\mathbf{1}_y - \mathbf{1}_x$ and the linear operator $I_{xy}^{0,*}$ sends a function f in $V_0^1(x)$ to $f(y)(\mathbf{1}_y - \mathbf{1}_x)$. Therefore, by Lemma 5.20, Definition 6.16 and Lemma 10.18, we must show that, if u is a Γ-invariant non-negative function on X_1 such that, for any $x \sim y$ in X, one has

$$u(x,y) + \frac{d(x)-1}{d(x)} \geq (1 + u(x,y) + u(y,x))\left(1 - \frac{1}{\left(u(x,y) + \frac{d(x)-1}{d(x)}\right)S(x)}\right), \tag{10.1}$$

where

$$S(x) = \sum_{z\sim x} \frac{1}{u(x,z) + \frac{d(x)-1}{d(x)}},$$

then necessarily $u = 0$.

Let us prove this claim. We let u and S be as above. From (10.1), we get, for (x, y) in X_1,

$$1 + u(x,y) + u(y,x) \geq \left(u(x,y) + \frac{d(x)-1}{d(x)}\right)\left(u(y,x) + \frac{1}{d(x)}\right)S(x). \tag{10.2}$$

By setting

$$S(x,y) = \sum_{\substack{z\sim x \\ z\neq y}} \frac{1}{u(x,z) + \frac{d(x)-1}{d(x)}},$$

we have

$$\left(u(x,y) + \frac{d(x)-1}{d(x)}\right)S(x) = 1 + \left(u(x,y) + \frac{d(x)-1}{d(x)}\right)S(x,y)$$

and (10.2) becomes

$$1 - \frac{1}{d(x)} + u(xy) \geq \left(u(x,y) + \frac{d(x)-1}{d(x)}\right)\left(u(y,x) + \frac{1}{d(x)}\right)S(x,y),$$

or, equivalently, as $u(x,y) + \frac{d(x)-1}{d(x)} > 0$,

$$\frac{1}{S(x,y)} \geq u(y,x) + \frac{1}{d(x)}. \tag{10.3}$$

Set $m = \max_{(x,y)\in X_1} u(x,y)$. For $x \sim y$ in X_1, we have

$$S(x,y) \geq \frac{d(x)-1}{m + \frac{d(x)-1}{d(x)}},$$

hence, from (10.3),

$$m + \frac{1}{d(x)} \leq \frac{m + \frac{d(x)-1}{d(x)}}{d(x)-1} = \frac{m}{d(x)-1} + \frac{1}{d(x)}.$$

As $d(x) \geq 3$, we get $m = 0$, which should be proved. □

We have just proved that the harmonic kernel is an image kernel or, equivalently by Corollary 8.19, that the space H_0^ω is dense in the Hilbert space of distributions H^{χ,χ^-} associated to (χ, χ^-). In Proposition 11.21 below, we will show that these two spaces are actually equal.

10.7. Non-negativity of the canonical pseudokernel

Before concluding this chapter, for further use, we relate non-negativity of a dual kernel with non-negativity of its canonical pseudokernel.

PROPOSITION 10.19. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with canonical k-pseudokernel L.*

If (K, K^-) is non-negative, then L is non-negative.

If (K, K^-) is Γ-invariant and L is non-negative, then there exists a Γ-invariant $(k-1)$-pseudokernel M, with associated k-dual kernel (J, J^-), such that $(J + K, J^- + K^-)$ is non-negative.

See Definition 6.11 and 6.12 for the canonical pseudokernel of a dual kernel.

The first statement of the Proposition is obvious. We focus on the second one. As usual, we denote by K^j, $j \geq k-1$, the dual prekernels obtained from (K, K^-) by successive orthogonal extensions. A first step towards the proof is

LEMMA 10.20. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with canonical k-pseudokernel L. If L is non-negative, then, for every $h \geq 0$, the $(h+k)$-pseudokernel $K^{h+k} - K^{-}{}^{>h+1}$ is non-negative.*

Note that in this lemma and its proof, we use the convention of Remark 6.13 to identify dual prekernels with a subspace of pseudokernels.

PROOF. We prove by induction on $h \geq 0$ that the statement is true for any $k \geq 2$.

For $h = 0$, this is the definition of a non-negative k-dual kernel.

If the statement is true for $h \geq 0$, we fix a k-dual kernel (K, K^-) and we consider its orthogonal extension (K^+, K). By Lemma 6.14, the canonical $(k+1)$-pseudokernel of (K^+, K) is L^+, which is non-negative. By the induction assumption, which we apply to (K^+, K), the $(h+k)$-pseudokernel $K^{h+k+1} - K^{>^{h+1}}$ is non-negative. By assumption, $L = K - K^{->}$ is non-negative, hence $L^{>^{h+1}} = K^{>^{h+1}} - K^{->^{h+2}}$ is non-negative. The conclusion follows. □

By going to the limit as $h \to \infty$, this gives

COROLLARY 10.21. *Let $k \geq 2$ and (K, K^-) be a Γ-invariant k-dual kernel with canonical k-pseudokernel L. Assume that L is non-negative. Let w be a Γ-invariant weight function of (K, K^-) and θ be in H_0^ω.*

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have

$$\Phi_w(\theta, \theta) \geq q_{xy}^{K^-}(N_{xy}^{\ell-1,*}\theta, N_{xy}^{\ell-1,*}\theta),$$

where $q_{xy}^{K^-}$ is the symmetric bilinear form on $V_0^{\ell-1}(xy)$ associated with K^-.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any x in X, we have

$$\Phi_w(\theta, \theta) \geq q_x^{K^-}(N_x^{\ell,*}\theta, N_x^{\ell,*}\theta),$$

where $q_x^{K^-}$ is the symmetric bilinear form on $V_0^\ell(x)$ associated with K^-.

In this Corollary, we have used freely the language of Section 3 and Section 5.

PROOF. Assume k is even. By Lemma 10.20, for every $h \geq \ell$, we have $q_x^{K^{2h}}(N_x^{h,*}\theta, N_x^{h,*}\theta) \geq q_{xy}^{K^-}(N_{xy}^{\ell-1,*}\theta, N_{xy}^{\ell-1,*}\theta)$. The result now follows from Corollary 8.9.

The proof in the odd case is analogous. □

The next lemma will show that the right hand-side of the inequalities in Corollary 10.21 can be chosen to be close to 0.

LEMMA 10.22. *Let θ be in H_0^ω. Then, as γ leaves finite subsets of Γ, the distribution $\gamma\theta$ converges weakly to 0 in the Hilbert space H_0^ω.*

PROOF. Indeed, by construction (see Section 3.1), the space H_0^ω may be seen as a closed subspace of $\ell^2(X_1)$ where the statement clearly holds true. □

We can now conclude.

PROOF OF PROPOSITION 10.19. By definition, if (K, K^-) is non-negative, then L is non-negative.

Conversely, suppose (K, K^-) is Γ-invariant and L is non-negative. Let us assume for example that k is even, $k = 2\ell$, $\ell \geq 1$. We fix $x \sim y$ in X and θ in H_0^ω. As Φ_w is Γ-invariant, for every γ in Γ, by Corollary 10.21, we have

$$\Phi_w(\theta, \theta) \geq q_{xy}^{K^-}(N_{xy}^{\ell-1,*}\gamma\theta, N_{xy}^{\ell-1,*}\gamma\theta).$$

As the bounded linear map $N_{xy}^{\ell-1,*}$ on H_0^ω has finite-dimensional range, by Lemma 10.22, $N_{xy}^{\ell-1,*}\gamma\theta$ goes to 0 as γ leaves finite subsets of Γ and we get $\Phi_w(\theta, \theta) \geq 0$, that is, Φ_w is non-negative. By Theorem 8.17, there exists a non-negative k-dual kernel (H, H^-) which admits w as a weight function. By Theorem 9.1, $(H - K, H^- - K^-)$ is a associated to a $(k-1)$-pseudokernel, which should be proved. $\square$

CHAPTER 11

Admissible kernels

We have described the image kernels. These are the dual kernels which are the image kernels associated to a non-negative bilinear form Φ_w, where w is a symmetric Γ-invariant function on X_k. We will now focus on the case where Φ_w is coercive, that is where Φ_w defines on H_0^ω the same topology as the standard scalar product.

We will need to use again part of the language that was introduced in Chapter 4. Recall in particular that a k-Euclidean field is a k-quadratic field whose associated bilinear forms are positive definite (see Definition 4.18). To such a field, we have associated a k-dual kernel in Chapter 5, where such dual kernels are called Euclidean dual kernels (see Definition 5.12 and Definition 5.13). The data of a Euclidean field or of the associated Euclidean dual kernel are equivalent.

Definition 11.1. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field, with associated Euclidean k-dual kernel (K, K^-). We shall say that p and (K, K^-) are admissible if there exists a symmetric Γ-invariant function w on X_k such that Φ_w is coercive and (K, K^-) is the image dual kernel of w.

The purpose of this chapter is to give a criterion for a Γ-invariant Euclidean field to be admissible which only involves finite-dimensional spaces.

11.1. Convolution operators

In this section, we relate the fact that a Euclidean field is admissible with the fact that a certain convolution operator is bounded in $\ell^2(X_1)$. This will require us tu use again the language of Chapter 4.

Recall that X_* stands for the space of pairs (x, y) in X^2 with $x \neq y$ and X_1 for the pairs (x, y) in X^2 with $x \sim y$. If φ is a function on X_*, we will associate to φ an operator P_φ acting on skew-symmetric functions on X_1 as follows. Given a finitely supported skew-symmetric

function ψ on X_1, we set, for (x, y) in X_1,

$$P_\varphi\psi(x,y) = \sum_{\substack{(a,b)\in X_1\\ y,b\in[xa]}} \varphi(x,a)\psi(b,a) - \sum_{\substack{(a,b)\in X_1\\ x,b\in[ya]}} \varphi(y,a)\psi(b,a) - \frac{1}{2}(\varphi(x,y)+\varphi(y,x))\psi(x,y), \tag{11.1}$$

which by construction is a skew-symmetric function on X_1. Note that, if φ is symmetric, for any $x \neq y$ in X, if x_1 and y_1 are the neighbours of x and y on $[xy]$, we have

$$P_\varphi(\mathbf{1}_{yy_1} - \mathbf{1}_{y_1y})(x, x_1) = \varphi(x, y). \tag{11.2}$$

The operator P_φ was defined in order to warrant this latter property. Note also that if φ is Γ-invariant, the operator P_φ commutes with the action of Γ. In this case, we call P_φ the convolution operator of φ.

We let $\ell^2_-(X_1)$ denote the Hilbert space of skew-symmetric square-summable functions on X_1. By (standard) abuse of language, we shall say that P_φ is bounded on $\ell^2_-(X_1)$ if there exists a constant $C > 0$ such that, for every finitely supported skew-symmetric function ψ on X_1, one has $\|P_\varphi\psi\|_2 \leq C\|\psi\|_2$.

Now, let $k \geq 2$ and p be a k-Euclidean field. In Chapter 4 (see in particular Section 4.6), we have associated to p a symmetric function φ_p^∞ on $X_* = \{(x, y) \in X^2 | x \neq y\}$. This function describes the scalar product obtained from p on $\overline{\mathcal{D}}(\partial X)$ by successive orthogonal extensions. Here comes a criterion for p to be admissible.

PROPOSITION 11.2. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then p is admissible if and only if the convolution operator $P_{\varphi_p^\infty}$ is bounded in $\ell^2_-(X_1)$.*

PROOF. First assume that p is admissible. Then, by definition, there exists a Γ-invariant symmetric function w on X_k such that the Euclidean dual kernel (K, K^-) associated to p (see Section 5.1) is the image kernel of Φ_w (see Definition 8.14). Then, let Θ be the self-adjoint operator of H_0^ω which represents Φ_w. As Φ_w is coercive, Θ is invertible. For θ in $\mathcal{D}(\partial X)$, let θ^* be the element of H_0^ω which represents the bounded linear functional $T \mapsto T(\theta)$ on H_0^ω . By Theorem 8.6 and Theorem 8.17, saying that (K, K^-) is the image kernel of Φ_w amounts to saying that, for any θ_1, θ_2 in $\mathcal{D}(\partial X)$, one has

$$p^\infty(\theta_1, \theta_2) = \langle \Theta^{-1}\theta_1^*, \theta_2^* \rangle.$$

Now let $\mathcal{P}$ be the linear map defined in Section 3.1: by Lemma 3.5, $\mathcal{P}$ is an isomorphism from H_0^ω onto a closed subspace of $\ell^2_-(X_1)$. Let

Π be the orthogonal projection from $\ell^2_-(X_1)$ onto this subspace. By construction of the map $\mathcal{P}$, for any $x \sim y$ in X, for any T in H^ω_0, we have

$$T(\mathbf{1}_{U_{xy}}) = \mathcal{P}T(x,y) = \langle \mathcal{P}T, \mathbf{1}_{(x,y)} \rangle = \frac{1}{2}\langle \mathcal{P}T, \mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)} \rangle, \tag{11.3}$$

hence $\mathcal{P}(\mathbf{1}^*_{U_{xy}}) = \Pi(\frac{1}{2}(\mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}))$. Let Υ be the bounded operator of $\ell^2_-(X_1)$ such that, for ψ in $\ell^2_-(X_1)$, we have $\Upsilon\psi = \mathcal{P}(\Theta^{-1}T)$ where T is the distribution T in H^ω_0 with $\mathcal{P}(T) = \Pi\psi$. By construction, for any $a \sim b$ and $x \sim y$ in X with $b, y \in [ax]$, we have

$$\begin{aligned}\langle \Upsilon(\mathbf{1}_{(a,b)} - \mathbf{1}_{(b,a)}), (\mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}) \rangle &= 4p^\infty(\mathbf{1}_{U_{ab}}, \mathbf{1}_{U_{xy}}) \\ &= -4\varphi^\infty_p(a,x) = -2\langle P_{\varphi^\infty_p}(\mathbf{1}_{(a,b)} - \mathbf{1}_{(b,a)}), (\mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}) \rangle,\end{aligned}$$

where we have used (11.2). By linearity, we get $2P_{\varphi^\infty_p}\psi = -\Upsilon\psi$ for any skew-symmetric finitely supported function ψ on X_1 and hence $P_{\varphi^\infty_p}$ is bounded.

Let us keep the same notation and prove the converse statement. We now assume that $P_{\varphi^\infty_p}$ is a bounded endomorphism of $\ell^2_-(X_1)$. Note in particular that (11.2) implies that $P_{\varphi^\infty_p}$ is self-adjoint. Besides, by the description of the space $\mathcal{P}H^\omega_0$ in Lemma 3.5 and still by (11.2), a direct computation shows that the range of $P_{\varphi^\infty_p}$ is contained in $\mathcal{P}H^\omega_0$. Therefore, there exists a bounded self-adjoint operator Ξ of H^ω_0 such that, for any ψ in $\ell^2_-(X_1)$, one has $P_{\varphi^\infty_p}\psi = \mathcal{P}(\Xi T)$ where T is the distribution T in H^ω_0 with $\mathcal{P}(T) = \Pi\psi$. Pick θ in $\mathcal{D}(\partial X)$. By (11.3) above, as the $\mathbf{1}_{U_{xy}}$, $x \sim y \in X$, span $\mathcal{D}(\partial X)$ as a vector space, we can find a finitely supported skew-symmetric function ψ on X_1 with $\Pi\psi = \mathcal{P}\theta^*$. Now, (11.2) gives

$$p^\infty(\theta,\theta) = -\langle P_{\varphi^\infty_p}\psi, \psi \rangle = -\langle \Xi\theta^*, \theta^* \rangle.$$

In particular, there exists $C > 0$ such that

$$p^\infty(\theta,\theta) \leq C \left\|\theta^*\right\|^2 \tag{11.4}$$

for any θ in $\mathcal{D}(\partial X)$.

Let (K, K^-) be the Euclidean k-dual kernel associated to p as in Section 5.1 and H^{K,K^-} be the Hilbert space of distributions associated to (K, K^-) as in Section 5.4. We claim that the latter inequality implies the inclusion $H^{K,K^-} \subset H^\omega_0$ as spaces of distributions. Indeed, by Corollary 5.19, the space H^{K,K^-} is exactly the topological dual space of the space $\overline{\mathcal{D}}(\partial X)$, equipped with the scalar product p^∞. Hence, if T is a distribution in H^{K,K^-}, we can find $C' > 0$ with $T(\theta)^2 \leq C' p^\infty(\theta,\theta)$, $\theta \in \mathcal{D}(\partial X)$. From (11.4), we get $T(\theta)^2 \leq CC' \left\|\theta^*\right\|^2$, hence, for any skew-symmetric finitely supported

function ψ on X_1, $\langle \mathcal{P}T, \psi\rangle^2 \leq CC' \|\Pi\psi\|^2 \leq CC' \|\psi\|^2$. Therefore, $\mathcal{P}T$ belongs to $\ell^2(X_1)$, that is, T belongs to H_0^ω as claimed.

By Theorem 8.6, we know that we have $H_0^\omega \subset H^{K,K^-}$ and that the inclusion map is bounded. We have just proved that this inclusion map is surjective, so that by the open mapping theorem it is an isomophism of Banach spaces. Therefore, still by Theorem 8.6, if w is a weight function of (K, K^-), the bilinear form Φ_w is coercive. Finally, we note that, by Lemma B.7, as the dual kernel (K, K^-) is exact, the bilinear forms associated to (K, K^-) are the images of the scalar product of H^{K,K^-} by the natural surjective maps (see Definition 5.12 and Definition 5.13 for the notion of an exact kernel). We have shown that H^{K,K^-} is equal to H_0^ω and that the scalar product is a coercive bilinear form Φ_w, so that by definition, the Euclidean field is admissible. □

11.2. Quadratic pseudofields

We will now look for a condition to ensure that the convolution operator associated to a quadratic type function obtained by successive orthogonal extensions is bounded. This condition will use a recursive formula for such quadratic type functions. To state this formula, we will need to use a new vector space which can be seen as a concrete version of the dual space of the space $\mathcal{L}_k$ of Γ-invariant k-pseudokernels.

Recall that, for any $\ell \geq 0$ and any x in X (resp. any $x \sim y$ in X), the space $\overline{V}^\ell(x)$ (resp. $\overline{V}^\ell(xy)$) is the quotient space of the space of functions on $S^\ell(x)$ (resp. $S^\ell(xy)$) by the line of constant functions.

Fix $k \geq 1$. If k is odd, $k = 2\ell+1$, $\ell \geq 0$, a k-quadratic pseudofield is a family $(s_{xy})_{(x,y)\in X_1}$ such that, for any (x, y) in X_1, s_{xy} is a symmetric bilinear form on $\overline{V}^\ell(xy)$. If k is even, $k = 2\ell$, $\ell \geq 1$, a k-quadratic pseudofield is a family $(s_{xy})_{(x,y)\in X_1}$ such that, for any (x, y) in X_1, s_{xy} is a symmetric bilinear form on $\overline{V}^\ell(x)$. The space of all Γ-invariant k-quadratic pseudofields is denoted by $\mathcal{M}_k$. Let us identify $\mathcal{M}_k$ with the dual space of $\mathcal{L}_k$.

Let $s = (s_{xy})_{(x,y)\in X_1}$ be in $\mathcal{M}_k$ and L be in $\mathcal{L}_k$, that is s is a Γ-invariant k-quadratic pseudofield and L is a Γ-invariant k-pseudokernel. For any (x, y) in X_1, L defines a symmetric bilinear form r^L_{xy} on the dual space of the space where s_{xy} is defined. By making use of the quadratic duality from Appendix C, we get a well-defined real number $\langle r^L_{xy}, s_{xy}\rangle$ which comes from the duality between these spaces. Now, to define a duality between $\mathcal{L}_k$ and $\mathcal{M}_k$, we need to average these numbers over $\Gamma\backslash X_1$. As in Section 10.4, we just have to be careful to deal with the case where Γ fixes some edges in X.

Recall that, for any x in X, we denote by Γ_x the stabilizer of x in Γ, which is a finite subgroup of Γ. If s is in $\mathcal{M}_k$ and L is in $\mathcal{L}_k$, we set

$$\langle L, s\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle r^L_{xy}, s_{xy}\rangle. \tag{11.5}$$

By Lemma 10.11, we can also write

$$\langle L, s\rangle = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\sum_{y\sim x}\langle r^L_{xy}, s_{xy}\rangle. \tag{11.6}$$

From now on, we shall use this duality to identify $\mathcal{M}_k$ with the dual space of $\mathcal{L}_k$.

11.3. Operations on quadratic pseudofields

We now define the dual operations to the natural operations on pseudokernels introduced in Section 6.2.

Definition 11.3. Let $k \geq 1$ and s be a k-quadratic pseudofield.

If k is even, for any $x \sim y$ in X, we set $s^\vee_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}$. The k-quadratic pseudofield $s^\vee$ is called the reversal of s. The map $s \mapsto s^\vee$ is a linear automorphisms of the space of k-quadratic pseudofields.

If k is odd, for any $x \sim y$ in X, we set $s^\vee_{xy} = s_{yx}$. The k-quadratic pseudofield $s^\vee$ is called the inversion of s. This map $s \mapsto s^\vee$ is an involution of the space of k-quadratic pseudofields.

Definition 11.4. We define the direct restriction of quadratic pseudofields as follows. Let $k \geq 1$ and s be a $(k+1)$-quadratic pseudofield.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we set $s^<_{xy} = (I^\ell_{xy})^\star s_{xy}$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set $s^<_{xy} = (J^\ell_{xy})^\star s_{xy}$.

Remark 11.5. If V is a vector space and W is a subspace of V, the natural restriction map $\mathcal{Q}(V) \to \mathcal{Q}(W)$ is surjective. Hence direct restriction maps $(k+1)$-quadratic pseudofields onto k-quadratic pseudofields.

As in Lemma 6.8, we show

Lemma 11.6. *Let $k \geq 1$ and s be a $(k+2)$-quadratic pseudofield. We have $s^{<<\vee} = s^{\vee<<}$.*

Finally, we formulate the dual definition to Definition 6.7:

DEFINITION 11.7. Let $k \geq 1$ and s be a $(k+1)$-quadratic pseudofield. We define the reduction s^- of s by $s^- = s^{\vee<}$.

As announced, these operations are dual to the operations on quadratic pseudofields.

LEMMA 11.8. *Let $k \geq 1$, L be a Γ-invariant k-pseudokernel, s be a Γ-invariant k-quadratic pseudofield and t be a Γ-invariant $(k+1)$-quadratic pseudofield. We have*

$$\langle L^\vee, s\rangle = \langle L, s^\vee\rangle \text{ and } \langle L^>, t\rangle = \langle L, t^<\rangle.$$

The proof is closely related to the one of Lemma 10.12.

PROOF. Let $(r_{xy})_{(x,y)\in X_1}$ be the biliner forms associated with L.

Suppose k is odd, $k = 2\ell+1$, $\ell \geq 0$. By the duality formula (11.5), $\langle L^\vee, s\rangle$ is the sum over $\Gamma\backslash X_1$ of the Γ-invariant function

$$(x,y) \mapsto \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle r_{yx}, s_{xy}\rangle$$

whereas $\langle L, s^\vee\rangle$ is the sum over $\Gamma\backslash X_1$ of the Γ-invariant function

$$(x,y) \mapsto \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle r_{xy}, s_{yx}\rangle$$

and the first formula follows as the involution $(x,y) \mapsto (y,x)$ of X_1 induces an involution of $\Gamma\backslash X_1$. Besides, by the duality formula (11.6), $\langle L^>, t\rangle$ is the sum over $\Gamma\backslash X$ of the Γ-invariant function on X

$$x \mapsto \frac{1}{|\Gamma_x|}\sum_{y\sim x}\langle (I^{\ell,*}_{xy})^\star r_{xy}, t_{xy}\rangle = \frac{1}{|\Gamma_x|}\sum_{y\sim x}\langle r_{xy}, (I^\ell_{xy})^\star t_{xy}\rangle$$
$$= \frac{1}{|\Gamma_x|}\sum_{y\sim x}\langle r_{xy}, t^<_{xy}\rangle,$$

where the first equality follows from Lemma C.2. Thus, the second formula holds.

Suppose k is even, $k = 2\ell$, $\ell \geq 1$. By the duality formula (11.6), $\langle L^\vee, s\rangle$ is the sum over $\Gamma\backslash X$ of the Γ-invariant function

$$x \mapsto \frac{1}{|\Gamma_x|}\sum_{y\sim x}\sum_{\substack{z\sim x\\ z\neq y}}\langle r_{xz}, s_{xy}\rangle = \frac{1}{|\Gamma_x|}\sum_{\substack{y,z\sim x\\ y\neq z}}\langle r_{xz}, s_{xy}\rangle = \frac{1}{|\Gamma_x|}\sum_{z\sim x}\langle r_{xz}, s^\vee_{xz}\rangle$$

and the equality $\langle L^\vee, s\rangle = \langle L, s^\vee\rangle$ follows. Finally, by the duality formula (11.5) and Lemma C.2, $\langle L^>, t\rangle$ is the sum over $\Gamma\backslash X_1$ of the Γ-invariant function on X_1

$$(x,y) \mapsto \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle (J^{\ell,*}_{xy})^\star r_{xy}, s_{xy}\rangle = \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle r_{xy}, (J^\ell_{yx})^\star s_{xy}\rangle,$$

which is also equal to $\langle L, s^< \rangle$. □

As a direct consequence of Lemma 11.8, we get

COROLLARY 11.9. *The reduction operator $s \mapsto s^-, \mathcal{M}_{k+1} \to \mathcal{M}_k$ is the adjoint operator of the orthogonal extension operator $L \mapsto L^+, \mathcal{L}_k \to \mathcal{L}_{k+1}$.*

11.4. Quadratic transfer operators

Recall that our aim is to give a recursive formula for quadratic type functions obtained by successive orthogonal extensions. This formula will involve the powers of a linear operator acting on Γ-invariant quadratic pseudofields that we will now define. We call this operator the quadratic transfer operator as it is analogous to the transfer operators of hyperbolic dynamics studied for example in [**40**].

First, we need to introduce the dual operators to the ones of Section 4.2 in the presence of a Euclidean quadratic field.

Let $k \geq 2$ and p be a k-Euclidean quadratic field. Let x, y be in X with $x \sim y$. With p (and its orthogonal extensions) come Euclidean structures on the spaces $\overline{V}^\ell(x)$ and $\overline{V}^\ell(xy)$, for any $\ell \geq 0$. In particular, the injective linear operators $I^\ell_{xy} : \overline{V}^\ell(xy) \to \overline{V}^{\ell+1}(x)$ and $J^\ell_{xy} : \overline{V}^\ell(x) \to \overline{V}^\ell(xy)$ admit adjoint operators with respect to these Euclidean structures. We denote these adjoint operators as

$$I^{\ell,\dagger p}_{xy} : \overline{V}^{\ell+1}(x) \to \overline{V}^\ell(xy)$$
$$\text{and } J^{\ell,\dagger p}_{xy} : \overline{V}^\ell(xy) \to \overline{V}^\ell(x).$$

These are surjective operators which heavily depend on p.

LEMMA 11.10. *Let $k \geq 2$ and p be a k-Euclidean field. For any $\ell \geq \frac{k}{2}$, for any $x \sim y$ in X, we have*

$$J^{\ell,\dagger p}_{yx} J^\ell_{xy} = I^{\ell-1}_{yx} I^{\ell-1,\dagger p}_{xy}.$$

For any $\ell \geq \frac{k-1}{2}$, for any x in X and any y, z in $S^1(x)$ with $y \neq z$, we have

$$I^{\ell,\dagger p}_{xz} I^\ell_{xy} = J^\ell_{xz} J^{\ell,\dagger p}_{xy}.$$

PROOF. In the first case, this is a direct consequence of the fact that, under the assumptions, the scalar product of the space $\overline{V}^\ell(xy)$ is obtained from the scalar products on the subspaces $J^\ell_{xy}\overline{V}^\ell(x)$ and $J^\ell_{yx}\overline{V}^\ell(y)$ by orthogonal extension.

In the same way, in the second case, this follows from the fact that the scalar product of the space $\overline{V}^{\ell+1}(x)$ is obtained from the scalar

products on the subspaces $I_{xy}^{\ell}\overline{V}^{\ell}(xy)$, $y \sim x$, through orthogonal extension. □

If s is a $(k-1)$-quadratic pseudfield, we define a k-quadratic pseudofield $s^{>_p}$ as follows. If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we set $s^{>_p}_{xy} = (I_{xy}^{\ell-1,\dagger p})^{\star} s_{xy}$. If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we set $s^{>_p}_{xy} = (J_{xy}^{\ell,\dagger p})^{\star} s_{xy}$.

The defintion directly gives

LEMMA 11.11. *Let $k \geq 2$, p be a k-Euclidean field and s be a $(k-1)$-quadratic pseudofield. Then we have $s^{>_p<} = s$.*

As in Lemma 11.6, we have

LEMMA 11.12. *Let $k \geq 3$, p be a k-Euclidean field and s be a $(k-2)$-quadratic pseudofield. Then we have*

$$s^{\vee>_{p^-}>_p} = s^{>_{p^-}>_p\vee}.$$

Finally, from Definition 11.4 and Lemma 11.10, we get

COROLLARY 11.13. *Let $k \geq 2$, p be a k-Euclidean field with orthogonal extension p^+ and s be a k-quadratic field. We have*

$$s^{>_{p^+}\vee<} = s^{<\vee>_p}.$$

Let us now define the quadratic transfer operator T_p.

DEFINITION 11.14. Let $k \geq 2$ and p be a k-Euclidean field. We define the quadratic transfer operator T_p, acting on $(k-1)$-quadratic pseudofields: if s is a $(k-1)$-quadratic pseudofield, we set

$$T_p s = s^{\vee>_p\vee<}.$$

In other words, if k is even, $k = 2\ell$, $\ell \geq 1$, we have, for $x \sim y$ in X,

$$(T_p s)_{xy} = \sum_{\substack{z\sim x \\ z\neq y}} (I_{xz}^{\ell-1,\dagger p} I_{xy}^{\ell-1})^{\star} s_{zx}.$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X,

$$(T_p s)_{xy} = (J_{yx}^{\ell,\dagger p} J_{xy}^{\ell})^{\star} \sum_{\substack{z\sim y \\ z\neq x}} s_{yz}.$$

We will show later that, when p is Γ-invariant, it is is admissible if and only if the spectral radius of T_p on $\mathcal{M}_{k-1}$ is < 1. As a first step towards this result, let us study the behaviour of T_p under orthogonal extensions. The following statement follows directly from Corollary 11.13

COROLLARY 11.15. *Let $k \geq 2$, p be a k-Euclidean field with orthogonal extension p^+ and s be a k-quadratic pseudofield. Then, we have $T_{p^+}s = s^{\vee<\vee>_p}$, hence in particular $(T_{p^+}s)^- = T_p(s^-)$.*

COROLLARY 11.16. *Assume p to be Γ-invariant. Then the spectrum of T_{p^+} in $\mathcal{M}_k$ is the union of $\{0\}$ and the spectrum of T_p in $\mathcal{M}_{k-1}$.*

PROOF. Since the orthogonal extension operator is injective on pseudokernels, by Corollary 11.9, the reduction map $\varpi_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is surjective. Now, Corollary 11.15 implies that T_{p^+} is 0 on the null space of ϖ_k and that the endomorphism induced by T_{p^+} on $\mathcal{M}_k/\ker \varpi_k \simeq \mathcal{M}_{k-1}$ is conjugated to T_p. The result follows. □

11.5. Computing quadratic type functions

We will now give a formula for the quadratic type function associated to a Euclidean field. Thanks to this formula, we will be able to relate the question whether the associated convolution operator is bounded to the domination of the spectral radius of the quadratic transfer operator.

PROPOSITION 11.17. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Assume that the associated quadratic transfer operator has spectral radius < 1 on the finite-dimensional vector space $\mathcal{M}_{k-1}$ of Γ-invariant $(k-1)$-quadratic pseudofields. Then p is admissible.*

As we already said, the converse is also true, but we will prove it only later.

We now give our formula for the quadratic type function. To state it, we need to introduce a new notation in order to avoid some possible confusions. For $x \neq z$ in X and $\ell = d(x,z)$, we let $\mathbf{1}_z^x$ denote the indicator function of $\{z\}$, viewed as an element of the space $\overline{V}^\ell(x)$. In the same way, for $x \sim y$ and z in X, if $\ell = \min(d(x,z), d(x,y))$, we let $\mathbf{1}_z^{xy}$ denote the indicator function of $\{z\}$, viewed as an element of the space $\overline{V}^\ell(xy)$.

LEMMA 11.18. *Let $k \geq 2$ and p be a k-Euclidean field. Let a, b be in X with $j = d(a,b) \geq k$ and let $c_0 = a, c_1, \ldots, c_j = b$ be the geodesic path from a to b.*

If k is even, $k = 2\ell$, $\ell \geq 1$, we have

(11.7)

$$\varphi_p^\infty(a,b) = -p_{c_\ell}(\mathbf{1}_a^{c_\ell}, I^{\ell-1}_{c_\ell c_{\ell+1}} I^{\ell-1,\dagger p}_{c_{\ell+1}c_\ell} I^{\ell-1}_{c_{\ell+1}c_{\ell+2}} \cdots I^{\ell-1}_{c_{j-\ell-1}c_{j-\ell}} I^{\ell-1,\dagger p}_{c_{j-\ell}c_{j-\ell-1}} \mathbf{1}_b^{c_{j-\ell}}).$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we have

$$\varphi_p^\infty(a,b) = -p_{c_\ell c_{\ell+1}}(\mathbf{1}_a^{c_\ell c_{\ell+1}}, J^\ell_{c_{\ell+1}c_\ell} J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+2}c_{\ell+1}} \cdots J^\ell_{c_{j-\ell}c_{j-\ell-1}} J^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}}). \tag{11.8}$$

Given p a k-Euclidean field, for a, b in X with $d(a,b) = k$, we directly know how to compute the scalar product between $\mathbf{1}_a$ and $\mathbf{1}_b$ with respect to p. The formulas in the lemma say that, when the distance is greater than k, in order to compute the scalar product between $\mathbf{1}_a$ and $\mathbf{1}_b$ with respect to large orthogonal extensions of the Euclidean field p, we can use the operators of Section 4.2 and their Euclidean adjoint operators with respect to p to let the vector $\mathbf{1}_b$ travel along the segment $[ab]$ until it is close enough to $\mathbf{1}_a$.

Proof. We fix j and we prove this result by descending induction on k with $2 \leq k \leq j$. For $k = j$, the result is the very definition of $\varphi_p^\infty(a,b) = \varphi_p(a,b)$. Now, assume $k \leq j-1$ and the result is true for $k+1$. By definition, we have $\varphi_p^\infty(a,b) = \varphi_{p^+}^\infty(a,b)$, so that we can apply the induction assumption to compute this number.

If k is even, $k = 2\ell$, the induction assumption tells us that $-\varphi_p^\infty(a,b)$ is the $p^+_{c_\ell c_{\ell+1}}$-scalar product of the vectors $\mathbf{1}_a^{c_\ell c_{\ell+1}}$ and

$$J^\ell_{c_{\ell+1}c_\ell} J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+2}c_{\ell+1}} \cdots J^\ell_{c_{j-\ell}c_{j-\ell-1}} J^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}}$$

in the space $\overline{V}^\ell(c_\ell c_{\ell+1})$. Now, we have

$$\mathbf{1}_a^{c_\ell c_{\ell+1}} = J^\ell_{c_\ell c_{\ell+1}} \mathbf{1}_a^{c_\ell} \text{ and } \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}} = J^\ell_{c_{j-\ell}c_{j-\ell-1}} \mathbf{1}_b^{c_{j-\ell}}.$$

Therefore, by the definition of the adjoint operators, $-\varphi_p^\infty(a,b)$ is the p_{c_ℓ}-scalar product of the vectors $\mathbf{1}_a^{c_\ell}$ and

$$J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+1}c_\ell} J^{\ell,\dagger p}_{c_\ell c_{\ell+1}} J^\ell_{c_{\ell+2}c_{\ell+1}} \cdots J^\ell_{c_{j-\ell}c_{j-\ell-1}} J^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell}} J^\ell_{c_{j-\ell}c_{j-\ell-1}} \mathbf{1}_b^{c_{j-\ell}}$$

in the space $\overline{V}^\ell(c_\ell)$. By Lemma 11.10, for $\ell \leq h \leq j-\ell-1$, we have

$$J^{\ell,\dagger p}_{c_h c_{h+1}} J^\ell_{c_{h+1}c_h} = I^{\ell-1}_{c_h c_{h+1}} I^{\ell-1,\dagger p}_{c_{h+1}c_h},$$

and (11.7) follows.

In the same way, if k is odd, $k = 2\ell+1$, the induction assumption tells us that $-\varphi_p^\infty(a,b)$ is the $p^+_{c_\ell}$-scalar product of the vectors $\mathbf{1}_a^{c_{\ell+1}}$ and

$$I^\ell_{c_{\ell+1}c_{\ell+2}} I^{\ell,\dagger p}_{c_{\ell+2}c_{\ell+1}} I^\ell_{c_{\ell+2}c_{\ell+3}} \cdots I^\ell_{c_{j-\ell-2}c_{j-\ell-1}} I^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell-2}} \mathbf{1}_b^{c_{j-\ell-1}}$$

in the space $\overline{V}^{\ell+1}(c_{\ell+1})$. As we have

$$\mathbf{1}_a^{c_{\ell+1}} = I^\ell_{c_{\ell+1}c_\ell} \mathbf{1}_a^{c_\ell c_{\ell+1}} \text{ and } \mathbf{1}_b^{c_{j-\ell-1}} = I^\ell_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}},$$

by the definition of the adjoint operators, $-\varphi_p^\infty(a,b)$ is the $p_{c_\ell c_{\ell+1}}$-scalar product of the vectors $\mathbf{1}_a^{c_\ell c_{\ell+1}}$ and

$$I^{\ell,\dagger p}_{c_{\ell+1}c_\ell} I^{\ell}_{c_{\ell+1}c_{\ell+2}} I^{\ell,\dagger p}_{c_{\ell+2}c_{\ell+1}} I^{\ell}_{c_{\ell+2}c_{\ell+3}} \cdots I^{\ell,\dagger p}_{c_{j-\ell-1}c_{j-\ell-2}} I^{\ell}_{c_{j-\ell-1}c_{j-\ell}} \mathbf{1}_b^{c_{j-\ell-1}c_{j-\ell}}$$

in the space $\overline{V}^\ell(c_\ell c_{\ell+1})$. By Lemma 11.10, for $\ell \le h \le j-\ell-2$, we have

$$I^{\ell,\dagger p}_{c_{h+1}c_h} I^{\ell}_{c_{h+1}c_{h+2}} = J^{\ell}_{c_{h+1}c_h} J^{\ell,\dagger p}_{c_{h+1}c_{h+2}}$$

and (11.8) follows. □

The main idea of the proof of Proposition 11.17 is to use quadratic transfer operators in order to give a simpler form of (11.7) and (11.8).

Proof of Proposition 11.17. Assume k is even, $k = 2\ell$, $\ell \ge 1$ and let us define a Γ-invariant $(k-1)$-quadratic pseudofield s. For any $x \sim y$ in X, we set s_{xy} to be the symmetric bilinear form on $\overline{V}^{\ell-1}(xy)$ such that, for f in $\overline{V}^{\ell-1}(xy)$, one has

$$s_{xy}(f,f) = \sum_{\substack{a \in S^\ell(x) \\ y \notin [xa]}} p_y(I^{\ell-1}_{xy} f, \mathbf{1}_a)^2 = \sum_{\substack{a \in S^\ell(x) \\ y \notin [xa]}} p^-_{xy}(f, I^{\ell-1,\dagger p}_{xy} \mathbf{1}_a)^2.$$

By using Definition 11.14, where the quadratic transfer operator is constructed, and Lemma 11.18, we get, for any b in $S^\ell(y)$ with $x \notin [yb]$, for any $n \ge 0$,

$$(T_p^n s)_{xy}(I^{\ell-1,\dagger p}_{yx} \mathbf{1}_b, I^{\ell-1,\dagger p}_{yx} \mathbf{1}_b) = \sum_{\substack{a \in S^{k+n+1}(b) \\ [xy] \subset [ab]}} \varphi_p^\infty(a,b)^2.$$

Therefore, if T_p has spectral radius < 1, we can find $\rho < 1$ such that

$$\sum_{(a,b) \in \Gamma \backslash X_*} \rho^{-d(a,b)} \varphi_p^\infty(a,b)^2 < \infty$$

(recall that $\Gamma\backslash X$ is finite). By Haagerup inequality (see Proposition E.3), the convolution operator $P_{\varphi_p^\infty}$ is bounded in $\ell^2_-(X_1)$. By Proposition 11.2, the Euclidean field p is admissible.

Assume k is odd, $k = 2\ell+1$, $\ell \ge 1$. Now, we define a Γ-invariant $(k-1)$-quadratic pseudofield s by setting, for every $x \sim y$ in X and every f in $\overline{V}^\ell(x)$,

$$s_{xy}(f,f) = \sum_{\substack{b \in S^\ell(y) \\ x \notin [yb]}} p_{xy}(J^{\ell}_{xy} f, \mathbf{1}_b)^2 = \sum_{\substack{b \in S^\ell(y) \\ x \notin [yb]}} p^-_{x}(f, J^{\ell,\dagger p}_{xy} \mathbf{1}_b)^2.$$

Using again Definition 11.14 and Lemma 11.18 yields, for any a in $S^{\ell+1}(x)$ with $y \notin [xa]$, for any $n \geq 0$, if z is the neighbour of x on $[ax]$,

$$(T_p^n s)_{xy}(J_{xz}^{\ell,\dagger p}\mathbf{1}_a, J_{xz}^{\ell,\dagger p}\mathbf{1}_a) = \sum_{\substack{b\in S^{k+n+1}(a)\\ [xy]\subset[ab]}} \varphi_p^\infty(a,b)^2.$$

As above, if T_p has spectral radius < 1, we can find $\rho < 1$ such that

$$\sum_{(a,b)\in\Gamma\backslash X_*} \rho^{-d(a,b)}\varphi_p^\infty(a,b)^2 < \infty$$

and Proposition E.3 and Proposition 11.2 give the conclusion. □

11.6. The harmonic field

As an example and for further use, we will apply the previous constructions and results to the harmonic kernel from Sections 5.5 and 10.6.

We start by giving a more explicit definition of the adjoint operators in case $k = 2$. We keep using the notation of Lemma 11.18.

LEMMA 11.19. *Let p be a 2-Euclidean quadratic field. For any $x \sim y$ in X and any f in $\overline{V}^1(x)$, we have*

$$I_{xy}^{0,\dagger p} f = \frac{p_x(f, \mathbf{1}_y^x)}{p_{xy}^-(\mathbf{1}_y^{xy}, \mathbf{1}_y^{xy})}\mathbf{1}_y^{xy}.$$

PROOF. Indeed, in case $k = 2$, $\overline{V}^0(xy)$ is a line which is spanned by $\mathbf{1}_y^{xy}$, I_{xy}^0 sends $\mathbf{1}_y^{xy}$ to $\mathbf{1}_y^x$ and we have by definition $p_x(\mathbf{1}_y^x, \mathbf{1}_y^x) = p_{xy}^-(\mathbf{1}_y^{xy}, \mathbf{1}_y^{xy})$. □

COROLLARY 11.20. *Let p be a 2-Euclidean field. Then the associated quadratic type function φ_p^∞ on X_* may be computed as follows. Fix $x \neq y$ in X.*

If $d(x,y) = 1$, one has $\varphi_p^\infty(x,y) = \varphi_{p^-}(x,y) = p_{xy}^-(\mathbf{1}_x^{xy}, \mathbf{1}_x^{xy})$.

If $d(x,y) = 2$, one has $\varphi_p^\infty(x,y) = \varphi_p(x,y) = -p_z(\mathbf{1}_x^z, \mathbf{1}_y^z)$, where z is the middle point of $[xy]$.

In general, if $j = d(x,y) \geq 2$ and $x = z_0, z_1, \ldots, z_j = y$ is the geodesic path from x to y, one has

$$\varphi_p^\infty(x,y) = \frac{\prod_{h=1}^{j-1}\varphi_p(z_{h-1}, z_{h+1})}{\prod_{h=1}^{j-2}\varphi_{p^-}(z_h, z_{h+1})}.$$

These formulas are closely related to the ones appearing in the work of Młotkowski [**37**].

PROOF. For $j = 1, 2$ this is the definition of φ_p^∞. Now, for $j \geq 3$, note that we have, by Lemma 11.19,

$$I^{0,\dagger p}_{z_{j-1}z_{j-2}} \mathbf{1}_y^{z_{j-1}} = \frac{p_{z_{j-1}}(\mathbf{1}_y^{z_{j-1}}, \mathbf{1}_{z_{j-2}}^{z_{j-1}})}{p^-_{z_{j-1}z_{j-2}}(\mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}}, \mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}})} \mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}}$$
$$= \frac{\varphi_p(z_{j-2}, y)}{\varphi_{p^-}(z_{j-2}, z_{j-1})} \mathbf{1}_{z_{j-1}}^{z_{j-1}z_{j-2}},$$

where, in the second equality, we have used the relation $\mathbf{1}_{z_{j-2}}^{z_{j-1}z_{j-2}} + \mathbf{1}_{z_{j-1}}^{z_{j-1}z_{j-2}} = 0$. We obtain

$$I^0_{z_{j-2}z_{j-1}} I^{0,\dagger p}_{z_{j-1}z_{j-2}} \mathbf{1}_y^{z_{j-1}} = \frac{\varphi_p(z_{j-2}, y)}{\varphi_{p^-}(z_{j-2}, z_{j-1})} \mathbf{1}_{z_{j-1}}^{z_{j-2}}.$$

By Lemma 11.18, this gives

$$\varphi_p^\infty(x, y) = \varphi_p^\infty(x, z_{j-1}) \frac{\varphi_p(z_{j-2}, y)}{\varphi_{p^-}(z_{j-2}, z_{j-1})},$$

whence the result. □

Now, we go back to the harmonic kernel (χ, χ^-) of Section 5.5. We let π be the associated 2-Euclidean field, which we call the harmonic field.

PROPOSITION 11.21. *The harmonic field is admissible. The associated quadratic transfer operator T_π on the space $\mathcal{M}_1$ of Γ-invariant 1-quadratic pseudofields has spectral radius $\leq \frac{1}{2}$.*

PROOF. We will apply Proposition 11.17. To this aim, we need to say more on the quadratic transfer operator T_π. Let s be a 1-quadratic pseudofield and, for (x, y) in X_1, set $u(xy) = s_{xy}(\mathbf{1}_x^{xy}, \mathbf{1}_x^{xy}) = s_{xy}(\mathbf{1}_y^{xy}, \mathbf{1}_y^{xy})$. Let p be a 2-Euclidean field. By Lemma 11.19, for x in X and y, z neighbours of x, we have

$$I^{0,\dagger p}_{xz} I^0_{xy} \mathbf{1}_y^{xy} = \frac{p_x(\mathbf{1}_y^x, \mathbf{1}_z^x)}{p^-_{xz}(\mathbf{1}_z^{xz}, \mathbf{1}_z^{xz})} \mathbf{1}_z^{xz}.$$

Therefore, by Definition 11.14, we can identify T_p with the operator that sends a function u on X_1 to the function

$$(x, y) \mapsto \sum_{\substack{z \sim x \\ z \neq y}} \frac{p_x(\mathbf{1}_y^x, \mathbf{1}_z^x)^2}{p^-_{xz}(\mathbf{1}_z^{xz}, \mathbf{1}_z^{xz})^2} u(zx).$$

Now, for the harmonic field π, by Lemma 5.20, we have

$$\pi_x(\mathbf{1}_y^x, \mathbf{1}_y^x) = 1 \text{ and } \pi_x(\mathbf{1}_y^x, \mathbf{1}_z^x) = -\frac{1}{d(x) - 1} \text{ if } y \neq z.$$

Thus T_π may be seen as the operator that sends a function u on X_1 to the function

$$(x, y) \mapsto \frac{1}{(d(x)-1)^2} \sum_{\substack{z \sim x \\ z \neq y}} u(zx).$$

With respect to the uniform norm on functions on X_1, this operator has norm $\leq \sup_{x \in X} \frac{1}{d(x)-1} \leq \frac{1}{2}$, hence it has spectral radius $\leq \frac{1}{2}$. By Proposition 11.17, the Euclidean field π is admissible. □

11.7. Tangent dual kernels

We now aim at establishing the converse of Theorem 11.17, namely that if a Γ-invariant Euclidean kernel is admissible, its quadratic transfer operator has spectral radius < 1. Our argument is differential geometric and requires us to compute the tangent space of the space of Euclidean kernels, viewed as a submanifold of the vector space of dual kernels. This is the purpose of this section.

Fix $k \geq 2$. We denote by $\mathcal{P}_k$ the set of all Γ-invariant k-Euclidean fields. The space $\mathcal{P}_k$ is an open subset of the finite-dimensional vector space of all Γ-invariant k-quadratic fields. In particular, it comes with a natural manifold structure. As explained in Section 5.1, there is a natural injective map $\mathcal{P}_k \hookrightarrow \mathcal{K}_k$ where, as in Chapter 9, $\mathcal{K}_k$ stands for the space of Γ-invariant k-dual kernels. It will turn out that this map is an immersion and that we can describe the tangent spaces of its range. To do this we again need to define a family of linear operators.

Let p be a k-Euclidean field. Then, for $\ell \geq 0$ and $x \sim y$ in X, p defines a Euclidean structure on the spaces $V_0^\ell(x)$ and $V_0^\ell(xy)$ that is dual to the Euclidean structure on the spaces $\overline{V}^\ell(x)$ and $\overline{V}^\ell(xy)$. Now, we have linear surjective operators $I_{xy}^{\ell,*} : V_0^{\ell+1}(x) \to V_0^\ell(xy)$ and $J_{xy}^{\ell,*} : V_0^\ell(xy) \to V_0^\ell(x)$. We define the operators

$$I_{xy}^{\ell,*\dagger p} : V_0^\ell(xy) \to V_0^{\ell+1}(x) \tag{11.9}$$

$$\text{and } J_{xy}^{\ell,*\dagger p} : V_0^\ell(x) \to V_0^\ell(xy) \tag{11.10}$$

as being the adjoints of these operators with respect to the Euclidean structure p. They are injective operators which can also be seen as the adjoint operators, with respect to the duality, of the above introduced operators $I_{xy}^{\ell,\dagger p}$ and $J_{xy}^{\ell,\dagger p}$.

PROPOSITION 11.22. *Let $k \geq 2$. The natural map $\mathcal{P}_k \hookrightarrow \mathcal{K}_k$ is an immersion. Fix p in $\mathcal{P}_k$ and let $\mathrm{T}_p\mathcal{P}_k$ denote the tangent space of $\mathcal{P}_k$, viewed as a subspace of $\mathcal{K}_k$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, then $\mathrm{T}_p\mathcal{P}_k$ is the space of all Γ-invariant k-dual kernels whose associated bilinear forms $(q_x)_{x\in X}$ and $(q^-_{xy})_{x\sim y\in X}$ satisfy the relations

$$(I^{\ell-1,*\dagger p}_{xy})^\star q_x = q^-_{xy} = (I^{\ell-1,*\dagger p}_{yx})^\star q_y, \quad x \sim y \in X.$$

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, then $\mathrm{T}_p\mathcal{P}_k$ is the space of all Γ-invariant k-dual kernels whose associated bilinear forms $(q_{xy})_{x\sim y\in X}$ and $(q^-_x)_{x\in X}$ satisfy the relations

$$(J^{\ell,*\dagger p}_{xy})^\star q_{xy} = q^-_x = (J^{\ell,*\dagger p}_{xz})^\star q_{xz}, \quad x \in X, \quad y, z \sim x.$$

PROOF. As $\mathcal{P}_k$ is an open subspace of the vector space of all k-quadratic fields, its tangent space may be identified with this vector space. Now, fix p in $\mathcal{P}$. The Euclidean structures associated with p on the spaces $\overline{V}^\ell(x)$ and $\overline{V}^\ell(xy)$, $\ell \geq 0$, $x \sim y$ in X, give rise to isomorphisms between these spaces and the spaces $V^\ell_0(x)$ and $V^\ell_0(xy)$. These isomorphisms conjugate the linear maps I^ℓ_{xy} and J^ℓ_{xy} with the above defined linear maps $I^{\ell,*\dagger p}_{xy}$ and $J^{\ell,*\dagger p}_{xy}$. The conclusion follows from these facts and standard considerations on the tangent space of the space of scalar products on a finite-dimensional vector space. □

11.8. The adjoint quadratic transfer operator

Our goal is still to prove that a Euclidean field p is admissible if and only if the associated quadratic transfer operator T_p has spectral radius < 1. We will actually need to use the adjoint operator of the quadratic transfer operator which we will now describe. Recall that we have identified the dual space of the space $\mathcal{M}_k$ of k-quadratic pseudofields with the space $\mathcal{L}_k$ of k-pseudokernels. In order to describe the adjoint operator of T_p, we first introduce adjoint operations to the maps $s \mapsto s^{>_p}$.

Let $k \geq 2$ and p be a k-Euclidean field. Fix a k-pseudokernel $L = (L_{xy})_{(x,y)\in X_1}$ with associated bilinear forms $(r_{xy})_{(x,y)\in X_1}$. Using the notation of (11.9) and (11.10), we define a $(k-1)$-pseudokernel $L^{<_p}$ as follows:
— if k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, the bilinear form associated with $L^{<_p}_{xy}$ on $V^{\ell-1}_0(xy)$ is $(I^{\ell-1,*\dagger p}_{xy})^\star r_{xy}$.
— if k is odd, $k = 2\ell+1$, $\ell \geq 1$, for $x \sim y$ in X, the bilinear form associated with $L^{<_p}_{xy}$ on $V^\ell_0(x)$ is $(J^{\ell,*\dagger p}_{xy})^\star r_{xy}$.

Reasoning as in the proof of Lemma 11.8, we show

LEMMA 11.23. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. If L is a Γ-invariant k-pseudokernel and s is a Γ-invariant $(k-1)$-quadratic pseudofield, we have $\langle L^{<_p}, s\rangle = \langle L, s^{>_p}\rangle$.*

We can now determine the adjoint operator to the quadratic transfer operator of Definition 11.14.

COROLLARY 11.24. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The adjoint operator T_p^* of the quadratic transfer operator $T_p : \mathcal{M}_{k-1} \to \mathcal{M}_{k-1}$ is given by, for L in $\mathcal{L}_{k-1}$,*

$$T_p^* L = L^{>^\vee <_p{}^\vee}.$$

REMARK 11.25. We can write the definition of T_p^* by means of the notation of (11.9) and (11.10). Let $L = (L_{xy})_{(x,y)\in X_1}$ be in $\mathcal{L}_{k-1}$ and let $(r_{xy})_{(x,y)\in X_1}$ be the associated bilinear forms.

If k is even, $k = 2\ell$, $\ell \geq 1$, then $T_p^* L$ is the $(k-1)$-pseudokernel with associated bilinear forms defined by, for (x, y) in X_1,

$$(T_p^* r)_{xy} = \sum_{\substack{z\sim y \\ z\neq x}} (I_{yz}^{\ell-1,*} I_{yx}^{\ell-1,*\dagger p})^\star r_{yz} = (I_{yx}^{\ell-1,*\dagger p})^\star \sum_{\substack{z\sim y \\ z\neq x}} (I_{yz}^{\ell-1,*})^\star r_{yz}.$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, then $T_p^* L$ is the $(k-1)$-pseudokernel with associated bilinear forms defined by, for (x, y) in X_1,

$$(T_p^* r)_{xy} = \sum_{\substack{z\sim x \\ z\neq y}} (J_{zx}^{\ell,*} J_{xz}^{\ell,*\dagger p})^\star r_{zx}.$$

We summarize the computation of the tangent space of $\mathcal{P}_k$ and the definition of the adjoint quadratic transfer operator.

PROPOSITION 11.26. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean kernel. Then, as subspaces of $\mathcal{L}_{k-1}$, we have*

$$\mathcal{L}_{k-1} \cap \mathrm{T}_p \mathcal{P}_k = \ker(T_p^* - 1).$$

In the proposition and its proof, we use the construction of the k-dual kernel associated to a $(k-1)$-pseudokernel from Section 6.4 to consider $\mathcal{L}_{k-1}$ as a subspace of $\mathcal{K}_k$.

PROOF. We pick L in $\mathcal{L}_{k-1}$ which we view as a family $(r_{xy})_{(x,y)\in X_1}$ of symmetric bilinear forms.

Asume k is even, $k = 2\ell$, $\ell \geq 1$. Then by Definition 6.16 and Proposition 11.22, saying that r is tangent to $\mathcal{P}_k$ at p is saying that, for any (x, y) in X_1, one has

$$(I_{xy}^{\ell-1,*\dagger p})^\star \sum_{z\sim x} (I_{xz}^{\ell-1,*})^\star r_{xz} = r_{xy} + r_{yx}.$$

As, by construction, $I_{xy}^{\ell-1,*}I_{xy}^{\ell-1,*\dagger p}$ is the identity operator of $V_0^{\ell-1}(xy)$, this is equivalent to saying that

$$(I_{xy}^{\ell-1,*\dagger p})^\star \sum_{\substack{z\sim x\\ z\neq y}} (I_{xz}^{\ell-1,*})^\star r_{xz} = r_{yx},$$

which, by Corollary 11.24 and Remark 11.25, reads as $(T_p^* r)_{yx} = r_{yx}$.

Now, if k is odd, $k = 2\ell+1$, $\ell \geq 1$, by Definition 6.17 and Proposition 11.22, saying that r is tangent to $\mathcal{P}_k$ at p is saying that, for any (x,y) in X_1, one has

$$(J_{xy}^{\ell,*\dagger p})^\star ((J_{xy}^{\ell,*})^\star r_{xy} + (J_{yx}^{\ell,*})^\star r_{yx}) = \frac{1}{d(x)-1}\sum_{z\sim x} r_{xz}.$$

As above, by construction, $J_{xy}^{\ell,*}J_{xy}^{\ell,*\dagger p}$ is the identity operator of $V_0^\ell(x)$, so that this is equivalent to saying that

$$(J_{yx}^{\ell,*}J_{xy}^{\ell,*\dagger p})^\star r_{yx} = \frac{1}{d(x)-1}\sum_{\substack{z\sim x\\ z\neq y}} r_{xz} - \frac{d(x)-2}{d(x)-1} r_{xy}. \tag{11.11}$$

For (x,y) in X_1, we sum (11.11) applied to the pairs (x,z) with $z\sim x$, $z\neq y$. We get

$$\sum_{\substack{z\sim x\\ z\neq y}} (J_{zx}^{\ell,*}J_{xz}^{\ell,*\dagger p})^\star r_{zx} = \frac{1}{d(x)-1}\sum_{\substack{z\sim x\\ z\neq y}}\sum_{\substack{t\sim x\\ t\neq z}} r_{xt} - \frac{d(x)-2}{d(x)-1}\sum_{\substack{z\sim x\\ z\neq y}} r_{xz} = r_{xy}.$$

Thus, if r is in $\mathrm{T}_p\mathcal{P}_k$, we have $T_p^* r = r$. Conversely, if $T_p^* r = r$, the same computation shows that (11.11) holds and hence that r is in the tangent space $\mathrm{T}_p\mathcal{P}_k$. □

11.9. The weight map as a diffeomorphism

We are ready to state and prove

THEOREM 11.27. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean kernel. Then p is admissible if and only if the quadratic transfer operator T_p has spectral radius < 1 on the space $\mathcal{M}_{k-1}$ of $(k-1)$-quadratic pseudofields.*

In the course of the proof, we shall use a classical generalization of the Perron-Frobenius Theorem. Recall that, if V is a finite-dimensional vector space, a closed convex cone $\mathcal{C} \subset V$ is said to be proper if it does not contain any vector line.

LEMMA 11.28. *Let V be a finite-dimensional real vector space and T be an endomorphism of V which preserves a proper closed convex cone*

$\mathcal{C}$ of V with nonempty interior, that is, $T\mathcal{C} \subset \mathcal{C}$. Then the spectral radius of T is an eigenvalue of T associated to an eigenvector in $\mathcal{C}$.

PROOF. If T is nilpotent, there is nothing to prove. Else, let $\rho > 0$ be the spectral radius of T. By replacing T with $\rho^{-1}T$, we can assume $\rho = 1$. Let V' be the subspace of V whose complexification is the sum of all eigenspaces of T associated with eigenvalues of modulus 1 of T. Then T preserves V' and the closure in $\mathrm{GL}(V')$ of the sub-semigroup spanned by the restriction T' of T to V' is a compact subgroup K of $\mathrm{GL}(V')$.

Fix any norm on V. It follows from the Jordan reduction of T that there exists a proper subspace W of V with $T^{-1}W = W$ and, for any v in $V \smallsetminus W$, any limit point in V of $\frac{1}{\|T^n v\|}T^n v$ belongs to V'.

Now, as $\mathcal{C}$ has non-empty interior, we can pick such a v in $\mathcal{C}$. Therefore, the closed convex cone $\mathcal{C}' = \mathcal{C} \cap V'$ is non-zero. Pick v' in $\mathcal{C}'$. Then $v'' = \int_K kv' \mathrm{d}k$, the average of kv' with respect to the Haar measure of K, is K-invariant. To conclude, it suffices to prove that $v'' \neq 0$. But, by the Hahn-Banach Theorem, $\mathcal{C}'$ being a proper closed convex cone in V', there exists a linear functional φ on V' which is positive on $\mathcal{C}' \smallsetminus \{0\}$. As, for any k in K, kv' belongs to $\mathcal{C}'$, we have $\varphi(kv') > 0$, hence $\varphi(v'') > 0$ and $v'' \neq 0$. The result follows. □

In our case, the quadratic transfer operators preserve a natural convex cone. We say that a quadratic pseudofield is non-negative if all the associated symmetric bilinear forms are non-negative. From the fact that T_p is defined by taking sums of pull-back maps between vector spaces, we directly get

LEMMA 11.29. *Let $k \geq 2$, p be a k-Euclidean field and s be a non-negative $(k-1)$-quadratic pseudofield. Then $T_p s$ is non-negative.*

For $k \geq 1$, let $\mathcal{M}_k^+ \subset \mathcal{M}_k$ be the cone of Γ-invariant non-negative k pseudofields. This is a proper closed convex cone in $\mathcal{M}_k$ with non-empty interior.

PROOF OF THEOREM 11.27. Proposition 11.17 says that if T_p has spectral radius < 1, then p is admissible. Let us prove the converse statement. We will use the results of Chapter 8 to show that the weight map is a local diffeomorphism from the space of admissible kernels to the space of cohomology classes of functions and then conclude by using the study of the weight map from Chapter 9 and Proposition 11.26. Let us do this precisely.

As in Chapter 9, we let $\mathcal{W}_k$ stand for the space of cohomology classes of Γ-invariant symmetric functions on X_k and $W_k : \mathcal{K}_k \to \mathcal{W}_k$ for the weight map. We also let $\iota_k : \mathcal{P}_k \hookrightarrow \mathcal{K}_k$ denote the natural injection.

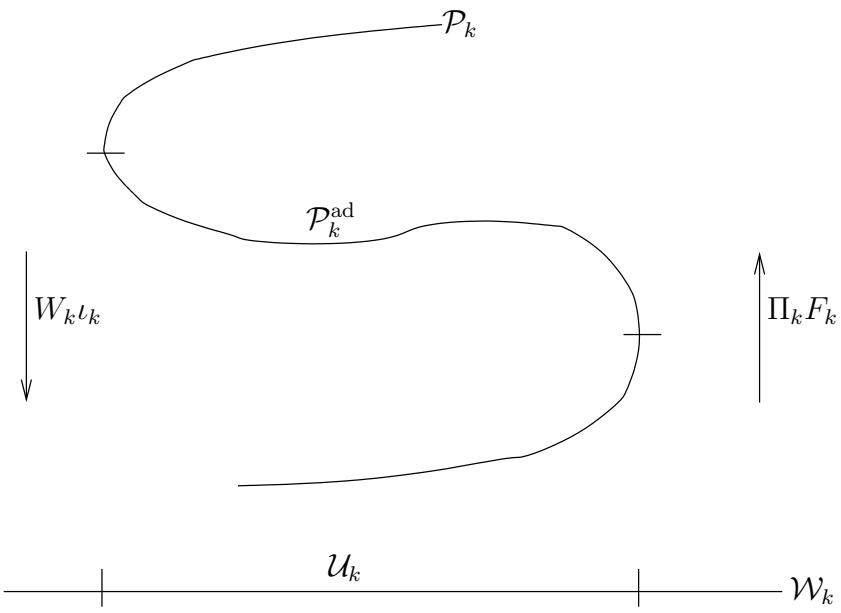

FIGURE 1. The objects in the proof of Theorem 11.27

Let us introduce maps that are related to the Hilbert space H_0^ω. We let $\mathcal{Q}^\infty(H_0^\omega)$ denote the space of continuous quadratic forms on the Hilbert space H_0^ω and $\mathcal{Q}^\infty_{++}(H_0^\omega) \subset \mathcal{Q}^\infty(H_0^\omega)$ denote the open subset of coercive positive quadratic forms. The image map $\Pi_k : \mathcal{Q}^\infty_{++}(H_0^\omega) \to \mathcal{P}_k$ is well-defined. Finally, we have a linear map $F_k : W_k \to \mathcal{Q}^\infty(H_0^\omega)$ that sends the cohomology class of a function w to the quadratic form Φ_w. We let $\mathcal{U}_k = F_k^{-1}\mathcal{Q}^\infty_{++}(H_0^\omega)$ be the open set of cohomology classes of those w such that Φ_w is coercive. To summarize, we have maps:

$$W_k\iota_k : \mathcal{P}_k \to \mathcal{W}_k \text{ and } \Pi_k F_k : \mathcal{U}_k \to \mathcal{P}_k \tag{11.12}$$

and we want to describe the set $\mathcal{P}_k^{\mathrm{ad}} = \Pi_k F_k(\mathcal{U}_k)$ of admissible Γ-invariant k-Euclidean kernels. This situation is pictured in Figure 1.

Here comes the key observation of the proof, that is, Theorem 8.17 says that

$$W_k\iota_k\Pi_k F_k w = w \tag{11.13}$$

for any w in $\mathcal{U}_k$. Now, let us notice that all the maps involved in (11.12) are smooth. Indeed, W_k and F_k are linear maps defined on finite-dimensional vector spaces and ι_k is smooth by Proposition 11.22, whereas Π_k is smooth by Proposition A.16. Therefore, (11.13) gives, by the chain-rule,

$$\mathrm{d}_{\Pi_k F_k w}(W_k\iota_k)\mathrm{d}_w(\Pi_k F_k) = \mathrm{Id}_{\mathcal{W}_k} \tag{11.14}$$

for w in $\mathcal{U}_k$. Set $p = \Pi_k F_k w$. As the map W_k is linear, we have $\mathrm{d}_p(W_k \iota_k) = W_k \mathrm{d}_p \iota_k$ and (11.14) implies that $W_k(\mathrm{T}_p \mathcal{P}_k) = \mathcal{W}_k$, that is W_k maps $\mathrm{T}_p \mathcal{P}_k$ onto $\mathcal{W}_k$.

Let us show that, for p in $\mathcal{P}_k^{\mathrm{ad}}$, W_k actually induces a linear isomorphism from $\mathrm{T}_p \mathcal{P}_k$ onto $\mathcal{W}_k$. As we have just shown this map to be surjective, this amounts to proving that both $\mathcal{P}_k$ and $\mathcal{W}_k$ have the same dimension. To do this, let π be the harmonic field, as in Section 11.6, which is a Γ-invariant 2-Euclidean field. We write π^k for the $(k-2)$-th orthogonal extension of π: this is a Γ-invariant k-Euclidean field. Proposition 11.21 says that π, and hence π^k, is admissible, and also that the quadratic transfer operator T_π has spectral radius $\leq \frac{1}{2}$ on $\mathcal{M}_1$. Then, it follows from Corollary 11.16, that T_{π^k} has spectral radius $\leq \frac{1}{2}$ on $\mathcal{M}_{k-1}$. By duality, the adjoint quadratic transfer operator $T^*_{\pi^k}$ has spectral radius $\leq \frac{1}{2}$ on $\mathcal{L}_{k-1}$. By Proposition 11.26, we have therefore $\mathcal{L}_{k-1} \cap \mathrm{T}_{\pi^k} \mathcal{P}_k = \{0\}$. By Corollary 9.2, $\mathcal{L}_{k-1}$ is exactly the null space of W_k, so that we have just shown that W_k is injective on $\mathrm{T}_{\pi^k} \mathcal{P}_k$, and hence that $\mathcal{P}_k$ and $\mathcal{W}_k$ have the same dimension.

Now we know that for p in $\mathcal{P}_k^{\mathrm{ad}}$, W_k is injective on $\mathrm{T}_p \mathcal{P}_k$, that is, still by Corollary 9.2, $\mathcal{L}_{k-1} \cap \mathrm{T}_p \mathcal{P}_k = \{0\}$. By Proposition 11.26, 1 is not an eigenvalue of the quadratic transfer operator T_p. By Lemma 11.29, the operator T_p preserves the cone $\mathcal{M}^+_{k-1} \subset \mathcal{M}_{k-1}$ of Γ-invariant nonnegative $(k-1)$-quadratic pseudofields. Hence, by Lemma 11.28, since 1 is not an eigenvalue of T_p, the spectral radius of T_p is $\neq 1$. As $\mathcal{Q}^\infty_{++}(H^\omega_0)$ is convex and F_k is linear, the open set $\mathcal{U}_k = F_k^{-1} \mathcal{Q}^\infty_{++}(H^\omega_0) \subset \mathcal{W}_k$ is convex and hence $\mathcal{P}_k^{\mathrm{ad}} = \Pi_k F_k \mathcal{U}_k$ is connected, since Π_k is continuous on $\mathcal{Q}^\infty_{++}(H^\omega_0)$ by Proposition A.16. As T_p depends continuously on p on $\mathcal{P}_k$, so does the spectral radius of T_p. Now, for π^k the $(k-2)$-th orthogonal extension of the harmonic kernel, we have shown above that T_{π^k} has spectral radius < 1. Therefore, for any p in $\mathcal{P}_k^{\mathrm{ad}}$, T_p has spectral radius < 1, which should be proved. □

As a Corollary of the proof, we get

Corollary 11.30. *Let $k \geq 2$. Then the space $\mathcal{W}_k$ has the same dimension as the space of Γ-invariant k-quadratic fields. The set $\mathcal{P}_k^{\mathrm{ad}}$ of admissible Γ-invariant k-Euclidean fields is open in $\mathcal{P}_k$. The weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ induces a smooth diffeomorphism from $\mathcal{P}_k^{\mathrm{ad}}$ onto its image.*

Remark 11.31. The fact that $\mathcal{W}_k$ and the space of Γ-invariant k-quadratic fields have the same dimension also follows from the duality between these spaces established in Proposition 12.2 below.

Part 3

Non-negativity of weights

CHAPTER 12

The weight pairing

In the next chapters, we will show that, for $k \geq 2$, if w is a symmetric function on X_k and the bilinear form Φ_w is non-negative, then w is cohomologous to a non-negative function. This statement will be shown by using a dual property in spaces of quadratic pseudofields. In this chapter we start by describing a bilinear pairing between the space of Γ-invariant k-quadratic fields and the space of Γ-invariant k-dual kernels.

12.1. Invariant quadratic type functions

The construction of this pairing will rely on certain duality properties on the space of Γ-invariant functions on X_k, $k \geq 2$. To introduce these properties, we go back to the point of view of quadratic type functions from Section 4.1 and say a little more about Γ-invariant ones.

First, we have a natural surjectivity result:

LEMMA 12.1. *Let $k \geq 2$. The reduction map $\varphi \mapsto \varphi^-$ maps Γ-invariant quadratic type functions on X_k onto Γ-invariant quadratic type functions on X_{k-1}.*

To prove this, let us introduce some notation which extends the objects of Section 10.4. As usual, for $x \neq y$ in X, we let x_1 and y_1 denote the neighbours of x and y on $[xy]$.

Fix $k \geq 1$. We let $\mathscr{V}_k$ denote the finite-dimensional vector space of Γ-invariant functions on X_k. We define $S_k : \mathscr{V}_k \to \mathscr{V}_k$ as being the natural symmetry operator, $S_k v(x,y) = v(y,x)$, $v \in \mathscr{V}_k$, $(x,y) \in X_k$, and we set $\mathscr{V}_k^+ \subset \mathscr{V}_k$ and $\mathscr{V}_k^- \subset \mathscr{V}_k$ to be respectively the space of symmetric and skew-symmetric functions. We also let L_k and R_k be the left and right augmentation operators $\mathscr{V}_k \to \mathscr{V}_{k+1}$ defined by, for v in $\mathscr{V}_k$ and (x,y) in X_{k+1},

$$L_k v(x,y) = v(x_1,y) \text{ and } R_k v(x,y) = v(x,y_1).$$

Note that one has $S_{k+1}L_k = R_k S_k$ and $L_{k+1}R_k = R_{k+1}L_k$. Lastly, we equip $\mathscr{V}_k$ with the S_k-invariant scalar product $\langle .,. \rangle$ defined by, for u, v

in $\mathscr{V}_k$,

$$\langle u, v\rangle = \sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} u(x,y)v(x,y). \tag{12.1}$$

A direct computation using as usual Lemma 10.11 shows that, with respect to this scalar product, the adjoint maps of L_k and R_k are the reduction operators $L_k^\dagger$ and $R_k^\dagger$, defined by, for v in $\mathscr{V}_{k+1}$ and (x,y) in X_k,

$$L_k^\dagger v(x,y) = \sum_{\substack{z\sim x\\ z\neq x_1}} v(z,y) \text{ and } R_k^\dagger v(x,y) = \sum_{\substack{t\sim y\\ t\neq y_1}} v(x,t).$$

In particular, by Definition 4.2, a function φ in $\mathscr{V}_{k+1}$ has quadratic type if and only if it is symmetric and the function $L_k^\dagger\varphi$ in $\mathscr{V}_k$ is symmetric. By Lemma 4.3, one then has $L_k^\dagger\varphi = R_k^\dagger\varphi = \varphi^-$.

Let $\mathcal{F}_k$, $k \geq 2$, denote the finite-dimensional vector space of Γ-invariant k-quadratic fieds.

Proof of Lemma 12.1. Recall the reduction map $p \mapsto p^-$ of Section 4.2. For $k \geq 3$, we denote by $\rho_k : \mathcal{F}_k \to \mathcal{F}_{k-1}$ the reduction map of Γ-invariant k-quadratic fields. This is a linear map. If $k \geq 3$, by Proposition 4.21, for any Γ-invariant $(k-1)$-Euclidean field p, one has $(p^+)^- = p$. Hence the space $\rho_k(\mathcal{F}_k)$ contains the open subset of Euclidean fields in $\mathcal{F}_{k-1}$ (which is nonempty by Proposition 5.21). We get $\rho_k(\mathcal{F}_k) = \mathcal{F}_{k-1}$. The conclusion follows, by the identification of quadratic fields with quadratic type functions in Proposition 4.11 and Lemma 4.13.

It remains to prove the case where $k = 2$. Recall that $\mathscr{V}_2^+ \subset \mathscr{V}_2$ is the space of symmetric functions. We claim that $L_1^\dagger$ maps $\mathscr{V}_2^+$ onto $\mathscr{V}_1$. This amounts to proving that the adjoint map of the restriction of $L_1^\dagger$ to $\mathscr{V}_2^+$ is injective. As the orthogonal projection of $\mathscr{V}_2$ onto $\mathscr{V}_2^+$ is $\frac{1}{2}(1 + S_2)$, this adjoint map is $\frac{1}{2}(1 + S_2)L_1$. Let now u be in $\mathscr{V}_1$ with $\frac{1}{2}(1 + S_2)L_1u = 0$. For any x in X and $y \neq z$ in $S^1(x)$, we have $u(x,y) + u(x,z) = 0$. Pick a third neighbour t of x (which exists by assumption). We have $u(x,y) = -u(x,z) = u(x,t) = -u(x,y)$, hence $u = 0$ as required. Thus $L_1^\dagger(\mathscr{V}_2^+) = \mathscr{V}_1$. In particular, if φ is a symmetric function on X_1, we can find a symmetric function ψ on X_2 with $L_1^\dagger\psi = \varphi$. By definition, ψ has quadratic type and $\psi^- = \varphi$. □

Now, we can show that, among symmetric functions, the orthogonal complement of Γ-invariant quadratic type functions on X_k is the space of Γ-invariant functions that are coboundaries.

PROPOSITION 12.2. *Let $k \geq 1$ and w be a symmetric Γ-invariant function on X_k. Then one has $\langle w, \varphi \rangle = 0$ for every quadratic type function φ on X_k if and only if there exists a skew-symmetric Γ-invariant function v on X_{k-1} such that, for any (x, y) in X_k, $w(x, y) = v(x, y_1) - v(x_1, y)$.*

Note that we have exceptionnally denoted by X_0 the diagonal in $X \times X$. Skew-symmetric functions on X_0 are zero! The Proposition says that the space of Γ-invariant quadratic type functions on X_k may be seen as the dual space to the space of cohomology classes of symmetric Γ-invariant functions on X_k.

REMARK 12.3. In particular, Proposition 12.2 implies that the dimension of the space of Γ-invariant quadratic type functions on X_k grows exponentially with k. Since by Proposition 4.11, the set $\mathcal{P}_k$ of Γ-invariant Euclidean fields may be seen as an open subset in the latter space, as k grows, we are building more and more Γ-invariant scalar products on $\overline{\mathcal{D}}(\partial X)$, hence more and more unitary representations of Γ.

The proof of Proposition 12.2 relies on a classical phenomenon in duality that we state in the framework of Euclidean spaces.

LEMMA 12.4. *Let V and W be Euclidean spaces and $T : V \to W$ be a linear map and $X \subset W$ be a subspace. Then the orthogonal complement of $T^{-1}X$ is given by*

$$(T^{-1}X)^{\perp} = T^{\dagger}(X^{\perp}),$$

where $T^{\dagger} : W \to V$ is the Euclidean adjoint operator of T.

PROOF. If $Y \subset W$ is a subspace, we claim that $(T^{\dagger}Y)^{\perp} = T^{-1}(Y^{\perp})$. Indeed, for v in V, we have

$$v \in (T^{\dagger}Y)^{\perp} \Leftrightarrow (\forall y \in Y \quad \langle v, T^{\dagger}y \rangle = 0) \Leftrightarrow (\forall y \in Y \quad \langle Tv, y \rangle = 0) \Leftrightarrow Tv \in Y^{\perp}.$$

The resut follows by taking $Y = X^{\perp}$. □

PROOF OF PROPOSITION 12.2. If $k = 1$, a quadratic type function on X_1 is simply a symmetric function. Thus, by assumption, we have $\langle w, w \rangle = 0$, hence $w = 0$ and we are done.

Assume $k \geq 2$. Recall that $\mathscr{V}_k^+$ and $\mathscr{V}_k^-$ are the spaces of symmetric and skew-symmetric functions in $\mathscr{V}_k$. Thus, by Definition 4.2, the space of Γ-invariant quadratic type functions on X_k is

$$\mathscr{V}_k^+ \cap (L_{k-1}^{\dagger})^{-1} \mathscr{V}_{k-1}^+.$$

As $\mathscr{V}_k^-$ is the orthogonal complement of $\mathscr{V}_k^+$ in $\mathscr{V}_k$, Lemma 12.4 implies that w belongs to the space

$$\mathscr{V}_k^- + L_{k-1}(\mathscr{V}_{k-1}^-),$$

that is, we may write $w = u + L_{k-1}v$ where u and v are skew-symmetric functions on X_k and X_{k-1}. Now, w being symmetric, we get

$$w = S_k w = S_k u + S_k L_{k-1} v = -u + R_{k-1} S_{k-1} v = -u - R_{k-1} v,$$

hence

$$w = \frac{1}{2} w + \frac{1}{2} S_k w = \frac{1}{2} L_{k-1} v - \frac{1}{2} R_{k-1} v$$

and the result follows. □

12.2. The weight formula

For $k \geq 2$, we now introduce the weight pairing between Γ-invariant k-quadratic fields and Γ-invariant k-dual kernels. One of the main features of this pairing is that it can be defined by two natural formulas. We first prove that these two definitions are equivalent.

THEOREM 12.5. *Let $k \geq 2$. Let p be a Γ-invariant k-quadratic field and $\varphi_p : X_k \to \mathbb{R}$ be the associated quadratic type function. Let also (K, K^-) be a Γ-invariant k-dual kernel and $w : X_k \to \mathbb{R}$ be a Γ-invariant weight function of (K, K^-). If k is even, we have*

(12.2)

$$\sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle p_x, q_x^K \rangle - \frac{1}{2} \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}^-, q_{xy}^{K^-} \rangle = \frac{1}{2} \langle \varphi_p, w \rangle.$$

If k is odd, we have

(12.3)

$$\frac{1}{2} \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}, q_{xy}^K \rangle - \sum_{x \in \Gamma\backslash X} \frac{d(x) - 1}{|\Gamma_x|} \langle p_x^-, q_x^{K^-} \rangle = \frac{1}{2} \langle \varphi_p, w \rangle.$$

The reader should compare (12.2) and (12.3) with the formulas in Lemma 5.9.

As usual, for any $\ell \geq 1$, if K is a 2ℓ-dual prekernel, for x in X, we have denoted by q_x^K the symmetric bilinear form associated with K_x on $V_0^\ell(x)$. If K is a $(2\ell+1)$-dual prekernel, for $x \sim y$ in X, we have denoted by q_{xy}^K the symmetric bilinear form associated with K_{xy} on $V_0^\ell(xy)$. See Chapter 4 for the notions of a quadratic field and the associated quadratic type function. See Definition 7.4 for the notion of a weight function of a dual kernel. As in Section 12.1, we have denoted by $\langle ., . \rangle$ the natural scalar product on the space of Γ-invariant functions on X_k which has been defined by Equation (12.1). As in Appendix C,

we have also denoted by $\langle ., . \rangle$ the natural duality between the space of symmetric bilinear forms on a vector space and on its dual space.

DEFINITION 12.6. The bilinear pairing defined between dual kernels and quadratic fields in Theorem 12.5 will be called the weight pairing. We denote it by $(p, K, K^-) \mapsto [p, (K, K^-)]$.

From the elementary properties of Γ-invariant quadratic type functions, we get a nice compatibility property of the weight pairing with orthogonal extensions. Recall the notion of the reduction of a quadratic field from Section 4.2.

COROLLARY 12.7. *Let $k \geq 2$. Let p be a Γ-invariant $(k+1)$-quadratic field with reduction p^- and (K, K^-) be a Γ-invariant k-dual kernel with orthogonal extension (K^+, K). We have*

$$[p, (K^+, K)] = [p^-, (K, K^-)].$$

PROOF. We keep the notation from Section 12.1. Let $w : X_k \to \mathbb{R}$ be a Γ-invariant weight function for (K, K^-). Then, by Corollary 9.14, the function $\frac{1}{2}(R_k w + L_k w)$ is a weight function for (K^+, K). Now, note that, by Lemma 4.13, the quadratic type functions associated to p^- and p satisfy the relation $\varphi_{p^-} = (\varphi_p)^- = R_k^\dagger \varphi_p = L_k^\dagger \varphi_p$. Thus, we get

$$\begin{aligned}[p, (K^+, K)] = \frac{1}{4}\langle \varphi_p, R_k w + L_k w \rangle &= \frac{1}{4}\langle R_k^\dagger \varphi_p + L_k^\dagger \varphi_p, w \rangle \\ &= \frac{1}{2}\langle \varphi_{p^-}, w \rangle = [p^-, (K, K^-)],\end{aligned}$$

which should be proved. □

We now give the

PROOF OF THEOREM 12.5. We will use the study of the weight map in Chapter 9 to deduce the general case of Proposition 12.5 from particular ones.

First, we assume that (K, K^-) is the k-dual kernel associated with a $(k-1)$-pseudokernel L. In that case, we will prove that both hand-sides of (12.2) and (12.3) are zero.

Indeed, on one hand, Theorem 9.1 tells us that w is a coboundary, that is, there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that, for any (x, y) in X_1, one has $w(x, y) = v(x, y_1) - v(x_1, y)$ where x_1 and y_1 are the neighbours of x and y on $[xy]$. Thus, by Proposition 12.2, we have $\langle \varphi_p, w \rangle = 0$.

On the other hand, assume first that k is even, $k = 2\ell$. For $x \sim y$ in X, we let as usual r_{xy}^L be the symmetric bilinear form associated with

L_{xy} on $V_0^{\ell-1}(xy)$. By construction, we have $q_x^K = \sum_{y\sim x}(I_{xy}^{\ell-1,*})^\star r_{xy}^L$, hence, again using Lemma 10.11,

$$\sum_{x\in\Gamma\backslash X}\frac{1}{|\Gamma_x|}\langle p_x, q_x^K\rangle = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_x, (I_{xy}^{\ell-1,*})^\star r_{xy}^L\rangle.$$

By Lemma C.2, for $x\sim y$ in X, we have

$$\langle p_x, (I_{xy}^{\ell-1,*})^\star r_{xy}^L\rangle = \langle (I_{xy}^{\ell-1})^\star p_x, r_{xy}^L\rangle = \langle p_{xy}^-, r_{xy}^L\rangle,$$

hence

$$\sum_{x\in\Gamma\backslash X}\frac{1}{|\Gamma_x|}\langle p_x, q_x^K\rangle = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy}^-, r_{xy}^L\rangle.$$

Now, we also have, for $x\sim y$ in X, $q_{xy}^{K^-} = r_{xy}^L + r_{yx}^L$ and thus

$$\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy}^-, q_{xy}^{K^-}\rangle = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy}^-, r_{xy}^L\rangle$$

and the left hand-side of (12.2) is zero.

In the same way, if k is odd, $k = 2\ell+1$, by Lemma C.2 and Lemma 10.11, we have

$$\begin{aligned}\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy}, q_{xy}^K\rangle &= \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_x^-, r_{xy}^L\rangle\\ &= \sum_{x\in\Gamma\backslash X}\frac{1}{|\Gamma_x|}\sum_{y\sim x}\langle p_x^-, r_{xy}^L\rangle = \sum_{x\in\Gamma\backslash X}\frac{d(x)-1}{|\Gamma_x|}\langle p_x^-, q_x^{K^-}\rangle,\end{aligned}$$

where as usual, for $x\sim y$ in X, r_{xy}^L stand for the symmetric bilinear form associated with L_{xy} on $V_0^\ell(x)$.

This finishes the case where (K, K^-) is the dual kernel associated to a pseudokernel. In the general case, now, we will use again the dual kernel $(K^w, 0)$ from Section 9.1. Recall from Corollary 9.5 that w is a weight function of $(K^w, 0)$. Therefore, by Theorem 9.1, the dual kernel $(K, K^-) - (K^w, 0)$ is associated to a certain pseudokernel. As we have just shown that (12.2) and (12.3) are true for pseudokernels, it suffices to show that they are true when $(K, K^-) = (K^w, 0)$.

In that case, assume first that k is even, $k = 2\ell$. Then, for any x in X and y, z in $S^\ell(x)$, we have

$$\begin{aligned}&K_x(y,z) = w(y,z) \quad x\in[yz]\\ &K_x(y,z) = 0 \quad \text{else.}\end{aligned}$$

Therefore, by Lemma C.5,

$$\langle p_x, q_x^K \rangle = -\frac{1}{2} \sum_{\substack{(y,z)\in S^\ell(x)\times S^\ell(x)\\ x\in[yz]}} w(y,z) p_x(\mathbf{1}_y, \mathbf{1}_z)$$

$$= \frac{1}{2} \sum_{\substack{(y,z)\in S^\ell(x)\times S^\ell(x)\\ x\in[yz]}} w(y,z)\varphi_p(y,z).$$

Now, (12.2) follows from Lemma 10.11.

In the same way, if k is odd, $k = 2\ell + 1$, for any $x \sim y$ in X and z, t in $S^\ell(x)$, we have

$$K_{xy}(z,t) = w(z,t) \quad [xy] \subset [yz]$$
$$K_{xy}(z,t) = 0 \quad \text{else.}$$

Therefore, by Lemma C.5,

$$\langle p_{xy}, q_{xy}^K \rangle = \frac{1}{2} \sum_{\substack{(z,t)\in S^\ell(xy)\times S^\ell(xy)\\ [xy]\subset[zt]}} w(y,z)\varphi_p(z,t)$$

and (12.3) again follows from Lemma 10.11. □

CHAPTER 13

Radical pseudofields

Recall that quadratic pseudofields were introduced in Chapter 11.2. For $k \geq 1$, the space $\mathcal{M}_k$ of Γ-invariant k-quadratic pseudofields may be seen as the dual space of the space $\mathcal{L}_k$ of Γ-invariant k-pseudokernels. In this chapter, we introduce a new property of quadratic pseudofields, which will play a role in certain duality formulas.

13.1. Radical quadratic pseudofields and the shoot map

We define a new subclass of quadratic pseudofields.

DEFINITION 13.1. Let $k \geq 2$ and s be a k-quadratic pseudofield. We say that s is radical if $s^< = s^{\vee<\vee}$. If $k = 1$, by convention, any quadratic pseudofield is said to be radical.

The space of Γ-invariant radical k-quadratic pseudofields is denoted by $\mathcal{M}_k^1$.

The reduction of a radical quadratic pseudofield is again radical.

LEMMA 13.2. *Let $k \geq 2$ and s be a radical k-quadratic pseudofield. Then the $(k-1)$-quadratic pseudofield $s^- = s^{\vee<}$ is radical.*

PROOF. If $k = 2$, there is nothing to prove. If $k \geq 3$, we get

$$s^{-<} = s^{\vee<<} = s^{<<\vee} = s^{\vee<\vee<\vee} = s^{-\vee<\vee},$$

where we have used Lemma 11.6. The result follows. □

We will now use radical quadratic pseudofields to define quadratic fields.

LEMMA 13.3. *Let $k \geq 1$ and s be a radical k-quadratic pseudofield.*
If k is even, for x in X, set $p_x = \sum_{y \sim x} s_{xy}$.
If k is odd, for $x \sim y$, set $p_{xy} = s_{xy} + s_{yx}$.
In both cases, p is a quadratic field.

See Definition 4.7 and Definition 4.8 for the precise definition of a quadratic field.

DEFINITION 13.4. Let $k \geq 1$, s be a radical k-quadratic pseudofield and p be as in Lemma 13.3. Then the quadratic field p is called the shoot of s. The linear map $P_k : \mathcal{M}_k^1 \to \mathcal{F}_k$ that sends a Γ-invariant radical k-quadratic pseudofield to its shoot is called the shoot map.

REMARK 13.5. We can consider quadratic fields as being also quadratic pseudofields. Indeed, for $k \geq 1$, given a k-quadratic field p, we can identify p with the k-quadratic pseudofield r defined by, for $x \sim y$ in X, $r_{xy} = p_x$ if k is even, and $r_{xy} = p_{xy}$ if k is odd. Then, if s is a radical k-quadratic pseudofield, the shoot of s is $s + s^\vee$.

PROOF OF LEMMA 13.3. If $k = 1$, there is nothing to prove, p being a quadratic field only meaning that $p_{xy} = p_{yx}$, $x \sim y \in X$.

If $k \geq 2$, first note that, as s is radical, we have

$$(s + s^\vee)^< = s^< + s^{\vee <} = s^{\vee <\vee} + s^{\vee <} = s^{-\vee} + s^-.$$

If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we have $p_x = (s + s^\vee)_{xy}$, hence,

$$(I_{xy}^{\ell-1})^\star p_x = (I_{xy}^{\ell-1})^\star (s + s^\vee)_{xy} = (s + s^\vee)^<_{xy} = (s^- + s^{-\vee})_{xy} = (I_{yx}^{\ell-1})^\star p_y,$$

that is, p is a quadratic field.

Similarly, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we have $p_{xy} = (s + s^\vee)_{xy}$, hence

$$(J_{xy}^{\ell})^\star p_{xy} = (J_{xy}^{\ell})^\star (s + s^\vee)_{xy} = (s + s^\vee)^<_{xy} = (s^- + s^{-\vee})_{xy} = (J_{xz}^{\ell})^\star p_{xz},$$

for any $z \sim x$. Again, this meens that p is a quadratic field. □

The proof also gives

COROLLARY 13.6. *Let $k \geq 2$, s be a radical k-quadratic pseudofield and p be the shoot of s. Then p^- is the shoot of s^-.*

13.2. Reduction of radical pseudofields

As an illustration of the use of the above introduced formalism, we now prove that the reduction map is surjective over radical quadratic pseudofields.

PROPOSITION 13.7. *Let $k \geq 1$. Then the reduction map $s \mapsto s^-$ maps the space $\mathcal{M}_{k+1}^1$ of Γ-invariant radical $(k+1)$-quadratic pseudofields onto the space $\mathcal{M}_k^1$ of Γ-invariant radical k-quadratic pseudofields.*

PROOF. We will actually show the dual statement, namely that the adjoint map is injective. We prove this by induction on $k \geq 1$.

If $k = 1$, as every 1-quadratic pseudofield is radical, we must show that, if L and M are in $\mathcal{L}_1$, that is, if L and M are Γ-invariant 1-pseudokernels, and $L^{>\vee} = M^{\vee>\vee} - M^{>}$, then $L = 0$. Indeed, by Proposition 6.9, we have $M = 0 = M^{\vee} - L$, hence $L = 0$.

If $k \geq 2$ and the result holds for $k-1$, we must now show that, if L and M are in $\mathcal{L}_k$, and $L^{>\vee} = M^{\vee>\vee} - M^{>}$, then $L = N^{\vee>\vee} - N^{>}$ for some N in $\mathcal{L}_{k-1}$. Indeed, Proposition 6.9 now says that there exists N in $\mathcal{L}_{k-1}$ with

$$M^{\vee} - L = N^{>} \text{ and } M = N^{\vee>}$$

and the conclusion follows. □

13.3. Crossed duality

We now prove that the shoot map and the canonical map from Section 6.3 are dual to each other, through the weight pairing of Section 12.2.

LEMMA 13.8. *Let $k \geq 2$. Let s in $\mathcal{M}_k^1$ be a Γ-invariant radical k-quadratic pseudofield and set $p = P_k s$ to be the shoot of s. Let (K, K^-) in $\mathcal{K}_k$ be a Γ-invariant k-dual kernel and set $L = C_k(K, K^-)$ to be the canonical k-pseudokernel of (K, K^-). We have*

$$[p, (K, K^-)] = \langle s, L \rangle.$$

PROOF. Assume k is even, $k = 2\ell$, $\ell \geq 1$. As usual, for $x \sim y$ in V, denote by q_x, q^-_{xy} and r_{xy} the symmetric bilinear forms associated with K_x, K^-_{xy} and L_{xy}. By Definition 6.11, we have $r_{xy} = q_x - (I^{\ell-1,*}_{xy})^\star q^-_{xy}$. By Theorem 12.5, we have

$$[p, (K, K^-)] = \sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle p_x, q_x \rangle - \frac{1}{2} \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p^-_{xy}, q^-_{xy} \rangle.$$

By definition of the shoot map and by Lemma 10.11, we have

$$\sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle p_x, q_x \rangle = \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, q_x \rangle.$$

Besides, for $x \sim y$ in X, we have, as $q^-_{xy} = q^-_{yx}$,

$$\begin{aligned} \langle p^-_{xy}, q^-_{xy} \rangle = \langle s^<_{xy} + s^{\vee<}_{xy}, q^-_{xy} \rangle = \langle s^<_{xy} + s^{<\vee}_{xy}, q^-_{xy} \rangle = \langle s^<_{xy} + s^<_{yx}, q^-_{xy} \rangle \\ = \langle s_{xy}, (I^{\ell-1,*}_{xy})^\star q^-_{xy} \rangle + \langle s_{yx}, (I^{\ell-1,*}_{yx})^\star q^-_{xy} \rangle, \end{aligned}$$

where the latter follows from the definition of the direct restriction map and Lemma C.2. As $q^-_{xy} = q^-_{yx}$, we get

$$\sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p^-_{xy}, q^-_{xy} \rangle = 2 \sum_{(x,y) \in \Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, (I^{\ell-1,*}_{xy})^\star q^-_{xy} \rangle$$

and the conclusion follows.

Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$, and write now, for $x \sim y$ in X, q_{xy}, q_x^- and r_{xy} for the symmetric bilinear forms associated with K_{xy}, K_x^- and L_{xy}. Definition 6.12 gives $r_{xy} = q_{xy} - (J_{xy}^{\ell,*})^\star q_x^-$, whereas Theorem 12.5 says that

$$[p,(K,K^-)] = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy},q_{xy}\rangle - \sum_{x\in\Gamma\backslash X}\frac{d(x)-1}{|\Gamma_x|}\langle p_x^-,q_x^-\rangle.$$

As above, we have

$$\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle p_{xy},q_{xy}\rangle = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle s_{xy},q_{xy}\rangle.$$

Now, we write, by Corollary 13.6 and the definition of the direct restriction map, for x in X,

$$\begin{aligned}(d(x)-1)\langle p_x^-,q_x^-\rangle &= (d(x)-1)\sum_{y\sim x}\langle s_{xy}^{\vee<},q_x^-\rangle = \sum_{\substack{y,z\sim x\\ y\neq z}}\langle s_{xy}^{\vee<},q_x^-\rangle\\ &= \sum_{z\sim x}\langle s_{xz}^{\vee<\vee},q_x^-\rangle = \sum_{z\sim x}\langle s_{xz}^{<},q_x^-\rangle = \sum_{z\sim x}\langle (J_{xz}^{\ell})^\star s_{xz},q_{xz}\rangle\\ &= \sum_{z\sim x}\langle s_{xz},(J_{xz}^{\ell,*})^\star q_{xz}\rangle,\end{aligned}$$

where we have used the fact that the quadratic pseudofield s is radical. This gives, by Lemma 10.11,

$$\sum_{x\in\Gamma\backslash X}\frac{d(x)-1}{|\Gamma_x|}\langle p_x^-,q_x^-\rangle = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle s_{xy},(J_{xy}^{\ell,*})^\star q_{xy}\rangle,$$

and we are done. □

Let us translate Lemma 13.8 in terms of our linear maps. Denote by $I_k : \mathcal{M}_k^1 \hookrightarrow \mathcal{M}_k$ the natural inclusion. Recall that $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is the weight map. By Theorem 12.5, we can use the weight pairing to identify $\mathcal{F}_k$ with the dual space of $\mathcal{W}_k$ and we consider the adjoint operator W_k^* as a linear map from $\mathcal{F}_k$ to the dual space of $\mathcal{K}_k$.

Corollary 13.9. *Let $k \geq 2$. We have $W_k^* P_k = C_k^* I_{k+1}$.*

Proof. Indeed, if s, p, (K,K^-) and L are as in Lemma 13.8, by defintion of the weight pairing (see Definition 12.6), we have

$$[p,(K,K^-)] = \langle P_k s, W_k(K,K^-)\rangle = \langle W_k^* P_k s,(K,K^-)\rangle.$$

Besides, we have

$$\langle s,L\rangle = \langle I_{k+1}s, C_k(K,K^-)\rangle = \langle C_k^* I_{k+1}s,(K,K^-)\rangle.$$

The claim now follows from Lemma 13.8. □

13.4. Fibered surjectivity

From the duality between the shoot map and the canonical map, we get a strengthening of Proposition 13.7.

PROPOSITION 13.10. *Let $k \geq 1$. Then the shoot map P_k maps $\mathcal{M}_k^1$ onto $\mathcal{F}_k$. If $k \geq 2$, then the map $s \mapsto (P_k s, s^-)$ sends $\mathcal{M}_k^1$ onto the space*

$$\{(p,t) \in \mathcal{F}_k \times \mathcal{M}_{k-1}^1 | p^- = P_{k-1} t\}.$$

Note that the fact that the map of the statement takes values in the considered space follows from Corollary 13.6.

Again, we will prove Proposition 13.10 by establishing a dual statement. This will require us to use the following abstract description of dual spaces of fibered products.

LEMMA 13.11. *Let V, V_1, V_2 and W be finite-dimensional vector spaces. Assume we are given linear maps*

$$\begin{array}{ccc} V & \xrightarrow{\pi_2} & V_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \varpi_2} \\ V_1 & \xrightarrow[\varpi_1]{} & W \end{array}$$

such that $\varpi_1 \pi_1 = \varpi_2 \pi_2$.

Suppose π_1 and ϖ_1 are surjective. Then, ϖ_2 is surjective.

Besides, suppose the following property holds: for any linear functionals φ_1 on V_1 and φ_2 on V_2 with $\pi_1^* \varphi_1 = \pi_2^* \varphi_2$, there exists θ in W^* with $\varpi_1^* \theta = \varphi_1$ and $\varpi_2^* \theta = \varphi_2$. Then, the linear map $\pi = \pi_1 \oplus \pi_2$ maps V onto the fibered product

$$X = \{v_1 + v_2 \in V_1 \oplus V_2 | \varpi_1(v_1) = \varpi_2(v_2)\}.$$

PROOF. First, we claim that, if π_1 and ϖ_1 are surjective, then ϖ_2 is surjective. Indeed, if w is in W, as ϖ_1 is surjective, we may find v_1 in V_1 with $\varpi_1 v_1 = w$. As π_1 is surjective, we may find v in V with $\pi_1 v = v_1$. We get $\varpi_2 \pi_2 v = \varpi_1 \pi_1 v = w$ and we are done.

Assume besides the property of the statement holds and let X be the fibered product of V_1 and V_2 above W. The natural linear map $V_1^* \oplus V_2^* \to X^*$ is surjective and its null space is the space

$$Y = \{\varpi_1^* \theta - \varpi_2^* \theta | \theta \in W^*\} \subset V_1^* \oplus V_2^*.$$

As $\varpi_1 \pi_1 = \varpi_2 \pi_2$, the map π takes values in X. Therefore, to conclude, it suffices to show that the null space of π^* is Y. The latter is a direct consequence of the assumption. □

To deal with the boundary case $k = 2$ of Proposition 13.10, we shall need the following computation of a weight function.

LEMMA 13.12. *Let J be a 1-pseudokernel and K^- be a 1-dual prekernel. Set $K = J^> + J^{>\vee}$, so that K is a 2-dual prekernel. Then a function u on X_1 is (K, K^-)-compatible if and only if, for $x \sim y$ in X, one has*

$$u(x,y) + u(y,x) = K^-_{xy}(x,y)$$

and the associated weight function is then given by, for (x,y) in X_2,

$$w(x,y) = J_{zx}(x,z) + J_{zy}(y,z) - u(z,x) - u(z,y),$$

where z is the middle point of the segment $[xy]$. In particular, if J and K^- are Γ-invariant, for any p in $\mathcal{F}_2$, we have

$$[p,(K,K^-)] = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi_{p^-}(x,y)(J_{xy}(x,y) + J_{yx}(x,y) - K^-_{xy}(x,y)),$$

where φ_{p^-} is the quadratic type function on X_1 associated to the reduction p^- of p.

See Section 4.2 for the relation between quadratic type functions and quadratic fields.

PROOF. The formulas for u and w directly follow from Definition 7.2 and Definition 7.4. Now, by Theorem 12.5, for p in $\mathcal{F}_2$, we have

$$[p,(K,K^-)] = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_2} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi_p(x,y)w(x,y),$$

where φ_p is the quadratic type function on X_2 associated with p. Recall from Lemma 4.13 that, for (x,y) in X_1, we have

$$\varphi_{p^-}(x,y) = \sum_{\substack{z\sim x\\ z\neq y}} \varphi_p(z,y) = \sum_{\substack{z\sim y\\ z\neq x}} \varphi_p(x,z),$$

so that the formula for w together with Lemma 10.11 give

$$[p,(K,K^-)] = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi_{p^-}(x,y)(J_{xy}(x,y)+J_{yx}(x,y)-u(x,y)-u(y,x))$$

and the result follows. □

We will also use the following direct consequence of Lemma 5.1.

LEMMA 13.13. *Let A be a finite set and V_0 be the vector space of functions with zero sum on A. Let q be a symmetric bilinear form on V_0 and set $K(a,b) = q(\mathbf{1}_a - \mathbf{1}_b, \mathbf{1}_a - \mathbf{1}_b)$, $a, b \in A$. Let u be a function on A. Then the following are equivalent:*
(i) *one has $K(a,b) = u(a) + u(b)$, $a, b \in A$.*
(ii) *one has $q(f,g) = \sum_{a\in A} u(a)f(a)g(a)$, f, g in V_0.*

PROOF OF PROPOSITION 13.10. For $k = 1$, we just need to show that the shoot map is surjective : this is clear since, for p in $\mathcal{F}_1$, we have $P_1(s) = p$, where s is the 2-quadratic pseudofield defined by $s_{xy} = \frac{1}{2}p_{xy}$, $x \sim y \in X$.

For $k = 2$, we apply Lemma 13.11 to $V = \mathcal{M}_2^1$, $V_1 = \mathcal{M}_1^1 = \mathcal{M}_1$, $V_2 = \mathcal{F}_2$ and $W = \mathcal{F}_1$. The assumption hold: indeed, we have $P_2(s)^- = P_1(s^-)$ for s in $\mathcal{M}_2^1$ by Corollary 13.6; the reduction map $\mathcal{M}_2^1 \to \mathcal{M}_1^1$ is surjective by Proposition 13.7 and we have just shown that the shoot map $P_1 : \mathcal{M}_1^1 \to \mathcal{F}_1$ is surjective.

Therefore, by Lemma C.5, Lemma 13.8 and Lemma 13.11, it suffices to show that, if (K, K^-) in $\mathcal{K}_2$ and L, M in $\mathcal{L}_1$ are such that, with the notation of Remark 6.13,

$$C_2(K, K^-) = K - K^{->} = L^{>\vee} + M^{\vee>\vee} - M^{>}, \tag{13.1}$$

then L is a 1-predual kernel, that is, $L^\vee = L$, and

$$[p, (K, K^-)] = \frac{1}{2} \sum_{(x,y)\in\Gamma\backslash X_1} \varphi_{p^-}(x,y) L_{xy}(x,y), \quad p \in \mathcal{F}_2. \tag{13.2}$$

Indeed if (13.1) holds, the function $(x,y) \mapsto K_z(x,y), X_2 \to \mathbb{R}$ (where z is the middle point of the segment $[xy]$) is split in the sense of Section 9.4, that is, we may write

$$K_z(x,y) = v(x,z) + v(y,z), \quad (x,y) \in X_2,$$

where z is as above and v is a Γ-invariant function on X_1. By Lemma 13.13, there exists a 1-pseudokernel J in $\mathcal{L}_1$ with $K = J^> + J^{>\vee}$. From (13.1), we get

$$J^> + J^{>\vee} - K^{->} = L^{>\vee} + M^{\vee>\vee} - M^>,$$

hence, by Proposition 6.9,

$$J - K^- = -M \text{ and } J = L + M^\vee,$$

which gives in particular $L = K^- - M - M^\vee$ and therefore $L^\vee = L$ as required. Now, we have

$$J + J^\vee = 2K^- - M - M^\vee = K^- + L,$$

hence

$$J + J^\vee - K^- = L$$

and, as $(K, K^-) = (J^> + J^{>\vee}, K^-)$, (13.2) follows from Lemma 13.12.

For $k \geq 2$, we prove the statement by induction on k. We have just shown that it holds for $k = 2$. Now, we assume that $k \geq 3$ and that it has been established for $k - 1$. Again, we will apply Lemma 13.11 to $V = \mathcal{M}^1_k$, $V_1 = \mathcal{M}^1_{k-1}$, $V_2 = \mathcal{F}_k$ and $W = \mathcal{F}_{k-1}$. The assumption still hold: we have $P_k(s)^- = P_{k-1}(s^-)$ for s in $\mathcal{M}^1_k$ by Corollary 13.6; the reduction map $\mathcal{M}^1_k \to \mathcal{M}^1_{k-1}$ is surjective by Proposition 13.7 and the induction assumption implies that the shoot map $P_{k-1} : \mathcal{M}^1_{k-1} \to \mathcal{F}_{k-1}$ is surjective.

By Lemma C.5, Lemma 13.8 and Lemma 13.11, it now suffices to show that, if (K, K^-) in $\mathcal{K}_k$ and L, M in $\mathcal{L}_{k-1}$ are such that, still with the notation of Remark 6.13,

$$C_k(K, K^-) = K - K^{->} = L^{>\vee} + M^{\vee>\vee} - M^>, \tag{13.3}$$

then there exists a $(k-1)$-dual kernel (J, J^-) in $\mathcal{K}_{k-1}$, with orthogonal extension (J^+, J), A in $\mathcal{L}_{k-1}$ and N in $\mathcal{L}_{k-2}$ such that

$$C_{k-1}(J, J^-) = L + N^{\vee>\vee} - N^> \tag{13.4}$$

and (K, K^-) is the sum of (J^+, J) and the k-dual kernel associated to A. First we note that, with the language of Section 9.4, (13.3) implies that the pseudoweight of (K, K^-) is split. Thus, Proposition 9.15 and Lemma 9.17 say that there exists (J, J^-) in $\mathcal{K}_{k-1}$ and A in $\mathcal{L}_{k-1}$ such that (K, K^-) is the sum of (J^+, J) and the k-dual kernel associated with A. Let now B be the pseudokernel with $B^\vee = A$. By Lemma 6.14 and Lemma 6.19, we get

$$C_k(K, K^-) = C_{k-1}(J, J^-)^{>\vee} + B^{\vee>\vee} - B^>.$$

Together with (13.3), this gives

$$C_{k-1}(J, J^-)^{>\vee} + B^{\vee>\vee} - B^> = L^{>\vee} + M^{\vee>\vee} - M^>.$$

Thus, Proposition 6.9 says that there exists C in $\mathcal{L}_{k-2}$ with

$$C_{k-1}(J, J^-) + B^\vee = C^> + L + M^\vee \text{ and } C^{\vee>} - B = -M.$$

We get $C^{\vee>} = B - M$ and (13.4) holds with $N = -C$. □

13.5. The root of a quadratic field

Proposition 13.10 shows in particular that, for any $k \geq 1$, the shoot map $\mathcal{M}^1_k \to \mathcal{F}_k$ is surjective. We will now prove that it actually admits a natural section.

Recall from Lemma 4.15 that, if A is a finite set and $\overline{V}$ is the quotient of the space of functions on A by the constant ones, then a symmetric bilinear form p on $\overline{V}$ is completely determined by the numbers $p(\mathbf{1}_a, \mathbf{1}_b)$, $a, b \in A$, $a \neq b$.

DEFINITION 13.14. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and p be a k-quadratic field and φ_p be the associated quadratic type function on X_k. Define a k-quadratic pseudofield s as follows. For any $x \sim y$ in X and any $z \neq t$ in $S^\ell(x)$, we set

$$\begin{aligned} s_{xy}(\mathbf{1}_z, \mathbf{1}_t) &= 0 && \text{if } y \notin [xz] \cup [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) && \text{if } [xy] \subset [zt] \\ &= -\varphi_p(z,t) && \text{if } y \in [xz] \cap [xt]. \end{aligned}$$

Then s is called the root of p.

DEFINITION 13.15. (k odd) Let $k \geq 1$ be an odd integer, $k = 2\ell+1$, $\ell \geq 0$, and p be a k-quadratic field and φ_p be the associated quadratic type function on X_k. Define a k-quadratic pseudofield s as follows. For any $x \sim y$ in X and any $z \neq t$ in $S^\ell(xy)$, we set

$$\begin{aligned} s_{xy}(\mathbf{1}_z, \mathbf{1}_t) &= 0 && \text{if } y \in [xz] \cap [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) && \text{if } [xy] \subset [zt] \\ &= -\varphi_p(z,t) && \text{if } y \notin [xz] \cup [xt]. \end{aligned}$$

Then s is called the root of p.

PROPOSITION 13.16. *Let $k \geq 1$, p be a k-quadratic field and s be the root of p. Then s is a radical quadratic pseudofield and p is the shoot of s. If $k \geq 2$, s^- is the root of p^-.*

If p is in $\mathcal{F}_k$, we write $R_k p$ for the root of p. Proposition 13.16 says that R_k sends $\mathcal{F}_k$ into $\mathcal{M}_k^1$ and that $P_k R_k$ is the identity map of $\mathcal{F}_k$.

PROOF. Let $x \sim y$ be in X.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. From the definition of s, we get, for $z \neq t$ in $S^\ell(x)$,

$$\begin{aligned} s^\vee_{xy}(\mathbf{1}_z, \mathbf{1}_t) &= -\varphi_p(z,t) && \text{if } y \notin [xz] \cup [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) && \text{if } [xy] \subset [zt] \\ &= 0 && \text{if } y \in [xz] \cap [xt]. \end{aligned} \tag{13.5}$$

In particular, $s_{xy} + s^\vee_{xy} = p_x$. We also get, for $z \neq t$ in $S^{\ell-1}(xy)$, if $y \in [xz] \cap [xt]$,

$$s^{\vee<}_{xy}(\mathbf{1}_z, \mathbf{1}_t) = s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = 0,$$

whereas, if $x \in [yz]$ and $y \in [xt]$,

$$s^{\vee<}_{xy}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} s^\vee_{xy}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2} \sum_{\substack{u \sim z \\ u \notin [xz]}} \varphi_p(u,t) = -\frac{1}{2}\varphi_{p^-}(z,t)$$

and, if $x \in [yz] \cap [yt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z \\ u\notin[xz]}} \sum_{\substack{v\sim t \\ v\notin[xt]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_v) = -\sum_{\substack{u\sim z \\ u\notin[xz]}} \sum_{\substack{v\sim t \\ v\notin[xt]}} \varphi_p(u,v) = -\varphi_p(z,t).$$

In other words, $s^{\vee<}$ is the root of p^-. Finally, still for $z \neq t$ in $S^{\ell-1}(xy)$, if y belongs to $[xz] \cap [xt]$, we have

$$s_{xy}^{<}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_{p^-}(z,t);$$

if $x \in [yz]$ and $y \in [xt]$,

$$s_{xy}^{<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z \\ u\notin[xz]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2} \sum_{\substack{u\sim z \\ u\notin[xz]}} \varphi_p(u,t) = -\frac{1}{2}\varphi_{p^-}(z,t);$$

and, if $x \in [yz] \cap [yt]$,

$$s_{xy}^{<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z \\ u\notin[xz]}} \sum_{\substack{v\sim t \\ v\notin[xt]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_v) = 0.$$

Comparing the formulas for $s_{yx}^{\vee<}$ and $s_{xy}^{<}$ shows that $s^{<} = s^{\vee<\vee}$, that is, s is radical.

Assume now k is odd, $k = 2\ell + 1$, $\ell \geq 0$. We get, for $z \neq t$ in $S^{\ell}(xy)$,

$$\begin{aligned} s_{xy}^{\vee}(\mathbf{1}_z, \mathbf{1}_t) &= -\varphi_p(z,t) && \text{if } y \in [xz] \cap [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) && \text{if } [xy] \subset [zt] \\ &= 0 && \text{if } y \notin [xz] \cup [xt], \end{aligned}$$

so that in particular $s_{xy} + s_{xy}^{\vee} = p_{xy}$. Also, if $\ell \geq 1$, for $z \neq t$ in $S^{\ell}(x)$, if $y \in [xz] \cap [xt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = s_{xy}^{\vee}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_p(z,t);$$

if $y \in [xt]$ and $y \notin [xz]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z \\ u\notin[xz]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2}\varphi_{p^-}(z,t);$$

and, if $y \notin [xz] \cup [xt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z \\ u\notin[xz]}} \sum_{\substack{v\sim t \\ v\notin[xt]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_v) = 0.$$

Thus, $s^{\vee<}$ is the root of p^-. In particular, $s^{\vee<\vee}$ is given by (13.5) applied to p^-. Finally, again for $z \neq t$ in $S^\ell(xy)$, if $y \in [xz] \cap [zt]$, we have

$$s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = s_{xy}(\mathbf{1}_z, \mathbf{1}_t) = 0;$$

if $y \in [xt]$ and $y \notin [xz]$,

$$s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2}\varphi_{p^-}(z,t);$$

and, if $y \notin [xz] \cup [xt]$,

$$s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} \sum_{\substack{v \sim t \\ v \notin [xt]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_v) = -\sum_{\substack{u \sim z \\ u \notin [xz]}} \sum_{\substack{v \sim t \\ v \notin [xt]}} \varphi_p(u,v) = -\varphi_{p^-}(u,v).$$

Comparing the latter three formulas with (13.5) applied to p^-, we get $s^< = s^{\vee<\vee}$, that is, s is radical. □

13.6. Roots and pseudoweights

Recall that in Section 9.2, we have drawn a link between the canonical map, the weight map and the pseudoweight construction for pseudokernels. In particular, there is a natural linear map that sends a pseudokernel in $\mathcal{L}_k$ to the cohomology class of the symmetrization of its pseudoweight, which is an element of $\mathcal{W}_k$. The next Lemma says that the root map R_k may be seen as the adjoint map of this latter map, when the space $\mathcal{W}_k$ is identified with the dual space of $\mathcal{F}_k$ through the weight pairing.

PROPOSITION 13.17. *Let $k \geq 1$, p be in $\mathcal{F}_k$ and $s = R_k p$ be the root of p. Let L be in $\mathcal{L}_k$ and v be the pseudoweight of L. Then we have*

$$\langle s, L \rangle = \frac{1}{2} \sum_{(x,y) \in \Gamma \backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} \varphi_p(x,y) v(x,y).$$

We start with a version of Proposition 13.17 for non-necessarily Γ-invariant objects.

LEMMA 13.18. *Let $k \geq 1$, p be a k-quadratic field and s be the root of p. Let L be a finitely supported k-pseudokernel and v be the pseudoweight of L. Then v is a finitely supported function on X_k and we have*

$$\sum_{(x,y) \in X_1} \langle s_{xy}, L_{xy} \rangle = \frac{1}{2} \sum_{(x,y) \in X_k} \varphi_p(x,y) v(x,y).$$

Saying that L is finitely supported is saying that the set $\{(x,y) \in X_1 | L_{xy} \neq 0\}$ is finite. Recall that the pseudoweight of a pseudokernel is introduced in Definition 9.8.

PROOF. We will check that the formula holds when L varies in a generating subset of the vector space of finitely supported k-pseudokernels.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. We fix $x \sim y$ and $z \neq t$ in $S^\ell(x)$. Then, we define a $(k+1)$-pseudokernel L by setting

$$L_{ab}(c,d) = \mathbf{1}_{a=x}\mathbf{1}_{b=y}\mathbf{1}_{\{c,d\}=\{z,t\}}, \quad a \sim b \in X, \quad c \neq d \in S^\ell(a).$$

In other words, the symmetric bilinear form associated with L_{ab} on $V_0^\ell(a)$ is 0 if $(a,b) \neq (x,y)$, and the symmetric bilinear form associated with L_{xy} on $V_0^\ell(x)$ is

$$(\varphi, \psi) \mapsto -\frac{1}{2}\varphi(z)\psi(t) - \frac{1}{2}\varphi(t)\psi(z).$$

In particular, by Lemma C.5, we have

$$\sum_{(a,b)\in X_1} \langle s_{ab}, L_{ab}\rangle = -s_{xy}(\mathbf{1}_z, \mathbf{1}_t) \tag{13.6}$$

To conclude, we now need to compute the pseudoweight v of L. Corollary 6.23 and Definition 6.23 give

$$v(a,b) = \sum_{\substack{c\in S^\ell(a)\\ a\notin[bc]}} L_{cc_-}(a,b), \quad (a,b) \in X_k,$$

where as usual, for c as above, c_- is the neighbour of c on $[ac]$. By using the precise definition of L, this gives

$$v(a,b) = \mathbf{1}_{a=z}\mathbf{1}_{y\in[xz]}\mathbf{1}_{t\in[xb]} + \mathbf{1}_{a=t}\mathbf{1}_{y\in[xt]}\mathbf{1}_{z\in[xb]}, \quad (a,b) \in X_k. \tag{13.7}$$

Now we apply (13.6) and (13.7) to the three cases in Definition 13.14. If $y \notin [xz] \cup [xt]$, we get $v = 0$ and $\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle = 0$. If $y \in [xz]$ and $y \notin [xt]$, we get $v = \mathbf{1}(z,t)$, hence

$$\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) = \varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z, \mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle.$$

If $y \in [xz] \cap [xt]$, set $h = k - d(z,t) \geq 2$. As φ_p is a quadratic type function, we have

$$\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) = \sum_{\substack{b\in S^h(t)\\ [tb]\cap[xt]=\{t\}}} \varphi_p(z,b) + \sum_{\substack{b\in S^h(z)\\ [zb]\cap[xz]=\{z\}}} \varphi_p(t,b)$$
$$= 2\varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle.$$

The case where k is even follows.

Assume k is odd, $k = 2\ell+1$, $\ell \geq 0$. We now fix $x \sim y$ and $z \neq t$ in $S^\ell(xy)$. We define a k-pseudokernel L by setting

$$L_{ab}(c,d) = \mathbf{1}_{a=x}\mathbf{1}_{b=y}\mathbf{1}_{\{c,d\}=\{z,t\}}, \quad a \sim b \in X, \quad c \neq d \in S^\ell(ab).$$

Still by Lemma C.5, we have

$$\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle = -s_{xy}(\mathbf{1}_z,\mathbf{1}_t) \tag{13.8}$$

Again, we compute the pseudoweight v of L. Corollary 6.23 and Definition 6.23 give

$$v(a,b) = \sum_{\substack{c\in S^{\ell+1}(a)\\ a\notin[bc]}} L_{c-c}(a,b), \quad (a,b)\in X_k.$$

This gives

$$v(a,b) = \mathbf{1}_{a=z}\mathbf{1}_{y\notin[xz]}\mathbf{1}_{t\in[xb]} + \mathbf{1}_{a=t}\mathbf{1}_{y\notin[xt]}\mathbf{1}_{z\in[xb]}, \quad (a,b)\in X_k. \tag{13.9}$$

We now apply (13.8) and (13.9) to the three cases in Definition 13.15. If $y \in [xz]\cap[xt]$, we get $v = 0$ and $\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle = 0$. If $y \notin [xz]$ and $y \in [xt]$, we get $v = \mathbf{1}(z,t)$, hence

$$\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) = \varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle.$$

If $y \notin [xz]\cup[xt]$, set $h = k - d(z,t) \geq 1$. As φ_p is a quadratic type function, we have

$$\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) = \sum_{\substack{b\in S^h(t)\\ [tb]\cap[xt]=\{t\}}} \varphi_p(z,b) + \sum_{\substack{b\in S^h(z)\\ [zb]\cap[xz]=\{z\}}} \varphi_p(t,b)$$
$$= 2\varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle$$

and the case where k is odd follows. □

We will deduce Proposition 13.17 from Lemma 13.18. To this aim, we need to show that every Γ-invariant pseudokernel can be obtained from a finitely supported one.

LEMMA 13.19. *Let $k \geq 1$ and L be a finitely supported k-pseudokernel. Set $\overline{L} = \sum_{\gamma\in\Gamma} \gamma L$. Then the map $L \mapsto \overline{L}$ maps the space of finitely supported k-pseudokernels onto $\mathcal{L}_k$.*

PROOF. Indeed, fix $S \subset X_1$ a section of the quotient map $X_1 \to \Gamma\backslash X_1$, that is, $X_1 = \Gamma S$ and $\Gamma s \cap S = \{s\}$ for any s in S. If M is in $\mathcal{L}_k$, for $(x, y) \in X_1$, set $L_{xy} = \frac{1}{|\Gamma_x \cap \Gamma_y|} M_{xy}$ if (x, y) is in S and $L_{xy} = 0$ else. Then L is a finitely supported k-pseudokernel and $\overline{L} = M$. □

To relate summation formulas on X and on $\Gamma\backslash X$, we shall use the standard

LEMMA 13.20. *Let A be a set and G be a group acting on A such that, for every a in A, its stabilizer G_a in G is finite. If φ is a finitely supported function on A, set*

$$\overline{\varphi}(a) = \sum_{g\in G} \varphi(ga), \quad a \in A.$$

Then, we have

$$\sum_{a\in A} \varphi(a) = \sum_{a\in G\backslash A} \frac{1}{|G_a|} \overline{\varphi}(a).$$

PROOF OF PROPOSITION 13.17. By Lemma 13.19, it suffices to prove the result for Γ-invariant k-pseudokernels which are of the form $\overline{L}$ where L is a finitely supported k-pseudokernel. For such a L, let v be its pseudoweight. If p is in $\mathcal{F}_k$ and s is the root of p, Lemma 13.18 gives

$$\sum_{(x,y)\in X_1} \langle s_{xy}, L_{xy}\rangle = \frac{1}{2} \sum_{(x,y)\in X_k} \varphi_p(x, y)v(x, y).$$

Now, the pseudoweight of $\overline{L}$ is $\overline{v} = \sum_{\gamma\in\Gamma} \gamma v$, and Lemma 13.20 gives

$$\sum_{(x,y)\in X_1} \langle s_{xy}, L_{xy}\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, \overline{L}_{xy}\rangle = \langle s, \overline{L}\rangle$$

and

$$\sum_{(x,y)\in X_k} \varphi_p(x, y)v(x, y) = \sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} \varphi_p(x, y)\overline{v}(x, y).$$

The result follows. □

CHAPTER 14

Tight quadratic fields

For $k \geq 2$, recall that $\mathcal{W}_k$ stands for the vector space of cohomology classes of Γ-invariant symmetric functions on X_k. We say that the cohomology class of such a function w is non-negative if the associated bilinear form Φ_w on H_0^ω is non-negative and we let $\mathcal{W}_k^+$ be the closed convex cone of non-negative cohomology classes. By Theorem 8.17, the convex cone $\mathcal{W}_k^+$ is the image of the convex cone $\mathcal{K}_k^+$ of non-negative Γ-invariant k-dual kernels by the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$.

Now Propositions 4.11 and 12.2 say that the space $\mathcal{F}_k$ of Γ-invariant k-quadratic fields may be considered as the dual space of $\mathcal{W}_k$ through the weight pairing. We say that a p in $\mathcal{F}_k$ is tight if, for any non-negative (K, K^-) in $\mathcal{K}_k$, one has $[p, (K, K^-)] \geq 0$. We denote by $\mathcal{F}_k^{\mathrm{tgt}} \subset \mathcal{F}_k$ the convex cone of tight Γ-invariant k-quadratic fields. It can be seen as the dual cone of $\mathcal{W}_k^+$ (see Section 14.2 below).

Recall from Section 11.9 that a k-quadratic pseudofield s is non-negative if all the symmetric bilinear forms s_{xy}, $x \sim y \in X$, are non-negative. We denote by $\mathcal{M}_k^+ \subset \mathcal{M}_k$ the closed convex cone of Γ-invariant non-negative k-quadratic pseudofields and we set $(\mathcal{M}_k^1)^+ = \mathcal{M}_k^1 \cap \mathcal{M}_k^+$ to be the closed convex cone of Γ-invariant non-negative radical k-quadratic pseudofields.

THEOREM 14.1. *Let $k \geq 2$. Then a Γ-invariant k-quadratic field is tight if and only if it is the shoot of a Γ-invariant non-negative radical k-quadratic pseudofield. In other words, we have $\mathcal{F}_k^{\mathrm{tgt}} = P_k((\mathcal{M}_k^1)^+)$.*

The proof of this result is the objective of this chapter.

14.1. Non-negative extensions

In this section, we prove a technical result that will allow us to show that certain pseudokernels are non-negative by knowing that some extensions of them are.

PROPOSITION 14.2. *Let $k \geq 2$ and L, M be in $\mathcal{L}_k$. The $(k+1)$-pseudokernel $L^> + M^{>\vee}$ is non-negative if and only if there exists A, B in $\mathcal{L}_{k-1}$ and C, D in $\mathcal{L}_k^+$ with $L = A^> + C$, $M = B^> + D$ and $A + B^\vee$ non-negative.*

The proof relies on an adaptation of the argument in Lemma 6.10.

LEMMA 14.3. *Let $W_0, W_1, \dots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. Assume $q_1, \dots q_d$ to be symmetric bilinear forms on $W_1, \dots, W_d$ and set $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$. Assume q is positive definite. Then the following holds.

For $1 \leq i \leq d$, the restriction r_i of q_i to $U_i = \ker \varpi_i$ is positive definite and we have $W_i = U_i \oplus V_i$ where V_i is the q_i-orthogonal subspace to U_i, that is,

$$V_i = \{v \in W_i | \forall u \in U_i \quad q_i(u, v) = 0\}.$$

Let $\alpha_i : W_i \to U_i$ be the projection in this decomposition. Then, there exists a unique symmetric bilinear form p_i on W_0 with $q_i = \alpha_i^\star r_i + \varpi_i^\star p_i$.

The symmetric bilinear form $p = p_1 + \cdots + p_d$ on W_0 is positive definite.

PROOF. Fix $1 \leq i \leq d$ and pick $v_i \neq 0$ in V_i with $\varpi_i(v_i) = 0$. For $j \neq i$, we set $v_j = 0$, so that the vector $v = (v_j)_{1 \leq j \leq d}$ belongs to W and we have $\pi_i(v) = v_i$. By definition, we have $r_i(v_i, v_i) = q_i(v_i, v_i) = q(v, v) > 0$ and r_i is positive definite. In particular, it is non-degenerate, so that $U_i \cap V_i = \{0\}$. Now, the natural map $w_i \mapsto q_i(w_i, .)$ from W_i to the dual space W_i^* induces an injective linear map $W_i/U_i \hookrightarrow V_i^*$, so that we have $\dim W_i - \dim U_i \leq \dim V_i$, that is, $\dim W_i \leq \dim(U_i \oplus V_i)$ and we get $W_i = U_i \oplus V_i$.

Let $\alpha_i : W_i \to U_i$ and $\beta_i : W_i \to V_i$ be the projections in this decomposition. As U_i and V_i are q_i-orthogonal, we have $q_i = \alpha_i^\star r_i + \beta_i^\star s_i$, where s_i is the restriction of q_i to V_i. Now, ϖ_i induces an isomorphism from U_i onto W_0, so that there exists a unique symmetric bilinear form p_i on W_0 with $\beta_i^\star s_i = \varpi_i^\star p_i$.

Finally, we set $V \subset W$ to be the subspace defined by

$$V = \{w \in W | \forall 1 \leq i \leq d \quad \pi_i(w) \in V_i\}.$$

By construction, the restriction of q to V is equal to $\pi_0^\star(p_1 + \cdots + p_d)$, hence $p_1 + \cdots + p_d$ is positive definite on W_0. □

The proof of Proposition 14.2 also uses an elementary geometric property of convex cones. Recall that a closed convex cone in a finite-dimensional vector space is said to be proper if it contains no vector line.

LEMMA 14.4. *Let V be a finite-dimensional vector space and $\mathcal{C} \subset V$ be a proper closed convex cone. Then, for every u in V, the set $\mathcal{C} \cap (u - \mathcal{C})$ is compact in V.*

PROOF. We prove the converse statement. Assume that there exists u in V such that $\mathcal{C} \cap (u - \mathcal{C})$ is not compact and let us show that $\mathcal{C}$ is not proper. Equip V with a norm $\|.\|$. Let (v_n) be a sequence of elements of $\mathcal{C}$ with $u - v_n \in \mathcal{C}$ for any n and $\|v_n\| \xrightarrow[n\to\infty]{} \infty$. Then, after extracting, we can assume that there exists w in V with $\|v_n\|^{-1} v_n \xrightarrow[n\to\infty]{} w$. As v_n is in $\mathcal{C}$, we have $w \in \mathcal{C}$. As $u - v_n$ is in $\mathcal{C}$ and $\|v_n\| \xrightarrow[n\to\infty]{} \infty$, we also have $-w \in \mathcal{C}$. The Lemma follows. □

PROOF OF PROPOSITION 14.2. Note that one direction is obvious. We prove the other one. Let L, M be in $\mathcal{L}_k$ with $L^> + M^{>\vee}$ non-negative.

We will first establish the case where $L^> + M^{>\vee}$ belongs to the interior of the cone $\mathcal{L}_k^+$, and then use an approximation argument to deal with the general case. Thus, we first assume that, for any $x \sim y$ in X, the symmetric bilinear form associated to $(L+M^\vee)_{xy}$ by Lemma 5.1 is positive definite. In that case, Proposition 4.5, Proposition 4.6 and Lemma 14.3 tell us that we may write $L = A^> + C$ and $M = B^> + D$ where C and D are in $\mathcal{L}_k^+$ and A and B are in $\mathcal{L}_{k-1}$ and that the $(k-1)$-pseudokernel $A + B^\vee$ is non-negative, which should be proved.

Now, we will use an approximation argument to conclude. We pick a Γ-invariant exact k-dual kernel (K, K^-) (see Definition 5.12 and Definition 5.13 for this notion). For example, (K, K^-) can be the k-dual kernel obtained by orthogonal extension from the harmonic kernel (see Section 5.5, Section 10.6 and Section 11.6). We consider the k-dual prekernel K as a k-pseudokernel and the $(k-1)$-dual prekernel K^- as a $(k-1)$-pseudokernel (as in Remark 6.13). We note that the orthogonal extension (K^+, K) of (K, K^-) may be defined by $K^+ = K^> + (K - K^{->})^{>\vee}$. For $0 < \varepsilon \leq 1$, we set $L_\varepsilon = L + \varepsilon K$ and $M_\varepsilon = M + \varepsilon(K - K^{->})$, so that $L_\varepsilon^> + M_\varepsilon^{>\vee} = L^> + M^{>\vee} + \varepsilon K^+$ and the symmetric bilinear forms associated to this $(k+1)$-pseudokernel are positive definite. Therefore, there exists C_ε and D_ε in $\mathcal{L}_k^+$ and A_ε and B_ε in $\mathcal{L}_{k-1}$ such that $A_\varepsilon + B_\varepsilon^\vee$ is non-negative and $L_\varepsilon = A_\varepsilon^> + C_\varepsilon$ and $M_\varepsilon = B_\varepsilon^> + D_\varepsilon$. In particular, we have

$$\begin{aligned} L^> + M^{>\vee} + \varepsilon K^+ = L_\varepsilon^> + M_\varepsilon^{>\vee} = A_\varepsilon^{>>} + B_\varepsilon^{>>\vee} + C_\varepsilon^> + D_\varepsilon^{>\vee} \\ = (A_\varepsilon + B_\varepsilon^\vee)^{>>} + C_\varepsilon^> + D_\varepsilon^{>\vee} \end{aligned}$$

(where we have used Lemma 6.8). We get, for $0 < \varepsilon \leq 1$,

$$C_\varepsilon^> \in \mathcal{L}_{k+1}^+ \cap (L^> + M^{>\vee} + K^+ - \mathcal{L}_{k+1}^+).$$

As $\mathcal{L}_{k+1}^+$ clearly is a proper closed convex cone in the finite-dimensional vector space $\mathcal{L}_{k+1}$, by Lemma 14.4, the set $\mathcal{L}_{k+1}^+ \cap (L^> + M^{>\vee} + K^+ - \mathcal{L}_{k+1}^+)$ is compact. As the extension map $L \mapsto L^>$ is injective, the set $\{C_\varepsilon | 0 < \varepsilon \leq 1\}$ is bounded in $\mathcal{L}_k$. In the same way, for any $0 < \varepsilon \leq 1$, we have

$$D_\varepsilon^{>\vee} \in \mathcal{L}_{k+1}^+ \cap (L^> + M^{>\vee} + K^+ - \mathcal{L}_{k+1}^+),$$

hence the set $\{D_\varepsilon | 0 < \varepsilon \leq 1\}$ is bounded in $\mathcal{L}_k$. Therefore, we can find a sequence $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and C, D in $\mathcal{L}_k^+$ with $C_{\varepsilon_n} \xrightarrow[n\to\infty]{} C$ and $D_{\varepsilon_n} \xrightarrow[n\to\infty]{} D$. For any n, we have $L_{\varepsilon_n} = A_{\varepsilon_n}^> + C_{\varepsilon_n}$. Hence, A_{ε_n} has a limit A in $\mathcal{L}_{k-1}$ and $L = A^> + C$. In the same way, B_{ε_n} has a limit B in $\mathcal{L}_{k-1}$ and $M = B^> + D$. As for any n, $A_{\varepsilon_n} + B_{\varepsilon_n}^\vee$ is non-negative, so is $A + B^\vee$ and we are done. □

Let us draw a first consequence of this Lemma that will be useful shortly.

COROLLARY 14.5. *Let $k \geq 1$ and L be in $\mathcal{L}_k$. If $L^{\vee>\vee} - L^>$ is non-negative, then $L = 0$.*

PROOF. We prove this result by induction on $k \geq 1$.

If $k = 1$, for $x \sim y$ in X, we set $u(x, y) = L_{xy}(x, y)$. By the assumption, for any x in X and f in $V_0^1(x)$, we have

$$\sum_{\substack{z\sim x \\ z\neq y}} u(z, x) f(z)^2 \geq u(x, y) f(y)^2. \tag{14.1}$$

For $y \sim x$, applying this property to the function

$$f = \mathbf{1}_y - \frac{1}{d(x) - 1} \sum_{\substack{z\sim x \\ z\neq y}} \mathbf{1}_z,$$

we get

$$\frac{1}{(d(x) - 1)^2} \sum_{\substack{z\sim x \\ z\neq y}} u(z, x) \geq u(x, y). \tag{14.2}$$

Define a linear operator T on the space $\mathscr{V}_1$ of Γ-invariant functions on X_1 by setting

$$Tv(x, y) = \frac{1}{(d(x) - 1)^2} \sum_{\substack{z\sim x \\ z\neq y}} v(z, x), \quad v \in \mathscr{V}_1, \quad x \sim y \in X.$$

As in the proof of Proposition 11.21, since $d(x) \geq 3$ for any x in X, the operator T has operator norm $\leq \frac{1}{2}$ when $\mathscr{V}_1$ is equipped with the supremum norm. In particular, $T^n \xrightarrow[n\to\infty]{} 0$. Now, (14.2) reads as $Tu \geq u$. As T maps non-negative functions to non-negative functions, we get $T^n u \geq u$ for any integer $n \geq 0$, which gives $u \leq 0$. Fix x in X and choose y, z and t to be three pairwise different neighbours of x: this is possible as $d(x) \geq 3$. By applying (14.1) to the function $f = \mathbf{1}_z - \mathbf{1}_t$, we obtain

$$u(z,x) + u(t,x) \geq 0.$$

As $u \leq 0$, we get $u = 0$ as required.

Assume $k \geq 2$ and the result holds for $k-1$. By Proposition 14.2, we can find A, B in $\mathcal{L}_{k-1}$ and C, D in $\mathcal{L}_k^+$ with $-L = A^> + C$ and $L^\vee = B^> + D$ and $A + B^\vee$ is non-negative. We get

$$B^{\vee>\vee} - B^> \geq -A^{>\vee} - B^> = (L+C)^\vee + (-L^\vee + D) = C^\vee + D \geq 0.$$

By the induction assumption, we get $B = 0$. In particular A is non-negative. As $-L = A^> + C$, L is non-positive, hence so is $L^\vee$. As $L^\vee = D$, $L^\vee$ is non-negative. We get $L = 0$ as required. □

14.2. Convex cones

We recall some notions of convex geometry and we show that there exists a duality between the notion of a non-negative quadratic pseudofield and the one of a non-negative pseudokernel.

Let V be a finite dimensional vector space with dual space V^*. Assume $\mathcal{C} \subset V$ is a closed convex cone. The dual cone of $\mathcal{C}$ is the set of φ in V^* with $\langle \varphi, x \rangle \geq 0$ for any x in $\mathcal{C}$. This is a closed convex cone and an application of the geometric form of Hahn-Banach theorem gives $\mathcal{C}^{**} = \mathcal{C}$.

We know how to transport these objects under linear maps.

Lemma 14.6. *Let V, W be finite dimensional vector spaces and $T : V \to W$ be a linear map. Let $\mathcal{C}$ be a closed convex cone in V with $\mathcal{C} \cap \ker T = \{0\}$. Then $T\mathcal{C}$ is closed in W.*

Proof. As every linear map is the composition of an injective linear map with a surjective one, and as the result is obvious in the injective case, we can assume that T is surjective. Note also that it suffices to prove the result when the null space of T has dimension 1, the general case following from a direct induction argument.

In other words, we are reduced to proving that, for any v in $V \setminus (\mathcal{C} \cup (-\mathcal{C}))$, the set $\mathbb{R}v + \mathcal{C}$ is closed in V. Let (t_n) be a sequence in $\mathbb{R}$

and (u_n) be a sequence in $\mathcal{C}$ with $u_n - t_n v \xrightarrow[n\to\infty]{} w$ for w in V. If (t_n) is bounded, we may assume that (t_n) is convergent and the conclusion follows. If not, after extracting a subsequence, and up to replacing v by $-v$, we may assume that $t_n \xrightarrow[n\to\infty]{} +\infty$. In particular, we have $t_n^{-1} u_n \xrightarrow[n\to\infty]{} v$, hence v belongs to $\mathcal{C}$, a contradiction. □

Remark 14.7. The assumption in Lemma 14.6 is necessary. Let $\mathcal{C} \subset \mathbb{R}^3$ be the cone

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | x^2 \geq y^2 + z^2 \text{ and } x \geq 0\}$$

and T be the linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$, $(x, y, z) \mapsto (x - y, z)$. A direct computation shows that we have

$$T\mathcal{C} = \{(u, v) \in \mathbb{R}^2 | u > 0\} \cup \{(0, 0)\}.$$

Thus, the convex cone $\mathcal{C}$ is closed, but $T\mathcal{C}$ is not.

Corollary 14.8. *Let V, W be finite dimensional vector spaces and $T : V \to W$ be a linear map. Let $\mathcal{C}$ be a closed convex cone in V with $\mathcal{C} \cap \ker T = \{0\}$, then we have $(T\mathcal{C})^* = (T^*)^{-1}(\mathcal{C}^*)$.*

Here comes a first case where we can determine a dual cone.

Lemma 14.9. *Let V be a finite-dimensional vector space. Then, with respect to the quadratic duality, the dual cone of the convex cone $\mathcal{Q}_+(V)$ of non-negative symmetric bilinear forms on V is the cone $\mathcal{Q}_+(V^*)$.*

See Appendix C for the definition of the quadratic duality.

Proof. Let q in $\mathcal{Q}(V^*)$. If q belongs to the dual convex cone of $\mathcal{Q}_+(V)$, then for every φ in V^*, as φ^2 belongs to $\mathcal{Q}_+(V)$, we have $q(\varphi, \varphi) = \langle q, \varphi^2 \rangle \geq 0$. Conversely, every non-negative symmetric bilinear form p on V may be written as $p = \varphi_1^2 + \cdots + \varphi_r^2$ for some $\varphi_1, \ldots, \varphi_r$ in V^*. Thus, if q is in $\mathcal{Q}_+(V^*)$, we have

$$\langle q, p \rangle = \langle q, \varphi_1^2 \rangle + \cdots + \langle q, \varphi_r^2 \rangle = q(\varphi_1, \varphi_1) + \cdots + q(\varphi_r, \varphi_r) \geq 0,$$

hence q belongs to the dual cone of $\mathcal{Q}_+(V)$. □

14.3. Non-negative quadratic pseudofields and pseudokernels

From Lemma 14.9 and the definition of the duality between quadratic pseudofields and pseudokernels in Section 11.2, we get

Corollary 14.10. *Let $k \geq 1$. Then the convex cone $\mathcal{M}_k^+$ of non-negative Γ-invariant k-quadratic pseudofields is the dual cone of the convex cone $\mathcal{L}_k^+$ of non-negative Γ-invariant k-pseudokernels.*

We will now finish the proof of Theorem 14.1. This relies on the notions introduced above and on Proposition 14.2.

To check the assumption of Lemma 14.6, we will use the easy

LEMMA 14.11. *Let $k \geq 2$ and s be in $(\mathcal{M}_k^1)^+$ a non-negative Γ-invariant radical $(k+1)$-quadratic pseudofield. If $P_k s = 0$, then $s = 0$.*

PROOF. Indeed, if a sum of a non-negative symmetric bilinear forms is zero, each of them is zero. □

PROOF OF THEOREM 14.1. Let us denote by $I_k : \mathcal{M}_k^1 \hookrightarrow \mathcal{M}_k$ the inclusion map. Then the null space of the adjoint map $I_k^* : \mathcal{L}_k \to (\mathcal{M}_k^1)^*$ is the space $\{M^{\vee > \vee} - M^> | M \in \mathcal{L}_{k-1}\}$. Corollary 14.5 and Lemma 14.6 imply that $I_k^* \mathcal{L}_k^+$ is closed in $(\mathcal{M}_k^1)^*$ and Corollary 14.8 implies that the dual cone of $I_k^* \mathcal{L}_k^+$ is $(\mathcal{M}_k^1)^+$.

Now, Theorem 9.1 and Corollary 6.20 say that the map $I_k^* C_k$ gives rise to a linear embedding $\overline{C}_k : \mathcal{W}_k \to (\mathcal{M}_k^1)^*$. Corollary 13.9 says that, when $\mathcal{F}_k$ is identified with the dual space of $\mathcal{W}_k$ through the weight pairing, the adjoint operator of $\overline{C}_k$ is the shoot map $P_k : \mathcal{M}_k^1 \to \mathcal{F}_k$. By Lemma 14.6 and Lemma 14.11, the convex cone $P_k((\mathcal{M}_k^1)^+)$ is closed in $\mathcal{F}_k$. By Corollary 14.8, its dual cone in $\mathcal{W}_k$ is $\overline{C}_k^{-1} I_k^* \mathcal{L}_k^+$. The result follows since Proposition 10.19 implies that $\overline{C}_k^{-1} I_k^* \mathcal{L}_k^+ = \mathcal{W}_k^+$ □

14.4. Reduction of nonnegative radical quadratic pseudofields

To conclude this chapter, we will now prove a refinement of Proposition 13.7, namely that reduction is surjective over non-negative radical quadratic pseudofields.

PROPOSITION 14.12. *Let $k \geq 2$. Then the reduction map $s \mapsto s^-$ maps the convex cone $(\mathcal{M}_{k+1}^1)^+$ of non-negative Γ-invariant radical $(k+1)$-quadratic pseudofields onto the convex cone space $(\mathcal{M}_k^1)^+$ of non-negative Γ-invariant radical k-quadratic pseudofields.*

Again, the proof will follow from a duality argument. To check the assumption in Lemma 14.6, we will need

LEMMA 14.13. *Let $k \geq 1$ and s be a nonnegative radical $(k+1)$-quadratic pseudofield. If $s^- = 0$, we have $s = 0$.*

PROOF. Note that, if V is a real vector space and $V_1, \ldots, V_d$ are subspaces which span V, then a nonnegative symmetric bilinear form on V is zero if and only if its restriction to V_i is zero for any $1 \leq i \leq d$.

Now, assume k is odd, $k = 2\ell+1$, $\ell \geq 0$. In that case, by Proposition 4.5, for x in X, the space $\overline{V}^{\ell+1}(x)$ is spanned by the spaces $I_{xy}^{\ell}\overline{V}^{\ell}(xy)$, $y \sim x$. Fix $y \sim x$. As $s^{\vee<} = s^{-} = 0$, we have

$$\sum_{\substack{z\sim x\\ z\neq y}} (I_{xy}^{\ell})^{\star} s_{xz} = 0,$$

hence, as all these symmetric bilinear forms on $\overline{V}^{\ell}(xy)$ are non-negative, for all $z \sim x$, $z \neq y$, $(I_{xy}^{\ell})^{\star} s_{xz} = 0$. As s is radical, we have $s^{<} = s^{\vee<\vee} = 0$, that is $(I_{xy}^{\ell})^{\star} s_{xy} = 0$. We have proved that, for all x in X, for any $y, z \sim x$, $(I_{xy}^{\ell})^{\star} s_{xz} = 0$. By the remark above, we get $s = 0$ as required. The proof in the odd case is analogous. □

Here comes the main argument of the proof.

LEMMA 14.14. *Let $k \geq 2$ and L be in $\mathcal{L}_k$. Assume that there exists M in $\mathcal{L}_k$ such that $L^{>\vee} + M^{\vee>\vee} - M^{>}$ is non-negative. Then, there exists N in $\mathcal{L}_{k-1}$ such that $L + N^{\vee>\vee} - N^{>\vee}$ is non-negative.*

PROOF. Indeed, Proposition 14.2 implies that there exists A, B in $\mathcal{L}_{k-1}$ and C, D in $\mathcal{L}_k^+$ such that $A + B^{\vee}$ is non-negative and $-M = A^{>} + C$ and $L + M^{\vee} = B^{>} + D$. We get

$$L = B^{>} + D - M^{\vee} \geq B^{>} - M^{\vee} = B^{>} + A^{>\vee} + C^{\vee} \geq B^{>} - B^{\vee>\vee}$$

and the result follows. □

EXAMPLE 14.15. In general, the statement of Lemma 14.14 is false for $k = 1$. Indeed, let A be a finite set with at least three elements. There exists a function $u : A \to \mathbb{R}$ which takes a negative value but which is such that $\sum_{a\in A} u(a) f(a)^2 \geq 0$ for any function f on A with $\sum_{a\in A} f(a) = 0$. Let Γ be the group of Example 2.1 associated with A. Then, the data of u determines the data of L in $\mathcal{L}_1 \smallsetminus \mathcal{L}_1^+$ such that $L^{\vee} = L$ and $K = L^{>} + L^{>\vee}$ is nonnegative. Let w be a weight function for (K, L) and Φ_w be the associated quadratic form on H_0^{ω}. For (x, y) in X_1, denote by $v(x, y) = v(y, x)$ the value of u on the element of A associated with the edge (x, y). From Lemma 5.9 and Corollary 8.9, a direct computation gives, for any θ in H_0^{ω},

$$\Phi_w(\theta, \theta) = \frac{1}{2} \sum_{(x,y)\in X_1} v(x, y)\theta(U_{xy})^2 = \frac{1}{2} \sum_{x\in X} \sum_{y\sim x} v(x, y)\theta(U_{xy})^2.$$

In particular, Φ_w is non-negative, so that, by Theorem 8.17 and Theorem 9.1, there exists a 2-pseudokernel M such that $(K, L) + M$ is a

non-negative 2-dual kernel, and hence, the canonical 2-pseudokernel N of $(K, L) + M$ is non-negative. By Lemma 6.19, since $L^\vee = L$, we have

$$N = L^> + M^{>\vee} - M^{\vee>} = L^{>\vee} + L^> - L^{\vee>\vee} + M^{>\vee} - M^{\vee>}.$$

In other words, setting $A = M^\vee - L$, we get that $L^> + A^{\vee>\vee} - A^>$ is non-negative although L is not.

Proof of Proposition 14.12. Note that we have the inclusion

$$((\mathcal{M}^1_{k+1})^+)^{\vee<} \subset (\mathcal{M}^1_k)^+$$

by definition and we only need to prove the reverse inclusion. By Lemma 14.6 and Lemma 14.13, $((\mathcal{M}^1_{k+1})^+)^{\vee<}$ is closed in $\mathcal{M}^1_k$, so that we can prove the result by looking at the dual cones. By Corollary 14.5, Corollary 14.8 and Corollary 14.10, the dual cone of $(\mathcal{M}^1_k)^+$ is $I^*_k(\mathcal{M}^+_k)$. Again by Corollary 14.8, the dual cone of $((\mathcal{M}^1_{k+1})^+)^{\vee<}$ is the set

$$(((\mathcal{M}^1_{k+1})^+)^{\vee<})^* = I^*_k\{L \in \mathcal{L}_k | I^*_{k+1}(L^{>\vee}) \in I^*_{k+1}(\mathcal{M}^+_{k+1})\}.$$

Thus, Lemma 14.14 says that we have $(((\mathcal{M}^1_{k+1})^+)^{\vee<})^* = ((\mathcal{M}^1_k)^+)^*$ and the Proposition follows. □

CHAPTER 15

Non-negative bilinear forms and non-negative functions

In this chapter, as an application of Theorem 14.1, we show that, for w a symmetric Γ-invariant function on X_k, $k \geq 2$, if the bilinear form Φ_w is non-negative, then w can be assumed to have non-negative values.

THEOREM 15.1. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume the symmetric bilinear form Φ_w associated to w on H_0^ω is non-negative. Then, w is cohomologous to a function with non-negative values, that is, there exists a skew-symmetric Γ-invariant function v on X_{k-1} such that, for any (x, y) in X_k, one has*

$$w(x, y) \geq v(x, y_1) + v(x_1, y).$$

See Section 3 for the definition of H_0^ω and Φ_w.

The proof of this result will rely on a duality argument. Theorem 14.1 gives a dual characterization of the functions w such that Φ_w is non-negative. We will establish a dual characterization of the functions w such that w is cohomologous to a function with non-negative values and compare the two characterizations in order to conclude. The latter one will be formulated in analogy with Livšic Theorem in hyperbolic dynamics (see [**26**, Theorem 19.2.1]), as in [**36**]. In particular, its proof will require us to give a more powerful version of the closing Lemma than Lemma 2.6. To show this enhanced closing Lemma, we will begin by establishing an equidistribution result in $\Gamma\backslash X_1$ to ensure that Γ contains sufficiently many hyperbolic elements.

15.1. Patterson-Sullivan functions

In this Section, we adapt Patterson-Sullivan theory (see for example [**42**]) to our language. The following results will be no surprise for the experts of this domain. In the context of the proof of Theorem 15.1, we will use them to get a proof of the strong closing Lemma 15.9 below.

PROPOSITION 15.2. *There exists a positive Γ-invariant function u on X_1 and a number $\rho \geq 2$ such that, for every $x \sim y$ in X, one has*

$$\sum_{\substack{z \sim y \\ z \neq x}} u(y,z) = \rho u(x,y).$$

The number ρ is unique and the function u is unique up to multiplication by a positive real number.

Let $\mathscr{V}_1$ be the space of functions on X_1 that are Γ-invariant. We define a linear endomorphism R of $\mathscr{V}_1$ by setting

$$Rv(x,y) = \sum_{\substack{z \sim y \\ z \neq x}} v(y,z), \quad v \in \mathscr{V}_1, \quad x \sim y \in X. \tag{15.1}$$

The number ρ is the spectral radius of R.

REMARK 15.3. If X is regular, that is, if the number $d(x)$ of neighbours of a vertex x of X is independent of x with constant value d, we have $\rho = d - 1$ and we can take $u = 1$ in Proposition 15.2. The reader who is only interested in this case can skip Section 15.1 entirely.

REMARK 15.4. This number ρ only depends on X, not on Γ. Indeed, as u is positive by Proposition 15.2, there exists $C > 1$ such that, for any $n \geq 1$ and $x \sim y$ in X, one has

$$\begin{aligned} \frac{1}{C}\rho^n u(xy) = \frac{1}{C} R^n u(xy) \leq |\{z \in X | d(x,z) = n+1, y \in [xz]\}| \\ \leq C R^n u(xy) = C\rho^n u(xy). \end{aligned}$$

For this reason, we call ρ the growth rate of X.

The function u is said to be a Patterson-Sullivan function of Γ.

We will need to state new results in the language of Section 2.2.

LEMMA 15.5. *Let $x \neq y$ be in X. There exists a hyperbolic element of Γ whose axis contains the segment $[xy]$. Equivalently, the set of pairs of fixed points of hyperbolic elements of Γ is dense in $\partial^2 X$.*

PROOF. Recall that U_{xy} is the set of ξ in ∂X such that y belongs to $[x\xi)$. By Proposition 2.3, there exists ξ in U_{yx} and η in U_{xy} such that the orbit $\Gamma(\xi, \eta)$ is dense in $\partial^2 X$. Set $k = d(x,y)$ and

$$\Gamma_k = \{g \in \Gamma | d(x, gx) \geq k \text{ and } d(y, gy) \geq k\}.$$

By assumption, $\Gamma \smallsetminus \Gamma_k$ is finite. As every element in X has at least three neighbours, ∂X has no isolated points and $\Gamma_k(\xi, \eta)$ is still dense in $\partial^2 X$. Therefore, we can find g in Γ such that $d(x, gx) \geq k$, $d(y, gy) \geq k$,

FIGURE 1. The first case in the proof of Lemma 15.5

$g\xi \in U_{yx}$ and $g\eta \in U_{xy}$. Let us show that g is hyperbolic and that $[xy]$ is contained in the axis of g.

Indeed, note that the assumption imply the inclusion

$$[xy] \subset ((g\xi)(g\eta)).$$

Now, let x_1 and y_1 be the neighbours of x and y on $[xy]$. Consider the points gx and gy on the geodesic line $((g\xi)(g\eta))$.

If gy belongs to the geodesic ray $[y(g\eta))$, as gx belongs to the geodesic ray $[(gy)(g\xi))$ and $d(gx, gy) = d(x, y)$, gx belongs to the geodesic ray $[x(g\eta))$ (see Figure 1). As $d(x, gx) \geq k$, we have $[xy] \subset [x(gx)]$ and in particular, x_1 is the neighbour of x on $[x(gx)]$. Since gx and gy both belong to $[y(g\eta))$, we have $gx_1 \notin [x(gx)]$. Therefore, by Lemma 2.5, g is hyperbolic and $[x(gx)]$ is contained in the axis of g. All the more so is $[xy]$.

If gy does not belong to the geodesic ray $[y(g\eta))$, as $d(gy, y) \geq k$, we must have $gy \in [x(g\xi))$. Since gx belongs to $[(gy)(g\xi))$, we get that gx belongs to $[x(g\xi))$ and, as above, we conclude that g is hyperbolic and that $[xy]$ is contained in the axis of g. □

Given two edges (a, b) and (x, y) in X_1, say that the edge (x, y) is visible from the edge (a, b) if b and x belong to the segment $[ay]$.

COROLLARY 15.6. *Let $a \sim b$ and $x \sim y$ be in X. There exists γ in Γ such $(\gamma x, \gamma y)$ is visible from (a, b).*

PROOF. By Lemma 2.4, there exists g in Γ such that a does not belong to the segments $[b(gx)]$ and $[b(gy)]$. If (gx, gy) is visible from (a, b), we can set $\gamma = g$ and we are done.

If no, that is, if gy belongs to the segment $[b(gx)]$, the situation is as in Figure 2. Then, we chose a neighbour $z \neq gx$ of gy that does not belong to $[b(gy)]$ (which exists as $d(gy) \geq 3$). By Lemma 15.5, we can find a hyperbolic element h of Γ whose axis contains $[z(gx)]$. As h acts as a non trivial translation on its axis, up to replacing h by its inverse, hgx is the neighbour of hgy on the segment $[(hgy)z]$. Thus, (hgx, hgy) is visible from (a, b) and we can set $\gamma = hg$. □

After these geometric preliminaries, we can prove the Proposition. This relies on classical arguments from the theory of Markov chains.

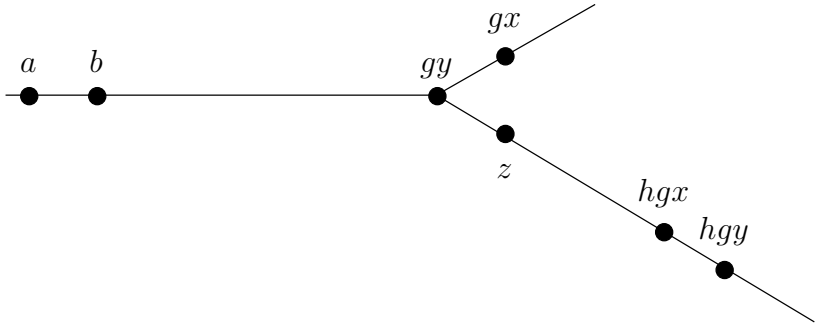

FIGURE 2. The second case in the proof of Corollary 15.6

PROOF OF PROPOSITION 15.2. Let R be as in (15.1). Let $\rho \geq 0$ be the spectral radius of R. As any x in X admits at least 3 neighbours, we have $R^n\mathbf{1} \geq 2^n\mathbf{1}$, $n \geq 0$, hence $\rho \geq 2$. As R maps non-negative functions to non-negative functions, by Perron-Frobenius Theorem (see for example Lemma 11.28), there exists a non zero non-negative element u in $\mathscr{V}_1$ which is an eigenvector of R with eigenvalue ρ, that is, $Ru = \rho u$.

Let v be any non-zero non-negative function on X_1 and λ be a real number with $Rv = \lambda v$. In particular, we have $\lambda \geq 0$. We claim that λ is positive and that v only takes positive values. Indeed, let $x \sim y$ be such that $v(x, y) > 0$. Fix $a \sim b$ in X and let γ be as in Corollary 15.6. Set $n = d(a, \gamma x)$. We have

$$\lambda^n v(a, b) = R^n v(a, b) \geq v(\gamma x, \gamma y) = v(x, y) > 0,$$

hence $\lambda \neq 0$ and v is positive. In particular, u is positive.

Besides, as v is positive, we can find $\varepsilon > 0$ with $v \geq \varepsilon u$. Fix $x \sim y$ in X. For any $n \geq 0$, we get

$$\lambda^n v(x, y) = R^n v(x, y) \geq \varepsilon R^n u(x, y) = \rho^n \varepsilon u(x, y),$$

hence $\lambda = \rho$. Let us show that v is a multiple of u, which finishes the proof. This is an application of the maximum principle. Set $\alpha = \sup_{X_1} \frac{v}{u}$ and let $Y \subset X_1$ be the set of (x, y) in X_1 with $v(x, y) = \alpha u(x, y)$. This is a Γ-invariant subset of X_1.

Fix (x, y) in Y. We claim that we have $(y, z) \in Y$ for all $z \sim y$, $z \neq x$. Indeed, we have

$$\alpha = \frac{v(x, y)}{u(x, y)} = \sum_{\substack{z \sim y \\ z \neq x}} \frac{v(y, z)}{\rho u(x, y)} = \sum_{\substack{z \sim y \\ z \neq x}} \frac{v(y, z)}{u(y, z)} \frac{u(y, z)}{\rho u(x, y)}$$

and $\sum_{\substack{z\sim y\\ z\neq x}} \frac{u(y,z)}{\rho u(x,y)} = 1$. As, for all z as above $v(y,z) \leq \alpha u(y,z)$, we get $v(y,z) = \alpha u(y,z)$.

Since Y is Γ-invariant, Corollary 15.6 shows that $Y = X_1$ and we are done. □

For x, y in X, say that x and y are evenly related if the distance $d(x,y)$ is an even integer. This relation is an equivalence relation with two equivalence classes. Denote by Y this set of equivalence classes, so that Y has two elements. We shall say that the action of Γ on X is bipartite if the induced action on Y is trivial. Equivalently, the action of Γ on X is bipartite if, for any x in X and $\gamma \in \Gamma$, the distance $d(x, \gamma x)$ is even.

For u in $\mathscr{V}_1$, let $u^\vee$ be the function $(x,y) \mapsto u(y,x)$ on X_1. We equip $\mathscr{V}_1$ with the usual scalar product

$$(u,v) \mapsto \langle u, v\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} u(x,y)v(x,y). \tag{15.2}$$

The proof of Proposition 15.2 yields a description of the other eigenvalues of R with modulus ρ. We split the statement according to whether the action of Γ on X is bipartite or not.

Corollary 15.7. *Let ρ be the growth rate of X and u be a Patterson-Sullivan function on X_1.*

Assume that the action of Γ on X is not bipartite. Then, ρ is the unique complex eigenvalue of R with modulus ρ. In particular, for any v in $\mathscr{V}_1$, we have

$$\rho^{-n} R^n v \xrightarrow[n\to\infty]{} \frac{\langle v, u^\vee\rangle}{\langle u, u^\vee\rangle} u. \tag{15.3}$$

Assume that the action of Γ on X is bipartite. Then, the complex eigenvalues of R with modulus ρ are ρ and $-\rho$. Let χ be a function on X that is constant with value 1 on one of the classes of the even distance equivalence relation, and -1 on the other. Then, the function

$$\chi u : (x,y) \mapsto \chi(x)u(x,y), X_1 \to \mathbb{R}$$

is, up to a scalar multiple, the unique eigenvector of R with eigenvalue $-\rho$. In particular, for any v in $\mathscr{V}_1$, we have

$$\rho^{-n} R^n v - \frac{1}{\langle u, u^\vee\rangle}(\langle v, u^\vee\rangle u - (-1)^n \langle v, (\chi u)^\vee\rangle \chi u) \xrightarrow[n\to\infty]{} 0. \tag{15.4}$$

Proof. We start by studying the complex eigenvectors of R associated with eigenvalues of modulus ρ. Let v be a Γ-invariant complex

valued function on X_1 such that $Rv = \rho e^{i\theta} v$ for some real number θ. For $x \sim y$ in X, we have

$$\frac{v(x,y)}{u(x,y)} = e^{-i\theta} \sum_{\substack{z \sim y \\ z \neq x}} \frac{v(y,z)}{\rho u(x,y)} = e^{-i\theta} \sum_{\substack{z \sim y \\ z \neq x}} \frac{v(y,z)}{u(y,z)} \frac{u(y,z)}{\rho u(x,y)}. \tag{15.5}$$

As $Ru = \rho u$, by convexity, we have

$$\frac{|v(x,y)|}{u(x,y)} \leq \sum_{\substack{z \sim y \\ z \neq x}} \frac{|v(y,z)|}{u(y,z)}.$$

By using the maximum principle as in the proof of Proposition 15.2, we get that $|v|$ is a multiple of u, and we can assume $|v| = u$. Set $\varphi = \frac{v}{u}$, so that φ has constant modulus 1. From (15.5) and the fact that $Ru = \rho u$ we get that, for any x, y, z in X with $x \sim y$, $y \sim z$ and $x \neq z$, we have

$$\varphi(x,y) = e^{-i\theta} \varphi(y,z). \tag{15.6}$$

In particular, if t is a neighbour of y with $t \neq x$ and $t \neq z$ (which exists as $d(y) \geq 3$), we have

$$\varphi(t,y) = e^{-i\theta} \varphi(y,z) = e^{-i\theta} \varphi(y,x).$$

Therefore, we have shown that $\varphi(y,z) = \varphi(y,x)$ for any y in X and any two neighbours x, z of y. We write $\psi(y)$ for this value. Then, (15.6) implies that, for any $x \sim y$ in X, we have

$$\psi(x) = e^{-i\theta} \psi(y).$$

By reversing the roles of x and y, we also have $\psi(y) = e^{-i\theta} \psi(x)$, which gives $e^{2i\theta} = 1$, that is, $e^{i\theta} \in \{-1, 1\}$. If $e^{i\theta} = 1$, then ψ is constant. If $e^{i\theta} = -1$, then ψ is a multiple of the function χ of the statement.

Assume the action of Γ is not bipartite. Equivalently, the function χ is not Γ-invariant. In this case, we have just proved that any complex eigenvalue of R other than ρ has modulus $< \rho$ and that the eigenspace associated to ρ is $\mathbb{R}u$. Set L to be the endomorphism of $\mathscr{V}_1$ defined by

$$Lv(x,y) = \sum_{\substack{z \sim x \\ z \neq y}} v(z,x) = (Rv^\vee)^\vee(x,y), \quad v \in \mathscr{V}_1, \quad x \sim y \in X. \tag{15.7}$$

Then, the eigenspace of L associated to ρ is $\mathbb{R}u^\vee$. By using Lemma 10.11, one can show that L and R are adjoint to each other with respect to the scalar product $\langle ., . \rangle$. As u is positive, we have $\langle u, u^\vee \rangle > 0$ and hence the orthogonal complement to the line $\mathbb{R}u^\vee$ is an R-invariant subspace of $\mathscr{V}_1$ which does not contain u. In particular, the spectral

radius of R in this orthogonal coomplement is $< \rho$ and (15.3) follows by elementary linear algebra.

Assume the action of Γ is bipartite. Then, the function χ is Γ-invariant and R admits two eigenvalues with modulus ρ, which are ρ itself, with eigenspace $\mathbb{R}u$, and $-\rho$, with eigenspace $\mathbb{R}\chi u$. Let still L be as in (15.7). Then $\mathbb{R}u^\vee$ is the eigenline of L associated to the eigenvalue ρ and $\mathbb{R}(\chi u)^\vee$ is the eigenline of L associated to the eigenvalue $-\rho$. Therefore, R has spectral radius $< \rho$ in the invariant subspace

$$\{v \in \mathscr{V}_1 | \langle v, u\rangle = \langle v, (\chi u)^\vee\rangle = 0\}.$$

Again, (15.4) follows by elementary linear algebra and by noticing that $\langle \chi u, (\chi u)^\vee\rangle = -\langle u, u^\vee\rangle$. □

15.2. Cohomology classes with non-negative values

In order to prove Theorem 15.1, we give a characterization of functions that are cohomologous to functions with non-negative values in the spirit of Livšic Theorem in hyperbolic dynamics (see [**26**, Theorem 19.2.1]).

Let us use the language of Section 2.2. Let γ be a hyperbolic element in Γ and (ξ, η) be the invariant geodesic line of γ. Choose a parametrization $(x_h)_{h\in\mathbb{Z}}$ of $(\xi\eta)$ and denote by $\ell \geq 1$ the translation length of γ, so that $\gamma x_h = x_{h+\ell}$, $h \in \mathbb{Z}$. For $k \geq 1$ and w a Γ-invariant function on X_k, we set

$$\sum_{[\gamma]} w = \sum_{h=0}^{\ell-1} w(x_h, x_{h+k}).$$

Indeed, this number only depends on w and the conjugacy class $[\gamma]$ of γ in Γ.

Recall that $\mathscr{S}$ stands for the space of parametrized geodesic lines of X and $T : \mathscr{S} \to \mathscr{S}$ for the time shift. In Section 2.5, we have identified the space of Γ-invariant functions on X_k, $k \geq 1$, with a subspace of the space of smooth functions on $\Gamma\backslash\mathscr{S}$. If w is a Γ-invariant function on X_k, we associate to w the function

$$(x_h)_{h\in\mathbb{Z}} \mapsto w(x_0, x_k), \mathscr{S} \to \mathbb{R}.$$

As in Section 2.3, two smooth functions f and g on $\Gamma\backslash\mathscr{S}$ are said to be cohomologous if $f - g = h - h \circ T$ for some smooth function h on $\Gamma\backslash\mathscr{S}$. The following statement is a direct analogue of the main result of [**36**]. The proof will follow the same lines.

Proposition 15.8. *Let $k \geq 2$ and w be a Γ-invariant function on X_k. The following are equivalent:*

(i) *For any T-invariant Borel probability measure μ on $\Gamma\backslash\mathscr{S}$, one has*

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu \geq 0,$$

where f is a smooth function on $\Gamma\backslash\mathscr{S}$ that is cohomologous to the function associated to w.
(ii) *For any hyperbolic element γ of Γ, one has*

$$\sum_{[\gamma]} w \geq 0.$$

(iii) *The function w is cohomologous to a non-negative function, that is, there exists a Γ-invariant function v on X_{k-1} such that, for any (x, y) in X_k, one has*

$$w(x, y) \geq v(x, y_1) - v(x_1, y).$$

If w is symmetric on X_k, the function v can be assumed to be skew symmetric.

The proof will use the following version of the closing Lemma, which is a reinforcement of Lemma 2.6:

LEMMA 15.9. *There exists $c \geq 0$ with the following property: for any $x \neq y$ in X, there exists a hyperbolic element γ in Γ with invariant geodesic line $(\xi\eta)$ such that*

$$[xy] \subset [x(\gamma x)] \subset (\xi\eta) \text{ and } d(y, \gamma x) \leq c.$$

PROOF. We will deduce this fact from the equidistribution statements in Corollary 15.7. We keep the notation of this result: $\mathscr{V}_1$ is the space of Γ-invariant functions on X_1, equipped with the usual scalar product from (15.2), R is the operator defined in (15.1), ρ is the growth rate of X and u is a Patterson-Sullivan function as in Proposition 15.2. By Corollary 15.7, for v in $\mathscr{V}_1$ with $\langle v, u^\vee \rangle = 1$, we have

$$\rho^{-2n} R^{2n} v + \rho^{-2n-1} R^{2n+1} v \xrightarrow[n\to\infty]{} \frac{2}{\langle u, u^\vee \rangle} u. \tag{15.8}$$

As $u^\vee$ takes only positive values, the set

$$K = \{v \in \mathscr{V}_1 | v \geq 0 \text{ and } \langle v, u^\vee \rangle = 1\}$$

is compact. Therefore, the convergene of linear operators in (15.8) is uniform on K. Thus, as u is positive, there exists $n \geq 1$ such that, for any non zero $v \geq 0$ in $\mathscr{V}_1$, the function $\rho^{-2n} R^{2n} v + \rho^{-2n-1} R^{2n+1} v$ is positive everywhere. Let $x \neq y$ be in X and x_1 and y_1 be the neighbours of x and y on $[xy]$. Applying the previous property to the function $v = \sum_{\gamma\in\Gamma} \mathbf{1}_{\gamma(x,x_1)}$, tells us that there exists γ in Γ such that

$d(\gamma x_1, y) \leq 2n+1$ and $(\gamma x, \gamma x_1)$ is visible from (y_1, y) (that is, we have $[y(\gamma x)] \subset [y_1(\gamma x_1)]$). In particular, we have $[xy] \subset [x(\gamma x)]$, $x_1 \in [x(\gamma x)]$ and $\gamma x_1 \notin [x(\gamma x)]$, so that, by Lemma 2.5, γ is hyperbolic and its axis contains $[x(\gamma x)]$, hence $[xy]$. The Lemma follows by taking $c = 2n$. □

Now that we have an adapted closing Lemma, we can borrow the main idea of the proof of Proposition 15.8 from [**36**].

Proof of Proposition 15.8. *(iii)⇒(i).* Assume v is as in the statement. Let f and h be the smooth functions on $\mathscr{S}$ associated to w and v. For $s = (x_h)_{h\in\mathbb{Z}}$ in $\mathscr{S}$, we have $f(s) = w(x_0, x_k)$ and $h(s) = v(x_0, x_{k-1})$ so that, by assumption, $f \geq h - h\circ T$. The conclusion follows.

(i)⇒(ii). Let γ be a hyperbolic element with translation length $\ell \geq 1$ and $s = (x_h)_{h\in\mathbb{Z}}$ be a parametrization of the invariant geodesic line of γ. We have $T^\ell s = \gamma s$, hence the probability measure on $\Gamma\backslash\mathscr{S}$,

$$\mu = \frac{1}{\ell}\sum_{h=0}^{\ell-1} \delta_{T^h\Gamma s}$$

is T-invariant. By construction, we have

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu = \frac{1}{\ell}\sum_{[\gamma]} w,$$

where f is the smooth function associated with w. The conclusion follows.

(ii)⇒(iii). As in Section 9.1, for $j \geq k$ and (x, y) in X_j, we write

$$\sum_{[xy]} w = \sum_{h=0}^{j-k} w(x_h, x_{h+k}),$$

where $x_0 = x, x_1, \ldots, x_{j-1}, x_j = y$ is the geodesic path from x to y. Let c be as in Lemma 15.9. By this Lemma, for x, y in X with $d(x, y) \geq k$, we have

$$\sum_{[xy]} w \geq \sum_{[\gamma]} w - c \sup|w|,$$

where γ is a certain hyperbolic element. By assumption, we get

$$\inf_{\substack{x,y\in X\\ d(x,y)\geq k}} \sum_{[xy]} w > -\infty.$$

We use this property to define the function v. For (x, y) in X_{k-1}, we set

$$v(x,y) = \inf_{\substack{z\in X\\ y\in[xz]\\ d(x,z)\geq k}} \sum_{[xz]} w.$$

We claim that v satifies the required property. Indeed, let (x, y) be in X_k and x_1 and y_1 be the neighbours of x and y on $[xy]$. Note that we have

$$\{z \in X | y \in [x_1 z] \text{ and } d(x_1, z) \geq k\} \\ \subset \{z \in X | y_1 \in [xz] \text{ and } d(x, z) \geq k + 1\}$$

and that, for such a z, we have

$$w(x,y) + \sum_{[x_1 z]} w = \sum_{[xz]} w.$$

We get

$$w(x,y) + v(x_1,y) \geq v(x,y_1)$$

as required. If w is symmetric, we also have

$$w(x,y) + v(y_1,x) \geq v(y,x_1),$$

hence

$$2w(x,y) + v(x_1,y) - v(y,x_1) \geq v(x,y_1) - v(y_1,x),$$

that is, we may replace v by a skew-symmetric function. □

15.3. Bilinear forms and invariant distributions

The proof of Theorem 15.1 will use a duality property. Thus, we will now study distributions on $\Gamma\backslash\mathscr{S}$.

We start by describing general phenomena. Let U be a locally compact totally discontinuous topological space, endowed with a proper action of a discrete group G. If φ is a smooth function with compact support on U, the function $\overline{\varphi}$ defined by

$$\overline{\varphi}(u) = \sum_{g\in G} \varphi(gu), \quad u \in U,$$

is smooth with compact support on $G\backslash U$. The map $\varphi \mapsto \overline{\varphi}$ sends the space $\mathcal{D}(U)$ of compactly supported smooth functions on U onto $\mathcal{D}(G\backslash U)$. The adjoint map $\mathcal{D}^*(G\backslash U) \to \mathcal{D}^*(U)$ is injective and its range is exactly the space $\mathcal{D}^*(U)^G$ of G-invariant distributions on U.

In our concrete situation, the space $\mathscr{S}$ of parametrized geodesic lines is equipped with two commuting group actions. One is the natural action of Γ. The other is the action of $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ defined by the time

shift T and the time reversal $\iota : (x_h)_{h\in\mathbb{Z}} \mapsto (x_{-h})_{h\in\mathbb{Z}}$. The quotient of $\mathscr{S}$ by the $\mathbb{Z}$-action generated by T may be identified with $\partial^2 X$ by the map that forgets the parametrization of a geodesic line.

Say that a distribution on $\partial^2 X$ is symmetric if it is invariant under the map $(\xi, \eta) \mapsto (\eta, \xi)$. We have just defined a natural identification of the space of (ι, T)-invariant distributions in $\mathcal{D}^*(\Gamma\backslash\mathscr{S})$ with the space of Γ-invariant symmetric distributions in $\mathcal{D}^*(\partial^2 X)$. We claim that the latter space can be identified with the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ of symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. Indeed, this comes from the following extension of Lemma 4.15:

LEMMA 15.10. *Assume U is a compact totally discontinuous topological space. Set $U_2 = \{(x, y) \in U^2 | x \neq y\}$ and $\iota : U_2 \to U_2$ to be the natural involution $(x, y) \mapsto (y, x)$. Write $\overline{\mathcal{D}}(U)$ for the quotient of the space $\mathcal{D}(U)$ by the constant functions. For φ, ψ in $\mathcal{D}(U)$ and (x, y) in U_2, set*

$$\Xi(\varphi, \psi) = \frac{1}{2}(\varphi(x) - \varphi(y))(\psi(x) - \psi(y)).$$

Then $\Xi(\varphi, \psi)$ is a smooth compactly supported function on U_2, that is, it belongs to $\mathcal{D}(U_2)$. If θ is a distribution on U_2, let q_θ be the symmetric bilinear form on $\overline{\mathcal{D}}(U)$ defined by

$$q_\theta(\varphi, \psi) = \theta(\Xi(\varphi, \psi)), \quad \varphi, \psi \in \overline{\mathcal{D}}(U).$$

Then, the map $\theta \mapsto q_\theta$ is a linear isomorphism from the space $\mathcal{D}^*(U_2)^\iota$ of ι-invariant distributions on U_2 onto the space $\mathcal{Q}(\overline{\mathcal{D}}(U))$ of symmetric bilinear forms on $\overline{\mathcal{D}}(U)$.

PROOF. Note that, as φ and ψ are smooth, that is, locally constant, the function $\Xi(\varphi, \psi)$ has compact support in U_2 and the bilinear form q_θ is well-defined.

Let $U = \bigsqcup_{a\in A} U_a$ be a partition of U into finitely many closed open subsets. Then, the space $V \subset \mathcal{D}(U)$ of functions that are constant on each of the U_a may be identified with the space of functions on the finite set A. The space $\mathcal{D}(U)$ is the union of all such subspaces V. The Lemma follows by applying Lemma 4.15 to each of them. □

Thus, in view of Proposition 4.1, we can identify the space of symmetric distributions in $\mathcal{D}^*(\partial^2 X)$ with the space of quadratic type functions on X_*. This identification is given by the following formula.

COROLLARY 15.11. *Let θ be a symmetric distribution on $\partial^2 X$ and φ be the associated quadratic type function on X_*. For (x, y) in X_*, we have*

$$\varphi(x, y) = \frac{1}{2}\theta(\mathbf{1}_{V_{xy}}),$$

where

$$V_{xy} = \{(\xi, \eta) \in \partial^2 X | [xy] \subset (\xi\eta)\}. \tag{15.9}$$

REMARK 15.12. With the notation of the Corollary, one easily checks that the function $(x, y) \mapsto \frac{1}{2}\theta(\mathbf{1}_{V_{xy}})$ is of quadratic type.

PROOF. Let q be the symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ associated with θ in Lemma 15.10. Recall from Section 4.1, that we have set

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}$$

and

$$\varphi(x, y) = -q(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{yx}}).$$

For $\xi \neq \eta$ in ∂X, we have

$$(\mathbf{1}_{U_{xy}}(\xi) - \mathbf{1}_{U_{xy}}(\eta))(\mathbf{1}_{U_{yx}}(\xi) - \mathbf{1}_{U_{yx}}(\eta)) = -\mathbf{1}_{V_{xy}}(\xi, \eta).$$

The conclusion follows by Lemma 15.10. □

We conclude this discussion by giving a formula for computing the value of a distribution on a given function thanks to these identifications.

LEMMA 15.13. *Fix $k \geq 1$.*

Let θ be a (ι, T)-invariant distribution on $\mathscr{S}$ with associated quadratic type function φ on X_*. Let w be a finitely supported function on X_k and f be the smooth compactly supported function on $\mathscr{S}$

$$f : (x_h)_{h\in\mathbb{Z}} \mapsto w(x_0, x_k), \mathscr{S} \to \mathbb{R}.$$

Then we have

$$\langle \theta, f \rangle = \frac{1}{2} \sum_{(x,y)\in X_k} w(x, y)\varphi(x, y).$$

Let θ be a (ι, T)-invariant distribution on $\Gamma\backslash\mathscr{S}$ with associated quadratic type function φ on X_*. Let w be a Γ-invariant function on X_k and f be a smooth function on $\Gamma\backslash\mathscr{S}$ that belongs to the cohomology class associated to w. We have

$$\langle \theta, f \rangle = \frac{1}{2} \sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} w(x, y)\varphi(x, y).$$

PROOF. We establish the first formula. It suffices to prove the claim when w is the indicator function $\mathbf{1}_{\{(a,b),(b,a)\}} = \mathbf{1}_{(a,b)} + \mathbf{1}_{(b,a)}$ for some (a, b) in X_k. As in the statement, let f be the associated function on $\mathscr{S}$ and identify the quotient of $\mathscr{S}$ by $T^{\mathbb{Z}}$ with $\partial^2 X$. Then the function $\overline{f} = \sum_{n\in\mathbb{Z}} f \circ T^n$ is the indicator function of the set V_{ab} defined in (15.9) and the formula follows by Corollary 15.11.

As in the proof of Proposition 13.17, the second formula follows from the first by using Lemma 13.20. □

Let us describe the behaviour of Radon measures under our chain of identifications. Recall that, if U is a second countable totally discontinous locally compact space, a distribution θ on U is a Radon measure if and only if, for any non-negative function φ in $\mathcal{D}(U)$, one has $\theta(\varphi) \geq 0$. If $\mathcal{V}$ is a collection of compact open subsets of U which form a basis of the topology of U, then θ is a Radon measure if and only if one has $\theta(\mathbf{1}_V) \geq 0$ for any V in $\mathcal{V}$. If G is a discrete group acting properly on U, the natural identification between $\mathcal{D}^*(G\backslash U)$ and $\mathcal{D}^*(U)^G$ induces an identification of the space of Radon measures on $G\backslash U$ with the space of G-invariant Radon measures on U.

In our case, the sets $(V_{xy})_{(x,y)\in X_1}$ of (15.9) form a basis of the topology of the quotient space of $\partial^2 X$ by the action of ι. Thus, by using Corollary 15.11, we get

LEMMA 15.14. *Let φ be a quadratic type function on X_*. Then the associated symmetric distribution on $\partial^2 X$ is a Radon measure if and only if φ takes non-negative values. If φ is Γ-invariant, the associated (ι, T)-invariant distribution on $\Gamma\backslash\mathscr{S}$ is a Radon measure if and only φ takes non-negative values.*

Note that, as $\Gamma\backslash\mathscr{S}$ is compact, a Radon measure on this space is the same as a finite Borel measure.

15.4. Radon measures and tight quadratic fields

Theorem 15.1 will be a consequence of the following dual statement which uses the vocabulary that we have just introduced.

PROPOSITION 15.15. *Let μ be a (ι, T)-invariant Borel probability measure on $\Gamma\backslash\mathscr{S}$ and φ be the associated quadratic type function on X_*. Fix $k \geq 2$ and let p be the k-quadratic field that is the natural image of φ in $\mathcal{F}_k$. Then p is tight.*

See Section 4 for the relations between quadratic type functions and quadratic fields. See Section 14 for the definition of tight quadratic fields.

The proof relies on the construction of the root map in Section 13.5 and the following elementary remark.

LEMMA 15.16. *Let A be a finite set. Let V be the space of real-valued functions on A and $\overline{V} = V/\mathbb{R}$ be its quotient by the space of*

constant functions. Let p be a symmetric bilinear form on $\overline{V}$. For $a \neq b$ in A, set

$$\varphi(a,b) = -p(\mathbf{1}_a, \mathbf{1}_b).$$

Assume that φ takes non-negative values on $A_2 = \{(a,b) \in A^2 | a \neq b\}$. Then, p is non-negative as a bilinear form.

Proof. Indeed, by a direct computation (or by Lemma C.5), for f in $\overline{V}$, one has

$$p(f,f) = \frac{1}{2} \sum_{(a,b)\in A_2} \varphi(a,b)(f(a) - f(b))^2.$$

□

Proof of Proposition 15.15. By Lemma 15.14, as μ is a positive measure, the function φ has non-negative values. Let s be the root of p. By Definition 13.14, Definition 13.15 and Lemma 15.16, the quadratic pseudofield s is non-negative. By Proposition 13.16, s is radical and the shoot of s is p. Therefore p is tight by Theorem 14.1. □

As announced, we will deduce Theorem 15.1 from Proposition 15.15.

Proof of Theorem 15.1. As in the statement, let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k such that the associated bilinear form Φ_w is non-negative on H_0^ω. We will show that w is cohomologous to a non-negative function by applying Proposition 15.8. Thus, we take a T-invariant Borel probability measure μ on $\Gamma\backslash\mathscr{S}$ and we will prove that $\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu \geq 0$, where f is the smooth function on $\Gamma\backslash\mathscr{S}$ associated with w. As w is symmetric and μ is T-invariant, we have $\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu = \int_{\Gamma\backslash\mathscr{S}} f \circ \iota \mathrm{d}\mu$, so that we may replace μ by $\frac{1}{2}(\mu + \iota_*\mu)$ and assume that μ is (ι, T)-invariant. Let φ be the quadratic type function associated to μ as in Section 15.3. By Proposition 15.15, the restriction of φ to X_k is the quadratic type function associated to a tight k-quadratic field p in $\mathcal{F}_k$. By the definition of the weight pairing in Theorem 12.5 and by Lemma 15.13, if (K, K^-) is a k-dual kernel with w as a weight function, we have

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu = [p, (K,K)^-].$$

As p is tight and Φ_w is non-negative, we get $\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu \geq 0$. As this is true for any T-invariant measure μ, w is cohomologous to a non-negative function by Proposition 15.8. □

Part 4

Hilbert spaces of metrics

CHAPTER 16

The admissible Riemannian metric

In this chapter, for $k \geq 2$, we will use the weight pairing of Chapter 12 to define a natural Riemannian metric on the space $\mathcal{P}_k^{\mathrm{ad}}$ of admissible Γ-invariant k-Euclidean fields which was studied in Chapter 11. The orthogonal extension embedding $\mathcal{P}_k^{\mathrm{ad}} \hookrightarrow \mathcal{P}_{k+1}^{\mathrm{ad}}$ will be proved to a Riemannian immersion. This metric may be seen as an analogue of the natural Riemannian metric on the space of positive definite symmetric bilinear forms of a finite-dimensional vector space (see [**27**]).

16.1. The weight tensor

Given $k \geq 2$, we will first use the weight pairing of Section 12.2 to define a natural smooth section g of the vector bundle $\mathcal{Q}(\mathrm{T}\mathcal{P}_k)$ of symmetric bilinear forms on the tangent space of $\mathcal{P}_k$. It turns out that $\mathcal{P}_k^{\mathrm{ad}}$ is precisely a connected component of the set of p such that g_p is positive.

We now give the precise definition of g. Recall that in Section 5.1, we have defined an embedding from the space of k-Euclidean fields into the space of k-dual kernels. In case of Γ-invariant Euclidean fields, the spaces are finite-dimensional and we have studied this embedding from the point of view of differential geometry. In particular, we have shown in Proposition 11.22 that it is a smooth map. As in Section 11.9, let us now denote this map by $\iota_k : \mathcal{P}_k \hookrightarrow \mathcal{K}_k$. As before, we also denote by $\mathcal{F}_k$ the space of Γ-invariant k-quadratic fields.

Let p in $\mathcal{P}_k$ be a Γ-invariant k-Euclidean field. If q and r are Γ-invariant k-quadratic fields (which we view as tangent vectors to $\mathcal{P}_k$), we set

$$g_p(q, r) = -[q, \mathrm{d}_p\iota_k(r)],$$

where $[.,.]$ is the weight pairing. We call g the weight tensor on $\mathcal{P}_k$. The reason why we don't mention the dependance on k in our notation for the weight tensor will become clear in the next sections.

LEMMA 16.1. *Let $k \geq 2$. For any p in $\mathcal{P}_k$, g_p is a symmetric bilinear form on $\mathcal{F}_k$.*

If k is even, $k = 2\ell$, $\ell \geq 1$, let q, r be in $\mathcal{F}_k$. For x in X, let A_x and B_x be the p_x-symmetric endomorphisms of $\overline{V}^{\ell}(x)$ which represent

q_x and r_x with respect to p_x. For $x \sim y$ in X, let A^-_{xy} and B^-_{xy} be the p^-_{xy}-symmetric endomorphisms of $\overline{V}^{\ell-1}(xy)$ which represent q^-_{xy} and r^-_{xy} with respect to p^-_{xy}. One has

$$g_p(q,r) = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\operatorname{tr}(A_xB_x) - \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x\cap\Gamma_y|}\operatorname{tr}(A^-_{xy}B^-_{xy}).$$

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, let q, r be in $\mathcal{F}_k$. For $x \sim y$ in X, let A_{xy} and B_{xy} be the p_{xy}-symmetric endomorphisms of $\overline{V}^{\ell}(xy)$ which represent q_{xy} and r_{xy} with respect to p_{xy}. For x in X, let A^-_x and B^-_x be the p^-_x-symmetric endomorphisms of $\overline{V}^{\ell}(x)$ which represent q^-_x and r^-_x with respect to p^-_x. One has

$$g_p(q,r) = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x\cap\Gamma_y|}\operatorname{tr}(A_{xy}B_{xy}) - \sum_{x\in\Gamma\backslash X}\frac{d(x)-1}{|\Gamma_x|}\operatorname{tr}(A^-_xB^-_x).$$

PROOF. The formulas are a direct consequence of the definition of the tensor g, Theorem 12.5 and Lemma C.4. Symmetry follows. □

REMARK 16.2. Note that, with the notation of Section 11.4, for $x \sim y$ in X, one has, if k is even, $I^{\ell-1,\dagger p}_{xy}A_xI^{\ell-1}_{xy} = A^-_{xy} = I^{\ell-1,\dagger p}_{yx}A_yI^{\ell-1}_{yx}$ and, if k is odd, $A^-_x = J^{\ell,\dagger p}_{xy}A_{xy}J^{\ell}_{xy}$.

16.2. Derivative of the orthogonal extension

We will prove that the weight tensor is natural in the sense that, for $k \geq 2$, the weight tensor of $\mathcal{P}_k$ is the pull-back of the one of $\mathcal{P}_{k+1}$ by the orthogonal extension map. This will require us to first prove that this map is smooth.

Let $\eta_k : \mathcal{P}_k \to \mathcal{P}_{k+1}$ denote the orthogonal extension map $p \mapsto p^+$.

PROPOSITION 16.3. *Let $k \geq 2$. The orthogonal extension map η_k is smooth. Let p be in $\mathcal{P}_k$ and q in $\mathcal{F}_k$. We have the following formulas for describing the Γ-invariant $(k+1)$-quadratic field $\mathrm{d}_p\eta_k(q)$. If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we have*

$$\mathrm{d}_p\eta_k(q)_{xy} = (J^{\ell,\dagger p}_{xy})^\star q_x + (J^{\ell,\dagger p}_{yx})^\star q_y - (M^{\ell-1,\dagger p}_{xy})^\star q^-_{xy}.$$

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, for x in X, we have

$$\mathrm{d}_p\eta_k(q)_x = \sum_{y\sim x}(I^{\ell,\dagger p}_{xy})^\star q_{xy} - (d(x)-1)(M^{\ell,\dagger p}_x)^\star q^-_x.$$

The linear maps M^ℓ_x, $x \in X$, $\ell \geq 1$, and M^ℓ_{xy}, $x \sim y \in X$, $\ell \geq 0$, have been defined in Section 4.2. See Lemma 4.4 for their main properties. We have denoted their adjoint linear maps with respect to

the Euclidean structures associated to p in the usual way, as in Section 11.4.

Proposition 16.3 immediately follows from the following abstract result. As usual, for a vector space V, we denote by $\mathcal{Q}(V)$ the space of symmetric bilinear forms on V and by $\mathcal{Q}_{++}(V) \subset \mathcal{Q}(V)$ the set of positive definite forms.

LEMMA 16.4. *Let X be a finite-dimensional vector space, $d \geq 2$ an integer and $X_0, X_1, \ldots, X_d$ be subspaces of X. Assume that, for any $1 \leq i \neq j \leq d$, one has $X_i \cap X_j = X_0$ and that $X/X_0 = \bigoplus_{i=1}^d X_i/X_0$. Set $\mathcal{F}$ to be the space of all $q = (q_0, q_1, \ldots, q_d)$ in $\mathcal{Q}(X_0) \times \mathcal{Q}(X_1) \times \ldots \times \mathcal{Q}(X_d)$ with $(q_i)_{|V_i} = q_0$, $1 \leq i \leq d$, and $\mathcal{P} \subset \mathcal{F}$ to be the set of those p in $\mathcal{F}$ such that each of the p_i, $0 \leq i \leq d$, is positive definite. Then the orthogonal extension map $\eta : \mathcal{P} \to \mathcal{Q}_{++}(X)$ is smooth. If p is in $\mathcal{Q}$ and q is in $\mathcal{F}$, one has*

$$\mathrm{d}_p\eta(q) = P_1^\star q_1 + \cdots + P_d^\star q_d - (d-1)P_0^\star q_0,$$

where, for $0 \leq i \leq d$, P_i is the $\eta(p)$-orthogonal projection $X \to X_i$.

PROOF. As in Section C.2, for a vector space V, we let

$$\delta_V : \mathcal{Q}_{++}(V) \to \mathcal{Q}_{++}(V^*)$$

denote the natural smooth diffeomorphism between scalar products on V and on its dual space V^*.

For $0 \leq i \leq d$, we let $\pi_i : X^* \to X_i^*$ be the restriction map. Set $\mathcal{K} = \mathcal{Q}(X_0^*) \times \mathcal{Q}(X_1^*) \times \ldots \times \mathcal{Q}(X_d^*)$ and let $\sigma : \mathcal{K} \to \mathcal{Q}(X^*)$ be the linear map

$$(r_0, r_1, \ldots, r_d) \mapsto \pi_1^\star r_1 + \cdots + \pi_d^\star r_d - (d-1)\pi_0^\star r_0.$$

We also set $\mathcal{D} : \mathcal{P} \to \mathcal{K}$ to be the product map

$$(p_0, p_1, \ldots, p_d) \mapsto (\delta_{X_0}(p_0), \delta_{X_1}(p_1), \ldots, \delta_{X_d}(p_d)).$$

By Lemma 5.3, we may write η as the product map $\eta = \delta_X^{-1}\sigma\mathcal{D}$. The result now follows from the chain-rule and Lemma C.3. □

For $k \geq 3$, we still denote by $\rho_k : \mathcal{F}_k \mapsto \mathcal{F}_{k-1}$ the reduction map of Γ-invariant k-quadratic fields. This is a linear map.

COROLLARY 16.5. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. We have $\rho_{k+1}\mathrm{d}_p\eta_k = \mathrm{Id}_{\mathcal{F}_k}$. The map η_k is a closed immersion.*

PROOF. By Proposition 4.21, we have $(p^+)^- = p$ for any p in $\mathcal{P}_k$, hence, by differentiating this identity, $\rho_{k+1}\mathrm{d}_p\eta_k = \mathrm{Id}_{\mathcal{F}_k}$ (which can also be checked directly by using the formulas in Proposition 16.3). In particular, $\mathrm{d}_p\eta_k$ is injective. That it has closed range follows from the characterization of orthogonal extensions in Lemma 8.16. □

16.3. Naturality of the weight tensor

We can now examine the behaviour of the weight tensor under orthogonal extension.

PROPOSITION 16.6. *Let $k \geq 2$ and p in $\mathcal{P}_k$ be a Γ-invariant k-Euclidean field. Chose Γ-invariant quadratic fields q in $\mathcal{F}_k$ and r in $\mathcal{F}_{k+1}$. One has*

$$g_{p^+}(\mathrm{d}_p\eta_k(q), r) = g_p(q, \rho_{k+1}r).$$

PROOF. By definition, we have

$$g_{p^+}(\mathrm{d}_p\eta_k(q), r) = -[r, \mathrm{d}_{p^+}\iota_{k+1}\mathrm{d}_p\eta_k(r)] = -[r, \mathrm{d}_p(\iota_{k+1}\eta_k)(q)].$$

Now, Proposition 5.2 tells us that the dual kernel associated to the orthogonal extension of p is the orthogonal extension of the dual kernel associated to p. By differentiating this property at p, we get that the $(k+1)$-dual kernel $\mathrm{d}_p(\iota_{k+1}\eta_k)(q)$ is the orthogonal extension of the k-dual kernel $\mathrm{d}_p(\iota_k)(q)$. By Corollary 12.7, this gives

$$g_{p^+}(\mathrm{d}_p\eta_k(q), r) = -[\rho_{k+1}r, \mathrm{d}_p(\iota_k)(r)] = g_p(q, \rho_{k+1}r),$$

which should be proved. □

In case $\rho_{k+1}r = 0$, Proposition 16.6 gives

COROLLARY 16.7. *Let $k \geq 2$, p be in $\mathcal{P}_k$, q be in $\mathcal{F}_k$ and r be in $\mathcal{F}_{k+1}$ with $\rho_{k+1}(r) = r^- = 0$. One has $g_{p^+}(\mathrm{d}_p\eta_k(q), r) = 0$.*

In case r belongs to $\mathrm{d}_p\eta_k(\mathcal{F}_k)$, Proposition 16.6 gives

COROLLARY 16.8. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. One has $(\mathrm{d}_p\eta_k)^\star g_{p^+} = g_p$.*

In other words, the pull-back of the weight tensor of $\mathcal{P}_{k+1}$ by the orthogonal extension map is the weight tensor of $\mathcal{P}_k$.

PROOF. By Corollary 16.5, for q in $\mathcal{F}_k$, one has $\rho_{k+1}\mathrm{d}_p\eta_k(q) = q$, and the result follows by by Proposition 16.6. □

We will later use the following consequence of these results:

COROLLARY 16.9. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. Assume that g_p is positive on $\mathcal{F}_k$. Then g_{p^+} is positive on $\mathcal{F}_{k+1}$.*

PROOF. By Corollary 16.5, we have $\mathcal{F}_{k+1} = \ker\rho_{k+1} \oplus \mathrm{d}_p\eta_k(\mathcal{F}_k)$. By Corollary 16.7, these two subspaces are g_{p^+}-orthogonal to each other. Now, by the assumption and Corollary 16.8, g_{p^+} is positive on the space $\mathrm{d}_p\eta_k(\mathcal{F}_k)$, whereas by Lemma 16.1, it is positive on $\ker\rho_{k+1}$. The result follows. □

16.4. Positivity and admissibility

We can use the previous results to give a new criterion for a Euclidean field to be admissible.

THEOREM 16.10. *Let $k \geq 2$. The set $\mathcal{P}_k^{\mathrm{ad}} \subset \mathcal{P}_k$ of admissible Γ-invariant k-Euclidean fields is a connected component of the set of p in $\mathcal{P}_k$ such that the symmetric bilinear form g_p on $\mathcal{F}_k$ is positive.*

See Definition 11.1 for the notion of an admissible kernel. It may be true that $\mathcal{P}_k^{\mathrm{ad}}$ is actually equal to the set of p in $\mathcal{P}_k$ such that g_p is positive.

Theorem 16.10 implies in particular that g induces on $\mathcal{P}_k^{\mathrm{ad}}$ the structure of a Riemannian manifold.

We start the proof with a general positivity result.

LEMMA 16.11. *Let A be a finite set with n elements, $n \geq 3$, V be the space of real-valued functions on A and $\overline{V}$ be its quotient by the line of constant functions. For f in V set*

$$p(f,f) = \sum_{a \in A} f(a)^2 - \frac{1}{n-1} \sum_{\substack{(a,b) \in A^2 \\ a \neq b}} f(a) f(b)$$

and view p as a scalar product on $\overline{V}$. Then, for any non-zero p-symmetric endomorphism S of $\overline{V}$, we have

$$\mathrm{tr}(S^2) > \frac{1}{2} \sum_{a \in A} p(S\mathbf{1}_a, \mathbf{1}_a)^2.$$

PROOF. Equip as usual V with the standard scalar product q defined by, for f in V, $q(f,f) = \sum_{a \in A} f(a)^2$ and let P be the q-orthogonal projection on $V_0 = \{f \in V | \sum_a f(a) = 0\}$ which is the q-orthogonal complement of constant functions. A direct computation shows that, for f in V, one has $p(f,f) = \frac{n}{n-1} q(Pf, f)$. In particular, p-symmetric endomorphisms of $\overline{V}$ may be identified with q-symmetric endomorphisms S of V such that $S\mathbf{1} = 0$. For any such S, set $\Phi(S) = \mathrm{tr}(S^2) - \frac{1}{2} \sum_{a \in A} p(S\mathbf{1}_a, \mathbf{1}_a)^2$. One has

$$\Phi(S) = \sum_{(a,b) \in A^2} q(S\mathbf{1}_a, \mathbf{1}_b)^2 - \frac{n^2}{2(n-1)^2} \sum_{a \in A} q(S\mathbf{1}_a, \mathbf{1}_a)^2. \tag{16.1}$$

If $n \geq 4$, we have $\frac{n^2}{2(n-1)^2} < 1$ and the result follows. It remains to deal with the case where $n = 3$. Then, denote by a, b, c the three elements of A and set $u = q(S\mathbf{1}_b, \mathbf{1}_c)$, $v = q(S\mathbf{1}_c, \mathbf{1}_a)$ and $w = q(S\mathbf{1}_a, \mathbf{1}_b)$. As

$S\mathbf{1} = 0$, we get from (16.1),

$$\begin{aligned}\Phi(S) &= 2(u^2+v^2+w^2) - \frac{1}{8}((u+v)^2+(v+w)^2+(w+u)^2)\\ &= \frac{7}{4}(u^2+v^2+w^2) - \frac{1}{4}(uv+vw+wu)\end{aligned}$$

and a direct computation shows that this quadratic form on $\mathbb{R}^3$ is positive definite. □

To prove Theorem 16.10, we will again use the harmonic field which was studied in Sections 5.5, 10.6 and 11.6. Recall from Proposition 11.21 that π is an admissible 2-Euclidean field.

COROLLARY 16.12. *The symmetric bilinear form g_π is positive on $\mathcal{F}_2$.*

PROOF. By construction, for x in X and f in $\overline{V}^1(x)$, we have

$$\pi_x(f,f) = \sum_{y\sim x} f(y)^2 - \frac{1}{n(x)-1}\sum_{\substack{y,z\sim x\\ y\neq z}} f(y)f(z),$$

and we can therefore aim at applying Lemma 16.12 to the set $S^1(x)$ and the bilinear form π_x.

Fix q in $\mathcal{F}_k$ and, as in Lemma 16.1, for x in X, let us denote by A_x the endomorphism of $\overline{V}^1(x)$ which represents q_x with respect to p_x. By Lemma 10.11, Lemma 11.19, Lemma 16.1 and Remark 16.2, we have

$$g_\pi(q,q) = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\left(\operatorname{tr}(A_x^2) - \frac{1}{2}\sum_{y\sim x}\pi_x(A_x\mathbf{1}_y,\mathbf{1}_y)^2\right),$$

which, by Lemma 16.12, is positive as soon as $q \neq 0$. □

Next, we will use the weight formula to characterize those p such that g_p is non-degenerate on $\mathcal{F}_k$.

LEMMA 16.13. *Let $k \geq 2$ and p be in $\mathcal{P}_k$. Then the symmetric bilinear form g_p is non-degenerate on $\mathcal{F}_k$ if and only if the quadratic transfer operator of p does not admit 1 as an eigenvalue.*

See Definition 11.14 for the description of the quadratic transfer operator.

PROOF. Assume that the quadratic transfer operator of p admits 1 as an eigenvalue. By Proposition 11.26, there exists a non-zero q in $\mathcal{F}_k$ such that $\mathrm{d}_p\iota_k(q)$ is a pseudokernel. Then, by Corollary 9.2, Proposition 12.2 and Theorem 12.5, for any r in $\mathcal{F}_k$, one has $g_p(q,r) = 0$, hence g_p is degenerate.

Conversely, assume q is a non-zero element in $\mathcal{F}_k$ and $g_p(q,r) = 0$ for any r in $\mathcal{F}_k$. Chose a Γ-invariant weight function w for the Γ-invariant k-dual kernel $\mathrm{d}_p\iota_k(q)$. By Proposition 12.2 and Theorem 12.5, for any r in $\mathcal{F}_k$, one has $\langle w, \varphi_r\rangle = 0$. By Proposition 4.11, for any Γ-invariant quadratic type function φ on X_k, one has $\langle w, \varphi\rangle = 0$, hence, by Proposition 12.2, there exists a Γ-invariant skew-symmetric function v on X_{k-1} such that $w(x,y) = v(x,y_1) - v(x_1,y)$ for (x,y) in X_k. By Theorem 9.1, the k-dual kernel $\mathrm{d}_p\iota_k(q)$ is the one associated with a $(k-1)$-pseudokernel. Again by Proposition 11.26, the quadratic transfer operator of p admits 1 as an eigenvalue. □

Proof of Theorem 16.10. Note that, being the image of a convex set by a continuous map, the set $\mathcal{P}_k^{\mathrm{ad}}$ of admissible kernels is connected.

By Corollary 16.9 and Corollary 16.12, the symmetric form g_{π^k} is positive on $\mathcal{F}_k$, where π^k denotes the $(k-2)$-th orthogonal extension of the harmonic kernel. By Proposition 11.21, π^k belongs to $\mathcal{P}_k^{\mathrm{ad}}$ and, by Theorem 11.27 and Lemma 16.13, for any p in $\mathcal{P}_k^{\mathrm{ad}}$, the symmetric form g_p is non-degenerate. Therefore, as $\mathcal{P}_k^{\mathrm{ad}}$ is connected, g_p is positive for any p in $\mathcal{P}_k^{\mathrm{ad}}$.

Let $\mathcal{P}'_k$ be the connected component of g_{π^k} in the set of those p in $\mathcal{P}_k$ such that g_p is positive. We have just shown the inclusion $\mathcal{P}_k^{\mathrm{ad}} \subset \mathcal{P}'_k$.

Conversely, by Corollary 11.16 and Proposition 11.21, the quadratic transfer operator T_{π^k} of the Euclidean field π^k has spectral radius < 1. By Lemma 16.13, the spectral radius of T_p is $\neq 1$ for any p in $\mathcal{P}'_k$. As the spectral radius is continuous and $\mathcal{P}'_k$ is connnected by assumption, the spectral radius of T_p is < 1 for any p in $\mathcal{P}'_k$, hence p is admissible by Theorem 11.27. Thus we get $\mathcal{P}'_k \subset \mathcal{P}_k^{\mathrm{ad}}$ as required. □

Let $k \geq 2$ and p be an admissible Γ-invariant k-Euclidean field. Thanks to Proposition 16.6 and Theorem 16.10, we can associate to p and its orthogonal extensions a scalar product on each of the spaces $\mathcal{F}_j$, $j \geq k$, thus equiping this family of spaces with the structure of a straight Euclidean projective system, in the sense of Definition B.6. In Theorem 19.5 below, we will give a description of the projective limit of this system which shows in particular that, as a topological vector space, it is independent of the choice of p. Before to prove this result, we will develop a theory which may be seen as a kind of skew version of the theory of quadratic fields and dual kernels. The formulas which will appear in this theory will later play a role in the analysis of the spectral theory of Euclidean fields.

CHAPTER 17

Skew quadratic fields

We have shown in Proposition 13.16 that the root map is a section of the shoot map. Thus, the space of radical quadratic pseudofields splits in a natural way as the direct sum of the image of the root map and of the null space of the shoot map. We will now develop a study of the elements of this null space, which we will call skew quadratic fields. This theory will play the role of a skew-symmetric counterpart to the theory of quadratic fields.

17.1. Skew quadratic fields and the shoot map

We begin by defining formally skew quadratic fields.

DEFINITION 17.1. Let $k \geq 1$. If $k = 1$, a 1-skew quadratic field is a 1-quadratic pseudofield s with $s^\vee = -s$. If $k \geq 2$, a k-skew quadratic field is a k-quadratic pseudofield s with $s^\vee = -s$ and $s^{<\vee} = -s^<$.

We can relate skew quadratic fields with the null space of the shoot map.

LEMMA 17.2. *Let $k \geq 1$ and s be a k-quadratic pseudofield. Then s is a k-skew quadratic field if and only if s is radical and the shoot of s is 0.*

The proof is immediate.

REMARK 17.3. In the next chapters, we adopt the following notational convention. If u is a function on X_1, for $k \geq 1$, L a k-pseudokernel and s a k-quadratic pseudofield, we let uL and us be the pseudokernel and the quadratic pseudofield defined by $(x, y) \mapsto u(x, y)L_{xy}$ and $(x, y) \mapsto u(x, y)s_{xy}$. We get $(uL)^> = uL^>$ and $(us)^< = us^<$ as well as $\langle us, L\rangle = \langle s, uL\rangle$ when u, L and s are Γ-invariant. As in Section 15.1, we also write $u^\vee$ for the function $(x, y) \mapsto u(y, x)$ on X_1, so that, if k is odd, we have $(uL)^\vee = u^\vee L^\vee$ and $(us)^\vee = u^\vee s^\vee$. Finally, if v is a function on X, we use the same letter v to denote the function $(x, y) \mapsto v(x)$ on X_1. Then, $v^\vee$ stands for the function $(x, y) \mapsto v(y)$.

REMARK 17.4. As in Remark 13.5, we can consider the space of k-quadratic fields as a subspace of the space of k-quadratic pseudofields. Then, with this identification, if $k \geq 2$ is even (resp. odd), a k-quadratic pseudofield s is the one associated with a k-quadratic field if and only if $s^\vee = (d-1)s$ and $s^{<\vee} = s^<$ (resp. $s^\vee = s$ and $s^{<\vee} = (d-1)s^<$). Thus, the notion of a skew quadratic field may be seen as the skew-symmetric counterpart of the notion of a quadratic field. We will develop the study of skew quadratic fields by keeping this idea in mind.

For $k \geq 1$, we let $\mathcal{G}_k$ denote the space of Γ-invariant k-skew quadratic fields. By Proposition 13.16 and Lemma 17.12, we have $\mathcal{M}_k^1 = \mathcal{G}_k \oplus R_k\mathcal{F}_k$.

The notion of a skew quadratic field behaves well under direct restriction. Lemma 11.6 gives

LEMMA 17.5. *Let $k \geq 2$ and s be a k-skew quadratic field. Then $s^<$ is a $(k-1)$-skew quadratic field.*

The orthogonal subspace of the space of k-skew quadratic fields in the space of k-pseudokernels is the image of the canonical map.

LEMMA 17.6. *Let $k \geq 1$ and s be k-quadratic pseudofield.*

If $k = 1$, then s is a 1-skew quadratic field if and only if, for every L in $\mathcal{L}_1$, one has $\langle s, L + L^\vee\rangle = 0$.

If $k \geq 2$, then s is a k-skew quadratic field if and only if, for every (K, K^-) in $\mathcal{K}_k$, one has $\langle s, C_k(K, K^-)\rangle = 0$.

PROOF. If $k = 1$, this directly follows from Definition 17.1, the inversion map $L \mapsto L^\vee$ being an involution of the space of 1-pseudokernels.

If $k \geq 2$, this is a direct consequence of Lemma 13.8 and Lemma 17.2. □

We get an analogue for skew quadratic fields of Lemma 12.1 for quadratic fields.

COROLLARY 17.7. *Let $k \geq 2$. Then direct restriction maps $\mathcal{G}_k$ onto $\mathcal{G}_{k-1}$.*

PROOF. As usual, this follows from the dual injectivity statement.

If $k = 2$, we must show that, if L is a 1-pseudokernel and $L^>$ is the canoncial 2-pseudokernel of a 2-dual kernel (K, K^-), then $L^\vee = L$. Indeed, with the convention of Remark 6.13, we have $L^> = K - K^{->}$, hence

$$(L + K^-)^{>\vee} = K^\vee = (d-1)K = (d-1)(L + K^-).$$

Lemma 6.9 gives $L = -K^-$ and we are done.

If $k \geq 3$, we must show that, if L is a $(k-1)$-pseudokernel and $L^>$ is a canonical pseudokernel, then L already is a canonical pseudokernel. Indeed, assume (K, K^-) is a k-dual kernel and $L^> = K - K^{->}$.

If k is even, we again have

$$(L + K^-)^{>\vee} = K^\vee = (d-1)K = (d-1)(L + K^-)$$

and Lemma 6.9 now says that there exists a $(k-2)$-pseudokernel J with

$$L + K^- = J^> \text{ and } (d-1)(L + K^-) = J^{\vee>}.$$

In particular, we have $J^\vee = (d-1)J$, that is, J is a $(k-2)$-dual prekernel. We get $L + K^- = J^>$, hence L is the canonical $(k-1)$-pseudokernel of $-(K^-, J)$.

If k is odd, we have

$$(L + K^-)^{>\vee} = K^\vee = K = L + K^-$$

and Lemma 6.9 says that there exists a $(k-2)$-pseudokernel J with

$$L + K^- = J^> = J^{\vee>}.$$

Thus J is a $(k-2)$-dual prekernel and L is the canonical $(k-1)$-pseudokernel of $-(K^-, J)$. □

17.2. Orthogonal extension of skew quadratic fields

Still in analogy with the case of quadratic fields, we now show how the choice of a Euclidean field allows to define a section of direct restriction of skew quadratic fields.

We make use of the notation of Section 11.4. Let $k \geq 2$ and p be a k-Euclidean field. As in Remark 13.5, let us identify k-quadratic fields with a subspace of k-quadratic pseudofields. The computation of the derivative of the orthogonal extension map $\eta_k : \mathcal{P}_k \to P_{k+1}$ in Proposition 16.3 can be stated as, for $x \sim y$ in X, p in $\mathcal{P}_k$ and q in $\mathcal{F}_k$,

$$\begin{aligned} \mathrm{d}_p\eta_k(q) &= q^{>_{p^+}} + q^{>_{p^+}{}^\vee} - q^{->_p>_{p^+}} \quad \text{if } k \text{ is even.} \\ &= q^{>_{p^+}} + q^{>_{p^+}{}^\vee} - (d-1)q^{->_p>_{p^+}} \quad \text{if } k \text{ is odd.} \end{aligned}$$

We will define orthogonal extension of skew quadratic fields in analogy with this formula.

Definition 17.8. Let $k \geq 2$, p be a k-Euclidean field and s be a k-skew quadratic field. If k is even, we set

$$s^{+_p} = -s^{>_{p^+}} + s^{>_{p^+}{}^\vee} + s^{<>_p>_{p^+}}. \tag{17.1}$$

If k is odd, we set

$$s^{+_p} = -s^{>_{p^+}} + \frac{1}{d-1} s^{>_{p^+}{}^\vee} + \frac{1}{d-1} s^{<>_p>_{p^+}}. \tag{17.2}$$

We call s^{+_p} the p-orthogonal extension of s.

PROPOSITION 17.9. *Let $k \geq 2$, p be a k-Euclidean field and s be a k-skew quadratic field. Then s^{+_p} is a $(k+1)$-skew quadratic field and $s^{+_p<} = -s$, that is, $s^{+_p-} = s^{+_p\vee<} = s$.*

To prove this, we will need to compute the square of the reversal map.

LEMMA 17.10. *Let $k \geq 2$ be even and s be a k-quadratic pseudofield and L be a k-pseudokernel. We have $s^{\vee\vee} = (d-1)s + (d-2)s^\vee$ and $L^{\vee\vee} = (d-1)L + (d-2)L^\vee$.*

In case the tree X is not homogeneous, this Lemma uses the notation introduced in Remark 17.3.

PROOF. Indeed, for $x \sim y$ in X, we have

$$\begin{aligned} s^{\vee\vee}_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} s^\vee_{xz} = \sum_{\substack{z\sim x\\ z\neq y}} \sum_{\substack{t\sim x\\ t\neq z}} s_{xt} = \sum_{t\sim x} |\{z \in S^1(x) | z \notin \{y,t\}\}| s_{xt} \\ = (d(x)-1)s_{xy} + (d(x)-2)\sum_{\substack{t\sim x\\ t\neq y}} s_{xt}. \end{aligned}$$

The proof for pseudokernels is the same. □

PROOF OF PROPOSITION 17.9. If k is even, we have

$$\begin{aligned} s^{+_p\vee} = -s^{>_{p^+}{}^\vee} + s^{>_{p^+}} + s^{<>_p>_{p^+}{}^\vee} = -s^{>_{p^+}{}^\vee} + s^{>_{p^+}} + s^{<\vee>_p>_{p^+}} \\ = -s^{>_{p^+}{}^\vee} + s^{>_{p^+}} - s^{<>_p>_{p^+}} = -s^{+_p\vee}, \end{aligned}$$

where we have used Corollary 11.12 and the fact that $s^{<\vee} = -s^<$. Besides, we have

$$s^{+_p<} = -s^{>_{p^+}<} + s^{>_{p^+}\vee<} + s^{<>_p>_{p^+}<}.$$

Lemma 11.11 gives $s^{>_{p^+}<} = s$ and $s^{<>_p>_{p^+}<} = s^{<>_p}$ and Corollary 11.13 gives $s^{>_{p^+}\vee<} = s^{<\vee>_p} = -s^{<>_p}$. We get $s^{+_p<} = -s$ as required. In particular, $s^{+_p<\vee} = -s^{+_p<}$ and s^{+_p} is a $(k+1)$-skew quadratic field.

If k is odd, we have

$$\begin{aligned} s^{+_p\vee} &= -s^{>_{p^+}\vee} + \frac{1}{d-1}s^{>_{p^+}\vee\vee} + \frac{1}{d-1}s^{<>_p>_{p^+}\vee} \\ &= -s^{>_{p^+}\vee} + s^{>_{p^+}} + \frac{d-2}{d-1}s^{>_{p^+}\vee} + \frac{1}{d-1}s^{<\vee>_p>_{p^+}} \\ &= s^{>_{p^+}} - \frac{1}{d-1}s^{>_{p^+}\vee} - \frac{1}{d-1}s^{<>_p>_{p^+}}, \end{aligned}$$

where we have used Corollary 11.12, Lemma 17.10 and the fact that $s^{<\vee} = -s^{<}$. Now, we have

$$s^{+_p<} = -s^{>_{p^+}<} + \frac{1}{d-1}s^{>_{p^+}\vee<} + \frac{1}{d-1}s^{<>_p>_{p^+}<}.$$

Lemma 11.11 still gives $s^{>_{p^+}<} = s$ and $s^{<>_p>_{p^+}<} = s^{<>_p}$ and Corollary 11.13 gives $s^{>_{p^+}\vee<} = s^{<\vee>_p} = -s^{<>_p}$. We get $s^{+_p<} = -s$ as required and again, $s^{+_p<\vee} = -s^{+_p<}$, so that s^{+_p} is a $(k+1)$-skew quadratic field. □

17.3. Skew quadratic fields and harmonic maps

For any $k \geq 2$, Lemma 12.1 shows that reduction of quadratic fields maps $\mathcal{F}_k$ onto $\mathcal{F}_{k-1}$ and the constructions in Section 4.1 show that the projective limit of the projective system $(\mathcal{F}_k)_{k\geq 1}$ may be identified with the space of Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. We now develop an analogue interpretation of the projective limit of the projective system $(\mathcal{G}_k)_{k\geq 1}$.

If V is a vector space, say that a skew-symmetric map $\varphi : X_1 \to V$ is harmonic if, for any x in X, one has $\sum_{y\sim x}\varphi(x,y) = 0$. Recall that $\mathcal{Q}(V)$ stands for the space of symmetric bilinear forms on V.

Proposition 17.11. *Let $\varphi : X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ be a harmonic skew-symmetric map. For any $k \geq 1$ and $x \sim y$ in X, define*

$$\begin{aligned} s_{k,xy} &= (N_x^\ell)^\star\varphi(x,y) \quad k = 2\ell, \ell \geq 1. \\ s_{k,xy} &= (N_{xy}^\ell)^\star\varphi(x,y) \quad k = 2\ell+1, \ell \geq 0. \end{aligned}$$

Then s_k is a k-skew quadratic field and $s_{k+1}^< = s_k$. One has $\varphi = 0$ if and only if $s_k = 0$ for any $k \geq 1$. Conversely, if, for any $k \geq 1$, s_k is a k-skew quadratic field and $s_{k+1}^< = s_k$, then $(s_k)_{k\geq 1}$ can be obtained in this way. In particular, the projective limit of the projective system $(\mathcal{G}_k)_{k\geq 1}$ may be identified with the space of Γ-equivariant harmonic skew-symmetric maps $X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$.

The natural linear map $N_x^\ell : \overline{V}^\ell(x) \to \overline{\mathcal{D}}(\partial X)$, $x \in X$, $\ell \geq 1$, is defined in Section 4.4. The analogue linear map $N_{xy}^\ell : \overline{V}^\ell(xy) \to \overline{\mathcal{D}}(\partial X)$, $x \sim y \in X$, $\ell \geq 0$, is defined in Section 8.3.

PROOF. The proof is a direct consequence of the relation (8.7). □

17.4. $(d-1)$-radical quadratic pseudofields

We are still developing the theory of skew quadratic fields in analogy with the theory of quadratic fields. It turns out that, in this context, we can define a skew analogue of radical quadratic pseudofields and the shoot map. This is the purpose of the following two sections.

DEFINITION 17.12. Let $k \geq 1$ and s be a k-pseudofield. If $k \geq 3$ is odd, we say that s is $(d-1)$-radical if one has $s^{\vee<\vee} = (d-1)s^<$. If $k \geq 2$ is even, we say that s is $(d-1)$-radical if one has $s^{\vee<\vee} = (d^\vee - 1)s^<$. If $k = 1$, by convention, every 1-pseudofield is said to be $(d-1)$-radical.

For $k \geq 1$, the space of Γ-invariant $(d-1)$-radical k-pseudofields is denoted by $\mathcal{M}_k^{(d-1)}$. We get an analogue of Lemma 13.2 and Proposition 13.7.

PROPOSITION 17.13. *Let $k \geq 2$ and s be a $(d-1)$-radical k-pseudofield. If k is odd, then $s^{\vee<}$ is $(d-1)$-radical. If k is even, then $\frac{1}{d-1}s^{\vee<}$ is $(d-1)$-radical. In both cases, the associated linear map $\mathcal{M}_k^{(d-1)} \to \mathcal{M}_{k-1}^{(d-1)}$ is onto.*

PROOF. Assume k is odd. By assumption, we have $s^{\vee<\vee} = (d-1)s^<$. We get

$$s^{\vee<\vee<\vee} = ((d-1)s^<)^{<\vee} = ((d-1)s^{<<})^\vee = (d^\vee - 1)s^{\vee<<},$$

where we have used Lemma 11.6. This tells us exactly that $s^{\vee<}$ is $(d-1)$-radical.

Assume k is even (and $k \geq 4$, else there is nothing to prove). We now have $s^{\vee<\vee} = (d^\vee - 1)s^<$, hence, as $k-1$ is odd,

$$\left(\frac{1}{d-1}s^{\vee<}\right)^{\vee<\vee} = \left(\frac{1}{d^\vee - 1}s^{\vee<\vee}\right)^{<\vee} = s^{<<\vee} = s^{\vee<<}$$
$$= (d-1)\left(\frac{1}{d-1}s^{\vee<}\right)^<,$$

the latter again following from Lemma 11.6. Thus, the $(k-1)$-pseudofield $\frac{1}{d-1}s^{\vee<}$ is $(d-1)$-radical.

Now, we must prove the surjectivity property. As usual, this will follow from a dual injectivity property.

First, assume $k = 1$. In that case, we must show that, if L and M are in $\mathcal{L}_1$ and

$$\frac{1}{d-1}L^{>\vee} = (d^\vee - 1)M^> - M^{\vee>\vee},$$

then $L = 0$. Indeed, in this case, Lemma 6.9 says that $M = 0$ and $\frac{1}{d-1}L = -M^\vee$, hence $L = 0$.

Assume $k \geq 2$ is even. In that case, let L and M be in $\mathcal{L}_k$ and

$$L^{>\vee} = (d-1)M^> - M^{\vee>\vee}.$$

Lemma 6.9 says that there exists N in $\mathcal{L}_{k-1}$ with

$$L + M^\vee = N^> \text{ and } (d-1)M = N^{\vee>}.$$

As k is even, we have $(d-1)M^\vee = N^{\vee>\vee}$, hence

$$L = N^> - \frac{1}{d-1}N^{\vee>\vee} = (d^\vee - 1)\left(\frac{1}{d^\vee - 1}N\right)^> - \left(\frac{1}{d^\vee - 1}N\right)^{\vee<\vee},$$

and the injectivity property follows.

Assume $k \geq 3$ is odd and let now L and M be in $\mathcal{L}_k$ with

$$\frac{1}{d-1}L^{>\vee} = (d^\vee - 1)M^> - M^{\vee>\vee},$$

so that, still by Lemma 6.9, there exists N in $\mathcal{L}_{k-1}$ with

$$\frac{1}{d-1}L + M^\vee = N^> \text{ and } (d^\vee - 1)M = N^{\vee>}.$$

As k is odd, we get $(d-1)M^\vee = N^{\vee>\vee}$, hence $L = (d-1)N^> - N^{\vee>\vee}$, which should be proved. □

17.5. The skew shoot map

The shoot map was defined in Definition 13.4. It maps radical pseudofields to quadratic fields. Skew quadratic fields appeared naturally as radical pseudofields in the null space of the shoot map. Thus, thanks to Proposition 13.10, we have an exact sequence

$$0 \to \mathcal{G}_k \to \mathcal{M}_k^1 \xrightarrow{P_k} \mathcal{F}_k \to 0.$$

We now introduce the skew shoot map which sends $(d-1)$-radical pseudofields to skew quadratic fields. The null space of the skew shoot map will be the space of quadratic fields and, with the convention of Remark 17.4, we will have an exact sequence

$$0 \to \mathcal{F}_k \to \mathcal{M}_k^{(d-1)} \xrightarrow{Q_k} \mathcal{G}_k \to 0.$$

LEMMA 17.14. *Let $k \geq 1$ and t be a $(d-1)$-radical k-pseudofield.*
If k is odd, set $s = t^{\vee} - t$.
If k is even, set $s = \frac{1}{d-1} t^{\vee} - t$.
Then, s is a k-skew quadratic field.

DEFINITION 17.15. Let $k \geq 1$, t be a $(d-1)$-radical k-pseudofield and s be as in Lemma 17.14. Then the skew quadratic field s is called the skew shoot of t. The linear map $Q_k : \mathcal{M}_k^{(d-1)} \to \mathcal{G}_k$ that sends a Γ-invariant $(d-1)$-radical k-pseudofield to its skew shoot is called the skew shoot map.

PROOF OF LEMMA 17.14. By construction, we have $s^{\vee} = -s$. If $k = 1$, we are done.

If $k \geq 2$ and k is even, as t is $(d-1)$-radical, we have

$$s^{<} = \frac{1}{d-1} t^{\vee <} - t^{<} = \frac{1}{d-1} t^{\vee <} - \left(\frac{1}{d-1} t^{\vee <} \right)^{\vee},$$

hence $s^{<\vee} = -s^{<}$ and s is a k-skew quadratic field.

If $k \geq 3$ and k is odd, we have

$$s^{<} = t^{\vee <} - t^{<} = t^{\vee <} - \frac{1}{d-1} t^{\vee <\vee},$$

hence again $s^{<\vee} = -s^{<}$ and s is a k-skew quadratic field. □

The proof also yields

COROLLARY 17.16. *Let $k \geq 2$, t be a $(d-1)$-radical k-pseudofield and s be the skew shoot of t.*
If k is even, the skew shoot of $\frac{1}{d-1} t^{\vee <}$ is $-s^{<}$.
If k is odd, the skew shoot of $t^{\vee <}$ is $-s^{<}$.

Finally, we show that the null space of the skew shoot map is the space of quadratic fields, with the convention of Remark 13.5 for embedding quadratic fields inside pseudofields. We have the following counterpart to Lemma 17.2.

LEMMA 17.17. *Let $k \geq 1$ and s be a k-pseudofield. Then s is a k-quadratic field if and only if s is $(d-1)$-radical and the skew shoot of s is 0.*

The proof is straightforward.

CHAPTER 18

Skew dual kernels

We continue to develop the analogy between the theory of skew quadratic fields and the theory of quadratic fields by introducing skew dual kernels which are related to skew quadratic fields as dual kernels are related to quadratic fields. We then define a pairing between skew dual kernels and skew quadratic fields. When given in addition a Euclidean field, this allows us to introduce a natural scalar product on the space of skew quadratic fields, which we call the skew weight metric.

18.1. Skew dual kernels and their orthogonal extensions

We define skew dual kernels in analogy with dual kernels. Their orthogonal extensions are defined by copying the formula for the orthogonal extension of skew quadratic fields.

DEFINITION 18.1. Let $k \geq 2$. A k-skew dual kernel is a pair (H, H^-) where H is a k-pseudokernel, H^- is a $(k-1)$-pseudokernel and $H^\vee = -H$ and $(H^-)^\vee = -H$.

The space of Γ-invariant k-skew dual kernels is denoted by $\mathcal{J}_k$.

By mimicking (17.1) and (17.2), we define the orthogonal extension of skew dual kernels.

DEFINITION 18.2. (k even) Let $k \geq 2$ be an even integer and (H, H^-) be a k-skew dual kernel. We set

$$H^+ = -H^> + H^{>\vee} - H^{->>}.$$

Then the $(k+1)$-skew dual kernel (H^+, H) is called the orthogonal extension of (H, H^-).

DEFINITION 18.3. (k odd) Let $k \geq 2$ be an odd integer and (H, H^-) be a k-skew dual kernel. We set

$$H^+ = -H^> + \frac{1}{d-1} H^{>\vee} - \frac{1}{d-1} H^{->>}.$$

Then the $(k+1)$-skew dual kernel (H^+, H) is called the orthogonal extension of (H, H^-).

As for dual kernels, there is a natural way for associating a skew dual kernel to a pseudokernel.

DEFINITION 18.4. (k even) Let $k \geq 2$ be an even integer and L be $(k-1)$-pseudokernel. We set

$$H = L^{>} - \frac{1}{d-1} L^{>\vee} \text{ and } H^{-} = L^{\vee} - L.$$

We call (H, H^{-}) the k-skew dual kernel associated to L.

DEFINITION 18.5. (k odd) Let $k \geq 2$ be an odd integer and L be $(k-1)$-pseudokernel. We set

$$H = L^{>} - L^{>\vee} \text{ and } H^{-} = L^{\vee} - (d-1)L.$$

We call (H, H^{-}) the k-skew dual kernel associated to L.

This defines an injective map from $\mathcal{L}_{k-1}$ to $\mathcal{J}_k$.

LEMMA 18.6. *Let $k \geq 1$ and L be a k-pseudokernel. If the $(k+1)$-skew dual kernel associated to L is zero, then L is zero.*

PROOF. We prove this by induction on k.

If $k = 1$, we have $L^{>} = \frac{1}{d-1} L^{>\vee}$ and Lemma 6.9 gives $L = 0$.

Assume $k \geq 2$ and the result holds for $k-1$. If k is even, we have $L^{>} = L^{>\vee}$, so that Lemma 6.9 says that there exists a $(k-1)$-pseudokernel M with $L = M^{>} = M^{\vee>}$, hence $M = M^{\vee}$. As we have $L^{\vee} = (d-1)L$, we get $M^{>\vee} = (d-1)M^{>}$, that is, the $(k-1)$-skew dual kernel associated to M is 0. By induction, we have $M = 0$, hence $L = 0$. The proof in the odd case is analogous. □

We get a compatibility relation with orthogonal extension.

LEMMA 18.7. *Let $k \geq 2$ and L be a $(k-1)$-pseudokernel. Let (H, H^{-}) be the k-skew dual kernel associated to L and (H^{+}, H) be its orthogonal extension.*

If k is even, (H^{+}, H) is the $(k+1)$-skew dual kernel associated to $\frac{1}{d-1} L^{>\vee}$.

If k is odd, (H^{+}, H) is the $(k+1)$-skew dual kernel associated to $L^{>\vee}$.

PROOF. Assume k is even. Let (J, J^{-}) be the $(k+1)$-skew dual kernel associated with $\frac{1}{d-1} L^{>\vee}$. By Lemma 17.10, we get

$$J^{-} = \frac{1}{d-1} L^{>\vee\vee} - L^{>\vee} = L^{>} + \frac{d-2}{d-1} L^{>\vee} - L^{>\vee} = L^{>} - \frac{1}{d-1} L^{>\vee} = H.$$

Besides, by definition of the orthogonal extension, we have

$$\begin{aligned} H^+ &= -\left(L^> - \frac{1}{d-1}L^{>\vee}\right)^> + \left(L^> - \frac{1}{d-1}L^{>\vee}\right)^{>\vee} - (L^\vee - L)^{>>} \\ &= -L^{>>} + \frac{1}{d-1}L^{>\vee>} + L^{>>\vee} - \frac{1}{d^\vee - 1}L^{>\vee>\vee} - L^{\vee>>} + L^{>>} \\ &= \left(\frac{1}{d-1}L^{>\vee}\right)^> - \left(\frac{1}{d-1}L^{>\vee}\right)^{>\vee} = J, \end{aligned}$$

where we have used Lemma 6.8.

Assume now k is odd and let (J, J^-) be the $(k+1)$-skew dual kernel associated with $L^{>\vee}$. We have $J^- = L^> - L^{>\vee} = H$ and

$$\begin{aligned} H^+ = -(L^> - L^{>\vee})^> + \frac{1}{d-1}(L^> - L^{>\vee})^{>\vee} \\ - \frac{1}{d-1}(L^\vee - (d-1)L)^{>>}. \end{aligned}$$

By again using Lemma 6.8, we get $H^+ = L^{>\vee>} - \frac{1}{d-1}L^{\vee>>} = J$ and we are done. □

18.2. The skew weight pairing

We now introduce a pairing between Γ-invariant skew quadratic fields and Γ-invariant skew dual kernels which is an analogue of the weight pairing introduced in Definition 12.6.

Definition 18.8. Let $k \geq 2$, s be a Γ-invariant k-skew quadratic field and (H, H^-) be a Γ-invariant k-skew dual kernel.

If k is even, we set

$$[s, (H, H^-)] = \left\langle \frac{d-1}{d}s, H \right\rangle + \left\langle \frac{1}{2}s^<, H^- \right\rangle.$$

If k is odd, we set

$$[s, (H, H^-)] = \left\langle \frac{1}{2}s, H \right\rangle + \left\langle \frac{1}{d}s^<, H^- \right\rangle.$$

We call the pairing $[.,.]$ the skew weight pairing.

Remark 18.9. Note that there is no analogue of the weight map in the theory of skew quadratic fields and skew dual kernels.

The skew weight pairing is degenerated on k-skew dual kernels associated to $(k-1)$-pseudokernels.

LEMMA 18.10. *Let $k \geq 2$, s be a Γ-invariant k-skew quadratic field, L be a Γ-invariant $(k-1)$-pseudokernel and (H, H^-) be the k-skew dual kernel associated to L. We have $[s, (H, H^-)] = 0$.*

We shall see later that the converse is true: if a k-skew dual kernel has zero weight pairing with every k-skew quadratic field, then it is the skew dual kernel associated to a $(k-1)$-pseudokernel.

PROOF. Assume k is even. Definition 18.4 and Definition 18.8 give

$$[s, (H, H^-)] = \left\langle \frac{d-1}{d} s, L^> - \frac{1}{d-1} L^{>\vee} \right\rangle + \left\langle \frac{1}{2} s^<, L^\vee - L \right\rangle.$$

We have

$$\left\langle \frac{d-1}{d} s, \frac{1}{d-1} L^{>\vee} \right\rangle = \left\langle \frac{1}{d} s, L^{>\vee} \right\rangle = \left\langle -\frac{1}{d} s^<, L \right\rangle,$$

since $s^\vee = -s$. As $s^{<\vee} = -s^<$, we get

$$[s, (H, H^-)] = \left\langle \frac{d-1}{d} s^< + \frac{1}{d} s^<, L \right\rangle - \langle s^<, L \rangle = 0.$$

Assume k is odd. Definition 18.4 and Definition 18.8 now give

$$[s, (H, H^-)] = \left\langle \frac{1}{2} s, L^> - L^{>\vee} \right\rangle + \left\langle \frac{1}{d} s^<, L^\vee - (d-1)L \right\rangle.$$

As above, we have

$$\left\langle \frac{1}{2} s, L^> - L^{>\vee} \right\rangle = \langle s^<, L \rangle$$

and

$$\left\langle \frac{1}{d} s^<, L^\vee - (d-1)L \right\rangle = \left\langle \frac{1}{d} s^{<\vee} - \frac{d-1}{d} s^<, L \right\rangle = -\langle s^<, L \rangle$$

and the result follows. □

We have an analogue of Corollary 12.7.

PROPOSITION 18.11. *Let $k \geq 2$, s be in $\mathcal{G}_{k+1}$ and (H, H^-) be in $\mathcal{J}_k$. Set (H^+, H) to be the orthogonal extension of (H, H^-). We have*

$$[s, (H^+, H)] = -[s^<, (H, H^-)].$$

PROOF. Assume k is even. Definition 18.2 and Definition 18.8 give

$$\begin{aligned}[s,(H^+,H)] &= \left\langle \frac{1}{2}s, -H^{>} + H^{>\vee} - H^{->>} \right\rangle + \left\langle \frac{1}{d}s^{<}, H \right\rangle \\ &= -\frac{1}{2}\left\langle s^{<}, H\right\rangle + \frac{1}{2}\left\langle s^{\vee<}, H\right\rangle - \frac{1}{2}\left\langle s^{<<}, H^{-}\right\rangle + \left\langle \frac{1}{d}s^{<}, H \right\rangle \\ &= -\left\langle s^{<}, H\right\rangle - \frac{1}{2}\left\langle s^{<<}, H^{-}\right\rangle + \left\langle \frac{1}{d}s^{<}, H \right\rangle \\ &= -[s,(H,H^-)],\end{aligned}$$

where we have used that $s^\vee = -s$.

Assume k is odd. Definition 18.3 and Definition 18.8 now give

$$\begin{aligned}[s,(H^+,H)] &= \left\langle \frac{1}{d}s, -(d-1)H^{>} + H^{>\vee} - H^{->>} \right\rangle + \left\langle \frac{1}{2}s^{<}, H \right\rangle \\ &= -\left\langle \frac{d-1}{d}s^{<}, H \right\rangle + \left\langle \frac{1}{d}s^{\vee<}, H \right\rangle - \left\langle \frac{1}{d}s^{<<}, H^{-} \right\rangle + \left\langle \frac{1}{2}s^{<}, H \right\rangle \\ &= -[s,(H,H^-)],\end{aligned}$$

where we have used that $s^\vee = -s$ and $\frac{d-1}{d} + \frac{1}{d} - \frac{1}{2} = \frac{1}{2}$. □

18.3. The skew canonical pseudokernel

We now introduce the analogue of the canonical map of dual kernels. This will allow us in particular to determine the null space of the weight pairing.

DEFINITION 18.12. Let $k \geq 2$ and (H, H^-) be a k-skew dual kernel. The skew canonical k-pseudokernel L of (H, H^-) is defined by

$$L = H + H^{->}.$$

We write $D_k : \mathcal{J}_k \to \mathcal{L}_k$ for the linear map that sends a Γ-invariant k-skew dual kernel to its skew canonical k-pseudokernel. We call this map the skew canonical map. As in the case of dual kernels, we have a compatibility property with orthogonal extension.

LEMMA 18.13. *Let $k \geq 2$ and (H, H^-) be a k-skew dual kernel with skew canonical k-pseudokernel L and orthogonal extension (H^+, H).*

If k is even, the skew canonical $(k+1)$-pseudokernel of (H^+, H) is $L^{>\vee}$.

If k is odd, the skew canonical $(k+1)$-pseudokernel of (H^+, H) is $\frac{1}{d-1}L^{>\vee}$.

PROOF. Let M be the skew canonical $(k+1)$-pseudokernel of the $(k+1)$-skew dual kernel (H^+, H).

Assume k is even. In that case, Definition 18.2 gives

$$M = (-H^{>} + H^{>\vee} - H^{->>}) + H^{>} = H^{>\vee} + H^{-\vee>>} = (H + H^{->})^{>\vee} = L^{>\vee},$$

where we have used that $H^{-\vee} = -H^{-}$ and Lemma 6.8.

Assume k is odd. Now, Definition 18.3 gives

$$M = (-H^{>} + \frac{1}{d-1}H^{>\vee} - \frac{1}{d-1}H^{->>}) + H^{>} = \frac{1}{d-1}(H^{>\vee} + H^{-\vee>>}) = \frac{1}{d-1}L^{>\vee}.$$

□

As for dual kernels (see Lemma 6.19), the skew canonical k-pseudokernel of the k-skew dual kernel associated to a $(k-1)$-pseudokernel may be computed directly.

LEMMA 18.14. *Let $k \geq 2$, M be a $(k-1)$-pseudokernel and (H, H^-) be the k-skew dual kernel associated to $M^\vee$. Let L be the skew canonical k-pseudokernel of (H, H^-).*

If k is even, we have $L = M^{>} - \frac{1}{d-1}M^{\vee>\vee}$.

If k is odd, we have $L = (d-1)M^{>} - M^{\vee>\vee}$.

PROOF. Assume k is even. Then Definition 18.4 and Definition 18.12 give

$$L = M^{\vee>} - \frac{1}{d-1}M^{\vee>\vee} + (M - M^{\vee})^{>} = M^{>} - \frac{1}{d-1}M^{\vee>\vee}.$$

Assume k is odd. Definition 18.5 and Definition 18.12 now give

$$L = M^{\vee>} - M^{\vee>\vee} + (M^{\vee\vee} - (d-1)M^{\vee})^{>} = (d-1)M^{>} - M^{\vee>\vee},$$

where we have used Lemma 17.10. □

Again as for dual kernels, the skew canonical map is injective.

LEMMA 18.15. *Let $k \geq 2$ and (H, H^-) be a k-skew dual kernel. If the skew canonical k-pseudokernel of (H, H^-) is 0, then $(H, H^-) = 0$.*

PROOF. We will prove this by induction on k.

Assume $k = 2$. Then, we have $H^{->} = -H = H^{\vee} = -H^{->\vee}$. Lemma 6.9 gives $H^- = 0$, and hence $H = 0$.

Assume $k \geq 3$ and the results holds for $k-1$. Again, we have $H^{->} = -H = H^{\vee} = -H^{->\vee}$. Lemma 6.9 now says that there exists a $(k-1)$-pseudokernel J with $H^- = -J^{>} = J^{\vee>}$, so that $J^{\vee} = -J$

and (H^-, J) is a $(k-1)$-skew dual kernel with zero skew canonical k-pseudokernel. The induction assumption says that $H^- = 0$, hence also $H = 0$, which should be proved. □

Lemma 18.13 and Lemma 18.15 give directly

COROLLARY 18.16. *Let $k \geq 2$, (H, H^-) be a k-skew dual kernel, L be its skew canonical k-pseudokernel and M be a $(k-1)$-pseudokernel.*

If k is even, we have $L = M^> - \frac{1}{d-1} M^{\vee > \vee}$ if and only if (H, H^-) is the k-skew dual kernel associated to $M^\vee$.

If k is odd, we have $L = (d-1)M^> - M^{\vee > \vee}$ if and only if (H, H^-) is the k-skew dual kernel associated to $M^\vee$.

18.4. Crossed duality

We will show that the skew canonical map may be seen as the adjoint of the skew shoot map with respect to the skew weight pairing. This will allow us to describe the null space of this pairing in the space of skew dual kernels.

LEMMA 18.17. *Let $k \geq 2$. Let t in $\mathcal{M}_k^{(d-1)}$ be a Γ-invariant $(d-1)$-radical k-pseudofield and set $s = Q_k t$ to be the skew shoot of t. Let (H, H^-) in $\mathcal{J}_k$ be a Γ-invariant k-skew dual kernel and set $L = D_k(K, K^-)$ to be the skew canonical k-pseudokernel of (K, K^-). We have*

$$[s, (H, H^-)] = -\langle t, L \rangle.$$

PROOF. Again, this is a straightforward computation. If k is even, Definition 17.15 and Definition 18.8 give

$$[s, (H, H^-)] = \left\langle \frac{1}{d} t^\vee - \frac{d-1}{d} t, H \right\rangle + \frac{1}{2} \left\langle \frac{1}{d-1} t^{\vee <} - t^<, H^- \right\rangle.$$

As $H^\vee = -H$, we have $\left\langle \frac{1}{d} t^\vee - \frac{d-1}{d} t, H \right\rangle = -\langle t, H \rangle$. As $H^{-\vee} = -H^-$ and t is $(d-1)$-radical, we have

$$\left\langle \frac{1}{d-1} t^{\vee <}, H^- \right\rangle = -\left\langle \frac{1}{d^\vee - 1} t^{\vee < \vee} - t^<, H^- \right\rangle$$
$$= -\langle t^<, H^- \rangle = -\langle t, H^{->} \rangle.$$

We get $[s, (H, H^-)] = -\langle t, H \rangle - \langle t, H^{->} \rangle = -\langle t, L \rangle$, as required.

If k is odd, still by Definition 17.15 and Definition 18.8, we have

$$[s, (H, H^-)] = \left\langle \frac{1}{2} t^\vee - \frac{1}{2} t, H \right\rangle + \left\langle \frac{1}{d} t^{\vee <} - \frac{1}{d} t^<, H^- \right\rangle.$$

As $H^\vee = -H$, we have $\left\langle \frac{1}{2}t^\vee - \frac{1}{2}t, H\right\rangle = -\langle t, H\rangle$. As $H^{-\vee} = -H^-$ and t is $(d-1)$-radical, we have

$$\left\langle \frac{1}{d}t^{\vee <}, H^-\right\rangle = -\left\langle \frac{1}{d}t^{\vee <\vee}, H^-\right\rangle = -\left\langle \frac{d-1}{d}t^<, H^-\right\rangle$$
$$= -\left\langle \frac{d-1}{d}t, H^{->}\right\rangle.$$

Again, we get $[s,(H,H^-)] = -\langle t, H + H^{->}\rangle = -\langle t, L\rangle$, as required. □

This allows us to determine the null space of the skew weight pairing in skew dual kernels.

COROLLARY 18.18. *Let $k \geq 2$ and (H,H^-) be in $\mathcal{J}_k$. Then one has $[s,(H,H^-)] = 0$ for every s in $\mathcal{G}_k$ if and only if (H,H^-) is the k-skew dual kernel associated to a $(k-1)$-pseudokernel.*

PROOF. The *if* part was established in Lemma 18.10. Conversely, by Lemma 18.17, if $[s,(H,H^-)] = 0$ for any s in $\mathcal{G}_k$, then $D_k(H,H^-)$ belongs to the orthogonal subspace to $\mathcal{M}_k^{(d-1)}$ in $\mathcal{L}_k$. The conclusion follows from Definition 18.1 and Corollary 18.16. □

By reasoning as for Proposition 13.10, from Proposition 17.13 and Lemma 18.17, one can show

PROPOSITION 18.19. *Let $k \geq 1$. Then Q_k maps $\mathcal{M}_k^{(d-1)}$ onto $\mathcal{G}_k$. If $k \geq 2$ and k is even (resp. odd), then the map $t \mapsto (Q_k t, \frac{1}{d-1}t^{\vee <})$ (resp. $t \mapsto (Q_k t, t^{\vee <})$) sends $\mathcal{M}_k^{(d-1)}$ onto the space*

$$\{(s,u) \in \mathcal{G}_k \times \mathcal{M}_{k-1}^{(d-1)} | -s^< = Q_{k-1}u\}.$$

18.5. The skew weight metric

We conclude this Section by using the above constructed objects to equip the space $\mathcal{G}_k$ of Γ-invariant k-skew quadratic fields with a scalar product that is constructed in analogy with the weight metric of Section 16.

Let $k \geq 2$ and p be a k-Euclidean field. Given a k-pseudokernel L, recall that in Section 11.8, we have defined a $(k-1)$-pseudokernel $L^{<_p}$. This construction is obtained by applying to the linear forms associated to L the adjoint operators of the natural operators, with respect to the Euclidean structures coming from p.

If s is a k-skew quadratic field, we let H be the k-pseudokernel obtained in the following way:
— if k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, then H_{xy} is the function on $S^\ell(x)^2$ associated by Lemma 5.1 to the bilinear form on

$V_0^\ell(x)$ obtained from s_{xy} by the identification between $V_0^\ell(x)$ and $\overline{V}^\ell(x)$ through the scalar product p_x.
— if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, then H_{xy} is the function on $S^\ell(xy)^2$ associated by Lemma 5.1 to the bilinear form on $V_0^\ell(xy)$ obtained from s_{xy} by the identification between $V_0^\ell(xy)$ and $\overline{V}^\ell(xy)$ through the scalar product p_{xy}.

In both cases, we have $H^\vee = -H$. We set $H^- = -H^{<_p}$. As $s^{<\vee} = -s^<$, one easily checks that $H^{<_p\vee} = -H^{<_p}$. Therefore, the pair (H, H^-) is a k-skew dual kernel.

Suppose now p and s are Γ-invariant. If r is an other Γ-invariant k-skew quadratic field, we set

$$h_p(r, s) = [r, (H, H^-)].$$

PROPOSITION 18.20. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The bilinear form h_p on $\mathcal{G}_k$ is symmetric and positive definite. If r is in $\mathcal{G}_{k+1}$ and s is in $\mathcal{G}_k$, we have*

$$h_{p^+}(r, s^{+_p}) = -h_p(r^<, s).$$

We will call h_p the skew weight metric on $\mathcal{G}_k$.

PROOF. This is a consequence of the previous constructions of the orthogonal extension and the skew weight pairing.

Indeed, let r, s be in $\mathcal{G}_k$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, let A_{xy} and B_{xy} be the endomorphisms that represent the bilinear forms s_{xy} and r_{xy} with respect to the scalar product p_x on $\overline{V}^\ell(x)$ and let A^-_{xy} and B^-_{xy} be the endomorphisms that represent the bilinear forms $s^<_{xy}$ and $r^<_{xy}$ with respect to the scalar product p^-_{xy} on $\overline{V}^{\ell-1}(xy)$. Then the Definition 18.8 of the weight pairing and that of h_p give

(18.1)

$$h_p(r, s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \left(\frac{d(x) - 1}{d(x)} \operatorname{tr}(A_{xy}B_{xy}) - \frac{1}{2} \operatorname{tr}(A^-_{xy}B^-_{xy}) \right).$$

Symmetry follows. Besides, as $d(x) \geq 3$ for any x in X, we get $\frac{d(x)-1}{d(x)} > \frac{1}{2}$ and therefore, if $r = s \neq 0$, the latter is positive.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, let A_{xy} and B_{xy} be the endomorphisms that represent the bilinear forms s_{xy} and r_{xy} with respect to the scalar product p_{xy} on $\overline{V}^\ell(xy)$ and let A^-_{xy} and B^-_{xy} be the endomorphisms that represent the bilinear forms $s^<_{xy}$ and $r^<_{xy}$ with respect to the scalar product p^-_x on $\overline{V}^\ell(x)$. Still by

Definition 18.8, we may write

(18.2)

$$h_p(r,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\left(\frac{1}{2}\operatorname{tr}(A_{xy}B_{xy}) - \frac{1}{d(x)}\operatorname{tr}(A^-_{xy}B^-_{xy})\right).$$

Symmetry and positivity follow.

It remains to prove the adjointness property of orthogonal extension. We take r in $\mathcal{G}_{k+1}$ and s is in $\mathcal{G}_k$ and, as in the construction of h_p, we let H be the k-pseudokernel with $H^\vee = -H$ that is associated with s by the duality coming from p. By this construction, we have

$$h_p(r^<, s) = [r^<, (H, H^-)],$$

where $H^- = -H^{<_p}$. By comparing the Definition 17.8 of the orthogonal extension of s and Definitions 18.2 and 18.3 of the orthogonal extension (H^+, H) of (H, H^-), we get

$$h_p(r, s^{+_p}) = [r, (H^+, H)].$$

The conclusion now follows from Proposition 17.9 and Proposition 18.11. □

Recall from Corollary 17.7 that direct restriction is a surjective linear map $\mathcal{G}_{k+1} \to \mathcal{G}_k$.

Corollary 18.21. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The scalar product h_p on $\mathcal{G}_k$ is the Euclidean image of h_{p^+} on $\mathcal{G}_{k+1}$ by direct restriction.*

Euclidean images of nonnegative symmetric bilinear forms under surjective maps were defined in Appendix A.

Proof. We need to show that, with respect to the Euclidean structures h_{p^+} and h_p, the adjoint map of direct restriction $\mathcal{G}_{k+1} \to \mathcal{G}_k$ is an isometric embedding $\mathcal{G}_k \to \mathcal{G}_{k+1}$. Now, Proposition 18.20 precisely says that this adjoint map is the map $s \mapsto -s^{+_p}, \mathcal{G}_k \to \mathcal{G}_{k+1}$ and Proposition 17.9 ensures that, for s in $\mathcal{G}_k$, one has $-s^{+_p<} = s$. The conclusion follows. □

CHAPTER 19

Projective limits of the weight metric and the skew weight metric

Let still $k \geq 2$ and p be a Γ-invariant k-Euclidean field. With the language of Appendix B, it follows from Corollary 18.21 above that the skew weight metrics associated to p and its orthogonal extensions allow to equip the projective system $(\mathcal{G}_j)_{j\geq k}$ with the structure of a straight nonnegative projective system. In the same way, if p is admissible, Proposition 16.6 and Theorem 16.10 ensure that the weight metrics associated to p and its orthogonal extensions allow to equip the projective system $(\mathcal{F}_j)_{j\geq k}$ with the structure of a straight nonnegative projective system. The purpose of this chapter is to give a concrete description of the Euclidean projective limits of both these systems.

19.1. The projective system of skew quadratic fields

We start with the case of skew quadratic fields. This case is easier and will motivate the formulation of the case of quadratic fields.

We recall some facts about Hilbert-Schmidt quadratic forms which can be deduced from the analagous properties of Hilbert-Schmidt operators (see for example [**46**]). Let H be a real Hilbert space with scalar product p and (v_i) be a Hilbert basis of H. A symmetric bilinear form q on H is said to be a Hilbert-Schmidt form if one has

$$\sum_{i,j} q(v_i, v_j)^2 < \infty.$$

This property is independent on the choice of the Hilbert basis. It implies q to be bounded with respect to the Hilbert norm on H. The space of Hilbert-Schmidt symmetric bilinear forms q is denoted by $\mathcal{Q}^2(H)$. The bilinear form $p_2^* : (q, r) \mapsto \sum_{i,j} q(v_i, v_j)r(v_i, v_j)$ is positive definite on $\mathcal{Q}^2(H)$ and it also does not depend of the basis. The space $\mathcal{Q}^2(H)$ is complete with respect to the associated Euclidean norm. If H has finite dimension, this scalar product may be written as $(q, r) \mapsto \mathrm{tr}(AB)$ where A and B are the endomorphisms which represent q and r with respect to p.

Now, let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. As in Section 4.6 and 4.5, the successive orthogonal extensions of p define a scalar product on the space $\overline{\mathcal{D}}(\partial X)$ of classes of smooth functions on ∂X modulo the constant functions. The completion of $\overline{\mathcal{D}}(\partial X)$ with respect to this scalar product is denoted by H^p. By abuse of notation, we still denote by p the scalar product of H^p. As $\overline{\mathcal{D}}(\partial X)$ is dense in H^p, we may consider the space $\mathcal{Q}^2(H^p)$ of Hilbert-Schmidt symmetric bilinear forms on H^p as a subspace of the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ of symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$.

Recall from Proposition 17.11 that we may identify the projective limit of the system $(\mathcal{G}_k)_{k\geq 1}$ with the space of Γ-invariant harmonic skew symmetric maps $X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$.

PROPOSITION 19.1. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then the Euclidean projective limit of the projective system*

$$(\mathcal{G}_j)_{j\geq k},$$

equipped with the skew weight metric, is the space of Γ-invariant harmonic skew symmetric maps $X_1 \to \mathcal{Q}^2(H^p)$, equipped with the scalar product

$$(r,s) \mapsto \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \frac{d(x)-2}{2d(x)} p_2^*(r_{xy}, s_{xy}).$$

See Appendix B for the language of Euclidean projective limits.

The proof uses the following direct consequence of the definition of the Hilbert-Schmidt norm:

LEMMA 19.2. *Let H be a Hilbert space and $(H_\ell)_{\ell\geq 0}$ be an increasing sequence of finite-dimensional subspaces of H such that $\bigcup_{\ell\geq 0} H_\ell$ is dense in H. Then, the Euclidean projective limit of the system $(\mathcal{Q}^2(H_\ell))_{\ell\geq 0}$ is the space $\mathcal{Q}^2(H)$.*

PROOF OF PROPOSITION 19.1. Set $C = 6\sup_{(x,y)\in X_1} |\Gamma_x \cap \Gamma_y|$.

If k is even, for r, s in $\mathcal{G}_k$, we can write (18.1) as

$$h_p(s,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \left(\frac{d(x)-1}{d(x)} (p_x)_2^*(r_{xy}, s_{xy}) - \frac{1}{2} (p_{xy}^-)_2^*(r_{xy}^<, s_{xy}^<) \right). \tag{19.1}$$

As $d(x) \geq 3$ for any x in X, this gives

$$h_p(s,s) \leq \sum_{(x,y)\in\Gamma\backslash X_1} (p_x)_2^*(r_{xy}, s_{xy}) \leq C h_p(s,s). \tag{19.2}$$

In the same way, if k is odd, for r, s in $\mathcal{G}_k$, (18.2) gives

$$h_p(s,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\left(\frac{1}{2}(p_{xy})_2^*(r_{xy},s_{xy}) - \frac{1}{d(x)}(p_x^-)_2^*(r_{xy}^<, s_{xy}^<)\right). \tag{19.3}$$

As above, we get

$$h_p(s,s) \leq \sum_{(x,y)\in\Gamma\backslash X_1} (p_{xy})_2^*(r_{xy},s_{xy}) \leq C h_p(s,s). \tag{19.4}$$

Together with Lemma 19.2, (19.2) and (19.4) imply that a Γ-invariant harmonic skew symmetric map $s : X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ belongs to the Euclidean projective limit of $(\mathcal{G}_j)_{j\geq k}$ if and only if, for any (x,y) in X_1, s_{xy} belongs to $\mathcal{Q}^2(H^p)$. For such an s and $j \geq k$, let s^j be the image of s in $\mathcal{G}_j$. Still by Lemma 19.2, we get, for (x,y) in X_1,

$$(p_x^j)_2^*(s_{xy}^j, s_{xy}^j) \xrightarrow[\substack{j\to\infty \\ j \text{ even}}]{} p_2^*(s_{xy},s_{xy})$$
$$\text{and } (p_{xy}^j)_2^*(s_{xy}^j, s_{xy}^j) \xrightarrow[\substack{j\to\infty \\ j \text{ odd}}]{} p_2^*(s_{xy},s_{xy}),$$

where p^j is the $(j-k)$-th orthogonal extension of p. By using the relation $\frac{d(x)-1}{d(x)} - \frac{1}{2} = \frac{d(x)-2}{2d(x)} = \frac{1}{2} - \frac{1}{d(x)}$, we get from (19.1) and (19.3),

$$h_{p^j}(s^j,s^j) \xrightarrow[j\to\infty]{} \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\frac{d(x)-2}{2d(x)} p_2^*(s_{xy},s_{xy}).$$

The conclusion follows in view of Lemma B.3 and Definition B.4. □

19.2. The weight pairing and the standard pairing

Our objective will now be to give a description of the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric associated to an admissible Γ-invariant k-Euclidean field, as in Section 16. To do this, we will use the abstract formalism of Appendix D to relate the weight pairing and the weight metric to the language of traces on Von Neumann algebras.

Indeed, by definition, the Hilbert space H_0^ω of Section 3.1 is Γ-isomorphic to a closed Γ-invariant subspace of the space $\ell^2(X_1)$. By assumption, Γ has finitely many orbits in X_1 and the stabilizers of the elements of X_1 in Γ are finite. Thus, with the language of Appendix D, the action of Γ on X_1 is standard. Therefore, we can apply to the representation of Γ in the Hilbert space H_0^ω the results of this Appendix. In particular, in the present Section, we relate the weight

pairing of Definition 12.6 with the standard quadratic pairing $\langle ., . \rangle_\Gamma$ of Section D.3.

Recall that by construction, we have a dense embedding of $\overline{\mathcal{D}}(\partial X)$ in the topological dual space $(H_0^\omega)'$ of H_0^ω. This gives a natural injection of $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ inside $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, where $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ is the space of continuous Γ-invariant symmetric bilinear forms on $(H_0^\omega)'$. Besides, in view of the results of Section 4, the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ may be identified with the projective limit of the projective system $(\mathcal{F}_k)_{k\geq 1}$ of Γ-invariant quadratic fields.

PROPOSITION 19.3. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, $w : X_k \to \mathbb{R}$ be a Γ-invariant weight function for (K, K^-) and Φ_w be the associated symmetric bilinear form on H_0^ω. Let also Π be in $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ and p be the image of Π by the natural map $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma \to \mathcal{F}_k$. Then we have*

$$\langle \Pi, \Phi_w \rangle_\Gamma = [p, (K, K^-)].$$

The bilinear form Φ_w is defined in Section 3.2 and the notion of a k-dual kernel is introduced in Section 5.1. The weight functions associated to a k-dual kernel are constructed in Section 7.2. The weight pairing is defined in Section 12.2.

PROOF. Define a bilinear form Ψ_w on $\ell^2(X_1)$ by setting, for ρ, θ in $\ell^2(X_1)$,

$$\Psi_w(\rho, \theta) = -\frac{1}{2} \sum_{(x,y)\in X_k} w(x, y)\rho(x, x_1)\theta(y, y_1), \tag{19.5}$$

where, as usual, for (x, y) in X_k, x_1 and y_1 are the neighbours of x and y in the segment $[xy]$. By Lemma D.11, Ψ_w is a well-defined bounded bilinear form on $\ell^2(X_1)$. As w is a symmetric function, Ψ_w is symmetric. Let $I : H_0^\omega \to \ell^2(X_1)$ be the natural injection. Then (3.3) implies $I^\star \Psi_w = \Phi_w$.

On the other hand, identify $\ell^2(X_1)$ with its topological dual space by the usual scalar product and let $I^* : \ell^2(X_1) \to (H_0^\omega)'$ denote the adjoint map of I. Recall that, for (x, y) in X_1, U_{xy} denotes the closed open subset of ∂X,

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}.$$

By the construction of the embedding I in Section 3.1, for θ in H_0^ω, when θ is viewed as a distribution on ∂X, we have

$$I\theta(x, y) = \theta(\mathbf{1}_{U_{xy}}).$$

This gives

$$I^*(\mathbf{1}_{(x,y)}) = \mathbf{1}_{U_{xy}},$$

where we have identified $\overline{\mathcal{D}}(\partial X)$ with a subspace of $(H_0^\omega)'$. Therefore, by (19.5), (D.3) and the definition of the standard pairing in Section D.3, we get

$$\begin{aligned}(19.6)\quad &\langle (I^*)^\star \Pi, \Psi_w\rangle_\Gamma = \\ &-\frac{1}{2}\sum_{((a,a'),(b,b'))\in\Gamma\backslash X_1^2} \frac{\Pi(\mathbf{1}_{U_{aa'}},\mathbf{1}_{U_{bb'}})}{|\Gamma_a\cap\Gamma_{a'}\cap\Gamma_b\cap\Gamma_{b'}|}\sum_{(x,y)\in X_k} w(x,y)\mathbf{1}_{\substack{x=a,x_1=a',\\ y=b,y_1=b'}} \\ &\qquad = -\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_k}\frac{1}{|\Gamma_x\cap\Gamma_y|}\Pi(\mathbf{1}_{U_{xx_1}},\mathbf{1}_{U_{yy_1}})w(x,y),\end{aligned}$$

where we also have used Lemma 10.11. By the weight formula, that is, Theorem 12.5, we have

$$-\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_k}\frac{1}{|\Gamma_x\cap\Gamma_y|}\Pi(\mathbf{1}_{U_{xx_1}},\mathbf{1}_{U_{yy_1}})w(x,y) = [p,(K,K^-)].$$

The conclusion follows since, by Lemma D.9, we have

$$\langle (I^*)^\star\Pi,\Psi_w\rangle_\Gamma = \langle \Pi, I^\star\Psi_w\rangle_\Gamma = \langle \Pi,\Phi_w\rangle_\Gamma.$$

□

Proposition 19.3 yields the following corollary which we were not able to prove by a more direct method.

COROLLARY 19.4. *For $k\geq 2$, any admissible Γ-invariant k-Euclidean field is tight.*

Admissible Euclidean fields are defined in Section 11; tight quadratic fields are defined in Section 14.

PROOF. Let p be an admissible k-Euclidean field. Then the successive orthogonal extensions of p define a scalar product p^∞ on the dual space of H_0^ω. Let w be a Γ-invariant function on X_k such that Φ_w is non-negative and (K,K^-) be a non-negative k-dual kernel such that w is a weight function of (K,K^-). By Proposition 19.3, we have

$$[p,(K,K^-)] = \langle p^\infty,\Phi_w\rangle_\Gamma.$$

By Proposition D.10, we get $[p,(K,K^-)]\geq 0$. The conclusion follows. □

19.3. The projective system of quadratic fields

To conclude, for $k\geq 2$ and p an admissible Γ-invariant k-Euclidean field, we will use the previous constructions to describe the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the

weight metric, by means of the standard scalar products of Section D.4.

As above, we identify the projective limit of the system $(\mathcal{F}_k)_{k\geq 1}$ with the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ of Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. Let $k \geq 2$ and p be a Γ-invariant admissible k-Euclidean field. We still set H^p to be the completion of $\overline{\mathcal{D}}(\partial X)$ with respect to the scalar product obtained from p by its successive orthogonal extensions as in Section 4. Then we have a natural embedding of $\mathcal{Q}^\infty(H^p)^\Gamma$ inside $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$.

Assume now that p is admissible as in Definition 11.1, so that, by Theorem 8.6 and Theorem 8.17, the space H^p may be considered as the topological dual space of the Hilbert space H_0^ω. As the representation of Γ in H_0^ω is standard, so is its representation on H^p and we may define the standard scalar product associated to p on $\mathcal{Q}^\infty(H^p)^\Gamma$, which we still write as p_Γ^* as in Definition D.13.

THEOREM 19.5. *Let $k \geq 2$ and p be an admissible Γ-invariant k-Euclidean field. Then the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric g_p, is the completion of $\mathcal{Q}^\infty(H^p)^\Gamma$ with respect to the standard scalar product p_Γ^*. More precisely, when both of them are seen as subspaces of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, the Euclidean projective limit contains $\mathcal{Q}^\infty(H^p)^\Gamma$ as a dense subspace and the restriction of the scalar product of the Euclidean projective limit to $\mathcal{Q}^\infty(H^p)^\Gamma$ is p_Γ^*.*

We first prove a density result.

LEMMA 19.6. *The space*

$$\bigcup_{k\geq 1}\{\Phi_w | w : X_k \to \mathbb{R}, w \text{ is } \Gamma\text{-invariant}\}$$

is dense in $\mathcal{Q}^\infty(H_0^\omega)^\Gamma$ with respect to the topology induced by a standard scalar product.

The fact that the statement does not depend on the choice of the standard scalar product is a consequence of Corollary D.16.

PROOF. As in the proof of Proposition 19.3, we identify $\ell^2(X_1)$ and its dual space by the usual scalar product and we let $I : H_0^\omega \to \ell^2(X_1)$ denote the natural inclusion. We equip H_0^ω with the induced scalar product of the usual one on $\ell^2(X_1)$. Therefore, by Lemma D.14, the natural map $(I^*)^\star : \mathcal{Q}^\infty((H_0^\omega)')^\Gamma \to \mathcal{Q}^\infty(\ell^2(X_1))^\Gamma$. is an isometric embedding for the standard scalar products. Let Π be in $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$.

In view of the proof of Proposition 19.3, for $x \neq y$ in X, if x_1 and y_1 are the neighbours of x and y in $[xy]$, we have

$$\begin{aligned}(I^*)^\star\Pi(\mathbf{1}_{(x,x_1)}, \mathbf{1}_{(y,y_1)}) &= (I^*)^\star\Pi(\mathbf{1}_{(x_1,x)}, \mathbf{1}_{(y_1,y)}) \\ &= -(I^*)^\star\Pi(\mathbf{1}_{(x_1,x)}, \mathbf{1}_{(y,y_1)}) \\ &= -(I^*)^\star\Pi(\mathbf{1}_{(x,x_1)}, \mathbf{1}_{(y_1,y)}) \\ &= \Pi(\mathbf{1}_{U_{x_1x}}, \mathbf{1}_{U_{y_1y}}) \\ &= -\psi(x,y),\end{aligned} \tag{19.7}$$

where, as in Section 4.1, ψ is the quadratic type function associated to Π, when Π is viewed as a symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$.

We apply Lemma D.18 to the Γ-action on X_1 which is standard by assumption. This tells us that we may identify $\mathcal{Q}^\infty(\ell^2(X_1))^\Gamma$ with a space of Γ-invariant symmetric functions on X_1^2; then, every element φ in the completion of $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ with respect to the standard scalar product may be seen as a Γ-invariant function on X_1^2 that is square-summable in $\Gamma\backslash X_1^2$. Due to (19.7), this function satisfies, for $x \neq y$ in X,

$$\begin{aligned}\varphi((x,x_1),(y,y_1)) &= \varphi((x_1,x),(y_1,y)) \\ &= \varphi((x_1,x),(y,y_1)) \\ &= \varphi((x,x_1),(y_1,y)).\end{aligned} \tag{19.8}$$

Let φ be such a function and assume moreover that, for any $k \geq 1$ and any symmetric Γ-invariant function w on X_k, φ is orthogonal to Ψ_w with respect to the standard quadratic pairing, where Ψ_w is defined as in the proof of Proposition 19.3. Then, by reasoning as in (19.6), we get

$$\sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi((x,x_1),(y,y_1))w(x,y) = 0.$$

As this is true for any $k \geq 1$ and any function w, for any $x \neq y$ in X, we get $\varphi((x,x_1),(y,y_1)) = 0$. By (19.8), we get $\varphi = 0$.

By construction, for $k \geq 1$ and w a Γ-invariant function on X_k, we have $I^\star\Psi_w = \Phi_w$. Therefore, by Lemma D.9, we have just shown that, with respect to the standard pairing, in the completion of $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$, the orthogonal subspace of

$$\bigcup_{k\geq 1}\{\Phi_w | w : X_k \to \mathbb{R}, w \text{ is } \Gamma\text{-invariant}\}$$

is reduced to $\{0\}$. By Corollary D.17, the Lemma follows. □

We will need the infinite dimensional version of Lemma C.3, whose proof is obtained in the same way. Recall that, if H is a Hilbert space,

$\mathcal{Q}^\infty_{++}(H)$ is the set of coercive symmetric bilinear forms on H, that is, the set of scalar products which define the topology of H.

LEMMA 19.7. *Let H be a Hilbert space with scalar product p and topological dual space H'. Write $\delta_H : \mathcal{Q}^\infty_{++}(H) \to \mathcal{Q}^\infty_{++}(H')$ for the natural bijection and $\theta_p : H \to H', v \mapsto p(v, .)$. Then δ_H is smooth and the differential of δ_H at p is the linear map*

$$\mathrm{d}(\delta_H)_p : \mathcal{Q}^\infty(H) \to \mathcal{Q}^\infty(H'), q \mapsto -(\theta_p^{-1})^\star q.$$

The proof of Theorem 19.5 will use an elementary argument of Hilbert spaces geometry that we formulate in the language of Appendix B.

LEMMA 19.8. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a straight Euclidean projective system and $H \subset \varprojlim_{\ell\geq 0} X_\ell$ be its Euclidean projective limit, equipped with its natural scalar product q. Let L be another Hilbert space and $\varphi : L \to H$ be a continuous linear map. For $\ell \geq 0$, set $\varphi_\ell = \pi_\ell\varphi : L \to X_\ell$ and assume that the adjoint map of φ_ℓ is an isometry onto its image in L. Then, the adjoint of φ is also an isometry onto its image.*

PROOF. Let $\ell \geq 0$. As $(\pi_\ell)_* q = q_\ell$, the adjoint $\pi_\ell^\dagger : X_\ell \to H$ of π_ℓ is an isometry onto its image in H. As $\varphi^\dagger\pi_\ell^\dagger = \varphi_\ell^\dagger$, $\varphi^\dagger$ induces an isometry from $\pi_\ell^\dagger X_\ell$ onto its image in L. The conclusion follows as $\bigcup_\ell \pi_\ell^\dagger X_\ell$ is dense in H. □

PROOF OF THEOREM 19.5. The strategy of the proof is to aim at applying Lemma 19.8 to the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric, and to the Hilbert space L which is the completion of $\mathcal{Q}^\infty(H^p)^\Gamma$ with respect to the standard metric. Therefore, we will construct linear maps $\mathcal{F}_j \to \mathcal{Q}^\infty(H^p)^\Gamma$ which will be adjoint to the natural maps $\mathcal{Q}^\infty(H^p)^\Gamma \to \mathcal{F}_j$. We now begin this construction.

Write $\delta : \mathcal{Q}^\infty_{++}(H_0^\omega) \to \mathcal{Q}^\infty_{++}((H_0^\omega)')$ for the natural bijection. Recall that $\mathcal{P}_k$ stands for the set of Γ-invariant k-Euclidean fields and $\mathcal{K}_k$ for the vector space of Γ-invariant k-dual kernels. By Proposition 11.22 the natural map $\iota_k : \mathcal{P}_k \hookrightarrow \mathcal{K}_k$ is a smooth immersion.

Let first p be any element of $\mathcal{P}_k$ and p^∞ be the scalar product on $\overline{\mathcal{D}}(\partial X)$ obtained from p by successive orthogonal extenions. The construction of the k-dual kernel $(K, K^-) = \iota_k(p)$ in Section 5.1 and the definition of the Hilbert space H^{K,K^-} in Section 5.4 ensure that H^{K,K^-} is the space of distributions which are bounded with respect to p and that the natural bilinear form q^{K,K^-} on H^{K,K^-} is dual to p^∞. If p is admissible, then by definition, $H^{K,K^-} = H_0^\omega$ and the previous can be written as

$$p^\infty = \delta(q^{K,K^-}) = \delta(q^{\iota_k(p)}). \tag{19.9}$$

Recall that $\mathcal{W}_k$ is the space of cohomology classes of Γ-invariant symmetric functions on X_k and that $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is the weight map. As in the proof of Theorem 11.27, let $F_k : \mathcal{W}_k \to \mathcal{Q}^\infty_{++}(H^\omega_0)$ denote the map that sends the cohomology class of a function w to Φ_w. Let still p be admissible and $(K, K^-) = \iota_k(p)$. By Theorem 8.6, we have $q^{K,K^-} = \Phi_w$ where w is a Γ-invariant weight function of $(K, K)^-$. Thus, by using (19.9), we get

$$p^\infty = \delta F_k W_k \iota_k(p).$$

Finally, denote by $\pi_k : \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{F}_k$ the natural surjective linear map. Since by construction $\pi_k(p^\infty) = p$, we have established that, for p in $\mathcal{P}^{\rm ad}_k$,

$$p = \pi_k \delta F_k W_k \iota_k(p). \tag{19.10}$$

In the above relations, the linear maps π_k, F_k and W_k are linear and continuous. Indeed, W_k and F_k are defined on finite-dimensional vector spaces, whereas π_k is defined on $\mathcal{Q}^\infty((H^\omega_0)')^\Gamma$ by evaluating bilinear forms on elements of H^ω_0. Besides, the map ι_k is smooth by Proposition 11.22 and the map δ is smooth by Lemma 19.7. Write $U_p : H^p = (H^\omega_0)' \to H^\omega_0$ for the isomorphism associated to p. By Lemma 19.7, we get, after differentiating in (19.10), for p in $\mathcal{P}^{\rm ad}_k$ and q in $\mathcal{F}_k$,

$$q = -\pi_k U_p^\star F_k W_k \mathrm{d}_p \iota_k(q). \tag{19.11}$$

In view of this, we define a linear map $\theta_k : \mathcal{F}_k \to \mathcal{Q}^\infty(H^p)^\Gamma$ by setting, for q in $\mathcal{F}_k$,

$$\theta_k q = -U_p^\star F_k W_k \mathrm{d}_p \iota_k(q), \tag{19.12}$$

so that (19.11) reads as

$$\pi_k \theta_k q = q. \tag{19.13}$$

We claim that, for Π in $\mathcal{Q}^\infty(H^p)^\Gamma$, we have the adjointness property

$$p^*_\Gamma(\Pi, \theta_k q) = g_p(\pi_k \Pi, q), \tag{19.14}$$

where g_p is the weight metric. Indeed, in view of the definition of the standard scalar product p^*_Γ in Section D.4 and by (19.12), we have,

$$p^*_\Gamma(\Pi, \theta_k q) = -\langle \Pi, F_k W_k \mathrm{d}_p \iota_k(q) \rangle_\Gamma = -[\pi_k \Pi, \mathrm{d}_p \iota_k(q)],$$

where the second equality follows from Proposition 19.3. By the definition of g_p in Section 16.1, (19.14) follows.

Let us use these constructions to show that, when viewed as a subspace of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, the space $\mathcal{Q}^\infty(H^p)^\Gamma$ is contained in the Euclidean

projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric. Indeed, for Π in $\mathcal{Q}^\infty(H^p)^\Gamma$, we have, by (19.14),

$$g_p(\pi_k\Pi, \pi_k\Pi) = p^*_\Gamma(\theta_k\pi_k\Pi, \Pi).$$

By Cauchy Schwartz inequality, we get

$$g_p(\pi_k\Pi, \pi_k\Pi)^2 \leq p^*_\Gamma(\theta_k\pi_k\Pi, \theta_k\pi_k\Pi)p^*_\Gamma(\Pi, \Pi).$$

Still by (19.14), we have

$$p^*_\Gamma(\theta_k\pi_k\Pi, \theta_k\pi_k\Pi) = g_p(\pi_k\theta_k\pi_k\Pi, \pi_k\Pi) = g_p(\pi_k\Pi, \pi_k\Pi),$$

where we have used (19.13). Thus, $g_p(\pi_k\Pi, \pi_k\Pi) \leq p^*_\Gamma(\Pi, \Pi)$. By applying this property to the successive orthogonal extensions of p, we get that $\mathcal{Q}^\infty(H^p)^\Gamma$ is contained in the Euclidean projective limit H of the projective system $(\mathcal{F}_j)_{j\geq k}$, and that this inclusion has norm ≤ 1.

Therefore, this inclusion induces a bounded linear map $\varphi : L \to H$, where L is the completion of $\mathcal{Q}^\infty(H^p)^\Gamma$ with respect to the standard scalar product. By (19.14), for $j \geq k$, the adjoint map of $\pi_j\varphi$ is θ_j and by (19.13), θ_j is an isometry onto its image. By Lemma 19.8, to conclude it suffices to prove that the null space of φ is reduced to $\{0\}$. This will follow from Lemma 19.6. Indeed, recall that by Theorem 9.1, the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is surjective. Hence, by Proposition 19.3, for Π in L, if $\varphi\Pi = 0$, then, for any $j \geq 1$ and any w in $\mathcal{W}_j$, we have $\langle \Pi, \Phi_w\rangle_\Gamma = 0$. By Lemma 19.6, this implies $\Pi = 0$ as required. □

Theorem 19.5 yields the following corollary which we were not able to prove in a more direct way.

Corollary 19.9. *Let $k \geq 2$ and p, q be admissible Γ invariant Euclidean fields. There exists $C \geq 1$ such that, for any $j \geq k$, one has*

$$\frac{1}{C} g_{p^j} \leq g_{q^j} \leq C g_{p^j}$$

where g_{p^j} and g_{q^j} are the weight metrics associated with the $(j-k)$-th orthogonal extensions of p and q on $\mathcal{F}_j$.

Proof. This follows directly from Corollary D.16 and Theorem 19.5. □

Part 5

The spectral transform

CHAPTER 20

Pseudofunctions

From now on, we assume X is regular (or homogeneous): for every x in X, the number $d(x)$ of neighbours of x is constant with value $q+1$, where q is an integer ≥ 2.

Our aim until Chapter 31 will be to answer the spectral problem of Section 1.5 for the action of Γ on the completion H^p of $\overline{\mathcal{D}}(\partial X)$ with respect to the scalar product obtained by successive orthogonal extensions of a given Γ-invariant Euclidean field p. In Section 1.5, the spectral problem was set by means of the space $\mathcal{F}(X, H^p)^\Gamma$ of Γ-equivariant maps $X \to H^p$. It will actually be more convenient to use instead the space $\mathcal{F}(X_1, H^p)^\Gamma$ of Γ-equivariant maps $X_1 \to H^p$.

The purpose of this chapter is to define pseudofunctions, which are truncated maps $X_1 \to \overline{\mathcal{D}}(\partial X)$ in the same way as pseudokernels can be thought of as truncated maps $X_1 \to \mathcal{D}(\partial^2 X)$. The space of all Γ-equivariant maps $X_1 \to \overline{\mathcal{D}}(\partial X)$ is denoted by $\mathcal{H}_\infty$. For any $k \geq -1$, we introduce the notion of a k-pseudofunction. The boundary cases $k \in \{-1, 0, 1\}$ are related to the objects appearing in the proof of the Ihara trace formula, Theorem 1.4. The space of all Γ-invariant k-pseudofunctions is denoted by $\mathcal{H}_k$: this is a finite-dimensional vector space with a natural embedding into the space $\mathcal{H}_\infty$ and $\mathcal{H}_\infty$ may be written as the increasing union $\mathcal{H}_\infty = \bigcup_{k \geq -1} \mathcal{H}_k$. We define operations on pseudofunctions in analogy with the operations on pseudokernels appearing in Section 6.2. We use these operations to define the polyextension map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$, where $\mathcal{H}_k^{(\mathbb{N})}$ is the space of finitely supported sequences of elements of $\mathcal{H}_k$. The range of the polyextension map is the span in $\mathcal{H}_\infty$ of the image of $\mathcal{H}_k$ by the natural operators associated to the geometry of the tree. The polyextension map allows to encode the action of these operators by operators acting on sequences of real numbers, which we call the model operators. Model operators appear implicitely in the study of the harmonic analysis on X and X_1 in [**21**]. The polyextension map is not injective but its null space may be determined in an explicit manner.

20.1. Pseudofunctions and the special cases $k = -1, 0, 1$

We start by defining k-pseudofunctions for $k \geq 1$. Recall that, for x in X and $\ell \geq 0$, we write $S^\ell(x)$ for the sphere with radius ℓ and center x and $\overline{V}^\ell(x)$ for the quotient of the space of real valued functions on $S^\ell(x)$ by the constant functions. In the same way, for $x \sim y$ in X, we set

$$S^\ell(xy) = \{z \in X \,|\, \min(d(x,z), d(y,z)) = \ell\}$$

and we let $\overline{V}^\ell(xy)$ be the quotient of the space of real valued functions on $S^\ell(xy)$ by the constant functions.

Definition 20.1. Let $k \geq 1$. If k is even, $k = 2\ell$, $\ell \geq 1$, a k-pseudofunction is a family $(H_{xy})_{(x,y)\in X_1}$ where, for any $x \sim y$ in X, H_{xy} is an element of $\overline{V}^\ell(x)$. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, a k-pseudofunction is a family $(H_{xy})_{(x,y)\in X_1}$ where, for any $x \sim y$ in X, H_{xy} is an element of $\overline{V}^\ell(xy)$.

The finite dimensional vector space of Γ-invariant k-pseudofunctions is denoted by $\mathcal{H}_k$.

In case $k = 1$, for $x \sim y$ in X, the space $\overline{V}^0(xy)$ has dimension 1. Therefore, if H is a 1-pseudofunction, we can write $H_{xy} = u(xy)\mathbf{1}_y$, $x \sim y \in X$, for some uniquely defined function u on X_1. In the sequel, we shall use this convention to identify 1-pseudofunctions with functions on X_1. In particular, as in Section 1.6, the space $\mathcal{H}_1$ will also be seen as the space of Γ-invariant functions on X_1.

Following this identification, let us define 0-pseudofunctions and (-1)-pseudofunctions.

By convention, we say that a 0-pseudofunction is a pair $(0, u)$, where u is a function on X and we write $\mathcal{H}_0$ for the space of Γ-invariant 0-pseudofunctions.

Recall from Section 1.6, that a function u on X is said to be constant on neighbours if, for any x in X and for any neighbours y, z of X, one has $u(y) = u(z)$. The space of functions that are constant on neighbours has dimension 2. The space of Γ-invariant functions that are constant on neighbours has dimension 1 or 2. As in Section 1.6 and Section 15.1, the action of Γ on X (or if this makes sense, the associated quotient graph) is called bipartite in case this dimension is 2. There is a natural involution on the space of functions on X that are constant on neighbours: if u is such a function, we define the opposite of u as the function whose value on x in X is the value of u on neighbours of x.

Again by convention, a (-1)-pseudofunction is a pair $(-1, u)$, where u is a function on X that is constant on neighbours and we write $\mathcal{H}_{-1}$ for the space of Γ-invariant (-1)-pseudofunctions.

REMARK 20.2. We have slightly changed the definitions of $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ with respect to the ones used in Section 1.6. The reason for these formal modifications will be clear in the next Section.

20.2. Operations on pseudofunctions

We define natural algebraic operations on pseudofunctions which are analogues of the ones defined on pseudokernels in Section 6.2.

DEFINITION 20.3. Let $k \geq 1$ and H be a k-pseudofunction.

If k is even, for any $x \sim y$ in X, we set $H^\vee_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} H_{xz}$. We call $H^\vee$ the reversal of H. The map $H \mapsto H^\vee$ is a linear automorphism of the space of k-pseudofunctions.

If k is odd, for any $x \sim y$ in X, we set $H^\vee_{xy} = H_{yx}$. We call $H^\vee$ the inversion of H. The map $H \mapsto H^\vee$ is an involution of the space of k-pseudofunctions.

Note that, if $k = 1$, and H is the 1-pseudofunction associated to the function u on X_1, then $H^\vee$ is the 1-pseudofunction associated to the function $(x, y) \mapsto -u(yx)$ on X_1. By convention, if $k = 0$ and H is the 0-pseudofunction associated to the function u on X, we set $H^\vee$ to be the 0-pseudofunction associated to the function $x \mapsto -u(x)$ on X. If $k = -1$ and H is the (-1)-pseudofunction associated to the function u on X that is constant on neighbours, we set $H^\vee$ to be the (-1)-pseudofunction associated to the function $-v$ on X, where v is the opposite of u. These choices will be justified by Lemma 20.6 below.

If $k \geq -1$ is odd, the inversion is an involution of $\mathcal{H}_k$ and we will use the notation

$$\mathcal{H}_{k,+} = \{H \in \mathcal{H}_k | H^\vee = H\} \text{ and } \mathcal{H}_{k,-} = \{H \in \mathcal{H}_k | H^\vee = -H\}.$$

In particular, we have $\mathcal{H}_k = \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-}$. If k is even (and ≥ 2), the reversal is not an involution, but a direct computation gives (see Lemma 17.10):

LEMMA 20.4. *Let $k \geq 0$ be even and H be a k-pseudofunction. We have $H^{\vee\vee} = qH + (q-1)H^\vee$.*

In other words, if $k \geq 0$ is even, the endomorphism $H \mapsto H^\vee$ of $\mathcal{H}_k$ is diagonalizable, with eigenvalues q and -1. We then set

$$\mathcal{H}_{k,+} = \{H \in \mathcal{H}_k | H^\vee = qH\} \text{ and } \mathcal{H}_{k,-} = \{H \in \mathcal{H}_k | H^\vee = -H\}.$$

We still have $\mathcal{H}_k = \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-}$.

Let us now define extensions of pseudofunctions. For $x \sim y$ in X, we use the notation of Section 4.2 for the natural injections:

$$I^\ell_{xy} : \overline{V}^\ell(xy) \to \overline{V}^{\ell+1}(x), \qquad \ell \geq 0,$$
$$J^\ell_{xy} : \overline{V}^\ell(x) \to \overline{V}^\ell(xy), \qquad \ell \geq 1.$$

Definition 20.5. Let $k \geq 1$ and H be a k-pseudofunction. We define the direct extension $H^>$ of H, which is a $(k+1)$-pseudofunction, as follows.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we set $H^>_{xy} = I^\ell_{xy} H_{xy}$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set $H^>_{xy} = J^\ell_{xy} H_{xy}$.

Direct extension is a linear embedding from the space of k-pseudofunctions into the space of $(k+1)$-pseudofunctions.

By convention, if $k = 0$ and H is the 0-pseudofunction associated to the function u on X, we set $H^>$ to be the 1-pseudofunction associated to the function $(x, y) \mapsto u(x)$ on X_1. If $k = -1$ and H is the (-1)-pseudofunction associated to the function u on X, we set $H^>$ to be the 0-pseudofunction associated to the function u on X.

By analogy with the case of pseudokernels (see Definition 6.7), if H is a pseudofunction, we will sometimes write H^+ instead of $H^{>\vee}$ and call it the orthogonal extension of H.

As for pseudokernels (see Lemma 6.8), we have a commutation property.

Lemma 20.6. *Let $k \geq -1$ and H be a k-pseudofunction. We have $L^{\vee>>} = L^{>>\vee}$.*

Proof. In case $k \geq 1$, this is proved as Lemma 6.8. Let us prove the cases where $k = 0$ and $k = -1$. This will justify our conventions.

If $k = 0$, assume that H is the 0-pseudofunction associated with the function u on X. Then, we need to prove that $H^{>>\vee} = -H^{>>}$. Indeed, for every $x \sim y$ in X, we have $H^>_{xy} = u(x)\mathbf{1}_y$ (in $\overline{V}^0(xy)$), hence $H^{>>}_{xy} = u(x)\mathbf{1}_y$ (in $\overline{V}^1(x)$). We get

$$H^{>>\vee}_{xy} = u(x) \sum_{\substack{z \sim x \\ z \neq y}} \mathbf{1}_z = -u(x)\mathbf{1}_y = -H^{>>}_{xy},$$

which should be proved.

If $k = -1$, assume that H is the (-1)-pseudofunction associated with the function u on X, where u is constant on neighbours. Let v be the opposite of u. Then, on one hand, $H^{>>}$ is the 1-pseudofunction

associated with the function $(x,y) \mapsto u(x)$ on X_1, hence $H^{>>\vee}$ is associated with the function $(x,y) \mapsto -u(y)$. On the other hand, $H^\vee$ is the (-1)-pseudofunction associated with the function $-v$ on X, hence $H^{\vee>>}$ is associated with the function $(x,y) \mapsto -v(x)$. The conclusion follows since by definition, for (x,y) in X_1, one has $u(y) = v(x)$. □

We will later need to know that the eigenspaces of the $\vee$ operator are not reduced to $\{0\}$.

COROLLARY 20.7. *For $k \geq -1$, one has $\mathcal{H}_{k,-} \neq \{0\}$. If k is large enough, one has $\mathcal{H}_{k,+} \neq \{0\}$.*

PROOF. Let H_{-1} and $H_0 = H_{-1}^>$ be respectively the (-1)-pseudofunction and the 0-pseudofunction associated with the constant function with value 1 on X. By construction, one has $H_{-1}^\vee = -H_{-1}$ and $H_0^\vee = -H_0$. For $k \geq 1$, define by induction a k-pseudofunction H_k by setting $H_k = H_{k-1}^>$. By Lemma 20.6, we get $H_k^\vee = -H_k$ for any $k \geq -1$, hence $\mathcal{H}_{k,-} \neq \{0\}$.

Let us now show that we have $\mathcal{H}_{k,+} \neq \{0\}$ if k is large enough.

Assume that k is even, $k = 2\ell$, $\ell \geq 1$. Let $S \subset X$ be a system of representatives for the action of Γ: thus, we have $\Gamma S = X$ and $\Gamma x \cap S = \{x\}$ for x in S. Then, an element H in $\mathcal{H}_{k,+}$ may be seen as a family $(H_x)_{x \in S}$, where, for x in S, H_x is a Γ_x-invariant element of $\overline{V}^\ell(x)$. Fix x in S. If ℓ is large enough, the finite group Γ_x has more than one orbit in the sphere $S^\ell(x)$. Therefore, the space $\overline{V}^\ell(x)$ contains a non zero Γ_x-invariant element. We get $\mathcal{H}_{k,+} \neq \{0\}$.

Assume that k is odd, $k = 2\ell + 1$, $\ell \geq 0$. Let Y be the set of non ordered pairs $\{x,y\}$ with x,y in X and $x \sim y$. The set Y may be seen as the quotient of X_1 by the natural involution $(x,y) \mapsto (y,x)$. For $\{x,y\}$ in Y, we let $\Gamma_{\{x,y\}}$ be the stabilizer of $\{x,y\}$ in Γ. Let now $S \subset Y$ be a system of representatives for the action of Γ. An element H in $\mathcal{H}_{k,+}$ may be seen as a family $(H_{\{x,y\}})_{\{x,y\} \in S}$, where, for $\{x,y\}$ in S, $H_{\{x,y\}}$ is a $\Gamma_{\{x,y\}}$-invariant element of $\overline{V}^\ell(xy)$. Fix $\{x,y\}$ in S. As above, if ℓ is large enough, the finite group $\Gamma_{\{x,y\}}$ has more than one orbit in the sphere $S^\ell(xy)$. Again, this implies $\mathcal{H}_{k,+} \neq \{0\}$. □

Let us now state an analogue of Proposition 6.9.

PROPOSITION 20.8. *Let $k \geq -1$ and G, H be k-pseudofunctions. Assume that we have $G^> = H^{>\vee}$.*

If $k \geq 0$, there exists a $(k-1)$-pseudofunction F with $G = F^{\vee>}$ and $H = F^>$.

If $k = -1$, we have $G + H = 0$.

PROOF. The cases $k = 0, -1$ directly follow from the definitions.

Assume $k = 1$. Let u and v be the functions on X_1 associated with G and H. For $x \sim y$ in X, we have, in $\overline{V}^1(x)$,

$$u(xy)\mathbf{1}_y = I^0_{xy} G_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} I^0_{xz} H_{xz} = \sum_{\substack{z\sim x\\ z\neq y}} v(xz)\mathbf{1}_z.$$

As $\mathbf{1}_y = -\sum_{\substack{z\sim x\\ z\neq y}} \mathbf{1}_z$, this gives

$$\sum_{\substack{z\sim x\\ z\neq y}} (v(xz) + u(xy))\mathbf{1}_z = 0.$$

Thus, for any $y, z \sim x$, $y \neq z$, we have $v(xz) + u(xy) = 0$. As $q \geq 2$, there exists a function w on X such that, for any $x \sim y$ in X, one has $v(xy) = w(x) = -u(xy)$. The conclusion follows by our conventions on 0-pseudofunctions.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. For $x \sim y$ in X, we have, in $\overline{V}^\ell(xy)$,

$$J^\ell_{xy} G_{xy} = J^\ell_{yx} H_{yx}.$$

In particular, for every z in $S^{\ell-1}(x)$ with $y \notin [xz]$, the function $G_{xy} \in \overline{V}^\ell(x)$ is constant on the set

$$\{w \in S^\ell(x) | z \in [xw]\},$$

that is, we may write $G_{xy} = I^{\ell-1}_{xy} F_{yx}$ where F_{yx} is in $\overline{V}^{\ell-1}(xy)$. In other words, we have $G = F^{\vee>}$. As $G^> = H^{>\vee}$, we get, by using Lemma 20.6, $H^{>\vee} = F^{\vee>>} = F^{>>\vee}$, hence $H = F^>$.

The proof in the odd case is analogous. □

20.3. The polyextension map

The spaces $\mathcal{H}_k$, $k \geq -1$, are embedded into each other through the direct extension map. We will now study their direct limit and encode some elements of this direct limit by sequences of pseudofunctions.

We shall say that a ∞-pseudofunction is any map $X_1 \to \overline{\mathcal{D}}(\partial X)$. The space of Γ-invariant ∞-pseudofunctions is denoted by $\mathcal{H}_\infty$. Recall from Sections 4.4 and 8.3 that we have defined natural operators

$$N^\ell_x : \overline{V}^\ell(x) \hookrightarrow \overline{\mathcal{D}}(\partial X) \qquad x \in X, \ell \geq 1$$
$$\text{and } N^\ell_{xy} : \overline{V}^\ell(xy) \hookrightarrow \overline{\mathcal{D}}(\partial X) \qquad x \sim y \in X, \ell \geq 0.$$

For $k \geq 1$, if H is a k-pseudofunction, we set $H^{>^\infty}$ to be the ∞-pseudofunction defined by, for $x \sim y$ in X,

$$\begin{aligned} H^{>^\infty}_{xy} &= N^\ell_x H_{xy} && k = 2\ell, \ell \geq 1 \\ H^{>^\infty}_{xy} &= N^\ell_{xy} H_{xy} && k = 2\ell + 1, \ell \geq 0. \end{aligned}$$

One easily checks that one has

$$(H^>)^{>^\infty} = H^{>^\infty}. \tag{20.1}$$

Thus, one can extend the construction in case $k = 0$ or $k = -1$ by using (20.1) as a definition.

Let $i \geq 0$, $k \geq -1$ and H be a k-pseudofunction. We denote by $H^{>^i}$ and H^{+^i} the $(k+i)$-pseudofunctions obtained from H respectively by i sucessive direct extensions and by i successive orthogonal extensions. Note that, by Lemma 20.6, we have

$$H^{+^i>>} = H^{>>+^i}. \tag{20.2}$$

This formalism allows us to compare the different notions of extension through the following generalization of Proposition 20.8.

PROPOSITION 20.9. *Let $k \geq 0$ and $(H_i)_{i\geq 0}$ be a sequence of k-pseudofunctions with $H_i = 0$ for i large enough. Set*

$$H = \sum_{i \geq 0} H_i^{+^i>^\infty}$$

Then $H = 0$ if and only if there exists a sequence $(G_i)_{i\geq 0}$ of $(k-1)$-pseudofunctions with $G_i = 0$ for i large enough such that

$$\begin{aligned} H_0 &= G_1^{\vee>\vee} + G_0^{\vee>} \\ H_i &= G_{i+1}^{\vee>\vee} - G_{i-1}^{>} && i \geq 1. \end{aligned} \tag{20.3}$$

The proof uses the following consequence of Proposition 20.8.

LEMMA 20.10. *Let $i \geq 1$, $k \geq 0$ and H be a k-pseudofunction. Assume that there exists a $(k+i-1)$-pseudofunction G with $H^{+^i} = G^>$. Then, there exists a $(k-1)$-pseudofunction F with $H = F^>$ and $G = F^{\vee+^{i-1}>}$.*

PROOF. We show the result by induction on $i \geq 1$. If $i = 1$, this is Proposition 20.8. Assume $i \geq 2$ and the result holds for $i-1$. By Proposition 20.8, there exists a $(k+i-2)$-pseudofunction G_1 with $H^{+^{i-1}} = G_1^>$ and $G = G_1^{\vee>}$. The induction assumption says that there exists a $(k-1)$-pseudofunction F with $H = F^>$ and $G_1 = F^{\vee+^{i-2}>}$. We get $G = F^{\vee+^{i-2}>\vee>} = F^{\vee+^{i-1}>}$ and the conclusion follows. □

Proof of Proposition 20.9. Assume that (20.3) holds. Then, a direct computation gives

$$H = G_1^{\vee+>^\infty} + G_0^{\vee>^\infty} + \sum_{i\geq 1} G_{i+1}^{\vee+^{i+1}>^\infty} - G_{i-1}^{>+^i>^\infty} \\ = \sum_{i\geq 0} G_i^{\vee+^i>^\infty} - G_i^{>+^{i+1}>^\infty}.$$

For $i \geq 0$, Lemma 20.6, (20.1) and (20.2) give

$$G_i^{\vee+^i>^\infty} - G_i^{>+^{i+1}>^\infty} = (G_i^{\vee+^i>>} - G_i^{>+^{i+1}})^{>^\infty} = (G_i^{>>\vee+^i} - G_i^{>+^{i+1}})^{>^\infty} \\ = 0.$$

We get $H = 0$ as required.

Conversely, we may assume that there exists i with $H_i \neq 0$. Then, we set $j = \max\{i \geq 0 | H_i \neq 0\}$ and we prove the statement by induction on j. If $j = 0$, there is nothing to prove. If $j = 1$, this is Proposition 20.8. Thus, we assume $j \geq 2$ and the statement holds for $j-1$. By assumption, we have the following equality of $(k+j)$-pseudofunctions:

$$-H_j^{+^j} = \sum_{i=0}^{j-1} H_i^{+^i>^{j-i}} = \left(\sum_{i=0}^{j-1} H_i^{+^i>^{j-i-1}} \right)^{>}.$$

Lemma 20.10 tells us that we can find a $(k-1)$-pseudofunction A with $H_j = A^{>}$ and

$$\sum_{i=0}^{j-1} H^{+^i>^{j-i-1}} + A^{\vee+^{j-1}>} = 0. \tag{20.4}$$

We define a new sequence $(J_i)_{i\geq 0}$ of k-pseudofunctions by setting

$$\begin{aligned} J_i &= H_i && 0 \leq i \leq j-3 \\ J_{j-2} &= H_{j-2} + A^{\vee>\vee} \\ J_{j-1} &= H_{j-1} \\ J_i &= 0 && i \geq j. \end{aligned}$$

We want to apply the induction assumption to this new sequence. Indeed, we compute

$$\sum_{i=0}^{j-1} J_i^{+^i>^{j-1-i}} = \sum_{i=0}^{j-1} H_i^{+^i>^{j-1-i}} + A^{\vee>\vee+^{j-2}>} = 0,$$

where the last equality follows from (20.4). Thus, the induction assumption tells us that there exists a sequence $(F_i)_{i\geq 0}$ of $(k-1)$-pseudofunctions, with finitely many non zero terms, such that

$$J_0 = F_1^{\vee>\vee} + F_0^{\vee>} \tag{20.5}$$
$$\text{and } J_i = F_{i+1}^{\vee>\vee} - F_{i-1}^{>}, \qquad i \geq 1.$$

We set $G_i = F_i$ for $i \neq j-1$ and $G_{j-1} = F_{j-1} - A$. One easily checks that (20.3) holds. Indeed, for $i \notin \{j-2, j\}$, it is equivalent to (20.5) since $H_i = J_i$. For $i = j-2$, it also follows from (20.5) since $H_{j-2} - J_{j-2} = -A^{\vee>\vee} = (G_{j-1} - F_{j-1})^{\vee>\vee}$. Finally, for $i = j$, (20.5) gives $F_{j+1}^{\vee>\vee} - F_{j-1}^{>} = 0$, hence $G_{j+1}^{\vee>\vee} - G_{j-1}^{>} = A^{>} = H_j$ as required. □

This result leads us to define two natural linear maps.

Definition 20.11. Let $k \geq 0$. For $(H_i)_{i\geq 0}$ in $\mathcal{H}_k^{(\mathbb{N})}$, we set

$$E_k(H_i)_{i\geq 0} = \sum_{i\geq 0} H_i^{+^i>^\infty} \in \mathcal{H}_\infty.$$

We call the linear map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$ the k-polyextension map.

Definition 20.12. Let $k \geq 0$. For $(G_i)_{i\geq 0}$ in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, we set

$$D_k(G_i)_{i\geq 0} \in \mathcal{H}_k^{(\mathbb{N})}$$

to be the sequence $(H_i)_{i\geq 0}$ given by (20.3). We call the linear map $D_k : \mathcal{H}_{k-1}^{(\mathbb{N})} \to \mathcal{H}_k^{(\mathbb{N})}$ the k-default map.

20.4. Injectivity of the default map

To complete the preceding picture, we will show that the default map D_k is injective. More generally, we have

Proposition 20.13. *Let $k \geq -1$ and $(G_i)_{i\geq 0}$ be a sequence of k-pseudofunctions. Assume that we have*

$$G_1^{\vee>\vee} = -G_0^{\vee>} \tag{20.6}$$
$$G_{i+1}^{\vee>\vee} = G_{i-1}^{>} \qquad i \geq 1. \tag{20.7}$$

Then, if k is odd, there exists a (-1)-pseudofunction J such that, for any $i \geq 0$, one has

$$G_{2i} = G_{2i+1} = (-1)^i J^{\vee^i>^{k+1}}. \tag{20.8}$$

If k is even, there exists a (-1)-pseudofunction J such that, for any $i \geq 0$, one has

$$G_{2i} = (-1)^i J^{\vee^{i+1}>^{k+1}} \text{ and } G_{2i+1} = (-1)^{i+1} J^{\vee^i>^{k+1}}. \tag{20.9}$$

Conversely, in both cases, if $(G_i)_{i\geq 0}$ *is of the form in* (20.8) *or* (20.9)*, then* (20.6) *and* (20.7) *hold.*

PROOF. First we check that the condition is sufficient. Let J be a (-1)-pseudofunction.

Assume k is odd and (20.8) holds. Then, we have $G_0 = G_1 = J^{>^{k+1}}$. Hence, by Lemma 20.6, $G_0^{\vee>} = G_1^{\vee>} = J^{\vee>^{k+2}}$ and, as the $\vee$-operator is (-1) on 0-pseudofunctions,

$$G_1^{\vee>\vee} = J^{\vee>^{k+2}\vee} = J^{\vee>\vee>^{k+1}} = -J^{\vee>^{k+2}} = -G_0^{\vee>}.$$

By the same arguments, for $i \geq 1$,

$$\begin{aligned} G_{2i}^{\vee>\vee} - G_{2i-2}^{>} = G_{2i+1}^{\vee>\vee} - G_{2i-1}^{>} &= (-1)^i J^{\vee^i>^{k+1}\vee>\vee} - (-1)^{i-1} J^{\vee^{i-1}>^{k+2}} \\ &= (-1)^{i+1} J^{\vee^{i+1}>^{k+2}} - (-1)^{i-1} J^{\vee^{i-1}>^{k+2}} = 0, \end{aligned}$$

which should be proved.

Assume k is even and (20.9) holds. On one hand, we have

$$G_1^{\vee>\vee} + G_0^{\vee>} = -J^{>^{k+1}\vee>\vee} + J^{\vee>^{k+1}\vee>} = J^{\vee>^{k+2}} - J^{\vee>^{k+2}} = 0.$$

On the other hand, for $i \geq 1$,

$$\begin{aligned} G_{2i+1}^{\vee>\vee} - G_{2i-1}^{>} &= (-1)^{i+1} J^{\vee^i>^{k+1}\vee>\vee} - (-1)^i J^{\vee^{i-1}>^{k+2}} \\ &= (-1)^{i+2} J^{\vee^{i+1}>^{k+2}} - (-1)^i J^{\vee^{i-1}>^{k+2}} = 0 \end{aligned}$$

and, for $i \geq 0$,

$$\begin{aligned} G_{2i+2}^{\vee>\vee} - G_{2i}^{>} &= (-1)^{i+1} J^{\vee^{i+2}>^{k+1}\vee>\vee} - (-1)^i J^{\vee^{i+1}>^{k+2}} \\ &= (-1)^{i+2} J^{\vee^{i+3}>^{k+2}} - (-1)^i J^{\vee^{i+1}>^{k+2}} = 0. \end{aligned}$$

Conversely, whether k is odd or even, let $(G_i)_{i\geq 0}$ be a sequence of k-pseudofunctions such that (20.6) and (20.7) hold. We will prove by induction on $k \geq -1$ that $(G_i)_{i\geq 0}$ is of the form in (20.8) or (20.9), depending on the parity of k.

If $k = -1$, the equations read as $G_0 = G_1$ and $G_{i-1} + G_{i+1}^{\vee} = 0$ for $i \geq 1$. The conclusion directly follows by taking $J = G_0$.

Assume $k \geq 0$ and the result is true for $k-1$. By Proposition 20.8, as (20.6) holds, there exists a $(k-1)$-pseudofunction H_0 with

$$G_1^{\vee} = H_0^{>} \text{ and } G_0^{\vee} = -H_0^{\vee>}. \tag{20.10}$$

Still by Proposition 20.8, as (20.7) holds, for $i \geq 1$, there exists a $(k-1)$-pseudofunction H_i with

$$G_{i+1}^{\vee} = H_i^{>} \text{ and } G_{i-1} = H_i^{\vee>}. \tag{20.11}$$

From (20.10) and (20.11) in case $i = 1$, we get $G_0^\vee = -H_0^{\vee>}$ and $G_0 = H_1^{\vee>}$, hence

$$H_1^{\vee>\vee} + H_0^{\vee>} = 0. \tag{20.12}$$

Besides, by (20.10) and (20.11), we have, for all $i \geq 1$, $G_i^\vee = H_{i-1}^{>}$, whereas, by (20.11), $G_i = H_{i+1}^{\vee>}$. Thus, we get

$$H_{i+1}^{\vee>\vee} = H_{i-1}^{>}. \tag{20.13}$$

By (20.12) and (20.13), we can apply the induction assumption to the sequence $(H_i)_{i\geq 0}$.

If k is even, this tells us that there exists a (-1)-pseudofunction J with, for $i \geq 0$, $H_{2i} = H_{2i+1} = (-1)^i J^{\vee^i>^k}$. By applying (20.11), we get, thanks to Lemma 20.6,

$$G_{2i} = H_{2i+1}^{\vee>} = (-1)^i J^{\vee^i>^k\vee>} = (-1)^i J^{\vee^{i+1}>^{k+1}}$$
$$G_{2i+1} = H_{2i+2}^{\vee>} = (-1)^{i+1} J^{\vee^{i+1}>^k\vee>} = (-1)^{i+1} J^{\vee^i>^{k+1}}.$$

If k is odd, there exists a (-1)-pseudofunction J with, for $i \geq 0$, $H_{2i} = (-1)^i J^{\vee^{i+1}>^k}$ and $H_{2i+1} = (-1)^{i+1} J^{\vee^i>^k}$. We get, as the $\vee$ operator is -1 on 0-pseudofunctions,

$$G_{2i} = H_{2i+1}^{\vee>} = (-1)^{i+1} J^{\vee^i>^k\vee>} = (-1)^i J^{\vee^i>^{k+1}}$$
$$G_{2i+1} = H_{2i+2}^{\vee>} = (-1)^{i+1} J^{\vee^i>^k\vee>} = (-1)^i J^{\vee^i>^{k+1}}.$$

The Proposition follows by induction. □

From Proposition 20.9 and Proposition 20.13, we get

COROLLARY 20.14. *Let $k \geq 0$. Then, the k-default map D_k is injective and the null space of the k-polyextension map E_k is the range of D_k. In other words, we have an exact sequence*

$$0 \longrightarrow \mathcal{H}_{k-1}^{(\mathbb{N})} \xrightarrow{D_k} \mathcal{H}_k^{(\mathbb{N})} \xrightarrow{E_k} \mathcal{H}_\infty.$$

PROOF. Let $(G_i)_{i\geq 0}$ be a sequence of $(k-1)$-pseudofunctions such that $G_i = 0$ for large enough i. Assume that (20.6) and (20.7) hold and let J be the (-1)-pseudofunction as in Proposition 20.13. As $G_i = 0$ for large enough i, we have $J = 0$, hence $G = 0$. In particular, this tells us that the default map D_k is injective.

Besides, let H be in $\mathcal{H}_k^{(\mathbb{N})}$ and assume that $E_k H = 0$. The above tells us that the sequence $(G_i)_{i\geq 0}$ of $(k-1)$-pseudofunctions obtained by applying Proposition 20.9 to H is uniquely determined by H. In particular, it is Γ-invariant, that is, H belongs to the range of the default map D_k, as should be proved. □

20.5. Operations on ∞-pseudofunctions

We will now define natural operations on $\mathcal{H}_\infty$ and show that they can be transfered to operations on $\mathcal{H}_k^{(\mathbb{N})}$ thanks to the polyextension map.

Let H be an ∞-pseudofunction. By analogy with Definition 20.3, we define the reversal RH and the inversion SH of H by

$$RH_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} H_{xz} \text{ and } SH_{xy} = H_{yx}, \quad x \sim y \in X.$$

The maps R and S define linear operators of $\mathcal{H}_\infty$ and we have $S^2 = 1$ and, as in Lemma 20.4, $R^2 = q + (q-1)R$. From these definitions, we directly get

LEMMA 20.15. *Let $k \geq -1$ and H be a k-pseudofunction. If k is even, we have $R(H^{>^\infty}) = H^{\vee>^\infty}$. If k is odd, we have $S(H^{>^\infty}) = H^{\vee>^\infty}$.*

We will now define analogues of these maps on $\mathcal{H}_k^{(\mathbb{N})}$.

DEFINITION 20.16. (k even) Let $k \geq 0$ be an even integer, $k = 2\ell$, $\ell \geq 0$, and $H = (H_i)_{i\geq 0}$ be a sequence of k-pseudofunctions.

The reversal RH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned}
(RH)_0 &= H_0^\vee \\
(RH)_i &= H_{i-1} + (q-1)H_i && i \text{ even}, i \geq 2 \\
(RH)_i &= qH_{i+1} && i \text{ odd}.
\end{aligned}$$

The inversion SH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned}
(SH)_i &= H_{i+1} && i \text{ even} \\
(SH)_i &= H_{i-1} && i \text{ odd}.
\end{aligned}$$

DEFINITION 20.17. (k odd) Let $k \geq -1$ be an odd integer, $k = 2\ell + 1$, $\ell \geq -1$, and $H = (H_i)_{i\geq 0}$ be a sequence of k-pseudofunctions.

The reversal RH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned}
(RH)_i &= qH_{i+1} && i \text{ even} \\
(RH)_i &= H_{i-1} + (q-1)H_i && i \text{ odd}.
\end{aligned}$$

The inversion SH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned}(SH)_0 &= H_0^\vee \\ (SH)_i &= H_{i-1} && i \text{ even}, i \geq 2 \\ (SH)_i &= H_{i+1} && i \text{ odd}.\end{aligned}$$

One directly checks that the relations $R^2 = q + (q-1)R$ and $S^2 = 1$ still hold. Besides, these maps are compatible with the polyextension map.

LEMMA 20.18. *Let $k \geq 0$ and $H = (H_i)_{i\geq 0}$ be in $\mathcal{H}_k^{(\mathbb{N})}$. We have*

$$E_k(RH) = RE_k(H) \text{ and } E_k(SH) = SE_k(H).$$

PROOF. Assume k is even. By definition, we get

$$E_k(H) = \sum_{i\geq 0} H^{+^i>^\infty} = H_0^{>^\infty} + \sum_{i\geq 1} (H_{2i-1}^{+^{2i-1}>} + H_{2i}^{+^{2i}})^{>^\infty}.$$

Now, H_0 belongs to $\mathcal{H}_k$ and, for $i \geq 1$, $H_{2i-1}^{+^{2i-1}>} + H_{2i}^{+^{2i}}$ belongs to $\mathcal{H}_{k+2i}$, so that Lemma 20.15 gives $RH_0^{>^\infty} = H_0^{\vee>^\infty}$ and

$$R(H_{2i-1}^{+^{2i-1}>} + H_{2i}^{+^{2i}})^{>^\infty} = (H_{2i-1}^{+^{2i-1}>\vee} + H_{2i}^{+^{2i}\vee})^{>^\infty}.$$

Using Lemma 20.4, we get

$$\begin{aligned}H_{2i-1}^{+^{2i-1}>\vee} + H_{2i}^{+^{2i}\vee} &= H_{2i-1}^{+^{2i}} + H_{2i}^{+^{2i-1}>\vee\vee} \\ &= qH_{2i}^{+^{2i-1}>} + (H_{2i-1} + (q-1)H_{2i})^{+^{2i}},\end{aligned}$$

which gives $RE_k(H) = Ek(RH)$.

In the same way, we have

$$E_k(H) = \sum_{i\geq 0} (H_{2i}^{+^{2i}>} + H_{2i+1}^{+^{2i+1}})^{>^\infty}.$$

As, for $i \geq 0$, $H_{2i}^{+^{2i}>} + H_{2i+1}^{+^{2i+1}}$ belongs to $\mathcal{H}_{k+2i+1}$, Lemma 20.15 gives

$$SE_k(H) = \sum_{i\geq 0} (H_{2i+1}^{+^{2i}>} + H_{2i}^{+^{2i+1}})^{>^\infty},$$

which should be proved.

The proof in case k is odd is analogous. □

Finally, to describe the behaviour of these operators R and S in the null space of the polyextension map, we define a new pair of operators R' and S' as follows.

DEFINITION 20.19. (k even) Let $k \geq 0$ be an even integer, $k \geq 2\ell$, $\ell \geq 0$, and $G = (G_i)_{\geq 0}$ be a sequence of k-pseudofunctions. The antireversal of G is the sequence $R'G$ of k-pseudofunctions defined by

$$\begin{aligned}(R'G)_0 &= (q-1)G_0 - G_0^\vee \\ (R'G)_i &= G_{i-1} + (q-1)G_i && i \text{ even}, i \geq 2 \\ (R'G)_i &= qG_{i+1} && i \text{ odd}.\end{aligned}$$

DEFINITION 20.20. (k odd) Let $k \geq -1$ be an odd integer, $k \geq 2\ell + 1$, $\ell \geq -1$, and $G = (G_i)_{\geq 0}$ be a sequence of k-pseudofunctions. The antiinversion of G is the sequence $S'G$ of k-pseudofunctions defined by

$$\begin{aligned}(S'G)_0 &= -G_0^\vee \\ (S'G)_i &= G_{i-1} && i \text{ even}, i \geq 2 \\ (S'G)_i &= G_{i+1} && i \text{ odd}.\end{aligned}$$

Note that, in the even case, the maps R and R' only differ by their action on the first component. So do the maps S and S' in the odd case.

These maps allow to describe the action of the reversal and the inversion on the range of the default map.

LEMMA 20.21. *Let $k \geq 0$ and $G = (G_i)_{i\geq 0}$ be in $\mathcal{H}_{k-1}^{(\mathbb{N})}$. If k is even, we have*

$$D_k(RG) = RD_k(G) \text{ and } D_k(S'G) = SD_k(G).$$

If k is odd, we have

$$D_k(R'G) = RD_k(G) \text{ and } D_k(SG) = SD_k(G).$$

PROOF. These are straightforward computations. For example, we deal with the reversion action in case k is even. We set $H = D_kG$. For $i = 0$, we have $H_0 = G_1^{\vee>\vee} + G_0^{\vee>}$, hence

$$\begin{aligned}(RH)_0 = G_1^{\vee>\vee\vee} + G_0^{\vee>\vee} = (G_0 + (q-1)G_1)^{\vee>\vee} + qG_1^{\vee>} \\ = (RG)_1^{\vee>\vee} + (RG)_0^{\vee>}\end{aligned}$$

as required. For $i \geq 1$, we have $H_i = G_{i+1}^{\vee>\vee} - G_{i-1}^{>}$, hence, if i is odd,

$$(RH)_i = qH_{i+1} = qG_{i+2}^{\vee>\vee} - qG_i^{>} = (RG)_{i+1}^{\vee>\vee} - (RG)_{i-1}^{\vee>\vee};$$

and if i is even,

$$\begin{aligned}(RH)_i = H_{i-1} + (q-1)H_i = G_i^{\vee>\vee} - G_{i-2}^{>} + (q-1)G_{i+1}^{\vee>\vee} - (q-1)G_{i-1}^{>} \\ = (RG)_{i+1}^{\vee>\vee} - (RG)_{i-1}^{>}.\end{aligned}$$

The other three computations are analogous. □

20.6. Pseudofunctions in the bipartite case

In this Section, we assume that Γ is bipartite. In that case, the spaces of pseudofunctions may be equipped with an additional natural operation. We introduce this operation and explain how it is related to the previous constructions.

As in Section 15.1, we fix a function $\chi : X \to \{1, -1\}$ such that, for every $x \sim y$ in X, one has $\chi(y) = -\chi(x)$. This function is uniquely defined up to a sign change and saying that Γ is bipartite amounts to saying that χ is Γ-invariant.

For $k \geq 1$ and H a k-pseudofunction, we define the twist $H^\wr$ of H as being the k-pseudofunction defined by

$$H^\wr_{xy} = \chi(x) H_{xy} \quad x \sim y \in X.$$

If $k = 0$ (resp. $k = -1$) and H is the k-pseudofunction associated to the function u on X (resp. to the function u on X that is constant on neighbours), we let $H^\wr$ be the k-pseudofunction associated with the function χu.

We directly get

Lemma 20.22. *Let $k \geq -1$ and H be a k-pseudofunction. We have $H^{\wr >} = H^{> \wr}$. If k is even, we have $H^{\wr \vee} = H^{\vee \wr}$. If k is odd, we have $H^{\wr \vee} = -H^{\vee \wr}$.*

In the same way, if H is an ∞-pseudofunction, we set UH to be the ∞-pseudofunction defined by

$$(UH)_{xy} = \chi(x) H_{xy} \quad x \sim y \in X.$$

As in Lemma 20.22, we have the relations

$$UR = RU \text{ and } US = -SU.$$

If $k \geq -1$ and H is a k-pseudofunction, we have $U(H^{>\infty}) = H^{\wr >\infty}$.

To check the compatibility properties of this operation with the polyextension map, we introduce an operator on sequences. For $k \geq -1$ and $(H_i)_{i \geq 0}$ a sequence of k-pseudofunctions, we set UH to be the sequence defined by

$$\begin{aligned}
(UH)_i &= (-1)^{\frac{i}{2}} H_i^\wr && i \text{ even} \\
(UH)_i &= (-1)^{\frac{i+1}{2}} H_i^\wr && i \text{ odd}, k \text{ even} \\
(UH)_i &= (-1)^{\frac{i-1}{2}} H_i^\wr && i \text{ odd}, k \text{ odd}.
\end{aligned}$$

A direct computation using Definition 20.11 and Definition 20.12 yields

LEMMA 20.23. *Let $k \geq -1$. For H in $\mathcal{H}_k^{(\mathbb{N})}$, we have*

$$E_k U H = U E_k H.$$

For G in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, we have

$$D_k U G = -U D_k G \qquad k \text{ even}$$
$$D_k U G = U D_k G \qquad k \text{ odd}.$$

CHAPTER 21

Preliminaries for the study of the model operators

Lemma 20.18 above suggests that, in order to study the natural operations on $\mathcal{H}_\infty$, we need to analyse operators defined on sequences of real numbers by formulas as in Definition 20.16 and 20.17. We call these operators the model operators. The purpose of the next three Chapters is to develop a precise spectral analysis of the model operators.

Our general strategy for doing this relies on considerations of abstract harmonic analysis. Indeed, the involutive algebra generated by two self-adjoint elements R and S with the relations $R^2 = qR+(q-1)R$ and $S^2 = 1$ is the algebra of the infinite diedral group. In particular, any $*$-representation of this algebra on a Hilbert space may be decomposed (maybe continuously) into irreducible components and these irreducible components have dimension at most 2, as the infinite diedral group has a normal abelian subgroup of index 2.

It turns out that, for the model operators, this decomposition may be constructed in a very explicit way. We shall build this decomposition. We will need to split these construction according to the parity of k and to the eigenvalues of the $\vee$ operator in the formulas from Definition 20.16 and 20.17. Thus, we need to consider four different cases which will be carried out in the Sections 22.1, 22.3, 23.1 and 23.2.

In the present preliminary Chapter, we introduce notation and facts that we will use in the construction of the spectral theory of model operators. Some of them will also be later required in the spectral analysis of Euclidean fields.

21.1. Algebraic preliminaries

We introduce a general algebraic framework for the actions of operators R and S which satisfy the properties of the R and S operators on ∞-pseudofunctions.

Let R and S be two symbols and $\mathcal{A}$ be the real algebra spanned by R and S with the relations $S^2 = 1$ and $R^2 = q + (q-1)R$. In other words, $\mathcal{A}$ is the quotient of the tensor algebra of the vector space $\mathbb{R}R \oplus \mathbb{R}S$ by the ideal spanned by $S^2 - 1$ and $R^2 - q - (q-1)R$. We

equip $\mathcal{A}$ with the unique involutive anti-automorphism $A \mapsto A^*$ such that $S^* = S$ and $R^* = R$.

Set $S' = \frac{1}{q+1}(2R - (q-1))$. A direct computation shows that $(S')^2 = 1$. Therefore, $\mathcal{A}$ may be seen as the group algebra of the group G generated by S and S', which is isomorphic to the infinite diedral group. In particular, the element $T = SS'$ is unitary and the element P defined by

$$P = \frac{1}{2}(T + T^{-1}) = \frac{1}{q+1}(RS + SR - (q-1)S)$$

is self-adjoint. Set $\mathcal{B}$ to be the subalgebra spanned by T. By construction, $\mathcal{B}$ is the algebra of Laurent polynmials $\mathbb{R}[T, T^{-1}]$.

LEMMA 21.1. *We have $\mathcal{A} = \mathcal{B} \oplus S\mathcal{B}$. The center $\mathcal{C}$ of $\mathcal{A}$ is the subalgebra spanned by P, which is isomorphic to the polynomial algebra $\mathbb{R}[P]$.*

The explicit form of the center will play a role for determining spectral measures in Chapter 22 and 23.

PROOF. The first property is a direct translation of the fact that $\mathcal{A}$ is the group algebra of G and that G may be written as $G = T^{\mathbb{Z}} \sqcup ST^{\mathbb{Z}}$. Now, let C be in the center $\mathcal{C}$ of $\mathcal{A}$ and write

$$C = \sum_n a_n T^n + S \sum_n b_n T^n$$

for some sequences (a_n) and (b_n) in $\mathbb{R}^{(\mathbb{Z})}$. We have

$$C = SCS = \sum_n a_n T^{-n} + S \sum_n b_n T^{-n},$$

hence $a_n = a_{-n}$ and $b_n = b_{-n}$ for n in $\mathbb{Z}$. Besides, as

$$S'ST^nS' = (SS)(S'S)(S'S')T^nS' = S(SS')(SS')(S'T^nS') = ST^{2-n},$$

we get

$$C = S'CS' = \sum_n a_n T^{-n} + S \sum_n b_n T^{2-n},$$

hence, for n in $\mathbb{Z}$, $b_n = b_{2-n} = b_{n-2}$. As b_n is zero for $|n|$ large, we get $b_n = 0$ for any n. The result easily follows. □

We shall also need the existence of nice complementary subspaces for some natural left ideals in $\mathcal{A}$.

COROLLARY 21.2. *One has the following decompositions:*

$$\begin{aligned}\mathcal{A} &= \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(R-q)\\ &= \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(R+1)\\ &= \mathcal{C} \oplus R\mathcal{C} \oplus \mathcal{A}(S-1)\\ &= \mathcal{C} \oplus R\mathcal{C} \oplus \mathcal{A}(S+1)\end{aligned}$$

PROOF. Let us show first the decomposition

$$\mathcal{A} = \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(S'-1). \tag{21.1}$$

Indeed, Lemma 21.1 gives (by exchanging the roles of S and S')

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{B}S'.$$

By using the identities, for B_0, B_1 in $\mathcal{B}$,

$$B_0 + B_1S' = (B_0 + B_1) + B_1(S'-1) = (B_0 - B_1) + B_1(S'+1),$$

we get

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{B}(S'-1) = \mathcal{B} \oplus \mathcal{B}(S'+1). \tag{21.2}$$

Still by Lemma 21.1, as $\mathcal{B}$ is the algebra of Laurent polynomials $\mathcal{B} = \mathbb{R}[T, T^{-1}]$ and $\mathcal{C}$ is the polynomial algebra $\mathcal{C} = \mathbb{R}[T + T^{-1}]$, we get

$$\mathcal{B} = \mathcal{C} \oplus \mathcal{C}(T - T^{-1}).$$

As we have $T - T^{-1} = -(T + T^{-1}) + 2T = -2P + 2T$, we get

$$\mathcal{B} = \mathcal{C} \oplus \mathcal{C}T,$$

hence, from the first first equality in (21.2),

$$\mathcal{A} = \mathcal{C} \oplus \mathcal{C}T \oplus \mathcal{B}(S'-1). \tag{21.3}$$

In particular, since $T = S + S(S'-1)$, this gives

$$\mathcal{A} = \mathcal{C} + \mathcal{C}S + \mathcal{A}(S'-1). \tag{21.4}$$

To conclude the proof of (21.1), we need to show that this sum is a direct sum. Thus, we take A in $\mathcal{A}$ and C_0, C_1 in $\mathcal{C}$ such that we have

$$C_0 + C_1S + A(S'-1) = 0$$

and we will show that $C_0 = C_1 = A(S'-1) = 0$. Indeed, still as $T = S + S(S'-1)$, we have

$$C_0 + C_1T + (A - C_1S)(S'-1) = 0.$$

Use the second equality in (21.2) to find B_0, B_1 in $\mathcal{B}$ with $A - C_1S = B_0 + B_1(S'+1)$, so that

$$(A - C_1S)(S'-1) = B_0(S'-1). \tag{21.5}$$

We get

$$C_0 + C_1 T + B_0(S' - 1) = 0,$$

so that (21.3) gives $C_0 = C_1 = 0$ and $B_0(S' - 1) = 0$. By (21.5), we get $A(S' - 1) = 0$ and (21.1) follows.

By changing the roles and signs of S and S', we also get

$$\begin{aligned}\mathcal{A} &= \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(S' + 1) \\ &= \mathcal{C} \oplus S'\mathcal{C} \oplus \mathcal{A}(S - 1) \\ &= \mathcal{C} \oplus S'\mathcal{C} \oplus \mathcal{A}(S + 1).\end{aligned}$$

The result follows by using the relations $(q + 1)(S' - 1) = 2(R - q)$, $(q + 1)(S' + 1) = 2(R + 1)$ and $\mathcal{C} \oplus S'\mathcal{C} = \mathcal{C} \oplus R\mathcal{C}$. □

As T spans a normal subgroup of index 2 in G, any self-adjoint representation of $\mathcal{A}$ in a Hilbert space H may be decomposed into a direct integral of irreducible representations of $\mathcal{A}$ and these irreducible representations have dimension 1 or 2.

There are four representations in dimension 1 which are determined by letting R (resp. S) be the scalars q or -1 (resp. 1 or -1). Let us now give explicit generators for the irreducible representations in dimension 2. The four different versions of the generators will be adapted to the four different sets of model operators that we will consider in Chapter 22 and Chapter 23.

The $++$ and $+-$ matrices. Set $s_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and, for t in $\mathbb{C}$,

$$r_{++}(t) = \begin{pmatrix} q & (q+1)t \\ 0 & -1 \end{pmatrix} \qquad a_{++}(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$$

$$r_{+-}(t) = \begin{pmatrix} -1 & (q+1)t \\ 0 & q \end{pmatrix} \qquad a_{+-}(t) = \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}.$$

The $-+$ and $--$ matrices. Set $r_- = \begin{pmatrix} q-1 & q \\ 1 & 0 \end{pmatrix}$ and, for t in $\mathbb{C}$,

$$s_{-+}(t) = \begin{pmatrix} 1 & (q+1)t - (q-1) \\ 0 & -1 \end{pmatrix}$$

$$a_{-+}(t) = \begin{pmatrix} 2 & (q+1)t - (q-1) \\ (q+1)t - (q-1) & q^2 + 1 - (q^2-1)t \end{pmatrix}$$

$$s_{--}(t) = \begin{pmatrix} -1 & (q+1)t + (q-1) \\ 0 & 1 \end{pmatrix}$$

$$a_{--}(t) = \begin{pmatrix} 2 & -(q+1)t - (q-1) \\ -(q+1)t - (q-1) & q^2 + 1 + (q^2-1)t \end{pmatrix}.$$

The proof of the following Lemma follows from straightforward computations.

LEMMA 21.3. *Let t be in $\mathbb{C}$. The action of $\mathcal{A}$ on $\mathbb{C}^2$ defined by letting R act as $r_{++}(t)$ (resp. $r_{+-}(t)$, resp. r_-, resp. r_-) and S act as s_+ (resp. s_+, resp. $s_{-+}(t)$, resp. $s_{--}(t)$) has a scalar commutant. In all four cases, the central element P of $\mathcal{A}$ acts as the scalar matrix $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.*

This representation is self-adjoint with respect to the symmetric bilinear form defined by $a_{++}(t)$ (resp. $a_{+-}(t)$, resp. $a_{-+}(t)$, resp. $a_{--}(t)$) on $\mathbb{C}^2$ and this symmetric bilinear form is the unique one with this property, up to a scalar multiple. If t is real, this symmetric bilinear form is positive definite on $\mathbb{R}^2$ if and only if $|t| < 1$. If $|t| > 1$, it has signature $(1, 1)$.

For $t^2 \neq 1$, these actions are all irreducible and isomorphic to each other.

21.2. An abstract spectral theorem

In the sequel, we will encounter self-adjoint representations of the algebra $\mathcal{A}$. To study their spectral theory, we will need the following abstract form of the spectral theorem [**44**, Theorem 12.23].

LEMMA 21.4. *Let V be a finite dimensional real vector space. Let p be a non-negative symmetric bilinear form on the space $V[t]$ of polynomial functions $\mathbb{R} \to V$ and assume that the operator $f(t) \mapsto tf(t)$ on $V[t]$ is bounded and symmetric with respect to p. Then, there exists a compactly supported Radon measure μ on $\mathbb{R}$ and a μ-integrable function $\pi : \mathbb{R} \to \mathcal{Q}_+(V)$ such that, for any f, g in $V[t]$, one has*

$$p(f, g) = \int_{\mathbb{R}} \pi(t)(f(t), g(t))\mathrm{d}\mu(t).$$

As usual, we wrote $\mathcal{Q}_+(V)$ for the space of non-negative symmetric bilinear forms on V. When applied to the concrete examples of representations of the algebra $\mathcal{A}$ from Section 21.1, this gives

COROLLARY 21.5. *Let p be non-negative symmetric bilinear form on $\mathbb{R}^2[t]$ such that the operators $f(t) \mapsto r_{++}(t)f(t)$ (resp. $f(t) \mapsto r_{-+}(t)f(t)$, resp. $f(t) \mapsto r_-f(t)$, resp. $f(t) \mapsto r_-f(t)$) as well as $f(t) \mapsto s_+f(t)$ (resp. $f(t) \mapsto s_+f(t)$, resp. $f(t) \mapsto s_{-+}(t)f(t)$, resp. $f(t) \mapsto s_{--}(t)f(t)$) are bounded and self-adjoint with respect to p. Then, there exists a Radon measure μ on $[-1, 1]$ such that, for any*

f, g in $\mathbb{R}^2[t]$, one has

$$p(f,g) = \int_{[-1,1]} \pi(t)(f(t), g(t))\mathrm{d}\mu(t),$$

where, for $-1 \leq t \leq 1$, $\pi(t)$ is the symmetric bilinear form with matrix $a_{++}(t)$ (resp. $a_{+-}(t)$, resp. $a_{-+}(t)$, resp $a_{--}(t)$) on $\mathbb{R}^2$.

Proof. We show the $++$ case, the other ones being analogous.

By Lemma 21.3, for f in $\mathbb{R}^2[t]$, and t in $\mathbb{R}$, we have

$$\frac{1}{q+1}(r_{++}(t)s_+ + s_+r_{++}(t) - (q-1)s_+)f(t) = tf(t).$$

Therefore, by Lemma 21.4, there exists a compactly supported Radon measure μ on $\mathbb{R}$ and a μ-integrable function $\pi : \mathbb{R} \to \mathcal{Q}_+(\mathbb{R}^2)$ such that, for any f, g in $\mathbb{R}^2$, one has

$$\pi(f,g) = \int_{\mathbb{R}} \pi(t)(f(t), g(t))\mathrm{d}\mu(t).$$

By assumption, for any polynomial functions $f, g : \mathbb{R} \to \mathbb{R}^2$, we have

$$\int_{\mathbb{R}} \pi(t)(r_{++}(t)f(t), g(t))\mathrm{d}\mu(t) = \int_{\mathbb{R}} \pi(t)(f(t), r_{++}(t)g(t))\mathrm{d}\mu(t)$$
$$\int_{\mathbb{R}} \pi(t)(s_+f(t), g(t))\mathrm{d}\mu(t) = \int_{\mathbb{R}} \pi(t)(f(t), s_+g(t))\mathrm{d}\mu(t).$$

Therefore, for μ-almost any t in $\mathbb{R}$, the operators $r_{++}(t)$ and s_+ are self-adjoint with respect to the symmetric bilinear form $\pi(t)$. By Lemma 21.3, there exists a μ-integrable function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with support in $[-1, 1]$ such that, for t in $\mathbb{R}$,

$$\pi(t)(v, w) = \varphi(t)v^* a_{++}(t)w, \quad v, w \in \mathbb{R}^2$$

(where v^* is the transpose of the column vector v in $\mathbb{R}^2$). The conclusion follows by replacing μ with the finite measure $\varphi\mu$. □

21.3. Spectral parametrization

As when studying spectral theory on X (see [**21**, Chapter II]), the formulas that will allow to decompose the model operators rely on the use of a certain rational function on $\mathbb{C}$. Set as usual

$$\mathbb{H} = \{t \in \mathbb{C} | \Im t > 0\} \text{ as well as } \mathbb{H}_q = \{u \in \mathbb{C} | \Im u > 0, |u| > \sqrt{q}\}.$$

We will write $\overline{\mathbb{H}}$ and $\overline{\mathbb{H}}_q$ for the closures of these open subsets of $\mathbb{C}$, that is,

$$\overline{\mathbb{H}} = \{t \in \mathbb{C} | \Im t \geq 0\} \text{ and } \overline{\mathbb{H}}_q = \{u \in \mathbb{C} | \Im u \geq 0, |u| \geq \sqrt{q}\}.$$

We will always denote by $\mathcal{I}_q$ the critical interval

$$\mathcal{I}_q = \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}\right].$$

By a result of Kesten [**32**] (see also [**21**, Proposition II.6.3] whose statement is unfortunately mistakenly written), the critical interval is the spectrum of the natural Markov operator on the space $\ell^2(X)$ of square-integrable functions of X. We retrieve this fact in Section 22.2 below.

We shall repeatedly use

LEMMA 21.6. *The function*

$$u \mapsto t = \frac{1}{q+1}\left(u + \frac{q}{u}\right)$$

maps $\mathbb{C}^*$ *onto* $\mathbb{C}$. *It induces a biholomorphism from* $\{u \in \mathbb{C} | \, |u| > \sqrt{q}\}$ *onto* $\mathbb{C} \smallsetminus \mathcal{I}_q$ *which sends* $\mathbb{H}_q$ *onto* $\mathbb{H}$. *It also induces a homeomorphism from* $\overline{\mathbb{H}}_q$ *onto* $\overline{\mathbb{H}}$. *Finally, it maps the circle* $\{u \in \mathbb{C} | |u| = \sqrt{q}\}$ *onto* $\mathcal{I}_q$.

The proof is straightforward.

From now on, we will stick to the notational convention that t and u are complex numbers which are related through the equation

$$u^2 - (q+1)tu + q = 0.$$

Note that the other root of the equation is $\frac{q}{u}$.

To determine spectral measures of the model operators, we will use a standard technique from spectral analysis (see for example [**20**, Theorem X.6.1]), which relies on the

LEMMA 21.7. *Let* μ *be a Borel measure on* $\mathbb{R}$ *with* $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$. *Then the Poisson transform of* μ *may be defined on* $\mathbb{H}$ *by the formula*

$$\mathcal{P}\mu(z) = \frac{1}{\pi}\int_{\mathbb{R}} \Im\left(\frac{1}{t-z}\right) \mathrm{d}\mu(t) \quad z \in \mathbb{H}.$$

PROOF. Indeed, for z in $\mathbb{H}$ and t in $\mathbb{R}$, if $z = x + iy$, $x, y \in \mathbb{R}$, we have

$$\frac{1}{\pi}\Im\left(\frac{1}{t-z}\right) = \frac{1}{\pi}\frac{y}{(t-x)^2 + y^2},$$

which is the standard formula for the Poisson kernel of the upper half plane (see [**1**, Chapter 7]). □

This Lemma will be applied to a particular harmonic function which plays a central role in the spectral theory of the tree X. We denote by μ_q the absolutely continuous measure with density $\frac{q+1}{2\pi}\frac{\sqrt{4q-(q+1)^2t^2}}{1-t^2}$ on the interval $\mathcal{I}_q = \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}\right]$. We will show later that μ_q is a

probability measure by getting it as the spectral measure associated to a unit vector and a self-adjoint operator on a Hilbert space. This construction will rely on

LEMMA 21.8. *The Poisson transform of μ_q on $\mathbb{H}$ may be defined as follows. For any t in $\mathbb{H}$, let u be the unique element of $\mathbb{H}_q$ with $u^2-(q+1)tu+q=0$. Then, we have*

$$\mathcal{P}\mu_q(t)=\frac{q+1}{\pi}\Im\left(\frac{u}{1-u^2}\right).$$

PROOF. For t in $\overline{\mathbb{H}}$, we set $F(t)=\frac{q+1}{\pi}\Im\left(\frac{u}{1-u^2}\right)$, where u is the unique element of $\overline{\mathbb{H}}_q$ with $q+u^2=(q+1)t$. As the function $G:u\mapsto\Im\left(\frac{u}{1-u^2}\right)$ is harmonic on $\mathbb{H}_q$ and continuous on $\overline{\mathbb{H}}_q$, by Lemma 21.6, the function F is harmonic on $\mathbb{H}$ and continuous on $\overline{\mathbb{H}}$. As $G(u)\xrightarrow[u\to\infty]{}0$, by standard properties of harmonic functions (see for example [**1**, Theorem 7.5]), we may write F as the Poisson transform of the measure $\nu=F(t)\mathrm{d}t$ on $\mathbb{R}$. Let us show that $\nu=\mu_q$.

As G vanishes on the set $\{u\in\mathbb{R}|\,|u|\geq\sqrt{q}\}$, F vanishes on the set $\{t\in\mathbb{R}|(q+1)\,|t|\geq 2\sqrt{q}\}$. Therefore, it suffices to determine the value of G on the set

$$\partial\mathbb{H}_q\smallsetminus\mathbb{R}=\{u\in\mathbb{C}|\Im u>0,|u|=\sqrt{q}\}.$$

Thus, let u in $\mathbb{C}$ with $|u|=\sqrt{q}$ and set $t=\frac{1}{q+1}(u+q/u)=\frac{2}{q+1}\Re u$. We get

$$\begin{aligned}2i\Im\left(\frac{u}{1-u^2}\right)&=\frac{u}{1-u^2}-\frac{q/u}{1-q^2/u^2}=\frac{u-q^2/u-q/u+qu}{(1-u^2)(1-q^2/u^2)}\\&=(q+1)\frac{u-q/u}{q^2+1-u^2-q^2/u^2}=\frac{1}{q+1}\frac{u-q/u}{1-t^2},\end{aligned}$$

which gives

$$\Im\left(\frac{u}{1-u^2}\right)=\frac{1}{q+1}\frac{\Im u}{1-t^2}.$$

The conclusion follows as, if $\Im u\geq 0$, we have

$$\Im u=\sqrt{q-(\Re u)^2}=\frac{1}{2}\sqrt{4q-(q+1)^2t^2}.$$

□

21.4. Hilbert spaces of sequences

The construction of the spectral decomposition of the model operators will rely on the application of spectral constructions in certain Hilbert spaces of sequences. We now define precisely these spaces.

For x in $\mathbb{R}^{(\mathbb{N})}$ and y in $\mathbb{R}^{\mathbb{N}}$, we set

$$\langle x, y\rangle_+ = \sum_{i\geq 0} q^i(x_{2i}y_{2i} + x_{2i+1}y_{2i+1})$$
$$\langle x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i-1}y_{2i-1} + x_{2i}y_{2i})$$

We denote the associated real Hilbert spaces of sequences by H_+ and H_-, that is, H_+ (resp. H_-) is equal to the space

$$\left\{ x \in \mathbb{R}^{\mathbb{N}} \left| \sum_{i\geq 0} q^{\frac{i}{2}} x_i^2 < \infty \right. \right\} < \infty,$$

which we equip with the scalar product $\langle ., .\rangle_+$ (resp. $\langle ., .\rangle_-$). We stress out that the Hilbert spaces H_+ and H_- have the same underlying vector space, but not the same scalar product.

Spectral analysis will require us to use the complexifications $H_{+,\mathbb{C}}$ and $H_{-,\mathbb{C}}$ of H_+ and H_-. In this framework, we recall our non standard convention (see Section 1.9): for any x in $\mathbb{C}^{(\mathbb{N})}$ and y in $\mathbb{C}^{\mathbb{N}}$, we still set

$$\langle x, y\rangle_+ = \sum_{i\geq 0} q^i(x_{2i}y_{2i} + x_{2i+1}y_{2i+1})$$
$$\langle x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i-1}y_{2i-1} + x_{2i}y_{2i}).$$

So that we can define $H_{+,\mathbb{C}}$ (resp. $H_{-,\mathbb{C}}$) as the space of complex sequences

$$\left\{ x \in \mathbb{C}^{\mathbb{N}} \left| \sum_{i\geq 0} q^{\frac{i}{2}} |x_i|^2 < \infty \right. \right\} < \infty,$$

equipped with the Hermitian scalar product $(x, y) \mapsto \langle \overline{x}, y\rangle_+$ (resp. $(x, y) \mapsto \langle \overline{x}, y\rangle_-$), where $\overline{x}$ is obtained by taking the complex conjugates of the coordinates of x.

CHAPTER 22

The even model operators

In this Chapter, we study operators on scalar sequences defined in analogy with the operators of Definition 20.16. We actually split the definition of the model operators according to the eigenvalue of the $\vee$ operator, which in the even case, can be q or -1. For these model operators, we build a complete spectral theory.

22.1. The ++ model operators

In this Section, following Definition 20.16, we consider the operators R_{++} and S_{++} defined on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$\begin{aligned} (R_{++}x)_0 &= qx_0 \\ (R_{++}x)_i &= x_{i-1} + (q-1)x_i \qquad && i \text{ even}, i \geq 2 \\ (R_{++}x)_i &= qx_{i+1} && i \text{ odd}. \end{aligned}$$

and

$$\begin{aligned} (S_{++}x)_i &= x_{i+1} && i \text{ even} \\ (S_{++}x)_i &= x_{i-1} && i \text{ odd}. \end{aligned}$$

These operators preserve the space of sequences with only finitely many non zero entries. They satisfy the relations $R_{++}^2 = q + (q-1)R_{++}$ and $S_{++}^2 = 1$. In other words, they define a representation of the algebra $\mathcal{A}$ of Section 21.1 in the space of sequences of real numbers. In particular, we set $P_{++} = \frac{1}{q+1}(R_{++}S_{++} + S_{++}R_{++} - (q-1)S_{++})$.

The scalar product of H_+ is adapted to the study of this situation.

LEMMA 22.1. *The operators R_{++} and S_{++} are bounded and self-adjoint in H_+.*

Thus, these operators define a self-adjoint representation of the algebra $\mathcal{A}$ of Section 21.1 in H_+.

PROOF. The fact that the operators are bounded is easy. From the formulas, it is clear that S_{++} is self-adjoint. Let us check that R_{++}

also is. Indeed, for x, y in H_+, we have

$$\langle x, y\rangle_+ = x_0y_0 + \sum_{i\geq 1} q^{i-1}(x_{2i-1}y_{2i-1} + qx_{2i}y_{2i}).$$

Therefore,

$$\begin{aligned}\langle R_{++}x, y\rangle_+ &= qx_0y_0 + \sum_{i\geq 1} q^{i-1}(qx_{2i}y_{2i-1} + q(x_{2i-1} + (q-1)x_{2i})y_{2i}) \\ &= qx_0y_0 + \sum_{i\geq 1} q^{i}(x_{2i}y_{2i-1} + x_{2i-1}y_{2i} + (q-1)x_{2i}y_{2i}).\end{aligned}$$

The conclusion follows as the latter formula is symmetric in x and y. □

Recall that $\mathbf{1}_0$ stands for the sequence (x_i) with $x_0 = 1$ and $x_i = 0$ for $i \geq 1$. In this Section, we prove the following result which defines a spectral analysis for the operators R_{++} and S_{++}.

PROPOSITION 22.2. *There exists a unique linear map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}_0}(t) = \begin{pmatrix}1\\0\end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{++}x}(t) = r_{++}(t)\widehat{x}(t) \text{ and } \widehat{S_{++}x}(t) = s_+\widehat{x}(t), \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{++} in H_+ is the interval $\mathcal{I}_q$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

$$\langle x, y\rangle_+ = \int_{\mathcal{I}_q} \widehat{x}(t)^* a_{++}(t)\widehat{y}(t)\mathrm{d}\mu_q(t). \tag{22.1}$$

We have used the notation introduced in Section 21.1 for the $++$ matrices. Recall that v^* means the transpose of a column vector v in $\mathbb{R}^2$ and that the measure μ_q is the absolutely continuous measure with density function $t \mapsto \frac{q+1}{2\pi}\frac{\sqrt{4q-(q+1)^2t^2}}{1-t^2}$ on $\mathcal{I}_q$ (see Section 21.3). Note that the statement implies that μ_q is a probability measure.

An other way of stating the integral formula would be to say that the transform $x \mapsto \widehat{x}$ induces an isometry from H_+ onto the space of classes of measurable functions $v : \mathcal{I}_q \to \mathbb{R}^2$ with $\int_{\mathcal{I}_q} v(t)^* a_{++}(t)v(t)\mathrm{d}\mu_q(t) < \infty$.

We begin the proof of Proposition 22.2 by showing that the images of $\mathbf{1}_0$ under R_{++} and S_{++} span the set of sequences with finitely many non zero entries.

LEMMA 22.3. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

PROOF. This is a standard triagularization argument. Indeed, the formulas defining the operators R_{++} and S_{++} imply by a straightforward induction that, for $i \geq 0$,

$$\begin{aligned} ((R_{++}S_{++})^i \mathbf{1}_0)_{2i} = 1 \text{ and } ((R_{++}S_{++})^i \mathbf{1}_0)_j = 0, \qquad & j > 2i; \\ (S_{++}(R_{++}S_{++})^i \mathbf{1}_0)_{2i+1} = 1 \text{ and } (S_{++}(R_{++}S_{++})^i \mathbf{1}_0)_j = 0, \quad & j > 2i+1. \end{aligned}$$

The result follows. □

Now, let us construct the joint spectral theory of R_{++} and S_{++}. For u in $\mathbb{C}^*$, we let $a_{++}(u)$ and $b_{++}(u)$ be the sequences of complex numbers defined by, for $i \geq 0$,

$$a_{++}(u)_{2i} = b_{++}(u)_{2i+1} = -qu^{-i} \text{ and } a_{++}(u)_{2i+1} = b_{++}(u)_{2i} = u^{1-i}.$$

These sequences are built in order to satisfy the following relations.

LEMMA 22.4. *For u in $\mathbb{C}^*$, we have $S_{++}a_{++}(u) = b_{++}(u)$ and, for any $i \geq 1$,*

$$\begin{aligned} (R_{++}a_{++}(u))_i &= \left(\frac{q}{u} + u\right) b_{++}(u)_i + q a_{++}(u)_i \\ \text{and } (R_{++}b_{++}(u))_i &= -b_{++}(u)_i. \end{aligned}$$

The above relations do not work for $i = 0$. To correct this, we introduce new sequences as follows. For t in $\mathbb{C}$, we choose as in Lemma 21.6, some u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$. We assume that we have $(q+1)^2t^2 \neq 4q$, so that $u^2 \neq q$, and we set

$$\begin{aligned} \alpha_{++}(t) &= \frac{1}{u^2 - q} a_{++}(u) + \frac{u^2}{q(q-u^2)} a_{++}\left(\frac{q}{u}\right) \\ \beta_{++}(t) &= S\alpha_{++}(t). \end{aligned}$$

The notation is justified by the fact that, since the right hand-side is invariant by the involution $u \mapsto \frac{q}{u}$, it only depends on t. By construction, one has

$$\alpha_{++}(t)_0 = 1 \text{ and } \beta_{++}(t)_0 = 0. \tag{22.2}$$

Besides, note that if t is real, both $\alpha_{++}(t)$ and $\beta_{++}(t)$ are real sequences.

We can now get the missing case in Lemma 22.4. Indeed, we have the following relation between these sequences and the operators introduced in Section 21.1.

LEMMA 22.5. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{++}\alpha_{++}(t) \\ S_{++}\beta_{++}(t) \end{pmatrix} = s_+ \begin{pmatrix} \alpha_{++}(t) \\ \beta_{++}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{++}\alpha_{++}(t) \\ R_{++}\beta_{++}(t) \end{pmatrix} = r_{++}(t) \begin{pmatrix} \alpha_{++}(t) \\ \beta_{++}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{++}(t)\rangle_+ \\ \langle x, \beta_{++}(t)\rangle_+ \end{pmatrix}.$$

Note again, that if t is real and x has real coefficients, the vector $\widehat{x}(t)$ has real coordinates. Besides, (22.2) reads as

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{22.3}$$

From Lemma 22.1 and Lemma 22.5, we directly get

LEMMA 22.6. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{++}x}(t) = r_{++}(t)\widehat{x}(t) \text{ and } \widehat{S_{++}x}(t) = s_+\widehat{x}(t).$$

Let us describe in more details the map $x \mapsto \widehat{x}$.

LEMMA 22.7. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism from the space $\mathbb{R}^{(\mathbb{N})}$ onto the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

PROOF. Indeed, (22.3) implies that the function $\widehat{\mathbf{1}_0}$ is polynomial, since it is constant. Besides, by Lemma 22.6, the space of x in $\mathbb{R}^{(\mathbb{N})}$ such that the function $\widehat{x}$ is polynomial is stable under the action of the algebra $\mathcal{A}$. Thus, by Lemma 22.3, this space is equal to $\mathbb{R}^{(\mathbb{N})}$.

Now, by Lemma 21.3, Equation (22.3) and Lemma 22.6, for $i \geq 0$, we have

$$\widehat{P^i_{++}\mathbf{1}_0}(t) = \begin{pmatrix} t^i \\ 0 \end{pmatrix} \text{ and } \widehat{S_{++}P^i_{++}}\mathbf{1}_0(t) = \begin{pmatrix} 0 \\ t^i \end{pmatrix}. \tag{22.4}$$

Thus, the map $x \mapsto \widehat{x}$ maps $\mathbb{R}^{(\mathbb{N})}$ onto the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.

It remains to prove that this map is injective. By Lemma 22.3, we have $\mathbb{R}^{(\mathbb{N})} = \mathcal{A}\mathbf{1}_0$. Besides, recall from Corollary 21.2 the decomposition $\mathcal{A} = \mathbb{R}[P] \oplus S\mathbb{R}[P] \oplus \mathcal{A}(R-q)$. As $R_{++}\mathbf{1}_0 = q\mathbf{1}_0$, we get

$$\mathbb{R}^{(\mathbb{N})} = (\mathbb{R}[P_{++}] \oplus S_{++}\mathbb{R}[P_{++}])\mathbf{1}_0.$$

Therefore, if x is in $\mathbb{R}^{(\mathbb{N})}$, we can write $x = f(P_{++})\mathbf{1}_0 + S_{++}g(P_{++})\mathbf{1}_0$ where f and g are polynomial functions on $\mathbb{R}$. By (22.4), for t in $\mathbb{R}$, we have

$$\widehat{x}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

and injectivity follows. □

The sequences $\alpha_{++}(t)$ and $\beta_{++}(t)$ are uniquely defined by the relations in Lemma 22.5.

COROLLARY 22.8. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{++}\gamma \\ S_{++}\delta \end{pmatrix} = s_+ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{++}\gamma \\ R_{++}\delta \end{pmatrix} = r_{++}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0\alpha_{++}(t_0)$ and $\delta = \gamma_0\beta_{++}(t_0)$.

PROOF. Define a linear map $\varphi : \mathbb{R}^2[t] \to \mathbb{R}^2$ as follows. For $\begin{pmatrix} f \\ g \end{pmatrix}$ in $\mathbb{R}^2[t]$, by Lemma 22.7, there exists a unique x in $\mathbb{R}^{(\mathbb{N})}$ with $\widehat{x} = \begin{pmatrix} f \\ g \end{pmatrix}$. Then, we set $\varphi \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \langle \gamma, x \rangle_+ \\ \langle \delta, x \rangle_+ \end{pmatrix}$. From Lemma 22.6 and the assumption, we have

$$\varphi\left(r_{++}(t) \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}\right) = \begin{pmatrix} \langle \gamma, R_{++}x \rangle_+ \\ \langle \delta, R_{++}x \rangle_+ \end{pmatrix} = \begin{pmatrix} \langle R_{++}\gamma, x \rangle_+ \\ \langle R_{++}\delta, x \rangle_+ \end{pmatrix} = r_{++}(t_0)\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \tag{22.5}$$

and in the same way,

$$\varphi\left(s_+ \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}\right) = \begin{pmatrix} \langle \gamma, S_{++}x \rangle_+ \\ \langle \delta, S_{++}x \rangle_+ \end{pmatrix} = s_+\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}. \tag{22.6}$$

By Lemma 21.3, we get

$$\varphi \begin{pmatrix} tf(t) \\ tg(t) \end{pmatrix} = \begin{pmatrix} \langle \gamma, P_{++}x \rangle_+ \\ \langle \delta, P_{++}x \rangle_+ \end{pmatrix} = t_0\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

and hence, if $f(t_0) = g(t_0) = 0$, $\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = 0$. Therefore, there exists a linear map $m : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for any f, g in $\mathbb{R}[t]$, $\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = m \begin{pmatrix} f(t_0) \\ g(t_0) \end{pmatrix}$. Now, (22.5) and (22.6) imply that m commutes with the matrices $r_{++}(t_0)$ and s_+. Therefore, by Lemma 21.3, m is a scalar

matrix, that is, there exists a real number λ with, for any f, g in $\mathbb{R}[t]$, $\varphi\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \lambda \begin{pmatrix} f(t_0) \\ g(t_0) \end{pmatrix}$. By Lemma 22.7, we get, for any x in $\mathbb{R}^{(\mathbb{N})}$,

$$\langle \gamma, x\rangle_+ = \lambda \langle \alpha_{++}(t), x\rangle_+ \text{ and } \langle \delta, x\rangle_+ = \lambda \langle \beta_{++}(t), x\rangle_+.$$

The conclusion follows. □

So far, we have constructed the spectral transform $x \mapsto \widehat{x}$ in Proposition 22.2. It remains to establish the Plancherel formula (22.1). To this aim, we will use Corollary 21.5. Indeed, Lemma 22.7 tells us that we can identify $\mathbb{R}^{(\mathbb{N})}$ with the space $\mathbb{R}^2[t]$, whereas Lemma 22.6 tells us that the operators R_{++} and S_{++} then act as the matrices $r_{++}(t)$ and s_+ of Section 21.1. Therefore, to get (22.1), it suffices to compute the measure μ provided by Corollary 21.5. This computation will use the notation of Section 21.3 and the standard method following from Lemma 21.7, which relies on the computation of the resolvent function $\langle \mathbf{1}_0, (P_{++} - t)^{-1}\mathbf{1}_0\rangle_+$ for t in $\mathbb{H}$. To achieve this computation, we introduce a last family of sequences in $\mathbb{C}^{(\mathbb{N})}$. For u in $\mathbb{C}$, $u \notin \{-1, 0, 1\}$, we set

$$c_{++}(u)_{2i} = \frac{(q+1)u^{1-i}}{1-u^2} \text{ and } c_{++}(u)_{2i+1} = \frac{(q+1)u^{-i}}{1-u^2} \quad i \geq 0.$$

Note that $c_{++}(u)$ belongs to $H_{+,\mathbb{C}}$ if and only if $|u| > \sqrt{q}$. A direct computation gives

LEMMA 22.9. *Let u be in $\mathbb{C}$ with $u \notin \{-1, 0, 1\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then we have*

$$P_{++}c_{++}(u) = tc_{++}(u) + \mathbf{1}_0.$$

We gather the arguments of this Section.

PROOF OF PROPOSITION 22.2. The uniqueness of the transform $x \mapsto \widehat{x}$ follows from Lemma 22.3. The fact that it satisfies the required properties was obtained in (22.3) and Lemma 22.6. We know that it induces a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$ thanks to Lemma 22.7.

It remains to show the Plancherel formula (22.1). Let μ be the measure from Corollary 21.5 applied to the scalar product on $\mathbb{R}^2[t]$ which is obtained by pulling back the scalar product $\langle ., .\rangle_+$ under the inverse of the transform $x \mapsto \widehat{x}$. Let t be in $\mathbb{H}$. By Lemma 21.3 and Lemma 22.6, as for s in $\mathbb{R}$, the matrix $a_{++}(s)$ has upper left coefficient 1, we have,

$$\langle \mathbf{1}_0, (P_{++} - t)^{-1}\mathbf{1}_0\rangle_+ = \int_{[-1,1]} \frac{1}{s-t} \mathrm{d}\mu(s).$$

By Lemma 22.9, this gives, for u in $\mathbb{H}_q$ with $u^2-(q+1)tu+q=0$,

$$\int_{[-1,1]} \frac{1}{s-t}\mathrm{d}\mu(s) = \langle \mathbf{1}_0, c_{++}(u)\rangle_+ = (q+1)\frac{u}{1-u^2},$$

hence, by Lemma 21.7,

$$\mathcal{P}\mu(t) = \frac{q+1}{\pi}\Im\left(\frac{u}{1-u^2}\right).$$

The conclusion now follows from Lemma 21.8. □

22.2. Spectral measures in $\ell^2(X)$

To motivate the reader, and in the hope that this will make our strategy more understandable, we explain how Proposition 22.2 can be used in order to recover the computation of spectral measures in $\ell^2(X)$ for the natural Markov operator. This result is due to Kesten [**32**]. It plays a key role in the spectral theory developed in [**21**].

We temporarily come back to the language of trees. We equip the space $\ell^2(X)$ with the bounded self-adjoint operator Q defined by, for f in $\ell^2(X)$,

$$Qf(x) = \frac{1}{q+1}\sum_{y\sim x} f(y), \quad x\in X.$$

Corollary 22.10 (Kesten [**32**]). *For any a in X, the spectral measure of $\mathbf{1}_a$ with respect to Q is the measure μ_q.*

Proof. In this proof, we use the letters R and S for the operators on functions on X_1 defined by, for any $g: X_1\to\mathbb{R}$,

$$Rg(x,y) = \sum_{\substack{z\sim x\\ z\neq y}} g(x,z) \text{ and } Sg(x,y) = g(y,x), \quad x\sim y\in X.$$

As usual, we set $P=\frac{1}{q+1}(RS+SR-(q-1)S)$. For any function f on X define the function Lf on X_1 by

$$Lf(x,y) = f(x), \quad x\sim y\in X.$$

When f is in $\ell^2(X)$, we get

$$\|Lf\|_2^2 = \sum_{(x,y)\in X_1} Lf(x,y)^2 = (q+1)\sum_{x\in X} f(x)^2 = (q+1)\|f\|_2^2.$$

Besides, for (x, y) in X_1, we have

$$\begin{aligned}(q+1)PLf(x,y) &= (RS+SR-(q-1)S)Lf(x,y)\\ &= \sum_{\substack{z\sim x\\ z\neq y}} Lf(z,x) + \sum_{\substack{z\sim y\\ z\neq x}} Lf(y,z) - (q-1)Lf(y,x)\\ &= \sum_{\substack{z\sim x\\ z\neq y}} f(z) + qf(y) - (q-1)f(y) = \sum_{z\sim x} f(z).\end{aligned}$$

We get $PL = LQ$. Thus, determining spectral measures of the elements of $\ell^2(X)$ with respect to Q is equivalent to determine the spectral measures of their images by L with respect to P.

Now, we fix a in X and we define an orthogonal sequence $(g_i)_{i\geq 0}$ of elements of $\ell^2(X_1)$ as follows. For any $i \geq 0$ and (x, y) in X_1, we set

$$g_{2i}(x,y) = \mathbf{1}_{\substack{d(x,a)=i\\ d(y,a)=i+1}} \text{ and } g_{2i+1}(x,y) = \mathbf{1}_{\substack{d(x,a)=i+1\\ d(y,a)=i}}.$$

We have $\|g_{2i}\|_2^2 = \|g_{2i+1}\|_2^2 = (q+1)q^i$. Besides, the definition immediately gives $Sg_{2i} = g_{2i+1}$, whereas a direct computation yields

$$\begin{aligned} Rg_0 &= qg_0 & \\ Rg_i &= qg_{i-1} + (q-1)g_i & i \text{ even}, i \geq 2\\ Rg_i &= g_{i+1} & i \text{ odd}.\end{aligned} \tag{22.7}$$

For a sequence $x = (x_i)_{i\geq 0}$ of real numbers, we set Cx to be the function $\sum_{i\geq 0} x_i g_i$ on X_1. With the notation of Section 22.1, the relations (22.7) give $RC = CR_{++}$ and $SC = CS_{++}$. Moreover, the computation of the norm of the g_i, $i \geq 0$, implies that, for x in H_+, we have $\|Cx\|_2^2 = (q+1)\|x\|_+^2$. As $C\mathbf{1}_0 = g_0 = L\mathbf{1}_a$, the conclusion follows from Proposition 22.2. □

22.3. The $+-$ model operators

We come back to the study of the model operators. We now consider the case of the other eigenvalue of the $\vee$ operator in Definition 20.16, so that we define the operators R_{+-} and S_{+-} acting on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$\begin{aligned}(R_{+-}x)_0 &= -x_0 & \\ (R_{+-}x)_i &= x_{i-1} + (q-1)x_i & i \text{ even}, i \geq 2\\ (R_{+-}x)_i &= qx_{i+1} & i \text{ odd}.\end{aligned}$$

and

$$(S_{+-}x)_i = x_{i+1} \qquad i \text{ even}$$
$$(S_{+-}x)_i = x_{i-1} \qquad i \text{ odd.}$$

Again, these operators preserve the space of sequences with only finitely many non zero entries. They satisfy the relations $R_{+-}^2 = q+(q-1)R_{++}$ and $S_{+-}^2 = 1$. Thus we still have a representation of the algebra $\mathcal{A}$ of Section 21.1 in the space of sequences of real numbers. We now set $P_{+-} = \frac{1}{q+1}(R_{+-}S_{+-} + S_{+-}R_{+-} - (q-1)S_{+-})$.

The same proof as above gives

LEMMA 22.11. *The operators R_{+-} and S_{+-} are bounded and self-adjoint in H_+.*

The spectral analysis for this new self adjoint representation of the algebra $\mathcal{A}$ is now defined as follows.

PROPOSITION 22.12. *There exists a unique linear map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{+-}x}(t) = r_{+-}(t)\widehat{x}(t) \text{ and } \widehat{S_{+-}x}(t) = s_+\widehat{x}(t), \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{+-} in H_+ is the set $\mathcal{I}_q \cup \{-1, 1\}$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

$$\langle x, y\rangle_+ = \frac{1}{q}\int_{\mathcal{I}_q} \widehat{x}(t)^* a_{+-}(t)\widehat{y}(t)\mathrm{d}\mu_q(t) + \frac{q-1}{2q}(\widehat{x}(-1)^* a_{+-}(-1)\widehat{y}(-1) + \widehat{x}^*(1)a_{+-}(1)\widehat{y}(1)). \tag{22.8}$$

The existence of a discrete component in the spectral measure corresponds to the existence of joint eigenvectors for the operators R_{+-} and S_{+-} in H_+.

We start proving Proposition 22.12. As for the ++-model, we can show

LEMMA 22.13. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

We now adapt the construction of the spectral transform by setting, for u in $\mathbb{C}^*$ and $i \geq 0$,

$$a_{+-}(u)_{2i} = b_{+-}(u)_{2i+1} = u^{-i} \text{ and } a_{+-}(u)_{2i+1} = b_{+-}(u)_{2i} = u^{1-i}.$$

A direct computation gives

LEMMA 22.14. *For u in $\mathbb{C}^*$, we have $S_{+-}a_{+-}(u) = b_{+-}(u)$ and, for any $i \geq 1$,*

$$(R_{+-}a_{+-}(u))_i = \left(\frac{q}{u} + u\right) b_{+-}(u)_i - a_{+-}(u)_i$$
$$\text{and } (R_{+-}b_{+-}(u))_i = qb_{+-}(u)_i.$$

Again, in order to have the relations above working for every $i \geq 0$, we introduce new sequences. For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we chose u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$ and we set

$$\alpha_{+-}(t) = \frac{q}{q-u^2}a_{+-}(u) + \frac{u^2}{u^2-q}a_{+-}\left(\frac{q}{u}\right)$$
$$\beta_{+-}(t) = S\alpha_{+-}(t).$$

As before, these are functions of t, since they are invariant under the involution $u \mapsto \frac{q}{u}$. We still have

$$\alpha_{+-}(t)_0 = 1 \text{ and } \beta(t)_{+-}0 = 0. \tag{22.9}$$

We now get

LEMMA 22.15. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{+-}\alpha_{+-}(t) \\ S_{+-}\beta_{+-}(t) \end{pmatrix} = s_+ \begin{pmatrix} \alpha_{+-}(t) \\ \beta_{+-}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{+-}\alpha_{+-}(t) \\ R_{+-}\beta_{+-}(t) \end{pmatrix} = r_{+-}(t) \begin{pmatrix} \alpha_{+-}(t) \\ \beta_{+-}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{+-}(t)\rangle_+ \\ \langle x, \beta_{+-}(t)\rangle_+ \end{pmatrix}.$$

Note again, that if t is real and x has real coefficients, the vector $\widehat{x}(t)$ has real coordinates and that (22.9) gives

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{22.10}$$

From Lemma 22.11 and Lemma 22.15, we directly get

LEMMA 22.16. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{+-}x}(t) = r_{+-}(t)\widehat{x}(t) \text{ and } \widehat{S_{+-}x}(t) = s_+\widehat{x}(t).$$

As in the $++$ case, we get

LEMMA 22.17. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism between the space $\mathbb{R}^{(\mathbb{N})}$ and the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

The proof is analogous to the one of Lemma 22.7. Note that it uses the relation $\mathcal{A} = \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(R+1)$ of Corollary 21.2. In the same way, we also get

COROLLARY 22.18. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{+-}\gamma \\ S_{+-}\delta \end{pmatrix} = s_+ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{+-}\gamma \\ R_{+-}\delta \end{pmatrix} = r_{+-}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0 \alpha_{+-}(t_0)$ and $\delta = \gamma_0 \beta_{+-}(t_0)$.

We now start establishing the Plancherel formula (22.8). To this aim, we will compute the measure μ associated to the scalar product $\langle ., . \rangle_+$ on $\mathbb{R}^{(\mathbb{N})}$ through Corollary 21.5 and the new spectral transform. The novelty will be the presence of an atomic part. For u in $\mathbb{C}$, $u \notin \{-q, 0, q\}$, we set

$$c_{+-}(u)_{2i} = \frac{(q+1)u^{1-i}}{q^2 - u^2} \text{ and } c_{+-}(u)_{2i+1} = \frac{q(q+1)u^{-i}}{u^2 - q^2} \quad i \geq 0.$$

A direct computation gives

LEMMA 22.19. *Let u be in $\mathbb{C}$ with $u \notin \{-q, 0, q\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, we have*

$$P_{+-} c_{+-}(u) = t c_{+-}(u) + \mathbf{1}_0.$$

We may now conclude.

PROOF OF PROPOSITION 22.12. The existence and uniqueness of the spectral transform $x \mapsto \widehat{x}$ and its algebraic properties follow from Lemma 22.13, (22.10), Lemma 22.16 and Lemma 22.17.

It remains to show the Plancherel formula (22.8). Let μ be the measure from Corollary 21.5 applied to the scalar product $\langle ., . \rangle_+$. By Lemma 21.3 and Lemma 22.16, we have, for t in $\mathbb{H}$,

$$\langle \mathbf{1}_0, (P_{+-} - t)^{-1} \mathbf{1}_0 \rangle_+ = \int_{[-1,1]} \frac{1}{s - t} \mathrm{d}\mu(s).$$

By Lemma 22.19, this gives, for u in $\mathbb{H}_q$ with $u^2 - (q+1)tu + q = 0$,

$$\int_{[-1,1]} \frac{1}{s - t} \mathrm{d}\mu(s) = \langle \mathbf{1}_0, c_{+-}(u) \rangle_+ = (q+1) \frac{u}{q^2 - u^2}.$$

Here comes the main difference with the ++ case: the holomorphic function $u \mapsto (q+1)\frac{u}{q^2-u^2}$ on $\mathbb{H}_q$ has no continuous extension to $\overline{\mathbb{H}}_q$.

To correct this, we will remove the singularities as $u = -q$ and $u = q$. Indeed, a direct computation shows that

$$(q+1)\frac{u}{q^2-u^2} = \frac{q+1}{q}\frac{u}{1-u^2} + \frac{q-1}{2q}\left(\frac{1}{1-t} - \frac{1}{1+t}\right),$$

hence, by Lemma 21.7 and Lemma 21.8,

$$\begin{aligned}\mathcal{P}\mu(t) &= \frac{q+1}{q\pi}\Im\left(\frac{u}{1-u^2}\right) + \frac{q-1}{2q\pi}\Im\left(\frac{1}{1-t} - \frac{1}{1+t}\right)\\ &= \frac{1}{q}\mathcal{P}\mu_q(t) + \frac{q-1}{2q}(\mathcal{P}\delta_1(t) + \mathcal{P}\delta_{-1}(t)),\end{aligned}$$

where δ stands for Dirac measures. The conclusion follows. □

22.4. The + twist

We complete the description of the even model operators by describing the additional structure related to the twist operator of bipartite graphs introduced in Section 20.6.

Let $(x_i)_{i\geq 0}$ be a sequence of real numbers. We set

$$\begin{aligned}(U_+x)_i &= (-1)^{\frac{i}{2}}x_i && i \text{ even}\\ (U_+x)_i &= (-1)^{\frac{i+1}{2}}x_i && i \text{ odd}.\end{aligned}$$

This defines a unitary operator of H_+. Besides, a direct computation yields

$$\begin{aligned}R_{++}U_+ &= U_+R_{++} & R_{+-}U_+ &= U_+R_{+-}\\ S_{++}U_+ &= -U_+S_{++} & S_{+-}U_+ &= -U_+S_{+-}.\end{aligned}$$

By using the definitions, we get

Lemma 22.20. *Let t be in $\mathbb{C}$. We have*

$$\begin{aligned}U_+\alpha_{++}(t) &= \alpha_{++}(-t) & U_+\alpha_{+-}(t) &= \alpha_{+-}(-t)\\ U_+\beta_{++}(t) &= -\beta_{++}(-t) & U_+\beta_{+-}(t) &= -\beta_{+-}(-t).\end{aligned}$$

CHAPTER 23

The odd model operators

We now study operators on sequences defined in analogy with the operators from Definition 20.17. This study will be lead as in Sections 22.1 and 22.3. Again, we split the definition of the model operators according to the eigenvalue of the $\vee$ operator, which in the odd case, can be 1 or -1.

23.1. The $-+$ model operators

We mimic Definition 20.17, and we consider the operators R_{-+} and S_{-+} defined on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$\begin{aligned} (R_{-+}x)_i &= qx_{i+1} && i \text{ even} \\ (R_{-+}x)_i &= x_{i-1} + (q-1)x_i && i \text{ odd.} \end{aligned}$$

and

$$\begin{aligned} (S_{-+}x)_0 &= x_0 \\ (S_{-+}x)_i &= x_{i-1} && i \text{ even}, i \geq 2 \\ (S_{-+}x)_i &= x_{i+1} && i \text{ odd.} \end{aligned}$$

We still have the relations $R_{-+}^2 = q + (q-1)R_{-+}$ and $S_{-+}^2 = 1$. Thus, we get a representation of the algebra $\mathcal{A}$ of Section 21.1 in the space of sequences of real numbers. We set $P_{-+} = \frac{1}{q+1}(R_{-+}S_{-+} + S_{-+}R_{-+} - (q-1)S_{-+})$.

Now, we change the Hilbert space and use H_- instead.

LEMMA 23.1. *The operators R_{-+} and S_{-+} are bounded and self-adjoint in H_-.*

We hence have a self-adjoint representation of the algebra $\mathcal{A}$ in H_-.

PROOF. One easily checks that the operators are bounded. Recall that, for any x, y in H_-, we have

$$\langle x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i-1}y_{2i-1} + x_{2i}y_{2i}) = \sum_{i\geq 0} q^i(x_{2i}y_{2i} + qx_{2i+1}y_{2i+1}).$$

We get, from the first formula,

$$\langle S_{-+}x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i}y_{2i-1} + x_{2i-1}y_{2i})$$

and, from the second formula,

$$\begin{aligned}\langle R_{-+}x, y\rangle_- &= \sum_{i\geq 0} q^i(qx_{2i+1}y_{2i} + q(x_{2i} + (q-1)x_{2i+1})y_{2i+1}) \\ &= \sum_{i\geq 0} q^{i+1}(x_{2i+1}y_{2i} + x_{2i}y_{2i+1} + (q-1)x_{2i+1}y_{2i+1}).\end{aligned}$$

The conclusion follows. □

As above, we define a spectral analysis for the operators R_{-+} and S_{-+}.

PROPOSITION 23.2. *There exists a unique map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}_0}(t) = \begin{pmatrix}1\\0\end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{-+}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{-+}x}(t) = s_{-+}(t)\widehat{x}(t), t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{-+} in H_- is the set $\mathcal{I}_q \cup \{-1\}$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

(23.1)

$$\langle x, y\rangle_- = \frac{2}{q+1}\int_{\mathcal{I}_q} \widehat{x}(t)^* a_{-+}(t)\widehat{y}(t)\mathrm{d}\mu_q(t) + \frac{q-1}{q+1}\widehat{x}(-1)^* a_{-+}(-1)\widehat{y}(-1).$$

We start the construction of these objects.

LEMMA 23.3. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

PROOF. This now follows from the following property which is obtained by a straightforward induction: for $i \geq 0$,

$$\begin{aligned}((S_{-+}R_{-+})^i\mathbf{1}_0)_{2i} &= 1 \\ \text{and } ((S_{-+}R_{-+})^i\mathbf{1}_0)_j &= 0, \qquad j > 2i; \\ (R_{-+}(S_{-+}R_{-+})^i\mathbf{1}_0)_{2i+1} &= 1 \\ \text{and } (R_{-+}(S_{-+}R_{-+})^i\mathbf{1}_0)_j &= 0, \qquad j > 2i+1.\end{aligned}$$

□

We define sequences in analogy with the previous cases. For u in $\mathbb{C}^*$, we let $a_{-+}(u)$ and $b_{-+}(u)$ be the sequences of complex numbers defined by, for $i \geq 0$,

$$a_{-+}(u)_{2i} = -qu^{-i} \qquad a_{-+}(u)_{2i+1} = (u - q + 1)u^{-i}$$
$$b_{-+}(u)_{2i} = u^{1-i} \qquad b_{-+}(u)_{2i+1} = -u^{-i}.$$

By construction, one has

LEMMA 23.4. *For u in $\mathbb{C}^*$, we have*

$$R_{-+}b_{-+}(u) = a_{-+}(u) \text{ and } R_{-+}a_{-+}(u) = qb_{-+}(u) + (q-1)a_{-+}(u).$$

and, for any $i \geq 1$,

$$(S_{-+}b_{-+}(u))_i = -b_{-+}(u)_i$$
$$\text{and } (S_{-+}a_{-+}(u))_i = a_{-+}(u)_i + \left(u + \frac{q}{u} - q + 1\right) b_{-+}(u)_i.$$

As in the even cases, to get the formulas above working for every $i \geq 0$, we introduce new sequences, defined as follows. For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we take u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$ and we set

$$\alpha_{-+}(t) = \frac{1}{u^2 - q}a_{-+}(u) + \frac{u^2}{q(q-u^2)}a_{-+}\left(\frac{q}{u}\right)$$
$$\beta_{-+}(t) = \frac{1}{u^2 - q}b_{-+}(u) + \frac{u^2}{q(q-u^2)}b_{-+}\left(\frac{q}{u}\right).$$

As usual, one has

$$\alpha_{-+}(t)_0 = 1 \text{ and } \beta_{-+}(t)_0 = 0. \tag{23.2}$$

Besides, note that if t is real, both $\alpha_{-+}(t)$ and $\beta_{-+}(t)$ are real sequences.

We can now get the missing case in Lemma 23.4. Indeed, we have the following relation between these sequences and the operators introduced in Section 21.1.

LEMMA 23.5. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{-+}\alpha_{-+}(t) \\ S_{-+}\beta_{-+}(t) \end{pmatrix} = s_{-+}(t) \begin{pmatrix} \alpha_{-+}(t) \\ \beta_{-+}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{-+}\alpha_{-+}(t) \\ R_{-+}\beta_{-+}(t) \end{pmatrix} = r_- \begin{pmatrix} \alpha_{-+}(t) \\ \beta_{-+}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{-+}(t)\rangle_- \\ \langle x, \beta_{-+}(t)\rangle_- \end{pmatrix}.$$

By (23.2) we get

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{23.3}$$

From Lemma 23.1 and Lemma 23.5, we directly get

LEMMA 23.6. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{-+}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{-+}x}(t) = s_{-+}(t)\widehat{x}(t).$$

We have again defined an isomorphism onto polynomial functions with values in $\mathbb{R}^2$.

LEMMA 23.7. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism between the space $\mathbb{R}^{(\mathbb{N})}$ and the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

PROOF. This is proved as Lemma 22.7. Indeed, by Lemma 23.6, for any $i \geq 0$, we have

$$\widehat{P_{-+}^i\mathbf{1}_0}(t) = \begin{pmatrix} t^i \\ 0 \end{pmatrix} \text{ and } \widehat{R_{-+}P_{-+}^i}\mathbf{1}_0(t) = \begin{pmatrix} (q-1)t^i \\ t^i \end{pmatrix},$$

which shows that the spectral transform maps $\mathbb{R}^{(\mathbb{N})}$ into polynomial functions and that it is surjective. To prove injectivity, one uses the decomposition $\mathcal{A} = \mathbb{R}[P] \oplus R\mathbb{R}[P] \oplus \mathcal{A}(S-1)$ from Corollary 21.2. □

We still get a uniqueness result:

COROLLARY 23.8. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{-+}\gamma \\ S_{-+}\delta \end{pmatrix} = s_{-+}(t_0)\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{-+}\gamma \\ R_{-+}\delta \end{pmatrix} = r_-\begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0\alpha_{-+}(t_0)$ and $\delta = \gamma_0\beta_{-+}(t_0)$.

We now focus on the Plancherel formula (23.1). As above, to compute the resolvent function $t \mapsto \langle \mathbf{1}_0, (P_{-+}-t)^{-1}\mathbf{1}_0\rangle$, we introduce a last family of sequences in $\mathbb{C}^{(\mathbb{N})}$. For u in $\mathbb{C}$, $u \notin \{-q, 0, 1\}$, we set

$$c_{-+}(u)_{2i} = \frac{(q+1)u^{1-i}}{(q+u)(1-u)} \text{ and } c_{-+}(u)_{2i+1} = \frac{(q+1)u^{-i}}{(q+u)(1-u)} \quad i \geq 0.$$

We still have

LEMMA 23.9. *Let u be in $\mathbb{C}$ with $u \notin \{-q, 0, 1\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, we have*

$$P_{-+}c_{-+}(u) = tc_{-+}(u) + \mathbf{1}_0.$$

We are now ready to give the

Proof of Proposition 23.2. The existence and uniqueness of the spectral transform $x \mapsto \widehat{x}$ and its algebraic properties follow from Lemma 23.3, (23.3), Lemma 23.6 and Lemma 23.7.

To show the Plancherel formula (23.1), we let μ be the measure from Corollary 21.5 applied to the scalar product $\langle ., . \rangle_-$. By Lemma 21.3, Lemma 23.6 and Lemma 23.9, we have, for t in $\mathbb{H}$ and u in $\mathbb{H}_q$ with $u^2 - (q+1)tu + q = 0$,

$$\int_{[-1,1]} \frac{1}{s-t} \mathrm{d}\mu(s) = \frac{(q+1)u}{(q+u)(1-u)}.$$

We remove the singularity of this function at $u = -q$. A direct computation shows that

$$\frac{(q+1)u}{(q+u)(1-u)} = \frac{2u}{1-u^2} - \frac{q-1}{q+1}\frac{1}{1+t}.$$

Hence, by Lemma 21.7 and Lemma 21.8,

$$\mathcal{P}\mu(t) = \frac{2}{q+1}\mathcal{P}\mu_q(t) + \frac{q-1}{q+1}\mathcal{P}\delta_{-1}(t).$$

The conclusion follows. □

23.2. The −− model operators

We describe the final set of model operators. Following Definition 20.17, we define operators R_{--} and S_{--} on sequences $x = (x_i)_{i \geq 0}$ of real numbers by

$$\begin{aligned} (R_{--}x)_i &= q x_{i+1} && i \text{ even} \\ (R_{--}x)_i &= x_{i-1} + (q-1)x_i && i \text{ odd.} \end{aligned}$$

and

$$\begin{aligned} (S_{--}x)_0 &= -x_0 \\ (S_{--}x)_i &= x_{i-1} && i \text{ even}, i \geq 2 \\ (S_{--}x)_i &= x_{i+1} && i \text{ odd.} \end{aligned}$$

They still define a representation of the algebra $\mathcal{A}$ of Section 21.1 in the space of sequences of real numbers and we set $P_{--} = \frac{1}{q+1}(R_{--}S_{--} + S_{--}R_{--} - (q-1)S_{--})$. As for Lemma 23.1, we show

Lemma 23.10. *The operators R_{--} and S_{--} are bounded and self-adjoint in H_-.*

We hence have a self-adjoint representation of the algebra $\mathcal{A}$ in H_-. The spectral analysis of the operators R_{--} and S_{--} states as

PROPOSITION 23.11. *There exists a unique map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{--}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{--}x}(t) = s_{--}(t)\widehat{x}(t), \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{--} in H_- is the set $\mathcal{I}_q \cup \{1\}$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

(23.4)

$$\langle x, y\rangle_- = \frac{2}{q+1}\int_{\mathcal{I}_q} \widehat{x}(t)^* a_{--}(t)\widehat{y}(t)\mathrm{d}\mu_q(t) + \frac{q-1}{q+1}\widehat{x}(1)^* a_{--}(1)\widehat{y}(1).$$

As for Lemma 23.3, we show

LEMMA 23.12. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

We now set, for u in $\mathbb{C}^*$ and $i \geq 0$,

$$a_{--}(u)_{2i} = qu^{-i} \qquad a_{--}(u)_{2i+1} = (u+q-1)u^{-i}$$
$$b_{--}(u)_{2i} = u^{1-i} \qquad b_{--}(u)_{2i+1} = u^{-i}.$$

By construction, one has

LEMMA 23.13. *For u in $\mathbb{C}^*$, we have*

$$R_{--}b_{--}(u) = a_{--}(u) \text{ and } R_{--}a_{--}(u) = qb_{--}(u) + (q-1)a_{--}(u).$$

and, for any $i \geq 1$,

$$(S_{--}b_{--}(u))_i = b_{--}(u)_i$$
$$\text{and } (S_{--}a_{--}(u))_i = -a_{--}(u)_i + \left(u + \frac{q}{u} + q - 1\right) b_{--}(u)_i.$$

To get the relations above working for every $i \geq 0$, for t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we take u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$ and we set

$$\alpha_{--}(t) = \frac{1}{q-u^2}a_{--}(u) + \frac{u^2}{q(u^2-q)}a_{--}\left(\frac{q}{u}\right)$$
$$\beta_{--}(t) = \frac{1}{q-u^2}b_{--}(u) + \frac{u^2}{q(u^2-q)}b_{--}\left(\frac{q}{u}\right).$$

One still has

$$\alpha_{--}(t)_0 = 1 \text{ and } \beta_{--}(t)_0 = 0. \tag{23.5}$$

We get the missing case in Lemma 23.13.

LEMMA 23.14. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{--}\alpha_{--}(t) \\ S_{--}\beta_{--}(t) \end{pmatrix} = s_{--}(t) \begin{pmatrix} \alpha_{--}(t) \\ \beta_{--}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{--}\alpha_{-+}(t) \\ R_{--}\beta_{--}(t) \end{pmatrix} = r_- \begin{pmatrix} \alpha_{--}(t) \\ \beta_{--}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{--}(t)\rangle_- \\ \langle x, \beta_{--}(t)\rangle_- \end{pmatrix}.$$

By (23.5) we have

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{23.6}$$

By Lemma 23.10 and Lemma 23.14, we get

LEMMA 23.15. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{--}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{--}x}(t) = s_{--}(t)\widehat{x}(t).$$

We still have

LEMMA 23.16. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism between the space $\mathbb{R}^{(\mathbb{N})}$ and the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

COROLLARY 23.17. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{--}\gamma \\ S_{--}\delta \end{pmatrix} = s_{--}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{--}\gamma \\ R_{--}\delta \end{pmatrix} = r_- \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0\alpha_{--}(t_0)$ and $\delta = \delta_0\beta_{--}(t_0)$.

We now prove the Plancherel formula (23.4). To compute the resolvent function $t \mapsto \langle \mathbf{1}_0, (P_{--} - t)^{-1}\mathbf{1}_0\rangle$, we set, for u in $\mathbb{C}$, $u \notin \{-1, 0, q\}$,

$$c_{--}(u)_{2i} = \frac{(q+1)u^{1-i}}{(q-u)(1+u)} \text{ and } c_{--}(u)_{2i+1} = \frac{(q+1)u^{-i}}{(u-q)(1+u)} \quad i \geq 0.$$

LEMMA 23.18. *Let u be in $\mathbb{C}$ with $u \notin \{-1, 0, q\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then we have*

$$P_{--}c_{--}(u) = tc_{--}(u) + \mathbf{1}_0.$$

We conclude by the

PROOF OF PROPOSITION 23.11. This is proved as the previous analogous results by using the relation, for $u \in \mathbb{C}$, $u \notin \{-1, 0, 1, q\}$ and $t = \frac{1}{q+1}(u + \frac{q}{u})$,

$$\frac{(q+1)u}{(q-u)(1+u)} = \frac{2u}{1-u^2} + \frac{q-1}{q+1}\frac{1}{1-t}.$$

□

23.3. The − twist

As in Section 22.4, we describe for the odd model operators the additional structure related to the twist operator of bipartite graphs from Section 20.6.

Let $(x_i)_{i\geq 0}$ be a sequence of real numbers. We set

$$(U_- x)_i = (-1)^{\frac{i}{2}} x_i \qquad i \text{ even}$$
$$(U_- x)_i = (-1)^{\frac{i-1}{2}} x_i \qquad i \text{ odd}.$$

This defines a unitary operator of H_-. We now get

$$R_{-+}U_- = U_- R_{-+} \quad R_{--}U_- = U_- R_{--} \quad S_{-+}U_- = -U_- S_{--}.$$

By a direct computation, we have

LEMMA 23.19. *Let t be in $\mathbb{C}$. We have*

$$U_- \alpha_{-+}(t) = -\alpha_{--}(-t) \text{ and } U_- \beta_{-+}(t) = -\beta_{--}(-t).$$

CHAPTER 24

Spectral transforms and the default map

We go back to the framework of Section 20.3 and Section 20.5. There, we defined an action of the algebra $\mathcal{A}$ on the space of ∞-pseudofunctions as well as two actions of the algebra $\mathcal{A}$ on spaces of sequences of pseudofunctions.

We will now transport the results on the model operators to these two actions and see how they are compatible with the default map.

24.1. Spectral transforms

We set the definition of the model operators in order to mimick the one of the actions of $\mathcal{A}$ on spaces of sequences given in Definitions 20.16, 20.17, 20.19 and 20.20. Therefore, we can transport back the definition of the spectral transforms associated to the different model operators to get spectral transforms of sequences of pseudofunctions. We begin with the actions defined in Definitions 20.16 and 20.17.

Let $k \geq 0$, t be in $\mathbb{R}$ and $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ be in $\mathcal{H}_k^2$. If k is even, we set

$$\mathfrak{S}_t H = \begin{pmatrix} q^{-1} H_1^\vee \\ H_0^\vee - (q-1) H_0 \end{pmatrix} \tag{24.1}$$
$$\mathfrak{R}_t H = \begin{pmatrix} H_0^\vee + q^{-1}(q+1) t H_1^\vee \\ (q-1) H_1 - H_1^\vee \end{pmatrix}.$$

If k is odd, we set

$$\mathfrak{R}_t H = \begin{pmatrix} H_1^\vee \\ (q-1) H_1 + q H_0^\vee \end{pmatrix} \tag{24.2}$$
$$\mathfrak{S}_t H = \begin{pmatrix} -H_0^\vee \\ H_1^\vee + (q+1) t H_0^\vee - (q-1) H_0 \end{pmatrix}.$$

Note that, even when the operators do not depend on t, we mention t in the notation. This will avoid splitting certain statements according to the parity of k.

In both cases, a direct computation gives

LEMMA 24.1. *Let $k \geq 0$, t be in $\mathbb{R}$ and H be in $\mathcal{H}_k^2$. We have*

$$\mathfrak{R}_t \mathfrak{S}_t H + \mathfrak{S}_t \mathfrak{R}_t H - (q-1)\mathfrak{S}_t H = (q+1)tH.$$

REMARK 24.2. The reader may wonder why our definition of the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ is not consistent with our choices for the matrices in Section 21.1. Indeed, for example, when k is even, the action of $\mathfrak{R}_t$ on $\mathcal{H}_{k,-}^2$ is not the one given by the matrix $r_{+-}(t)$. The reason for this, and for the choice of normalization in the definition of the operators $\mathfrak{R}'_t$ and $\mathfrak{S}'_t$ in (24.3) and (24.4) below, is that it will allow for a very simple formulation of Proposition 24.5 which describes the image under the spectral transform of the range of the default map of Definition 20.12.

PROPOSITION 24.3. *Let $k \geq 0$. There exists a unique linear map $H \mapsto \widehat{H}(t)$ from $\mathcal{H}_k^{(\mathbb{N})}$ to the space $\mathcal{H}_k^2[t]$ of $\mathcal{H}_k^2$ valued polynomial functions with the following properties:*
(i) *For any H in $\mathcal{H}_k$, one has*

$$\widehat{H\mathbf{1}_0}(t) = \begin{pmatrix} q^{-1}H^\vee \\ 0 \end{pmatrix} \qquad \text{if } k \text{ is even}$$
$$= \begin{pmatrix} 0 \\ H \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

(ii) *For any H in $\mathcal{H}_k^{(\mathbb{N})}$, one has*

$$\widehat{RH}(t) = \mathfrak{R}_t \widehat{H}(t) \text{ and } \widehat{SH}(t) = \mathfrak{S}_t \widehat{H}(t).$$

This map is a linear isomorphism from $\mathcal{H}_k^{(\mathbb{N})}$ onto $\mathcal{H}_k^2[t]$.

This result is a more or less direct consequence of Propositions 22.2, 22.12, 23.2 and 23.11, as we will soon check. Now, we would like to describe the image under the spectral transform $H \mapsto \widehat{H}$ of the range of the default map of Definition 20.12. To this aim, we first consider the actions introduced in Definitions 20.19 and 20.20.

Let $k \geq 0$, t be in $\mathbb{R}$ and $G = \begin{pmatrix} G_0 \\ G_1 \end{pmatrix}$ in $\mathcal{H}_k^2$. If k is even, we set

$$\mathfrak{S}'_t G = \begin{pmatrix} q^{-1}(q-1)G_1 - q^{-1}G_1^\vee \\ -G_0^\vee \end{pmatrix} \tag{24.3}$$
$$\mathfrak{R}'_t G = \begin{pmatrix} (q-1)G_0 - G_0^\vee + q^{-1}(q^2-1)tG_1 - q^{-1}(q+1)tG_1^\vee \\ G_1^\vee \end{pmatrix}.$$

If k is odd, we set

$$\mathfrak{R}'_t G = \begin{pmatrix} -G_1^\vee \\ (q-1)G_1 - qG_0^\vee \end{pmatrix} \tag{24.4}$$
$$\mathfrak{S}'_t G = \begin{pmatrix} G_0^\vee \\ -G_1^\vee - (q+1)tG_0^\vee - (q-1)G_0 \end{pmatrix}.$$

PROPOSITION 24.4. *Let $k \geq 0$. There exists a unique linear map $G \mapsto \breve{G}(t)$ from $\mathcal{H}_k^{(\mathbb{N})}$ to the space $\mathcal{H}_k^2[t]$ of $\mathcal{H}_k^2$ valued polynomial functions with the following properties:*
(i) *For any G in $\mathcal{H}_k$, one has*

$$\widetilde{G\mathbf{1}}_0(t) = \begin{pmatrix} q^{-1}G^\vee - q^{-1}(q-1)G \\ 0 \end{pmatrix} \qquad \text{if } k \text{ is even}$$
$$= \begin{pmatrix} 0 \\ -G \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

(ii) *For any G in $\mathcal{H}_k^{(\mathbb{N})}$, one has*

$$\widetilde{R'G}(t) = \mathfrak{R}'_t \breve{G}(t) \text{ and } \widetilde{SG}(t) = \mathfrak{S}'_t \breve{G}(t).$$

This map is a linear isomorphism from $\mathcal{H}_k^{(\mathbb{N})}$ onto $\mathcal{H}_k^2[t]$.

Again, we will see that this result is essentially a translation of Propositions 22.2, 22.12, 23.2 and 23.11.

Let us now state a less evident result, which will be the principal objective of this Chapter. Its purpose is to explicitely transport the default map of Definition 20.12 by the spectral transforms. Its simple formulation justifies our choices of normalization in Propositions 24.3 and 24.4.

PROPOSITION 24.5. *Let $k \geq 1$, G be in $\mathcal{H}_{k-1}^{(\mathbb{N})}$ and set $H = D_k G$. Fix t in $\mathbb{R}$. If k is even, we have*

$$\widehat{H}(t) = \begin{pmatrix} \breve{G}_0(t)^> \\ \breve{G}_1(t)^> \end{pmatrix} - \begin{pmatrix} q^{-1}(q+1)t & q^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \breve{G}_0(t)^{\vee >\vee} \\ \breve{G}_1(t)^{\vee >\vee} \end{pmatrix}.$$

If k is odd, we have

$$\widehat{H}(t) = \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} \breve{G}_0(t)^> \\ \breve{G}_1(t)^> \end{pmatrix} - \begin{pmatrix} \breve{G}_0(t)^{\vee >\vee} \\ \breve{G}_1(t)^{\vee >\vee} \end{pmatrix}.$$

As in Definition 20.12, we write D_k for the default map.

REMARK 24.6. Note that the matrix $\begin{pmatrix} q^{-1}(q+1)t & q^{-1} \\ -1 & 0 \end{pmatrix}$ is the inverse of the matrix $\begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix}$. The latter matrix already appeared in the proof of the Ihara trace formula, Theorem 1.4.

24.2. Simple pseudofields and duality

The proof of Proposition 24.5 will rely on a duality argument. Thus, for $k \geq 0$, we will need to introduce the dual space of the space $\mathcal{H}_k$. As usual, we construct this space as a space of families of objects parametrized by X_1.

Let $k \geq 1$. If k is even, $k = 2\ell$, $\ell \geq 1$, as in Section 5.1, we write $V_0^\ell(x)$, $x \in X$, for the space of real valued functions on the sphere $S^\ell(x)$, the sum of whose values is 0. Then, a k-simple pseudofield is a family $(s_{xy})_{(x,y)\in X_1}$ where, for any $x \sim y$ in X, s_{xy} is an element of $V_0^\ell(x)$.

If k is odd, $k = 2\ell+1$, $\ell \geq 0$, still as in Section 5.1, we write $V_0^\ell(xy)$, $x \sim y \in X$, for the space of real valued functions on the sphere $S^\ell(xy)$, the sum of whose values is 0. Then, a k-simple pseudofield is a family $(s_{xy})_{(x,y)\in X_1}$ where, for any $x \sim y$ in X, s_{xy} is an element of $V_0^\ell(xy)$.

As for 1-pseudofunctions, we have a natural identification of 1-simple pseudofields with functions on X_1. Indeed, if u is a function on X_1, we can associate to u the 1-simple pseudofield s such that

$$s_{xy} = u(xy)(\mathbf{1}_y - \mathbf{1}_x), \quad x \sim y \in X.$$

We define a 0-simple pseudofield as a function on X. We do not introduce a notion of a (-1)-simple pseudofield.

The finite-dimensional vector space of Γ-invariant k-simple pseudofields is denoted by $\mathcal{S}_k$.

Recall from Section 5.1 that, for $\ell \geq 1$ and x in X, we have identified the dual space of $\overline{V}^\ell(x)$ with $V_0^\ell(x)$. In the same way, for $\ell \geq 0$ and $x \sim y$ in X, we have identified the dual space of $\overline{V}^\ell(xy)$ with $V_0^\ell(xy)$. We use this convention and our usual construction to identify $\mathcal{S}_k$ with the dual space of $\mathcal{H}_k$. If $k \geq 1$, s is in $\mathcal{S}_k$ and H is in $\mathcal{H}_k$, we set

$$\langle s, H\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, H_{xy}\rangle.$$

If $k = 0$, and if s in $\mathcal{S}_0$ is associated with the function u on X and H in $\mathcal{H}_0$ is associated with the function v on X, we set

$$\langle s, H\rangle = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} u(x)v(x).$$

24.3. Operations on simple pseudofields

As usual, we define natural operations on pseudofields.

Let $k \geq 0$ and s be a k-simple pseudofield. If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set $s^\vee_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}$. If k is odd, $k = 2\ell+1$, $\ell \geq 0$, for any $x \sim y$ in X, we set $s^\vee_{xy} = s_{yx}$. If $k = 0$, we set $s^\vee = -s$.

We get, by using Lemma 10.11,

LEMMA 24.7. *Let $k \geq 0$, s be in $\mathcal{S}_k$ and H be in $\mathcal{H}_k$. We have $\langle s^\vee, H\rangle = \langle s, H^\vee\rangle$.*

If $k \geq 0$ is even, we set

$$\mathcal{S}_{k,+} = \{s \in \mathcal{S}_k | s^\vee = qs\} \text{ and } \mathcal{S}_{k,-} = \{s \in \mathcal{S}_k | s^\vee = -s\}.$$

If k is odd, we set

$$\mathcal{S}_{k,+} = \{s \in \mathcal{S}_k | s^\vee = s\} \text{ and } \mathcal{S}_{k,-} = \{s \in \mathcal{S}_k | s^\vee = -s\}.$$

In both cases, we have $\mathcal{S}_k = \mathcal{S}_{k,+} \oplus \mathcal{S}_{k,-}$.

Finally, if $k \geq 1$, and s is a k-simple pseudofield, we define the direct restriction $s^<$ of s, which is a $(k-1)$-simple pseudofield, as follows.

— If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set

$$s^<_{xy} = I^{\ell-1,*}_{xy} s_{xy},$$

where, as in Section 5.3, $I^{\ell-1,*}_{xy} : V^\ell_0(x) \to V^{\ell-1}_0(xy)$ is the adjoint of the natural operator $I^{\ell-1}_{xy} : \overline{V}^{\ell-1}(xy) \to \overline{V}^\ell(x)$.

— If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any $x \sim y$ in X, we set

$$s^<_{xy} = J^{\ell,*}_{xy} s_{xy},$$

where, as in Section 5.3, $J^{\ell,*}_{xy} : V^\ell_0(xy) \to V^\ell_0(x)$ is the adjoint of the natural operator $J^\ell_{xy} : \overline{V}^\ell(x) \to \overline{V}^\ell(xy)$.

— Lastly, if $k = 1$ and if v is the function on X_1 such that, for any $x \sim y$ in X, $s_{xy} = v(xy)(\mathbf{1}_y - \mathbf{1}_x)$, we let $s^<$ be the 0-simple pseudofield associated with the function u defined by

$$u(x) = \sum_{y\sim x} v(xy), \quad x \in X. \tag{24.5}$$

Again, we have

LEMMA 24.8. *Let $k \geq 1$, s be in $\mathcal{S}_k$ and H be in $\mathcal{H}_{k-1}$. We have $\langle s^<, H\rangle = \langle s, H^>\rangle$.*

As usual, the double commutation property holds.

LEMMA 24.9. *Let $k \geq 2$ and s be a k-simple pseudofield. We have $s^{<<\vee} = s^{\vee<<}$.*

PROOF. If $k \geq 3$, this directly follows from Lemma 4.4. Let us study the case $k = 2$.

In case $k = 2$, let u and v be the functions on X which are associated with the 0-simple pseudofields $s^{<<\vee}$ and $s^{\vee<<}$. We must show that $u = v$.

For $x \sim y$ in X, s_{xy} is a function in $V_0^1(x)$ and, by the definition of the objects in Section 5.3, we have $s_{xy}^<(y) = s_{xy}(y)$, hence $s_{xy}^< = s_{xy}(y)(\mathbf{1}_y - \mathbf{1}_x)$. Therefore, by (24.5), for x in X,

$$u(x) = -\sum_{y\sim x} s_{xy}(y).$$

Besides, for $x \sim y$ in X, we have, $s_{xy}^\vee(y) = \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}(y)$. Again by (24.5), this gives, for x in X,

$$v(x) = \sum_{y\sim x}\sum_{\substack{z\sim x\\ z\neq y}} s_{xz}(y) = \sum_{z\sim x}\sum_{\substack{y\sim x\\ y\neq z}} s_{xz}(y) = -\sum_{z\sim x} s_{xz}(z),$$

where the latter follows from the fact that s_{xz} belongs to $V_0^1(x)$ for $z \sim x$. We get $u = v$ as required. □

Contrarily to what happens in the quadratic case, direct restriction may be injective on eigenspaces of the $\vee$ operator.

LEMMA 24.10. *Let $k \geq 2$ and s be a k-simple pseudofield with $s^< = 0$. If k is even and $s^\vee = qs$, then $s = 0$. If k is odd and $s^\vee = s$ or $s^\vee = -s$, then $s = 0$.*

PROOF. Assume k is even, $k = 2\ell$, $\ell \geq 1$. Let s be a k-simple pseudofield with $s^\vee = qs$. For $y, z \sim x$, we have $s_{xy} = s_{xz}$. We write s_x for this element of $V_0^\ell(x)$ that only depends on x. If $s^< = 0$, for any $y \sim x$, when seen as a linear functional on $\overline{V}^\ell(x)$, s_x is 0 on the space $I_{xy}^{\ell-1}\overline{V}^{\ell-1}(xy)$. By Proposition 4.5, as y runs among the neighbours of x, these spaces span $\overline{V}^\ell(x)$, hence $s_x = 0$ as required.

If k is odd, we proceed in the same way by using Proposition 4.6 □

24.4. Duality on spaces of sequences

Now that, for $k \geq 0$, we have introduced the dual space of the space of Γ-invariant k-pseudofunctions, we can define a duality at the level of spaces of sequences. We adopt the same convention as the one used for defining Hilbert spaces of sequences of real numbers adapted to the study of the model operators in Section 21.4.

Thus, for $k \geq 0$, if $s = (s_i)_{i\geq 0}$ is in $\mathcal{S}_k^{\mathbb{N}}$ and $H = (H_i)_{i\geq 0}$ is in $\mathcal{H}_k^{(\mathbb{N})}$, we set

$$\begin{aligned}\langle s, H\rangle &= \sum_{i\geq 0} q^i(\langle s_{2i}, H_{2i}\rangle + \langle s_{2i+1}, H_{2i+1}\rangle) && \text{if } k \text{ is even}\\ &= \sum_{i\geq 0} q^i(\langle s_{2i}, H_{2i}\rangle + q\langle s_{2i+1}, H_{2i+1}\rangle) && \text{if } k \text{ is odd.}\end{aligned}$$

Besides, we use the same symbols as in Definitions 20.16, 20.17, 20.19 and 20.20 for the analogous operations on sequences of simple pseudofields. Thus, for $k \geq 0$ and s in $\mathcal{S}_k$, if k is even, we set

$$\begin{aligned}(Rs)_0 &= s_0^\vee\\ (R's)_0 &= (q-1)s_0 - s_0^\vee\\ (Rs)_i = (R's)_i &= s_{i-1} + (q-1)s_i && i \text{ even}, i \geq 2\\ (Rs)_i = (R's)_i &= qs_{i+1} && i \text{ odd.}\\ (Ss)_i &= s_{i+1} && i \text{ even}\\ (Ss)_i &= s_{i-1} && i \text{ odd.}\end{aligned}$$

If k is odd, we set

$$\begin{aligned}(Rs)_i &= qs_{i+1} && i \text{ even}\\ (Rs)_i &= s_{i-1} + (q-1)s_i && i \text{ odd.}\\ (Ss)_0 &= s_0^\vee\\ (S's)_0 &= -s_0^\vee\\ (Ss)_i = (S's)_i &= s_{i-1} && i \text{ even}, i \geq 2\\ (Ss)_i = (S's)_i &= s_{i+1} && i \text{ odd.}\end{aligned}$$

As in Lemmas 22.1, 22.11, 23.1 and 23.10, we get

LEMMA 24.11. *Let $k \geq 0$, s be in $\mathcal{S}_k^{\mathbb{N}}$ and H be in $\mathcal{H}_k^{(\mathbb{N})}$. We have*

$$\langle Rs, H\rangle = \langle s, RH\rangle \text{ and } \langle Ss, H\rangle = \langle s, SH\rangle.$$

If k is even, we have $\langle R's, H\rangle = \langle s, R'H\rangle$ and if k is odd, we have $\langle S's, H\rangle = \langle s, S'H\rangle$.

If $s = (s_i)_{i\geq 0}$ is a sequence of simple pseudofields, we will write $s^\vee$ for the sequence $(s_i^\vee)_{i\geq 0}$. All the operators that we have defined on spaces of sequences commute with this operation.

LEMMA 24.12. *Let $k \geq 0$ and $s = (s_i)_{i\geq 0}$ be a sequence of k-simple pseudofields. We have $R(s^\vee) = (Rs)^\vee$ and $S(s^\vee) = (Ss)^\vee$. If k is even, we have $R'(s^\vee) = (R's)^\vee$ and if k is odd, we have $S'(s^\vee) = (S's)^\vee$.*

24.5. Construction of the spectral transforms: the even case

We will build the spectral transform of sequences of pseudofunctions by analogy with the case of sequences of real numbers. We will use freely the notation of Sections 22.1, 22.3, 23.1 and 23.2.

Fix t in $\mathbb{R}$. Let $k \geq 0$ be even. As in Section 22.1, if s is in $\mathcal{S}_{k,+}$, we set $\alpha_s(t)$ and $\beta_s(t)$ to be the sequences in $\mathcal{S}_k^{\mathbb{N}}$ defined by, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{++}(t)_i s \text{ and } \beta_s(t)_i = \beta_{++}(t)_i s.$$

In the same way, as in Section 22.3, if s is in $\mathcal{S}_{k,-}$, we set $\alpha_s(t)$ and $\beta_s(t)$ to be the sequences in $\mathcal{S}_k^{\mathbb{N}}$ defined by, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{+-}(t)_i s \text{ and } \beta_s(t)_i = \beta_{+-}(t)_i s.$$

Finally, if s is in $\mathcal{S}_k$ and $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we set $\alpha_s(t) = \alpha_{s_+}(t) + \alpha_{s_-}(t)$ and $\beta_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$. Note that, by Lemma 22.7 and Lemma 22.17, the coordinates of these sequences are polynomial functions of t.

We get

LEMMA 24.13. *Let $k \geq 0$ be even and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

(24.6)
$$\begin{pmatrix} S\alpha_s(t) \\ S\beta_s(t) \end{pmatrix} = \begin{pmatrix} \beta_s(t) \\ \alpha_s(t) \end{pmatrix} \text{ and } \begin{pmatrix} R\alpha_s(t) \\ R\beta_s(t) \end{pmatrix} = \begin{pmatrix} \alpha_{s^\vee}(t) + (q+1)t\beta_s(t) \\ \beta_{(q-1)s-s^\vee}(t) \end{pmatrix}.$$

Conversely, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S\gamma \\ S\delta \end{pmatrix} = \begin{pmatrix} \delta \\ \gamma \end{pmatrix} \text{ and } \begin{pmatrix} R\gamma \\ R\delta \end{pmatrix} = \begin{pmatrix} \gamma^\vee + (q+1)t\delta \\ (q-1)\delta - \delta^\vee \end{pmatrix}. \tag{24.7}$$

Then, we have $\gamma = \alpha_s(t)$ and $\delta = \beta_s(t)$ where $s = \gamma_0$.

PROOF. This is a direct translation of the analogous results for the model operators. First, we show (24.6). By construction of the objects, it suffices to prove the result when s is an eigenvector of the $\vee$ operator.

Then, if $s^\vee = qs$, we have $(q-1)s - s^\vee = -s$, hence

$$\begin{pmatrix} \alpha_{s^\vee}(t) + (q+1)t\beta_s(t) \\ \beta_{(q-1)s-s^\vee}(t) \end{pmatrix} = \begin{pmatrix} q\alpha_s(t) + (q+1)t\beta_s(t) \\ -\beta_s(t) \end{pmatrix} = r_{++}(t) \begin{pmatrix} \alpha_s(t) \\ \beta_s(t) \end{pmatrix}$$

and (24.6) follows from Lemma 22.6. In the same way, if $s^\vee = -s$, we have $(q-1)s - s^\vee = qs$, hence

$$\begin{pmatrix} \alpha_{s^\vee}(t) + (q+1)t\beta_s(t) \\ \beta_{(q-1)s-s^\vee}(t) \end{pmatrix} = r_{+-}(t) \begin{pmatrix} \alpha_s(t) \\ \beta_s(t) \end{pmatrix}$$

and (24.6) follows from Lemma 22.16.

Now, we take γ and δ in $\mathcal{S}_k^{\mathbb{N}}$ such (24.7) holds. We write $\gamma = \gamma_+ + \gamma_-$ and $\delta = \delta_+ + \delta_-$ where $\gamma_+^\vee = q\gamma_+$, $\gamma_-^\vee = -\gamma_-$, $\delta_+^\vee = q\delta_+$ and $\delta_-^\vee = -\delta_-$. By using Lemma 24.12, we get

$$\begin{pmatrix} S\gamma_+ \\ S\delta_+ \end{pmatrix} = s_+ \begin{pmatrix} \gamma_+ \\ \delta_+ \end{pmatrix} \text{ and } \begin{pmatrix} R\gamma_+ \\ R\delta_+ \end{pmatrix} = r_{++}(t) \begin{pmatrix} \gamma_+ \\ \delta_+ \end{pmatrix}.$$

From Lemma 22.8, we get $\gamma_+ = \alpha_{\gamma_{+,0}}(t)$ and $\delta_+ = \beta_{\gamma_{+,0}}(t)$. In the same way, Lemma 22.18 ensures that $\gamma_- = \alpha_{\gamma_{-,0}}(t)$ and $\delta_- = \beta_{\gamma_{-,0}}(t)$. The conclusion follows. □

Still by analogy with the case of the model operators, but with a change of normalization that will be justified later, for H in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\widehat{H}(t) = \begin{pmatrix} \widehat{H}_0(t) \\ \widehat{H}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t) \rangle \\ \langle s, \widehat{H}_1(t) \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), H \rangle \\ \langle \beta_s(t), H \rangle \end{pmatrix}. \tag{24.8}$$

Proof of Proposition 24.3 in case k is even. Take H in $\mathcal{H}_k$. By the constructions in Sections 22.1 and 22.3, for s in $\mathcal{S}_k$, we have $\alpha_s(t)_0 = s$ and $\beta_s(t)_0 = 0$. Thus, for t in $\mathbb{R}$, we get, from (24.8),

$$\begin{pmatrix} \langle s, \widehat{(H\mathbf{1}_0)}_0(t)^\vee - (q-1)\widehat{(H\mathbf{1}_0)}_0(t) \rangle \\ \langle s, \widehat{(H\mathbf{1}_0)}_1(t) \rangle \end{pmatrix} = \begin{pmatrix} \langle s, H \rangle \\ 0 \end{pmatrix},$$

which gives $\widehat{H\mathbf{1}_0}(t) = \begin{pmatrix} q^{-1}H^\vee \\ 0 \end{pmatrix}$ as required.

Let us now show that the equivariance properties actually hold. Fix t in $\mathbb{R}$ and H in $\mathcal{H}_k^{(\mathbb{N})}$. Let $J = \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}$ and $K = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$ be in $\mathcal{H}_k^2$ such

that, for s in $\mathcal{S}_k$,

$$\begin{pmatrix}\langle s, J_0\rangle\\ \langle s, J_1\rangle\end{pmatrix} = \begin{pmatrix}\langle \alpha_s(t), H\rangle\\ \langle \beta_s(t), H\rangle\end{pmatrix} \text{ and } \begin{pmatrix}\langle s, K_0\rangle\\ \langle s, K_1\rangle\end{pmatrix} = \begin{pmatrix}\langle \alpha_s(t), RH\rangle\\ \langle \beta_s(t), RH\rangle\end{pmatrix},$$

so that $\widehat{H}(t) = \begin{pmatrix}q^{-1}J_0^\vee\\ J_1\end{pmatrix}$ and $\widehat{RH}(t) = \begin{pmatrix}q^{-1}K_0^\vee\\ K_1\end{pmatrix}$. From Lemma 24.11 and Lemma 24.13,

$$\begin{aligned}\begin{pmatrix}\langle s, K_0\rangle\\ \langle s, K_1\rangle\end{pmatrix} &= \begin{pmatrix}\langle R\alpha_s(t), H\rangle\\ \langle R\beta_s(t), H\rangle\end{pmatrix}\\ &= \begin{pmatrix}0 & (q+1)t\\ 0 & (q-1)\end{pmatrix}\begin{pmatrix}\langle \alpha_s(t), H\rangle\\ \langle \beta_s(t), H\rangle\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\langle \alpha_{s^\vee}(t), H\rangle\\ \langle \beta_{s^\vee}(t), H\rangle\end{pmatrix}\\ &= \begin{pmatrix}0 & (q+1)t\\ 0 & (q-1)\end{pmatrix}\begin{pmatrix}\langle s, J_0\rangle\\ \langle s, J_1\rangle\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\langle s^\vee, J_0\rangle\\ \langle s^\vee, J_1\rangle\end{pmatrix}.\end{aligned}$$

We get $K_0 = J_0^\vee + (q+1)tJ_1$ and $K_1 = (q-1)J_1 - J_1^\vee$, hence

$$\widehat{RH}(t) = \begin{pmatrix}q^{-1}K_0^\vee\\ K_1\end{pmatrix} = \begin{pmatrix}J_0 + q^{-1}(q-1)J_0^\vee + q^{-1}(q+1)tJ_1^\vee\\ (q-1)J_1 - J_1^\vee\end{pmatrix}.$$

As $J_1 = \widehat{H}_1(t)$, we get $\widehat{RH}_1(t) = (q-1)\widehat{H}_1(t) - \widehat{H}_1(t)^\vee$. As

$$J_0 = \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t),$$

a straightforward computation gives

$$\widehat{RH}_0(t) = \widehat{H}_0(t)^\vee + q^{-1}(q+1)t\widehat{H}_1(t)$$

and the equivariance property for the operator R holds.

In the same way, we let $L = \begin{pmatrix}L_0\\ L_1\end{pmatrix}$ be in $\mathcal{H}_k^2$ such that, for s in $\mathcal{S}_k$, $\begin{pmatrix}\langle s, L_0\rangle\\ \langle s, L_1\rangle\end{pmatrix} = \begin{pmatrix}\langle \alpha_s(t), SH\rangle\\ \langle \beta_s(t), SH\rangle\end{pmatrix}$, so that $\widehat{SH}(t) = \begin{pmatrix}q^{-1}L_0^\vee\\ L_1\end{pmatrix}$. Lemma 24.11 and Lemma 24.13 now give,

$$\begin{pmatrix}\langle s, L_0\rangle\\ \langle s, L_1\rangle\end{pmatrix} = \begin{pmatrix}\langle S\alpha_s(t), H\rangle\\ \langle S\beta_s(t), H\rangle\end{pmatrix} = \begin{pmatrix}\langle \beta_s(t), H\rangle\\ \langle \alpha_s(t), H\rangle\end{pmatrix} = \begin{pmatrix}\langle s, J_1\rangle\\ \langle s, J_0\rangle\end{pmatrix}.$$

Reasoning as above, we get $\widehat{SH}_1(t) = J_0 = \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t)$ and $\widehat{SH}_0(t) = q^{-1}J_1^\vee = q^{-1}\widehat{H}_1(t)^\vee$.

The uniqueness statement follows from the analogous ones in Proposition 22.2 and Proposition 22.12. □

To construct the $\check{\cdot}$ transform, we do the same constructions, but the roles of $\mathcal{S}_{k,+}$ and $\mathcal{S}_{k,-}$ are exchanged. Thus, if s is in $\mathcal{S}_{k,+}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{+-}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{+-}(t)_i s.$$

If s is in $\mathcal{S}_{k,-}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{++}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{++}(t)_i s.$$

As above, if s is in $\mathcal{S}_k$ and $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we set $\alpha'_s(t) = \alpha'_{s_+}(t) + \alpha'_{s_-}(t)$ and $\beta'_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$. By Lemma 22.7 and Lemma 22.17, the coordinates of these sequences are polynomial functions of t.

We now get

LEMMA 24.14. *Let $k \geq 0$ be even and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

$$\begin{pmatrix} S\alpha'_s(t) \\ S\beta'_s(t) \end{pmatrix} = \begin{pmatrix} \beta'_s(t) \\ \alpha'_s(t) \end{pmatrix}$$

$$\text{and } \begin{pmatrix} R'\alpha'_s(t) \\ R'\beta'_s(t) \end{pmatrix} = \begin{pmatrix} \alpha'_{(q-1)s-s^\vee}(t) + (q+1)t\beta'_s(t) \\ \beta'_{s^\vee}(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S\gamma \\ S\delta \end{pmatrix} = \begin{pmatrix} \delta \\ \gamma \end{pmatrix} \text{ and } \begin{pmatrix} R'\gamma \\ R'\delta \end{pmatrix} = \begin{pmatrix} (q-1)\gamma - \gamma^\vee + (q+1)t\delta \\ \delta^\vee \end{pmatrix}.$$

Then, we have $\gamma = \alpha'_s(t)$ and $\delta = \beta'_s(t)$ where $s = \gamma_0$.

Now, for G in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\breve{G}(t) = \begin{pmatrix} \breve{G}_0(t) \\ \breve{G}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, \breve{G}_0(t)^\vee \rangle \\ \langle s, -\breve{G}_1(t) \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha'_s(t), G \rangle \\ \langle \beta'_s(t), G \rangle \end{pmatrix}. \tag{24.9}$$

PROOF OF PROPOSITION 24.4 IN CASE k IS EVEN. This can be obtained from Lemma 24.14 and (24.9) as in the proof of Proposition 24.3. □

24.6. Construction of the spectral transforms: the odd case

We proceed to analogous constructions in the odd case. Let $k \geq 0$ be odd.

For t in $\mathbb{R}$, as in Section 23.1, if s is in $\mathcal{S}_{k,+}$, we set, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{-+}(t)_i s \text{ and } \beta_s(t)_i = \beta_{-+}(t)_i s.$$

In the same way, as in Section 23.2, if s is in $\mathcal{S}_{k,-}$, we set, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{--}(t)_i s \text{ and } \beta_s(t)_i = \beta_{--}(t)_i s.$$

And, for s in $\mathcal{S}_k$, $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we set $\alpha_s(t) = \alpha_{s_+}(t) + \alpha_{s_-}(t)$ and $\beta_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$.

From Lemma 23.6, Lemma 23.8, Lemma 23.15 and Lemma 23.17, we get

LEMMA 24.15. *Let $k \geq 0$ be odd and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

$$\begin{pmatrix} S\alpha_s(t) \\ S\beta_s(t) \end{pmatrix} = \begin{pmatrix} \alpha_{s^\vee}(t) + \beta_{(q+1)ts-(q-1)s^\vee}(t) \\ -\beta_{s^\vee}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R\alpha_s(t) \\ R\beta_s(t) \end{pmatrix} = \begin{pmatrix} (q-1)\alpha_s(t) + q\beta_s(t) \\ \alpha_s(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S\gamma \\ S\delta \end{pmatrix} = \begin{pmatrix} \gamma^\vee + (q+1)t\delta - (q-1)\delta^\vee \\ -\delta^\vee \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R\gamma \\ R\delta \end{pmatrix} = \begin{pmatrix} (q-1)\gamma + q\delta \\ \gamma \end{pmatrix}.$$

Then, we have $\gamma = \alpha_s(t)$ and $\delta = \beta_s(t)$ where $s = \gamma_0$.

For H in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\widehat{H}(t) = \begin{pmatrix} \widehat{H}_0(t) \\ \widehat{H}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, \widehat{H}_1(t) \rangle \\ \langle s, \widehat{H}_0(t)^\vee \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), H \rangle \\ \langle \beta_s(t), H \rangle \end{pmatrix}. \tag{24.10}$$

PROOF OF PROPOSITION 24.3 IN CASE k IS ODD. This follows from Lemma 24.15 and (24.10). □

Again, to construct the $\check{\cdot}$ transform, we exchange the roles of $\mathcal{S}_{k,+}$ and $\mathcal{S}_{k,-}$. If s is in $\mathcal{S}_{k,+}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{--}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{--}(t)_i s.$$

If s is in $\mathcal{S}_{k,-}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{-+}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{-+}(t)_i s.$$

If s is in $\mathcal{S}_k$ and $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we still set $\alpha'_s(t) = \alpha'_{s_+}(t) + \alpha'_{s_-}(t)$ and $\beta'_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$.

We get again

LEMMA 24.16. *Let $k \geq 0$ be odd and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

$$\begin{pmatrix} S'\alpha'_s(t) \\ S'\beta'_s(t) \end{pmatrix} = \begin{pmatrix} -\alpha'_{s^\vee}(t) + \beta'_{(q+1)ts+(q-1)s^\vee}(t) \\ \beta'_{s^\vee}(t) \end{pmatrix}$$
$$\begin{pmatrix} R\alpha'_s(t) \\ R\beta'_s(t) \end{pmatrix} = \begin{pmatrix} (q-1)\alpha'_s(t) + q\beta'_s(t) \\ \alpha'_s(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S'\gamma \\ S'\delta \end{pmatrix} = \begin{pmatrix} -\gamma^\vee + (q+1)t\delta + (q-1)\delta^\vee \\ \delta^\vee \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R\gamma \\ R\delta \end{pmatrix} = \begin{pmatrix} (q-1)\gamma + q\delta \\ \gamma \end{pmatrix}.$$

Then, we have $\gamma = \alpha'_s(t)$ and $\delta = \beta'_s(t)$ where $s = \gamma_0$.

Finally, for G in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\breve{G}(t) = \begin{pmatrix} \breve{G}_0(t) \\ \breve{G}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, -\breve{G}_1(t) \rangle \\ \langle s, \breve{G}_0(t)^\vee \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha'_s(t), G \rangle \\ \langle \beta'_s(t), G \rangle \end{pmatrix}. \tag{24.11}$$

PROOF OF PROPOSITION 24.4 IN CASE k IS ODD. This is now a consequence of Lemma 24.16 and (24.11). □

24.7. Twist and spectral transform

In this Section, we assume Γ to be bipartite. Then, we can use the language of Section 20.6. We will describe the action of the twist operator on the spectral transform.

For $k \geq 0$, we let $s \mapsto s^\wr$ be the adjoint operations to the twist operator on k-simple pseudofields. More concretely, if $k \geq 1$ and s is a k-simple pseudofield, for $x \sim y$ in X, we set

$$s^\wr_{xy} = \chi(x) s_{xy}.$$

If $k = 0$ and s is the 0-simple pseudofield associated to the function u on X, we set $s^\wr$ to be 0-simple pseudofield associated to the function χu.

Still by duality with Section 20.6, we set, for $k \geq 0$ and s in $\mathcal{S}_k^{\mathbb{N}}$,

$$(Us)_i = (-1)^{\frac{i}{2}} s_i^\wr \qquad i \text{ even}$$
$$(Us)_i = (-1)^{\frac{i+1}{2}} s_i^\wr \qquad i \text{ odd}, k \text{ even}$$
$$(Us)_i = (-1)^{\frac{i-1}{2}} s_i^\wr \qquad i \text{ odd}, k \text{ odd}.$$

By using Lemma 22.20 and Lemma 23.19, we get

LEMMA 24.17. *Let $k \geq 0$ and s be in $\mathcal{S}_k$. For t in $\mathbb{C}$, we have*

$$\begin{aligned} U\beta_s(t) &= -\beta_{s^\wr}(-t) \\ U\alpha_s(t) &= \alpha_{s^\wr}(-t) && \text{if } k \text{ is even} \\ U\alpha_s(t) &= -\alpha_{s^\wr}(-t) && \text{if } k \text{ is odd.} \end{aligned}$$

Thanks to Lemma 20.22 and (24.8) and (24.10), this yields

LEMMA 24.18. *Let $k \geq 0$ and H be in $\mathcal{H}_k^{(\mathbb{N})}$. For t in $\mathbb{C}$, we have*

$$\widehat{UH}(t) = \begin{pmatrix} \widehat{H}_0(t)^\wr \\ -\widehat{H}_1(t)^\wr \end{pmatrix}.$$

24.8. The adjoint of the default map

Recall that, for $k \geq 0$, the default map $D_k : \mathcal{H}_{k-1}^{(\mathbb{N})} \to \mathcal{H}_k^{(\mathbb{N})}$ was defined in Definition 20.12. By Corollary 20.14, the range of the default map is the null space of the polyextension map of Definition 20.11. In Section 24.4 above, we have identified the dual space of $\mathcal{H}_k^{(\mathbb{N})}$ with $\mathcal{S}_k^{\mathbb{N}}$. Therefore, the default map gives rise to an adjoint linear map $D_k^* : \mathcal{S}_k^{\mathbb{N}} \to \mathcal{S}_{k-1}^{\mathbb{N}}$. We can give a direct description of D_k^*.

LEMMA 24.19. *Let $k \geq 1$ and s be in $\mathcal{S}_k^{\mathbb{N}}$. If k is even, we have*

$$\begin{aligned} (D_k^* s)_0 &= s_0^{<\vee} - s_1^{<} \\ (D_k^* s)_i &= \frac{1}{q} s_{i-1}^{\vee<\vee} - s_{i+1}^{<}, && i \geq 1. \end{aligned}$$

If k is odd, we have

$$\begin{aligned} (D_k^* s)_0 &= s_0^{<\vee} - q s_1^{<} \\ (D_k^* s)_i &= s_{i-1}^{\vee<\vee} - q s_{i+1}^{<}, && i \geq 1. \end{aligned}$$

PROOF. This is a direct computation. For example, we explicit the even case. If k is even, for G in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, we have by definition

$$\begin{aligned} \langle D_k^* s, H \rangle &= \langle s, D_k H \rangle = \langle s_0, G_1^{\vee>\vee} + G_0^{\vee>} \rangle \\ &\quad + \sum_{i=1}^{\infty} q^{i-1} (\langle s_{2i-1}, G_{2i}^{\vee>\vee} - G_{2i-2}^{>} \rangle + q \langle s_{2i}, G_{2i+1}^{\vee>\vee} - G_{2i-1}^{>} \rangle). \end{aligned}$$

By using Lemma 24.7 and Lemma 24.8, we get

$$\begin{aligned} \langle D_k^* s, H \rangle &= \langle s_0^{<\vee} - s_1^{<}, G_0 \rangle \\ &\quad + \sum_{i=1}^{\infty} \langle q^{i-1} s_{2i-2}^{\vee>\vee} - q^i s_{2i}^{<}, G_{2i-1} \rangle + \langle q^{i-1} s_{2i-1}^{\vee>\vee} - q^i s_{2i+1}^{<}, G_{2i} \rangle. \end{aligned}$$

The conclusion follows. The odd case is analogous. □

24.9. The spectral default

We now prove Proposition 24.5 by using the precise definition of the spectral transforms given in Section 24.5 and Section 24.6. The main point of the proof is the following delicate computation.

LEMMA 24.20. *Let $k \geq 1$ be even, s be in $\mathcal{S}_k^{\mathbb{N}}$ and t be in $\mathbb{R}$. We have*

$$D_k^* \beta_s(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee <}}(t). \tag{24.12}$$

If k is even, we have

$$D_k^* \alpha_s(t) = \alpha'_{s^{<\vee}}(t) + \beta'_{s^{\vee <\vee}{}_{-(q-1)s^{<\vee}}{}_{-(q+1)ts^<}}(t). \tag{24.13}$$

If k is odd, we have

$$D_k^* \alpha_s(t) = \alpha'_{s^{<\vee}{}_{-(q-1)s^<}}(t) + \beta'_{s^{\vee <\vee}{}_{-(q+1)ts^<}}(t). \tag{24.14}$$

One way to prove this is to use the explicit definitions of the objects given in Sections 22.1, 22.3, 23.1, 23.2, 24.5 and 24.6 and the formulas for the operator D_k^* given in Lemma 24.19. We will instead use an argument based on the uniqueness property in Lemmas 24.14 and 24.16.

We will split the proof according to the parity of k.

PROOF OF LEMMA 24.20 IN CASE k IS EVEN. Looking at (24.12) and (24.13) suggests to introduce $\delta = D_k^* \beta_s(t) + (D_k^* \alpha_s(t))^\vee$ and $\gamma = R\delta$. As $R^2 = q + (q-1)R$, we get the second relation in the uniqueness part of Lemma 24.16. We will conclude by proving that the first relation also holds. By Lemma 20.21, Lemma 24.12 and Lemma 24.13, we have

$$S'\delta = D_k^* S\beta_s(t) + (D_k^* S\alpha_s(t))^\vee = D_k^* \alpha_s(t) + (D_k^* \beta_s(t))^\vee = \delta^\vee.$$

It remains to compute $S'\gamma$. Again by Lemma 20.21, Lemma 24.12 and Lemma 24.13, we have

$$\begin{aligned} \gamma = R\delta &= D_k^* R\beta_s(t) + (D_k^* R\alpha_s(t))^\vee \\ &= D_k^* \beta_{(q-1)s-s^\vee}(t) + (D_k^* \alpha_{s^\vee}(t))^\vee + (q+1)t(D_k^* \beta_s(t))^\vee, \end{aligned} \tag{24.15}$$

hence

$$S'\gamma = D_k^* \alpha_{(q-1)s-s^\vee}(t) + (D_k^* \beta_{s^\vee}(t))^\vee + (q+1)t(D_k^* \alpha_s(t))^\vee.$$

Applying the $\vee$ operator to (24.15) and summing with the last equation gives

$$\begin{aligned} S'\gamma + \gamma^\vee &= (q-1)D_k^* \alpha_s(t) + (q-1)(D_k^* \beta_s(t))^\vee \\ &\quad + (q+1)t(D_k^* \alpha_s(t))^\vee + (q+1)tD_k^* \beta_s(t) = (q-1)\delta^\vee + (q+1)t\delta, \end{aligned}$$

that is, we have $S'\gamma = -\gamma^\vee + (q+1)t\delta + (q-1)\delta^\vee$. Therefore, by Lemma 24.16, we have $\gamma = \alpha'_{\gamma_0}(t)$ and $\delta = \beta'_{\gamma_0}(t)$. Let us determine γ_0. By construction in Sections 22.1, 22.3 and 24.5, we have

$$\alpha_s(t)_0 = \beta_s(t)_1 = s \text{ and } \alpha_s(t)_1 = \beta_s(t)_0 = 0.$$

By using Lemma 24.19 and (24.15), we get

$$\begin{aligned}\gamma_0 = -((q-1)s - s^\vee)^< + s^{\vee<} + (q+1)t(-s^<)^\vee \\ = 2s^{\vee<} - (q-1)s^< - (q+1)ts^{<\vee}.\end{aligned}$$

Thus, we have shown that

$$D_k^*\beta_s(t) + (D_k^*\alpha_s(t))^\vee = \delta = \beta'_{2s^{\vee<} - (q-1)s^< - (q+1)ts^{<\vee}}(t).$$

Applying this identity to $s^\vee$ yields

$$(D_k^*\alpha_{s^\vee}(t))^\vee = \beta'_{2qs^< + (q-1)s^{\vee<} - (q+1)ts^{\vee<\vee}}(t) - D_k^*\beta_{s^\vee}(t)$$

and therefore, by using the value for γ given by (24.15),

(24.16)
$$\begin{aligned}D_k^*\beta_{(q-1)s-2s^\vee}(t) + \beta'_{2qs^< + (q-1)s^{\vee<} - (q+1)ts^{\vee<\vee}}(t) + (q+1)t(D_k^*\beta_s(t))^\vee \\ = \gamma = \alpha'_{2s^{\vee<} - (q-1)s^< - (q+1)ts^{<\vee}}(t).\end{aligned}$$

To conclude, we will temporarily split the proof according to the eigenvalues of the $\vee$ operator in $\mathcal{S}_k$.

If $s^\vee = qs$, (24.16) says

$$-D_k^*\beta_s(t) + t(D_k^*\beta_s(t))^\vee = \alpha'_{s^{<} - ts^{<\vee}}(t) - q\beta'_{s^{<} - ts^{<\vee}}(t).$$

As the linear map $r \mapsto r - tr^\vee$ is injective on $\mathcal{S}_{k-1}$ for all $t \notin \{-1, 1\}$ and as the functions are polynomial in t, we get

$$D_k^*\beta_s(t) = -\alpha'_{s^<}(t) + q\beta'_{s^<}(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{24.17}$$

If $s^\vee = -s$, (24.16) says

$$D_k^*\beta_s(t) + t(D_k^*\beta_s(t))^\vee = -\alpha'_{s^< + ts^{<\vee}}(t) - \beta'_{s^< + ts^{<\vee}}(t).$$

As above, we get

$$D_k^*\beta_s(t) = -\alpha'_{s^<}(t) - \beta'_{s^<}(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{24.18}$$

Joining the two cases, we get from (24.17) and (24.18) that (24.12) holds for any s in $\mathcal{S}_k$. We now apply the operator S' to this identity. By Lemma 20.21, Lemma 24.12, Lemma 24.13 and Lemma 24.16, this gives

$$D_k^*\alpha_s(t) = \alpha'_{s^{<\vee}}(t) - \beta'_{(q+1)ts^< + (q-1)s^{<\vee}}(t) + \beta'_{s^{\vee<\vee}}(t)$$

and (24.13) follows. □

PROOF OF LEMMA 24.20 IN CASE k IS ODD. We now consider (24.12) and (24.14) and we set $\delta = qD_k^*\beta_s(t) + (D_k^*\alpha_s(t))^\vee$ and $\gamma = S\delta$, so that the first relation in the uniqueness part of Lemma 24.14 holds. As above, we will conclude by proving that the second relation also holds. By Lemma 20.21, Lemma 24.12 and Lemma 24.15, we have

$$\begin{aligned} R'\delta &= qD_k^*R\beta_s(t) + (D_k^*R\alpha_s(t))^\vee \\ &= qD_k^*\alpha_s(t) + (D_k^*(q\beta_s(t) + (q-1)\alpha_s(t)))^\vee \\ &= qD_k^*\alpha_s(t) + (q-1)(D_k^*\alpha_s(t))^\vee + q(D_k^*(\beta_s(t)))^\vee = \delta^\vee. \end{aligned}$$

We now compute $R'\gamma$. Still by Lemma 20.21, Lemma 24.12 and Lemma 24.15, we have

$$\begin{aligned} \gamma = S\delta &= qD_k^*S\beta_s(t) + (D_k^*S\alpha_s(t))^\vee \\ &= -qD_k^*\beta_{s^\vee}(t) + (D_k^*\alpha_{s^\vee}(t))^\vee + (D_k^*\beta_{(q+1)ts-(q-1)s^\vee}(t))^\vee, \end{aligned} \tag{24.19}$$

hence

$$\gamma^\vee - (q-1)\gamma = -q(D_k^*\beta_{s^\vee}(t))^\vee + qD_k^*\alpha_{s^\vee}(t) + qD_k^*\beta_{(q+1)ts}(t).$$

and

$$\begin{aligned} R'\gamma = &-qD_k^*\alpha_{s^\vee}(t) + q(D_k^*\beta_{s^\vee}(t))^\vee \\ &+ (q-1)(D_k^*\alpha_{s^\vee}(t))^\vee + (D_k^*\alpha_{(q+1)ts-(q-1)s^\vee}(t))^\vee \\ &= -qD_k^*\alpha_{s^\vee}(t) + q(D_k^*\beta_{s^\vee}(t))^\vee + (D_k^*\alpha_{(q+1)ts}(t))^\vee \end{aligned}$$

Summing up the last two relations gives

$$R'\gamma + \gamma^\vee - (q-1)\gamma = (D_k^*\alpha_{(q+1)ts}(t))^\vee + qD_k^*\beta_{(q+1)ts}(t) = (q+1)t\delta.$$

That is, we have $R'\gamma = -\gamma^\vee + (q-1)\gamma + (q+1)t\delta$. Therefore, by Lemma 24.14, we have $\gamma = \alpha'_{\gamma_0}(t)$ and $\delta = \beta'_{\gamma_0}(t)$. We compute γ_0. By construction in Sections 23.1, 23.2 and 24.6, we have

$$\begin{aligned} \alpha_s(t)_0 &= s & \alpha_s(t)_1 &= q^{-1}(q-1)s \\ \beta_s(t)_0 &= 0 & \beta_s(t)_1 &= q^{-1}s. \end{aligned}$$

By using Lemma 24.19 and (24.19), we get

$$\begin{aligned} \gamma_0 = q^{\vee<} + (s^{\vee<\vee} - (q-1)s^{\vee<})^\vee + ((q-1)s^{\vee<} - (q+1)ts^<)^\vee \\ = 2qs^{\vee<} + (q-1)s^{\vee<\vee} - (q+1)ts^{<\vee}. \end{aligned}$$

We have shown that

$$qD_k^*\beta_s(t) + (D_k^*\alpha_s(t))^\vee = \delta = \beta'_{2qs^{\vee<}+(q-1)s^{\vee<\vee}-(q+1)ts^{<\vee}}(t).$$

Applying this identity to $s^\vee$, we get

$$(D_k^*\alpha_{s^\vee}(t))^\vee = \beta'_{2qs^<+(q-1)s^{<\vee}-(q+1)ts^{\vee<\vee}}(t) - qD_k^*\beta_{s^\vee}(t),$$

whence, by using the value for γ given by (24.19),

(24.20)
$$-2qD_k^*\beta_{s^\vee}(t)+\beta'_{2qs^<+(q-1)s^{<\vee}-(q+1)ts^{\vee<\vee}}(t)+(D_k^*\beta_{(q+1)ts-(q-1)s^\vee}(t))^\vee \\ =\gamma=\alpha'_{2qs^{\vee<}+(q-1)s^{\vee<\vee}-(q+1)ts^{<\vee}}(t).$$

To conclude, we will again temporarily split the proof according to the eigenvalues of the $\vee$ operator in $\mathcal{S}_k$.

If $s^\vee=s$, (24.20) says

$$-2qD_k^*\beta_s(t)+((q+1)t-(q-1))(D_k^*\beta_s(t))^\vee \\ =\alpha'_{2qs^<-((q+1)t-(q-1))s^{<\vee}}(t)-\beta'_{2qs^<-((q+1)t-(q-1))s^{<\vee}}(t).$$

As the eigenvalues of the $\vee$ operator on $\mathcal{S}_{k-1}$ are q and -1, the linear map $r\mapsto 2qr-((q+1)t-(q-1))r^\vee$ is injective on $\mathcal{S}_{k-1}$ for all $t\notin\{-1,1\}$. As the functions are polynomial in t, we get

$$D_k^*\beta_s(t)=-\alpha'_{s^<}(t)+\beta'_{s^<}(t)=-\alpha'_{s^<}(t)+\beta'_{s^{\vee<}}(t). \tag{24.21}$$

If $s^\vee=-s$, (24.20) says

$$2qD_k^*\beta_s(t)+((q+1)t+(q-1))(D_k^*\beta_s(t))^\vee \\ =-\alpha'_{2qs^<+((q+1)t+(q-1))s^{<\vee}}(t)-\beta'_{2qs^<+((q+1)t+(q-1))s^{<\vee}}(t).$$

As above, we get

$$D_k^*\beta_s(t)=-\alpha'_{s^<}(t)-\beta'_{s^<}(t)=-\alpha'_{s^<}(t)+\beta'_{s^{\vee<}}(t). \tag{24.22}$$

Joining the two cases, we get from (24.21) and (24.22) that (24.12) holds for any s in $\mathcal{S}_k$. We now apply the operator R' to this identity. By Lemma 20.21, Lemma 24.12, Lemma 24.15 and Lemma 24.14, this gives

$$D_k^*\alpha_s(t)=\alpha'_{s^{<\vee}-(q-1)s^<}(t)-\beta'_{(q+1)ts^<}(t)+\beta'_{s^{\vee<\vee}}(t)$$

and (24.14) follows. □

We conclude the Chapter by the proof of Proposition 24.5, which is a direct consequence of Lemma 24.20.

Proof of Proposition 24.5. Let G be in $\mathcal{H}_{k-1}^{(\mathbb{N})}$ and set $H=D_kG$. We write as usual $\breve{G}=\begin{pmatrix}\breve{G}_0\\ \breve{G}_1\end{pmatrix}$ and $\widehat{H}=\begin{pmatrix}\widehat{H}_0\\ \widehat{H}_1\end{pmatrix}$.

First assume that k is even. By the definition of the spectral transform in (24.8), we have, for t in $\mathbb{R}$ and s in $\mathcal{S}_k$,

$$\langle s,\widehat{H}_0(t)\rangle=\langle q^{-1}s^\vee,\widehat{H}_0(t)^\vee-(q-1)\widehat{H}_0(t)\rangle=q^{-1}\langle\alpha_{s^\vee}(t),H\rangle \\ =q^{-1}\langle\alpha_{s^\vee}(t),D_kG\rangle=q^{-1}\langle D_k^*\alpha_{s^\vee}(t),G\rangle.$$

Lemma 24.20 gives

$$D_k^*\alpha_{s^\vee}(t) = \alpha'_{s^{\vee<\vee}}(t) + \beta'_{s^{\vee\vee<\vee}-(q-1)s^{\vee<\vee}-(q+1)ts^{\vee<}}(t)$$
$$= \alpha'_{s^{\vee<\vee}}(t) + \beta'_{qs^{<\vee}-(q+1)ts^{\vee<}}(t),$$

hence, by the definition of the spectral transform in (24.11),

$$\langle s, \widehat{H}_0(t)\rangle = q^{-1}\langle \alpha'_{s^{\vee<\vee}}(t) + \beta'_{qs^{<\vee}-(q+1)ts^{\vee<}}(t), G\rangle$$
$$= -q^{-1}\langle s^{\vee<\vee}, \breve{G}_1(t)\rangle + \langle s^< - q^{-1}(q+1)ts^{\vee<\vee}, \breve{G}_0(t)\rangle.$$

We get $\widehat{H}_0(t) = -q^{-1}\breve{G}_1(t)^{\vee<\vee} + \breve{G}_0(t)^> - q^{-1}(q+1)t\breve{G}_0(t)^{\vee>\vee}$ as required. In the same way, the definitions (24.8) and (24.11) and Lemma 24.20 give

$$\langle s, \widehat{H}_1(t)\rangle = \langle \beta_s(t), H\rangle = \langle D_k^*\beta_s(t), G\rangle$$
$$= \langle -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t), G\rangle = \langle s^<, \breve{G}_1(t)\rangle + \langle s^{\vee<\vee}, \breve{G}_0(t)\rangle,$$

hence $\widehat{H}_1(t) = \breve{G}_1(t)^> + \breve{G}_0(t)^{\vee>\vee}$.

Assume now that k is odd. We use (24.9) and (24.10) and Lemma 24.20 to get

$$\langle s, \widehat{H}_0(t)\rangle = \langle \beta_{s^\vee}(t), H\rangle = \langle D_k^*\beta_{s^\vee}(t), G\rangle$$
$$= \langle -\alpha'_{s^{\vee<}}(t) + \beta'_{s^<}(t), G\rangle = -\langle s^{\vee<\vee}, \breve{G}_0(t)\rangle - \langle s^<, \breve{G}_1(t)\rangle,$$

hence $\widehat{H}_0(t) = -\breve{G}_0(t)^{\vee>\vee} - \breve{G}_1(t)^>$. Finally, the same arguments give

$$\langle s, \widehat{H}_1(t)\rangle = \langle \alpha_s(t), H\rangle = \langle D_k^*\alpha_s(t), G\rangle$$
$$= \langle \alpha'_{s^{<\vee}-(q-1)s^<}(t) + \beta'_{s^{\vee<\vee}-(q+1)ts^<}(t), G\rangle$$
$$= \langle s^{<\vee\vee} - (q-1)s^{<\vee}, \breve{G}_0(t)\rangle - \langle s^{\vee<\vee} - (q+1)ts^<, \breve{G}_1(t)\rangle$$
$$= q\langle s^<, \breve{G}_0(t)\rangle + \langle (q+1)ts^< - s^{\vee<\vee}, \breve{G}_1(t)\rangle,$$

that is, $\widehat{H}_1(t) = q\breve{G}_0(t)^> + (q+1)t\breve{G}_1(t)^> - \breve{G}_1(t)^{\vee>\vee}$ and the Proposition follows. □

Part 6

The simple transfer operator and the resolvent formula

CHAPTER 25

The simple transfer operator

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the successive orthogonal extensions of p define a scalar product p^∞ on $\overline{\mathcal{D}}(\partial X)$ (see Section 4.5). Recall from Section 20.3 that we write $\mathcal{H}_\infty$ for the space of Γ-invariant ∞-pseudofunctions, that is, the space of Γ-invariant maps $X_1 \to \overline{\mathcal{D}}(\partial X)$. By abuse of notation, we will still denote by p^∞ the scalar product associated with p on $\mathcal{H}_\infty$, which is defined by, for any H, J in $\mathcal{H}_\infty$,

$$p^\infty(H,J) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p^\infty(H_{xy}, J_{xy}). \tag{25.1}$$

Our aim is to establish a Plancherel formula for p^∞ in the spirit of (22.1), (22.8), (23.1) and (23.4). Following the strategy of the proofs of this formulas, we need to establish a formula for the resolvent of a certain self-adjoint operator acting on (the completion of) $\mathcal{H}_\infty$. To state this resolvent formula, we will use a linear operator that is an analogue of the quadratic transfer operator of Section 11.4. The purpose of the present Chapter is to define this operator and give some informations on its spectrum.

25.1. Adjoint operations and simple transfer operator

Our definition will be formulated in the spirit of Section 17.2. In particular, it requires for us to introduce the adjoint operation of direct extension.

Let $k \geq 2$ and p be a k-Euclidean field. Recall from Section 11.4 that, if k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y \in X$, we write $I_{xy}^{\ell-1,\dagger p} : \overline{V}^\ell(x) \to \overline{V}^{\ell-1}(xy)$ for the surjective linear map that is the adjoint with respect to p_x of the natural injective map $I_{xy}^{\ell-1} : \overline{V}^{\ell-1}(xy) \to \overline{V}^\ell(x)$. Then, if H is a k-pseudofunction, we let $H^{<_p}$ be the $(k-1)$-pseudofunction with $H_{xy}^{<_p} = I_{xy}^{\ell-1,\dagger p} H_{xy}$, $x \sim y \in X$. We equip $\mathcal{H}_k$ with the scalar product defined by

$$p(H,J) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_x(H_{xy}, J_{xy}), \quad H, J \in \mathcal{H}_k.$$

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any $x \sim y \in X$, we write $J_{xy}^{\ell,\dagger p} : \overline{V}^{\ell}(xy) \to \overline{V}^{\ell}(x)$ for the surjective linear map that is the adjoint with respect to p_{xy} of the natural injective map $J_{xy}^{\ell} : \overline{V}^{\ell}(x) \to \overline{V}^{\ell}(xy)$. Then, if H is a k-pseudofunction, we let $H^{<_p}$ be the $(k-1)$-pseudofunction with $H_{xy}^{<_p} = J_{xy}^{\ell,\dagger p} H_{xy}$, $x \sim y \in X$. We equip $\mathcal{H}_k$ with the scalar product defined by

$$p(H, J) = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_{xy}(H_{xy}, J_{xy}), \quad H, J \in \mathcal{H}_k.$$

By construction, we always have

$$H^{><_p} = H.$$

By using Lemma 10.11, we directly get the adjointness property.

LEMMA 25.1. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For any H in $\mathcal{H}_k$ and J in $\mathcal{H}_{k-1}$, we have*

$$p(H, J^{>}) = p^{-}(H^{<_p}, J).$$

Note that Lemma 11.10 may be translated into

LEMMA 25.2. *Let $k \geq 2$, p be a k-Euclidean field with orthogonal extension p^+ and H be a k-pseudofunction. We have*

$$H^{>\vee<_{p^+}} = H^{<_p \vee >}.$$

As in Section 4.5, we write p^+ for the orthogonal extension of p. Now, by analogy with Definition 11.14, we set

DEFINITION 25.3. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The simple transfer operator S_p of p is the linear endomorphism of $\mathcal{H}_{k-1}$ defined by

$$S_p H = H^{>\vee<_p \vee} \quad H \in \mathcal{H}_{k-1}.$$

By abuse of notation, we will usually also write $S_p : \mathcal{H}_k \to \mathcal{H}_k$ for the transfer operator of the orthogonal extension p^+ of p. By Lemma 25.2, for H is in $\mathcal{H}_k$, we have

$$S_p H = H^{<_p \vee > \vee}. \tag{25.2}$$

Thus, we get

LEMMA 25.4. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H in $\mathcal{H}_{k-1}$, we have*

$$(S_p H)^{>\vee} = S_p(H^{>\vee}).$$

Note that the $\vee$ operator is self-adjoint with respect to p, as follows directly from the definitions (and from Lemma 10.11). We write $S_p^\dagger$ for the adjoint of S_p with respect to p. We get

LEMMA 25.5. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H in $\mathcal{H}_{k-1}$, we have*

$$S_p^\dagger H = H^{\vee > \vee <_p}.$$

In particular, the operators S_p and $S_p^\dagger$ are conjugated by the $\vee$ operator, that is, we have $S_p(H^\vee) = (S_p^\dagger H)^\vee$.

Recall from Section 20.6 that, when Γ is bipartite, we have introduced the twist operator of pseudofunctions. This operator is then self-adjoint with respect to p. By Lemma 20.22, we obtain

LEMMA 25.6. *Assume Γ is bipartite. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H in $\mathcal{H}_{k-1}$, we have*

$$S_p(H^\wr) = -(S_p H)^\wr \text{ and } S_p^\dagger(H^\wr) = -(S_p^\dagger H)^\wr.$$

25.2. Spectrum of the simple transfer operator

The resolvent formula for Euclidean fields will be written by using rational functions of the simple transfer operator. For this to make sense, we will need some information on the spectrum of S_p.

We start by exhibiting a subspace of $\mathcal{H}_{k-1}$ where the simple transfer operator has a very simple behaviour. Recall from Section 20.2 that we have introduced the notion of a (-1)-pseudofunction and that the space $\mathcal{H}_{-1}$ has dimension 1 or 2.

LEMMA 25.7. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let H be in $\mathcal{H}_{-1}$. Then, we have*

$$S_p\left(H^{>^k}\right) = -H^{\vee >^k} = S_p^\dagger\left(H^{>^k}\right).$$

In other words, the operators S_p and $S_p^\dagger$ preserve the space of (-1)-pseudofunctions and may be seen on that space as the operator that maps a function on X that is constant on neighbours to the opposite function. Note that this implies in particular that, contrarily to the quadratic transfer operator, the simple transfer operator always admits 1 as an eigenvalue.

PROOF. We prove the statement for S_p, the other proof being analogous by using Lemma 25.5. Assume that k is even. Then, by Lemma 20.6, we have

$$S_p\left(H^{>^k}\right) = H^{>^{k+1}\vee <_p \vee} = H^{>\vee >^k <_p \vee} = H^{>\vee >^{k-1}\vee}.$$

As $H^{>}$ is a 0-pseudofunction, by convention, $H^{>\vee} = -H^{>}$. Hence again by Lemma 20.6,

$$S_p\left(H^{>^k}\right) = -H^{>^k\vee} = -H^{\vee>^k}$$

as required. Now, if k is odd, we have

$$S_p\left(H^{>^k}\right) = H^{>^{k+1}\vee<_p\vee} = H^{\vee>^{k+1}<_p\vee} = H^{\vee>^k\vee} = H^{\vee>\vee>^{k-1}}$$

and, again, as $H^{\vee>}$ is a 0-pseudofunction, $H^{\vee>\vee} = -H^{\vee>}$. □

We notice that the spectrum of the simple transfer operator essentially does not depend on the space.

LEMMA 25.8. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the spectrum of S_p in $\mathcal{H}_k$ is the union of $\{0\}$ and the spectrum of S_p in $\mathcal{H}_{k-1}$.*

See Corollary 11.16 for the analogous result for quadratic transfer operators.

PROOF. By Lemma 25.4, for J in $\mathcal{H}_{k-1}$, we have $(S_pJ)^{>\vee} = S_p(J^{>\vee})$. By (25.2), we have $S_p\mathcal{H}_k \subset \mathcal{H}_{k-1}^{>\vee}$. The result follows. □

Now we prove a universal bound on the spectral radius.

PROPOSITION 25.9. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, S_p has spectral radius $< q$ in $\mathcal{H}_{k-1}$.*

The proof of the Proposition essentially relies on the following norm estimate for the adjoint simple transfer operator.

LEMMA 25.10. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H be in $\mathcal{H}_{k-1}$. Then, we have $\left\|S_p^\dagger H\right\|_p \leq q\left\|H\right\|_p$. Besides, if k is even (resp. odd), then equality holds, that is, $\left\|S_p^\dagger H\right\|_p = q\left\|H\right\|_p$ if and only if there exists J in $\mathcal{H}_{k-2}$ (resp. $\mathcal{H}_{k-3}$) with $J^\vee = qJ$ and $H = J^{>\vee}$ (resp. $H = J^{>>}$).*

We have denoted by $\|.\|_p$ the natural norm induced by p on $\mathcal{H}_{k-1}$, that is, $\left\|H\right\|_p = \sqrt{p^-(H,H)}$ for H in $\mathcal{H}_{k-1}$.

PROOF. Recall that, if $j \geq 1$ is odd, the $\vee$ operator is an isometry of $\mathcal{H}_j$ with respect to p, whereas if j is even, it is a self-adjoint operator with spectrum $\{-1, q\}$. In particular, for any H in $\mathcal{H}_{k-1}$, we have $\left\|H^{\vee>\vee}\right\|_p \leq q\left\|H\right\|_p$. As the $<_p$ operator is the adjoint of the isometric embedding given by direct extension $\mathcal{H}_{k-1} \hookrightarrow \mathcal{H}_k$, we get

$$\left\|S_p^\dagger H\right\|_p = \left\|H^{\vee>\vee<_p}\right\|_p \leq \left\|H^{\vee>\vee}\right\|_p \leq q\left\|H\right\|_p. \tag{25.3}$$

If $H^\vee = J^>$ for some J in $\mathcal{H}_{k-2}$, by Lemma 20.6, we have

$$S_p^\dagger H = H^{\vee>\vee<_p} = J^{>>\vee<_p} = J^{\vee>><_p} = J^{\vee>}. \tag{25.4}$$

Thus, if k is even and $J^\vee = qJ$, we get $S_p^\dagger H = qJ^>$ and

$$\left\|S_p^\dagger H\right\|_p = q\left\|J^>\right\|_p = q\left\|J^{>\vee}\right\|_p = q\left\|H\right\|_p.$$

If k is odd and $H = K^{>>}$ for some K in $\mathcal{H}_{k-3}$ with $K^\vee = qK$, by Lemma 20.6, we have

$$S_p^\dagger H = K^{>>\vee>\vee<_p} = qK^{>>>\vee<_p} = qK^{>\vee>><_p} = qK^{>\vee>}$$

and again

$$\left\|S_p^\dagger H\right\|_p = q\left\|K^{>\vee>}\right\|_p = q\left\|K^{>\vee}\right\|_p = q\left\|K^>\right\|_p = q\left\|K^{>>}\right\|_p = q\left\|H\right\|_p.$$

Conversely, assume $\left\|S_p^\dagger H\right\|_p = q\left\|H\right\|_p$, so that the two inequalities in (25.3) must be equalities. We get $\|H^{\vee>\vee<_p}\| = \|H^{\vee>\vee}\|_p$, so that $H^{\vee>\vee}$ belongs to $\mathcal{H}_{k-1}^>$. By Proposition 20.8, there exists J in $\mathcal{H}_{k-2}$ with $H^\vee = J^>$.

Assume that k is even. Then, by using (25.4), we get

$$\left\|J^\vee\right\|_p = \left\|J^{\vee>}\right\|_p = \left\|S_p^\dagger H\right\|_p = q\left\|H\right\|_p = q\left\|H^\vee\right\|_p = q\left\|J^>\right\|_p = q\left\|J\right\|_p.$$

Hence $J^\vee = qJ$ as required.

Assume that k is odd. Then, again by (25.4), we have

$$\left\|H^\vee\right\|_p = \left\|J^>\right\|_p = \left\|J\right\|_p = \left\|J^\vee\right\|_p = \left\|J^{\vee>}\right\|_p = \left\|S_p^\dagger H\right\|_p = q\left\|H\right\|_p.$$

Therefore, we obtain $H^\vee = qH$. As $H^\vee = J^>$, we have $J^{>\vee} = qJ^>$ which, by Proposition 20.8, implies that we may write $J = K^>$ for some K in $\mathcal{H}_{k-3}$ with $K^\vee = qK$. We have $H = q^{-1}J^> = q^{-1}K^{>>}$ and the conclusion follows. □

Proposition 25.9 will follow from Lemma 25.10 by an induction argument which relies on

LEMMA 25.11. *Let $i \geq 0$ and $\ell \geq 1$ be integer. Let H be a 2ℓ-pseudofunction and G be a $2(i+\ell)$-pseudofunction with $G^\vee = qG$. Assume that we have $H^{>^{2i+1}} = G^{>\vee}$. Then, there exists a $2(\ell-1)$-pseudofunction J with $J^\vee = qJ$ and $H = J^{>\vee>}$ and $G = J^{>^{2(i+1)}}$.*

PROOF. We fix $\ell \geq 1$ and we prove this statement by induction on $i \geq 0$.

For $i = 0$, we have $H^> = G^{>\vee}$. Thus, Proposition 20.8 gives a $(2\ell-1)$-pseudofunction K with $G = K^>$ and $H = K^{\vee>}$. As $G^\vee = qG$, we have $K^{>\vee} = qK^>$. Still by Proposition 20.8, there exists a $2(\ell-1)$-pseudofunction J with $J^\vee = qJ$ and $K = J^>$. We get $H = K^{\vee>} = J^{>\vee>}$ and $G = J^{>>}$ and we are done.

Now, assume that $i \geq 1$ and the result is true for $i-1$. As above, by Proposition 20.8, there exists a $(2i+2\ell-1)$-pseudofunction K with $G = K^{>}$ and $H^{>^{2i}} = K^{\vee>}$. As $G^{\vee} = qG$, there exists a $2(i+\ell-1)$-pseudofunction G_1 with $G_1^{\vee} = qG_1$ and $K = G_1^{>}$. We get $H^{>^{2i}} = K^{\vee>} = G_1^{>\vee>}$, hence $H^{>^{2i-1}} = G_1^{>\vee}$. Therefore, the induction assumption says that there exists a $2(\ell-1)$-pseudofunction J with $J^{\vee} = qJ$ and $H = J^{>\vee>}$ and $G_1 = J^{>^{2i}}$. As $G = G_1^{>>}$, we get $G = J^{>^{2(i+1)}}$ and the conclusion follows by induction. □

Proof of Proposition 25.9. Note that Lemma 25.10 directly implies the Proposition when $k = 2, 3$. Indeed from the definitions in Section 20.1, the $\vee$ operator on $\mathcal{H}_0$ is -1. Thus, if $k = 2, 3$, then $S_p^{\dagger}$ has norm $< q$ and the conclusion follows. The proof for arbitrary $k \geq 2$ will follows from the same reason by an induction argument.

If k is even, $k = 2\ell$, $\ell \geq 1$, we claim that, for any $1 \leq i \leq \ell$, if H is in $\mathcal{H}_{k-1}$ and $\left\| \left(S_p^{\dagger}\right)^i H \right\|_p = q^i \left\| H \right\|_p$, then $H = J^{>^{2i-1}\vee}$ for some J in $\mathcal{H}_{k-2i}$ with $J^{\vee} = qJ$. If $i = 1$, this is Lemma 25.10. If $i \leq \ell - 1$ and the claim is true for i, let us show that it is also true for $i+1$. Therefore, we assume that we have

$$\left\| \left(S_p^{\dagger}\right)^{i+1} H \right\|_p = q^{i+1} \left\| H \right\|_p .$$

We set $G = S_p^{\dagger} H$. By the inequality in Lemma 25.10, we have $\left\| \left(S_p^{\dagger}\right)^i G \right\|_p = q^i \left\| G \right\|_p$ and $\left\| G \right\|_p = q \left\| H \right\|_p$. Hence, on one hand, the induction assumption says that we may write $G = K^{>^{2i-1}\vee}$ for some K in $\mathcal{H}_{2(\ell-i)}$ with $K^{\vee} = qK$, whereas Lemma 25.10 says that we may write $H = L^{>\vee}$ for some L in $\mathcal{H}_{2(\ell-1)}$ with $L^{\vee} = qL$. From (25.4), we get $G = S_p^{\dagger} H = L^{\vee>} = qL^{>}$, hence $K^{>^{2i-1}\vee} = qL^{>}$, or equivalently, $K^{>^{2i-1}} = qL^{>\vee}$. As $L^{\vee} = qL$, we can apply Lemma 25.11. Then, we know that there exists J in $\mathcal{H}_{2(\ell-i-1)}$ with $J^{\vee} = qJ$ and $K = qJ^{>\vee>}$ and $L = J^{>^{2i}}$. As $H = L^{>\vee}$, this gives $H = J^{>^{2i+1}\vee}$, which should be proved.

In particular, applying this claim in case $i = \ell$, as the $\vee$ operator is -1 on $\mathcal{H}_0$, we get that $\left(S_p^{\dagger}\right)^{\ell}$ has norm $< q^{\ell}$ on $\mathcal{H}_{k-1}$, which implies the Proposition in the even case.

Assume that k is odd, $k = 2\ell+1$, $\ell \geq 1$. We now claim that for any $1 \leq i \leq \ell$, if H is in $\mathcal{H}_{k-1}$ and $\left\| \left(S_p^{\dagger}\right)^i H \right\|_p = q^i \left\| H \right\|_p$, then $H = J^{>^{2i}}$ for some J in $\mathcal{H}_{k-2i-1}$ with $J^{\vee} = qJ$. If $i = 1$, this is Lemma 25.10. If $i \leq \ell - 1$ and the claim is true for i, let us show that it is also true for

$i+1$. Again, we assume that we have

$$\left\| \left(S_p^\dagger\right)^{i+1} H \right\|_p = q^{i+1} \|H\|_p$$

and we set $G = S_p^\dagger H$, so that, by Lemma 25.10, we have $\left\| \left(S_p^\dagger\right)^i G \right\|_p = q^i \|G\|_p$ and $\|G\|_p = q \|H\|_p$. By the induction assumption, there exists K in $\mathcal{H}_{k-2i-1}$ with $K^\vee = qK$ and $K^{>^{2i}} = G$. By Lemma 25.10, there exists L in $\mathcal{H}_{k-3}$ with $L^\vee = qL$ and $L^{>>} = H$. We have, by Lemma 20.6,

$$K^{>^{2i}} = G = S_p^\dagger H = L^{>>\vee>\vee<_p} = qL^{>>>\vee<_p} = qL^{>\vee>><_p} = qL^{>\vee>},$$

hence $K^{>^{2i-1}} = qL^{>\vee}$. By Lemma 25.11, there exists J in $\mathcal{H}_{k-2i-3}$ with $J^\vee = qJ$ and $K = qJ^{>\vee>}$ and $L = J^{>^{2i}}$. We get $H = J^{>^{2(i+1)}}$ as required.

As above, the case $i = \ell$ implies that $\left(S_p^\dagger\right)^\ell$ has norm $< q^\ell$ on $\mathcal{H}_{k-1}$. The Proposition follows in the odd case. □

25.3. Spectrum in the admissible case

In this Section, we relate the simple transfer operator with the quadratic transfer operator of Chapter 11. We use this relation to show that, when p is admissible, the spectral radius of S_p is not too large. This fact will not be used later.

PROPOSITION 25.12. *Let $k \geq 2$ and p be an admissible Γ-invariant k-Euclidean field. Then S_p has spectral radius $< \sqrt{q}$ in $\mathcal{H}_{k-1}$.*

See Chapter 11 for the definition and equivalent characterizations of admissible Euclidean fields.

The proof of Proposition 25.12 relies on constructions from quadratic algebra which are explained in Appendix C. Recall that, if V is a finite-dimensional real vector space, we have a natural map $v \mapsto v^2, V \to \mathcal{Q}(V^*)$ from V to the space of symmetric bilinear forms on V^*. It is defined by setting $v^2(\varphi, \psi) = \varphi(v)\psi(v)$, for v in V and φ, ψ in V^*. This map satisfies Cauchy-Schwarz inequality:

LEMMA 25.13. *Let V be a finite-dimensional real vector space. Pick $v_1, \ldots, v_r$ in V and $t_1, \ldots, t_r$ in $\mathbb{R}$. We have*

$$\left(\sum_{i=1}^r t_i v_i \right)^2 \leq \left(\sum_{i=1}^r t_i^2 \right) \left(\sum_{i=1}^r v_i^2 \right),$$

meaning that the difference is a non-negative symmetric bilinear form on V^*.

PROOF. Indeed, by the standard Cauchy-Schwartz inequality, for any φ in V^*, one has

$$\left(\sum_{i=1}^{r} t_i \varphi(v_i)\right)^2 \leq \left(\sum_{i=1}^{r} t_i^2\right)\left(\sum_{i=1}^{r} \varphi(v_i)^2\right).$$

□

Now, recall the notion of a pseudokernel from Section 6.1. In Section 6.2, we have defined natural operations on pseudokernels that are analogous to the natural operations on pseudofunctions of Section 20.2. For $k \geq 1$, if H is a k-pseudofunction we define a k-pseudokernel $L = H^2$ as follows: if k is odd (resp. even), $k = 2\ell + 1$ (resp. $k = 2\ell$), $\ell \geq 0$ (resp. $\ell \geq 1$), for any $x \sim y$ in X, the symmetric bilinear form on $V_0^\ell(xy)$ (resp. $V_0^\ell(x)$) associated with L_{xy} is H_{xy}^2. In other words, for a, b in $S^\ell(x)$ (resp. $S^\ell(xy)$), we have

$$L_{xy}(a, b) = (H_{xy}(a) - H_{xy}(b))^2.$$

Besides, note that we have

$$(H^>)^2 = (H^2)^>. \tag{25.5}$$

This definition and Lemma 25.13 directly give

COROLLARY 25.14. *Let $k \geq 1$ and H be a k-pseudofunction. If k is odd, we have $(H^\vee)^2 = (H^2)^\vee$. If k is even, we have $(H^\vee)^2 \leq q(H^2)^\vee$, meaning that the difference is a non-negative pseudokernel.*

In the same way, let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. With the notation of Section 11.8, if H is a k-pseudofunction, we have

$$(H^{<_p})^2 = (H^2)^{<_p}. \tag{25.6}$$

We change the convention of Section 11.8 and we write T_p for the operator that was denoted by T_p^* there. We still call T_p the quadratic transfer operator. By Corollary 11.24, this is the endomorphism of $\mathcal{L}_{k-1}$ defined by, for L in $\mathcal{L}_{k-1}$, $T_p L = L^{>\vee<_p\vee}$.

We can now give the

PROOF OF PROPOSITION 25.12. By Corollary 11.24, for L in $\mathcal{L}_{k-1}$, we have $T_p L = L^{>\vee<_p\vee}$. By Definition 25.3 and Corollary 25.14, we get, for H in $\mathcal{H}_{k-1}$,

$$(S_p H)^2 = (H^{>\vee<_p\vee})^2 \leq q(H^2)^{>\vee<_p\vee} = qT_p(H^2),$$

where we also have used (25.5) and (25.6). Now, iterating this inequality, we get, for any $n \geq 0$, $(q^{-n/2} S_p^n H)^2 \leq T_p^n(H^2)$. As p is admissible,

by Theorem 11.27, T_p has spectral radius < 1, hence $T_p^n(H^2) \xrightarrow[n\to\infty]{} 0$. We get $q^{-n/2}S_p^n H \xrightarrow[n\to\infty]{} 0$ and the result follows. □

25.4. Exceptional eigenvalues

When p is not admissible the spectrum of the simple transfer operator may exit the open disk with radius $\sqrt{q}$ in $\mathbb{C}$, but in a controlled way.

PROPOSITION 25.15. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let $u \in \mathbb{C}$ be a spectral value of S_p in $\mathcal{H}_{k-1}$. If $|u| \geq \sqrt{q}$, then u is real and simple.*

Recall that saying that u is simple amounts to saying that $\mathcal{H}_{k-1} = \ker(S_p - u) \oplus (S_p - u)\mathcal{H}_{k-1}$ or equivalently $\ker(S_p - u)^2 = \ker(S_p - u)$.

The proof of Proposition 25.15 relies on studying the restriction of S_p to a certain subspace of $\mathcal{H}_{k-1}$. The p-orthogonal complement of this subspace will be described thanks to the following Lemmas.

LEMMA 25.16. *Let $k \geq -1$ be odd and H be a k-pseudofunction. The following are equivalent.*
(i) *There exists a (-1)-pseudofunction J with $H = J^{>^{k+1}}$.*
(ii) *One has $H^{>\vee} = -H^{>}$ and $H^{\vee>\vee} = -H^{\vee>}$.*

LEMMA 25.17. *Let $k \geq 0$ be even, H be a k-pseudofunction and ε be in $\{-1, 1\}$. The following are equivalent.*
(i) *There exists a (-1)-pseudofunction J with $H = J^{>^{k+1}}$ and $J^{\vee} = \varepsilon J$.*
(ii) *One has $H^{>\vee} = \varepsilon H^{>}$ and $H^{\vee>\vee} = \varepsilon H^{\vee>}$.*

PROOF. We prove the two results simultaneously.

(i)⇒(ii) If $k = -1$ or $k = 0$, this directly follows from the definitions. Now, Lemma 20.6 implies that if this true for k, this is also true for $k + 2$. The conclusion follows.

(ii)⇒(i) If $k = -1$, there is nothing to prove. If $k = 0$ and H is a 0-pseudofunction such that $H^{>} = \varepsilon H^{>\vee}$, $\varepsilon \in \{-1, 1\}$, by Proposition 20.8, there exists a (-1)-pseudofunction J with $H = J^{>}$. Then, we get

$$J^{>>} = H^{>} = \varepsilon H^{>\vee} = \varepsilon J^{>>\vee} = \varepsilon J^{\vee>>},$$

hence $J = \varepsilon J^{\vee}$ as required.

Let us conclude by a two steps induction argument. Assume $k \geq 1$ and the result holds for $k - 2$. Pick H in $\mathcal{H}_k$ and ε in $\{-1, 1\}$ and assume that we have $H^{>\vee} = \varepsilon H^{>}$ and $H^{\vee>\vee} = \varepsilon H^{\vee>}$. By Proposition 20.8, as $H^{>\vee} = \varepsilon H^{>}$, we can find a $(k-1)$-pseudofunction K with $K^{\vee} = \varepsilon K$ and $H = K^{>}$. In the same way, as $H^{\vee>\vee} = \varepsilon H^{\vee>}$, we can find a $(k-1)$-pseudofunction L with $L^{\vee} = \varepsilon L$ and $H^{\vee} = L^{>}$. Thus,

we have $L^{>} = K^{>\vee}$ and, again by Proposition 20.8, we can find a $(k-2)$-pseudofunction M with $M^{>} = K$ and $M^{\vee>} = L$. The relations $K^{\vee} = \varepsilon K$ and $L^{\vee} = \varepsilon L$ give $M^{>\vee} = \varepsilon M^{>}$ and $M^{\vee>\vee} = \varepsilon M^{\vee>}$ and the conclusion follows by induction. □

Note that, by Lemma 25.7, the spaces of pseudofunctions that appear in Lemma 25.16 and Lemma 25.17 are stable under the action of simple transfer operators and their adjoint operators. We will prove Proposition 25.15 by describing the action of transfer operators on the orthogonal complements of these subspaces.

To this aim, we introduce some notation. If $k \geq 2$ and p is a Γ-invariant k-Euclidean field, for H in $\mathcal{H}_{k,+}$, we set

$$A_p H = H^{<_p>} + H^{<_p>\vee} \text{ and } B_p H = H^{<_p\vee>} + H^{<_p\vee>\vee}.$$

By construction, A_p and B_p map $\mathcal{H}_{k,+}$ in $\mathcal{H}_{k,+}$ We summarize the information that we will need about these operators.

LEMMA 25.18. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The operators $A_p : \mathcal{H}_{k,+} \to \mathcal{H}_{k,+}$ and $B_p : \mathcal{H}_{k,+} \to \mathcal{H}_{k,+}$ are self-adjoint. For H in $\mathcal{H}_{k,+}$, we have*

$$\begin{aligned} p(H, A_p H) &\leq (q+1)p(H,H) && \text{if } k \text{ is even,} \\ &\leq 2p(H,H) && \text{if } k \text{ is odd.} \end{aligned}$$

Equality holds if and only if there exists G in $\mathcal{H}_{k-2,+}$ with $H = G^{>>}$.

PROOF. Assume k is even. For H, J in $\mathcal{H}_{k,+}$, we have

$$\begin{aligned} p(A_p H, J) = p(H^{<_p>}, J) + p(H^{<_p>\vee}, J) &= p(H^{<_p}, J^{<_p}) + p(H^{<_p>}, J^{\vee}) \\ &= (q+1)p(H^{<_p}, J^{<_p}), \end{aligned}$$

which is symmetric in H and J. As the $<_p$ operator is the adjoint of the isometric injection $>$, this gives $p(H, A_p H) \leq (q+1)p(H,H)$ and equality holds if and only if $H = F^{>}$ for some F in $\mathcal{H}_{k-1}$. As $H^{\vee} = qH$, we then get $F^{>\vee} = qF^{>}$, hence, by Proposition 20.8, there exists G in $\mathcal{H}_{k-2}$ with $F = G^{>}$ and $G^{\vee} = qG$, so that $H = G^{>>}$. Besides, we have

$$p(B_p H, J) = (q+1)p(H^{<_p\vee}, J^{<_p}),$$

which is also symmetric in H and J.

The proof in the odd case is analogous. □

The operators A_p and B_p may be used to describe the action of $S_p^{\dagger}$ in a large subspace of $\mathcal{H}_{k-1}$.

LEMMA 25.19. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and J, K be in $\mathcal{H}_{k,+}$. If k is even, we have*

$$S_p^\dagger\left(J^{<_p} + K^{<_p\vee}\right) = \left(B_p J + (A_p - 1)K\right)^{<_p} - J^{<_p\vee}.$$

If k is odd, we have

$$\begin{aligned} &S_p^\dagger\left(J^{<_p} + K^{<_p\vee}\right) \\ &\quad = \left(B_p J + (q(A_p - 1) + (q-1)B_p)K\right)^{<_p} - \left(J + (q-1)K\right)^{<_p\vee}. \end{aligned}$$

This Lemma is directly inspired by the proof of the Ihara trace formula [**28**] (see Section 1.6).

PROOF. This is a direct computation. If k is even, we have, by Lemma 25.5,

$$\begin{aligned} S_p^\dagger\left(J^{<_p} + K^{<_p\vee}\right) &= J^{<_p\vee><_p\vee><_p} + K^{<_p\vee\vee><_p\vee<_p} \\ &= J^{<_p\vee><_p\vee<_p} + K^{<_p><_p\vee<_p} \\ &= (B_p J)^{<_p} - J^{<_p\vee><_p} + (A_p K)^{<_p} - K^{<_p><_p} \\ &= (B_p J)^{<_p} - J^{<_p\vee} + (A_p K)^{<_p} - K^{<_p}, \end{aligned}$$

as required. If k is odd, in the same way,

$$\begin{aligned} S_p^\dagger\left(J^{<_p} + K^{<_p\vee}\right) &= J^{<_p\vee><_p\vee<_p} + K^{<_p\vee\vee><_p\vee<_p} \\ &= J^{<_p\vee><_p\vee<_p} + qK^{<_p><_p\vee<_p} + (q-1)K^{<_p\vee><_p\vee<_p} \\ &= (B_p J)^{<_p} - J^{<_p\vee><_p} + q(A_p K)^{<_p} - qK^{<_p><_p} \\ &\qquad + (q-1)(B_p K)^{<_p} - (q-1)K^{<_p\vee><_p} \\ &= (B_p J)^{<_p} - J^{<_p\vee} + q(A_p K)^{<_p} - qK^{<_p} \\ &\qquad + (q-1)(B_p K)^{<_p} - (q-1)K^{<_p\vee}. \end{aligned}$$

□

The formulas in Lemma 25.19 will give a control of eigenvalues of S_p due to the following phenomenon.

LEMMA 25.20. *Let V be a Euclidean space with scalar product $\langle ., . \rangle$. Let A and B be self-adjoint endomorphisms of V and $r, s \geq 1$ be real numbers with $\langle Av, v\rangle \leq r \|v\|^2$ for any v in V. Let C be the endomorphism of V^2 given in matrix form by*

$$C = \begin{pmatrix} B & s(A-1) + (s-1)B \\ -1 & -(s-1) \end{pmatrix}.$$

Let u be a complex eigenvalue of C. Then, if u is not real or u is not simple, we have $|u|^2 \leq s(r-1)$.

PROOF. As in Section 1.9, let $V_{\mathbb{C}}$ be the complexification of V and denote by $v \mapsto \overline{v}$ the complex conjugation in $V_{\mathbb{C}}$. We write $\langle ., . \rangle$ for the complex symmetric bilinear form of $V_{\mathbb{C}}$ whose restriction to V is the scalar product of V. Thus, the Hermitian form $(v, w) \mapsto \langle \overline{v}, w \rangle$ is a Hermitian scalar product on $V_{\mathbb{C}}$. For v in $V_{\mathbb{C}}$, we still write $\|v\| = \sqrt{\langle \overline{v}, v \rangle}$.

Let u be a complex non real eigenvalue of C. Then, there exists v, w in $V_{\mathbb{C}}$ which are not both 0 such that

$$Bv + s(A-1)w + (s-1)Bw = uv \tag{25.7}$$

$$-v - (s-1)w = uw. \tag{25.8}$$

The second equation gives $v = -(u + s - 1)w$. In particular, $w \neq 0$ and we can assume $\|w\| = 1$. Besides, we can eliminate v from (25.7) to get

$$-uBw + s(A-1)w = -u(u+s-1)w,$$

that is,

$$u^2 w + u(s - 1 - B)w + s(A-1)w = 0. \tag{25.9}$$

By taking the product with $\overline{w}$, as $\|w\| = 1$, we get the scalar equation

$$u^2 + u\langle \overline{w}, (s-1-B)w \rangle + s\langle \overline{w}, (A-1)w \rangle = 0. \tag{25.10}$$

We claim that the coefficients of this equation are real numbers. Indeed, for example, as the endomorphism A is real and self-adjoint, we have,

$$\overline{\langle \overline{w}, Aw \rangle} = \langle w, \overline{Aw} \rangle = \langle w, A\overline{w} \rangle = \langle Aw, \overline{w} \rangle = \langle \overline{w}, Aw \rangle,$$

hence $\langle \overline{w}, Aw \rangle$ and also $\langle \overline{w}, Bw \rangle$ are real numbers. Since by assumption, u is not a real number, the roots of (25.10) are u and $\overline{u}$ and we have

$$|u|^2 = s\langle \overline{w}, (A-1)w \rangle \leq s(r-1)$$

as required.

Now, let u be a real eigenvalue of C that is not simple. Then, we may find v, w, v', w' in V with v, w not both 0 and

$$C \begin{pmatrix} v \\ w \end{pmatrix} = u \begin{pmatrix} v \\ w \end{pmatrix} \text{ and } C \begin{pmatrix} v' \\ w' \end{pmatrix} = u \begin{pmatrix} v' \\ w' \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix}.$$

As above, (25.7) and (25.8) hold and we can assume that w is a unit vector. Besides, we now get

$$\begin{aligned} Bv' + s(A-1)w' + (s-1)Bw' &= uv' + v \\ -v' - (s-1)w' &= uw' + w. \end{aligned} \tag{25.11}$$

Thus, we have

$$v = -(u+s-1)w \text{ and } v' = -(u+s-1)w' - w,$$

so that (25.11) gives

$$-uBw' - Bw + s(A-1)w' = -u((u+s-1)w' + w) - (u+s-1)w,$$

hence

$$u^2w' + u(s-1-B)w' + s(A-1)w' + 2uw + (s-1-B)w = 0.$$

Let w'' be the orthogonal projection of w' on the orthogonal complement of $\mathbb{R}w$ in V. As (25.9) holds, we still have

$$u^2w'' + u(s-1-B)w'' + s(A-1)w'' + 2uw + (s-1-B)w = 0.$$

Since $\langle w, w''\rangle = 0$, by taking the scalar product with w in this equation, we obtain

$$\langle w, (-uB + sA)w''\rangle + 2u + \langle w, (s-1-B)w\rangle = 0. \tag{25.12}$$

As A and B are self-adjoint, by using (25.9), we get

$$\begin{aligned}\langle w, (-uB + sA)w''\rangle &= \langle (-uB + sA)w, w''\rangle \\ &= \langle (s - (s-1)u - u^2)w, w''\rangle = 0.\end{aligned}$$

Therefore, (25.12) yields

$$2u + \langle w, (s-1-B)w\rangle = 0,$$

that is, u is a double root of (25.10). We get

$$u^2 = s\langle w, (A-1)w\rangle \leq s(r-1)$$

as required. □

To deal with certain subtle equality cases in the proof of Proposition 25.15, we will need an enhanced version of Proposition 20.8.

LEMMA 25.21. *Let a, b in $\mathbb{R}$, $k \geq -1$ and H be in $\mathcal{H}_{k,+}$. Assume that we have*

$$aH^{>>} + bH^{>\vee>} = H^{>\vee>\vee}. \tag{25.13}$$

Then, if k is even, we have $H = 0$; if k is odd, we have $H = 0$ or $a - b + 1 = 0$.

PROOF. Assume $k = -1$. Then, as the $\vee$ operator is -1 on 0-pseudofunctions, the equation reads as

$$(a-b)H^{>>} = -H^{>>\vee} = -H^{\vee>>} = -H^{>>},$$

the latter holding by the assumption that H is in $\mathcal{H}_{-1,+}$. In other words we have $(a - b + 1)H = 0$ and the conclusion follows.

If $k = 0$, the conclusion holds trivially since $\mathcal{H}_{0,+} = \{0\}$.

We will prove the general case by a two steps induction. Thus, we assume that $k \geq 1$ and that the result is true for $k-2$. Let H be in $\mathcal{H}_{k,+}$ such that (25.13) holds. Thus, we have

$$(aH^{>} + bH^{>\vee})^{>} = H^{>\vee>\vee}.$$

By Proposition 20.8, there exists J in $\mathcal{H}_k$ with $H^{>\vee} = J^{>}$. Again applying Proposition 20.8, we find K in $\mathcal{H}_{k-1}$ with $H = K^{>}$. As H is in $\mathcal{H}_{k,+}$, a final application of Proposition 20.8 allows to find an element L in $\mathcal{H}_{k-2,+}$ with $H = L^{>>}$. Now, as H satisfies (25.13), Lemma 20.6 gives

$$aL^{>>} + bL^{>\vee>} = L^{>\vee>\vee}$$

and the conclusion follows from the induction assumption. □

We can now use these tools to give the

Proof of Proposition 25.15. We will actually prove the analogous result for the adjoint operator $S_p^\dagger$.

Consider the linear map

$$\Delta_p : \mathcal{H}_{k,+}^2 \to \mathcal{H}_{k-1}, (J, K) \mapsto J^{<_p} + K^{<_p\vee}.$$

By elementary duality arguments, the p-orthogonal complement in $\mathcal{H}_{k-1}$ of the range of Δ_p is the space of H in $\mathcal{H}_{k-1}$ such that $H^{>} + H^{>\vee} = 0$ and $H^{\vee>} + H^{\vee>\vee} = 0$. By Lemma 25.7, Lemma 25.16 and Lemma 25.17, the latter space is stable under $S_p^\dagger$ and the spectrum of $S_p^\dagger$ in this space is contained in $\{-1, 1\}$. Now, Lemma 25.19 says that the range of Δ_p is also stable under $S_p^\dagger$. We will study the spectrum of $S_p^\dagger$ in $\Delta_p \mathcal{H}_{k,+}^2$ by means of Lemma 25.20.

First, assume that k is even. Then, for J, K in $\mathcal{H}_{k,+}$, Lemma 25.19 may be written as

$$S_p^\dagger \Delta_p \begin{pmatrix} J \\ K \end{pmatrix} = \Delta_p \begin{pmatrix} B_p & A_p - 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} J \\ K \end{pmatrix}. \tag{25.14}$$

Thanks to Lemma 25.18, we can apply Lemma 25.20 to the space $\mathcal{H}_{k,+}$ and the operator $C_p = \begin{pmatrix} B_p & A_p - 1 \\ -1 & 0 \end{pmatrix}$ by taking $r = q + 1$ and $s = 1$. This tells us that all complex eigenvalues of C_p in $\mathcal{H}_{k,+}^2$ which have modulus $> \sqrt{q}$ are real and simple. A fortiori, by (25.14), the same holds for the eigenvalues of $S_p^\dagger$ in $\Delta_p \mathcal{H}_{k,+}^2$. To conclude, we must investigate the delicate case of eigenvalues with modulus exactly $\sqrt{q}$. This will require us to have a closer look at the proof of Lemma 25.20.

Thus, let u in $\mathbb{C}$ with $|u| = \sqrt{q}$ and assume by contradiction that u is an eigenvalue of C_p in $\mathcal{H}_{k,+}^2$ and that, if u is real (that is, if $u \in \{-\sqrt{q}, \sqrt{q}\}$), this eigenvalue is not simple. By the proof of Lemma

25.20 (see in particular (25.9) and (25.10)), there exists $K \neq 0$ in the complexification $\mathcal{H}_{k,+,\mathbb{C}}$ of $\mathcal{H}_{k,+}$ with

(25.15)

$$u^2K - uB_pK + (A_p - 1)K = 0 \text{ and } p((A_p - 1)K, \overline{K}) = qp(K, \overline{K}).$$

By Lemma 25.18, the second equation implies that there exists L in $\mathcal{H}_{k-2,+,\mathbb{C}}$ with $K = L^{>>}$. We get by definition,

$$A_pK = L^{>><_p>} + L^{>><_p>\vee} = L^{>>} + L^{>>\vee} = (q+1)L^{>>}$$

and

$$B_pK = L^{>><_p\vee>} + L^{>><_p\vee>\vee} = L^{>\vee>} + L^{>\vee>\vee},$$

so that, by (25.15), we have

$$\left(u + \frac{q}{u}\right) L^{>>} - L^{>\vee>} = L^{>\vee>\vee}.$$

By Lemma 25.21, we get $L = 0$, which contradicts the assumption that $K \neq 0$. The Proposition follows in the even case.

We now assume that k is odd. For J, K in $\mathcal{H}_{k,+}$, Lemma 25.19 now gives

$$S_p^\dagger \Delta_p \begin{pmatrix} J \\ K \end{pmatrix} = \Delta_p \begin{pmatrix} B_p & q(A_p - 1) + (q-1)B_p \\ -1 & -(q-1) \end{pmatrix} \begin{pmatrix} J \\ K \end{pmatrix}. \tag{25.16}$$

By Lemma 25.18, we can apply Lemma 25.20 to the space $\mathcal{H}_{k,+}$ and the operator $C_p = \begin{pmatrix} B_p & q(A_p - 1) + (q-1)B_p \\ -1 & -(q-1) \end{pmatrix}$ by taking $r = 2$ and $s = q$. This again tells us that all complex eigenvalues of C_p in $\mathcal{H}^2_{k,+}$ which have modulus $> \sqrt{q}$ are real and simple. By (25.16), the same holds for the eigenvalues of $S_p^\dagger$ in $\Delta_p \mathcal{H}^2_{k,+}$. As above, we will use the proof of Lemma 25.20 to study the case of eigenvalues with modulus exactly $\sqrt{q}$.

Let u in $\mathbb{C}$ with $|u| = \sqrt{q}$ and assume by contradiction that u is an eigenvalue of C_p in $\mathcal{H}^2_{k,+}$ and that, if u is real, this eigenvalue is not simple. Still by the proof of Lemma 25.20, there exists $K \neq 0$ in $\mathcal{H}_{k,+,\mathbb{C}}$ with

(25.17)

$$u^2K + u(q - 1 - B_p)K + q(A_p - 1)K = 0 \text{ and } p((A_p - 1)K, \overline{K}) = p(K, \overline{K}).$$

By Lemma 25.18, the second equation implies that there exists L in $\mathcal{H}_{k-2,+,\mathbb{C}}$ with $K = L^{>>}$. This gives

$$A_pK = L^{>><_p>} + L^{>><_p>\vee} = L^{>>} + L^{>>\vee} = 2L^{>>}$$

and, as above,

$$B_pK = L^{>><_p\vee>} + L^{>><_p\vee>\vee} = L^{>\vee>} + L^{>\vee>\vee}.$$

Hence, (25.17) yields

$$\left(u + \frac{q}{u} + (q-1)\right) L^{>>} - L^{>\vee>} = L^{>\vee>\vee}.$$

As $K \neq 0$, we have $L \neq 0$, so that, by Lemma 25.21, we get

$$u + \frac{q}{u} + q + 1 = 0.$$

The two roots of this equation are $-q$ and -1. This contradicts the assumption that u has modulus $\sqrt{q}$ and the Proposition follows in the odd case. □

25.5. Exceptional quadratic forms

The proof of Proposition 25.15 shows a particular structure of the eigenspaces of $S_p^\dagger$ associated to eigenvalues with modulus $> \sqrt{q}$. This will allow us to prove a positivity result on those spaces, which will later play a role in the statement of the Plancherel formula for Euclidean fields.

PROPOSITION 25.22. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let u in $\mathbb{R}$ be an eigenvalue of $S_p^\dagger$ with $|u| > \sqrt{q}$. Then, the symmetric bilinear form $(H, J) \mapsto p(H, J^\vee)$ is anisotropic on the eigenspace*

$$\{H \in \mathcal{H}_{k-1} | S_p^\dagger H = uH\}.$$

More precisely, if k is even, it is positive definite if $u > \sqrt{q}$ and negative definite if $u < \sqrt{q}$. If k is odd, it is always positive definite.

Recall that a symmetric bilinear form p on a real vector space V is said to be anisotropic, or equivalently definite, if, for any $v \neq 0$ in V, one has $p(v, v) \neq 0$. In that case, p is either positive definite or negative definite.

PROOF. Let H be in $\mathcal{H}_{k-1}$ with $S_p^\dagger H = uH$. Assume k is even. Then, by the proof of Proposition 25.15 and by Lemma 25.19, we can find J, K in $\mathcal{H}_{k,+}$ with

$$\begin{aligned} J^{<_p} + K^{<_p\vee} &= H \\ B_p J + (A_p - 1)K &= uJ \\ -J &= uK. \end{aligned}$$

We get

$$\begin{aligned} p(H,H^\vee) &= p(J^{<_p},J^{<_p\vee}) + 2p(J^{<_p},K^{<_p}) + p(K^{<_p},K^{<_p\vee}) \\ &= (u^2+1)p(K^{<_p},K^{<_p\vee}) - 2up(K^{<_p},K^{<_p}) \\ &= \frac{u^2+1}{q+1}p(K,B_pK) - \frac{2u}{q+1}p(K,A_pK). \end{aligned}$$

Note that we have $B_pJ = (1-u^2)K - A_pK$, hence $B_pK = (u-u^{-1})K + u^{-1}A_pK$. We obtain

$$(q+1)p(H,H^\vee) = (u-u^{-1})((u^2+1)p(K,K) - p(K,A_pK)).$$

By assumption, we have $u^2 > q$ and by Lemma 25.18, we have $p(K,A_pK) \leq (q+1)p(K,K)$. The conclusion follows.

Assume now that k is odd. Again by Proposition 25.15 and Lemma 25.19, we get J,K in $\mathcal{H}_{k,+}$ with

$$J^{<_p} + K^{<_p\vee} = H$$

$$B_pJ + q(A_p-1)K + (q-1)B_pK = uJ \tag{25.18}$$

$$-J - (q-1)K = uK \tag{25.19}$$

We have $J = -(u+q-1)K$, hence

$$H = -(u+q-1)K^{<_p} + K^{<_p\vee} \text{ and } H^\vee = qK^{<_p} - uK^{<_p\vee}.$$

We obtain

$$\begin{aligned} p(H,H^\vee) = -q(u+q-1)p(K^{<_p},K^{<_p}) + (u(u+q-1)+q)p(K^{<_p},K^{<_p\vee}) \\ - up(K^{<_p},K^{<_p\vee\vee}). \end{aligned}$$

As $K^{<_p\vee\vee} = qK^{<_p} + (q-1)K^{<_p\vee}$, this gives

$$\begin{aligned} p(H,H^\vee) &= -q(2u+q-1)p(K^{<_p},K^{<_p}) + (u^2+q)p(K^{<_p},K^{<_p\vee}) \\ &= -\frac{q(2u+q-1)}{2}p(K,A_pK) + \frac{u^2+q}{2}p(K,B_pK). \end{aligned}$$

Since (25.18) and (25.19) yield

$$B_pK = (u+q-1-qu^{-1})K + qu^{-1}A_pK,$$

we finally get

$$\begin{aligned} &2p(H,H^\vee) = \\ &(u^2+q)(u+q-1-qu^{-1})p(K,K) + ((u^2+q)qu^{-1} - q(2u+q-1))p(K,A_pK) \\ &\qquad = (u+q-1-qu^{-1})((u^2+q)p(K,K) - qp(K,A_pK)). \end{aligned}$$

By Proposition 25.9, as u is an eigenvalue of $S_p^\dagger$, we must have $-q < u < -\sqrt{q}$ or $\sqrt{q} < u < q$. Since $u+q-1-qu^{-1} = u^{-1}(u-1)(u+q)$

this number is positive for all those values of u. Now, by Lemma 25.18, we have $p(K, A_p K) \leq 2p(K, K)$ and the conclusion follows. $\square$

CHAPTER 26

A resolvent formula for Euclidean fields

Our objective is still to give a Plancherel formula for $\mathcal{H}_\infty$, in the spirit of (22.1), (22.8), (23.1) and (23.4). To this aim, we will start by establishing a resolvent formula. Let us be more precise.

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. We have associated to p a scalar product on $\mathcal{H}_\infty$ by the natural formula (25.1). Let R and S be the operators acting on $\mathcal{H}_\infty$ which are defined in Section 20.5. One easily shows, by using Lemma 10.11, that the operators R and S are bounded and self-adjoint with respect to p. As usual, we set $P = \frac{1}{q+1}(RS+SR-(q-1)S)$. We still denote by p^∞ the scalar product of the completion $\mathcal{H}^p_\infty$ of $\mathcal{H}_\infty$ with respect to p^∞. We also denote by p^∞ the natural complex symmetric bilinear form on the complexification $\mathcal{H}^p_{\infty,\mathbb{C}}$ of this completion. Then, for H in $\mathcal{H}_\infty$ and t in the upper half-plane $\mathbb{H}$, the complex number $p^\infty(H, (P-t)^{-1}H)$ is well-defined. In this Chapter, we establish the following formulas for this number, which only involve operations in finite-dimensional spaces.

PROPOSITION 26.1. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field, u be in $\mathbb{H}_q$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Fix H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$.*

If k is even, we have

$$p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty}) = \frac{q+1}{q}\frac{u}{1-u^2}p(H, (qu+S_p)(u-S_p)^{-1}H)$$

$$p^\infty(J^{>^\infty}, (P-t)^{-1}J^{>^\infty}) = (q+1)\frac{u}{q^2-u^2}p(J, (u+qS_p)(u-S_p)^{-1}J)$$

$$p^\infty(H^{+>^\infty}, (P-t)^{-1}J^{>^\infty}) = -\frac{q+1}{q}p(H, S_p(u-S_p)^{-1}J)$$

$$p^\infty(J^{+>^\infty}, (P-t)^{-1}H^{>^\infty}) = (q+1)p(J, S_p(u-S_p)^{-1}H).$$

If k is odd, we have

$$p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty}) = \frac{(q+1)u}{(q+u)(1-u)} p(H, (u+S_p)(u-S_p)^{-1}H)$$
$$p^\infty(J^{>^\infty}, (P-t)^{-1}J^{>^\infty}) = \frac{(q+1)u}{(q-u)(1+u)} p(J, (u+S_p)(u-S_p)^{-1}J)$$
$$p^\infty(H^{+>^\infty}, (P-t)^{-1}J^{>^\infty}) = -(q+1)p(H, S_p(u-S_p)^{-1}J)$$
$$p^\infty(J^{+>^\infty}, (P-t)^{-1}H^{>^\infty}) = (q+1)p(J, S_p(u-S_p)^{-1}H).$$

Recall from Section 21.3 that $\mathbb{H}_q$ is the set of u in $\mathbb{C}$ with $\Im u > 0$ and $|u| > \sqrt{q}$. In particular, the formulas make sense as Proposition 25.15 implies that S_p has no eigenvalue in $\mathbb{H}_q$.

26.1. A formula for inverses

The proof of Proposition 26.1 will rely on an explicit construction of the vector $(P-t)^{-1}H^{>^\infty}$ in the completion of $\mathcal{H}_\infty$ with respect to p. This construction will be an analogue of the ones provided for the model operators. It will require us to show that certain series converge in $\mathcal{H}^p_{\infty,\mathbb{C}}$, which will be ensured by the following

LEMMA 26.2. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H be in $\mathcal{H}_k$. Then, for every $i \geq 0$, we have*

$$p(H^{+^{2i}}, H^{+^{2i}}) \leq q^{2i} p(H, H).$$

For brevity, we have denoted by p the scalar product obtained from p on $\mathcal{H}_{k+2i}$ by successive orthogonal extensions.

PROOF. Recall the notation of Sections 20.3 and 20.5. In particular, by Lemma 20.15, we have, if k is even,

$$H^{++>^\infty} = H^{>\vee>\vee>^\infty} = R(H^{>\vee>^\infty}) = RS(H^{>^\infty}).$$

If k is odd, we have $H^{++>^\infty} = SR(H^{>^\infty})$. By iterating, we get, for $i \geq 0$,

$$\begin{aligned} H^{+^{2i}>^\infty} &= (RS)^i(H^{>^\infty}) && \text{if } k \text{ is even} \\ &= (SR)^i(H^{>^\infty}) && \text{if } k \text{ is odd.} \end{aligned}$$

To conclude, recall that the operators R and S are self-adjoint in $\mathcal{H}_\infty$ with respect to p. As $S^2 = 1$, S is an isometry, whereas, as the eigenvalues of R are q and -1, R has norm q. Thus, both RS and SR have norm $\leq q$ in $\mathcal{H}_\infty$. The Lemma follows. □

Thus, we can ensure that the series defined in the following Lemma converge.

LEMMA 26.3. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H be in $\mathcal{H}_k$. Fix u in $\mathbb{C}$ with $|u| > q$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, for H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$, the vectors $(P-t)^{-1}H^{>^\infty}$ and $(P-t)^{-1}J^{>^\infty}$ may be defined by the following absolutely converging series in $\mathcal{H}^p_{\infty,\mathbb{C}}$. If k is even,*

$$(P-t)^{-1}H^{>^\infty} = \frac{q+1}{1-u^2}\sum_{i=0}^{\infty} u^{-i}(uH^{+^{2i}>^\infty} + H^{+^{2i+1}>^\infty})$$

$$(P-t)^{-1}J^{>^\infty} = \frac{q+1}{q^2-u^2}\sum_{i=0}^{\infty} u^{-i}(uJ^{+^{2i}>^\infty} - qJ^{+^{2i+1}>^\infty}).$$

If k is odd,

$$(P-t)^{-1}H^{>^\infty} = \frac{q+1}{(q+u)(1-u)}\sum_{i=0}^{\infty} u^{-i}(uH^{+^{2i}>^\infty} + H^{+^{2i+1}>^\infty})$$

$$(P-t)^{-1}J^{>^\infty} = \frac{q+1}{(q-u)(1+u)}\sum_{i=0}^{\infty} u^{-i}(uJ^{+^{2i}>^\infty} - J^{+^{2i+1}>^\infty}).$$

PROOF. The convergence of the series is a consequence of Lemma 26.2. The fact that they define $(P-t)^{-1}H^{>^\infty}$ follows from Lemma 20.18 and the analogous results for the model operators in Lemmas 22.9, 22.19, 23.9 and 23.18. □

26.2. Scalar product with large extensions

Now that we have constructed $(P-t)^{-1}H^{>^\infty}$ in Lemma 26.3, it remains to evaluate the quantity $p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty})$. This will use the following Lemma which relates this computation with the simple transfer operator.

LEMMA 26.4. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H, J be in $\mathcal{H}_k$. For every $i \geq 0$, we have*

$$p(H^{+^{2i}}, J^{>^{2i}}) = p(S_p^i H, J)$$

$$p(H^{+^{2i+1}}, J^{+>^{2i}}) = p(S_p^i H, J) \qquad \text{if } k \text{ is even}$$

$$p(H^{+^{2i+1}}, J^{+>^{2i}} - (q-1)J^{>^{2i+1}}) = qp(S_p^i H, J) \qquad \text{if } k \text{ is odd.}$$

PROOF. Let us prove the first formula. This follows from the properties of the operations on pseudofunctions and an induction argument. Indeed, assume the formula holds for $i \geq 0$ and let us still write S_p for the simple transfer operator of the double orthogonal extension p^{++}.

Then, we have, by using Lemma 25.4,

$$p(H^{+^{2i+2}}, J^{>^{2i+2}}) = p(S_p^i(H^{>\vee>\vee}), J^{>>}) = p((S_p^i H)^{>\vee>\vee}, J^{>>})$$
$$= p(S_p^i H, J^{>>\vee<_p\vee<_p}) = p(S_p^i H, J^{\vee>><_p\vee<_p}) = p(S_p^i H, J^{\vee>\vee<_p})$$
$$= p(S_p^i H, S_p^\dagger J) = p(S_p^{i+1} H, J).$$

Now we prove the other two formulas. Assume k is even. By the first formula, we have

$$p(H^{+^{2i+1}}, J^{+>^{2i}}) = p(S_p^i(H^{>\vee}), J^{>\vee}) = p((S_p^i H)^{>\vee}, J^{>\vee})$$
$$= p(S_p^i H, J^{>\vee\vee<_p}) = p(S_p^i H, J).$$

In the same way, if k is odd,

$$p(H^{+^{2i+1}}, J^{+>^{2i}} - (q-1)J^{>^{2i+1}}) = p(S_p^i H, (J^{>\vee} - (q-1)J^{>})^{\vee<_p})$$
$$= qp(S_p^i H, J).$$

□

When J is an eigenvector of the $\vee$ operator, we can also get a formula for other extensions.

COROLLARY 26.5. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H, J be in $\mathcal{H}_k$. For every $i \geq 0$, we have,*

$$p(H^{+^{2i+1}}, J^{>^{2i+1}}) = \frac{1}{q} p(S_p^{i+1} H, J) \qquad k \text{ even, } J^\vee = qJ$$
$$= p(S_p^{i+1} H, J) \qquad k \text{ odd, } J^\vee = J$$
$$= -p(S_p^{i+1} H, J) \qquad J^\vee = -J$$

and also

$$p(H^{+^{2i>}}, J^{+>^{2i}}) = \frac{1}{q} p(S_p^{i+1} H, J) \qquad k \text{ even, } J^\vee = qJ$$
$$= p(S_p^{i+1} H, J) \qquad k \text{ odd, } J^\vee = J$$
$$= -p(S_p^{i+1} H, J) \qquad J^\vee = -J.$$

PROOF. By Lemma 25.4 and Lemma 26.4, we have

$$p(H^{+^{2i+1}}, J^{>^{2i+1}}) = p(S_p^i(H^{>\vee}), J^{>}) = p((S_p^i H)^{>\vee}, J^{>})$$
$$= p((S_p^i H)^{>\vee<_p}, J).$$

The first set of formulas easily follows.

Let us study the second set. If $i = 0$, we have

$$p(H^{>}, J^{+}) = p(H^{>}, J^{>\vee}) = p(H^{>\vee<_p}, J)$$

and the conclusion follows. If $i \geq 1$, by Lemma 25.2, Lemma 25.4 and Lemma 26.4, we have

$$\begin{aligned} p(H^{+^{2i}>}, J^{+>^{2i}}) &= p(H^{+^{2i}}, J^{+>^{2i-1}}) \\ &= p(S_p^{i-1}(H^{>\vee>\vee}), J^{>\vee>}) \\ &= p((S_p^{i-1}H)^{>\vee>\vee}, J^{>\vee>}) \\ &= p((S_p^{i-1}H)^{>\vee>\vee<_p\vee<_p}, J) \\ &= p((S_p^{i-1}H)^{>\vee<_p\vee>\vee<_p}, J) = p((S_p^i H)^{>\vee<_p}, J). \end{aligned}$$

The second set of formulas follows. □

26.3. Geometric series and the resolvent formula

We can now conclude the proof of the resolvent formula which essentially relies on summing a geometric series.

Proof of Proposition 26.1. We assume that k is even and we compute $p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty})$. Take u in $\mathbb{C}$ with $|u| > q$ and set as usual $t = \frac{1}{q+1}(u + \frac{q}{u})$. By Lemma 26.3, Lemma 26.4 and Corollary 26.5, we get

$$p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty}) = \frac{q+1}{1-u^2}\sum_{i=0}^{\infty} u^{-i}\left(up(H, S_p^i H) + \frac{1}{q}p(H, S_p^{i+1}H)\right).$$

By Lemma 25.8 and Proposition 25.9, S_p has spectral radius $< q$ in $\mathcal{H}_k$. Thus, the series $\sum_{i\geq 0} u^{-i}S_p^i$ converges absolutely in the space of endomorphisms of $\mathcal{H}_{k,\mathbb{C}}$. We get

$$\begin{aligned} \sum_{i=0}^{\infty} u^{-i}\left(uS_p^i + \frac{1}{q}S_p^{i+1}\right) &= \frac{1}{q}(qu + S_p)(1 - u^{-1}S_p)^{-1} \\ &= \frac{u}{q}(qu + S_p)(u - S_p)^{-1} \end{aligned}$$

hence,

$$p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty}) = \frac{q+1}{q}\frac{u}{1-u^2}p(H, (qu+S_p)(u-S_p)^{-1}H).$$

We have shown that the two hand-sides of the first formula in Proposition 26.1 are equal for $|u| > q$. Now, as P is self-adjoint, the left hand-side is a holomorphic function of t as t varies in $\mathbb{H}$. Therefore, by Lemma 21.6, it is a holomorphic function of u as u varies in $\mathbb{H}_q$. By Proposition 25.15, the right hand-side is a holomorphic function of u as u varies in $\mathbb{H}_q$. Hence, by analytic continuation, the equality

holds for every u in $\mathbb{H}_q$. The computation of $p^\infty(J^{>^\infty}, (P-t)^{-1}J^{>^\infty})$ is analogous.

Still assume k is even and let us now compute the complex number $p^\infty(H^{+>^\infty}, (P-t)^{-1}J^{>^\infty})$. As above, we let t, u be in $\mathbb{C}$ with $|u| > q$ and $q + u^2 = (q+1)tu$. Again by Lemma 26.3, Lemma 26.4 and Corollary 26.5, we have

$$p^\infty(H^{+>^\infty}, (P-t)^{-1}J^{>^\infty}) = \frac{q+1}{q^2-u^2}\sum_{i=0}^{\infty} u^{-i}\left(\frac{u}{q}p(H, S_p^{i+1}J) - qp(H, S_p^i J)\right).$$

As $p(H, J) = 0$, we get

$$\begin{aligned} p^\infty(H^{+>^\infty}, (P-t)^{-1}J^{>^\infty}) &= \frac{q+1}{q^2-u^2}\sum_{i=0}^{\infty} u^{-i}\left(\frac{u}{q} - \frac{q}{u}\right)p(H, S_p^{i+1}J) \\ &= -\frac{q+1}{q}\sum_{i=1}^{\infty} u^{-i}p(H, S_p^i J) \\ &= -\frac{q+1}{q}p(H, S_p(u - S_p)^{-1}J) \end{aligned}$$

and we conclude as in the first case.

The remaing cases can be dealt with similarly. □

Later on, we will deduce a Plancherel formula from Proposition 26.1. For now, we can already say that it allows to control the spectrum of the operator P with respect to the scalar product associated to p on $\mathcal{H}_\infty$. We denote by Σ_p the finite set

$$\Sigma_p = \left\{ \frac{1}{q+1}\left(u + \frac{q}{u}\right) \middle| \, u \in \mathbb{R}, |u| > \sqrt{q}, \ker(S_p - u) \neq \{0\} \right\}. \tag{26.1}$$

We call Σ_p the exceptional spectrum. Note that, if Γ is bipartite, we have $\Sigma_p = -\Sigma_p$ by Lemma 25.6.

Corollary 26.6. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the spectrum of P in the completion $\mathcal{H}_\infty^p$ of $\mathcal{H}_\infty$ with respect to p is contained in the set*

$$\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}.$$

We will prove later in Corollary 31.2 that equality actually holds. Recall from Section 21.3 that $\mathcal{I}_q$ stands for the critical interval $\mathcal{I}_q = [-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$.

PROOF. By Lemma 21.6, the rational function $u \mapsto \frac{1}{q+1}(u + \frac{q}{u})$ induces a biholomorphism from the set $\{u \in \mathbb{C} || u| > \sqrt{q}\}$ onto $\mathbb{C} \smallsetminus \mathcal{I}_q$. Thus, Lemma 25.8, Proposition 25.15 and Proposition 26.1 imply that for any $j \geq k$ and any H in $\mathcal{H}_j$, the resolvent function $t \mapsto p^\infty(H^{>^\infty}, (P-t)^{-1}H^{>^\infty})$ admits an analytic continuation to the set $\mathbb{C} \smallsetminus (\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\})$. By standard properties of self-adjoint operators, this implies the result. □

In the sequel, we will build three families of nonnegative symmetric bilinear forms on $\mathcal{H}_k^2$ parametrized by each of the three sets $\mathcal{I}_q$, Σ_p and $\{-1, 1\}$. These families will allow us to write a Plancherel formula for Euclidean fields in Chapter 31.

Part 7

The spectral theorem for Euclidean fields

CHAPTER 27

u-radical pseudofields and spectral quadratic forms

In this Chapter, we introduce new algebraic objects that will be needed in order to construct the above mentioned families of symmetric bilinear forms.

27.1. u-radical pseudofields

We start by introducing new subspaces of the space of simple pseudofields. They are defined by an equation that will later turn out to be related to the formulas in Proposition 24.5. Since this relation relies on the spectral parametrization $u \mapsto t = \frac{1}{q+1}(u + \frac{q}{u})$ of Section 21.3, we will need to work with complex simple pseudofields: the space of complex simple pseudofields is the complexification of the space of simple pseudofields from section 24.2.

For $k \geq 1$, if k is even, $k = 2\ell$, $\ell \geq 1$, a complex k-simple pseudofield may be seen as a family $(s_{xy})_{(x,y)\in X_1}$ where, for $x \sim y$ in X, s_{xy} is a function $S^\ell(x) \to \mathbb{C}$ the sum of whose values is 0. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, a complex k-simple pseudofield may be seen as a family $(s_{xy})_{(x,y)\in X_1}$ where, for $x \sim y$ in X, s_{xy} is a function $S^\ell(xy) \to \mathbb{C}$ the sum of whose values is 0. For $k = 0$, a complex 0-simple pseudofield may be seen as a function $X \to \mathbb{C}$.

We extend to complex simple pseudofields all the operations defined for real pseudofields in Section 24.3.

Definition 27.1. Let $k \geq 1$, u be in $\mathbb{C}^*$, s be a complex k-simple pseudofield. We say that s is u-radical if $s^{\vee<\vee} = us^<$.

Remark 27.2. This definition is motivated by analogous notions previously introduced for quadratic pseudofields (see Definition 13.1 and Definition 17.12). It will turn out later that it is related to the formula that appears in Proposition 24.5 (see Lemma 28.3 below).

For $k \geq 1$, the space of Γ-invariant u-radical complex k-simple pseudofields is denoted by $\mathcal{S}^u_{k,\mathbb{C}}$. If u is real, the space of Γ-invariant u-radical k-simple pseudofields is denoted by $\mathcal{S}^u_k$.

LEMMA 27.3. *Let $k \geq 2$, u be be in $\mathbb{C}^*$ and s be a u-radical complex k-simple pseudofield. Then $s^{\vee<}$ is u-radical. The map $s \mapsto s^{\vee<}$ sends $\mathcal{S}^u_{k,\mathbb{C}}$ onto $\mathcal{S}^u_{k-1,\mathbb{C}}$.*

PROOF. Let s be as in the statement. By Lemma 24.9, we have

$$s^{\vee<\vee<\vee} = us^{<<\vee} = us^{\vee<<},$$

hence $s^{\vee<}$ is u-radical. Now, consider the linear map

$$s \mapsto s^{\vee<}, \mathcal{S}^u_{k,\mathbb{C}} \to \mathcal{S}^u_{k-1,\mathbb{C}}.$$

To show that it is surjective, we will show that the adjoint map is injective. Let H be in $\mathcal{H}_{k-1,\mathbb{C}}$ and assume that there exists J in $\mathcal{H}_{k-1,\mathbb{C}}$ with $H^{>\vee} = J^{\vee>\vee} - uJ^{>}$. Then, Proposition 20.8 says that there exists K in $\mathcal{H}_{k-2,\mathbb{C}}$ with $J^{\vee} - H = K^{>}$ and $uJ = K^{\vee>}$. We get $H = J^{\vee} - K^{>} = u^{-1}K^{\vee>\vee} - K^{>}$. Thus, we have shown that if $H^{>\vee}$ belongs to the orthogonal subspace to $\mathcal{S}^u_{k,\mathbb{C}}$ in $\mathcal{H}_{k,\mathbb{C}}$, then H belongs to the orthogonal subspace to $\mathcal{S}^u_{k-1,\mathbb{C}}$ in $\mathcal{H}_{k-1,\mathbb{C}}$, which is the desired statement. □

27.2. The u-opposition map

We describe a relation between u-radical pseudofiels and $\frac{q}{u}$-radical pseudofields.

DEFINITION 27.4. Let $k \geq 0$, u be in $\mathbb{C}^*$ and s be complex k-simple pseudofied. We define the u-opposite $I_u s$ of s as follows. If k is even, we set

$$I_u s = (q - u^2)s^{\vee} + u^2(q-1)s.$$

If k is odd, we set

$$I_u s = (q - u^2)s^{\vee} + u(q-1)s.$$

LEMMA 27.5. *Let u in $\mathbb{C}^*$, $k \geq 1$ and s be a u-radical complex k-simple pseudofield. Then $I_u s$ is $\frac{q}{u}$-radical.*

PROOF. This is a straightforward computation. If k is even, we get

$$\begin{aligned}
(I_u s)^{\vee<\vee} &= (q-u^2)s^{\vee\vee<\vee} + u^2(q-1)s^{\vee<\vee} \\
&= q(q-u^2)s^{<\vee} + (q-1)(q-u^2)s^{\vee<\vee} + u^2(q-1)s^{\vee<\vee} \\
&= q(q-u^2)s^{<\vee} + q(q-1)s^{\vee<\vee} \\
&= \frac{q}{u}(q-u^2)s^{\vee<} + q(q-1)us^{<} = \frac{q}{u}(I_u s)^{<}.
\end{aligned}$$

In the same way, if k is odd,

$$\begin{aligned}(I_u s)^{\vee<\vee} &= (q-u^2)s^{\vee\vee<\vee} + u(q-1)s^{\vee<\vee}\\ &= (q-u^2)s^{<\vee} + u(q-1)s^{\vee<\vee}\\ &= \frac{1}{u}(q-u^2)s^{\vee<\vee\vee} + u(q-1)s^{\vee<\vee}\\ &= \frac{q}{u}(q-u^2)s^{\vee<} + \left(\frac{q-1}{u}(q-u^2) + u(q-1)\right)s^{\vee<\vee}\\ &= \frac{q}{u}(q-u^2)s^{\vee<} + \frac{q}{u}(q-1)s^{\vee<\vee}\\ &= \frac{q}{u}(q-u^2)s^{\vee<} + q(q-1)s^{<} = \frac{q}{u}(I_u s)^{<}.\end{aligned}$$

□

When u is not one of the special values $-q$, -1, 1 or q, the opposition maps I_u and $I_{\frac{q}{u}}$ are essentially inverse to each other.

LEMMA 27.6. *Let u be in $\mathbb{C}^*$, $k \geq 0$ and s be a complex k-simple pseudofield. We have*

$$\begin{aligned}I_{\frac{q}{u}} I_u s &= \frac{q^2}{u^2}(q^2-u^2)(1-u^2) = (q+1)^2q^2(1-t^2) \qquad \text{if } k \text{ is even}\\ &= \frac{q}{u^2}(q^2-u^2)(1-u^2) = (q+1)^2q(1-t^2) \qquad \text{if } k \text{ is odd,}\end{aligned}$$

where $t = \frac{1}{q+1}(u + \frac{q}{u})$.

PROOF. If k is even, we have

$$\begin{aligned}I_{\frac{q}{u}} I_u s &= \left(q - \frac{q^2}{u^2}\right)((q-u^2)s^{\vee\vee} + u^2(q-1)s^{\vee})\\ &\qquad + \frac{q^2}{u^2}(q-1)((q-u^2)s^{\vee} + u^2(q-1)s)\\ &= \frac{q}{u^2}(u^2-q)(q(q-u^2)s + q(q-1)s^{\vee})\\ &\qquad + \frac{q^2}{u^2}(q-1)(q-u^2)s^{\vee} + q^2(q-1)^2 s\\ &= -\frac{q^2}{u^2}(q-u^2)^2 s + q^2(q-1)^2 s\end{aligned}$$

and

$$\begin{aligned}u^2(q-1)^2 - (q-u^2)^2 &= (u(q-1) - q + u^2)(u(q-1) + q - u^2)\\ &= (q+u)(u-1)(q-u)(u+1)\\ &= (q^2-u^2)(1-u^2) = \frac{(q+1)^2}{u^2}(1-t^2).\end{aligned} \tag{27.1}$$

The conclusion follows.

If k is odd, we get

$$\begin{aligned} I_{\frac{q}{u}} I_u s &= \left(q - \frac{q^2}{u^2}\right)((q-u^2)s^{\vee\vee} + u(q-1)s^{\vee}) \\ &\quad + \frac{q}{u}(q-1)((q-u^2)s^{\vee} + u(q-1)s) \\ &= -\frac{q}{u^2}(q-u^2)^2 s + \frac{q}{u}(u^2-q)(q-1)s^{\vee} \\ &\quad + \frac{q}{u}(q-1)(q-u^2)s^{\vee} + q(q-1)^2 s \\ &= q(q-1)^2 s - \frac{q}{u^2}(q-u^2)^2 s. \end{aligned}$$

Again, the conclusion follows from (27.1). □

COROLLARY 27.7. *Let u be in $\mathbb{C} \setminus \{-q, -1, 0, 1, q\}$. For $k \geq 1$, the u-opposition map I_u induces a linear isomorphism from the space of u-radical complex k-simple pseudofields onto the space of $\frac{q}{u}$-radical complex k-simple pseudofields.*

The opposition map is also compatible with restriction of u-radical pseudofields.

LEMMA 27.8. *Let u in $\mathbb{C}^*$, $k \geq 1$ and s be a u-radical complex k-simple pseudofield. If k is even, we have*

$$(I_u s)^{\vee <} = \frac{q}{u} I_u(s^{\vee <}).$$

If k is odd, we have

$$(I_u s)^{\vee <} = \frac{1}{u} I_u(s^{\vee <}).$$

PROOF. Again, this is a straightforward computation. For k even, we get

$$\begin{aligned} (I_u s)^{\vee <} &= (q-u^2)s^{\vee\vee <} + u^2(q-1)s^{\vee <} \\ &= q(q-u^2)s^{<} + q(q-1)s^{\vee <} \\ &= \frac{q}{u}(q-u^2)s^{\vee <\vee} + q(q-1)s^{\vee <} = \frac{q}{u} I_u(s^{\vee <}). \end{aligned}$$

For k odd, we get

$$\begin{aligned} (I_u s)^{\vee <} = (q-u^2)s^{<} + u(q-1)s^{\vee <} = \frac{1}{u}(q-u^2)s^{\vee <\vee} + u(q-1)s^{\vee <} \\ = \frac{1}{u} I_u(s^{\vee <}). \end{aligned}$$

□

27.3. Adjoint operations and transfer operator on pseudofields

We will now relate the theory of u-radical pseudofields to the spectral theory of the simple transfer operator S_p. To this aim, we introduce adjoint operations on pseudofields in analogy with the language of Section 25.1 (see Section 11.4 for the analogous construction for quadratic pseudofields).

Let $k \geq 1$ and p be a Γ-invariant k-Euclidean field. We use p to define as usual a natural scalar product on $\mathcal{S}_k$. Let r, s be in $\mathcal{S}_k$. If k is even, $k = 2\ell$, $\ell \geq 1$, let, for every x in X, p_x^* be the bilinear form dual to p_x on $V_0^\ell(x)$. We set,

$$p^*(r,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_x^*(r_{xy}, s_{xy}).$$

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 0$, let, for every $x \sim y$ in X, p_{xy}^* be the bilinear form dual to p_{xy} on $V_0^\ell(xy)$. We set,

$$p^*(r,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_{xy}^*(r_{xy}, s_{xy}).$$

Assume $k \geq 2$. We also introduce the adjoint operator of restriction of pseudofields. Let s be in $\mathcal{S}_{k-1}$ and let us define $s^{>_p}$ in $\mathcal{S}_k$. We use the notation of Section 11.7 and Section 17.2. If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we set $s_{xy}^{>_p} = I_{xy}^{\ell-1,*\dagger p} s_{xy}$. If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we set $s_{xy}^{>_p} = J_{xy}^{\ell,*\dagger p} s_{xy}$. As usual, we get

LEMMA 27.9. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For r in $\mathcal{S}_k$ and s in $\mathcal{S}_{k-1}$, we have $p^{-,*}(r^<, s) = p^*(r, s^{>_p})$.*

Still assume $k \geq 2$ and p is a Γ-invariant k-Euclidean field. In Section 25.1, we have defined the simple transfer operator S_p acting on the space of Γ-invariant $(k-1)$-pseudofunctions by the formula

$$S_p H = H^{>\vee<_p\vee}, \quad H \in \mathcal{H}_{k-1}.$$

The adjoint $S_p^\dagger$ of S_p with respect to the Euclidean structure may be defined by the formula

$$S_p^\dagger H = H^{\vee>\vee<_p}, \quad H \in \mathcal{H}_{k-1}.$$

Now, the adjoint of these operator with respect to the duality are linear endomorphisms of $\mathcal{S}_{k-1}$ which should formally be written as S_p^* and $S_p^{*\dagger} = S_p^{\dagger *}$. To avoid the latter heavy notation we write S_p instead. In other words, we set

$$S_p s = s^{>_p\vee<\vee}, \quad s \in \mathcal{S}_{k-1}.$$

The adjoint $S_p^\dagger$ of this operator with respect to the Euclidean structure on $\mathcal{S}_{k-1}$ is defined by

$$S_p^\dagger s = s^{\vee >_p \vee <}, \quad s \in \mathcal{S}_{k-1}.$$

Note that all these operators have the same spectrum and that we still have the relation

$$S_p(s^\vee) = (S_p^\dagger s)^\vee, \quad s \in \mathcal{S}_{k-1}.$$

Using these definitions, we get a direct relationship between the spectrum of S_p and the theory of u-radical pseudofields.

LEMMA 27.10. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Fix u in $\mathbb{C}^*$ and s in $\mathcal{S}_{k-1,\mathbb{C}}$. Then, $s^{>_p}$ is u-radical if and only if $S_p s = us$.*

PROOF. By construction, we have $s^{>_p <} = s$ and $s^{>_p \vee <\vee} = S_p s$. □

As for pseudofunctions, we also denote by S_p and $S_p^\dagger$ the linear endomorphisms of $\mathcal{S}_{k,\mathbb{C}}$ defined by

$$S_p s = s^{<\vee >_p \vee} \text{ and } S_p^\dagger s = s^{\vee <\vee >_p}, \quad s \in \mathcal{S}_{k,\mathbb{C}},$$

which can be thought of as the transfer operator and adjoint transfer operator associated to the orthogonal extension p^+ of p. We directly get

LEMMA 27.11. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For any s in $\mathcal{H}_k$, we have*

$$S_p^\dagger(s^<) = (S_p s)^< \text{ and } S_p(s^{<\vee}) = (S_p s)^{<\vee}.$$

27.4. The dual spectral bilinear form

We now introduce natural bilinear forms on spaces of u-radical simple pseudofields which are associated to the choice of a Euclidean field. These forms will play a key role in the statement of the Plancherel formula for Euclidean fields.

DEFINITION 27.12. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u in $\mathbb{C}^*$, we define the dual spectral bilinear form p_u^* associated to p on $\mathcal{S}_{k,\mathbb{C}}^u$ as the bilinear form

$$p_u^*(r,s) = p^*(r, I_u(s)) - p^{-,*}(r^<, I_u(s)^<), \quad r,s \in \mathcal{S}_{k,\mathbb{C}}^u.$$

This bilinear form is actually symmetric.

LEMMA 27.13. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u in $\mathbb{C}^*$, the dual spectral bilinear form p_u^* is symmetric on $\mathcal{S}_{k,\mathbb{C}}^u$.*

PROOF. This is again a direct computation. Let r, s be in $\mathcal{S}^u_{k,\mathbb{C}}$. As the $\vee$ operator is symmetric, we have $p^*(r, I_u(s)) = p^*(I_u(r), s)$. Now, if k is even, we have $r^{\vee<} = ur^{<\vee}$ and $s^{\vee<} = us^{<\vee}$, hence

$$\begin{aligned} p^{-,*}(r^<, I_u(s)^<) &= p^{-,*}(r^<, (q-u^2)s^{\vee<} + u^2(q-1)s^<) \\ &= p^{-,*}(r^<, u(q-u^2)s^{<\vee} + u^2(q-1)s^<) \\ &= p^{-,*}(u(q-u^2)r^{<\vee} + u^2(q-1)r^<, s^<) \\ &= p^{-,*}((q-u^2)r^{\vee<} + u^2(q-1)r^<, s^<) \\ &= p^{-,*}(I_u(r)^<, s^<). \end{aligned}$$

If k is odd, we have

$$qs^{\vee<} + (q-1)s^{\vee<\vee} = s^{\vee<\vee\vee} = us^{<\vee},$$

hence $qs^{\vee<} = us^{<\vee} - (q-1)us^<$, which gives

$$I_u(s)^< = (q-u^2)q^{-1}us^{<\vee} + u(q-1)(1-q^{-1})s^<.$$

As in the even case, we get $p^{-,*}(r^<, I_u(s)^<) = p^{-,*}(I_u(r)^<, s^<)$. This shows that p_u is symmetric. □

We have a non degeneracy criterion for p^*_u.

PROPOSITION 27.14. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u in $\mathbb{C}^*$, the bilinear form*

$$(r, s) \mapsto p^*(r, s) - p^{-,*}(r^<, s^<) = p^*(r - r^{<>_p}, s - s^{<>_p}) \tag{27.2}$$

is non degenerate on $\mathcal{S}^u_{k,\mathbb{C}} \times \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ if and only if neither u nor $\frac{q}{u}$ are eigenvalues of the simple transfer operator S_p. Therefore, if this is the case and if in addition $u \notin \{-q, -1, 1, q\}$, then p^*_u is non degenerate.

To show non degeneracy, we shall use the following elementary

LEMMA 27.15. *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ with characteristic $\neq 2$ and p be a non degenerate symmetric bilinear form on V. Let W be a subspace of V and assume that the restriction of p to W is non degenerate, so that the p-orthogonal projection $\pi : V \to W$ is well-defined. Let $U = W^{\perp_p}$ be the p-orthogonal complement of W and X and Y be complementary subspaces of U in V. Then the bilinear form $(x, y) \mapsto p(\pi(x), \pi(y))$ is non degenerate on $X \times Y$.*

PROOF OF PROPOSITION 27.14. Note that the second claim follows from the first and Corollary 27.7. We now prove the first claim.

First, assume for example that u is an eigenvalue of S_p. Then, by Lemma 27.10, there exists $s \neq 0$ in $\mathcal{S}_{k-1,\mathbb{C}}$ with $s^{>_p} \in \mathcal{S}^u_{k,\mathbb{C}}$. For r

in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have, by Lemma 27.9, $p^*(r, s^{>_p}) = p^{-,*}(r^<, s)$, hence the bilinear form in (27.2) is degenerate.

To prove the converse, we will apply the criterion in Lemma 27.15. Indeed, the linear map

$$s \mapsto s^{<>_p}, \mathcal{S}_{k,\mathbb{C}} \to \mathcal{S}_{k,\mathbb{C}},$$

is the p-orthogonal projection onto the subspace $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$, so that the linear map

$$s \mapsto s - s^{<>_p}, \mathcal{S}_{k,\mathbb{C}} \to \mathcal{S}_{k,\mathbb{C}},$$

is the p-orthogonal projection onto the p-orthogonal complement of $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$. Thus, by Lemma 27.15, to conclude, it suffices to show that $\mathcal{S}^u_{k,\mathbb{C}}$ and $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ are complentary subspaces to $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$ in $\mathcal{S}_{k,\mathbb{C}}$.

Indeed, as by assumption u is not an eigenvalue of S_p, Lemma 27.10 says that $\mathcal{S}^{>_p}_{k-1,\mathbb{C}} \cap \mathcal{S}^u_{k,\mathbb{C}} = \{0\}$. Now, by definition, $\mathcal{S}^u_{k,\mathbb{C}}$ is the null space of the linear map

$$s \mapsto s^{\vee<\vee} - us^<, \mathcal{S}_{k,\mathbb{C}} \to \mathcal{S}_{k-1,\mathbb{C}},$$

hence its codimension in $\mathcal{S}_{k,\mathbb{C}}$ is less than the dimension of $\mathcal{S}_{k-1,\mathbb{C}}$. Since it intersects $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$ trivially, this codimension is exactly equal to the dimension of $\mathcal{S}_{k-1,\mathbb{C}}$ and we get $\mathcal{S}_{k,\mathbb{C}} = \mathcal{S}^{>_p}_{k-1,\mathbb{C}} \oplus \mathcal{S}^u_{k,\mathbb{C}}$ as required. In the same way, as $\frac{q}{u}$ is not an eigenvalue of S_p, we get $\mathcal{S}_{k,\mathbb{C}} = \mathcal{S}^{>_p}_{k-1,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ and the conclusion follows by Lemma 27.15. □

The definition and the properties of the opposition operators yield the following adjointness formulas.

LEMMA 27.16. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}^*$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. If r is in $\mathcal{S}^u_{k,\mathbb{C}}$ and s is in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have*

$$p^*_{\frac{q}{u}}(I_u r, s) = p^*_u(r, I_{\frac{q}{u}} s).$$

If k is even, the latter is equal to

$$q^2(q+1)^2(1-t^2)(p^*(r,s) - p^{-,*}(r^<, s^<)).$$

If k is odd, it is equal to

$$q(q+1)^2(1-t^2)(p(r^*, s) - p^{-,*}(r^<, s^<)).$$

PROOF. Indeed, by Definition 27.12, we have

$$p^*_u(r, I_{\frac{q}{u}} s) = p^*(r, I_u I_{\frac{q}{u}} s) - p^{-,*}(r^<, (I_u I_{\frac{q}{u}} s)^<).$$

The conclusion follows from this identity and Lemma 27.6 and Lemma 27.13. □

27.5. The spectral bilinear form

We will now study the dual object to p_u^*, which will later turn out to be related to the resolvent formula for p obtained in Proposition 26.1.

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Pick u in $\mathbb{C} \smallsetminus \{-q, -1, 0, 1, q\}$ such that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Then, by Proposition 27.14, the dual spectral bilinear form p_u^* is non degenerate on $\mathcal{S}^u_{k,\mathbb{C}}$. Therefore, it defines by duality a symmetric bilinear form p_u on $\mathcal{H}_{k,\mathbb{C}}$ whose null space is the range of the map

$$G \mapsto G^{\vee > \vee} - uG^{>}, \mathcal{H}_{k-1,\mathbb{C}} \to \mathcal{H}_{k,\mathbb{C}}. \tag{27.3}$$

For H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have $p_u(H, J) = \langle r, J \rangle$ where r is the unique element of $\mathcal{S}^u_{k,\mathbb{C}}$ such that, for any s in $\mathcal{S}^u_{k,\mathbb{C}}$, one has $\langle s, H \rangle = p_u^*(r, s)$.

Definition 27.17. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u as above, the symmetric bilinear form p_u on $\mathcal{H}_{k,\mathbb{C}}$ is called the spectral bilinear form associated to u.

The purpose of this Section is to give a formula for computing p_u on eigenspaces of the operator $\vee$.

Proposition 27.18. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field, H and J be in $\mathcal{H}_{k,\mathbb{C}}$ and u be in $\mathbb{C} \smallsetminus \{-q, -1, 0, 1, q\}$ such that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$ and*

$$H' = (u - S_p)^{-1} \left(\frac{q}{u} - S_p \right)^{-1} H = (q + S_p^2 - (q+1)tS_p)^{-1} H.$$

If k is even, we have

$$\begin{aligned}
p_u(H, J) &= \frac{1}{q} \frac{1}{q^2 - u^2} p((q^2 - S_p^2)H', J) && H^\vee = qH, J^\vee = qJ \\
&= \frac{1}{qu} p(S_p H', J) && H^\vee = -H, J^\vee = qJ \\
&= -\frac{1}{u} p(S_p H', J) && H^\vee = qH, J^\vee = -J \\
&= \frac{1}{u^2 - 1} p((1 - S_p^2)H', J) && H^\vee = -H, J^\vee = -J.
\end{aligned}$$

If k is odd, we have

$$\begin{aligned} p_u(H,J) &= \frac{p((q-S_p)(1+S_p)H',J)}{(q-u)(u+1)} && H^\vee = H, J^\vee = J \\ &= \frac{1}{u}p(S_pH',J) && H^\vee = -H, J^\vee = J \\ &= -\frac{1}{u}p(S_pH',J) && H^\vee = H, J^\vee = -J \\ &= \frac{p((q+S_p)(1-S_p)H',J)}{(q+u)(u-1)} && H^\vee = -H, J^\vee = -J. \end{aligned}$$

This result relies on constructions in $\mathcal{S}_{k,\mathbb{C}}$. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}^*$ such that u and $\frac{q}{u}$ are not eigenvalues of S_p. Pick r in $\mathcal{S}_{k,\mathbb{C}}$. Then, by Proposition 27.14, there exists a unique element r_u in $\mathcal{S}^u_{k,\mathbb{C}}$ such that, for any s in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, one has

$$p^*(r,s) = p^*(r_u,s) - p^{-,*}(r_u^<, s^<).$$

In case r is an eigenvector of the $\vee$ operator, we can give a formula for defining r_u.

LEMMA 27.19. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field, r be in $\mathcal{S}_{k,\mathbb{C}}$ and u be in $\mathbb{C}^*$ such that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Set*

$$c = (u - S_p^\dagger)^{-1}\left(\frac{q}{u} - S_p^\dagger\right)^{-1}(r^<) \in \mathcal{S}_{k-1,\mathbb{C}}.$$

Then, r_u may be written as $r_u = r + a^{\vee>_p} + b^{\vee>_p\vee}$ where $b = (u - S_p^\dagger)c$ and $a \in \mathcal{S}_{k-1,\mathbb{C}}$ is defined as follows:

$$\begin{aligned} a &= \left(\frac{q^2}{u} - S_p^\dagger\right)c, && k \text{ even, } r^\vee = qr \\ a &= q\left(S_p^\dagger - \frac{1}{u}\right)c, && k \text{ even, } r^\vee = -r \\ a &= \left(q - 1 + \frac{q}{u} - S_p^\dagger\right)c, && k \text{ odd, } r^\vee = r \\ a &= \left(q - 1 - \frac{q}{u} + S_p^\dagger\right)c, && k \text{ odd, } r^\vee = -r. \end{aligned}$$

PROOF. We take a, b, c to be as in the statement and we set $s = r + a^{\vee>_p} + b^{\vee>_p\vee}$. We want to show that $s = r_u$. By definition, we need to show that s is u-radical and that $s - s^{<>_p} - r$ is p-orthogonal to $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$. First we show the latter. By construction, as $a^{\vee>_p<} = a^\vee$, we have

$$s - s^{<>_p} - r = b^{\vee>_p\vee} - r^{<>_p} - b^{\vee>_p\vee<>_p} = b^{\vee>_p\vee} - r^{<>_p} - (S_p^\dagger b)^{>_p}.$$

We have $b = (u - S_p^\dagger)c$, hence, $(\frac{q}{u} - S_p^\dagger)b = r^<$. This gives

$$s - s^{<>_p} - r = b^{\vee>_p\vee} - \frac{q}{u}b^{>_p},$$

hence $s - s^{<>_p} - r$ is p-orthogonal to $\mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$.

To conclude, we must prove that s is u-radical. First we compute

$$us^< = ur^< + ua^\vee + uS_p^\dagger b = ua^\vee + qb, \tag{27.4}$$

where we have again used the identity $(\frac{q}{u} - S_p^\dagger)b = r^<$. The computation of $s^{\vee<\vee}$ will depend on the parity of k.

If k is even, we have

$$\begin{aligned} s^{\vee<\vee} &= r^{\vee<\vee} + a^{\vee>_p\vee<\vee} + b^{\vee>_p\vee\vee<\vee} \\ &= r^{\vee<\vee} + (S_p^\dagger a)^\vee + qb + (q-1)(S_p^\dagger b)^\vee \\ &= r^{\vee<\vee} + (S_p^\dagger a)^\vee + qb + (q-1)\frac{q}{u}b^\vee - (q-1)r^{<\vee}. \end{aligned}$$

Together with (27.4), this gives

$$\begin{aligned} s^{\vee>} - us^{<\vee} &= (r^\vee - (q-1)r)^< + (S_p^\dagger - u)a + (q-1)\frac{q}{u}b \\ &= (r^\vee - (q-1)r)^< + (S_p^\dagger - u)\left(a - (q-1)\frac{q}{u}c\right). \end{aligned}$$

If $r^\vee = qr$, by using the relation $r^< = (u - S_p^\dagger)\left(\frac{q}{u} - S_p^\dagger\right)c$, we get

$$\begin{aligned} s^{\vee>} - us^{<\vee} &= r^< + (S_p^\dagger - u)\left(a - (q-1)\frac{q}{u}c\right) \\ &= (S_p^\dagger - u)\left(a + S_p^\dagger c - \frac{q^2}{u}c\right), \end{aligned}$$

which is 0 by assumption. In the same way, if $r^\vee = -r$, we get

$$\begin{aligned} s^{\vee>} - us^{<\vee} &= -qr^< + (S_p^\dagger - u)\left(a - (q-1)\frac{q}{u}c\right) \\ &= (S_p^\dagger - u)\left(a - qS_p^\dagger c + \frac{q}{u}c\right), \end{aligned}$$

which is also 0.

If k is odd, we have

$$s^{\vee<\vee} = r^{\vee<\vee} + (S_p^\dagger a)^\vee + qb + (q-1)b^\vee.$$

Thanks to (27.4), we get

$$s^{\vee<\vee} - us^< = \left(r^{\vee<} + (S_p^\dagger - u)a + (q-1)b\right)^\vee.$$

If $r^\vee = r$, we have

$$r^{\vee<} + (S_p^\dagger - u)a + (q-1)b = (S_p^\dagger - u)\left(a - \frac{q}{u}c + S_p^\dagger c - (q-1)c\right) = 0.$$

If $r^\vee = -r$, we have

$$r^{\vee<} + (S_p^\dagger - u)a + (q-1)b = (S_p^\dagger - u)\left(a + \frac{q}{u}c - S_p^\dagger c - (q-1)c\right) = 0.$$

The Lemma follows. □

We shall need a formula for computing certain values of the bilinear forms.

LEMMA 27.20. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let $f \in \mathbb{C}(x)$ be a rational function with no poles at 0 and on the spectrum of S_p. Then, for every r, s in $\mathcal{S}_{k,\mathbb{C}}$, we have*

$$p^{-,*}(f(S_p^\dagger)(r^<), s^{\vee<\vee}) = p^*(S_p f(S_p)r, s).$$

PROOF. First note that, by Lemma 25.8, $f(S_p)$ is a well-defined endomorphism of $\mathcal{S}_k$. Besides, as all spaces are finite dimensional, we can assume that f is a polynomial function. Then, from Lemma 27.11, we get $f(S_p^\dagger)(r^<) = (f(S_p)r)^<$, hence

$$\begin{aligned} p^{-,*}(f(S_p^\dagger)(r^<), s^{\vee<\vee}) = p^*(f(S_p)r, s^{\vee<\vee>_p}) &= p^*(f(S_p)r, S_p^\dagger s) \\ &= p^*(S_p f(S_p)r, s). \end{aligned}$$

□

We can now write formulas for the quantity $p^*(r_u, s) = p^*(r, s_{\frac{q}{u}})$, for $r, s \in S_{k,\mathbb{C}}$. Unfortunately, the formulas depend on the parity of k and the eigenvalue of the $\vee$ operator.

LEMMA 27.21. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field, r and s be in $\mathcal{S}_{k,\mathbb{C}}$ and u be in $\mathbb{C}^*$. Assume neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$ and*

$$r' = (u - S_p)^{-1}\left(\frac{q}{u} - S_p\right)^{-1} r = (q + S_p^2 - (q+1)tS_p)^{-1} r.$$

If k is even, we have

$$\begin{aligned} p^*(r_u, s) &= \frac{1}{q} p^*((q^2 - S_p^2)r', s) && r^\vee = qr, s^\vee = qs \\ &= \frac{u^2 - 1}{u} p^*(S_p r', s) && r^\vee = -r, s^\vee = qs \\ &= \frac{u^2 - q^2}{u} p^*(S_p r', s) && r^\vee = qr, s^\vee = -s \\ &= q p^*((1 - S_p^2)r', s) && r^\vee = -r, s^\vee = -s. \end{aligned}$$

If k is odd, we have

$$\begin{aligned} p^*(r_u, s) &= p^*((q - S_p)(1 + S_p)r', s) && r^\vee = r, s^\vee = s \\ &= \frac{(u+q)(u-1)}{u} p^*(S_p r', s) && r^\vee = -r, s^\vee = s \\ &= \frac{(u-q)(u+1)}{u} p^*(S_p r', s) && r^\vee = r, s^\vee = -s \\ &= p^*((q + S_p)(1 - S_p)r', s) && r^\vee = -r, s^\vee = -s. \end{aligned}$$

PROOF. We only establish the first two cases.

Assume k is even and $s^\vee = qs$. Let a, b, c be as in Lemma 27.19, so that we have $r_u = r + a^{\vee >_p} + b^{\vee >_p \vee}$. We write

$$\begin{aligned} p^*(r_u, s) &= p^*(r, s) + p^{-,*}(a, s^{<\vee}) + p^{-,*}(b, s^{\vee <\vee}) \\ &= p^*(r, s) + p^{-,*}\left(\frac{1}{q}a + b, s^{\vee <\vee}\right). \end{aligned} \tag{27.5}$$

If $r^\vee = qr$, Lemma 27.19 says that

$$\frac{1}{q}a + b = \left(u + \frac{q}{u} - \frac{q+1}{q} S_p^\dagger\right) c = \frac{q+1}{q}(qt - S_p)c.$$

By Lemma 27.20, we get

$$p^{-,*}\left(\frac{1}{q}a + b, s^{\vee <\vee}\right) = \frac{q+1}{q} p^*(S_p(qt - S_p)r', s).$$

We use the identity in $\mathbb{C}(x)$,

$$1 + \frac{q+1}{q} \frac{x(qt - x)}{q + x^2 - (q+1)tx} = \frac{1}{q} \frac{q^2 - x^2}{q + x^2 - (q+1)tx}$$

to deduce from (27.5) that $p^*(r_u, s) = \frac{1}{q} p^*((q^2 - S_p^2)r', s)$.

If $r^\vee = -r$, note that $p^*(r, s) = 0$, so that, from (27.5), we get $p^*(r_u, s) = p^{-,*}\left(\frac{1}{q}a + b, s^{\vee <\vee}\right)$. Now, Lemma 27.19 says that $\frac{1}{q}a + b = \frac{u^2-1}{u}c$. By Lemma 27.20, we get $p^*(r_u, s) = \frac{u^2-1}{u} p^*(S_p r', s)$ as required.

The other cases are obtained in the same way. □

To conclude, we will need another set of formulas.

LEMMA 27.22. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C} \smallsetminus \{-q, -1, 0, 1, q\}$. Pick r, s in $\mathcal{S}_{k,\mathbb{C}}$. Then, we have*

$$\begin{aligned} p^*(I_u^{-1} r, s) &= \frac{1}{q^2 - u^2} p^*(r, s) && k \text{ even, } s^\vee = qs \\ &= \frac{1}{q(u^2 - 1)} p^*(r, s) && k \text{ even, } s^\vee = -s \\ &= \frac{1}{(q - u)(u + 1)} p^*(r, s) && k \text{ odd, } s^\vee = s \\ &= \frac{1}{(q + u)(u - 1)} p^*(r, s) && k \text{ odd, } s^\vee = -s. \end{aligned}$$

PROOF. By Lemma 27.6, we have $I_u^{-1} r = \frac{u^2}{q^2(q^2-u^2)(u^2-1)} I_{\frac{q}{u}} r$ if k is even and $I_u^{-1} r = \frac{u^2}{q(q^2-u^2)(u^2-1)} I_{\frac{q}{u}} r$ if k is odd.

Assume for example k is even and $s^\vee = qs$. From the Definition 27.4 of the opposition map $I_{\frac{q}{u}}$ we have

$$\begin{aligned} p^*(I_{\frac{q}{u}} r, s) &= p^* \left(r, \left(q - \frac{q^2}{u^2} \right) s^\vee + \frac{q^2}{u^2}(q - 1)s \right) \\ &= \frac{1}{u^2} p^*(r, (q^2 u^2 - q^3 + q^3 - q^2)s) = \frac{q^2}{u^2}(u^2 - 1) p^*(r, s) \end{aligned}$$

and the conclusion follows.

The other cases are dealt with in the same way. □

We conclude the

PROOF OF PROPOSITION 27.18. Take H and J in $\mathcal{H}_{k,\mathbb{C}}$ and let r and s be the elements of $\mathcal{S}_{k,\mathbb{C}}$ such that, for any G in $\mathcal{H}_{k,\mathbb{C}}$, one has $p(H, G) = \langle r, G \rangle$ and $p(J, G) = \langle s, G \rangle$. For a in $\mathcal{S}^u_{k,\mathbb{C}}$, we have

$$\langle a, H \rangle = p^*(r, a) = p^*(r_{\frac{q}{u}}, a) - p^{-,*}(r^<_{\frac{q}{u}}, a^<) = p^*_u(I_u^{-1} r_{\frac{q}{u}}, a),$$

so that, by definition, we get $p_u(H, J) = p^*_u(I_u^{-1} r_{\frac{q}{u}}, s)$ and the Proposition follows from Lemma 27.21 and Lemma 27.22. □

27.6. u-radical pseudofields in the bipartite case

When Γ is bipartite, the twist operation of Section 20.6 and Section 24.7 interacts with the previous constructions. Let us write how this interaction works.

First, by the dual version of Lemma 20.22, we directly get,

LEMMA 27.23. *Assume Γ is bipartite. Let $k \geq 1$, u be in $\mathbb{C}^*$ and s be in $\mathcal{S}_{k,\mathbb{C}}$. Then s is u-radical if and only if $s^\wr$ is $(-u)$-radical. Besides, we have*

$$(I_u s)^\wr = (-1)^k I_{(-u)}(s^\wr).$$

Using these properties, straightforward computations yield

LEMMA 27.24. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}^*$. For r, s in $\mathcal{S}^u_{k,\mathbb{C}}$, we have*

$$p^*_{(-u)}(r^\wr, s^\wr) = (-1)^k p^*_u(r, s).$$

If $u^2 \notin \{1, q^2\}$ and neither u nor $\frac{q}{u}$ is an eigenvalue of S_p, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have

$$p_{(-u)}(H^\wr, J^\wr) = (-1)^k p_u(H, J).$$

Note that the condition on u in the latter statement is symmetric in view of Lemma 25.6.

CHAPTER 28

t-radical pairs and full spectral quadratic forms

In this Chapter we continue the algebraic constructions that are necessary in order to state the Plancherel formula for Euclidean fields.

28.1. t-radical pairs

We start by introducing a new subspace of $\mathcal{S}_k^2$ and relate it to the range of the default map of Section 20.3 thanks to Proposition 24.5.

DEFINITION 28.1. Let $k \geq 1$, t be in $\mathbb{C}$ and $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ be a pair of complex k-simple pseudofields. Then, s is said to be t-radical if one has

$$\begin{pmatrix} s_0^{\vee <\vee} \\ s_1^{\vee <\vee} \end{pmatrix} = \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} \begin{pmatrix} s_0^{<} \\ s_1^{<} \end{pmatrix}.$$

For $k \geq 1$, the space of t-radical pairs in $\mathcal{S}_{k,\mathbb{C}}^2$ is denoted by $\mathcal{S}_{k,\mathbb{C}}^{2,t}$. If t is real, the space of t-radical pairs in $\mathcal{S}_k^2$ is denoted by $\mathcal{S}_k^{2,t}$.

REMARK 28.2. Note that the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$ already appeared in the proof of the Ihara trace formula in Section 1.6.

We will relate the notion of a t-radical pair to the formulas in Proposition 24.5. To this aim, for $k \geq 0$, we introduce a natural duality between $\mathcal{S}_k^2$ and $\mathcal{H}_k^2$. For $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ in $\mathcal{S}_k^2$ and $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}_k^2$, we set

$$\langle s, H \rangle = \langle s_0, H_0 \rangle + \langle s_1, H_1 \rangle.$$

Then, Proposition 24.5 translates into

LEMMA 28.3. *Let $k \geq 1$, s be in $\mathcal{S}_k^2$ and t be in $\mathbb{R}$. Then s is t-radical if and only if, for every G in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, one has $\langle s, \widehat{D_k G}(t) \rangle = 0$.*

Recall that D_k is the default map that was defined in Section 20.3.

PROOF. Indeed, Proposition 24.5 gives, is k is even,

$$\langle s, \widehat{D_k G}(t)\rangle = \left\langle \begin{pmatrix} s_0^< \\ s_1^< \end{pmatrix} - \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}^{-1} \begin{pmatrix} s_0^{\vee<\vee} \\ s_1^{\vee<\vee} \end{pmatrix}, \breve{G}(t) \right\rangle$$

and if k is odd,

$$\langle s, \widehat{D_k G}(t)\rangle = \left\langle \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} \begin{pmatrix} s_0^< \\ s_1^< \end{pmatrix} - \begin{pmatrix} s_0^{\vee<\vee} \\ s_1^{\vee<\vee} \end{pmatrix}, \breve{G}(t) \right\rangle.$$

The Proposition follows as, by Proposition 24.4, the linear map $G \mapsto \breve{G}(t)$ sends $\mathcal{H}_{k-1}^{(\mathbb{N})}$ onto $\mathcal{H}_{k-1}^2$. □

In Lemma 28.3, we have established a relation between t-radical pairs and the range of the polyextension map. Now, from Lemma 20.21, Proposition 24.3 and Proposition 24.4, we should see a link between t-radical pairs and the operators of (24.1) and (24.2). Note that the adjoint operators of these operators may be defined by the following formulas. For $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ in $\mathcal{S}_k^2$, if k is even, we have

$$\mathfrak{S}_t^* s = \begin{pmatrix} s_1^\vee - (q-1)s_1 \\ q^{-1} s_0^\vee \end{pmatrix}$$
$$\mathfrak{R}_t^* s = \begin{pmatrix} s_0^\vee \\ q^{-1}(q+1)t s_0^\vee + (q-1)s_1 - s_1^\vee \end{pmatrix}.$$

If k is odd, we have

$$\mathfrak{R}_t^* s = \begin{pmatrix} q s_1^\vee \\ (q-1)s_1 + s_0^\vee \end{pmatrix}$$
$$\mathfrak{S}_t^* s = \begin{pmatrix} -s_0^\vee + (q+1)t s_1^\vee - (q-1)s_1 \\ s_1^\vee \end{pmatrix}.$$

LEMMA 28.4. *Let $k \geq 1$, t be in $\mathbb{C}$ and s be a t-radical pair of complex k-simple pseudofields. The pairs $\mathfrak{R}_t^* s$ and $\mathfrak{S}_t^* s$ are t-radical.*

PROOF. As mentioned above this can be seen as a consequence as Lemma 20.21, Proposition 24.3, Proposition 24.4 and Lemma 28.3. We can also get it by a straightforward computation. We shall use the fact that the inverse of the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$ is the matrix

$\begin{pmatrix} q^{-1}(q+1)t & -1 \\ q^{-1} & 0 \end{pmatrix}$. Assume k is even. We get

$$(\mathfrak{S}_t^* s)^{\vee<\vee} = \begin{pmatrix} qs_1^{<\vee} \\ s_0^{<\vee} + q^{-1}(q-1)s_0^{\vee<\vee} \end{pmatrix}$$

$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{S}_t^* s)^{<} = \begin{pmatrix} s_0^{\vee<} \\ (q-1)s_1^{<} - s_1^{\vee<} + q^{-1}(q+1)ts_0^{\vee<} \end{pmatrix}.$$

As s is t-radical, we have $s_0^{\vee<} = qs_1^{<\vee}$ and $-s_1^{\vee<} + q^{-1}(q+1)ts_0^{\vee<} = s_0^{<\vee}$. We get $(\mathfrak{S}_t^* s)^{\vee<\vee} = \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{S}_t^* s)^{<}$ as required. In the same way, we have

$$(\mathfrak{R}_t^* s)^{\vee<\vee} = \begin{pmatrix} qs_0^{<\vee} + (q-1)s_0^{\vee<\vee} \\ (q+1)ts_0^{<\vee} + q^{-1}(q^2-1)ts_0^{\vee<\vee} - qs_1^{<\vee} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{R}_t^* s)^{<}$$

$$= \begin{pmatrix} (q+1)ts_0^{\vee<} + q(q-1)s_1^{<} - qs_1^{\vee<} \\ -s_0^{\vee<} + q^{-1}(q+1)^2t^2 s_0^{\vee<} + (q^2-1)ts_1^{<} - (q+1)ts_1^{\vee<} \end{pmatrix}$$

$$= \begin{pmatrix} qs_0^{<\vee} + q(q-1)s_1^{<} \\ -s_0^{\vee<} + (q+1)ts_0^{<\vee} + (q^2-1)ts_1^{<} \end{pmatrix},$$

where we have used again the relation $-s_1^{\vee<} + q^{-1}(q+1)ts_0^{\vee<} = s_0^{<\vee}$. Still by using the relation $s_0^{\vee<} = qs_1^{<}$, we get $(\mathfrak{R}_t^* s)^{\vee<\vee} = \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{R}_t^* s)^{<}$.

Assume now k is odd, so that we write

$$(\mathfrak{R}_t^* s)^{\vee<\vee} = \begin{pmatrix} qs_1^{<\vee} \\ (q-1)s_1^{\vee<\vee} + s_0^{<\vee} \end{pmatrix}$$

$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{R}_t^* s)^{<} = \begin{pmatrix} q(q-1)s_1^{<} + qs_0^{\vee<} \\ -qs_1^{\vee<} + (q^2-1)ts_1^{<} + (q+1)ts_0^{\vee<} \end{pmatrix}.$$

As $s_0^{\vee<\vee} = qs_1^{<}$, we get $s_0^{\vee<} = s_1^{<\vee} - (q-1)s_1^{<}$, which gives equality of the first lines. Besides, from $s_1^{\vee<\vee} = -s_0^{<} + (q+1)ts_1^{<}$, we get $qs_1^{\vee<} + (q-1)s_1^{\vee<\vee} = -s_0^{<\vee} + (q+1)ts_1^{<\vee}$ which leads to the equality of the second lines. It remains to compute

$$(\mathfrak{S}_t^* s)^{\vee<\vee} = \begin{pmatrix} -s_0^{<\vee} + (q+1)ts_1^{<\vee} - (q-1)s_1^{\vee<\vee} \\ s_1^{<\vee} \end{pmatrix}$$

$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{S}_t^* s)^{<} = \begin{pmatrix} qs_1^{\vee<} \\ s_0^{\vee<} + (q-1)s_1^{<} \end{pmatrix}.$$

The relation $s_0^{\vee<\vee} = qs_1^<$ implies the equality of the second lines, whereas the relation $s_1^{\vee<\vee} = -s_0^< + (q+1)ts_1^<$ implies the equality of the first lines. □

28.2. t-radical pairs and u-radical pseudofields

We will now relate t-radical pairs to the theory of Chapter 27. This relies on diagonalizing the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$. Note that the eigenvalues of this matrix are the roots of the equation $q^2+u = (q+1)tu$. This fact was already used in Section 1.6.

To proceed to the diagonalization, we introduce a change of variables in spaces of pairs. Let $k \geq 0$ and fix u, t in $\mathbb{C}$ with $q^2+u = (q+1)tu$ and $u^2 \neq q$ or equivalently $(q+1)t^2 \neq 4q$. For $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}^2_{k,\mathbb{C}}$, we set

$$\delta_u H = \begin{pmatrix} q & u \\ q & \frac{q}{u} \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}. \tag{28.1}$$

As $u^2 \neq q$, this map is invertible and we get

$$\delta_u^{-1} H = \frac{1}{q-u^2} \begin{pmatrix} 1 & -q^{-1}u^2 \\ -u & u \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}. \tag{28.2}$$

The adjoint endomorphism δ_u^* of δ_u and its inverse $(\delta_u^*)^{-1} = (\delta_u^{-1})^*$ are defined by, for all $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ in $\mathcal{S}^2_{k,\mathbb{C}}$,

$$\delta_u^* s = \begin{pmatrix} q & q \\ u & \frac{q}{u} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \text{ and } (\delta_u^*)^{-1} s = \frac{1}{q-u^2} \begin{pmatrix} 1 & -u \\ -q^{-1}u^2 & u \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}.$$

LEMMA 28.5. *Let u be in $\mathbb{C}^*$, with $u^2 \neq q$, and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, for $k \geq 1$, a pair $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ of complex k-simple pseudofields is t-radical, if and only if the pseudofield $s_0 - us_1$ is u-radical and the pseudofield $s_0 - \frac{q}{u}s_1$ is $\frac{q}{u}$-radical. The map δ_u^* induces a linear isomorphism from $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ onto $\mathcal{S}^{2,t}_{k,\mathbb{C}}$.*

PROOF. Let T be the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$ and U be the matrix $\begin{pmatrix} q & q \\ u & \frac{q}{u} \end{pmatrix}$. Then U is invertible and a direct computation shows that U diagonalizes T, that is, we have $U^{-1}TU = \begin{pmatrix} u & 0 \\ 0 & \frac{q}{u} \end{pmatrix}$. The conclusion follows. □

28.3. Diagonalization map and natural operators

Let us transport the operators of Lemma 28.4 via the diagonalization map δ_u. For u in $\mathbb{C}^*$, by abuse of notation, we still write I_u for the endomorphism of $\mathcal{H}_{k,\mathbb{C}}$ that is adjoint to the u-opposition map of Definition 27.4. When $u^2 \neq q$, we define two endomorphisms ρ_u and σ_u of $\mathcal{H}^2_{k,\mathbb{C}}$ which we write in the following matrix forms.

If k is even, we set

$$\rho_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & I_u \\ -q^{-1}u^2 I_{\frac{q}{u}} & q(q-1)\end{pmatrix} \tag{28.3}$$

$$\text{and } \sigma_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & q^{-1}uI_u \\ -q^{-1}uI_{\frac{q}{u}} & u(q-1)\end{pmatrix}.$$

Their adjoint endomorphisms are defined by

$$\rho_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & -q^{-1}u^2 I_{\frac{q}{u}} \\ I_u & q(q-1)\end{pmatrix} \tag{28.4}$$

$$\text{and } \sigma_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & -q^{-1}uI_{\frac{q}{u}} \\ q^{-1}uI_u & u(q-1)\end{pmatrix}.$$

If k is odd, we set

$$\rho_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & uI_u \\ -uI_{\frac{q}{u}} & q(q-1)\end{pmatrix} \tag{28.5}$$

$$\text{and } \sigma_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & q^{-1}u^2 I_u \\ -I_{\frac{q}{u}} & u(q-1)\end{pmatrix}.$$

Their adjoint endomorphisms are defined by

$$\rho_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & -uI_{\frac{q}{u}} \\ uI_u & q(q-1)\end{pmatrix} \tag{28.6}$$

$$\text{and } \sigma_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & -I_{\frac{q}{u}} \\ q^{-1}u^2 I_u & u(q-1)\end{pmatrix}.$$

By using Lemma 27.6, one can check that in both cases, one has $\rho_u^2 = q + (q-1)\rho_u$ and $\sigma_u^2 = 1$. Besides, by Lemma 27.5, both ρ_u^* and σ_u^* preserve the space $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$.

Lemma 28.6. *Let $k \geq 0$, u be in $\mathbb{C} \smallsetminus \{-\sqrt{q}, 0, \sqrt{q}\}$ and $t = \frac{1}{q+1}(u + \frac{q}{u})$. One has $\delta_u \mathfrak{R}_t = \rho_u \delta_u$ and $\delta_u \mathfrak{S}_t = \sigma_u \delta_u$.*

Proof. Assume k is even and take H in $\mathcal{H}_{k,\mathbb{C}}$. We get

$$\delta_u \begin{pmatrix} H \\ -uH \end{pmatrix} = (q-u^2)\begin{pmatrix} H \\ 0 \end{pmatrix},$$

hence

$$\rho_u\delta_u\begin{pmatrix} H \\ -uH\end{pmatrix} = \begin{pmatrix} -(q-1)u^2H \\ -q^{-1}u^2 I_{\frac{q}{u}} H\end{pmatrix} = \begin{pmatrix} -(q-1)u^2H \\ (q-u^2)H^\vee - q(q-1)H\end{pmatrix},$$

where we have used Definition 27.4. On the other hand, by (24.1), we have

$$\mathfrak{R}_t\begin{pmatrix} H \\ -uH\end{pmatrix} = \begin{pmatrix} (1-\frac{u}{q}(q+1)t)H^\vee \\ uH^\vee - (q-1)uH\end{pmatrix} = \begin{pmatrix} -\frac{u^2}{q}H^\vee \\ uH^\vee - (q-1)uH\end{pmatrix}$$

and therefore

$$\delta_u\mathfrak{R}_t\begin{pmatrix} H \\ -uH\end{pmatrix} = \begin{pmatrix} -(q-1)u^2H \\ (q-u^2)H^\vee - q(q-1)H\end{pmatrix} = \rho_u\delta_u\begin{pmatrix} H \\ -uH\end{pmatrix}.$$

By symmetry, we also get $\rho_u\delta_u\begin{pmatrix} H \\ -\frac{q}{u}H\end{pmatrix} = \delta_u\mathfrak{R}_t\begin{pmatrix} H \\ -\frac{q}{u}H\end{pmatrix}$, hence, as $u^2 \neq q$, $\rho_u\delta_u = \delta_u\mathfrak{R}_t$. The other computations can be lead in the same way. □

28.4. Doubling of quadratic forms

We will now use the diagonalization map δ_u to construct symmetric bilinear forms on $\mathcal{H}_k^2$ with respect to which the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are self adjoint.

PROPOSITION 28.7. *Let $k \geq 0$ and t, u be in $\mathbb{C}$ with $q + u^2 = (q+1)tu$. Assume $(q+1)^2t^2 \neq 4q$ and $t^2 \neq 1$. Define $\mathcal{Q}^t_{k,\mathbb{C}}$ as the space of complex symmetric bilinear forms π on $\mathcal{H}^2_{k,\mathbb{C}}$ such that the linear operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to π.*

If k is even, for π in $\mathcal{Q}^t_{k,\mathbb{C}}$, set $\Phi_u\pi$ to be the symmetric bilinear form

$$(H, J) \mapsto \frac{1}{(q-u^2)^2}\pi\left(\begin{pmatrix} H \\ -uH\end{pmatrix}, \begin{pmatrix} J \\ -uJ\end{pmatrix}\right)$$

on $\mathcal{H}_{k,\mathbb{C}}$.

If k is odd, for π in $\mathcal{Q}^t_{k,\mathbb{C}}$, set $\Phi_u\pi$ to be the symmetric bilinear form

$$(H, J) \mapsto \frac{u}{q}\frac{1}{(q-u^2)^2}\pi\left(\begin{pmatrix} H \\ -uH\end{pmatrix}, \begin{pmatrix} J \\ -uJ\end{pmatrix}\right)$$

on $\mathcal{H}_{k,\mathbb{C}}$.

Then, the map Φ_u is a linear isomorphism from $\mathcal{Q}^t_{k,\mathbb{C}}$ onto the space of symmetric bilinear forms on $\mathcal{H}_{k,\mathbb{C}}$.

PROOF. Let us prove that the map Φ_u is injective. Pick π in $\mathcal{Q}^t_{k,\mathbb{C}}$ and assume that $\Phi_u\pi$ is 0. Set $\chi = (\delta_u^{-1})^\star\pi$. Then χ is a symmetric bilinear form on $\mathcal{H}^2_k$ and, by Lemma 28.6, the operators ρ_u and σ_u are

symmetric with respect to χ. By (28.2) and the assumption, χ has trivial restriction to $\mathcal{H}_k \times \{0\}$. Let us show that this implies $\chi = 0$.

If k is even, from (28.3), we see that the endomorphism of $\mathcal{H}^2_{k,\mathbb{C}}$ with matrix $\begin{pmatrix} 0 & I_u \\ 0 & q(q-1) \end{pmatrix}$ is symmetric with respect to χ. We get, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, as $\begin{pmatrix} 0 & I_u \\ 0 & q(q-1) \end{pmatrix} \begin{pmatrix} H \\ 0 \end{pmatrix} = 0$,

$$\chi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & I_u \\ 0 & q(q-1) \end{pmatrix} \begin{pmatrix} 0 \\ J \end{pmatrix}\right) = 0$$

hence $\chi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right) = 0$. Still by (28.3), the endomorphism $\begin{pmatrix} q(q-1) & 0 \\ I_{\frac{q}{u}} & 0 \end{pmatrix}$ is symmetric with respect to χ, so that we get

$$\chi\left(\begin{pmatrix} q(q-1) & 0 \\ I_{\frac{q}{u}} & 0 \end{pmatrix} \begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right) = 0,$$

hence $\chi\left(\begin{pmatrix} 0 \\ I_{\frac{q}{u}} H \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right) = 0$. Since by Lemma 27.6, $I_{\frac{q}{u}}$ is invertible, we get $\chi = 0$ as required.

If k is odd, from (28.5), we see that the endomorphisms of $\mathcal{H}^2_{k,\mathbb{C}}$ with matrices

$$\begin{pmatrix} 0 & uI_u \\ 0 & q(q-1) \end{pmatrix} \text{ and } \begin{pmatrix} q(q-1) & 0 \\ \frac{q}{u} I_{\frac{q}{u}} & 0 \end{pmatrix}$$

are symmetric with respect to χ and we proceed as in the previous case.

In both cases, we have shown that the map Φ_u is injective. Now, we will show that it surjective. We fix a symmetric bilinear form φ on $\mathcal{H}_{k,\mathbb{C}}$ and we set

$$\begin{aligned} \varphi' &= \frac{1}{q^2(q+1)^2(1-t^2)} I_u^\star \varphi && \text{if } k \text{ is even} \\ &= \frac{1}{q(q+1)^2(1-t^2)} I_u^\star \varphi && \text{if } k \text{ is odd.} \end{aligned}$$

By using Lemma 27.6, this definition ensures that, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, one has

$$\varphi(H, I_u J) = \varphi'(I_{\frac{q}{u}} H, J).$$

We define a linear endomorphism α_u on $\mathcal{H}^2_{k,\mathbb{C}}$ which is written in matrix form as

$$\alpha_u = \begin{pmatrix} 1 & -\frac{1}{q(q-1)} I_u \\ -\frac{1}{q(q-1)} I_{\frac{q}{u}} & 1 \end{pmatrix} \qquad \text{if } k \text{ is even} \tag{28.7}$$

$$= \begin{pmatrix} \frac{q}{u} & -\frac{1}{q-1} I_u \\ -\frac{1}{q-1} I_{\frac{q}{u}} & u \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

As α_u is self adjoint with respect to $\varphi \oplus \varphi'$, it defines a symmetric bilinear form χ on $\mathcal{H}^2_{k,\mathbb{C}}$, that is, for H, J in $\mathcal{H}^2_{k,\mathbb{C}}$,

$$\chi(H, J) = (\varphi \oplus \varphi')(H, \alpha_u J) = (\varphi \oplus \varphi')(\alpha_u H, J).$$

A direct computation using (28.3) and (28.5) shows that ρ_u and σ_u are symmetric with respect to χ. We set $\pi = \delta_u^\star \chi$, so that, by Lemma 28.6, $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to π. By construction, we have $\Phi_u \pi = \varphi$. □

28.5. The full dual spectral bilinear form

Given $k \geq 2$ and p a Γ-invariant k-Euclidean field, in Definition 27.17, we have introduced the spectral bilinear forms p_u associated with p. By Proposition 28.7 these objects define symmetric bilinear forms on $\mathcal{H}^2_k$ with respect to which the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric. In order to describe the null spaces of these forms, we start by constructing dual objects.

We keep using the notation (28.7) and we introduce a new operator $\beta_u : \mathcal{H}^2_{k,\mathbb{C}} \to \mathcal{H}^2_{k,\mathbb{C}}$. For u in $\mathbb{C}^*$, we set

$$\beta_u = \begin{pmatrix} 1 & \frac{1}{q(q-1)} I_u \\ \frac{1}{q(q-1)} I_{\frac{q}{u}} & 1 \end{pmatrix} \qquad \text{if } k \text{ is even}$$

$$= \begin{pmatrix} u & \frac{1}{q-1} I_u \\ \frac{1}{q-1} I_{\frac{q}{u}} & \frac{q}{u} \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

For $u^2 \neq q$, this operator is a multiple of the inverse of the operator α_u in (28.7). Indeed, by using Lemma 27.6, we get

$$\alpha_u \beta_u = \left(\frac{q - u^2}{u(q-1)} \right)^2 = \frac{(q+1)^2 t^2 - 4q}{(q-1)^2} \qquad \text{if } k \text{ is even} \tag{28.8}$$

$$= q \left(\frac{q - u^2}{u(q-1)} \right)^2 = q \frac{(q+1)^2 t^2 - 4q}{(q-1)^2} \qquad \text{if } k \text{ is odd.}$$

Besides, the adjoint operator β_u^* of β_u is defined by the matrix

$$\beta_u^* = \begin{pmatrix} 1 & \frac{1}{q(q-1)} I_{\frac{q}{u}} \\ \frac{1}{q(q-1)} I_u & 1 \end{pmatrix} \qquad \text{if } k \text{ is even} \tag{28.9}$$

$$= \begin{pmatrix} u & \frac{1}{q-1} I_{\frac{q}{u}} \\ \frac{1}{q-1} I_u & \frac{q}{u} \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

DEFINITION 28.8. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in $\mathbb{C}^*$ with $(q+1)^2 t^2 \neq 4q$. Choose u in $\mathbb{C}^*$ with $(q+1)tu = q+u^2$ and equip the space $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ with the bilinear form $p^*_u \oplus p^*_{\frac{q}{u}}$. We set $p_t^{2,*}$ to be the symmetric bilinear form on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$ such that the bilinear form $(\delta_u^*)^\star p_t^{2,*}$ is defined by the matrix β_u^* with respect to $p^*_u \oplus p^*_{\frac{q}{u}}$. We call $p_t^{2,*}$ the full dual spectral bilinear form associated with p on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$.

In other words, for r, s in $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have

$$p_t^{2,*}(\delta_u^* r, \delta_u^* s) = (p^*_u \oplus p^*_{\frac{q}{u}})(r, \beta_u^* s).$$

Note that symmetry of the matrices defining $p_t^{2,*}$ follows from Lemma 27.16.

Below, we summarize the properties of this bilinear form.

PROPOSITION 28.9. *Let $k \geq 2$ be even, p be a Γ-invariant k-Euclidean field and t be in $\mathbb{C}$ with $(q+1)^2 \neq 4q$. Then, for every r, s in $\mathcal{S}^{2,t}_{k,\mathbb{C}}$, we have*

$$p_t^{2,*}(r, \mathfrak{R}_t^* s) = p_t^{2,*}(\mathfrak{R}_t^* r, s) \text{ and } p_t^{2,*}(r, \mathfrak{S}_t^* s) = p_t^{2,*}(\mathfrak{S}_t^* r, s).$$

If $t^2 \neq 1$ and none of the two roots of the equation $(q+1)tu = q+u^2$ is an eigenvalue of S_p, then the full dual spectral bilinear form $p_t^{2,*}$ is non degenerate on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$.

If t is real, then $p_t^{2,*}$ is real.

If t belongs to the interior of the interval $\mathcal{I}_q$, then $p_t^{2,*}$ is real and positive definite on the real vector space $\mathcal{S}^{2,t}_k$.

If V is a real vector space and p is a complex symmetric bilinear form on the complexification $V_\mathbb{C}$ of V, then p is said to be real if $p(V, V) \subset \mathbb{R}$, that is, if p is obtained by complexification of a real symmetric bilinear form on V.

As in Section 21.3, we write $\mathcal{I}_q$ for the interval $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$.

To prove positivity, we shall use

LEMMA 28.10. *Let V and W be finite dimensional complex vector spaces, equipped with Hermitian scalar products p and q. Let a, d be in*

$\mathbb{R}$ and $B : W \to V$ and $C : V \to W$ be complex linear maps which are adjoint to each other with respect to p and q. Then, on $V \oplus W$, the Hermitian form defined by the matrix

$$\begin{pmatrix} a & B \\ C & d \end{pmatrix}$$

is positive if and only if a or d is positive and the endomorphism $ad - BC$ of V is positive.

PROOF. Note that BC is a non-negative Hermitian endomorphism of V since $p(BCv, v') = q(Cv, Cv')$ for v, v' in V.

Assume the matrix $\begin{pmatrix} a & B \\ C & d \end{pmatrix}$ defines a positive Hermitian form r on $V \times W$. Then, necessarily, a and d are positive. Now, pick $v \neq 0$ in V and let us show that $adp(v, v) > p(BCv, v)$. If $p(BCv, v) = 0$, there is nothing to prove. If $p(BCv, v) > 0$, note that we have

$$r\left(\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Cv \end{pmatrix}\right) = p(v, BCv).$$

Hence, by applying Cauchy-Schwarz inequality to the positive Hermitian form r, we get

$$\begin{aligned} p(v, BCv)^2 \leq r\left(\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} v \\ 0 \end{pmatrix}\right) r\left(\begin{pmatrix} 0 \\ Cv \end{pmatrix}, \begin{pmatrix} 0 \\ Cv \end{pmatrix}\right) \\ = ap(v, v)dq(Cv, Cv) = adp(v, v)p(v, BCv). \end{aligned}$$

As we have assumed $p(BCv, v) > 0$, we get $p(BCv, v) < adp(v, v)$ as required.

Conversely, assume a or d is positive and the Hermitian endomorphism $ad - BC$ is positive in V. Then, as BC is non-negative, necessarily, both a and d are positive. Now, take v in V and w in W which are not both 0. We must show the positivity of the following real number

$$s = \begin{pmatrix} v & w \end{pmatrix} \begin{pmatrix} a & B \\ C & d \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = ap(v, v) + 2q(Cv, w) + dq(w, w).$$

If $v = 0$, then $w \neq 0$ and $s = dq(w, w) > 0$. If $w = 0$, then $v \neq 0$ and $s = ap(v, v) > 0$. If $v \neq 0$ and $w \neq 0$, we have, by Cauchy-Schwarz inequality,

$$s \geq ap(v, v) - 2q(Cv, Cv)^{\frac{1}{2}} q(w, w)^{\frac{1}{2}} + dp(v, v).$$

As $ad - BC$ is positive, we have $q(Cv, Cv) = p(BCv, v) < adp(v, v)$ and we get

$$s > ap(v,v) - 2a^{\frac{1}{2}}d^{\frac{1}{2}}p(v,v)^{\frac{1}{2}}q(w,w)^{\frac{1}{2}} + dp(v,v) \\ = \left(a^{\frac{1}{2}}p(v,v)^{\frac{1}{2}} - d^{\frac{1}{2}}q(w,w)^{\frac{1}{2}}\right)^2 \geq 0.$$

The conclusion follows. □

Proof of Proposition 28.9. We choose u to be one of the roots of the equation $u^2+q = (q+1)tu$. Note that the assumption $(q+1)^2t^2 \neq 4q$ amounts to $u^2 \neq q$. The adjointness properties rely on elementary matrix computations in $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ using (28.4) and (28.6) and Lemma 28.6.

Assume that $t^2 \neq 1$, so that $u^2 \notin \{1, q^2\}$ and also that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Then, by Proposition 27.14, the bilinear forms p^*_u and $p^*_{\frac{q}{u}}$ are non degenerate. Therefore, the symmetric bilinear form $(\delta^*_u)^\star p^{2,*}_t$, which is defined by an invertible matrix with respect to $p^*_u \oplus p^*_{\frac{q}{u}}$ is non degenerate. By Lemma 28.5 and Definition 28.8, $p^{2,*}_t$ is non degenerate.

Assume that t is real, and let us show that $p^{2,*}_t$ is real, which is to say that, for r, s in $\mathcal{S}^{2,t}_{k,\mathbb{C}}$, one has

$$\overline{p^{2,*}_t(r,s)} = p^{2,*}_t(\overline{r}, \overline{s}). \tag{28.10}$$

As by assumption $(q+1)^2t^2 \neq 4q$, either t belongs to $\mathbb{R} \smallsetminus \mathcal{I}_q$ or t belongs to the interior of $\mathcal{I}_q$. In the first case, both u and $\frac{q}{u}$ are real numbers and all our previous constructions are real constructions, so that $p^{2,*}_t$ can easily be seen to be real.

It remains to deal with the case where t belongs to the interior of the interval $\mathcal{I}_q$, which we now assume. In particular, both u and $\frac{q}{u}$ are non real with modulus $\sqrt{q}$. Therefore, by Proposition 25.15, none of them is an eigenvalue of S_p, hence by the previous, $p^{2,*}_t$ is non degenerate. To show that the bilinear form $p^{2,*}_t$ is real, we will check the action of complex conjugation on all previous constructions. First, we note that, as t is real, one has $\overline{u} = \frac{q}{u}$. Therefore, we have $\overline{\mathcal{S}^u_{k,\mathbb{C}}} = \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ as subspaces of $\mathcal{S}_{k,\mathbb{C}}$. By the Definition 27.4 of the opposition map I_u, we have, for s in $\mathcal{S}^u_{k,\mathbb{C}}$,

$$\overline{I_u s} = I_{\frac{q}{u}}\overline{s} \tag{28.11}$$

and, by the Definition 27.12 of the dual spectral bilinear form p_u^*, we have, for r, s in $\mathcal{S}_{k,\mathbb{C}}^u$,

$$\overline{p_u^*(r,s)} = p_{\frac{q}{u}}^*(\overline{r},\overline{s}). \tag{28.12}$$

Finally, by the definition of the diagonalization map δ_u in (28.1), for r in $\mathcal{S}_{k,\mathbb{C}}^u$ and s in $\mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$,

$$\overline{\delta_u^* \begin{pmatrix} r \\ s \end{pmatrix}} = \delta_u^* \begin{pmatrix} \overline{s} \\ \overline{r} \end{pmatrix}. \tag{28.13}$$

Putting together (28.11), (28.12) and (28.13), we get (28.10) from the Definition 28.8 of $p_t^{2,*}$.

It remains to prove that, still for t in the interior of $\mathcal{I}_q$, the full dual spectral bilinear form $p_t^{2,*}$ is positive definite on the real vector space $\mathcal{S}_k^{2,t}$. We will show the equivalent statement that the Hermitian form $\tilde{p}_t^{2,*} : (r,s) \mapsto p_t^{2,*}(\overline{r},s)$ is positive definite on the complex vector space $\mathcal{S}_{k,\mathbb{C}}^{2,t}$. Again, this will require us to have a closer look at the definition of all the objects. First, as the real bilinear form $(r,s) \mapsto p^*(r,s) - p^{-,*}(r^<,s^<)$ is non-negative on $\mathcal{S}_k$, the Hermitian form $(r,s) \mapsto p^*(\overline{r},s) - p^{-,*}(\overline{r}^<,s^<)$ is non-negative on $\mathcal{S}_{k,\mathbb{C}}$. As by Proposition 25.15, neither u nor $\frac{q}{u}$ is an eigenvalue of S_p, by Proposition 27.14, the Hermitian form $(r,s) \mapsto p^*(\overline{r},s) - p^{-,*}(\overline{r}^<,s^<)$ is positive definite on $\mathcal{S}_k^u$. We denote it by $\tilde{p}_u^*$. To conclude, we will consider the space $\mathcal{S}_{k,\mathbb{C}}^u \oplus \mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$ which is equipped with the positive definite Hermitian form $\tilde{p}_u^* \oplus \tilde{p}_{\frac{q}{u}}^*$. On this space, there is a natural anti-automorphism γ_u which is $\gamma_u : (r,s) \mapsto (\overline{s},\overline{r})$ and (28.13) can be rewritten as

$$\overline{\delta_u^* s} = \delta_u^* \gamma_u s \quad s \in \mathcal{S}_{k,\mathbb{C}}^u \oplus \mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}. \tag{28.14}$$

By the Definition 27.12 of the dual spectral bilinear forms p_u^* and $p_{\frac{q}{u}}^*$, for r, r' in $\mathcal{S}_{k,\mathbb{C}}^u$ and s, s' in $\mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$, we have

$$(p_u^* \oplus p_{\frac{q}{u}}^*) \left(\begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} r' \\ s' \end{pmatrix} \right) = (\tilde{p}_u^* \oplus \tilde{p}_{\frac{q}{u}}^*) \left(\gamma_u \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} I_{\frac{q}{u}} s' \\ I_u r' \end{pmatrix} \right).$$

This, together with the definition of $p_t^{2,*}$ and (28.14), implies that

$$\tilde{p}_t^{2,*} \left(\delta_u^* \begin{pmatrix} r \\ s \end{pmatrix}, \delta_u^* \begin{pmatrix} r' \\ s' \end{pmatrix} \right) = (\tilde{p}_u^* \oplus \tilde{p}_{\frac{q}{u}}^*) \left(\begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} 0 & I_{\frac{q}{u}} \\ I_u & 0 \end{pmatrix} \beta_u^* \begin{pmatrix} s' \\ r' \end{pmatrix} \right). \tag{28.15}$$

To conclude, we will apply the criterion in Lemma 28.10.

If k is even, by Lemma 27.6 and (28.9), we have

$$\begin{pmatrix} 0 & I_{\frac{q}{u}} \\ I_u & 0 \end{pmatrix} \beta_u^* = \begin{pmatrix} q\frac{(q+1)^2}{q-1}(1-t^2) & I_{\frac{q}{u}} \\ I_u & q\frac{(q+1)^2}{q-1}(1-t^2) \end{pmatrix}.$$

As

$$\begin{aligned} q^2\frac{(q+1)^4}{(q-1)^2}(1-t^2)^2 - I_{\frac{q}{u}} I_u \\ = q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)((1-t^2)(q+1)^2 - (q-1)^2) \\ = q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)(4q-(q+1)^2t^2) > 0, \end{aligned}$$

by Lemma 28.10 and (28.15), the Hermitian form $\tilde{p}_t^{2,*}$ is positive definite.

If k is odd, by Lemma 27.6 and (28.9), we have

$$\begin{pmatrix} 0 & I_{\frac{q}{u}} \\ I_u & 0 \end{pmatrix} \beta_u^* = \begin{pmatrix} q\frac{(q+1)^2}{q-1}(1-t^2) & \frac{q}{u} I_{\frac{q}{u}} \\ uI_u & q\frac{(q+1)^2}{q-1}(1-t^2) \end{pmatrix}.$$

As

$$\begin{aligned} q^2\frac{(q+1)^4}{(q-1)^2}(1-t^2)^2 - qI_{\frac{q}{u}} I_u \\ = q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)((1-t^2)(q+1)^2 - (q-1)^2) \\ = q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)(4q-(q+1)^2t^2) > 0, \end{aligned}$$

again by Lemma 28.10 and (28.15), the Hermitian form $\tilde{p}_t^{2,*}$ is positive definite. □

28.6. The full spectral bilinear form

Given $k \geq 2$ and p a Γ-invariant k-Euclidean field, we now introduce the dual objects to the bilinear forms defined above.

Let t be a complex number such that $t^2 \notin \{1, \frac{4q}{(q+1)^2}\}$ and none of the roots of the equation $(q+1)tu = q + u^2$ is an eigenvalue of S_p. We have defined the full dual spectral bilinear form $p_t^{2,*}$ on the space $\mathcal{S}_{k,\mathbb{C}}^{2,t}$ of Γ-invariant t-radical pairs of complex k-simple pseudofields. In Proposition 28.9, we have shown that this bilinear form is non degenerate. Therefore, by duality, it defines a symmetric bilinear form p_t^2

on the space $\mathcal{H}^2_{k,\mathbb{C}}$ of Γ-invariant pairs of complex k-pseudofunctions whose null space is the range of the map

(28.16)

$$\mathcal{H}^2_{k-1,\mathbb{C}} \to \mathcal{H}^2_{k,\mathbb{C}}, G = \begin{pmatrix} G_0 \\ G_1 \end{pmatrix} \mapsto \begin{pmatrix} G_0^{\vee>\vee} \\ G_1^{\vee>\vee} \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} G_0^{>} \\ G_1^{>} \end{pmatrix}.$$

For H, J in $\mathcal{H}^2_{k,\mathbb{C}}$, we have $p^2_t(H,J) = \langle r, J\rangle$ where r is the unique element of $\mathcal{S}^{2,t}_{k,\mathbb{C}}$ such that, for any s in $\mathcal{S}^{2,t}_{k,\mathbb{C}}$, one has $\langle s, H\rangle = p^{2,*}_t(r,s)$.

DEFINITION 28.11. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For t as above, the symmetric bilinear form p^2_t on $\mathcal{H}^2_{k,\mathbb{C}}$ is called the full spectral bilinear form associated to t.

The full spectral bilinear form p^2_t will appear in the Plancherel formula for Euclidean fields on the continuous part of the spectrum of Corollary 26.6, which is the critical interval $\mathcal{I}_q$.

In Proposition 27.18, we have established formulas for computing the spectral bilinear form p_u, which are rather heavy to state. Fortunately, these formulas will be sufficient to characterize p^2_t.

PROPOSITION 28.12. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Fix t a complex number with $t^2 \notin \{1, \frac{4q}{(q+1)^2}\}$ and choose u to be a root of the equation $(q+1)tu = q + u^2$. Assume neither u nor $\frac{q}{u}$ is an eigenvalue of S_p. Then p^2_t enjoys the following properties.*

(i) *For every H, J in $\mathcal{H}^2_{k,\mathbb{C}}$, we have,*

$$p^2_t(\mathfrak{R}_t H, J) = p^2_t(H, \mathfrak{R}_t J) \text{ and } p^2_t(\mathfrak{S}_t H, J) = p^2_t(H, \mathfrak{S}_t J).$$

(ii) *For every H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have*

$$p^2_t\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) = (q-1)^2 u^2 p_u(H,J) \text{ if } k \text{ is even}$$
$$= (q-1)^2 u p_u(H,J) \text{ if } k \text{ is odd.}$$

Conversely, p^2_t is the unique bilinear form on $\mathcal{H}^2_{k,\mathbb{C}}$ that satisfies the two properties above.

PROOF. The uniqueness statement directly follows from Proposition 28.7.

(i) is the direct consequence of the analogous property established for $p^{2,*}_t$ in Proposition 28.9.

(ii) will follow from the explicit form of the matrices β_u in (28.9). Indeed, as $p^{2,*}_t$ is defined by the formula

$$p^{2,*}_t(\delta^*_u r, \delta^*_u s) = (p^*_u \oplus p^*_{\frac{q}{u}})(r, \beta^*_u s), \quad r, s \in \mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}},$$

the bilinear form p_t^2 is defined by the dual formula

$$p_t^2(\delta_u^{-1}H, \delta_u^{-1}J) = (p_u \oplus p_{\frac{q}{u}})(H, \beta_u^{-1}J), \quad H, J \in \mathcal{H}_{k,\mathbb{C}}^2.$$

For H in $\mathcal{H}_{k,\mathbb{C}}$, we get, from (28.2),

$$\delta_u^{-1}\begin{pmatrix} H \\ 0 \end{pmatrix} = \frac{1}{q-u^2}\begin{pmatrix} H \\ -uH \end{pmatrix}$$

and, from (28.7) and (28.8),

$$\begin{aligned} \beta_u^{-1}\begin{pmatrix} H \\ 0 \end{pmatrix} &= \left(\frac{(q-1)u}{q-u^2}\right)^2 \begin{pmatrix} H \\ -q^{-1}(q-1)^{-1}I_{\frac{q}{u}}H \end{pmatrix} \qquad \text{if } k \text{ is even} \\ &= \frac{1}{q}\left(\frac{(q-1)u}{q-u^2}\right)^2 \begin{pmatrix} \frac{q}{u}H \\ -(q-1)^{-1}I_{\frac{q}{u}}H \end{pmatrix} \qquad \text{if } k \text{ is odd.} \end{aligned}$$

The conclusion follows. □

28.7. t-radical pairs in the bipartite case

As in Section 27.6, if Γ is bipartite, the twist operation of Section 20.6 and Section 24.7 induces a symmetry in the previous constructions.

In analogy with Lemma 27.23, one can show

LEMMA 28.13. *Assume Γ is bipartite. Let $k \geq 1$, t be in $\mathbb{C}$ and $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ be in $\mathcal{S}_{k,\mathbb{C}}^2$. Then s is t-radical if and only if $\begin{pmatrix} s_0^\wr \\ -s_1^\wr \end{pmatrix}$ is $(-t)$-radical.*

Then, by direct computations, we get

LEMMA 28.14. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in $\mathbb{C}$, $t^2 \neq \frac{4q}{(q+1)^2}$. For r, s in $\mathcal{S}_{k,\mathbb{C}}^{2,t}$, we have*

$$p_{(-t)}^{2,*}\left(\begin{pmatrix} r_0^\wr \\ -r_1^\wr \end{pmatrix}, \begin{pmatrix} s_0^\wr \\ -s_1^\wr \end{pmatrix}\right) = p_t^{2,*}(r, s).$$

If moreover $t^2 \neq 1$ and none of the roots of the equation $q + u^2 = (q+1)tu$ is an eigenvalue of S_p, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have

$$p_{(-t)}^{2}\left(\begin{pmatrix} H_0^\wr \\ -H_1^\wr \end{pmatrix}, \begin{pmatrix} H_0^\wr \\ -H_1^\wr \end{pmatrix}\right) = p_t^{2}(H, J).$$

The last property could also be seen as a consequence of Lemma 27.23, Lemma 27.24 and the uniqueness part in Proposition 28.12

CHAPTER 29

Exceptional quadratic forms

For $k \geq 2$ and p a Γ-invariant k-Euclidean field, we have introduced the objects that will allow to write the Plancherel formula on the continuous part of the spectrum in Corollary 26.6, which is the critical interval $\mathcal{I}_q$. We now focus on the objects that will appear on the exceptional spectrum Σ_p defined in (26.1). Thus, we will now construct a family of quadratic forms parametrized by Σ_p.

29.1. The exceptional spectral bilinear form

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Recall from Proposition 25.15 that, if u is an eigenvalue of the simple transfer operator S_p such that $|u| > \sqrt{q}$, then u is real and simple. We then write Π_p^u for the unique projection of $\mathcal{H}_k$ onto $\ker(S_p - u)$ that commutes with S_p. In other words, for H in $\mathcal{H}_k$, we have $S_p \Pi_p^u H = u \Pi_p^u H$ and $H - \Pi_p^u H \in (S_p - u)\mathcal{H}_k$.

Definition 29.1. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. We define the exceptional spectral bilinear form p_u^{ex} associated to p and u on $\mathcal{H}_k$ as the bilinear form

$$p_u^{\mathrm{ex}}(H, J) = p(\Pi_p^u H, J^{<_p \vee >}), \quad H, J \in \mathcal{H}_k.$$

By Lemma 20.22 and Lemma 25.6, we have

Lemma 29.2. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. For H, J in $\mathcal{H}_k$, we have*

$$p_{(-u)}^{\mathrm{ex}}(H^\wr, J^\wr) = (-1)^{k+1} p_u^{\mathrm{ex}}(H, J).$$

The exceptional spectral bilinear form enjoys nonnegativity properties.

Proposition 29.3. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. The exceptional spectral bilinear form p_u^{ex} is symmetric. It is nonnegative if $u > \sqrt{q}$ or k is odd; it is nonpositive if $u < -\sqrt{q}$ and*

k is even. Its null space is exactly $(S_p - u)\mathcal{H}_k$. In particular, this null space contains the range of the map $\mathcal{H}_{k-1} \to \mathcal{H}_k, G \mapsto G^{\vee>\vee} - uG^{>}$.

Note that the adjoint operator $\Pi_p^{u,\dagger}$ of Π_p^u with respect to p is the projection onto $\ker(S_p^\dagger - u)$ that commutes with $S_p^\dagger$. The symmetry of p_u^{ex} relies on

LEMMA 29.4. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. For H in $\mathcal{H}_k$, we have*

$$\Pi_p^{u,\dagger}(H^{<_p\vee>}) = (\Pi_p^u H)^{<_p\vee>}.$$

PROOF. The definition of the operators in Section 25.1 gives $S_p H = H^{<_p\vee>\vee}$ and $S_p^\dagger H = H^{\vee<_p\vee>}$, hence

$$S_p^\dagger(H^{<_p\vee>}) = H^{<_p\vee>\vee<_p\vee>} = (S_p H)^{<_p\vee>}.$$

Therefore, if f is a polynomial function, we have $f(S_p^\dagger)(H^{<_p\vee>}) = (f(S_p)H)^{<_p\vee>}$. The conclusion follows as we can find a polynomial function f with $f(S_p) = \Pi_p^u$ and hence $f(S_p^\dagger) = \Pi_p^{u,\dagger}$. □

By abuse of notation, we also denote by Π_p^u and $\Pi_p^{u,\dagger}$ the projections of $\mathcal{H}_{k-1}$ onto $\ker(S_p - u)$ and $\ker(S_p^\dagger - u)$ that commute respectively with S_p and $S_p^\dagger$.

LEMMA 29.5. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. For H in $\mathcal{H}_k$, we have*

$$\Pi_p^{u,\dagger}(H^{<_p}) = (\Pi_p^u H)^{<_p}$$

and this element is 0 if and only if H belongs to $(S_p - u)\mathcal{H}_k$.

PROOF. Write $J = \Pi_p^u H$ and $H - J = S_p K - uK$ for some K in $\mathcal{H}_k$. Still by the definitions in Section 25.1, we have

$$H^{<_p} = J^{<_p} + K^{<_p\vee>\vee<_p} - uK^{<_p} = J^{<_p} + S_p^\dagger(K^{<_p}) - uK^{<_p}$$

and, in the same way,

$$S_p^\dagger(J^{<_p}) = J^{<_p\vee>\vee<_p} = (S_p J)^{<_p} = uJ^{<_p}.$$

We get $\Pi_p^{u,\dagger}(H^{<_p}) = J^{<_p} = (\Pi_p^u H)^{<_p}$ as required.

Assume now $J^{<_p} = 0$. As we have $J^{<_p\vee>\vee} = S_p J = uJ$ and $u \neq 0$, we get $J = 0$ and $H = S_p K - uK$, which should be proved. □

PROOF OF PROPOSITION 29.3. Let us prove the symmetry of the exceptional spectral bilinear form. For H, J in $\mathcal{H}_k$, we have, by Lemma 29.4,

$$\begin{aligned} p_u^{\mathrm{ex}}(H,J) &= p(\Pi_p^u H, J^{<_p\vee>}) = p(H, \Pi_p^{u,\dagger}(J^{<_p\vee>})) \\ &= p(H, (\Pi_p^u J)^{<_p\vee>}) = p(H^{<_p\vee>}, \Pi_p^u J) = p_u^{\mathrm{ex}}(J,H). \end{aligned}$$

Now, for the other statements, recall from Proposition 25.22, that the symmetric bilinear form $(H,J) \mapsto p(H, J^\vee)$ is positive definite on $\ker(S_p^\dagger - u)$ if $u > \sqrt{k}$ or k is odd and that it is negative definite if $u < -\sqrt{k}$ and k is even. For H, J in $\mathcal{H}_k$, we have, by Lemma 29.5,

$$\begin{aligned} p(\Pi_p^{u,\dagger}(H^{<_p}), \Pi_p^{u,\dagger}(J^{<_p})^\vee) &= p((\Pi_p^u H)^{<_p}, (\Pi_p^u J)^{<_p\vee}) \\ &= p(\Pi_p^u H, (\Pi_p^u J)^{<_p\vee>}) \\ &= p(H, \Pi_p^{u,\dagger}((\Pi_p^u J)^{<_p\vee>})) \\ &= p(H, (\Pi_p^u \Pi_p^u J)^{<_p\vee>}) \\ &= p(H, (\Pi_p^u J)^{<_p\vee>}) = p_u^{\mathrm{ex}}(H,J), \end{aligned}$$

where we have used Lemma 29.4 and the fact that $\Pi_p^u \Pi_p^u = \Pi_p^u$. The remainder of the Proposition is now a consequence of Proposition 25.22, Lemma 29.5 and the fact that, for G in $\mathcal{H}_{k-1}$, we have $G^{\vee>\vee} - uG^> = (S_p - u)(G^>)$. □

29.2. Values on $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$

In the study of the spectral theory of Euclidean fields, we will need more information on the behaviour of the exceptional spectral bilinear forms on eigenspaces of the $\vee$ operator. On these eigenspaces, we can rewrite the definition of the exceptional spectral bilinear forms.

COROLLARY 29.6. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field, u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$, H be in $\mathcal{H}_{k,+}$ and J be in $\mathcal{H}_{k,-}$. We have*

$$\begin{aligned} p_u^{\mathrm{ex}}(H,H) &= \frac{u}{q} p(\Pi_p^u H, H) && \text{if } k \text{ is even} \\ &= up(\Pi_p^u H, H) && \text{if } k \text{ is odd} \\ p_u^{\mathrm{ex}}(J,J) &= -up(\Pi_p^u J, J) \\ p_u^{\mathrm{ex}}(H,J) &= -up(\Pi_p^u H, J) \\ &= \frac{u}{q} p(H, \Pi_p^u J) && \text{if } k \text{ is even} \\ &= up(H, \Pi_p^u J) && \text{if } k \text{ is odd.} \end{aligned}$$

PROOF. By the definitions of the simple transfer operator S_p and of its adjoint operator $S_p^\dagger$ in Section 25.1, and by Definition 29.1, we get

$$p_u^{\mathrm{ex}}(J,J) = p(\Pi_p^u J, J^{<_p\vee>}) = -p(\Pi_p^u J, J^{\vee<_p\vee>}) = -p(\Pi_p^u J, S_p^\dagger J)$$
$$= -p(S_p\Pi_p^u J, J) = -up(\Pi_p^u J, J)$$

as required. The other formulas are obtained by the same technique. □

We also can show that the exceptional spectral bilinear forms are not 0 on the eigenspaces of the $\vee$ operator.

COROLLARY 29.7. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. Then, the restrictions of p_u^{ex} to $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$ are not 0.*

PROOF. Assume by contradiction that we have $p_u^{\mathrm{ex}}(H,J) = 0$ for any H, J in $\mathcal{H}_{k,+}$. Then, by Proposition 29.3, we have

$$\mathcal{H}_{k,+} \subset (S_p - u)\mathcal{H}_k.$$

As the p-orthogonal complement of $\mathcal{H}_{k,+}$ in $\mathcal{H}_k$ is $\mathcal{H}_{k,-}$, we get

$$\ker(S_p^\dagger - u) \subset \mathcal{H}_{k,-}.$$

Therefore, by assumption, we can find $H \neq 0$ in $\mathcal{H}_k$ with $S_p^\dagger H = uH$ and $H^\vee = -H$. As $S_p^\dagger H = H^{\vee<_p\vee>}$, we get $H^{<_p\vee>} = -uH$. Since $u \neq 0$, we have $H^{<_p} \neq 0$ and

$$H^{<_p\vee} = H^{<_p\vee><_p} = -uH^{<_p}.$$

By Proposition 25.9 and the assumption, we have $\sqrt{q} < |u| < q$. Thus, $(-u)$ does not belong to the set of eigenvalues of the $\vee$ operator, a contradiction.

One proves that p_u^{ex} is non zero on $\mathcal{H}_{k,-}$ in the same way. □

29.3. The full exceptional spectral bilinear form

We now apply the doubling procedure of Chapter 28.4 to the exceptional spectral bilinear forms. Recall from (26.1) that, for p in $\mathcal{P}_k$, Σ_p stands for the exceptional spectrum.

DEFINITION 29.8. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in Σ_p. In other words, we have $(q+1)^2t^2 > 4q^2$ and the unique real number u such that $q + u^2 = (q+1)tu$ and $u^2 > q$ is an eigenvalue of the simple transfer operator S_p. We denote by $p_t^{2,\mathrm{ex}}$ the unique bilinear form on $\mathcal{H}_k^2$ such that

(i) The operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ of (24.1) and (24.2) are symmetric with respect to $p_t^{2,\mathrm{ex}}$.
(ii) For H, J in $\mathcal{H}_k$, one has

$$p_t^{2,\mathrm{ex}}\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) = \left(u - \frac{q}{u}\right) p_u^{\mathrm{ex}}(H, J) \qquad \text{if } k \text{ is even}$$
$$= \left(1 - \frac{q}{u^2}\right) p_u^{\mathrm{ex}}(H, J) \qquad \text{if } k \text{ is odd.}$$

We call $p_t^{2,\mathrm{ex}}$ the full exceptional spectral bilinear form associated to p and t.

Note that the existence and uniqueness of $p_t^{2,\mathrm{ex}}$ are warranted by Proposition 28.7. Indeed, by the defintion of Σ_p in (26.1), we have $(q+1)^2t^2 > 4q$ and, by Proposition 25.9, we have $u^2 < q^2$, hence $t^2 < 1$.

From Lemma 29.2 and Definition 29.8, we directy get

LEMMA 29.9. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in Σ_p. For H, J in $\mathcal{H}_k^2$, we have*

$$p_{(-t)}^{\mathrm{ex}}\left(\begin{pmatrix} H_0^\wr \\ -H_1^\wr \end{pmatrix}, \begin{pmatrix} J_0^\wr \\ -J_1^\wr \end{pmatrix}\right) = p_t^{\mathrm{ex}}(H, J).$$

The full exceptional spectral bilinear is non-negative.

PROPOSITION 29.10. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in Σ_p. Then, the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ is nonnegative.*

PROOF. Let u be the root of the equation $q + u^2 = (q+1)tu$ with $|u| > \sqrt{q}$. The proof of the Proposition follows from a careful rereading of the proof of Proposition 28.7. Indeed, following this proof, we set, for H, J in $\mathcal{H}_k$,

$$p_{\frac{q}{u}}^{\mathrm{ex}}(H, J) = \frac{1}{q^2(q+1)^2(1-t^2)} p_u^{\mathrm{ex}}(I_u H, I_u J) \qquad \text{if } k \text{ is even}$$
$$= \frac{1}{q(q+1)^2(1-t^2)} p_u^{\mathrm{ex}}(I_u H, I_u J) \qquad \text{if } k \text{ is odd.}$$

Since Proposition 25.9 implies $t^2 < 1$, the symmetric bilinear form $p_{\frac{q}{u}}^{\mathrm{ex}}$ on $\mathcal{H}_k$ enjoys the same sign properties as p_u^{ex}. Now, from the proof of Proposition 28.7, we see that, if k is even, the bilinear form $(\delta_u^{-1})^\star p_t^{2,\mathrm{ex}}$ is defined by the matrix $u^{-1}(u^2-q)^{-1}\alpha_u$ of (28.7) with respect to $p_u^{\mathrm{ex}} \oplus p_{\frac{q}{u}}^{\mathrm{ex}}$; if k is odd, it is defined by the matrix $q^{-1}u^{-1}(u^2-q)^{-1}\alpha_u$. Note that the diagonal coefficients of α_u are 1 if k is even, and have the

same sign as u is k is odd. The conclusion follows from Lemma 27.6, the non-negativity criterion in Lemma 28.10 and Proposition 29.3. $\square$

CHAPTER 30

Special quadratic forms

Given $k \geq 2$ and a Γ-invariant k-Euclidean field p, for t in $\mathbb{C}$, we have constructed in Chapter 28 the full spectral bilinear form p_t^2 which will be used to write the Plancherel formula on the continuous part $\mathcal{I}_q$ of the spectrum in Corollary 26.6. Then, in Chapter 29, we have constructed the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ to write this formula at the points t in Σ_p. It remains to study the behaviour of the formula at the special points 1 and -1 of the spectrum: this is the purpose of this Chapter.

The constructions below rely on a direct analogy with the study of skew quadratic fields and skew dual kernels in Chapter 17 and Chapter 18.

30.1. Skew fields

We introduce a final algebraic object that is related to the behaviour of the theory of u-radical simple pseudofields at the degenerated values $u \in \{-q, -1, 1, q\}$. Its definition comes from a straightforward analogy with Section 17.1.

Definition 30.1. Let $k \geq 1$.

A k-skew field is a k-simple pseudofield s such that $s^\vee = -s$ and $s^{<\vee} = -s^<$.

If k is even, a reverse k-skew field is a k-simple pseudofield s such that $s^\vee = -s$ and $s^{<\vee} = s^<$.

If k is odd, a reverse k-skew field is a k-simple pseudofield s such that $s^\vee = s$ and $s^{<\vee} = -s^<$.

The space of Γ-invariant k-skew fields is denoted by $\mathcal{G}_k^1$. The space of Γ-invariant reverse k-skew fields is denoted by $\mathcal{G}_k^{(-1)}$.

From the dual version of Lemma 20.22, we get

Lemma 30.2. *Assume Γ is bipartite. Let $k \geq 1$ and s be a k-simple pseudofield. Then s is a k-skew field if and only if $s^\wr$ is a reverse k-skew field.*

Let us now relate these new objects to the formerly introduced ones. First, note that skew fields are 1-radical. Indeed, the definitions directly give.

LEMMA 30.3. *Let $k \geq 1$ and s be a k-simple pseudofield.*
If s is a k-skew field, then s is 1-radical.
If s is a reverse k-skew field, then s is (-1)-radical.

Skew fields can be built from q-radical simple pseudofields.

LEMMA 30.4. *Let $k \geq 1$ and s be a k-simple pseudofield.*
Assume k is even. If s is q-radical, then $qs - s^\vee$ is a k-skew field. If s is $(-q)$-radical, then $qs - s^\vee$ is a reverse k-skew field.
Assume k is odd. If s is q-radical, then $s - s^\vee$ is a k-skew field. If s is $(-q)$-radical, then $s + s^\vee$ is a reverse k-skew field.

PROOF. First, we assume that k is even. As $s^{\vee\vee} = qs+(q-1)s^\vee$, we have $(qs-s^\vee)^\vee = s^\vee - qs$. Besides, if s is q-radical, we have $s^{\vee<\vee} = qs^<$. We get

$$(qs - s^\vee)^{<\vee} = qs^{<\vee} - s^{\vee<\vee} = s^{\vee<} - qs^< = -(qs - s^\vee)^<,$$

that is, $qs - s^\vee$ is a k-skew field. If s is $(-q)$-radical, the same computation shows that $qs - s^\vee$ is a reverse k-skew field.

Now assume that k is odd. If s is q-radical, we have $s^{\vee<\vee} = qs^<$. As $s^{\vee<\vee\vee} = qs^{\vee<} + (q-1)s^{\vee<\vee}$, this gives $s^{<\vee} = s^{\vee<} + (q-1)s^<$. We get

$$(s - s^\vee)^{<\vee} = s^{<\vee} - s^{\vee<\vee} = s^{\vee<} + (q-1)s^< - qs^< = -(s - s^\vee)^<,$$

that is, $s - s^\vee$ is a k-skew field. Again, if s is $(-q)$-radical, the same computation shows that $s + s^\vee$ is a reverse k-skew field. □

Direct restriction preserves skew fields.

LEMMA 30.5. *Let $k \geq 2$ and s be a k-simple pseudofield. If s is a k-skew field, then $s^<$ is a $(k-1)$-skew field. If s is a reverse k-skew field, then $s^<$ is a reverse $(k-1)$-skew field. Direct restriction maps $\mathcal{G}_k^1$ onto $\mathcal{G}_{k-1}^1$ and $\mathcal{G}_k^{(-1)}$ onto $\mathcal{G}_{k-1}^{(-1)}$.*

PROOF. The fact that direct restriction preserves the space of k-skew fields and the one of reverse k-skew fields follows directly from the definitions and from Lemma 24.9.

For example, we show that the direct restriction maps $\mathcal{G}_k^1$ onto $\mathcal{G}_{k-1}^1$. As usual, we will prove that the adjoint map is injective. Thus, let H be in $\mathcal{H}_{k-1}$ and assume that $H^>$ is orthogonal to $\mathcal{G}_k^1$, that is, there exists J in $\mathcal{H}_{k,+}$ and K in $\mathcal{H}_{k-1,+}$ with $H^> = J + K^>$. If k is even (resp. odd), we get $(H-K)^{>\vee} = q(H-K)^>$ (resp. $(H-K)^{>\vee} = (H-K)^>$),

hence, by Proposition 20.8, there exists L in $\mathcal{H}_{k-2}$ with $L^\vee = qL$ (resp. $L^\vee = L$) and $H = K + L^>$. Therefore, H is orthogonal to $\mathcal{G}^1_{k-1}$ as should be proved. The proof for reverse skew fields is analogous. □

REMARK 30.6. By reasoning as in Proposition 17.11, one could show that the projective limit of the projective system $(\mathcal{G}^1_k)_{k\geq 1}$ (resp. $(\mathcal{G}^{(-1)}_k)_{k\geq 1}$ may be seen as the space of harmonic Γ-equivariant skew-symmetric (resp. symmetric) maps from X_1 to the space $\mathcal{D}_0(\partial X)$ of distributions θ on ∂X with $\theta(\mathbf{1}) = 0$.

REMARK 30.7. In view of the analogy between the language of skew fields and the one of skew quadratic fields in Chapter 17, it would be tempting to introduce the notions of a field and of a reverse field, by mimicking the language of Chapter 4. Thus, if $k \geq 1$ is even, we could define a k-field (resp. a reverse k-field) as a k-simple pseudofield s with $s^\vee = qs$ and $s^{<\vee} = s^<$ (resp. $s^{<\vee} = -s^<$). If k is odd, a k-field (resp. a reverse k-field) would be a k-simple pseudofield s with $s^\vee = s$ (resp. $s^\vee = -s$) and $s^{<\vee} = qs^<$. But, by Lemma 24.10, for $k \geq 2$, direct restriction would induce a linear isomorphism from the space of k-fields (resp. reverse k-fields) onto the space of $(k-1)$-fields (resp. reverse $(k-1)$-fields). Then, in view of the identification between 1-simple pseudofields and functions on X_1, the space of 1-fields (resp. reverse 1-fields), would just be the space of harmonic skew-symmetric (resp. symmetric) functions on X_1. Note that the Γ-invariant functions of this kind play a role in the proof of the Ihara trace formula in Theorem 1.4.

30.2. The special dual spectral bilinear forms

Given a Euclidean field, in analogy with the quadratic case (see Section 18.5), we build scalar products on the spaces of skew fields and reverse skew fields.

DEFINITION 30.8. (k even) Let $k \geq 2$ be even and p be a Γ-invariant k-Euclidean field. For r, s in $\mathcal{G}^1_k$, we set

$$p^{\mathrm{sp},*}_1(r,s) = \frac{q}{q+1}p^*(r,s) - \frac{1}{2}p^{-,*}(r^<, s^<).$$

For r, s in $\mathcal{G}^{(-1)}_k$, we also set

$$p^{\mathrm{sp},*}_{(-1)}(r,s) = \frac{q}{q+1}p^*(r,s) - \frac{1}{2}p^{-,*}(r^<, s^<).$$

We call $p^{\mathrm{sp},*}_1$ and $p^{\mathrm{sp},*}_{(-1)}$ the special dual spectral symmetric bilinear forms associated to p.

DEFINITION 30.9. (k odd) Let $k \geq 2$ be odd and p be a Γ-invariant k-Euclidean field. For r, s in $\mathcal{G}_k^1$, we set

$$p_1^{\mathrm{sp},*}(r,s) = \frac{1}{2}p^*(r,s) - \frac{1}{q+1}p^{-,*}(r^<, s^<).$$

For r, s in $\mathcal{G}_k^{(-1)}$, we also set

$$p_{(-1)}^{\mathrm{sp},*}(r,s) = \frac{1}{2}p^*(r,s) - \frac{1}{q+1}p^{-,*}(r^<, s^<).$$

We call $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ the special dual spectral symmetric bilinear forms associated to p.

These forms define scalar products.

LEMMA 30.10. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The special dual symmetric bilinear forms $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ are positive definite on $\mathcal{G}_k^1$ and $\mathcal{G}_k^{(-1)}$.*

PROOF. This directly follows from the fact that, as $q > 1$, we have $\frac{q}{q+1} > \frac{1}{2} > \frac{1}{q+1}$. □

In case Γ is bipartite, these forms are identified by the twist operator.

LEMMA 30.11. *Assume Γ is bipartite. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For r, s in $\mathcal{G}_k^1$, we have*

$$p_{(-1)}^{\mathrm{sp},*}(r^\wr, s^\wr) = p_1^{\mathrm{sp},*}(r,s).$$

30.3. The special spectral bilinear forms

We now use duality to build quadratic forms on spaces of pseudo-functions.

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Lemma 30.10 tells us that the special dual symmetric bilinear forms $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ are non degenerate on $\mathcal{G}_k^1$ and $\mathcal{G}_k^{(-1)}$. Therefore, by duality, they define symmetric bilinear forms p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$ on $\mathcal{H}_k$. For H, J in $\mathcal{H}_k$, we have $p_1^{\mathrm{sp}}(H,J) = \langle r, J\rangle$ and $p_{(-1)}^{\mathrm{sp}}(H,J) = \langle s, J\rangle$ where r (resp. s) is the unique element of $\mathcal{G}_k^1$ (resp. $\mathcal{G}_k^{(-1)}$) such that $\langle a, H\rangle = p_1^{\mathrm{sp},*}(r,a)$ (resp. $\langle a, H\rangle = p_{(-1)}^{\mathrm{sp},*}(s,a)$) for any a in $\mathcal{G}_k^1$ (resp. $\mathcal{G}_k^{(-1)}$).

DEFINITION 30.12. The symmetric bilinear forms p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$ are called the special symmetric bilinear forms associated to p.

We summarize the properties of p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$. Recall from Proposition 25.9 that the operators $(q - S_p)$ and $(q + S_p)$ are invertible.

PROPOSITION 30.13. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The special bilinear forms p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$ are non-negative on $\mathcal{H}_k$.*

The null space of p_1^{sp} is $\mathcal{H}_{k,+} + \mathcal{H}_{k-1,+}^{>}$. If k is even, the null space of $p_{(-1)}^{\mathrm{sp}}$ is $\mathcal{H}_{k,+} + \mathcal{H}_{k-1,-}^{>}$. If k is odd, the null space of $p_{(-1)}^{\mathrm{sp}}$ is $\mathcal{H}_{k,-} + \mathcal{H}_{k-1,+}^{>}$.

If k is even, for H, J in $\mathcal{H}_{k,-}$, one has

$$p_1^{\mathrm{sp}}(H,J) = (q+1)p((1+S_p)(q-S_p)^{-1}H, J)$$
$$p_{(-1)}^{\mathrm{sp}}(H,J) = (q+1)p((1-S_p)(q+S_p)^{-1}H, J).$$

If k is odd, for H, J in $\mathcal{H}_{k,-}$, one has

$$p_1^{\mathrm{sp}}(H,J) = 2p((q+S_p)(q-S_p)^{-1}H, J)$$

and, for H, J in $\mathcal{H}_{k,+}$, one has

$$p_{(-1)}^{\mathrm{sp}}(H,J) = 2p((q-S_p)(q+S_p)^{-1}H, J).$$

We will prove this statement in the same way as we proved Proposition 27.18 of which it can be seen as a degenerated version.

By Lemma 30.10, the dual special bilinear form $p_1^{\mathrm{sp},*}$ is non degenerate on $\mathcal{G}_k^1$. Therefore, for any r in $\mathcal{S}_{k,-}$, there exists r_1 in $\mathcal{G}_k^1$ such that, for any s in $\mathcal{G}_k^1$, one has

$$p^*(r,s) = p_1^{\mathrm{sp},*}(r_1, s).$$

In the same way, the dual special bilinear form $p_{(-1)}^{\mathrm{sp},*}$ is non degenerate on $\mathcal{G}_k^{(-1)}$. If k is even (resp. odd), for any r in $\mathcal{S}_{k,-}$ (resp. $\mathcal{S}_{k,+}$), there exists $r_{(-1)}$ in $\mathcal{G}_k^{(-1)}$ such that, for any s in $\mathcal{G}_k^{(-1)}$, one has

$$p^*(r,s) = p_{(-1)}^{\mathrm{sp},*}(r_{(-1)}, s).$$

Below, we give a formula to compute r_1 and $r_{(-1)}$. This is an analogue of Lemma 27.19.

LEMMA 30.14. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field.*

If k is even, for r in $\mathcal{S}_{k,-}$, we have

$$r_1 = \frac{q+1}{q} r - q a^{>_p} + a^{>_{p^\vee}} \text{ and } r_{(-1)} = \frac{q+1}{q} r + q b^{>_p} - b^{>_{p^\vee}}$$

where

$$a = \frac{q+1}{q}(q-S_p)^{-1}(r^{<^\vee}) \text{ and } b = \frac{q+1}{q}(q+S_p)^{-1}(r^{<^\vee}).$$

If k is odd, for r in $\mathcal{S}_{k,-}$, we have

$$r_1 = 2r - a^{>_p} + a^{>_{p^\vee}} \text{ where } a = 2(q-S_p)^{-1}(r^{<^\vee}).$$

For r in $\mathcal{S}_{k,+}$, we have

$$r_{(-1)} = 2r - b^{>_p} - b^{>_p\vee} \text{ where } b = 2(q+S_p)^{-1}(r^{<\vee}).$$

PROOF. We assume that k is even and we check the formula for r_1. Therefore, we set

$$a = \frac{q+1}{q}(q-S_p)^{-1}(r^{<\vee}) \text{ and } s = \frac{q+1}{q}r - qa^{>_p} + a^{>_p\vee}.$$

First we claim that s is a k-skew field. Indeed, on one hand, as $r^\vee = -r$, we have

$$s + s^\vee = -qa^{>_p} + a^{>_p\vee} - qa^{>_p\vee} + a^{>_p\vee\vee} = 0.$$

On the other hand, we have

$$\begin{aligned} s^< + s^{<\vee} &= \frac{q+1}{q}r^< - qa + a^{>_p\vee<} + \frac{q+1}{q}r^{<\vee} - qa^\vee + a^{>_p\vee<\vee} \\ &= \frac{q+1}{q}r^< - qa + (S_pa)^\vee + \frac{q+1}{q}r^{<\vee} - qa^\vee + S_pa. \end{aligned}$$

By definition, a satisfies the equation

$$S_pa + \frac{q+1}{q}r^{<\vee} = qa, \tag{30.1}$$

hence $s^< + s^{<\vee} = 0$.

Now, to conclude, we must show that $c = \frac{q}{q+1}s - \frac{1}{2}s^{<>_p} - r$ is p^*-orthogonal to $\mathcal{G}_k^1$. Let us compute this k-simple pseudofield. We get

$$\begin{aligned} c &= r - \frac{q^2}{q+1}a^{>_p} + \frac{q}{q+1}a^{>_p\vee} - \frac{q+1}{2q}r^{<>_p} + \frac{q}{2}a^{>_p} - \frac{1}{2}(S_pa)^{\vee>_p} - r \\ &= \frac{q}{q+1}a^{>_p\vee} - \frac{q+1}{2q}r^{<>_p} - \frac{q(q-1)}{2(q+1)}a^{>_p} - \frac{1}{2}(S_pa)^{\vee>_p}. \end{aligned}$$

Using (30.1) yields

$$c = \frac{q}{q+1}a^{>_p\vee} - \frac{q}{2}a^{\vee>_p} - \frac{q(q-1)}{2(q+1)}a^{>_p} = \frac{q}{q+1}(a^{>_p\vee} + a^{>_p}) - \frac{q}{2}(a^\vee + a)^{>_p},$$

hence c is indeed p^*-orthogonal to $\mathcal{G}_k^1$ as required and therefore $r_1 = s$. The formula for r_{-1} is obtained in the same way.

Assume now k is odd and let us again check the formula for r_1. Thus, we set

$$a = 2(q-S_p)^{-1}(r^{<\vee}) \text{ and } s = 2r - a^{>_p} + a^{>_p\vee}.$$

Again, we claim that s is a k-skew field. Indeed, as $r^\vee = -r$, we have $s^\vee = -s$. Besides,

$$s^< + s^{<\vee} = 2r^< - a + a^{>_p\vee<} + 2r^{<\vee} - a^\vee + a^{>_p\vee<\vee}.$$

As $S_p a = a^{>_p \vee <\vee}$, the definition of a yields

$$a^{>_p \vee <\vee} + 2r^{<\vee} = qa \text{ and } a^{>_p \vee <} + 2r^{<} = a^{\vee} - (q-1)a. \tag{30.2}$$

We get

$$s^{<} + s^{<\vee} = a^{\vee} - (q-1)a - a + qa - a^{\vee} = 0$$

as required.

As in the even case, to conclude, we need to show that the k-simple pseudofield c defined by $c = \frac{1}{2}s - \frac{1}{q+1}s^{<>_p} - r$ is p^*-orthogonal to $\mathcal{G}_k^1$. Indeed, the definitions of the objects give

$$\begin{aligned} c &= r - \frac{1}{2}a^{>_p} + \frac{1}{2}a^{>_p \vee} - \frac{2}{q+1}r^{<>_p} + \frac{1}{q+1}a^{>_p} - \frac{1}{q+1}a^{>_p \vee <>_p} - r \\ &= \frac{1}{2}a^{>_p \vee} - \frac{2}{q+1}r^{<>_p} - \frac{q-1}{2(q+1)}a^{>_p} - \frac{1}{q+1}a^{>_p \vee <>_p}. \end{aligned}$$

Thanks to (30.2), we have

$$\begin{aligned} c &= \frac{1}{2}a^{>_p \vee} - \frac{1}{q+1}(a^{\vee} - (q-1)a)^{>_p} - \frac{q-1}{2(q+1)}a^{>_p} \\ &= \frac{1}{2}a^{>_p \vee} - \frac{1}{q+1}a^{\vee >_p} + \frac{q-1}{2(q+1)}a^{>_p} \\ &= \frac{1}{2}(a^{>_p \vee} + a^{>_p}) - \frac{1}{q+1}(a^{\vee} + a)^{>_p}. \end{aligned}$$

Thus, c is p^*-orthogonal to $\mathcal{G}_k^1$ and hence $r_1 = s$. The same method yields the formula for r_{-1}. □

Proof of Proposition 30.13. By construction, the null space of p_1^{sp} is the orthogonal subspace of $\mathcal{G}_k^1$ in $\mathcal{H}_k$ and the null space of $p_{(-1)}^{\mathrm{sp}}$ is the orthogonal subspace of $\mathcal{G}_k^{(-1)}$ in $\mathcal{H}_k$. The description of these spaces in the Proposition directly follows from the definition of skew fields and reverse skew fields.

We now check the formulas. Assume that k is even and take r, s in $\mathcal{S}_{k,-}$. We keep the notation of Lemma 30.14. As $s^{\vee} = -s$, we get

$$p^*(r_1, s) = (q+1)p^*\left(\frac{1}{q}r + a^{>_p \vee}, s\right).$$

By Lemma 27.11 and Lemma 30.14, we have

$$a^{>_p \vee} = \frac{q+1}{q}((q - S_p)^{-1}r)^{<\vee >_p \vee} = \frac{q+1}{q}S_p(q - S_p)^{-1}r.$$

Now, we have

$$1 + (q+1)S_p(q - S_p)^{-1} = q(1 + S_p)(q - S_p)^{-1},$$

hence

$$p^*(r_1, s) = (q+1)p^*((1+S_p)(q-S_p)^{-1}r, s)$$

and the first formula follows. The other three are obtained in the same way. □

As usual, we have a symmetry property in the bipartite case. From Lemma 30.11, we get

LEMMA 30.15. *Assume Γ is bipartite and let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H, J in $\mathcal{H}_k$, we have*

$$p^{\mathrm{sp}}_{(-1)}(H^\wr, J^\wr) = p^{\mathrm{sp}}_1(H, J).$$

30.4. The full special spectral bilinear forms

We now use the special bilinear forms to build bilinear forms on $\mathcal{H}^2_k$ in the spirit of the doubling technique of Proposition 28.7.

DEFINITION 30.16. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ and $J = \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}$ in $\mathcal{H}^2_k$, we set

$$p_1^{2,\mathrm{sp}}(H, J) = p_1^{\mathrm{sp}}(qH_0 + H_1, qJ_0 + J_1)$$
$$\text{and } p_{(-1)}^{2,\mathrm{sp}}(H, J) = p_{(-1)}^{\mathrm{sp}}(qH_0 - H_1, qJ_0 - J_1).$$

The bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are called the full special spectral bilinear forms associated to p.

We summarize the properties of the full special spectral bilinear forms.

PROPOSITION 30.17. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The full special spectral bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are non-negative on $\mathcal{H}^2_k$. The null space of $p_1^{2,\mathrm{sp}}$ is the space*

$$\mathcal{H}^2_{k,+} + (\mathcal{H}^{>}_{k-1,+})^2 + \left\{ \begin{pmatrix} H \\ -qH \end{pmatrix} \middle| \, H \in \mathcal{H}_k \right\}.$$

If k is even, the null space of $p_{(-1)}^{2,\mathrm{sp}}$ is the space

$$\mathcal{H}^2_{k,+} + (\mathcal{H}^{>}_{k-1,-})^2 + \left\{ \begin{pmatrix} H \\ qH \end{pmatrix} \middle| \, H \in \mathcal{H}_k \right\}.$$

If k is odd, the null space of $p_{(-1)}^{2,\mathrm{sp}}$ is the space

$$\mathcal{H}^2_{k,-} + (\mathcal{H}^{>}_{k-1,+})^2 + \left\{ \begin{pmatrix} H \\ qH \end{pmatrix} \middle| \, H \in \mathcal{H}_k \right\}.$$

In both cases, the operators $\mathfrak{R}_1$ and $\mathfrak{S}_1$ are symmetric with respect to $p_1^{2,\mathrm{sp}}$ and the operators $\mathfrak{R}_{(-1)}$ and $\mathfrak{S}_{(-1)}$ are symmetric with respect to $p_{(-1)}^{2,\mathrm{sp}}$.

PROOF. The non-negativity properties and the description of the null spaces directly follows from Proposition 30.13 and Definition 30.16.

Assume k is even and let us show that the operators $\mathfrak{R}_1$ and $\mathfrak{S}_1$ are symmetric with respect to $p_1^{2,\mathrm{sp}}$. Recall from (24.1) that, for $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}_k^2$, we have $\mathfrak{S}_1 H = \begin{pmatrix} q^{-1} H_1^\vee \\ H_0^\vee - (q-1)H_0 \end{pmatrix}$. Therefore, for $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ and $J = \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}$ in $\mathcal{H}_k^2$,

$$p_1^{2,\mathrm{sp}}(\mathfrak{S}_1 H, J) = p_1^{\mathrm{sp}}(H_1^\vee + H_0^\vee - (q-1)H_0, qJ_0 + J_1).$$

By Proposition 30.13 $H_0 + H_0^\vee$ and $H_1 + H_1^\vee$ belong to the null space of p_1^{sp}. Thus, we get

$$p_1^{2,\mathrm{sp}}(\mathfrak{S}_1 H, J) = -p_1^{\mathrm{sp}}(qH_0 + H_1, qJ_0 + J_1).$$

By symmetry, we obtain $p_1^{2,\mathrm{sp}}(\mathfrak{S}_1 H, J) = p_1^{2,\mathrm{sp}}(H, \mathfrak{S}_1 J)$ as required. In the same way, still by (24.1), we have $\mathfrak{R}_1 H = \begin{pmatrix} H_0^\vee + q^{-1}(q+1)H_1^\vee \\ (q-1)H_1 - H_1^\vee \end{pmatrix}$, hence

$$\begin{aligned} p_1^{2,\mathrm{sp}}(\mathfrak{R}_1 H, J) &= p_1^{\mathrm{sp}}(qH_0^\vee + qH_1^\vee + (q-1)H_1, qJ_0 + J_1) \\ &= -p_1^{\mathrm{sp}}(qH_0 + H_1, qJ_0 + J_1). \end{aligned}$$

Again, we get $p_1^{2,\mathrm{sp}}(\mathfrak{R}_1 H, J) = p_1^{2,\mathrm{sp}}(H, \mathfrak{R}_1 J)$.

The other cases may be obtained by the same method. □

In the bipartite case, Definition 30.16 and Lemma 30.15 yield

LEMMA 30.18. *Assume Γ is bipartite and let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H, J in $\mathcal{H}_k^2$, we have*

$$p_{(-1)}^{2,\mathrm{sp}}\left(\begin{pmatrix} H_0^\wr \\ -H_1^\wr \end{pmatrix}, \begin{pmatrix} J_0^\wr \\ -J_1^\wr \end{pmatrix}\right) = p_1^{\mathrm{sp}}(H, J).$$

CHAPTER 31

A Plancherel formula for Euclidean fields

In this Chapter, we state and prove the Plancherel formula for a Γ-invariant Euclidean field p. We use this formula to compute the spectrum of the natural geometric operator of Section 1.5, acting on the space of Γ-invariant maps $X \to H^p$.

31.1. The formula and its spectral consequences

Given $k \geq 2$ and a Γ-invariant k-Euclidean field p, we use the previously introduced notions to write the following Plancherel formula for the non negative symmetric bilinear form $E_k^\star p$ obtained by pulling back the natural scalar product on $\mathcal{H}_\infty$ by the polyextension map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$.

THEOREM 31.1. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For any H, J in $\mathcal{H}_k^{(\mathbb{N})}$, we have*

(31.1)
$$p^\infty(E_k H, E_k J) = \frac{q+1}{2\pi(q-1)} \int_{\mathcal{I}_q} p_t^2(\widehat{H}(t), \widehat{J}(t)) \sqrt{4q - (q+1)^2 t^2} \mathrm{d}t$$
$$+ (q-1) \sum_{t \in \Sigma_p} p_t^{2,\mathrm{ex}}(\widehat{H}(t), \widehat{J}(t)) + \frac{q-1}{2(q+1)} p_1^{2,\mathrm{sp}}(\widehat{H}(1), \widehat{J}(1))$$
$$+ \frac{q-1}{2(q+1)} p_{(-1)}^{2,\mathrm{sp}}(\widehat{H}(-1), \widehat{J}(-1)).$$

The polyextension map E_k was defined in Section 20.3. The spectral transform $H \mapsto \widehat{H}, \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_k^2[t]$ was constructed in Proposition 24.3. For t in $\mathbb{C}$ with $(q+1)^2 t^2 \neq 4q$, the full spectral bilinear form p_t^2 on $\mathcal{H}_k^2$ was introduced in Definition 28.8. The exceptional spectrum Σ_p was introduced in (26.1) and, for t in Σ_p, we defined the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ on $\mathcal{H}_k^2$ in Definition 29.8. Finally, for $t = 1$ or $t = -1$, we defined the full special spectral bilinear form $p_t^{2,\mathrm{sp}}$ on $\mathcal{H}_k^2$ in Definition 30.16.

Note that, if Γ is bipartite, the symmetry properties in the spectral part of the formula which come from Lemma 28.14, Lemma 29.9 and

Lemma 30.18 are compatible with the equivariance property of the polyextension map established in Lemma 20.23 and the one of the spectral transform established in Lemma 24.18.

In analogy with the case of the model operators that we studied in Chapter 22 and Chapter 23, the proof of the formula in Theorem 31.1 will rely on the identification of the boundary values in the resolvent formula of Proposition 26.1.

Before beginning the proof, we use the formula to compute spectra of natural operators. First, we can complete Corollary 26.6.

COROLLARY 31.2. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the spectrum of P in the completion $\mathcal{H}_\infty^p$ of $\mathcal{H}_\infty$ with respect to p is the set*

$$\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}.$$

PROOF. Note that, by Lemma 25.8, we have $\Sigma_{p^+} = \Sigma_p$. Hence, we can replace p by a large orthogonal extension. In particular, by Corollary 20.7, we can assume that we have $\mathcal{H}_{k,+} \neq \{0\}$ and $\mathcal{H}_{k,-} \neq \{0\}$.

By Lemma 20.18, for H in $\mathcal{H}_k^{(\mathbb{N})}$, we have $PE_kH = E_kPH$. By Lemma 24.1 and Proposition 24.3, for t in $\mathbb{R}$, we have $\widehat{PH}(t) = t\widehat{H}(t)$. By Proposition 28.12, for t in the interior of $\mathcal{I}_q$, the full spectral bilinear form p_t^2 is positive definite. By Proposition 29.3, for t in Σ_p, the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ is non zero. By Proposition 30.17, the full special spectral bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are also non zero. Therefore, by Theorem 31.1, the spectrum of P in the closure of $E_k\mathcal{H}_k^{(\mathbb{N})}$ in $\mathcal{H}_\infty^p$ is $\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}$. The conclusion follows by applying this property to all orthogonal extensions of p. □

We can also use the Plancherel formula to prove our first statement.

PROOF OF COROLLARY 1.3. This is obtained as in Corollary 22.10 where we retrieved the computation of the spectral measures of the operator Q acting on $\ell^2(X)$.

Indeed, as in the proof of Corollary 22.10, we consider the natural operator $L : \mathcal{F}(X, H^p)^\Gamma \to \mathcal{F}(X_1, H^p)^\Gamma = \mathcal{H}_\infty^p$ defined by, for $x \sim y$ in X and f in $\mathcal{F}(X, H^p)^\Gamma$, $Lf(x, y) = f(x)$. We still have $PL = LQ$ and a direct computation using Lemma 10.11 shows that $p(Lf, Lf) = (q + 1)p(f, f)$. Therefore, by Corollary 31.2, the spectrum Σ_X of Q in $\mathcal{F}(X, H^p)^\Gamma$ is contained in $\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}$.

Conversely, by Proposition 28.12, for t in the interior of $\mathcal{I}_q$, the full spectral bilinear form p_t^2 is positive definite, hence we have $\mathcal{I}_q \subset \Sigma_X$. Besides, note that, by construction, for every f in $\mathcal{F}(X, H^p)^\Gamma$, we have

$RLf = qLf$. As by Proposition 30.17, the full special spectral bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are 0 on $\mathcal{H}^2_{j,+}$ for all even $j \geq k$, the special points (-1) and 1 do not belong to Σ_X. Finally, with no loss of generality, we can assume that k is even. For t in Σ_p, let u be the root of the equation $q + u^2 = (q+1)tu$ with $\sqrt{q} < |u| < q$. By Corollary 29.7, we know that the exceptional spectral bilinear form p_u^{ex} is non zero on $\mathcal{H}_{k,+}$. This implies that Σ_q is contained in Σ_X. The result follows. □

We now go back to proving Theorem 31.1.

31.2. Particular values of spectral quadratic forms

The resolvent formulas in Proposition 26.1 are obtained only for certain particular vectors. To deduce from these particular cases the general Plancherel formula in Theorem 31.1, we will need the following uniqueness criterion. Recall that the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ were introduced in (24.1) and (24.2).

PROPOSITION 31.3. *Let $k \geq -1$, u be in $\mathbb{C}^*$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Let ϖ be a symmetric bilinear form on $\mathcal{H}^2_{k,\mathbb{C}}$ such that the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to ϖ.*

Then, if k is even, ϖ is entirely determined by the bilinear forms

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,+,\mathbb{C}}$ and $\mathcal{H}_{k,-,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$ as well as the bilinear form

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$.

If k is odd, ϖ is entirely determined by the bilinear forms

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,+,\mathbb{C}}$ and $\mathcal{H}_{k,-,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$ as well as the bilinear form

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} -J \\ (q-1)J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$.

In particular, we can compute the values of the bilinear forms that appear in our other uniqueness criterion, that is, Proposition 28.7.

COROLLARY 31.4. *Let $k \geq -1$, u be in $\mathbb{C}^*$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Let ϖ be a symmetric bilinear form on $\mathcal{H}^2_{k,\mathbb{C}}$ such that the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to ϖ. Take H in $\mathcal{H}_{k,+,\mathbb{C}}$ and J in $\mathcal{H}_{k,-,\mathbb{C}}$.*

If k is even, we have

$$\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} H \\ -uH \end{pmatrix}\right) = \frac{q-1}{q+1}(u^2-1)\varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right)$$
$$\varpi\left(\begin{pmatrix} J \\ -uJ \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) = \frac{1}{q^2}\frac{q-1}{q+1}(q^2-u^2)\varpi\left(\begin{pmatrix} J \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)$$
$$\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) = (q-1)u\varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right).$$

If k is odd, we have

$$\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} H \\ -uH \end{pmatrix}\right) = \frac{q-1}{2u}(q+u)(u-1)\varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} 0 \\ H \end{pmatrix}\right)$$
$$\varpi\left(\begin{pmatrix} J \\ -uJ \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) = \frac{q-1}{2u}(q-u)(u+1)\varpi\left(\begin{pmatrix} 0 \\ J \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)$$
$$\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) = (q-1)\varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} -J \\ (q-1)J \end{pmatrix}\right).$$

The proofs rely on a case by case description of invariant bilinear forms.

LEMMA 31.5. *Fix t, u in $\mathbb{C}$ with $q + u^2 = (q+1)tu$. In each of the following six systems of matrix equations, the space of solutions is a complex line.*

The space of solutions of the system

$$\begin{pmatrix} q & 0 \\ (q+1)t & -1 \end{pmatrix} A = A \begin{pmatrix} q & (q+1)t \\ 0 & -1 \end{pmatrix} \tag{31.2}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the space $\mathbb{C}A_0$ where $A_0 = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ and we have

$$\begin{pmatrix} 1 & -u \end{pmatrix} A_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{q+1}(u^2-1).$$

The space of solutions of the system

$$\begin{pmatrix} -1 & 0 \\ -q^{-1}(q+1)t & q \end{pmatrix} B = B \begin{pmatrix} -1 & -q^{-1}(q+1)t \\ 0 & q \end{pmatrix} \tag{31.3}$$
$$\begin{pmatrix} 0 & -q \\ -q^{-1} & 0 \end{pmatrix} B = B \begin{pmatrix} 0 & -q^{-1} \\ -q & 0 \end{pmatrix}$$

is the space $\mathbb{C}B_0$ *where* $B_0 = \begin{pmatrix} q^2 & qt \\ qt & 1 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} B_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{q+1}(q^2-u^2).$$

The space of solutions of the system

$$\begin{pmatrix} q & 0 \\ (q+1)t & -1 \end{pmatrix} C = C \begin{pmatrix} -1 & -q^{-1}(q+1)t \\ 0 & q \end{pmatrix} \tag{31.4}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C = C \begin{pmatrix} 0 & -q^{-1} \\ -q & 0 \end{pmatrix}$$

is the space $\mathbb{C}C_0$ *where* $C_0 = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} C_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = (q-1)u.$$

The space of solutions of the system

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix} D = D \begin{pmatrix} 0 & 1 \\ q & q-1 \end{pmatrix} \tag{31.5}$$
$$\begin{pmatrix} -1 & (q+1)t-(q-1) \\ 0 & 1 \end{pmatrix} D = D \begin{pmatrix} -1 & 0 \\ (q+1)t-(q-1) & 1 \end{pmatrix}$$

is the space $\mathbb{C}D_0$ *where* $D_0 = \begin{pmatrix} q^2+1-(q^2-1)t & (q+1)t-(q-1) \\ (q+1)t-(q-1) & 2 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} D_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{u}(q+u)(u-1).$$

The space of solutions of the system

$$\begin{pmatrix} 0 & -q \\ -1 & q-1 \end{pmatrix} E = E \begin{pmatrix} 0 & -1 \\ -q & q-1 \end{pmatrix} \tag{31.6}$$
$$\begin{pmatrix} 1 & -(q+1)t-(q-1) \\ 0 & -1 \end{pmatrix} E = E \begin{pmatrix} 1 & 0 \\ -(q+1)t-(q-1) & -1 \end{pmatrix}$$

is the space $\mathbb{C}E_0$ *where* $E_0 = \begin{pmatrix} q^2+1+(q^2-1)t & (q+1)t+(q-1) \\ (q+1)t+(q-1) & 2 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} E_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{u}(q-u)(u+1).$$

The space of solutions of the system

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix} F = F \begin{pmatrix} 0 & -1 \\ -q & q-1 \end{pmatrix} \tag{31.7}$$

$$\begin{pmatrix} -1 & (q+1)t-(q-1) \\ 0 & 1 \end{pmatrix} F = F \begin{pmatrix} 1 & 0 \\ -(q+1)t-(q-1) & -1 \end{pmatrix}$$

is the space $\mathbb{C}F_0$ *where* $F_0 = \begin{pmatrix} q-1 & 1 \\ -1 & 0 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} F_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = q-1.$$

Proof of Proposition 31.3. The proof is a direct consequence of the explicit formulas in Lemma 31.5. For example, assume k is even. Then, by (24.1), the action of $\mathfrak{R}_t$ on $\mathcal{H}^2_{k,+,\mathbb{C}}$ is given by the matrix $\begin{pmatrix} q & 0 \\ (q+1)t & -1 \end{pmatrix}$ whereas the action of $\mathfrak{S}_t$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, as the space of solution to the system (31.2) in Lemma 31.5 has dimension 1 and is generated by the matrix A_0 whose top left coefficient is non zero, the restriction of ϖ to the space $\mathcal{H}^2_{k,+,\mathbb{C}} \times \mathcal{H}^2_{k,+,\mathbb{C}}$ is completely determined by the bilinear form

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,+,\mathbb{C}}$. In the same way we use (31.3) and (31.4) to prove that the restrictions of ϖ to the spaces $\mathcal{H}^2_{k,-,\mathbb{C}} \times \mathcal{H}^2_{k,-,\mathbb{C}}$ and $\mathcal{H}^2_{k,+,\mathbb{C}} \times \mathcal{H}^2_{k,-,\mathbb{C}}$ are determined respectively by the bilinear forms

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)$$

on $\mathcal{H}_{k,-,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$ and

$$(H, J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$.

We proceed similarly in the odd case by using (31.4), (31.5) and (31.6). □

Proof of Corollary 31.4. As above, if k is even, it follows from (31.2) in Lemma 31.5 that the restriction of ϖ to the space $\mathcal{H}^2_{k,+,\mathbb{C}}$ is given by a formula of the form

$$\varpi\left(\begin{pmatrix} H_0 \\ H_1 \end{pmatrix}, \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}\right) = \chi(H_0, J_0) + \chi(H_1, J_1) + t\chi(H_0, J_1) + t\chi(H_1, J_0),$$

for H_0, H_1, J_0, J_1 in $\mathcal{H}_{k,+,\mathbb{C}}$, where χ is a symmetric bilinear form on $\mathcal{H}_{k,+,\mathbb{C}}$. The formula for $\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} H \\ -uH \end{pmatrix}\right)$ follows. The other cases are dealt with in the same way. □

31.3. Imaginary parts of rational functions

As for the Plancherel formulas associated to the model operators (22.1), (22.8), (23.1) and (23.4), the proof of the Plancherel formula for Euclidean fields will rely on the identification of the boundary values of certain harmonic functions on $\mathbb{H}$. These harmonic functions are the imaginary parts of the resolvent functions which appear in Proposition 26.1. We will need purely algebraic formulas in order to compute these imaginary parts.

Lemma 31.6. *Let $\mathcal{A}$ be a real algebra and a be an element of $\mathcal{A}$. Let t be in $\mathcal{I}_q$ and u be in $\mathbb{C}^*$ with $q + u^2 = (q+1)tu$, so that $|u| = \sqrt{q}$. Assume that $u - a$ is invertible in the complexification $\mathcal{A}_\mathbb{C}$ of $\mathcal{A}$. Define elements of $\mathcal{A}_\mathbb{C}$ by setting*

$$\begin{aligned} b_{++} &= \frac{u}{1-u^2}(qu+a)(u-a)^{-1} \\ b_{+-} &= \frac{u}{q^2-u^2}(u+qa)(u-a)^{-1} \\ b_{-+} &= \frac{u}{(q+u)(1-u)}(u+a)(u-a)^{-1} \\ b_{--} &= \frac{u}{(q-u)(1+u)}(u+a)(u-a)^{-1} \\ c &= a(u-a)^{-1}. \end{aligned}$$

Then, the imaginary parts of these elements are given by

$$\begin{aligned}
\Im b_{++} &= \frac{1}{q+1}\frac{\Im u}{1-t^2}(q^2-a^2)(q-(q+1)ta+a^2)^{-1}\\
\Im b_{+-} &= \frac{1}{q+1}\frac{\Im u}{1-t^2}(1-a^2)(q-(q+1)ta+a^2)^{-1}\\
\Im b_{-+} &= \frac{2}{(q+1)^2}\frac{\Im u}{1-t^2}(1+a)(q-a)(q-(q+1)ta+a^2)^{-1}\\
\Im b_{--} &= \frac{2}{(q+1)^2}\frac{\Im u}{1-t^2}(1-a)(q+a)(q-(q+1)ta+a^2)^{-1}\\
\Im c &= -\Im(u)a(q-(q+1)ta+a^2)^{-1}.
\end{aligned}$$

PROOF. Note that the formulas make sense. Indeed, as a is real and $u-a$ is invertible, so is $\frac{q}{u}-a=\overline{u}-a$. Therefore, $q-(q+1)ta+a^2=(u-a)(\frac{q}{u}-a)$ is invertible.

Let us compute the imaginary part of b_{++}. We set $n_{++}=u(qu+a)$ and $d_{++}=(1-u^2)(u-a)$, so that $b_{++}=n_{++}d_{++}^{-1}$, hence

$$\Im b_{++}=\Im(n_{++}\overline{d_{++}})(d_{++}\overline{d_{++}})^{-1}.$$

By taking in account that $|u|=\sqrt{q}$, we get

$$\begin{aligned}
n_{++}\overline{d_{++}} &= u(qu+a)(1-\overline{u}^2)(\overline{u}-a)=(qu^2+ua-q^3-q\overline{u}a)(\overline{u}-a)\\
&= q^2u+qa-q^3\overline{u}-q\overline{u}^2a-qu^2a-ua^2+q^3a+q\overline{u}a^2,
\end{aligned}$$

which gives $\Im(n_{++}\overline{d_{++}})=(q+1)\Im(u)(q^2-a^2)$. Besides, we have

$$\begin{aligned}
d_{++}\overline{d_{++}} &= (1-u)(1-\overline{u})(1+u)(1+\overline{u})(u-a)(\overline{u}-a)\\
&= (q+1)^2(1-t^2)(q-(q+1)ta+a^2).
\end{aligned}$$

The conclusion follows. The other formulas are obtained in the same way. □

31.4. Isolated singularities of harmonic functions

To deal with the discrete part of the Plancherel formula in Theorem 31.1, we will need to recall standard properties of non-negative harmonic functions which can be found in [**1**, Chapter 7]. We use the notation of Section 21.3 for Poisson transforms.

LEMMA 31.7. *Let μ be a positive Borel measure on $\mathbb{R}$ and $a<b$ be in $\mathbb{R}$. Assume that $\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{1+t^2}<\infty$ so that the Poisson transform of μ is well-defined. Then, we have*

$$\int_a^b \mathcal{P}\mu(x+iy)\mathrm{d}x \xrightarrow[y\to 0]{} \frac{1}{2}\mu(\{a\})+\mu((a,b))+\frac{1}{2}\mu(\{b\}).$$

PROOF. This is a straightforward computation. By the definition of the Poisson transform and Fubini Theorem, for $y > 0$, we have

$$\int_a^b \mathcal{P}\mu(x+iy)\mathrm{d}x = \frac{1}{\pi}\int_a^b \int_{\mathbb{R}} \frac{y\mathrm{d}\mu(t)}{(x-t)^2+y^2}\mathrm{d}x$$
$$= \frac{1}{\pi}\int_{\mathbb{R}}\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2}\mathrm{d}\mu(t).$$

We will apply the Dominated Convergence Theorem to determine the limit of the above integral as $y \to 0$. To show the domination, we note that, for $t \le a-1$ and $a \le x \le b$, we have $(t-x)^2 \ge (t-a)^2 \ge 1$. We get, for $y > 0$,

$$(t-a)^2 + 1 \le 2(t-x)^2 + 2y^2,$$

hence

$$\frac{y}{(x-t)^2+y^2} \le \frac{2y}{(t-a)^2+1}$$

and

$$\left|\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2}\right| \le \frac{2(b-a)y}{(t-a)^2+1}. \tag{31.8}$$

Similarly, for $t \ge b+1$ and $y > 0$,

$$\left|\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2}\right| \le \frac{2(b-a)y}{(t-b)^2+1}. \tag{31.9}$$

Besides, for any t in $\mathbb{R}$ and $y > 0$, we have

$$\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2} = \arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}.$$

The latter quantity satisfies

$$\left|\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right| \le \pi \tag{31.10}$$

and

$$\begin{aligned}
&\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \xrightarrow[y\to 0]{} 0 && \text{if } t < a \text{ or } t > b\\
&\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \xrightarrow[y\to 0]{} \pi && \text{if } a < t < b\\
&\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \xrightarrow[y\to 0]{} \frac{\pi}{2} && \text{if } t = a \text{ or } t = b.
\end{aligned} \tag{31.11}$$

The bounds in (31.8), (31.9) and (31.10), together with the assumption that $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$, allow us to apply the Dominated Convergence Theorem to the convergence in (31.11). The conclusion follows. □

We shall use this property to remove isolated singularities of harmonic functions.

COROLLARY 31.8. *Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$. Fix s in $\mathbb{R}$ and assume that there exists a harmonic function F on $\mathbb{H}$ such that, for any z in $\mathbb{H}$,*

$$\mathcal{P}\mu(z) = \frac{1}{\pi}\Im\left(\frac{1}{s-z}\right) + F(z)$$

and that $F(x+iy) \underset{y\to 0}{\longrightarrow} 0$ uniformly for x in a neighborhood of s in $\mathbb{R}$. Then, there exists a positive Borel measure ν on $\mathbb{R}$ whose support does not contain s such that $\mu = \delta_s + \nu$ and one has $F = \mathcal{P}\nu$.

PROOF. Note that we have

$$\Im\left(\frac{1}{s-(x+iy)}\right) \underset{y\to 0}{\longrightarrow} 0 \tag{31.12}$$

uniformly for x in a compact subset of $\mathbb{R}$ that does not contain s. Therefore, by the assumption and by Lemma 31.7, there exists a neighborhood N of s in $\mathbb{R}$ such that, for any closed interval $I \subset N$ with $s \notin I$, one has $\mu(I) = 0$. Hence, we can write $\mu = \alpha\delta_s + \nu$ where $\alpha \geq 0$ and ν is a positive Radon measure whose support does not contain t.

As $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$, we still have $\int_{\mathbb{R}} \frac{\mathrm{d}\nu(t)}{1+t^2} < \infty$. Saying that the support of ν does not contain s amounts to saying that there exists $\varepsilon > 0$ such that, for ν-almost any t in $\mathbb{R}$, one has $|s-t| \geq \varepsilon$. For $y > 0$, this gives

$$\Im\left(\frac{1}{s-(t+iy)}\right) = \frac{y}{(s-t)^2+y^2} \leq \frac{2y}{(s-t)^2+\varepsilon^2},$$

hence, by (31.12) and the Dominated Convergence Theorem,

$$\mathcal{P}\nu(s+iy) \underset{y\to 0}{\longrightarrow} 0.$$

For $z \in \mathbb{H}$, we have

$$\frac{\alpha-1}{\pi}\Im\left(\frac{1}{s-z}\right) = F(z) - \mathcal{P}\nu(z).$$

By applying this identity to $z = s+iy$ and letting y go to 0, we get

$$\frac{\alpha-1}{y} \underset{y\to 0}{\longrightarrow} 0,$$

hence $\alpha = 1$. The conclusion follows. □

By applying elementary properties of holomorphic functions, we get

COROLLARY 31.9. *Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$. Fix s in $\mathbb{R}$ and assume that there exists an open neighborhood U of s in $\mathbb{C}$ and a holomorphic function F on $U \smallsetminus \{s\}$, with a simple pole at s, such that*

$$F(U \cap \mathbb{R}) \subset \mathbb{R} \text{ and } \mathcal{P}\mu(z) = \frac{1}{\pi}\Im F(z), \quad z \in U \cap \mathbb{H}.$$

Then, we have $\mu = \nu - a\delta_s$ where a is the residue of F at s and ν is a positive Borel measure on $\mathbb{R}$ whose support does not contain s.

The reader should beware the sign of a in the formula $\mu = \nu - a\delta_s$. In particular, the assumption imply $a \leq 0$.

31.5. Residue computations

When applying Corollary 31.9, we will need to compute the residues of certain meromorphic functions. They will always be given by the following elementary

LEMMA 31.10. *Let V be a finite-dimensional complex vector space and T be an endomorphism of V. Let s be a simple eigenvalue of T and Π be the projection on $\ker(T-s)$ with null space $(T-s)V$. Let $F \in \mathbb{C}[z,t]$ be a polynomial function. Then, the meromorphic operator valued function on $\mathbb{C}$*

$$G : z \mapsto F(z,T)(z-T)^{-1}$$

has at most a simple pole at s, with residue $F(s,s)\Pi$.

PROOF. Write $\Xi = 1 - \Pi$. As $T\Pi = s\Pi$, we get, for any z in $\mathbb{C}$ that is not an eigenvalue of T,

$$\begin{aligned} G(z) = F(z,T)(z-T)^{-1}\Pi + F(z,T)(z-T)^{-1}\Xi \\ = \frac{F(z,s)}{z-s}\Pi + F(z,T)(z-T)^{-1}\Xi. \end{aligned}$$

On one hand, by assumption, s is not an eigenvalue of T in $\Xi V = (T-s)V$. Thus, the meromorphic function $z \mapsto F(z,T)(z-T)^{-1}\Xi$ is holomorphic at s. On the other hand, the meromorphic function $z \mapsto \frac{F(z,s)}{z-s}\Pi$ has at most a simple pole at s, with residue $F(s,s)\Pi$. The conclusion follows. □

We will also need to use the change of coordinates of Section 21.3 in residue computations. This is possible thanks to an easy formula of complex analysis.

LEMMA 31.11. *Let U and V be open subsets of $\mathbb{C}$ and $\varphi : U \to V$ be a biholomorphism. Let w be in V and F be a holomorphic function on $V \smallsetminus \{w\}$ with a simple pole at w, with residue a. Set $z = \varphi^{-1}(w)$. Then, $F \circ \varphi$ has a simple pole at z, with residue $\varphi'(z)^{-1}a$.*

31.6. Proof of the Plancherel formula when $\sqrt{q}$ and $-\sqrt{q}$ are not eigenvalues of S_p

When $\sqrt{q}$ or $-\sqrt{q}$ is an eigenvalue of the simple transfer operator S_p, the resolvent formulas in Proposition 26.1 have no continuous extension in the neighborhood of the boundary points of the critical interval $\mathcal{I}_q$ in $\mathbb{C}$ and the proof of Theorem 31.1 is slightly more complicated. Therefore, we start by assuming that this is not the case.

PROOF OF THEOREM 31.1, FIRST CASE. We assume that $\sqrt{q}$ and $-\sqrt{q}$ are not eigenvalues of S_p. Recall that the definition of the scalar product on $\mathcal{H}_\infty$ in (25.1), as well as Lemma 10.11, imply that the operators R and S defined on $\mathcal{H}_\infty$ in Section 20.5 are self-adjoint with respect to this scalar product p^∞. Therefore, by Lemma 20.18, the natural operators R and S of Definition 20.16 and Definition 20.17, which act on $\mathcal{H}_k^{(\mathbb{N})}$, are symmetric with respect to the pull-back of p^∞ under the polyextension map, that is, for H, J in $\mathcal{H}_k^{(\mathbb{N})}$, we have

$$\begin{aligned} p^\infty(E_kRH, E_kJ) &= p^\infty(RE_kH, E_kJ) \\ &= p^\infty(E_kH, RE_kJ) = p^\infty(E_kH, E_kRJ) \end{aligned}$$

and, in the same way, $p^\infty(E_kSH, J) = p^\infty(E_kH, E_kSJ)$.

We use the spectral transform constructed in Proposition 24.3 to identify $\mathcal{H}_k^{(\mathbb{N})}$ with the space $\mathcal{H}_k^2[t]$ of polynomial functions $\mathbb{R} \to \mathcal{H}_k^2$. By Lemma 24.1, for t in $\mathbb{R}$, the operators of (24.1) and (24.2) satisfy

$$\mathfrak{R}_t\mathfrak{S}_t + \mathfrak{S}_t\mathfrak{R}_t - (q-1)\mathfrak{S}_t = (q+1)t. \tag{31.13}$$

Therefore, by using Lemma 21.4 as in the proof of Corollary 21.5, we get that there exists a Radon measure μ on $\mathbb{R}$ with support in $[-1, 1]$ and an integrable Borel map $t \mapsto \varpi_t, \mathbb{R} \to \mathcal{Q}_+(\mathcal{H}_k^2)$ from $\mathbb{R}$ to the space of non-negative symmetric bilinear forms on $\mathcal{H}_k^2$ such that
(i) for μ-almost any t in $\mathbb{R}$, the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ of (24.1) and (24.2) are symmetric with respect to ϖ_t.
(ii) for any H, J in $\mathcal{H}_k^{(\mathbb{N})}$, one has

$$p^\infty(E_kH, E_kJ) = \int_{\mathbb{R}} \varpi_t(\widehat{H}(t), \widehat{J}(t))\mathrm{d}\mu(t). \tag{31.14}$$

As we did for the model operators in Chapter 22 and Chapter 23, we will now use the resolvent formulas of Proposition 26.1 to compute

these spectral invariants. More precisely, we will relate the bilinear forms ϖ_t to the families of bilinear forms constructed in Chapters 27, 28, 29 and 30.

Indeed, the properties of the spectral transform in Proposition 24.3, together with (31.13) and Lemma 21.7, imply that, for H in $\mathcal{H}_k^{(\mathbb{N})}$, the function

$$z \mapsto \frac{1}{\pi} \Im p^\infty((P-z)^{-1}E_kH, E_kH)$$

on $\mathbb{H}$ is the Poisson transform of the measure $\varpi_t(\widehat{H}(t), \widehat{H}(t))\mathrm{d}\mu(t)$ on $\mathbb{R}$.

Let us have a closer look at the expression for the resolvent given in Proposition 26.1. First, we consider the right hand-side of these formulas as meromorphic functions of u in $\mathbb{C}$ with $|u| > \sqrt{q}$. By Proposition 25.15, all the eigenvalues of S_p in the set $\{u \in \mathbb{C} |\, |u| > \sqrt{q}\}$ are real and simple. By Proposition 25.9, all these eigenvalues have absolute value $< q$. Therefore, the right hand-side of the resolvent formulas define meromorphic functions of u, $|u| > \sqrt{q}$, with at most simple poles in the finite set

$$\{u \in \mathbb{R} |\, |u| > \sqrt{q}, \ker(S_p - u) \neq \{0\}\} \cup \{-q, q\}$$

and these functions are real on the real line. Now, consider the same formulas as meromorphic functions of t. Recall from Lemma 21.6 that the function $u \mapsto t = \frac{1}{q+1}(u + \frac{q}{u})$ induces a biholomorphism from the set $\{u \in \mathbb{C} |\, |u| > \sqrt{q}\}$ onto the set $\mathbb{C} \smallsetminus \mathcal{I}_q$ and that this biholomorphism maps the real set $\{u \in \mathbb{R} |\, |u| > \sqrt{q}\}$ onto $\mathbb{R} \smallsetminus \mathcal{I}_q$. Hence, the formulas in Proposition 26.1 define meromorphic functions of t in $\mathbb{C} \smallsetminus \mathcal{I}_q$ with at most simple poles in the set $\Sigma_p \cup \{-1, 1\}$, where Σ_p is defined in (26.1), and these functions are real on $\mathbb{R} \smallsetminus \mathcal{I}_q$.

Assume temporarily that k is even. By Proposition 24.3, for H in $\mathcal{H}_k$, the sequences $H\mathbf{1}_0$ and $H\mathbf{1}_1$ admit respectively as spectral transforms the constant functions with value $\begin{pmatrix} q^{-1}H^\vee \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ H \end{pmatrix}$. Besides, the Definition 20.11 of the polyextension map gives $E_k(H\mathbf{1}_0) = H^{>^\infty}$ and $E_k(H\mathbf{1}_1) = H^{+>^\infty}$. Thus, from the discussion above on the consequences of the formulas of Proposition 26.1, for H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$, the Poisson transforms of the positive measures

$$\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right)\mathrm{d}\mu(t) \text{ and } \varpi_t\left(\begin{pmatrix} J \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)\mathrm{d}\mu(t),$$

as well as the Poisson transform of the signed measure

$$\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)\mathrm{d}\mu(t),$$

are the imaginary parts of meromorphic functions of t in $\mathbb{C} \smallsetminus \mathcal{I}_q$ with at most simple poles in the set $\Sigma_p \cup \{-1, 1\}$, and which take real values on $\mathbb{R} \smallsetminus \mathcal{I}_q$. By Proposition 31.3, Lemma 31.7 and Corollary 31.9, this tells us that, after maybe changing the normalization, we can assume that there exists a Radon measure ν with support in $\mathcal{I}_q$ such that

$$\mu = \nu + \sum_{t \in \Sigma_p} \delta_t + \delta_1 + \delta_{-1}.$$

If k is odd, the analogous arguments imply the same conclusion.

In both cases, it remains to compute the values of ϖ_t, for t in $\Sigma_p \cup \{-1, 1\}$, and of the bilinear forms valued measure $\varpi_t \mathrm{d}\nu(t)$.

We first compute the values on Σ_p. As in (31.1), we will show that they are related to the full exceptional spectral bilinear forms of Chapter 29. By the reasoning above and by Corollary 31.9, it suffices to compute the residues of the meromorphic functions of Proposition 26.1 at their poles in the set $\Sigma_p \cup \{-1, 1\}$. For example, assume that k is even. Take H in $\mathcal{H}_{k,+}$. Again, Definition 20.11, Proposition 24.3 and Proposition 26.1 tell us that the Poisson transform of the positive measure

$$\varpi_t \left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix} \right) \mathrm{d}\mu(t)$$

is the imaginary part of the function on $\mathbb{H}$,

$$t \mapsto \frac{1}{\pi} \frac{q+1}{q} \frac{u}{1-u^2} p(H, (qu + S_p)(u - S_p)^{-1} H), \tag{31.15}$$

where $\Im u > 0$ and $q + u^2 = (q+1)tu$. Note that the right hand-side of the above is a meromorphic function of u, $|u| > \sqrt{q}$, with at most simple poles on the spectral values of S_p. Fix s in Σ_p and let v be the spectral value of S_p with $q + v^2 = (q+1)sv$ and $v > \sqrt{q}$. By Proposition 25.15 and Lemma 31.10, the residue of the right hand-side of (31.15) at v is

$$\frac{1}{\pi} \frac{q+1}{q} \frac{v}{1-v^2} p(H, (qv + S_p)\Pi_p^v H) = \frac{1}{\pi} \frac{(q+1)^2}{q} \frac{v^2}{1-v^2} p(H, \Pi_p^v H),$$

where, as in Section 29.1, Π_p^v stands for the projection of $\mathcal{H}_k$ onto $\ker(S_p - v)$ with null space $(S_p - v)\mathcal{H}_k$. The derivative of the function $u \mapsto t = \frac{1}{q+1}(u + \frac{q}{u})$ is the function $u \mapsto t = \frac{1}{q+1} \frac{u^2 - q}{u^2}$. Therefore, by Lemma 31.11, the residue at s of the right hand-side of (31.15), viewed as a function of t, is

$$\frac{1}{q+1} \frac{v^2 - q}{v^2} \frac{1}{\pi} \frac{(q+1)^2}{q} \frac{v^2}{1-v^2} p(H, \Pi_p^v H) = \frac{1}{\pi} \frac{q+1}{q} \frac{v^2 - q}{1-v^2} p(H, \Pi_p^v H).$$

By Corollary 31.9, we get

$$\varpi_s\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right)=\frac{q+1}{q}\frac{v^2-q}{v^2-1}p(H,\Pi_p^v H). \tag{31.16}$$

Take now J in $\mathcal{H}_{k,-}$. Still by Definition 20.11, Proposition 24.3 and Proposition 26.1, the Poisson transform of the positive measure

$$\varpi_t\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)\mathrm{d}\mu(t)$$

is the imaginary part of the function on $\mathbb{H}$,

$$t\mapsto\frac{1}{\pi}q^2(q+1)\frac{u}{q^2-u^2}p(J,(u+qS_p)(u-S_p)^{-1}J). \tag{31.17}$$

By Proposition 25.9, Proposition 25.15 and Lemma 31.10, the residue at v of the above, viewed as a function of u, is

$$\frac{1}{\pi}q^2(q+1)^2\frac{v^2}{q^2-v^2}p(J,\Pi_p^v J),$$

so that by Corollary 31.9 and Lemma 31.11,

$$\varpi_s\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)=q^2(q+1)\frac{q-v^2}{q^2-v^2}p(J,\Pi_p^v J). \tag{31.18}$$

Finally, again by Definition 20.11, Proposition 24.3 and Proposition 26.1, the Poisson transform of the signed measure

$$\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)\mathrm{d}\mu(t)$$

is the imaginary part of the function on $\mathbb{H}$,

$$t\mapsto\frac{q+1}{\pi}p(J,S_p(u-S_p)^{-1}H). \tag{31.19}$$

By Proposition 25.15 and Lemma 31.10, the residue at v of the above, viewed as a function of u, is

$$\frac{q+1}{\pi}vp(J,\Pi_p^v H),$$

so that by Corollary 31.9 and Lemma 31.11,

$$\varpi_s\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)=\frac{q-v^2}{v}p(J,\Pi_p^v H). \tag{31.20}$$

Recall that, by construction, the operators $\mathfrak{R}_s$ and $\mathfrak{S}_s$ are symmetric with respect to the symmetric bilinear form ϖ_s on $\mathcal{H}_k^2$. Therefore,

Corollary 31.4, together with (31.16), (31.18) and (31.20), gives

$$\varpi_s\left(\begin{pmatrix} H \\ -vH \end{pmatrix}, \begin{pmatrix} H \\ -vH \end{pmatrix}\right) = \frac{q-1}{q}(v^2-q)p(H, \Pi_p^v H)$$
$$\varpi_s\left(\begin{pmatrix} J \\ -vJ \end{pmatrix}, \begin{pmatrix} J \\ -vJ \end{pmatrix}\right) = (q-1)(q-v^2)p(J, \Pi_p^v J)$$
$$\varpi_s\left(\begin{pmatrix} H \\ -vH \end{pmatrix}, \begin{pmatrix} J \\ -vJ \end{pmatrix}\right) = (q-1)(q-v^2)p(J, \Pi_p^v H).$$

By comparing the latter with the formulas in Corollary 29.6, we get, from Definition 29.8, $\varpi_s = (q-1)p_s^{2,\mathrm{ex}}$ as in the statement of the Plancherel formula (31.1). The case where k is odd can be obtained by analogous computations.

Now, we compute the values of ϖ_1 and ϖ_{-1} which will be related to the full special spectral bilinear forms of Chapter 30. To simplify, we assume that k is even and we compute ϖ_1. We still let H be in $\mathcal{H}_{k,+}$ and J be in $\mathcal{H}_{k,-}$. By Proposition 25.9, the left hand-side of (31.15) and (31.19) is holomorphic at $u = q$. Thus, by Definition 20.11, Proposition 24.3, Proposition 26.1 and Lemma 31.7, we get

$$\varpi_1\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right) = \varpi_1\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right) = 0. \tag{31.21}$$

Besides, the residue at q of the right hand-side of (31.17), viewed as a function of u, is

$$-\frac{1}{2\pi}q^3(q+1)p(J, (S_p+1)(q-S_p)^{-1}J).$$

By the same arguments as above, we get

$$\varpi_1\left(\begin{pmatrix} J \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right) = \frac{1}{2}q^2(q-1)p(J, (S_p+1)(q-S_p)^{-1}J). \tag{31.22}$$

By comparing (31.21) and (31.22) with the formulas in Proposition 30.13 and the Definition 30.16 of the full special spectral quadratic forms, and lastly, by using the uniqueness result in Proposition 31.3, we get $\varpi_1 = \frac{q-1}{2(q+1)}p_1^{2,\mathrm{ex}}$ as required in (31.1). In the same way, one can show that $\varpi_{(-1)} = \frac{q-1}{2(q+1)}p_{(-1)}^{2,\mathrm{ex}}$. The case where k is odd can be dealt with analogously.

So far, we have determined the behaviour of the Plancherel formula (31.1) on the finite set $\Sigma_p \cup \{-1, 1\}$. To conclude, we will investigate its behaviour on the critical interval $\mathcal{I}_q$. As above, we assume for example that k is even and we pick H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$. As usual, for t in $\mathbb{H}$, write u for the unique solution of the equation $q+u^2 = (q+1)tu$ with $\Im u > 0$. Recall that we assume for the moment that neither $\sqrt{q}$ nor

$-\sqrt{q}$ is a spectral value of the simple transfer operator S_p. Therefore, Lemma 21.6, Proposition 25.15 and our former residue computations ensure that the function

$$t \mapsto \frac{q+1}{q}\frac{u}{1-u^2}p(H,(qu+S_p)(u-S_p)^{-1}H) - (q-1)\sum_{s\in\Sigma_q}\frac{1}{s-t}p_s^{2,\mathrm{ex}}\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right) \tag{31.23}$$

is holomorphic on $\mathbb{H}$ and admits a holomorphic continuation in the neighborhood of $\overline{\mathbb{H}}$. Besides this function goes to 0 as t goes to ∞, so that it is bounded on $\overline{\mathbb{H}}$. As the imaginary part of this holomorphic function is the Poisson transform of the positive measure

$$\pi\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right)\mathrm{d}\nu(t),$$

by [**1**, Theorem 7.5], this measure is absolutely continuous with respect to the Lebesgue measure and its density function is given by the imaginary part of the continuous extension of the right hand-side of (31.23). Since all the residue terms in this right hand-side are real, by the imaginary part computations in Lemma 31.6, this density function is the function

$$t \mapsto \frac{1}{2q}\frac{1}{1-t^2}p(H,(q^2-S_p^2)(q-(q+1)tS_p+S_p^2)^{-1}H)\sqrt{4q-(q+1)^2t^2}$$

on the interval $\mathcal{I}_q$. Reasoning in the same way and using the meromorphic function in (31.17), one shows that the positive measure

$$\pi\varpi_t\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)\mathrm{d}\nu(t),$$

is absolutely continuous with respect to the Lebesgue measure with density function

$$t \mapsto \frac{q^2}{2}\frac{1}{1-t^2}p(J,(1-S_p^2)(q-(q+1)tS_p+S_p^2)^{-1}J)\sqrt{4q-(q+1)^2t^2}$$

on $\mathcal{I}_q$. Finally, the same arguments imply, by using the the meromorphic function in (31.19), that the signed measure

$$\pi\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)\mathrm{d}\nu(t),$$

is absolutely continuous with respect to the Lebesgue measure with density function

$$t \mapsto -\frac{q+1}{2}p(J,S_p(q-(q+1)tS_p+S_p^2)^{-1}H)\sqrt{4q-(q+1)^2t^2}$$

on $\mathcal{I}_q$. By Proposition 31.3, up to a change of normalization, we can assume that ν is the measure $\sqrt{4q-(q+1)^2t^2}\mathrm{d}t$ on the interval $\mathcal{I}_q$ and that the bilinear forms ϖ_t, $t \in \mathcal{I}_q$, satisfy the relations:

$$\begin{aligned}
\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right) &= \frac{1}{2\pi q}\frac{1}{1-t^2}p(H,(q^2-S_p^2)H_t) \\
\varpi_t\left(\begin{pmatrix} J \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right) &= \frac{q^2}{2\pi}\frac{1}{1-t^2}p(J,(1-S_p^2)J_t) \\
\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right) &= -\frac{q+1}{2\pi}p(J,S_pH_t),
\end{aligned} \tag{31.24}$$

where we have set

$$H_t = (q-(q+1)tS_p+S_p^2)^{-1}H \text{ and } J_t = (q-(q+1)tS_p+S_p^2)^{-1}J.$$

Let us show that these formulas allow to relate these bilinear forms to the full spectral bilinear forms of Chapter 28. Take t to be an interior point of $\mathcal{I}_q$ and choose a root u of the equation $q+u^2=(q+1)tu$. As by construction, the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to ϖ_t, we get from (31.24), by Corollary 31.4,

$$\begin{aligned}
\varpi_t\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} H \\ -uH \end{pmatrix}\right) &= \frac{q^2-1}{2\pi q}\frac{u^2}{q^2-u^2}p(H,(q^2-S_p^2)H_t) \\
\varpi_t\left(\begin{pmatrix} J \\ -uJ \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= \frac{q^2-1}{2\pi}\frac{u^2}{u^2-1}p(J,(1-S_p^2)J_t) \\
\varpi_t\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= -\frac{q^2-1}{2\pi}up(J,S_pH_t),
\end{aligned} \tag{31.25}$$

where we used the relation

$$(q^2-u^2)(u^2-1) = (q+1)^2u^2(1-t^2).$$

By comparing (31.25) with the formulas for the spectral bilinear form p_u in Proposition 27.18, the uniqueness result in Proposition 28.12 gives $\varpi_t = \frac{1}{2\pi}\frac{q+1}{q-1}p_t^2$ as required. The same computations work in case k is odd. □

31.7. Proof of the Plancherel formula in the general case

If $\sqrt{q}$ or $-\sqrt{q}$ is a spectral value of the simple transfer operator S_p, we must modify the end of the argument of the preceding proof. Indeed, holomorphic functions as the one in (31.23) do not admit an analytic continuation to a neighborhood of $\overline{\mathbb{H}}$ since they have a singularity at at least one of the endpoints of the interval $\mathcal{I}_q$. However, it will turn out that these singularities are compensated by the vanishing at these endpoints of the coefficient $\sqrt{4q-(q+1)^2t^2}$ in (31.1).

To show this precisely, we start by describing some model harmonic functions.

LEMMA 31.12. *For z in $\mathbb{H}$, let w be the unique element of $\mathbb{H}$ with $\frac{1}{2}(w+w^{-1}) = z$ and set $F(z) = \Im[(1-w)^{-1}]$. Then the harmonic function F is the Poisson transform of the function $f : t \mapsto \frac{1}{2}\sqrt{\frac{1+t}{1-t}}$ on $[-1,1]$, that is, for $z = x+iy$ in $\mathbb{H}$, we have*

$$F(z) = \frac{1}{2\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{y\mathrm{d}t}{(x-t)^2+y^2}. \tag{31.26}$$

PROOF. As F is a non-negative harmonic function, by [**1**, Theorem 7.24], there exists a positive Borel measure μ on $\mathbb{R}$ with $\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$ and $c \geq 0$ such that, for $z = x+iy$ in $\mathbb{H}$, one has $F(z) = \mathcal{P}\mu(z) + cy$. As $F(z) \xrightarrow[z\to\infty]{} 0$, one has $c = 0$. Besides, one has $F(x+iy) \xrightarrow[y\to 0]{} 0$ uniformly for x in a compact set of $\mathbb{R}\smallsetminus[-1,1]$, so that, by Lemma 31.7, the measure μ has support in $[-1,1]$. A direct computation gives, that, uniformly for t in a compact set of $[-1,1)$, one has

$$F(t+iy) \xrightarrow[y\to 0]{} \frac{1}{2}\sqrt{\frac{1+t}{1-t}}.$$

Therefore, by Lemma 31.7, there exists $d \geq 0$ such that $\mu = \frac{1}{2}\sqrt{\frac{1+t}{1-t}}\mathbf{1}_{|t|\leq 1}\mathrm{d}t + d\delta_1$. Let G be the Poisson transform of the measure $\frac{1}{2}\sqrt{\frac{1+t}{1-t}}\mathbf{1}_{|t|\leq 1}\mathrm{d}t$, so that G is given by the right hand-side of (31.26). Then, G is a non-negative harmonic function and, for z in $\mathbb{H}$, we have

$$F(z) = G(z) + \frac{d}{\pi}\Im\left(\frac{1}{1-z}\right). \tag{31.27}$$

We claim that $d = 0$, which implies the Lemma.

Indeed, fix $s > 0$ and set $w = 1+s(1+i)$ and $z = \frac{1}{2}(w+w^{-1})$. On one hand, we have

$$F(z) = \Im\left(\frac{1}{1-w}\right) = \frac{1}{2s}. \tag{31.28}$$

On the other hand, we have

$$2(1-z) = 1-s(1+i)-\frac{1}{1+s(1+i)} = -\frac{s^2(1+i)^2}{1+s(1+i)} = -\frac{2is^2}{1+s(1+i)},$$

hence

$$\Im\left(\frac{1}{1-z}\right) = \frac{1+s}{s^2}. \tag{31.29}$$

Now, as $G \geq 0$, (31.27), (31.28) and (31.29) give

$$\frac{1}{2s} \geq \frac{d}{\pi} \frac{1+s}{s^2}.$$

Letting s go to 0 gives $d = 0$ as required. □

From this computation, we deduce a statement that will be directly applicable to the proof of Theorem 31.1.

COROLLARY 31.13. *Let ν be a finite positive Borel measure on $\mathcal{I}_q$. Let F be the holomorphic function on $\mathbb{H}_q$ with $F(u) \xrightarrow[u\to\infty]{} 0$ such that, for any u in $\mathbb{H}_q$, one has*

$$\mathcal{P}\nu(t) = \Im F(u),$$

where $t = \frac{1}{q+1}(u + \frac{q}{u})$. Assume that F admits a meromorphic extension to a neighborhood of $\overline{\mathbb{H}}_q$ with at most simple poles with real residues at $\sqrt{q}$ and $-\sqrt{q}$. Then the measure ν is absolutely continuous with respect to the Lebesgue measure, with density function

$$t \mapsto \Im F(u) = \Im F\left(\frac{q+1}{2}t + \frac{i}{2}\sqrt{4q - (q+1)^2 t^2}\right)$$

on $\mathcal{I}_q$.

PROOF. By assumption, there exist real numbers a and b and a holomorphic function G on a neighborhood of $\overline{\mathbb{H}}_q$ such that, for any u in $\mathbb{H}_q$, one has

$$F(u) = G(u) + \frac{a}{u - \sqrt{q}} + \frac{b}{u + \sqrt{q}}.$$

As G admits a continuous extension to the boundary and $G(u) \xrightarrow[u\to\infty]{} 0$, by [**1**, Theorem 7.5], the function $t \mapsto \Im G(u)$ on $\mathbb{H}$ is the Poisson transform of its restriction to $\mathbb{R}$, whereas, after a change of variable, Lemma 31.13 implies that the functions

$$t \mapsto \Im\left(\frac{1}{u - \sqrt{q}}\right) \text{ and } t \mapsto \Im\left(\frac{1}{u + \sqrt{q}}\right)$$

are the Poisson transforms of their restrictions to the interior of $\mathcal{I}_q$. The conclusion follows. □

PROOF OF THEOREM 31.1 IN THE GENERAL CASE. The proof can be lead exactly as in the case where $\sqrt{q}$ and $\sqrt{-q}$ are not eigenvalues of S_p. Indeed, Proposition 25.15 implies that $\sqrt{q}$ and $\sqrt{-q}$ are at most simple roots of the minimal polynomial of S_p and hence that we can apply Corollary 31.13 to functions such as the one in (31.23). □

Part 8

Cohomological equations

CHAPTER 32

Hölder continuous cohomology classes

The final goal of the book is the study of the inclusion $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \hookrightarrow \mathcal{Q}(\mathcal{H}_\infty)^{R,S}$ as explained in Section 1.7. We will in particular prove Corollary 1.5 which says that, for $k \geq 0$, the range of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ in $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ has finite codimension. Note that the space $\mathcal{AH}_k$ is the range of the polyextension map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$ of Definition 20.11.

This surjectivity property will actually follows from a dual injectivity statement. Indeed, as in Section 2.1, denote by $\mathscr{S}$ the space of parametrized geodesic lines of X and by $T : \mathscr{S} \to \mathscr{S}$ and $\iota : \mathscr{S} \to \mathscr{S}$ the time shift and the time reversal. Recall from Section 15.3, that the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ may be identified with the space $\mathcal{D}^*(\Gamma\backslash\mathscr{S})^{\iota,T}$ of distributions on $\Gamma\backslash\mathscr{S}$ that are invariant by both ι and T. In other words, this space may be seen as the dual space of the space of cohomology classes of smooth ι-invariant functions on $\Gamma\backslash\mathscr{S}$.

Our strategy for proving Corollary 1.5 will be to write the natural linear map

$$\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{Q}(\mathcal{AH}_k)^{R,S}$$

as the adjoint map of a map with values in the space of cohomology classes of smooth ι-invariant functions on $\Gamma\backslash\mathscr{S}$. To this aim, we need to construct a space whose dual space may be identified with $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$. This will use the spectral transform.

Indeed, with the aim of diagonalizing the operator P, in Chapter 24, we have constructed the spectral transform. For $k \geq 0$, this map induces a linear isomorphism between the space $\mathcal{AH}_k \subset \mathcal{H}_\infty$ and the quotient space of the space $\mathcal{H}_k^2[t]$ of all polynomial functions with values in $\mathcal{H}_k^2$ by the space of functions of the form

$$G(t)^{\vee>\vee} - \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} G(t)^>,$$

where $G(t)$ is in $\mathcal{H}_{k-1}^2[t]$.The spectral transform intertwines the action of R and S in $\mathcal{H}_\infty$ with the action of certain polynomial matrix operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ on $\mathcal{H}_k^2[t]$. In particular, it intertwines the action of P

and that of the multiplication by t, that is, it diagonalizes the action of P.

These properties allow to identify $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ with the dual space of a certain explicit quotient of the space $(\otimes^2\mathcal{H}_k^2)[t]$ of polynomial functions with values in the tensor square $\otimes^2\mathcal{H}_k^2$ of $\mathcal{H}_k^2$.

Therefore, to prove Corollary 1.5, we will construct a linear map $(\otimes^2\mathcal{H}_k^2)[t] \to \mathcal{D}(\Gamma\backslash\mathscr{S})$. Actually, by using an elementary reduction, we will only have to consider a certain linear map $\Omega_k : (\otimes^2\mathcal{H}_k)[t] \to \mathcal{D}(\Gamma\backslash\mathscr{S})$, which we will call the ultraweight map, since it shares some relations with the weight map of Section 9. Thanks to explicit formulas for the ultraweight map, we will be able to describe, up to a finite-dimensional subspace, the set of polynomial tensors H in $(\otimes^2\mathcal{H}_k)[t]$ such that $\Omega_k(H)$ is a coboundary with respect to the dynamics T on $\Gamma\backslash\mathscr{S}$. This will be the final result of the book, which in turn will lead to Corollary 1.5.

As a preliminary, in the next two chapters, we discuss how to solve cohomological equations with respect to the geodesic shift map T on $\Gamma\backslash\mathscr{S}$. Indeed, so far, we have only considered the cohomology relation among smooth functions on $\Gamma\backslash\mathscr{S}$ (see for example Chapter 2.3, Chapter 3.3, Chapter 12.1 and Chapter 15.2). We will now need to develop the language of cohomology classes for Hölder continuous functions, as in hyperbolic dynamics (see [**40**, Chapter 1]).

32.1. Hölder continuous functions and cohomology

We introduce the language of Hölder continuous function and the cohomology equivalence relation.

Recall from Section 2.1 that an element σ of $\mathscr{S}$ is a sequence $(\sigma_i)_{i\in\mathbb{Z}}$, where, for any i in $\mathbb{Z}$, we have $\sigma_{i+1} \sim \sigma_i$ and $\sigma_{i+1} \neq \sigma_{i-1}$. Given $0 < \alpha < 1$, we shall say that a function $f : \mathscr{S} \to \mathbb{R}$ is α-Hölder continuous if there exists $C > 0$ such that, for every σ and σ' in $\mathscr{S}$, and every $h \geq 0$, we have

$$(\forall i \quad |i| \leq h \quad \sigma_i = \sigma'_i) \Rightarrow |f(\sigma) - f(\sigma')| \leq C\alpha^h. \tag{32.1}$$

If there exists α in $(0, 1)$ such that f is α-Hölder continuous, we simply say that f is Hölder continuous.

Two Γ-invariant Hölder continuous functions f and g are said to be cohomologuous if there exists a Γ-invariant Hölder continuous function h on $\mathscr{S}$ such that $f - g = h - h \circ T$. We shall see later in Corollary 32.7 that, when f and g are smooth, then h must be smooth. Therefore, the cohomology relation on smooth functions is the same as the one defined in Section 2.3. A function is called a coboundary if it is cohomologuous

to 0. We will sometimes write $f \equiv g$ to say that f is cohomologous to g.

The following Livšic Theorem characterizes cohomology:

PROPOSITION 32.1. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Then the following are equivalent:*
(i) *f is a coboundary.*
(ii) *for any σ in $\mathscr{S}$ and $h \geq 1$ with $T^h\sigma \in \Gamma\sigma$, one has*

$$\sum_{i=0}^{h-1} f(T^i\sigma) = 0.$$

(iii) *for any T-invariant Borel probability measure μ on $\Gamma\backslash\mathscr{S}$, one has*

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu = 0.$$

(iv) *There exists a continuous function h on $\Gamma\backslash\mathscr{S}$ with $f = h - h \circ T$.*

PROOF. This is an adaptation of the classical argument from hyperbolic dynamics (see [**26**, Theorem 19.2.1]).

Note that the implications *(i)⇒(iv)*, *(iv)⇒(iii)* and *(iii)⇒(ii)* are obvious. We will now show *(ii)⇒(i)*.

Let f be a Hölder continuous function on $\mathscr{S}$ which satisfies *(ii)*. Fix C, α as in (32.1). We will build a Hölder continuous function h on $\mathscr{S}$ with $f = h - h \circ T$. By Proposition 2.3, there exists σ in $\mathscr{S}$ whose orbit under T is dense in $\Gamma\backslash\mathscr{S}$. In particular, for any $i \neq j \in \mathbb{Z}$, we have $T^i\sigma \neq T^j\sigma$. We start by defining h on $T^{\mathbb{Z}}\sigma$ by setting, for $j \geq 0$,

$$h(T^j\sigma) = -\sum_{i=0}^{j-1} f(T^i\sigma) \text{ and } h(T^{-j}\sigma) = \sum_{i=0}^{j-1} f(T^{i-j}\sigma).$$

We get $f = h - h \circ T$ on $T^{\mathbb{Z}}\sigma$. To conclude, we will show that h may be extended by continuity to all of $\Gamma\backslash\mathscr{S}$. By standard arguments of topology, it suffices to prove that h satisfies a uniform continuity property.

Therefore, we choose integers $i < j$ and we assume that $T^i\sigma$ and $T^j\sigma$ are closed in $\Gamma\backslash\mathscr{S}$. In other words, there exists $\ell \geq 1$ and γ in Γ with $\sigma_{j+k} = \gamma\sigma_{i+k}$ for all $k \in \mathbb{Z}$ with $|k| \leq \ell$. We defined a family $\tau = (\tau_k)_{k\in\mathbb{Z}}$ of points in X by setting, for k in $\mathbb{Z}$,

$$\tau_k = \gamma^q \sigma_{i+r},$$

where $0 \leq r < j - i$ and $k = q(j - i) + r$.

We claim that τ is a parametrized geodesic line. Indeed, since σ is a parametrized geodesic line, for every k in $\mathbb{Z} \smallsetminus (j-i)\mathbb{Z}$, τ_{k-1} and τ_{k+1}

are different neighbours of τ_k. If $k = q(j-i)$ for q in $\mathbb{Z}$, this is still the case: indeed, we have

$$\begin{aligned}\tau_{k-1} &= \gamma^{q-1}\sigma_{j-1}\\ \tau_k &= \gamma^q\sigma_i = \gamma^{q-1}\sigma_j\\ \tau_{k+1} &= \gamma^q\sigma_{i+1} = \gamma^{q-1}\sigma_{j+1},\end{aligned}$$

where we have used the fact that $\sigma_j = \gamma\sigma_i$ and $\sigma_{j+1} = \gamma\sigma_{i+1}$.

Let us use this construction and the assumption on f to show that $h(T^i\sigma)$ and $h(T^j\sigma)$ are close to each other. Indeed, by definition, we have

$$h(T^i\sigma) - h(T^j\sigma) = \sum_{k=0}^{j-i-1} f(T^{i+k}\sigma).$$

Assume first that $\ell < j-i$. Then, by construction, for any $-\ell \leq k \leq j-i+\ell$, we have $\sigma_{i+k} = \tau_k$. This and the assumption that f is Hölder continuous, tell us that, for $0 \leq k \leq j-i-1$, we have

$$\big|f(T^{i+k}\sigma) - f(T^k\tau)\big| \leq C\alpha^{\min(k,j-i-k)+\ell}.$$

Besides, as $T^{j-i}\tau = \tau$, by the assumption on f, we have

$$\sum_{k=0}^{j-i-1} f(T^k\tau) = 0.$$

We get

$$\big|h(T^i\sigma) - h(T^j\sigma)\big| \leq C\alpha^\ell \sum_{k=0}^{j-i-1} \alpha^{\min(k,j-i-k)} \leq 2C\alpha^\ell \sum_{k=0}^{\infty} \alpha^k = \frac{2C}{1-\alpha}\alpha^\ell.$$

If $\ell \geq j-i$, as the orbit of σ under T is dense in $\Gamma\backslash\mathscr{S}$, we can find an integer g with $|g-i| > \ell$ and $|g-j| > \ell$ and an element γ' in Γ such that, for any k with $|k| \leq \ell$, one has $\sigma_{g+k} = \gamma'\sigma_{i+k}$, and hence also $\sigma_{g+k} = \gamma'\gamma^{-1}\sigma_{j+k}$. By applying the previous case, we get

$$\big|h(T^i\sigma) - h(T^g\sigma)\big| \leq \frac{2C}{1-\alpha}\alpha^\ell \text{ and } \big|h(T^j\sigma) - h(T^g\sigma)\big| \leq \frac{2C}{1-\alpha}\alpha^\ell.$$

Hence

$$\big|h(T^i\sigma) - h(T^j\sigma)\big| \leq \frac{4C}{1-\alpha}\alpha^\ell.$$

Standard arguments of topology tell us that h admits a Hölder continuous extension to all of $\Gamma\backslash\mathscr{S}$. □

32.2. Functions on $\mathscr{S}_+$

In Chapter 33, we will establish a criterion for a certain type of Hölder continuous functions to be coboundaries. To this aim, we will need to study functions only depending on the future. We start by showing that every Hölder continuous function is cohomologuous to such a function.

As in Section 2.3, say that a function f on $\mathscr{S}$ is M-invariant if, for any σ and σ' in $\mathscr{S}$, if $\sigma_h = \sigma'_h$ for any $h \geq 0$, then $f(\sigma) = f(\sigma')$. Write $\mathscr{S}_+ = X \times \partial X$ and $\varpi : \mathscr{S} \to \mathscr{S}_+, \sigma \mapsto (\sigma_0, \sigma^+)$. An M-invariant function is a function of the form $f_+ \circ \varpi$, where f_+ is a function on $\mathscr{S}_+$.

We have an analogue of a classical result of hyperbolic dynamics.

LEMMA 32.2. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Then f is cohomologous to an M-invariant Hölder continuous function.*

If f is smooth, this is Lemma 2.14.

PROOF. If Γ is torsion free, this can be directly deduced from the analogue statement for subshifts of finite type (see [**40**, Proposition 1.2]). We prove the general case by following the same lines.

We start by choosing a section of the natural map $\mathscr{S} \to \mathscr{S}_+$ as follows. We pick a system of representatives $D \subset X_1$ for the action of Γ, that is, we have $X_1 = \Gamma D$ and $\Gamma(x, y) \cap D = \{(x, y)\}$ for any (x, y) in D. Then, we choose any map $\psi : D \to \partial X$ such that, for (x, y) in X_1, we have $y \notin [x\psi(x, y))$. We extend ψ to all X_1: for (x, y) in $X_1 \smallsetminus D$, we set $\psi(x, y) = \gamma\psi(\gamma^{-1}(x, y))$, where γ is an arbitrary element of Γ with $\gamma(x, y) \in D$. The map ψ is not Γ-equivariant in general, but it satisfies the following properties which will be enough for our purposes:

$$\forall (x, y) \in X_1, \quad y \notin [x\psi(x, y)) \text{ and } \psi(\Gamma(x, y)) \subset \Gamma\psi(x, y).$$

For (x, ξ) in $\mathscr{S}_+ = X \times \partial X$, we set $\rho(x, \xi)$ to be the parametrized geodesic line σ with $\sigma_0 = x$, $\sigma^+ = \xi$ and $\sigma^- = \psi(x, y)$, where $y = \sigma_1$ is the neighbour of x on $[x\xi)$.

Let σ be in $\mathscr{S}$. The element $\rho(\sigma_0, \sigma^+)$ has the same future as σ, that is $\varpi\rho(\sigma_0, \sigma^+) = \varpi(\sigma)$. We set

$$h(\sigma) = \sum_{k \geq 0} f(T^k\sigma) - f(T^k\rho(\sigma_0, \sigma^+)).$$

The series converges as f is Hölder continuous and the function h is Γ-invariant. We have

$$h(\sigma) - h(T\sigma) - f(\sigma) = \\ - f(\rho(\sigma_0, \sigma^+)) + \sum_{k\geq 0} f(T^k \rho(\sigma_1, \sigma^+)) - f(T^{k+1}\rho(\sigma_0, \sigma^+)).$$

As the right hand side of the latter is clearly M-invariant, it only remains to prove that h is Hölder continuous.

Indeed, let C and α be as in (32.1). Pick $\ell \geq 1$ and σ and τ in $\mathscr{S}$ with $\sigma_i = \tau_i$ for any $|i| \leq \ell$. Note that by construction, the same property holds when σ and τ are replaced by $\rho(\sigma_0, \sigma^+)$ and $\rho(\tau_0, \tau^+)$. We write

$$h(\sigma) - h(\tau) = \\ \sum_{k=0}^{m} f(T^k\sigma) - f(T^k\tau) - \sum_{k=0}^{m} f(T^k\rho(\sigma_0, \sigma^+)) - f(T^k\rho(\tau_0, \tau^+)) \\ + \sum_{k=m+1}^{\infty} f(T^k\sigma) - f(T^k\rho(\sigma_0, \sigma^+)) - \sum_{k=m+1}^{\infty} f(T^k\tau) - f(T^k\rho(\tau_0, \tau^+)),$$

where m is the floor integer of $\frac{\ell}{2}$. This gives

$$|h(\sigma) - h(\tau)| \leq 2C \sum_{k=0}^{m} \alpha^{\ell-k} + 2C \sum_{k=m+1}^{\infty} \alpha^k \leq \frac{4C}{1-\alpha}\alpha^m$$

and the conclusion follows. □

32.3. Transfer operator

We introduce a transfer operator acting on functions on $\Gamma\backslash\mathscr{S}_+$. The properties of this operator will allow us to develop criteria for Hölder continuous functions to be coboundaries.

Let f be a function on $\mathscr{S}_+ = X \times \partial X$. By analogy with [**40**, Chapter 2], for (x, ξ) in $\mathscr{S}_+$, we set

$$\mathscr{L} f(x, \xi) = \frac{1}{q} \sum_{\substack{y\sim x \\ y\notin[x\xi)}} f(y, \xi). \tag{32.2}$$

We call $\mathscr{L}$ the transfer operator (in the language of [**40**], we should say the transfer operator associated to the constant potential $\log q$). Note that $\mathscr{L}\mathbf{1} = \mathbf{1}$.

We will establish some standard spectral properties of the operator $\mathscr{L}$. Later, they will help us to solve cohomological equations.

First, we introduce a family of Banach spaces. Given α in $(0,1)$ and f a function on $\mathscr{S}_+$, we say that f is α-Hölder continuous if the function $f \circ \varpi$ on $\mathscr{S}$ is α-Hölder continuous. Note from (32.1), that the function f is α-Hölder continuous if and only if there exists $C > 0$ such that, for every $h \geq 0$, x in X and ξ, η in ∂X, we have

$$|[x\xi) \cap [x\eta)| \geq h + 1 \Rightarrow |f(x,\xi) - f(x,\eta)| \leq C\alpha^h.$$

The infimum of all C satisfying the above inequality is then called the α-Hölder constant of f and is denoted by $C_\alpha(f)$.

For α in $(0,1)$, we let $\mathscr{H}_\alpha^+$ denote the space of all Γ-invariant α-Hölder continuous functions on $\mathscr{S}_+$. The space $\mathscr{H}_\alpha^+$ is a Banach space with respect to the natural Hölder norm defined by, for any f in $\mathscr{H}_\alpha^+$,

$$\|f\|_\alpha = \sup |f| + C_\alpha(f).$$

As mentioned above, we have $\mathscr{L}\mathbf{1} = \mathbf{1}$. If Γ is not bipartite (see Section 20.1), we set $\overline{\mathscr{H}}_\alpha^+ = \mathscr{H}_\alpha^+ / \mathbb{R}\mathbf{1}$ to be the quotient of $\mathscr{H}_\alpha^+$ by the line of constant functions.

If Γ is bipartite, as in Section 15.1 and Section 20.6, we let $\chi : X \to \{-1, 1\}$ be such that, for any (x, y) in X_1, one has $\chi(x)\chi(y) = -1$. Saying that Γ is bipartite is saying that this function χ is Γ-invariant. We then set $\overline{\mathscr{H}}_\alpha^+$ to be the quotient space of $\mathscr{H}_\alpha^+$ by the 2-plane spanned by the function $\mathbf{1}$ and the function $(x, \xi) \mapsto \chi(x)$. Note that the latter function is an eigenvector of the operator $\mathscr{L}$ for the eigenvalue -1.

The transfer operator $\mathscr{L}$ has a spectral gap in $\overline{\mathscr{H}}_\alpha^+$.

PROPOSITION 32.3. *Let α be in $(0,1)$. The operator $\mathscr{L}$ has spectral radius < 1 in the quotient space $\overline{\mathscr{H}}_\alpha^+$. In particular, any function f in $\mathscr{H}_\alpha^+$ may be written as $f = g - \mathscr{L}g + c$ for some constant function c and some g in $\mathscr{H}_\alpha^+$. The constant c is uniquely defined by f and g is unique up to the addition of a constant function.*

In the course of the proof, we will need elementary facts from abstract functional analysis.

LEMMA 32.4. *Let V be a Banach space and $T : V \to V$ be a bounded linear operator.*
(i) *Assume W is a finite dimensional subspace of V with $TW \subset W$. Then the spectrum of T in V is the union of the spectrum of T in W and the spectrum of T in V/W.*
(ii) *Let $v \neq 0$ be a vector of V with $Tv = v$. Assume 1 is not a spectral value of T in $V/\mathbb{R}v$. Then $(1 - T)V$ is a closed subspace of V and $V = \mathbb{R}v \oplus (1 - T)V$.*

PROOF. *(i)* For $\lambda \in \mathbb{C}$, assume λ is not a spectral value of T in V. Then, λ is not an eigenvalue of T in V, hence not in W. As W is finite dimensional, the restriction of $\lambda - T$ to W is invertible, which means that $(\lambda - T)^{-1}$ preserves W. Then, it induces an endomorphism in V/W, which is the inverse of the one induced by $\lambda - T$, so that $\lambda - T$ is invertible in V/W.

Conversely, assume λ is neither a spectral value of T in W nor in V/W. As V is finite-dimensional, it admits a closed complementary subspace X and the decomposition $V = W \oplus X$ is an isomorphism of Banach spaces. In this decomposition, T may be written as a matrix $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is an endomorphism of W, B is a bounded linear map $X \to W$ and C is a bounded endomorphism of X. By assumption, both $\lambda - A$ and $\lambda - C$ are invertible. Then, $\lambda - T$ is invertible with inverse in matrix form

$$(\lambda - T)^{-1} = \begin{pmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1} B (\lambda - C)^{-1} \\ 0 & (\lambda - C)^{-1} \end{pmatrix}.$$

(ii) As above, we write $V = \mathbb{R}v \oplus X$, where X is a closed hyperplane. The endomorphism T may be written as

$$T(tv + x) = (t + \varphi(x))v + Sx, \quad t \in \mathbb{R}, \quad x \in X,$$

where S is a bounded endomoprhism of X and φ is a continuous linear functional on X. By assumption, 1 is not a spectral value of S. We set $\psi = \varphi \circ (S - 1)^{-1}$, so that ψ is also a continuous linear functional of X. For x in X, we have

$$T(\psi(x)v + x) = (\psi(x) + \varphi(x))v + Sx = \psi(Sx) + Sx,$$

hence the space $Y = \{\psi(x)v + x | x \in X\}$ is stable under T. One easily checks that Y is closed, that $V = \mathbb{R}v \oplus Y$ and that $(T - 1)V = Y$. □

PROOF OF PROPOSITION 32.3. We let $V \subset \mathscr{H}^+_\alpha$ be the space of functions on $\Gamma \backslash \mathscr{S}_+$ which are of the form $(x, \xi) \mapsto v(x, x_1)$, where x_1 is the neighbour of x on $[x\xi)$ and v is a Γ-invariant function on X_1. Then V is stable under $\mathscr{L}$ and, by Corollary 15.7, the operator $\mathscr{L}$ has spectral radius < 1 on the image of V in $\overline{\mathscr{H}}^+_\alpha$.

To conclude, we will show that $\mathscr{L}$ has spectral radius $\leq \alpha$ in $\mathscr{H}^+_\alpha / V$. To this aim, we change slightly the definition of the Hölder constant of a function: for f in $\mathscr{H}^+_\alpha$, we set

$$C^1_\alpha(f) = \sup_{\substack{(x,\xi,\eta) \in X \times \partial X \times \partial X \\ h \geq 1 \\ |[x\xi) \cap [x\eta)| \geq h+1}} \alpha^{-h} |f(x, \xi) - f(x, \eta)|,$$

that is, we consider the above supremum only when the intersection $[x\xi) \cap [x\eta)$ contains at least two points. One easily checks that the seminorm C^1_α induces a norm on $\mathscr{H}^+_\alpha/V$ which is equivalent to the quotient norm of the Hölder norm. Now, a direct computation shows that, for f in $\mathscr{H}^+_\alpha$, one has

$$C^1_\alpha(\mathscr{L} f) \leq \alpha C^1_\alpha(f),$$

hence $\mathscr{L}$ has spectral radius $\leq \alpha$ in $\mathscr{H}^+_\alpha/V$.

It follows from the first part of Lemma 32.4 that $\mathscr{L}$ has spectral radius < 1 in $\overline{\mathscr{H}}^+_\alpha$. The rest of the statement is a consequence of the second part of Lemma 32.4. □

32.4. Decomposition of Hölder continuous functions

Thanks to the spectral properties of the transfer operator $\mathscr{L}$, we get a way of decomposing a Hölder continuous function.

COROLLARY 32.5. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Then f may be written as*

$$g = h - h \circ T + g \circ \varpi + c,$$

where c is a constant function, h is a Hölder continuous function on $\Gamma\backslash\mathscr{S}$, g is a Hölder continuous function on $\Gamma\backslash\mathscr{S}_+$ and $\mathscr{L} g = 0$. The constant c and the function g are uniquely determined by f; the function h is uniquely determined up to the addition of a constant function.

PROOF. First, we prove the existence of the decomposition. By Lemma 32.2, we can find a Hölder continuous function h_1 on $\Gamma\backslash\mathscr{S}$ and a Hölder continuous function f_1 on $\Gamma\backslash\mathscr{S}_+$ with

$$f = h_1 - h_1 \circ T + f_1 \circ \varpi.$$

By Proposition 32.3, there exists a Hölder continuous function h_2 on $\Gamma\backslash\mathscr{S}_+$ and a constant c such that

$$\mathscr{L} f_1 = \mathscr{L} h_2 - h_2 + c.$$

Write $T_+ : \mathscr{S}_+ \to \mathscr{S}_+$ for the natural transformation, that is, for (x, ξ) in $\mathscr{S}_+$, $T_+(x, \xi) = (x_1, \xi)$, where x_1 is the neighbour of x on $[x\xi)$. We have $\varpi T = T_+ \varpi$ and, for any function φ on $\mathscr{S}_+$, $\mathscr{L}(\varphi \circ T_+) = \varphi$. Thus, if we set $g = f_1 - h_2 + h_2 \circ T_+ - c$, we get $\mathscr{L} g = 0$ and

$$f = g \circ \varpi + (h_1 + h_2 \circ \varpi) - (h_1 + h_2 \circ \varpi) \circ T + c$$

and the existence of the decomposition follows.

As for the uniqueness, let c be a constant, h be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$ and g be a Hölder continuous function on $\Gamma\backslash\mathscr{S}_+$ with $\mathscr{L}g = 0$. Assume that we have

$$g \circ \varpi + h - h \circ T + c = 0.$$

First, we claim that h is M-invariant, that is $h = h_1 \circ \varpi$ for some function h_1 on $\Gamma\backslash\mathscr{S}_+$. Indeed, let σ and τ be in $\mathscr{S}$ with $\varpi(\sigma) = \varpi(\tau)$. Then, as $h \circ T - h = g \circ \varpi + c$, we have

$$h(\sigma) - h(\tau) = h(T\sigma) - h(T\tau).$$

By iterating this identity, we get, for $k \geq 0$,

$$h(\sigma) - h(\tau) = h(T^k\sigma) - h(T^k\tau).$$

As h is continuous on the compact set $\Gamma\backslash\mathscr{S}$ and the parametrized geodesic lines $T^k\sigma$ and $T^k\tau$ get closer and closer to each other as $k \to \infty$, we get $h(\sigma) = h(\tau)$, hence $h = h_1 \circ \varpi$ for some Hölder continuous function h_1 on $\Gamma\backslash\mathscr{S}_+$.

Now, we have $g + h_1 - h_1 \circ T_+ + c = 0$. By applying the operator $\mathscr{L}$ to this identity, we get

$$\mathscr{L}h_1 - h_1 + c = 0.$$

By Proposition 32.3, h_1 is a constant function. In particular, we have $h_1 = h_1 \circ T$, hence $g + c = 0$. As $\mathscr{L}g = 0$, this gives $c = 0$, hence also $g = 0$, which should be proved. □

32.5. Solving the cohomological equation in subspaces

Later, we shall use the proof of Corollary 32.5 through the following *ad hoc* formulation.

COROLLARY 32.6. *Let V be a Banach space, $\Lambda : V \to V$ be a bounded operator and u be a non zero vector of V. Assume $\Lambda u = u$ and $1 - \Lambda$ is invertible in the quotient space $E/\mathbb{R}u$. Fix $0 < \alpha < 1$ and suppose we are given a bounded linear map $\Theta : E \to \mathscr{H}^+_\alpha$ with $\Theta\Lambda = \mathscr{L}\Theta$ and $\Theta u = \mathbf{1}$. Let v be in E and assume that the Hölder continuous function $\Theta(v) \circ \varpi$ is a coboundary on $\Gamma\backslash\mathscr{S}$, that is, there exists a Hölder continuous function h on $\Gamma\backslash\mathscr{S}$ with $\Theta(v)\circ\varpi = h - h\circ T$. Then, we must have*

$$h = \Theta(\Lambda w) \circ \varpi$$

for some w in V.

PROOF. Set $f = \Theta(v) \circ \varpi$ so that we have $f \circ \varpi = h - h \circ T$. As in the proof of Corollary 32.5, this tells us that $h = h_1 \circ \varpi$ for some

Hölder continuous function h_1 on $\Gamma\backslash\mathscr{S}_+$. We have $f = h_1 - h_1 \circ T_+$, hence, still as in the proof of Corollary 32.5,

$$\mathscr{L} f = \mathscr{L} h_1 - h_1.$$

Now, Lemma 32.4 and the assumption imply that there exists a real number c and a w in V with $v = \Lambda w - w + cu$. We have $\Lambda v = \Lambda^2 w - \Lambda w + cu$, hence, by applying Θ to this identity, we get

$$\mathscr{L} f = \mathscr{L}\Theta(\Lambda w) - \Theta(\Lambda w) + c.$$

From the uniqueness statement in Proposition 32.3, we get

$$h_1 - \Theta(\Lambda w) \in \mathbb{R}\mathbf{1} = \Theta(\mathbb{R}u),$$

and the conclusion follows. □

Thanks to this technical result, we can prove that smooth functions are cohomologous when viewed as Hölder continuous functions, if and only if they are cohomologous in the sense of Section 2.3 (see also Lemma 3.13). For $k \geq 1$, we let $\mathscr{V}_k$ stand for the space of Γ-invariant functions on X_k.

COROLLARY 32.7. *Let f be a smooth function on $\Gamma\backslash\mathscr{S}$. Assume that there exists a Hölder continuous function h on $\Gamma\backslash\mathscr{S}$ with $f = h - h \circ T$. Then, h is smooth.*

More precisely, given $k \geq 1$ and w in $\mathscr{V}_k$, the smooth function $\sigma \mapsto w(\sigma_0\sigma_k)$ on $\mathscr{S}$ is a coboundary if and only if there exists v in $\mathscr{V}_{k-1}$ such that, for any xy in X_k, one has

$$w(xy) = v(xy_1) - v(x_1 y),$$

where x_1 and y_1 are the neigbours of x and y on $[xy]$.

In the sequel, for $k \geq 1$, we shall say that two elements w and w' in $\mathscr{V}_k$ are cohomologous if there exists v in $\mathscr{V}_{k-1}$ such that, for any xy in X_k, one has

$$w(xy) - w'(xy) = v(xy_1) - v(x_1 y).$$

PROOF. Note that the second part of the statement follows from the first and Lemma 2.14. Thus, we only need to prove the second part. We will obtain it by using Corollary 32.6.

For $k \geq 1$, define a map Θ_k from $\mathscr{V}_k$ to the space of smooth functions on $\Gamma\backslash\mathscr{S}_+$ by setting, for w in $\mathscr{V}_k$ and (x, ξ) in $\mathscr{S}_+$,

$$\Theta_k w(x, \xi) = w(x, x_k),$$

where x_k is the unique element of $[x\xi)$ with $d(x, x_k) = k$. By compactness, the space of smooth functions on $\Gamma\backslash\mathscr{S}_+$ is $\bigcup_{k\geq 1} \Theta_k(\mathscr{V}_k)$.

We define an endomorphism Λ_k of $\mathscr{V}_k$ by setting, for w in $\mathscr{V}_k$ and (x, y) in X_k,

$$\Lambda_k w(x,y) = \frac{1}{q} \sum_{\substack{z \sim x \\ z \neq x_1}} w(z, y_1),$$

where x_1 and y_1 are the neighbours of x and y in $[xy]$. Note that we have $\mathscr{L}\Theta_k = \Theta_k \Lambda_k$.

Let $I_k : \mathscr{V}_k \to \mathscr{V}_{k+1}$ be the injection given by, for w in $\mathscr{V}_k$ and (x, y) in X_{k+1},

$$I_k w(x,y) = w(x, y_1),$$

where as above, y_1 is the neighbour of y in $[xy]$. We have $\Lambda_{k+1} I_k = I_k \Lambda_k$ and $\Theta_{k+1} I_k = \Theta_k$. Besides, we have $\Lambda_{k+1} \mathscr{V}_{k+1} \subset I_k \mathscr{V}_k$. Thus, for $k \geq 2$, the spectrum of Λ_k in $\mathscr{V}_k$ is the union of $\{0\}$ and the spectrum of Λ_1 in $\mathscr{V}_1$. By using Corollary 15.7, we get that $1 - \Lambda_k$ is invertible in $\mathscr{V}_k / \mathbb{R}\mathbf{1}$ for any $k \geq 1$. Hence, the assumption of Corollary 32.6 is satisfied.

Assume $k \geq 2$ and let w be in $\mathscr{V}_k$ such that the function $\Theta_k w$ is a coboundary when viewed as a Hölder continuous function $\Gamma \backslash \mathscr{S}$. By Corollary 32.6, the solutions h of the equation $\Theta_k w = h - h \circ T$ belong to the space

$$\Theta_k \Lambda_k \mathscr{V}_k \subset \Theta_k I_{k-1} \mathscr{V}_{k-1} = \Theta_{k-1} \mathscr{V}_{k-1}.$$

The conclusion follows.

It remains to deal with the case where $k = 1$. Then, let still w be in $\mathscr{V}_1$. If the smooth function $\sigma \mapsto w(\sigma_0 \sigma_1)$ on $\mathscr{S}$ is a coboundary, the above in case $k = 2$ tells us that there exists v in $\mathscr{V}_1$ such that, for any xy in X_1 and any neighbour $z \neq x$ of y, one has

$$w(xy) = v(xy) - v(yz).$$

In particular, for y in X, let z, z' be two neighbours of y. As $q \geq 2$, the vertex y admits a neighbour x that is neither z nor z', and we get

$$v(yz) = v(xy) - w(xy) = v(yz'),$$

hence, there exists u in $\mathscr{V}_0$ such that

$$v(yz) = u(y), \quad yz \in X_1.$$

For xy in X_1, we get

$$w(xy) = u(x) - u(y)$$

as required. □

REMARK 32.8. Let f be a Hölder continuous function on $\Gamma \backslash \mathscr{S}$ and θ be a T-invariant distribution. Assume that f is cohomologous to some smooth function g. Then, it follows from Corollary 32.7 above

that $\langle \theta, g \rangle$ is independent on the choice of g. In the sequel, we shall write this number as $\langle \theta, f \rangle$.

CHAPTER 33

The cohomological equation for endpoints series

We will use the cohomology theory of Hölder continuous functions to give a criterion for certain functions defined by series to be coboundaries.

33.1. Endpoints series

We define a particular construction of smooth functions on $\Gamma\backslash\mathscr{S}$. This type of functions will later appear naturally in the spectral theory of the action of Γ on $\overline{\mathcal{D}}(\partial X)$.

In the sequel, we will be concerned with the following finite-dimensional spaces. For $k \geq 0$, we let as above $\mathscr{V}_k$ be the space of all functions on $\Gamma\backslash X_k$. We shall also think to $\mathscr{V}_k$ as the space of all functions on X_k which are Γ-invariant. In the same way, for $h, k \geq 1$, we let $\mathscr{W}_{h,k}$ be the space of all functions on $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$. Equivalently, we shall also see $\mathscr{W}_{h,k}$ as the space of functions on $X_h \times X_k$ which are invariant under the product action of $\Gamma \times \Gamma$. When $h = k$, we will write $\mathscr{W}_k$ for $\mathscr{W}_{k,k}$.

For $\ell \geq h, k \geq 1$, we define the space $\mathscr{X}_{h,k,\ell}$ as follows: an element w of $\mathscr{X}_{h,k,\ell,\alpha}$ is a family $(w_j)_{j\geq\ell}$ where w_ℓ belongs to $\mathscr{V}_\ell$ and $(w_j)_{j\geq\ell+1}$ is a finitely supported sequence in $\mathscr{W}_{h,k}$.

Still for $\ell \geq h, k \geq 1$, let us construct a linear map $\Theta_{h,k,\ell} : \mathscr{X}_{h,k,\ell} \to \mathcal{D}(\Gamma\backslash\mathscr{S}_+)$. For $w = (w_j)_{j\geq\ell}$ in $\mathscr{X}_{h,k,\ell}$, we define the associated endpoints series $\Theta_{h,k,\ell}w$ as the function such that, for (x, ξ) in $\mathscr{S}_+$, one has

$$\Theta_{h,k,\ell}w(x,\xi) = w_\ell(x_0x_\ell) + \sum_{j=\ell+1}^{\infty} w_j(x_0x_h, x_{j-k}x_j), \tag{33.1}$$

where $(x_j)_{j\geq0}$ is the parametrization of the geodesic ray $[x\xi)$.

33.2. Accessible pairs

We will give a criterion for an endpoints series to be a coboundary in the sense of Chapter 32. To state this criterion, we need to introduce new subsets of $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$, $h, k \geq 1$. To avoid heavy formulas, for a, b in X, we will sometimes write ab instead of (a, b).

DEFINITION 33.1. Let $j \geq h, k \geq 1$. A pair $(\Gamma ab, \Gamma xy)$ in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is said to be j-accessible if there exists pq in X_j such that $pp_h \in \Gamma ab$ and $q_k p \in \Gamma xy$, where p_h and q_k are the elements of $[pq]$ wich lie at distance h from p and k from q. The set of j-accessible pairs in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is denoted by $((\Gamma\backslash X_h) \times (\Gamma\backslash X_k))_j$.

In other words, the pair $(\Gamma ab, \Gamma xy)$ is j-accessible if there exits a path of length j from Γab to Γxy.

When j is large, we can describe the set $((\Gamma\backslash X_h) \times (\Gamma\backslash X_k))_j$. The following lemma is a reformulation of the topological mixing property of the time shift on $\Gamma\backslash\mathscr{S}$.

LEMMA 33.2. *Assume Γ is not bipartite. Then, there exists an integer n such that, for every $h, k \geq 1$, and every $j \geq h + k + n$, every pair in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is j-accessible.*

Assume Γ is bipartite. Then, there exists an integer n such that, for every $h, k \geq 1$, and every $j \geq h+k+n$, a pair (ab, xy) in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is j-accessible if and only if the integral number $d(a, y)$ has the same parity as j.

PROOF. This is a consequence of the equidistribution statement in Corollary 15.7. We keep the notation of this result. Note that we have $\rho = q$ and that u and $u^\vee$ may be chosen to be the constant function $\mathbf{1}$. For ab in X_1, let v_{ab} denote the Γ-invariant function on X_1 defined by

$$v_{ab}(xy) = \sum_{\gamma\in\Gamma} \mathbf{1}_{ab=\gamma(xy)}, \quad xy \in X_1.$$

By Lemma 13.20, we have

$$\langle v_{ab}, \mathbf{1}\rangle = \sum_{xy\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} v_{ab}(xy) = 1.$$

If Γ is not bipartite, by Corollary 15.7, for ab in X_1, we have

$$\frac{1}{q^n} R^n v_{ab} \xrightarrow[n\to\infty]{} \frac{1}{\sum_{xy\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x\cap\Gamma_y|}}.$$

Therefore, there exists n such that, for every $m \geq n$ and every ab and xy in X_1, we can find a geodesic path $b_{-1} = a, b_0 = b, b_1, \ldots, b_m$ in X and γ in Γ with $b_{m-1} = \gamma x$ and $b_m = \gamma y$. In particular, the pair (ab, xy) is j-accessible in $(\Gamma\backslash X_1) \times (\Gamma\backslash X_1)$ for every $j \geq n + 1$.

Still when Γ is not bipartite, if (ab, xy) is in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ for some $h, k \geq 1$, we claim that (ab, xy) is j-accessible for every $j \geq h + k + n - 1$. Indeed, for such a j, we have $i = j - h - k + 1 \geq n$, hence, by the previous case, there exists pq in X_{i+1} and γ in Γ with $pp_1 = b_1 b$ and $q_1 q = \gamma(xx_1)$, where as usual, b_1, x_1, p_1 and q_1 are the

neighbours of b, x, p and q on $[ab]$, $[xy]$, $[pq]$ and $[pq]$. Then, we have $d(a,\gamma y)=j$ and $b,\gamma x\in[a(\gamma y)]$, so that the pair (ab,xy) is j-accessible in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$.

The proof in the non bipartite case follows the same lines by keeping in mind that the action of Γ preserves the classes of the even distance equivalence relation on X. □

By using the notion of an accessible pair, we can formulate the main result of this Chapter. This is a criterion for an endpoints series to be a coboundary.

PROPOSITION 33.3. *Let $\ell\geq h,k\geq 2$ and $w=(w_j)_{j\geq\ell}$ be in $\mathscr{X}_{h,k,\ell}$. Then the following are equivalent.*
(i) *The endpoints series $\Theta_{h,k,\ell}w$ is a coboundary, when wiewed as a smooth function on $\Gamma\backslash\mathscr{S}$.*
(ii) *There exists t in $\mathscr{V}_{\ell-1}$ and finitely supported sequences $(u_j)_{j\geq\ell}$ in $\mathscr{W}_{h,k-1}$ and $(v_j)_{j\geq\ell}$ in $\mathscr{W}_{h-1,k}$ such that, for every ab in X_ℓ,*

$$w_\ell(ab)=t(ab_1)-t(a_1b)+u_\ell(aa_h,b_{k-1}b)+v_\ell(aa_{h-1},b_kb)$$

and, for every $j\geq\ell+1$ and every j-accessible pair (ab,xy) in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$,

$$w_j(ab,xy)=u_j(ab,x_1y)+v_j(ab_1,xy)-u_{j-1}(ab,xy_1)-v_{j-1}(a_1b,xy).$$

In the statement above, for a,b in X and $i\leq d(a,b)$, we have denoted by a_i the point of the segment $[ab]$ that lies as distance i from a.

FIRST PART OF THE PROOF. We prove the easy case of the Proposition, that is, *(ii)⇒(i)*. Let u and v be as as in the statement. Fix σ in $\mathscr{S}$. By definition, for every $j\geq\ell+1$, the pair $(\sigma_0\sigma_h,\sigma_{j-k}\sigma_j)$ is j-accessible in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$. Therefore, by (33.1) and the assumption, we have

$$\begin{aligned}\Theta_{h,k,\ell}w(\sigma)&=t(\sigma_0\sigma_{\ell-1})-t(\sigma_1\sigma_\ell)+u_\ell(\sigma_0\sigma_h,\sigma_{\ell-k+1}\sigma_\ell)\\&+v_\ell(\sigma_0\sigma_{h-1},\sigma_{\ell-k}\sigma_\ell)+\sum_{j\geq\ell+1}u_j(\sigma_0\sigma_h,\sigma_{j-k+1}\sigma_j)+v_j(\sigma_0\sigma_{h-1},\sigma_{j-k}\sigma_j)\\&\qquad-u_{j-1}(\sigma_0\sigma_h,\sigma_{j-k}\sigma_{j-1})-v_{j-1}(\sigma_1\sigma_h,\sigma_{j-k}\sigma_j).\end{aligned}$$

By using the cancellation of the telescoping series, we get

$$\begin{aligned}\Theta_{h,k}w(\sigma)&=t(\sigma_0\sigma_{\ell-1})-t(\sigma_1\sigma_\ell)\\&+\sum_{j\geq\ell}v_j(\sigma_0\sigma_{h-1},\sigma_{j-k}\sigma_j)-v_j(\sigma_1\sigma_h,\sigma_{j+1-k}\sigma_{j+1})=h(\sigma)-h(T\sigma),\end{aligned}$$

where

$$h(\sigma) = t(\sigma_0\sigma_{\ell-1}) + \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j).$$

The result follows. □

The proof of the converse statement will last until the end of the Chapter.

33.3. Vanishing endpoints series

We begin by determining the null space of the endpoints series operator.

LEMMA 33.4. *Let $\ell \geq h, k \geq 1$ and w be in $\mathscr{X}_{h,k,\ell}$. Assume the endpoints series function $\Theta_{h,k,\ell}w$ vanishes on $\Gamma\backslash\mathscr{S}$. Then, there exists a finitely supported sequence $u = (u_j)_{j\geq\ell}$ in $\mathscr{W}_{h,k-1}$ such that, for every ab in X_h,*

$$w_\ell(ab) = u_\ell(aa_h, b_{k-1}b)$$

and, for every $j \geq \ell+1$ and every j-accessible pair (ab, xy) in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$,

$$w_j(ab, xy) = u_j(ab, x_1y) - u_{j-1}(ab, xy_1).$$

Note that the converse is true, in view of the first part of the proof of Proposition 33.3.

PROOF. The idea of the proof is that u may be defined by integrating w along certain paths. The fact that the integral does not depend on the path is warranted by the vanishing of the function $\Theta_{h,k,\ell}w$. Let us write this precisely.

For $j \geq h+k-1$, write $(X_h \times X_{k-1})_j$ for the set of pairs (ab, xy) such that $d(a, y) = j$ and b and x belong to $[ay]$. If $j \geq \ell$, we define a function u_j on $(X_h \times X_{k-1})_j$ by setting

$$u_j(ab, xy) = w_\ell(p_0p_\ell) + \sum_{i=\ell+1}^{j} w_i(ab, p_{i-k}p_i), \quad (ab, xy) \in (X_h \times X_{k-1})_j,$$

where $a = p_0, p_1, \ldots, p_j = y$ is the geodesic parametrization of the segment $[ay]$ (in particular, we have $p_h = b$ and $p_{j-k+1} = x$).

The function u_j is invariant under the diagonal action of Γ on $(X_h \times X_{k-1})_j$. We claim that the vanishing assumption on $\Theta_{h,k,\ell}w$ implies that u_j satisfies the following additional invariance property: for every (ab, xy) in $(X_h \times X_{k-1})_j$ and every γ in Γ, if $(\gamma(ab), xy)$ also belongs to $(X_h \times X_{k-1})_j$, then

$$u_j(ab, xy) = u_j(\gamma(ab), xy)$$

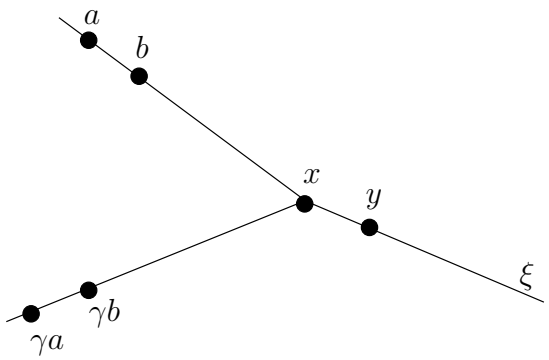

FIGURE 1. Proof of Lemma 33.4

(see Figure 1).

Indeed, for such ab, xy and γ, let $a = p_0, p_1, \dots, p_j = y$ be the geodesic parametrization of the segment $[ay]$ and choose a ξ in ∂X such that $[y\xi) \cap [ay] = [y\xi) \cap [(\gamma a)y] = \emptyset$. Finally, let $x_0 = x, x_1, \dots$ be the geodesic parametrization of the geodesic ray $[x\xi)$. We get

$$\Theta_{h,k,\ell} w(a,\xi) =$$

$$w_\ell(p_0 p_\ell) + \sum_{i=\ell+1}^{j} w_i(ab, p_{i-k}p_i) + \sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1})$$

$$= u_j(ab,xy) + \sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1})$$

and in the same way,

$$\Theta_{h,k,\ell} w(\gamma a,\xi) = u_j(\gamma(ab),xy) + \sum_{i=j+1}^{\infty} w_i(\gamma(ab), x_{i-j-1}x_{i-j+k-1}).$$

As $\Theta_{h,k,\ell} w = 0$ and the functions w_i, $i \geq \ell$, are invariant under the product action of $\Gamma \times \Gamma$ on $X_h \times X_k$, we obtain

$$u_j(\gamma(ab),xy) = -\sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1}) = u_j(ab,xy)$$

as required.

Note that by definition, the image of $(X_h \times X_{k-1})_j$ in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_{k-1})$ is the set $((\Gamma\backslash X_h) \times (\Gamma\backslash X_{k-1}))_j$ of j-accessible pairs. The previous tells us that we can consider u_j as a function on $((\Gamma\backslash X_h) \times (\Gamma\backslash X_{k-1}))_j$. We extend it as a function defined everywhere on $(\Gamma\backslash X_h) \times (\Gamma\backslash X_{k-1})$ by setting $u_j(ab,xy) = 0$ for any (ab,xy) in the complement

of $((\Gamma\backslash X_h)\times(\Gamma\backslash X_{k-1}))_j$ in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_{k-1})$. Note that as above, for (ab,xy) in $((\Gamma\backslash X_h)\times(\Gamma\backslash X_{k-1}))_j$, we have

$$u_j(ab,xy) = -\sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1}),$$

where $(x_i)_{i\geq 0}$ is the parametrization of a geodesic ray $[x\xi)$ with $[x\xi)\cap[ay] = [xy]$. Therefore, since the sequence $(w_j)_{j\geq \ell+1}$ is finitely supported, the sequence $(u_j)_{j\geq\ell}$ is also finitely supported.

Finally, on one hand, for ab in X_ℓ, we have

$$w_\ell(ab) = u_\ell(aa_h, b_{k-1}b).$$

On the other hand, if $j\geq \ell+1$ and (ab,xy) is a j-accessible pair in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$, we can find γ in Γ such that $d(a,\gamma y) = j$ and b and γx belong to $[a(\gamma y)]$. Then, we have

$$(ab,\gamma(x_1y))\in(X_k\times X_{k-1})_j \text{ and } (ab,\gamma(xy_1))\in(X_k\times X_{k-1})_{j-1}.$$

We get

$$\begin{aligned} w_j(ab,xy) = w_j(ab,\gamma(xy)) &= u_j(ab,\gamma(x_1y)) - u_{j-1}(ab,\gamma(xy_1)) \\ &= u_j(ab,x_1y) - u_{j-1}(ab,xy_1) \end{aligned}$$

as required. □

33.4. A symbolic transfer operator

To prove Proposition 33.3, we will apply Corollary 33.4 to the endpoints operator $\Theta_{h,k,\ell}$. To this aim, we will introduce a new operator $\Lambda_{h,k,\ell}$ acting on sequences spaces $\mathscr{X}_{h,k,\ell}$, $\ell\geq h,k\geq 2$, which will be semiconjugate to the transfer operator $\mathscr{L}$ via the linear map $\Theta_{h,k,\ell}$ from (33.1).

Let $\ell\geq h,k\geq 1$ and $w=(w_j)_{j\geq\ell}$ be a sequence where w_ℓ is a function on $\Gamma\backslash X_\ell$, and, for $j\geq\ell+1$, w_j is a function on $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$. We define a new sequence $\Lambda_{h,k}w$ by setting, for ab in X_ℓ,

$$(\Lambda_{h,k,\ell}w)_\ell(ab) = \frac{1}{q}\sum_{\substack{c\sim a\\ c\neq a_1}} (w_\ell(cb_1) + w_{\ell+1}(ca_{h-1}, b_kb)) \tag{33.2}$$

and, for $j\geq\ell+1$ and (ab,xy) in $X_h\times X_k$,

$$(\Lambda_{h,k,\ell}w)_j(ab,xy) = \frac{1}{q}\sum_{\substack{c\sim a\\ c\neq a_1}} w_{j+1}(cb_1,xy) \tag{33.3}$$

The definition of the transfer operator $\mathscr{L}$ in (32.2) and the one if the endpoints operator $\Theta_{h,k,\ell}$ in (33.1) directly give

LEMMA 33.5. *Let $\ell \geq h, k \geq 1$. For any w in $\mathscr{X}_{h,k,\ell}$, we have*

$$\mathscr{L}\Theta_{h,k,\ell} w = \Theta_{h,k,\ell}\Lambda_{h,k,\ell} w.$$

In order to apply Corollary 33.4, we will prove

LEMMA 33.6. *Let $\ell \geq h, k \geq 1$. Let u be the element of $\mathscr{X}_{h,k,\ell}$ such that $u_j = 0$ for $j \geq \ell + 1$ and that $u_\ell(ab) = 1$ for ab in X_ℓ. Then, we have $\Lambda_{h,k,\ell} u = u$ and $\Lambda_{h,k,\ell} - 1$ is invertible in $\mathscr{X}_{h,k,\ell}/\mathbb{R}u$.*

PROOF. For $i \geq 0$, let $\mathscr{X}^i_{h,k,\ell}$ be the space of sequences w in $\mathscr{X}_{h,k,\ell,\alpha}$ with $w_j = 0$ for any $j \geq \ell+i+1$, so that $\mathscr{X}_{h,k,\ell} = \bigcup_{i\geq 0} \mathscr{X}^i_{h,k,\ell}$. By (33.2) and (33.3), the space $\mathscr{X}^0_{h,k,\ell}$ is stable under $\Lambda_{h,k}$ and, for $i \geq 1$, we have $\Lambda_{h,k}\mathscr{X}^i_{h,k,\ell} \subset \mathscr{X}^{i-1}_{h,k,\ell}$. The latter implies in particular that $\Lambda_{h,k,\ell} - 1$ is invertible in the quotient space $\mathscr{X}_{h,k,\ell}/\mathscr{X}^0_{h,k,\ell}$.

Now, still by (33.2), we may identify $\mathscr{X}^0_{h,k,\ell}$ with $\mathscr{V}_\ell$ and the restriction of $\Lambda_{h,k,\ell}$ with the operator L defined by, for w in $\mathscr{V}_\ell$ and ab in X_ℓ,

$$Lw(ab) = \frac{1}{q} \sum_{\substack{c\sim a_1 \\ c\neq a_1}} w(cb_1).$$

From the proof of Corollary 32.7, we know that $L - 1$ is invertible in $\mathscr{V}_\ell/\mathbb{R}\mathbf{1}$. Thus, $\Lambda_{h,k,\ell} - 1$ is invertible in $\mathscr{X}^0_{k,\alpha}/\mathbb{R}u$. Therefore, $\Lambda_{h,k,\ell} - 1$ is invertible in $\mathscr{X}_{h,k,\ell,\alpha}/\mathbb{R}u$ as required. □

33.5. Solving the cohomological equation

Thanks to the study of the symbolic transfer operator, we can apply Corollary 32.6 and transport the cohomological equation back to an equation in sequences spaces.

SECOND PART OF THE PROOF OF PROPOSITION 33.3. We now prove the difficult case of the Proposition, that is, *(ii)⇒(i)*. Thus, we assume that we are given w in $\mathscr{X}_{h,k,\ell}$ such that the endpoint series $\Theta_{h,k,\ell} w$ is a coboundary, when viewed as a smooth function on $\Gamma\backslash\mathscr{S}$.

By Lemma 33.5 and Lemma 33.6, we can apply Corollary 32.6. This tells us that the solutions h of the equation

$$\Theta_{h,k,\ell} w = h - h \circ T \tag{33.4}$$

belong to $\Theta_{h,k,\ell}\Lambda_{h,k,\ell}\mathscr{X}_{h,k,\ell,\alpha}$.

As we have assumed $h \geq 2$, in view of the range of the operator $\Lambda_{h,k,\ell}$ in $\mathscr{X}_{h,k,\ell,\alpha}$ as described from (33.2) and (33.3), we can assume that there exists a function t on $\Gamma\backslash X_{\ell-1}$ and a finitely supported sequence

$v = (v_j)_{j\geq \ell}$ of functions on $(\Gamma\backslash X_{h-1}) \times (\Gamma\backslash X_k)$ such that, for any σ in $\mathscr{S}$,

$$h(\sigma) = t(\sigma_0\sigma_{\ell-1}) + \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j).$$

Thus, from (33.4), we get

$$\begin{aligned}
\Theta_{h,k,\ell} w(\sigma) &= t(\sigma_0\sigma_{\ell-1}) - t(\sigma_1\sigma_\ell) \\
&\quad + \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j) - v_j(\sigma_1\sigma_h, \sigma_{j-k+1}\sigma_{j+1}) \\
&= t(\sigma_0\sigma_{\ell-1}) - t(\sigma_1\sigma_\ell) + v_\ell(\sigma_0\sigma_{h-1}, \sigma_{\ell-k}\sigma_\ell) \\
&\quad + \sum_{j\geq\ell+1} v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j) - v_{j-1}(\sigma_1\sigma_h, \sigma_{j-k}\sigma_j) \\
&= \Theta_{h,k,\ell} w'(\sigma),
\end{aligned}$$

where $w' = (w'_j)_{j\geq\ell}$ is the element of $\mathscr{X}_{h,k,\ell,\alpha}$ defined by

$$w'_\ell(ab) = t(ab_1) - t(a_1 b) + v_\ell(aa_{h-1}, b_k b), \quad ab \in X_\ell$$

and, for $j \geq \ell + 1$,

$$w'_j(ab, xy) = v_j(ab_1, xy) - v_{j-1}(a_1 b, xy), \quad ab \in X_h, \quad xy \in X_k.$$

The result now follows from Lemma 33.4. □

Part 9

An abstract Plancherel formula

CHAPTER 34

The Plancherel formula on X_1

The purpose of this Chapter is to establish a Plancherel formula for functions on X_1. For functions on X, such a formula is obtained in [**21**]. Our formula is closely related to the latter.

34.1. Polynomial functions of the spectral parameter

We start by introducing remarkable polynomial functions which will play a role all along the remaining of the book. We use the notation of Section 21.3 for objects related to spectral analysis, so that below, $t = \frac{1}{q+1}(u + \frac{q}{u})$ will be the spectral parameter.

Let t be in $\mathbb{C}$ and write $(q+1)t = u + \frac{q}{u}$ for some u in $\mathbb{C}^*$. For $(q+1)^2t^2 \neq 4q$ (that is, $u^2 \neq q$), we set

$$\begin{aligned} A_0(t) &= 1, \\ A_j(t) &= u^j + \left(\frac{q}{u}\right)^j && j \geq 1, \\ B_j(t) &= \frac{u^j - \left(\frac{q}{u}\right)^j}{u - \frac{q}{u}} && j \geq 0. \end{aligned}$$

The reason for the choice of the value A_0 will become clear later.

These functions play a role in the spectral formulas of [**21**]. They are actually regular.

LEMMA 34.1. *For $j \geq 0$, the functions A_j and B_j are polynomial functions. For t in $\mathbb{C}$ and $j \geq 1$, the following relations hold:*

$$(q+1)tB_j(t) = B_{j+1}(t) + qB_{j-1}(t) \text{ and } B_{j+1}(t) = A_j(t) + qB_{j-1}(t).$$

PROOF. Let t, u be in $\mathbb{C}$ with $(q+1)tu = u^2+q$. For $(q+1)^2t^2 \neq 4q$, we have $B_0(t) = 0$ and $B_1(t) = 1$. Take $j \geq 1$. We have

$$\begin{aligned} (q+1)tB_j(t) &= \left(\frac{u + \frac{q}{u}}{u - \frac{q}{u}}\right)\left(u^j - \left(\frac{q}{u}\right)^j\right), \\ &= \frac{1}{u - \frac{q}{u}}\left(u^{j+1} - q\left(\frac{q}{u}\right)^{j-1} + qu^{j-1} - \left(\frac{q}{u}\right)^{j+1}\right) \\ &= B_{j+1}(t) + qB_{j-1}(t). \end{aligned}$$

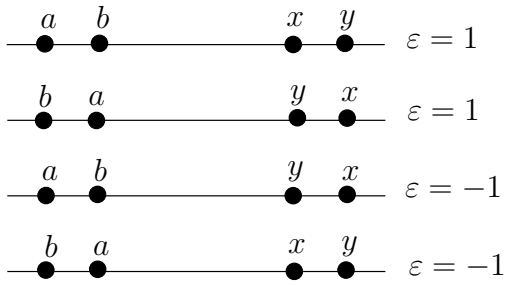

FIGURE 1. Construction of ε

In particular, an easy induction argument shows that B_j is a polynomial for any $j \geq 0$.

Besides, still for $j \geq 1$, we have

$$A_j(t) + qB_{j-1}(t) = u^j + \left(\frac{q}{u}\right)^j + q\frac{u^{j-1} - \left(\frac{q}{u}\right)^{j-1}}{u - \frac{q}{u}}$$
$$= \frac{1}{u - \frac{q}{u}}\left(u^{j+1} - qu^{j-1} + q\left(\frac{q}{u}\right)^{j-1} - \left(\frac{q}{u}\right)^{j+1} + qu^{j-1} - q\left(\frac{q}{u}\right)^{j-1}\right)$$
$$= B_{j+1}(t).$$

As B_{j+1} and B_{j-1} are polynomials, so is A_j. □

34.2. Geometric functions on $X_1 \times X_1$

We will use the above polynomial functions to define bilinear forms on the space $\mathcal{D}(X_1)$ of finitely supported functions on X_1. This definition will also require us to use some notation to describe the respective positions of two edges.

First, we introduce a notion of the distance between two elements of X_1. For ab and xy in X_1, we set

$$\delta(ab, xy) = \max(d(a, x), d(b, x), d(a, y), d(b, y)) - 1.$$

This number is non-negative and satisfies the relations

$$\delta(ab, xy) = \delta(xy, ab) = \delta(ba, xy).$$

One can check that δ actually defines a distance on the set of non oriented edges, that is, the quotient of X_1 by the involution $ab \mapsto ba$. We shall not use this fact.

We also introduce a function ε that checks whether two edges have compatible orientations or not. For ab in X_1, we set $\varepsilon(ab, ab) = 1$,

$\varepsilon(ab, ba) = -1$ and, for xy in X_1 with $\delta(ab, xy) \geq 1$,

$$\begin{aligned} \varepsilon(ab, xy) &= 1 && \text{if } b, x \in [ay] \text{ or } a, y \in [bx], \\ \varepsilon(ab, xy) &= -1 && \text{if } b, y \in [ax] \text{ or } a, x \in [by] \end{aligned}$$

(see Figure 1). We have the relations

$$\varepsilon(ab, xy) = \varepsilon(xy, ab) = -\varepsilon(ba, xy).$$

34.3. Spectral bilinear forms on $\mathcal{D}(X_1)$

We now begin the construction of the elements of the Plancherel formula.

For t in $\mathbb{C}$, we define a function χ_t on $X_1 \times X_1$ as follows. For ab in X_1, we set

$$\chi_t(ab, ab) = 1 \text{ and } \chi_t(ab, ba) = 0.$$

For ab and xy in X_1 with $j = \delta(ab, xy) \geq 1$, we set

$$\begin{aligned} \chi_t(ab, xy) &= \frac{1}{2q^j} A_j(t) && \text{if } \varepsilon(ab, xy) = 1 \\ \chi_t(ab, xy) &= \frac{q-1}{2q^j} B_j(t) && \text{if } \varepsilon(ab, xy) = -1. \end{aligned}$$

Later, we shall need the following easy bound:

LEMMA 34.2. *Let ab and xy be in X_1 and t, u be in $\mathbb{C}$ with $(q+1)tu = q + u^2$. We have*

$$\chi_t(ab, xy) \leq (\delta(ab, xy) + 1) \max\left(|u|^{-\delta(ab,xy)}, \left|\frac{q}{u}\right|^{-\delta(ab,xy)} \right).$$

PROOF. Indeed, for $j \geq 1$, we have

$$q^{-j} |A_j(t)| = \left| u^{-j} + \left(\frac{q}{u}\right)^{-j} \right| \leq 2 \max\left(|u|^{-j}, \left|\frac{q}{u}\right|^{-j} \right).$$

For $j \geq 0$, we have

$$q^{-j} |B_j(t)| = q^{-j} \left| \sum_{h=0}^{j-1} u^h \left(\frac{q}{u}\right)^{j-1-h} \right| \leq \frac{j}{q} \max\left(|u|^{-j}, \left|\frac{q}{u}\right|^{-j} \right).$$

□

By abuse of notation, we still write χ_t for the symmetric bilinear form on $\mathcal{D}(X_1)$ defined by

$$\chi_t(f, g) = \sum_{ab, xy \in X_1} \chi_t(ab, xy) f(ab) g(xy), \quad f, g \in \mathcal{D}(X_1).$$

This bilinear form will be used to describe the continuous part of the spectrum in the Plancherel formula. We first relate it to natural operations on functions on X_1.

For f a function on X_1 and xy in X_1, we write

$$Rf(xy) = \sum_{\substack{z\sim x\\ z\neq y}} f(xz) \text{ and } Sf(xy) = f(yx). \tag{34.1}$$

We have $R^2 = q + (q-1)R$ and $S^2 = 1$, so that we also set

$$P = \frac{1}{q+1}(RS + SR - (q-1)S),$$

as in Section 21.1. Then, P commutes with both R and S. A direct computation shows

LEMMA 34.3. *Let f, g be in $\mathcal{D}(X_1)$. For t in $\mathbb{C}$, we have*

$$\chi_t(Rf, g) = \chi_t(f, Rg),$$
$$\chi_t(Sf, g) = \chi_t(f, Sg)$$
$$\text{and } \chi_t(Pf, g) = t\chi_t(f, g).$$

34.4. Special spectral bilinear forms on $\mathcal{D}(X_1)$

To describe the discrete part of the spectrum in the Plancherel formula, we introduce two bilinear forms that are related to the special representations of the group of automorphisms of X (see [**21**]).

For ab, xy in X_1, we set

$$\chi_1^{\mathrm{sp}}(ab, xy) = \varepsilon(ab, xy)q^{-\delta(ab,xy)},$$
$$\chi_{(-1)}^{\mathrm{sp}}(ab, xy) = (-q)^{-\delta(ab,xy)}.$$

Again, for f, g in $\mathcal{D}(X_1)$, we write

$$\chi_1^{\mathrm{sp}}(f, g) = \sum_{ab,xy\in X_1} \chi_1^{\mathrm{sp}}(ab, xy)f(ab)g(xy),$$
$$\text{and } \chi_{(-1)}^{\mathrm{sp}}(f, g) = \sum_{ab,xy\in X_1} \chi_{(-1)}^{\mathrm{sp}}(ab, xy)f(ab)g(xy).$$

We get

LEMMA 34.4. *Let f, g be in $\mathcal{D}(X_1)$. We have*

$$\chi_1^{\mathrm{sp}}(Rf, g) = -\chi_1^{\mathrm{sp}}(f, g), \qquad \chi_{(-1)}^{\mathrm{sp}}(Rf, g) = -\chi_{(-1)}^{\mathrm{sp}}(f, g),$$
$$\chi_1^{\mathrm{sp}}(Sf, g) = -\chi_1^{\mathrm{sp}}(f, g), \qquad \chi_{(-1)}^{\mathrm{sp}}(Sf, g) = \chi_{(-1)}^{\mathrm{sp}}(f, g),$$
$$\chi_1^{\mathrm{sp}}(Pf, g) = \chi_1^{\mathrm{sp}}(f, g) \qquad \text{and } \chi_{(-1)}^{\mathrm{sp}}(Pf, g) = -\chi_{(-1)}^{\mathrm{sp}}(f, g).$$

34.5. Statement of the formula

Denote by $\langle ., . \rangle_2$ the standard scalar product on $\mathcal{D}(X_1)$, that is,

$$\langle f, g \rangle_2 = \sum_{ab \in X_1} f(ab) g(ab), \quad f, g \in \mathcal{D}(X_1).$$

The Plancherel formula reads as

PROPOSITION 34.5. *For any t in $\mathcal{I}_q$, the symmetric bilinear form χ_t is non-negative on $\mathcal{D}(X_1)$. So are the symmetric bilinear forms χ_1^{sp} and $\chi_{(-1)}^{\mathrm{sp}}$.*

Let f, g be in $\mathcal{D}(X_1)$. We have

$$\langle f, g \rangle_2 = \frac{2}{q+1} \int_{\mathcal{I}_q} \chi_t(f, g) \mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)} \chi_1^{\mathrm{sp}}(f, g) + \frac{q-1}{2(q+1)} \chi_{(-1)}^{\mathrm{sp}}(f, g). \tag{34.2}$$

As in Section 21.3, we wrote $\mathcal{I}_q = \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1} \right]$ and we let μ_q be the Borel probability measure on $\mathcal{I}_q$ which is absolutely continuous with respect to Lebesgue measure, with density $t \mapsto \frac{q+1}{2\pi} \frac{\sqrt{4q - (q+1)^2 t^2}}{1 - t^2}$.

Note that, by Lemma 34.1, for f, g in $\mathcal{D}(X_1)$, the function $t \mapsto \chi_t(f, g)$ is polynomial, so that the integral in (34.2) makes sense.

34.6. Functional relations

The proof of Proposition 34.5 essentially relies on the computation of the integrals of the functions A_j and B_j, $j \geq 0$, with respect to the measure μ_q. We will actually establish other related facts about them that we will need later.

We begin by defining a third family of polynomial functions. We set

$$C_0 = 1 \text{ and } C_j = B_{j+1} - B_{j-1}, \quad j \geq 1. \tag{34.3}$$

LEMMA 34.6. *For any $j \geq 0$, C_j has degree j and the familly $(C_j)_{j \geq 0}$ is an orthogonal basis of $\mathbb{R}[t]$ with respect to the scalar product of the Lebesgue space $\mathrm{L}^2(\mu_q)$. For any $j \geq 1$, we have*

$$\int_{\mathcal{I}_q} C_j(t)^2 \mathrm{d}\mu_q(t) = (q+1) q^{j-1}.$$

This directly follows from the study of the spherical transform of X in [**21**]. Below, we give a direct proof in our language.

PROOF. We have $B_0 = 0$ and $B_1 = 1$. This, together with the formula $(q+1)tB_j(t) = B_{j+1}(t) + qB_{j-1}(t)$, $j \geq 1$, $t \in \mathbb{R}$, from Lemma 34.1 shows that B_{j+1} has degree j for any $j \geq 0$. This implies in turn that C_j also has degree j. In particular $(C_j)_{j\geq 0}$ is a basis of the vector space $\mathbb{R}[t]$.

To check that this basis is actually orthogonal and to compute the integrals $\int_{\mathcal{I}_q} C_j(t)^2 \mathrm{d}\mu_q(t)$, $j \geq 1$, we will use the Plancherel formula for the model operators established in Proposition 22.2.

We adopt temporarily the notation of Section 22.1 and we claim that, for $j \geq 0$, the spectral transforms of the sequences $\mathbf{1}_{2j}$ and $\mathbf{1}_{2j+1}$ in $\mathbb{R}^{(\mathbb{N})}$ are defined by

$$\widehat{\mathbf{1}_{2j}}(t) = \begin{pmatrix} B_{j+1}(t) \\ -B_j(t) \end{pmatrix} \text{ and } \widehat{\mathbf{1}_{2j+1}}(t) = \begin{pmatrix} -B_j(t) \\ B_{j+1}(t) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{34.4}$$

This, we show by induction on j. For $j = 0$, by Proposition 22.2, we have $\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and, as $\mathbf{1}_1 = S_{++}\mathbf{1}_0$,

$$\widehat{\mathbf{1}_1}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now, if $j \geq 1$ and (34.4) holds for $j-1$, as $\mathbf{1}_{2j} = R_{++}\mathbf{1}_{2j-1}$, Proposition 22.2 gives

$$\begin{aligned} \widehat{\mathbf{1}_{2j}}(t) = \begin{pmatrix} q & (q+1)t \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -B_{j-1}(t) \\ B_j(t) \end{pmatrix} &= \begin{pmatrix} (q+1)tB_j(t) - qB_{j-1}(t) \\ -B_j(t) \end{pmatrix} \\ &= \begin{pmatrix} B_{j+1}(t) \\ -B_j(t), \end{pmatrix} \end{aligned}$$

where the latter identity follows from Lemma 34.1. Also, as $\mathbf{1}_{2j+1} = S_{++}\mathbf{1}_{2j}$, we get $\widehat{\mathbf{1}_{2j+1}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widehat{\mathbf{1}_{2j}}(t)$ and hence (34.4) holds for j. Therefore, it holds for any j.

Set $x_0 = \mathbf{1}_0$ and $x_j = \mathbf{1}_{2j} + \mathbf{1}_{2j-1}$, $j \geq 1$. From (34.3) and (34.4), we get, for any $j \geq 0$,

$$\widehat{x_j}(t) = \begin{pmatrix} C_j(t) \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Besides, for the scalar product of Section 21.4, the family $(x_j)_{j\geq 0}$ is orthogonal and we have

$$\langle x_0, x_0 \rangle_+ = 1 \text{ and } \langle x_j, x_j \rangle_+ = (q+1)q^{j-1}, \quad j \geq 1.$$

The conclusion now follows from the Plancherel formula of Proposition 23.2 and the definition of the matrix $a_{++}(t)$, $t \in \mathbb{R}$, in Section 21.1. □

We can express the families $(A_j)_{j\geq 0}$ and $(B_j)_{j\geq 0}$ in the orthogonal basis $(C_j)_{j\geq 0}$.

COROLLARY 34.7. *For any $j \geq 0$, we have*

$$A_j = C_j - (q-1) \sum_{\substack{0\leq h<j \\ j-h \text{ even}}} C_h$$

$$B_j = \sum_{\substack{0\leq h<j \\ j-h \text{ odd}}} C_h.$$

PROOF. The formula for $(B_j)_{j\geq 0}$ directly follows from the definition of $(C_j)_{j\geq 0}$. The formula for $(A_j)_{j\geq 0}$ is obtained from the latter and the relation $A_j = B_{j+1} - qB_{j-1}$ in Lemma 34.1. □

This implies the following integral computation that will be used in the proof of Proposition 34.5.

COROLLARY 34.8. *Let $j \geq 2$ be even. We have*

$$\int_{\mathcal{I}_q} A_j \mathrm{d}\mu_q = -(q-1) \text{ and } \int_{\mathcal{I}_q} B_j \mathrm{d}\mu_q = 0.$$

Let $j \geq 1$ be odd. We have

$$\int_{\mathcal{I}_q} A_j \mathrm{d}\mu_q = 0 \text{ and } \int_{\mathcal{I}_q} B_j \mathrm{d}\mu_q = 1.$$

We can now give the

PROOF OF PROPOSITION 34.5. We will check that, for every ab, xy in X_1, we have

$$\frac{2}{q+1}\int_{\mathcal{I}_q} \chi_t(ab,xy)\mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)}\chi_1^{\mathrm{sp}}(ab,xy) + \frac{q-1}{2(q+1)}\chi_{(-1)}^{\mathrm{sp}}(ab,xy) = \mathbf{1}_{ab=xy}. \tag{34.5}$$

Indeed, assume first $ab = xy$. Then, by construction, for any t in $\mathbb{R}$, we have $\chi_t(ab,ab) = 1$ and also $\chi_1^{\mathrm{sp}}(ab,ab) = 1 = \chi_{(-1)}^{\mathrm{sp}}(ab,ab)$. We get equality in (34.5).

Assume $ab = yx$. We have $\chi_t(ab,ba) = 0$ for any t in $\mathbb{R}$ whereas $\chi_1^{\mathrm{sp}}(ab,ab) = 1$ and $\chi_{(-1)}^{\mathrm{sp}}(ab,ab) = -1$. Again, (34.5) holds.

We now check the cases where $j = \delta(ab,xy) \geq 1$.

If $\varepsilon(ab, xy) = 1$, we have $\chi_t(ab, xy) = \frac{1}{2}q^{-j}A_j(t)$, hence, by Corollary 34.8,

$$\int_{\mathcal{I}_q} \chi_t(ab, xy)\mathrm{d}\mu_q(t) = -\frac{1}{4}q^{-j}(q-1)((-1)^j + 1).$$

The definitions of χ_1^{sp} and $\chi_{(-1)}^{\mathrm{sp}}$ give

$$\chi_1^{\mathrm{sp}}(ab, xy) = q^{-j} \text{ and } \chi_{(-1)}^{\mathrm{sp}}(ab, xy) = (-q)^j,$$

hence the left hand-side of (34.5) vanishes.

Finally, if $\varepsilon(ab, xy) = -1$, we have $\chi_t(ab, xy) = \frac{1}{2}(q-1)q^{-j}B_j(t)$, hence, by Corollary 34.8,

$$\int_{\mathcal{I}_q} \chi_t(ab, xy)\mathrm{d}\mu_q(t) = \frac{1}{4}q^{-j}(q-1)(1 - (-1)^j).$$

By definition,

$$\chi_1^{\mathrm{sp}}(ab, xy) = -q^{-j} \text{ and } \chi_{(-1)}^{\mathrm{sp}}(ab, xy) = (-q)^j,$$

and (34.5) is valid.

Therefore, we have shown that (34.5) holds for any value of ab and xy, that is, equivalently, (34.2) holds for any functions f, g in $\mathcal{D}(X_1)$. It only remains to prove that the bilinear forms χ_1^{sp}, $\chi_{(-1)}^{\mathrm{sp}}$ and χ_t, $t \in \mathcal{I}_q$, are non-negative. Fix f in $\mathcal{D}(X_1)$. Let φ be in $\mathbb{R}[t]$, a polynomial function. By (34.2), Lemma 34.3 and Lemma 34.4, we have

$$\frac{2}{q+1}\int_{\mathcal{I}_q} \varphi(t)^2\chi_t(f, f)\mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)}\varphi(1)^2\chi_1^{\mathrm{sp}}(f, f) + \frac{q-1}{2(q+1)}\varphi(-1)^2\chi_{(-1)}^{\mathrm{sp}}(f, f) = \langle\varphi(P)f, \varphi(P)f\rangle_2 \geq 0.$$

Elementary real analysis arguments show that we have $\chi_1^{\mathrm{sp}}(f, f) \geq 0$, $\chi_{(-1)}^{\mathrm{sp}}(f, f) \geq 0$ and $\chi_t(f, f) \geq 0$ for μ_q-almost any t in $\mathcal{I}_q$. The conclusion follows as $\chi_t(f, f)$ depends continuously on t. □

CHAPTER 35

Fundamental bilinear maps

Recall that we are aiming at the fine study of the map

$$\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{Q}(\mathcal{AH}_k)^{R,S}$$

Our strategy for proving Corollary 1.5 is to write the two linear spaces $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ as being dual spaces of certain other spaces and to write the above map as being the adjoint map of a certain linear map between these spaces. In this chapter we introduce a new object for realizing these dualities.

Indeed, in Section 15.3, we have established a natural correspondance between Γ-invariant bilinear forms on $\overline{\mathcal{D}}(\partial X)$ and (ι, T)-invariant distributions on the space $\Gamma\backslash\mathscr{S}$, where $\mathscr{S}$ is the space of parametrized geodesic lines of X, T is the time shift and ι the natural involution.

We will now construct a dual object that is a bilinear map from the space $\mathcal{H}_\infty$ of Γ-invariant ∞-pseudofunctions towards the space of smooth functions on $\Gamma\backslash\mathscr{S}$. We will then use the Plancherel formula for X_1, Proposition 34.5, to split this bilinear maps into spectral components.

35.1. Global fundamental bilinear map

We define the global fundamental bilinear map $\mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{D}(\Gamma\backslash\mathscr{S})$, where $\mathcal{H}_\infty$ is the space of Γ-invariant ∞-pseudofunctions introduced in Section 20.3.

Let $\sigma = (\sigma_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line (see Section 2.1). We shall always denote by σ^+ and σ^- the endpoints of σ in ∂X. More precisely, we will write σ^+ to be the endpoint of the geodesic ray $(\sigma_h)_{h\geq 0}$ and σ^- to be the endpoint of the geodesic ray $(\sigma_{-h})_{h\geq 0}$. We also write $\langle\sigma\rangle$ for the set $\{\sigma_h | h \in \mathbb{Z}\}$, where we forget the parametrization. Thus, we have $\langle\sigma\rangle = (\sigma^-\sigma^+)$.

We let X_1^σ be the subset of X_1 defined by

$$X_1^\sigma = \{xy \in X_1 | [x, \sigma_0] \cap \langle\sigma\rangle = \{\sigma_0\}\}$$

(see Figure 1). The main interest of this set is that it allows to define partitions of X_1 associated to the orbit of σ under the time shift T. Indeed, we have

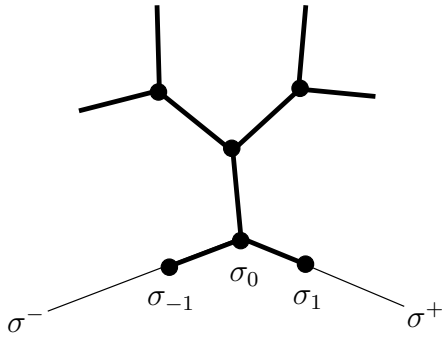

FIGURE 1. The set X_1^σ

LEMMA 35.1. *Let σ be a parametrized geodesic line. We have*

$$X_1 = \bigsqcup_{h\in\mathbb{Z}} X_1^{T^h\sigma},$$

that is, any xy in X_1 belongs to exactly one of the sets $X_1^{T^h\sigma}$, $h \in \mathbb{Z}$.

PROOF. Indeed, writing $\sigma = (\sigma_h)_{h\in\mathbb{Z}}$, we let h be the unique element of $\mathbb{Z}$ with

$$d(x, \sigma_h) = \min_{j\in\mathbb{Z}} d(x, \sigma_j).$$

□

Let H be an ∞-pseudofunction as in Section 20.3, that is, H is a map $xy \mapsto H_{xy}$ from X_1 to the space $\overline{\mathcal{D}}(\partial X)$. Thus, if xy is in X_1, H_{xy} is a smooth function on ∂X which is defined up to an additive constant. In particular, for any ξ, η in ∂X, the number

$$\Delta H_{xy}(\xi, \eta) = H_{xy}(\xi) - H_{xy}(\eta)$$

is well defined. If H is a k-pseudofunction for some $k \geq -1$, we write $\Delta H_{xy}(\xi, \eta)$ for $\Delta(H^{>^\infty})(\xi, \eta)$ (see Section 20.3 for the notation).

Now, recall that, as Γ has finitely many orbits in X_1, we have $\mathcal{H}_\infty = \bigcup_{k\geq -1} \mathcal{H}_k^{>^\infty}$. In particular, if H is a Γ-invariant ∞-pseudofunction, there exists $\ell \geq 0$ such that, for any $\xi \neq \eta$ in ∂X, we have $\Delta H_{xy}(\xi, \eta) = 0$ for any xy in X_1 with $d(x, (\xi\eta)) \geq \ell$. This justifies the following definition.

DEFINITION 35.2. We define the global fundamental bilinear map

$$\Phi : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{D}(\Gamma\backslash\mathscr{S})$$

as follows. For any H, J in $\mathcal{H}_\infty$ and any σ in $\mathscr{S}$, we set

$$\Phi(H,J)(\sigma) = \frac{1}{2} \sum_{xy \in X_1^\sigma} \Delta H_{xy}(\sigma^+, \sigma^-) \Delta J_{xy}(\sigma^+, \sigma^-).$$

Note that, for H, J as above, the function $\Phi(H, J)$ is ι-invariant.

The fundamental bilinear map has to be thought of as a map towards the space of cohomology classes of smooth functions on $\Gamma\backslash\mathscr{S}$, where cohomology classes were defined in Section 2.3 and Section 32.1 (both definitions being compatible by Corollary 32.7). In this sense, the next lemma says that the natural operators of the space of ∞-pseudofunctions are symmetric with respect to Φ.

LEMMA 35.3. *Let H, J be in $\mathcal{H}_\infty$. Then, we have $\Phi(RH, J) = \Phi(H, RJ)$ and the smooth functions $\Phi(SH, J)$ and $\Phi(H, SJ)$ are cohomologuous. So are the smooth functions $\Phi(PH, J)$ and $\Phi(H, PJ)$*

See Section 20.5 for the definition of R and S. As usual, we write $P = \frac{1}{q+1}(RS + SR - (q-1)S)$.

PROOF. In view of the definition of P, it suffices to prove the statements for R and S. Fix σ in $\mathscr{S}$. The fact that an edge xy of X_1 belongs to X_1^σ only depends on x. This directly implies that $\Phi(RH, J)(\sigma) = \Phi(H, RJ)(\sigma)$. Besides, we have

$$X_1^\sigma \smallsetminus \{yx | xy \in X_1^\sigma\} = \{\sigma_0\sigma_1, \sigma_0\sigma_{-1}\},$$

which implies

$$\begin{aligned} 2\Phi(SH,J)(\sigma) - 2\Phi(H,SJ)(\sigma) &= \Delta H_{\sigma_1\sigma_0}(\sigma^+,\sigma^-)\Delta J_{\sigma_0\sigma_1}(\sigma^+,\sigma^-) \\ &\quad + \Delta H_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\Delta J_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-) \\ &\quad - \Delta H_{\sigma_0\sigma_1}(\sigma^+,\sigma^-)\Delta J_{\sigma_1\sigma_0}(\sigma^+,\sigma^-) \\ &\quad - \Delta H_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-)\Delta J_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-) \\ &= \Psi(H,J)(T\sigma) - \Psi(H,J)(\sigma), \end{aligned}$$

where

$$\begin{aligned} \Psi(H,J)(\sigma) = \Delta H_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-)\Delta J_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-) \\ - \Delta H_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\Delta J_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-). \end{aligned}$$

The Lemma follows. □

35.2. Invariant distributions

We will now relate the fundamental bilinear map Φ to the constructions of Section 15.3. Let p be a Γ-invariant symmetric bilinear form on

$\overline{\mathcal{D}}(\partial X)$. As in Chapter 25, we still write p for the symmetric bilinear form on $\mathcal{H}_\infty$ defined by, for any (H, J) in $\mathcal{H}_\infty$,

$$p(H, J) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p(H_{xy}, J_{xy}).$$

We use the same notation for the associated bilinear forms on $\mathcal{H}_k$, $k \geq -1$.

PROPOSITION 35.4. *Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ and θ be the associated (ι, T)-invariant distribution on $\Gamma\backslash\mathscr{S}$. For any (H, J) in $\mathcal{H}_\infty$, we have*

$$p(H, J) = \langle \theta, \Phi(H, J)\rangle.$$

The proof is a consequence of the discussion in Section 15.3 and of the following purely combinatorial

LEMMA 35.5. *Let G be a discrete group, A be a discrete set and U be a totally discontinuous locally compact topological space. Assume we are given actions of G on A and U with the following properties: the action of G on A has finite stabilizers and finitely many orbits; the action of G on U is proper and cocompact.*

Pick a distribution θ on $G\backslash U$ and denote by $\widetilde{\theta}$ the associated G-invariant distribution on U. Let $\varphi : A \times U \to \mathbb{R}$ be a G-invariant function which is locally constant in the following uniform way: for every u in U, there exists a neighborhood V of u in U such that $\varphi(a, .)$ is constant on V for every a in A. Assume for every a in A, $\varphi(a, .)$ has compact support in U and, for every u in U, $\varphi(., u)$ has finite support in A. Then, the function $u \mapsto \sum_{a\in A} \varphi(a, u)$ is locally constant on U and we have

$$\sum_{a\in G\backslash A} \frac{1}{|G_a|} \left\langle \widetilde{\theta}, \varphi(a, .)\right\rangle = \left\langle \theta, \sum_{a\in A} \varphi(a, .)\right\rangle.$$

PROOF. Recall from Section 15.3 that $\widetilde{\theta}$ and θ are related as follows: if ψ is in $\mathcal{D}(U)$ and $\overline{\psi}(u) = \sum_{g\in G} \psi(gu)$ for u in U, then we have $\langle\widetilde{\theta}, \psi\rangle = \langle\theta, \overline{\psi}\rangle$.

Fix a system of representatives S for the action of G on A, that is, $S \subset A$ is such that $A = GS$ and $Ga \cap S = \{a\}$ for every a in S. By definition, we have

$$\sum_{a\in G\backslash A} \frac{1}{|G_a|} \left\langle \widetilde{\theta}, \varphi(a, .)\right\rangle = \sum_{a\in S} \frac{1}{|G_a|} \left\langle \widetilde{\theta}, \varphi(a, .)\right\rangle = \left\langle \widetilde{\theta}, \sum_{a\in S} \frac{1}{|G_a|} \varphi(a, .)\right\rangle.$$

For u in U, set $\psi(u) = \sum_{a\in S} \frac{1}{|G_a|}\varphi(a,u)$. The assumption on φ implies that ψ is in $\mathcal{D}(U)$. As S is a system of representatives for the action of G on A, for u in U, we have

$$\overline{\psi}(u) = \sum_{g\in G}\sum_{a\in S} \frac{1}{|G_a|}\varphi(ga,u) = \sum_{a\in A}\varphi(a,u).$$

The Lemma follows. □

Proof of Proposition 35.4. Still as in Section 15.3, we associate to θ a Γ-invariant and (ι,T)-invariant distribution $\widetilde{\theta}$ on $\mathscr{S}$. In the same way, as the quotient of $\mathscr{S}$ by the time shift T may be identified with $\partial^2 X$, we associate to $\widetilde{\theta}$ a distribution $\theta_{\partial^2 X}$ on $\partial^2 X$. Note that $\widetilde{\theta}$ and $\theta_{\partial^2 X}$ are Γ-invariant and symmetric.

Let H, J be in $\mathcal{H}_\infty$ and xy be in X_1. By Lemma 15.10, we have

$$p(H_{xy}, J_{xy}) = \frac{1}{2}\theta_{\partial^2 X}((\xi,\eta)\mapsto \Delta H_{xy}(\xi,\eta)\Delta J_{xy}(\xi,\eta)).$$

By Lemma 35.1, we get

$$p(H_{xy}, J_{xy}) = \frac{1}{2}\widetilde{\theta}\left(\sigma \mapsto \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\mathbf{1}_{xy\in X_1^\sigma}\right).$$

Thus, we obtain

$$p(H,J) = \frac{1}{2}\sum_{xy\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x\cap\Gamma_y|}\widetilde{\theta}\left(\sigma \mapsto \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\mathbf{1}_{xy\in X_1^\sigma}\right).$$

The Proposition follows by applying Lemma 35.5 to the action of Γ on $A = X_1$ and $U = \mathscr{S}$ and to the function

$$\varphi : X_1\times\mathscr{S}\to\mathbb{R}, (xy,\sigma)\mapsto \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\mathbf{1}_{xy\in X_1^\sigma}.$$

□

35.3. Spectral fundamental bilinear maps

We will use the Plancherel formula for X_1, Proposition 34.5, to decompose the global fundamental bilinear map Φ into spectral components. We start by introducing those components.

The spectral components will depend on a complex parameter t. We will only define these components as t ranges in the domain delimited by a certain ellipse in $\mathbb{C}$. We define $\mathcal{E}_q\subset\mathbb{C}$ as the set of those t in $\mathbb{C}$ such that

$$(\Re t)^2 + \left(\frac{q+1}{q-1}\Im t\right)^2 < 1.$$

Note that $\mathcal{E}_q \cap \mathbb{R} = (-1, 1)$. Elementary computations show

LEMMA 35.6. *Let t be in $\mathbb{C}$. Then t belongs to $\mathcal{E}_q$ if and only if the solutions of the equation $(q+1)tu = q + u^2$ satisfy $1 < |u| < q$.*

PROOF. For u in $\mathbb{C}^*$, write $u = \rho e^{i\theta}$ for some $\rho > 0$ and θ in $\mathbb{R}$. Then, we have

$$t = \frac{1}{q+1}\left(u + \frac{q}{u}\right) = \frac{\rho + \frac{q}{\rho}}{q+1}\cos\theta + \frac{\rho - \frac{q}{\rho}}{q+1}\sin\theta.$$

Thus, if $\rho = \sqrt{q}$, when u ranges in the circle $\{|u| = \sqrt{q}\}$, t ranges in the interval $\mathcal{I}_q$; if $\rho \neq \sqrt{q}$ and u ranges in the circle $\{|u| = \rho\}$, t ranges in the ellipse

$$\left(\frac{q+1}{\rho + \frac{q}{\rho}}\Re t\right)^2 + \left(\frac{q+1}{\rho - \frac{q}{\rho}}\Im t\right)^2 = 1.$$

The conclusion follows. □

COROLLARY 35.7. *Let H be in $\mathcal{H}_\infty$ and K be a compact subset of $\mathcal{E}_q$. We have*

$$\sup_{\substack{(\xi,\eta)\in\partial^2 X \\ ab\in X_1 \\ t\in K}} \sum_{xy\in X_1} |\chi_t(ab, xy)\Delta H_{xy}(\xi, \eta)| < \infty.$$

PROOF. By Lemma 35.6, we can find $\beta > 0$ such that, for any t in K and u in $\mathbb{C}$ with $(q+1)tu = q + u^2$, we have $1 + \beta \leq |u| \leq q - \beta$. We fix such t, u. As $\mathcal{H}_\infty = \bigcup_{\ell\geq 1} \mathcal{H}_{2\ell}^{>\infty}$, we can assume that H belongs to $\mathcal{H}_{2\ell}^{>\infty}$ for some $\ell \geq 1$. Let (ξ, η) be in $\partial^2 X$. For xy in X_1, we have $\Delta H_{xy}(\xi, \eta) = 0$ as soon as $d(x, (\xi\eta)) \geq \ell$. Note that we have

$$\sup_{j\geq 0} |\{xy \in X_1 | \delta(ab, xy) = j, d(x, (\xi\eta)) \leq \ell\}| < \infty.$$

Besides, as H is Γ-invariant, we have

$$\sup_{\substack{(\xi,\eta)\in\partial^2 X \\ xy\in X_1}} |\Delta H_{xy}(\xi, \eta)| < \infty.$$

By Lemma 34.2, we get, for some $C > 0$, for any (ξ, η) in $\partial^2 X$ and ab in X_1,

$$\sum_{xy\in X_1} |\chi_t(ab, xy)\Delta H_{xy}(\xi, \eta)| \leq C \sum_{j\geq 0} (j+1) \max\left(|u|^{-j}, \left|\frac{q}{u}\right|^{-j}\right).$$

As $1 + \beta \leq |u| \leq q - \beta$, the latter sum is uniformly bounded as t ranges in K. □

The convergence of these sums justifies the following

DEFINITION 35.8. Let t be in $\mathcal{E}_q$. For any H, J in $\mathcal{H}_\infty$ and any σ in $\mathscr{S}$, we set

$$\Phi_t(H,J)(\sigma) = \frac{1}{2} \sum_{\substack{ab \in X_1^\sigma \\ xy \in X_1}} \chi_t(ab,xy) \Delta H_{ab}(\sigma^+,\sigma^-) \Delta J_{xy}(\sigma^+,\sigma^-).$$

We consider Φ_t as a bilinear map from $\mathcal{H}_\infty \times \mathcal{H}_\infty$ to the space of functions on $\Gamma\backslash\mathscr{S}$, which we call the spectral fundamental bilinear map associated to t.

Note that, for t and H, J as above, the function $\Phi_t(H,J)$ is ι-invariant.

The spectral fundamental bilinear map creates Hölder continuous functions. As in Section 32.2, for $0 < \alpha < 1$, we still let $\mathscr{H}_\alpha$ denote the space of α-Hölder continuous functions on $\Gamma\backslash\mathscr{S}$, equipped with the natural norm

$$\|f\|_\alpha = \sup |f| + \sup_{\substack{h \geq 0 \\ \sigma,\tau \in \mathscr{S} \\ \forall |i| \leq h \quad \sigma_i = \tau_i}} \alpha^{-h} |f(\sigma) - f(\tau)|, \quad f \in \mathscr{H}_\alpha.$$

LEMMA 35.9. *Let H, J be in $\mathcal{H}_\infty$. For any t in $\mathcal{E}_q$, $\Phi_t(H,J)$ is a Hölder continuous function. More precisely, given an open subset Ω of $\mathbb{C}$, whose closure in $\mathbb{C}$ is contained in $\mathcal{E}_q$, there exists $0 < \alpha < 1$ such that $\Phi_t(H,J)$ belongs to $\mathscr{H}_\alpha$ for any t in Ω and the map $t \mapsto \Phi_t(H,J), \Omega \to \mathscr{H}_\alpha$ is analytic.*

PROOF. Let Ω be as in the statement. Then, by Lemma 35.6, we can find $\alpha < 1$ such that, for any t in Ω and u in $\mathbb{C}$ with $(q+1)ty = q + u^2$, we have $|u|^{-1} < \alpha$ and $\left|\frac{u}{q}\right| < \alpha$. The statement then follows by using Lemma 34.2 as in the proof of Corollary 35.7. □

As for the global fundamental bilinear map, we shall from now on think to the spectral fundamental bilinear maps as maps towards the space of cohomology classes of Hölder continuous functions. We get, by analogy with Lemma 34.3 and Lemma 35.3,

LEMMA 35.10. *Let H, J be in $\mathcal{H}_\infty$ and t be in $\mathcal{E}_q$. Then we have*

$$\Phi_t(RH, J) = \Phi_t(H, RJ)$$

and the Hölder continuous functions

$$\Phi_t(H,J) - \Phi_t(J,H),$$
$$\Phi_t(SH,J) - \Phi_t(H,SJ)$$
$$\text{and } \Phi_t(PH,J) - t\Phi_t(H,J)$$

are coboundaries.

For proving Lemma 35.10, to avoid lenghty dominations converging sums, we shall use

LEMMA 35.11. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Assume that there exists a family $(h_n)_{n\in\mathbb{Z}}$ of continuous functions on $\Gamma\backslash\mathscr{S}$ such that*

$$\sup_{\sigma\in\mathscr{S}}\sum_{n\in\mathbb{Z}}|h_n(\sigma)|<\infty \text{ and } \sup_{\sigma\in\mathscr{S}}\sum_{n\in\mathbb{Z}}|h_n(T^n\sigma)|<\infty$$

and such that $f=\sum_{n\in\mathbb{Z}}h_n\circ T^n-h_n$. Then, f is a coboundary.

PROOF. This is a consequence of Livšic Theorem, Proposition 32.1. Indeed, let μ be a T-invariant Borel probability measure on $\Gamma\backslash\mathscr{S}$. By the Dominated Convergence Theorem, the series $\sum_{n\in\mathbb{Z}}h_n\circ T^n-h_n$ converges in $\mathrm{L}^1(\Gamma\backslash\mathscr{S},\mu)$. As $\int_{\Gamma\backslash\mathscr{S}}(h_n\circ T^n-h_n)\mathrm{d}\mu=0$ for every n in $\mathbb{Z}$, we get $\int_{\Gamma\backslash\mathscr{S}}f\mathrm{d}\mu=0$. The conclusion follows from Proposition 32.1. □

PROOF OF LEMMA 35.10. The fact that $\Phi_t(RH,J)=\Phi_t(H,RJ)$ is obtained as in the proof of Lemma 35.3, by using the corresponding property of χ_t from Lemma 34.3.

Let us prove that $\Phi_t(H,J)-\Phi_t(J,H)$ is a coboundary. From Lemma 35.1, we get, for σ in $\mathscr{S}$,

$$\Phi_t(H,J)(\sigma)=\frac{1}{2}\sum_{h\in\mathbb{Z}}\sum_{\substack{ab\in X_1^\sigma\\ xy\in X_1^{T^h\sigma}}}\chi_t(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-).$$

By using the symmetry of χ_t, we get

$$\Phi_t(H,J)(\sigma)-\Phi_t(J,H)(\sigma)=\sum_{h\in\mathbb{Z}}\Xi_{t,h}(H,J)(T^h\sigma)-\Xi_{t,h}(H,J)(\sigma),$$

where, for h in $\mathbb{Z}$,

$$\Xi_{t,h}(H,J)(\sigma)=\frac{1}{2}\sum_{\substack{ab\in X_1^{T^{-h}\sigma}\\ xy\in X_1^\sigma}}\chi_t(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-).$$

Due to Corollary 35.7, we have

$$\sup_{\sigma\in\mathscr{S}}\sum_{h\in\mathbb{Z}}|\Xi_{t,h}(H,J)(\sigma)|<\infty \text{ and } \sup_{\sigma\in\mathscr{S}}\sum_{h\in\mathbb{Z}}\left|\Xi_{t,h}(H,J)(T^h\sigma)\right|<\infty.$$

Therefore, by Lemma 35.11, $\Phi_t(H,J)-\Phi_t(J,H)$ is a coboundary.

We now study $\Phi_t(SH, J) - \Phi_t(H, SJ)$. By using the relation

$$\chi_t(ba, xy) = \chi_t(ab, yx),$$

for ab, xy in X_1, and by reasoning as in the proof of Lemma 35.3, we obtain, for σ in $\mathscr{S}$,

$$\Phi_t(SH, J) - \Phi_t(H, SJ) = \Psi_t(H, J)(\sigma) - \Psi_t(H, J)(T\sigma),$$

with

$$\Psi_t(H, J)(\sigma) = \frac{1}{2} \sum_{xy \in X_1} \big(\Delta H_{\sigma_{-1}\sigma_0}(\sigma^+, \sigma^-) \chi_t(\sigma_{-1}\sigma_0, xy) - \Delta H_{\sigma_0\sigma_{-1}}(\sigma^+, \sigma^-) \chi_t(\sigma_0\sigma_{-1}, xy) \big) \big) \, \Delta J_{xy}(\sigma^+, \sigma^-).$$

By reasoning as in the proof of Lemma 35.9, one can show that the function $\Psi_t(H, J)$ is Hölder continuous on $\Gamma \backslash \mathscr{S}$. The conclusion follows.

Finally, notice from Lemma 34.3 that we have $\Phi_t(J, PH) = t\Phi(H, J)$. As the above implies that $\Phi_t(PH, J)$ is cohomologous to $\Phi_t(J, PH)$, the last statement follows. □

35.4. Special spectral fundamental bilinear maps

We still aim at using the Plancherel formula of X_1 from Proposition 34.5 in order to decompose the global fundamental bilinear map Φ. Thus, we need to introduce special components.

Recall the definition of δ in Section 34.2. The convergence in the formulas defining these special components is warranted by

LEMMA 35.12. *Let H be in $\mathcal{H}_\infty$. We have*

$$\sup_{\substack{(\xi,\eta) \in \partial^2 X \\ ab \in X_1}} \sum_{xy \in X_1} q^{-\delta(ab, xy)} |\Delta H_{xy}(\xi, \eta)| < \infty.$$

The proof is the same as the one of Corollary 35.7.

DEFINITION 35.13. For any H, J in $\mathcal{H}_\infty$ in $\mathcal{H}_\infty$, and any σ in $\mathscr{S}$, we set

$$\Phi_1^{\mathrm{sp}}(H, J)(\sigma) = \frac{1}{2} \sum_{\substack{ab \in X_1^\sigma \\ xy \in X_1}} \chi_1^{\mathrm{sp}}(ab, xy) \Delta H_{ab}(\sigma^+, \sigma^-) \Delta J_{xy}(\sigma^+, \sigma^-)$$

$$\Phi_{(-1)}^{\mathrm{sp}}(H, J)(\sigma) = \frac{1}{2} \sum_{\substack{ab \in X_1^\sigma \\ xy \in X_1}} \chi_{(-1)}^{\mathrm{sp}}(ab, xy) \Delta H_{ab}(\sigma^+, \sigma^-) \Delta J_{xy}(\sigma^+, \sigma^-)$$

We call Φ_1^{sp} and $\Phi_{(-1)}^{\mathrm{sp}}$ the special spectral fundamental bilinear maps.

Again, for H, J as above, the functions $\Phi_1^{\mathrm{sp}}(H,J)$ and $\Phi_{(-1)}^{\mathrm{sp}}(H,J)$ are ι-invariant.

These bilinear maps send $\mathcal{H}_\infty$ to $\mathscr{H}_{q^{-1}}$.

LEMMA 35.14. *Let H, J be in $\mathcal{H}_\infty$. Then, the functions $\Phi_1^{\mathrm{sp}}(H,J)$ and $\Phi_{(-1)}^{\mathrm{sp}}(H,J)$ are q^{-1}-Hölder continuous on $\Gamma\backslash\mathscr{S}$.*

PROOF. This directly follows from the definition of χ_1^{sp} and $\chi_{(-1)}^{\mathrm{sp}}$ in Section 34.4 and from the fact that, for H in $\mathcal{H}_\infty$, xy in X_1 and (ξ,η) in $\partial^2 X$, $\Delta H_{xy}(\xi,\eta)$ is 0 when xy is far enough from the geodesic line $(\xi\eta)$. □

In the same way as for Lemma 35.10, we show

LEMMA 35.15. *Let H, J be in $\mathcal{H}_\infty$. Then we have*

$$\Phi_1^{\mathrm{sp}}(RH,J) = -\Phi_1^{\mathrm{sp}}(H,J) \text{ and } \Phi_{(-1)}^{\mathrm{sp}}(RH,J) = -\Phi_{(-1)}^{\mathrm{sp}}(H,J)$$

and the Hölder continuous functions

$$\begin{aligned}
&\Phi_1^{\mathrm{sp}}(H,J) - \Phi_1^{\mathrm{sp}}(J,H), && \Phi_{(-1)}^{\mathrm{sp}}(H,J) - \Phi_{(-1)}^{\mathrm{sp}}(J,H),\\
&\Phi_1^{\mathrm{sp}}(SH,J) + \Phi_1^{\mathrm{sp}}(H,J), && \Phi_{(-1)}^{\mathrm{sp}}(SH,J) - \Phi_{(-1)}^{\mathrm{sp}}(H,J),\\
&\Phi_1^{\mathrm{sp}}(PH,J) - \Phi_1^{\mathrm{sp}}(H,J), && \Phi_{(-1)}^{\mathrm{sp}}(PH,J) + \Phi_{(-1)}^{\mathrm{sp}}(H,J),
\end{aligned}$$

are coboundaries.

35.5. The Plancherel formula for fundamental bilinear maps

From the Plancherel formula in X_1, we can decompose the global fundamental bilinear map.

PROPOSITION 35.16. *Let H, J be in $\mathcal{H}_\infty$. For σ in $\mathscr{S}$, we have*

$$\begin{aligned}
\Phi(H,J)(\sigma) = \frac{2}{q+1}\int_{\mathcal{I}_q} \Phi_t(H,J)(\sigma)\mathrm{d}\mu_q(t)\\
+ \frac{q-1}{2(q+1)}\Phi_1^{\mathrm{sp}}(H,J)(\sigma) + \frac{q-1}{2(q+1)}\Phi_{(-1)}^{\mathrm{sp}}(H,J)(\sigma).
\end{aligned}$$

PROOF. For ab in X_1 and $(\xi,\eta) \in \partial^2 X$, Corollary 35.7 and Lemma 35.12, together with the Dominated Convergence Theorem give, by the

same computation as in the proof of Proposition 34.5,

$$\Delta J_{ab}(\xi,\eta) = \frac{2}{q+1}\int_{\mathcal{I}_q}\sum_{xy\in X_1}\chi_t(ab,xy)\Delta J_{xy}(\xi,\eta)\mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)}\sum_{xy\in X_1}\chi_1^{\mathrm{sp}}(ab,xy)\Delta J_{xy}(\xi,\eta) + \frac{q-1}{2(q+1)}\sum_{xy\in X_1}\chi_{(-1)}^{\mathrm{sp}}(ab,xy)\Delta J_{xy}(\xi,\eta).$$

The conclusion then follows from the definition of the fundamental bilinear maps. □

Our goal is to use Proposition 35.16 to get a better understanding of the spectral theory of completions of $\overline{\mathcal{D}}(\partial X)$ with respect to Γ-invariant non-negative symmetric bilinear forms. We can already manage the case of representations associated to Radon measures. Recall from Proposition 15.15 that if ν is a finite (ι,T)-invariant Borel measure on $\Gamma\backslash\mathscr{S}$, the associated Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ is non-negative.

Corollary 35.17. *Let ν be a finite (ι,T)-invariant Borel measure on $\Gamma\backslash\mathscr{S}$ and p be the associated Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. Then, in the completion of $\mathcal{H}_\infty$ with respect to p, the spectrum of the operator P is $\mathcal{I}_q\cup\{-1,1\}$. The associated spectral measures are absolutely continuous with respect to the Lebesgue measure on $\mathcal{I}_q$.*

Proof. Let H,J be in $\mathcal{H}_\infty$. By Proposition 35.4, we have

$$p(H,J) = \int_{\Gamma\backslash\mathscr{S}}\Phi(H,J)\mathrm{d}\nu.$$

By Lemma 35.9, we can apply Fubini Theorem in the Plancherel formula of Proposition 35.16; this gives

$$p(H,J) = \frac{2}{q+1}\int_{\mathcal{I}_q}\int_{\Gamma\backslash\mathscr{S}}\Phi_t(H,J)\mathrm{d}\nu\mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)}\int_{\Gamma\backslash\mathscr{S}}\Phi_1^{\mathrm{sp}}(H,J)\mathrm{d}\nu + \frac{q-1}{2(q+1)}\int_{\Gamma\backslash\mathscr{S}}\Phi_{(-1)}^{\mathrm{sp}}(H,J)\mathrm{d}\nu.$$

As ν is T-invariant, by Lemma 35.10 and Lemma 35.15, for any polynomial function φ in $\mathbb{R}[t]$, we get

$$\begin{aligned} p(\varphi(P)H, J) = \frac{2}{q+1}\int_{\mathcal{I}_q}\varphi(t)\int_{\Gamma\backslash\mathscr{S}}\Phi_t(H,J)\mathrm{d}\nu\mathrm{d}\mu_q(t) \\ + \frac{q-1}{2(q+1)}\varphi(1)\int_{\Gamma\backslash\mathscr{S}}\Phi_1^{\mathrm{sp}}(H,J)\mathrm{d}\nu \\ + \frac{q-1}{2(q+1)}\varphi(-1)\int_{\Gamma\backslash\mathscr{S}}\Phi_{(-1)}^{\mathrm{sp}}(H,J)\mathrm{d}\nu. \end{aligned}$$

The conclusion follows by standard properties of spectral analysis of self-adjoint operators. □

Part 10

Endpoint series and weights

CHAPTER 36

Endpoints series and fundamental bilinear maps

In order to be able to study the spectral fundamental maps, we will now show that, up to a coboundary, they can be written as an endpoints series as in Section 33.1.

36.1. Weight of pseudofunctions

In this Section, for $k \geq 0$, we associate to every k-pseudofunction a function on X_k. This construction will later allow us to rewrite the definition of the spectral fundamental maps.

Definition 36.1. Let $k \geq 0$ be an integer and H be a k-pseudofunction. We define the weight $\omega_k(H)$, which is a function on X_k, as follows. Let ab be in X_k and $a_0 = a, a_1, \ldots, a_k = b$ be the geodesic parametrization of the segment $[ab]$. For c in X, $c \neq b$ we write c_- for the neighbour of c on $[bc]$.

If $k = 0$ and H is the 0-pseudofunction associated to the function u on X (see Section 20.1), we set $\omega_0(H) = u$.

If k is even, $k = 2\ell$, $\ell \geq 1$, we set

$$\omega_k(H)(ab) = \Delta H_{a_\ell a_{\ell+1}}(b, a) + \sum_{i=1}^{\ell-1} \sum_{\substack{c \in X \\ [a_{\ell+i}c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} \Delta H_{cc_-}(b, a_{2i}).$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, we set

$$\omega_k(H)(ab) = \Delta H_{a_\ell a_{\ell+1}}(b, a) + \sum_{i=1}^{\ell} \sum_{\substack{c \in X \\ [a_{\ell+i}c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} \Delta H_{cc_-}(b, a_{2i-1}).$$

Later, in Chapter 6, we will study the weight map ω_k. For the moment we show how it appears naturally when handling fundamental bilinear maps.

36.2. Initial bilinear maps

Here, we introduce a notation in order to reformulate the definition of the fundamental bilinear maps and we relate this notation with the weight defined above.

For σ in $\mathscr{S}$, we set

$$X^\sigma = \{x \in X | [x, \sigma_0] \cap \langle\sigma\rangle = \{\sigma_0\}\},$$

so that $X_1^\sigma = \{xy \in X_1 | x \in X^\sigma\}$.

Recall from Section 33.1 that, for $k \geq 1$, we let $\mathscr{V}_k$ stand for the space of Γ-invariant functions on X_k.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. If H is a k-pseudofunction, for σ in $\mathscr{S}$ and ab in X_1^σ with $d(a, \sigma_0) = r$, the quantity $\Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-)$ is zero if $r \geq \ell$. If $r < \ell$, this quantity only depends on the segment $[\sigma_{-(\ell-r)}, \sigma_{\ell-r}]$. Therefore, for $j \geq 0$, there exists a symmetric bilinear map

$$\kappa_{j,k} : \mathcal{H}_k \times \mathcal{H}_k \to \mathscr{V}_{j+k}$$

such that, for any H, J in $\mathcal{H}_k$ and σ in $\mathscr{S}$, one has

$$\sum_{ab \in X_1^\sigma} \Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{ab}^{>\infty}(\sigma^+, \sigma^-) = \kappa_{0,k}(H, J)(\sigma_{-\ell}, \sigma_\ell)$$

if $j = 0$ and

$$\sum_{\substack{a,x \in X^\sigma \\ d(a,x)=j}} \Delta H_{aa_1}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{xx_1}^{>\infty}(\sigma^+, \sigma^-)$$

$$+ \sum_{h=1}^{j} \sum_{\substack{a \in X^\sigma \\ x \in X^{T^h\sigma} \\ d(a,x)=j}} (\Delta H_{aa_1}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{xx_1}^{>\infty}(\sigma^+, \sigma^-) + \Delta H_{xx_1}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{aa_1}^{>\infty}(\sigma^+, \sigma^-))$$

$$= \kappa_{j,k}(H, J)(\sigma_{-\ell}, \sigma_{j+\ell})$$

if $j \geq 1$ (where as usual, for a, x as above, a_1 and x_1 are the neighbours of a and x on $[ax]$).

Assume now k is odd, $k = 2\ell+1$, $\ell \geq 0$. To deal with the symmetries associated with edges instead of vertices we introduce a new family of subsets of X_1. For σ in $\mathscr{S}$, we set

$$X_1^{\sigma,\sharp} = X_1^\sigma \cup \{\sigma_1\sigma_0\} \smallsetminus \{\sigma_0\sigma_{-1}\}.$$

If H is a k-pseudofunction, for σ in $\mathscr{S}$ and ab in $X_1^{\sigma,\sharp}$, $ab \notin \{\sigma_0\sigma_1, \sigma_1\sigma_0\}$, the quantity $\Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-)$ is zero if $r = \min(d(a, \langle\sigma\rangle), d(b, \langle\sigma\rangle)) \geq \ell$. If $r < \ell$, this quantity only depends on the segment $[\sigma_{-(\ell-r)}, \sigma_{\ell-r}]$.

Besides, the quantity $\Delta H^{>\infty}_{\sigma_0\sigma_1}(\sigma^+,\sigma^-)$ only depends on the segment $[\sigma_{-\ell},\sigma_{\ell+1}]$. Therefore, for $j\geq 0$, there exists a symmetric bilinear map

$$\kappa_{j,k}:\mathcal{H}_k\times\mathcal{H}_k\to\mathscr{V}_{j+k}$$

such that, for any H,J in $\mathcal{H}_k$ and σ in $\mathscr{S}$, one has

$$\sum_{ab\in X_1^{\sigma,\sharp}}\Delta H^{>\infty}_{ab}(\sigma^+,\sigma^-)\Delta J^{>\infty}_{ab}(\sigma^+,\sigma^-)=2\kappa_{0,k}(H,J)(\sigma_{-\ell},\sigma_{\ell+1})$$

if $j=0$ and

$$\sum_{\substack{ab,xy\in X_1^{\sigma,\sharp}\\ \delta(ab,xy)=j\\ b,y\in[ax]}}\Delta H^{>\infty}_{ab}(\sigma^+,\sigma^-)\Delta J^{>\infty}_{xy}(\sigma^+,\sigma^-)$$
$$+\sum_{h=1}^{j}\sum_{\substack{ab\in X_1^{\sigma,\sharp}\\ xy\in X_1^{T^h\sigma,\sharp}\\ \delta(ab,xy)=j\\ b,y\in[ax]}}(\Delta H^{>\infty}_{ab}(\sigma^+,\sigma^-)\Delta J^{>\infty}_{xy}(\sigma^+,\sigma^-)+\Delta H^{>\infty}_{xy}(\sigma^+,\sigma^-)\Delta J^{>\infty}_{ab}(\sigma^+,\sigma^-))$$
$$=2\kappa_{j,k}(H,J)(\sigma_{-\ell},\sigma_{j+\ell+1})$$

if $j\geq 1$.

When $j\geq k-1$, we have a better formula for these bilinear maps which will help us to write the fundamental bilinear maps as endpoints series.

LEMMA 36.2. *Let $k\geq 1$, $j\geq k-1$ and H,J be in $\mathcal{H}_k$. Then the function $\kappa_{j,k}(H,J)$ is cohomologous to the function*

$$X_{j+k}\to\mathbb{R}, xy\mapsto -\omega_k(H)(xx_k)\omega_k(J)(yy_k)-\omega_k(J)(xx_k)\omega_k(H)(yy_k),$$

where x_k and y_k are the elements of $[xy]$ at distance k from x and y.

PROOF. Assume k is even, $k=2\ell$, $\ell\geq 1$. Take σ in $\mathscr{S}$. As mentioned above, for a in X^σ with $d(a,\langle\sigma\rangle)\geq\ell$ and $b\sim a$, we have $\Delta H^{>\infty}_{ab}(\sigma^+,\sigma^-)=0$. Therefore, for $h\geq 0$, a in X^σ and x in $X^{T^h\sigma}$, if $d(a,x)=j$ and $\Delta H^{>\infty}_{aa_1}(\sigma^+,\sigma^-)\Delta J^{>\infty}_{xx_1}(\sigma^+,\sigma^-)\neq 0$, we have

$$h\geq j-2(\ell-1)=j-k+2\geq 1.$$

More precisely, we have

$$\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell}) = \sum_{0\le r,s\le \ell-1} \sum_{\substack{a\in S^r(\sigma_0)\cap X^\sigma \\ x\in S^s(\sigma_{j-r-s})\cap X^{T^{j-r-s}\sigma}}} \Delta H_{aa_1}(\sigma_{\ell-r},\sigma_{-(\ell-r)})\Delta J_{xx_1}(\sigma_{j+\ell-r-2s},\sigma_{j-r-\ell}) + \Delta H_{xx_1}(\sigma_{j+\ell-r-2s},\sigma_{j-r-\ell})\Delta J_{aa_1}(\sigma_{\ell-r},\sigma_{-(\ell-r)}).$$

For r,s as above, the smooth function

$$\sigma \mapsto \sum_{\substack{a\in S^r(\sigma_0)\cap X^\sigma \\ x\in S^s(\sigma_{j-r-s})\cap X^{T^{j-r-s}\sigma}}} \Delta H_{aa_1}(\sigma_{\ell-r},\sigma_{-(\ell-r)})\Delta J_{xx_1}(\sigma_{j+\ell-r-2s},\sigma_{j-r-\ell})$$

is cohomologous to

$$\sigma \mapsto \sum_{\substack{a\in S^r(\sigma_{\ell+r})\cap X^{T^{\ell+r}\sigma} \\ x\in S^s(\sigma_{j+\ell-s})\cap X^{T^{j+\ell-s}\sigma}}} \Delta H_{aa_1}(\sigma_{2\ell},\sigma_{2r})\Delta J_{xx_1}(\sigma_{j+2\ell-2s},\sigma_j).$$

The conclusion follows by using Definition 36.1.

Assume k is odd, $k=2\ell+1$, $\ell\ge 0$. Now, for ab in X_1 and σ in $\mathscr{S}$, we have $\Delta H_{ab}(\sigma^+,\sigma^-)=0$ as soon as $\min(d(a,\langle\sigma\rangle),d(b,\langle\sigma\rangle))\ge\ell$. For $j\ge k-1$, we will split the sum defining $\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1})$ according to whether $a=\sigma_0$ or not and whether $x=\sigma_{h+1}$ or not. Thus, we write

$$\kappa_{j,k}(H,J)=\kappa^{00}_{j,k}(H,J)+\kappa^{01}_{j,k}(H,J)+\kappa^{10}_{j,k}(H,J)+\kappa^{11}_{j,k}(H,J),$$

where, first,

$$\kappa^{00}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \Delta H_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell})\Delta J_{\sigma_{j+1}\sigma_j}(\sigma_{j+\ell+1},\sigma_{j-\ell}) + \Delta H_{\sigma_{j+1}\sigma_j}(\sigma_{j+\ell+1},\sigma_{j-\ell})\Delta J_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell}),$$

which is cohomologous to the function

$$\sigma\mapsto \Delta H_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0)\Delta J_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j) + \Delta H_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j)\Delta J_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0);$$

second,

$$\kappa^{01}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \sum_{1\le s\le\ell}\sum_{x\in S^s(\sigma_{j+1-s})\cap X^{T^{j+1-s}\sigma}} \Delta H_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell})\Delta J_{xx_-}(\sigma_{j+\ell+2-2s},\sigma_{j-\ell}) + \Delta H_{xx_-}(\sigma_{j+\ell+2-2s},\sigma_{j-\ell})\Delta J_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell}),$$

which is cohomologous to the function

$$\sigma \mapsto \sum_{1\leq s\leq \ell} \sum_{x\in S^s(\sigma_{j+\ell+1-s})\cap X^{T^{j+\ell+1-s}\sigma}} \Delta H_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0)\Delta J_{xx_-}(\sigma_{j+k+1-2s},\sigma_j) + \Delta H_{xx_-}(\sigma_{j+k+1-2s},\sigma_j)\Delta J_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0);$$

third,

$$\kappa^{10}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \sum_{1\leq r\leq \ell} \sum_{a\in S^r(\sigma_0)\cap X^\sigma} \Delta H_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1})\Delta J_{\sigma_{j+1-r}\sigma_{j-r}}(\sigma_{j+\ell+1-r},\sigma_{j-\ell-r}) + \Delta H_{\sigma_{j+1-r}\sigma_{j-r}}(\sigma_{j+\ell+1-r},\sigma_{j-\ell-r})\Delta J_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1}),$$

which is cohomologous to the function

$$\sigma \mapsto \sum_{1\leq r\leq \ell} \sum_{a\in S^r(\sigma_{\ell+r})\cap X^{T^{\ell+r}\sigma}} \Delta H_{aa_-}(\sigma_k,\sigma_{2r-1})\Delta J_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j) + \Delta H_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j)\Delta J_{aa_-}(\sigma_k,\sigma_{2r-1});$$

fourth

$$\kappa^{11}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \sum_{1\leq r,s\leq \ell} \sum_{\substack{a\in S^r(\sigma_0)\cap X^\sigma \\ x\in S^s(\sigma_{j+1-r-s})\cap X^{T^{j+1-r-s}\sigma}}} \Delta H_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1})\Delta J_{xx_-}(\sigma_{j+\ell+2-r-2s},\sigma_{j-\ell-r}) + \Delta H_{xx_-}(\sigma_{j+\ell+2-r-2s},\sigma_{j-\ell-r})\Delta J_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1}),$$

which is cohomologous to

$$\sigma \mapsto \sum_{1\leq r,s\leq \ell} \sum_{\substack{a\in S^r(\sigma_{\ell+r})\cap X^{T^{\ell+r}\sigma} \\ x\in S^s(\sigma_{j+\ell+1-s})\cap X^{T^{j+\ell+1-s}\sigma}}} \Delta H_{aa_-}(\sigma_k,\sigma_{2r-1})\Delta J_{xx_-}(\sigma_{j+k+1-2s},\sigma_j) + \Delta H_{xx_-}(\sigma_{j+k+1-2s},\sigma_j)\Delta J_{aa_-}(\sigma_k,\sigma_{2r-1}).$$

Using Definition 36.1 yields the conclusion. □

36.3. Endpoints series formulas for the spectral fundamental bilinear maps

In this section, we use the previous constructions to give an alternative formula for defining the spectral fundamental maps by means of an endpoints series as in Section 33.1.

We will need new families of polynomial functions besides the $(A_j)_{j\geq 0}$ and the $(B_j)_{j\geq 0}$ from Section 34.1 and the $(C_j)_{j\geq 0}$ from Section 34.6. For $j \geq 0$, we set

$$D_j = A_j - (q-1)B_j = C_j - (q-1)\sum_{0\leq h<j} C_h$$
$$E_j = A_j + (q-1)B_j = C_j - (q-1)\sum_{0\leq h<j} (-1)^{j-h} C_h$$
$$F_j = qA_j - (q-1)B_{j+1} = C_j - (q^2-1)\sum_{\substack{0\leq h<j \\ j-h \text{ even}}} C_h,$$

where the equalities follow from Corollary 34.7.

As in Section 20.1, for $k \geq -1$, we write $\mathcal{H}_k = \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-}$ the decomposition into eigenspaces of the $\vee$ operator. Recall that the spectral fundamental bilinear map Φ_t was introduced in Definition 35.8.

PROPOSITION 36.3. *Let $k \geq 2$ and t be in $\mathcal{E}_q$.*

Assume k is even. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous function $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^+_{t,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^+_{t,k}(H,J)(\sigma) = \frac{q+1}{4}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) + \frac{q(q+1)}{4}\sum_{j=1}^{k-2} q^{-j} C_j(t)\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$
$$-\frac{q(q+1)}{4}\sum_{j=k-1}^{\infty} q^{-j} C_j(t)(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

For any H, J in $\mathcal{H}_{k,-}$, the Hölder continuous function $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^-_{t,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^-_{t,k}(H,J)(\sigma) = \frac{q+1}{4q}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) - \frac{q+1}{4q}\sum_{j=1}^{k-2} q^{-j} F_j(t)\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$
$$+\frac{q+1}{4q}\sum_{j=k-1}^{\infty} q^{-j} F_j(t)(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

Assume k is odd. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous function $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ is cohomologous to the Hölder continuous

function $\Phi^+_{t,k}(H,J)$ *defined by, for* σ *in* $\mathscr{S}$*,*

$$\Phi^+_{t,k}(H,J)(\sigma) = \sum_{j=0}^{k-2} q^{-j} E_j(t) \kappa_{j,k}(H,J)(\sigma_0 \sigma_{j+k})$$

$$- \sum_{j=k-1}^{\infty} q^{-j} E_j(t) (\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

For any H, J *in* $\mathcal{H}_{k,-}$*, the Hölder continuous function* $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ *is cohomologous to the Hölder continuous function* $\Phi^-_{t,k}(H,J)$ *defined by, for* σ *in* $\mathscr{S}$*,*

$$\Phi^-_{t,k}(H,J)(\sigma) = \kappa_{0,k}(H,J)(\sigma_0\sigma_k) - \sum_{j=1}^{k-2} q^{-j} D_j(t) \kappa_{j,k}(H,J)(\sigma_0 \sigma_{j+k})$$

$$+ \sum_{j=k-1}^{\infty} q^{-j} D_j(t) (\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

The proof is a consequence of the formulas below. These allow to compute the spectral bilinear forms of Section 34.3 in eigenspaces of the operators R and S by means of our new families of polynomial functions.

LEMMA 36.4. *Fix* t *in* $\mathbb{R}$*.*
For a, x *in* X *and* $j = d(a,x)$*, we have*

$$\sum_{\substack{b\sim a \\ y\sim x}} \chi_t(ab,xy) = \frac{(q+1)^2}{2} \qquad j = 0$$

$$= \frac{q+1}{2q^{j-1}} C_j(t) \qquad j \geq 1.$$

For a *in* X *anb* $b, c \sim a$*,* $b \neq c$*, we have*

$$\chi_t(ab,ab) - \chi_t(ab,ac) = \frac{q+1}{2q}.$$

For a, x *in* X *with* $j = d(a,x) \geq 1$ *and* $b \sim a$*,* $y \sim x$ *with* $b, y \notin [ax]$*,*

$$\chi_t(ab,xy) - \chi_t(aa_1,xy) - \chi_t(ab,xx_1) + \chi_t(aa_1,xx_1) = -\frac{q+1}{2q^{j+1}} F_j(t).$$

For ab, xy in X_1 and $j = \delta(ab, xy)$, we have

$$\begin{aligned}
\chi_t(ab,xy) - \chi_t(ab,yx) &= \varepsilon(ab,xy) && j=0\\
&= \frac{1}{2q^j} D_j(t)\varepsilon(ab,xy) && j\geq 1\\
\chi_t(ab,xy) + \chi_t(ab,yx) &= 1 && j=0\\
&= \frac{1}{2q^j} E_j(t) && j\geq 1.
\end{aligned}$$

The proofs directly follow from the definition of χ_t in Section 34.3.

Proof of Proposition 36.3 in case k is even. For t in $\mathcal{E}_q$, H, J in $\mathcal{H}_\infty$ and σ in $\mathscr{S}$, we set

$$\begin{aligned}
\Phi'_t(H,J)(\sigma) = \frac{1}{2}\sum_{ab,xy\in X_1^\sigma} &\chi_t(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\
+\frac{1}{2}\sum_{h\geq 1}\sum_{\substack{ab\in X_1^\sigma\\ xy\in X_1^{T^h\sigma}}} &\chi_t(ab,xy)(\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\
&+\Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).
\end{aligned} \tag{36.1}$$

As in Corollary 35.9, one shows that $\Phi'_t(H,J)$ is a Hölder continuous function. In view of Definition 35.8, by Corollary 35.7 and Lemma 35.11, $\Phi'_t(H,J)$ and $\Phi_t(H,J)$ are cohomologous.

Assume $RH = qH$ and $RJ = qJ$. As, for $x \sim y$ in X, the elements H_{xy} and J_{xy} of $\overline{\mathcal{D}}(\partial X)$ only depend on x, we will write H_x and J_x for them below. In particular, from Lemma 36.4, we get

$$\begin{aligned}
\Phi'_t(H,J)(\sigma) = \frac{(q+1)^2}{4}\sum_{a\in X^\sigma} &\Delta H_a(\sigma^+,\sigma^-)\Delta J_a(\sigma^+,\sigma^-)\\
+\frac{q+1}{4}\sum_{a\neq x\in X^\sigma} &\frac{C_{d(a,x)}(t)}{q^{d(a,x)-1}}\Delta H_a(\sigma^+,\sigma^-)\Delta J_x(\sigma^+,\sigma^-)\\
+\frac{q+1}{4}\sum_{h\geq 1}\sum_{\substack{a\in X^\sigma\\ x\in X^{T^h\sigma}}} &\frac{C_{d(a,x)}(t)}{q^{d(a,x)-1}}(\Delta H_a(\sigma^+,\sigma^-)\Delta J_x(\sigma^+,\sigma^-)\\
&+\Delta H_x(\sigma^+,\sigma^-)\Delta J_a(\sigma^+,\sigma^-)).
\end{aligned}$$

Recall that $k \geq 2$ is an even integer, $k = 2\ell$, $\ell \geq 1$. Take H, J in $\mathcal{H}_{k,+}$. From the formula above, we write

$$\Phi'_t(H^{>^\infty}, J^{>^\infty})(\sigma_{-\ell}\sigma_\ell) = \frac{q+1}{4}\kappa_{0,k}(H,J)(\sigma) + \frac{q(q+1)}{4}\sum_{j=1}^{\infty} q^{-j}C_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell}).$$

The first case now follows from Lemma 36.2.

We now address the second case. We will follow the same lines as above. We start by noticing that, by Lemma 36.4, if H, J are in $\mathcal{H}_\infty$ with $RH = -H$ and $RJ = -J$, for a, x in X with $j = d(a,x) \geq 1$ and ξ, η in ∂X, we have

$$\sum_{\substack{b\sim a\\ y\sim x}} \chi_t(ab,xy)\Delta H_{ab}(\xi,\eta)\Delta J_{xy}(\xi,\eta) = -\frac{q+1}{2q^{j+1}}F_j(t)\Delta H_{aa_1}(\xi,\eta)\Delta J_{xx_1}(\xi,\eta).$$

When $a = x$, we get

$$\sum_{b,c\sim a} \chi_t(ab,ac)\Delta H_{ab}(\xi,\eta)\Delta J_{ac}(\xi,\eta) = \frac{q+1}{2q}\sum_{b\sim a}\Delta H_{ab}(\xi,\eta)\Delta J_{ab}(\xi,\eta).$$

Thus, when $RH = -H$ and $RJ = -J$, we may rewrite (36.1) as

$$\begin{aligned}\Phi'_t(H,J)(\sigma) = {} & \frac{q+1}{4q}\sum_{ab\in X_1^\sigma}\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)\\ & - \frac{q+1}{4}\sum_{a\neq x\in X^\sigma}\frac{F_{d(a,x)}(t)}{q^{d(a,x)+1}}\Delta H_{aa_1}(\sigma^+,\sigma^-)\Delta J_{xx_1}(\sigma^+,\sigma^-)\\ & - \frac{q+1}{4}\sum_{h\geq 1}\sum_{\substack{a\in X^\sigma\\ x\in X^{T^h\sigma}}}\frac{F_{d(a,x)}(t)}{q^{d(a,x)+1}}(\Delta H_{aa_1}(\sigma^+,\sigma^-)\Delta J_{xx_1}(\sigma^+,\sigma^-)\\ & \qquad + \Delta H_{xx_1}(\sigma^+,\sigma^-)\Delta J_{aa_1}(\sigma^+,\sigma^-)).\end{aligned}$$

For H, J in $\mathcal{H}_{k,-}$, we get

$$\Phi'_t(H^{>^\infty}, J^{>^\infty})(\sigma) = \frac{q+1}{4q}\kappa_{0,k}(H,K)(\sigma_{-\ell},\sigma_\ell) - \frac{q+1}{4q}\sum_{j=0}^{\infty} q^{-j}F_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell}).$$

Again, the conclusion follows from Lemma 36.2. □

PROOF OF PROPOSITION 36.3 IN CASE k IS ODD. The odd case will be dealt with in an analogue way. Given H, J in $\mathcal{H}_\infty$, we define a new Hölder continuous function that is cohomologous to $\Phi_t(H, J)$. For σ in $\mathscr{S}$, we set

(36.2)

$$\begin{aligned}\Phi''_t(H,J)(\sigma) = \frac{1}{2}\sum_{ab,xy\in X_1^{\sigma,\sharp}} \chi_t(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\ + \frac{1}{2}\sum_{h\geq 1}\sum_{\substack{ab\in X_1^{\sigma,\sharp}\\ xy\in X_1^{T^h\sigma,\sharp}}} \chi_t(ab,xy)(\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\ + \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).\end{aligned}$$

By (36.1), $\Phi''_t(H, J)$ and $\Phi'_t(H, J)$ are cohomologous.

Assume $SH = H$ and $SJ = J$. In view of Lemma 36.4, we have

$$\begin{aligned}\Phi''_t(H,J)(\sigma) = \frac{1}{2}\sum_{ab\in X_1^{\sigma,\sharp}} \Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)\\ + \frac{1}{2}\sum_{\substack{ab,xy\in X_1^{\sigma,\sharp}\\ \delta(ab,xy)\geq 1\\ b,y\in[ac]}} \frac{E_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}}\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\ + \frac{1}{2}\sum_{h\geq 1}\sum_{\substack{ab\in X_1^{\sigma,\sharp}\\ xy\in X_1^{T^h\sigma,\sharp}\\ b,y\in[ac]}} \frac{E_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}}(\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\ + \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).\end{aligned}$$

Recall that $k \geq 1$ is an odd integer, $k = 2\ell + 1$, $\ell \geq 0$. For H, J in $\mathcal{H}_{k,+}$, we get

$$\Phi''_t(H^{>\infty}, J^{>\infty})(\sigma) = \sum_{j\geq 0} q^{-j}E_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}).$$

Lemma 36.2 yields the first case of the Proposition.

For the second case, given H, J in $\mathcal{H}_\infty$ with $SH = -H$ and $SJ = -J$, we use Lemma 36.4 and (36.2) to write, for σ in $\mathscr{S}$,

$$\begin{aligned}\Phi_t''(H,J)(\sigma) = \frac{1}{2}\sum_{ab\in X_1^{\sigma,\sharp}} \Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)\\ - \frac{1}{2}\sum_{\substack{ab,xy\in X_1^{\sigma,\sharp}\\ \delta(ab,xy)\geq 1\\ b,y\in[ac]}} \frac{D_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}}\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\ - \frac{1}{2}\sum_{h\geq 1}\sum_{\substack{ab\in X_1^{\sigma,\sharp}\\ xy\in X_1^{T^h\sigma,\sharp}\\ b,y\in[ac]}} \frac{D_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}}(\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\\ + \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).\end{aligned}$$

For H, J in $\mathcal{H}_{k,-}$, we obtain

$$\begin{aligned}\Phi_t''(H^{>\infty},J^{>\infty})(\sigma) = \kappa_{0,k}(H,J)(\sigma_{-\ell}\sigma_{\ell+1})\\ - \sum_{j\geq 1} q^{-j}D_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}).\end{aligned}$$

Still by Lemma 36.2, the last case of the Proposition follows. □

36.4. Endpoints series formula for the special spectral fundamental maps

Recall that the special spectral fundamental bilinear maps $\Phi^{\mathrm{sp}}_{(-1)}$ and Φ^{sp}_1 were introduced in Definition 35.13. In the case of the special spectral fundamental maps, following the same strategy as above yields

Proposition 36.5. *Let $k \geq 1$.*

Assume k is even. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous functions $\Phi^{\mathrm{sp}}_{(-1)}\left(H^{>\infty}, J^{>\infty}\right)$ and $\Phi^{\mathrm{sp}}_1\left(H^{>\infty}, J^{>\infty}\right)$ are coboundaries. For any H, J in $\mathcal{H}_{k,-}$, the Hölder continuous functions $\Phi^{\mathrm{sp}}_{(-1)}\left(H^{>\infty}, J^{>\infty}\right)$ and $\Phi^{\mathrm{sp}}_1\left(H^{>\infty}, J^{>\infty}\right)$ are cohomologous to the Hölder continuous functions $\Phi^{\mathrm{sp},-}_{(-1),k}(H,J)$ and $\Phi^{\mathrm{sp},-}_{1,k}(H,J)$ defined by,

for σ in $\mathscr{S}$,

$$\Phi^{\mathrm{sp},-}_{(-1),k}(H,J)(\sigma) = \frac{q+1}{2q}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) - \frac{(q+1)^2}{2q}\sum_{j=1}^{k-2}(-q)^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+\frac{(q+1)^2}{2q}\sum_{j=k-1}^{\infty}(-q)^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j))$$

and

$$\Phi^{\mathrm{sp},-}_{1,k}(H,J)(\sigma) = \frac{q+1}{2q}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) - \frac{(q+1)^2}{2q}\sum_{j=1}^{k-2}q^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+\frac{(q+1)^2}{2q}\sum_{j=k-1}^{\infty}q^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

Assume k is odd. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous function $\Phi^{\mathrm{sp}}_1\left(H^{>^\infty}, J^{>^\infty}\right)$ is a coboundary; the Hölder continuous function $\Phi^{\mathrm{sp}}_{(-1)}\left(H^{>^\infty}, J^{>^\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^{\mathrm{sp},+}_{(-1),k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^{\mathrm{sp},+}_{(-1),k}(H,J)(\sigma) = 2\kappa_{0,k}(H,J)(\sigma_0,\sigma_k)+4\sum_{j=1}^{k-2}(-q)^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$-4\sum_{j=k-1}^{\infty}(-q)^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

For any H, J in $\mathcal{H}_{k,-}$, the Hölder continuous function $\Phi^{\mathrm{sp}}_{(-1)}\left(H^{>^\infty}, J^{>^\infty}\right)$ is a coboundary; the Hölder continuous function $\Phi^{\mathrm{sp}}_1\left(H^{>^\infty}, J^{>^\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^{\mathrm{sp},-}_{1,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^{\mathrm{sp},-}_{1,k}(H,J)(\sigma) = 2\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) - 4\sum_{j=1}^{k-2}q^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+4\sum_{j=k-1}^{\infty}q^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

The proof is analogous to that of Proposition 36.5, by replacing Lemma 36.4 with

LEMMA 36.6. *For a, x in X and $j = d(a, x)$, we have*

$$\sum_{\substack{b\sim a\\ y\sim x}} \chi_1^{\mathrm{sp}}(ab, xy) = 0 = \sum_{\substack{b\sim a\\ y\sim x}} \chi_{(-1)}^{\mathrm{sp}}(ab, xy).$$

For a in X anb $b, c \sim a$, $b \neq c$, we have

$$\chi_1^{\mathrm{sp}}(ab, ab) - \chi_1^{\mathrm{sp}}(ab, ac) = \chi_{(-1)}^{\mathrm{sp}}(ab, ab) - \chi_{(-1)}^{\mathrm{sp}}(ab, ac) = \frac{q+1}{q}.$$

For a, x in X with $j = d(a, x) \geq 1$ and $b \sim a$, $y \sim x$ with $b, y \notin [ax]$, we have

$$\chi_1^{\mathrm{sp}}(ab, xy) - \chi_1^{\mathrm{sp}}(aa_1, xy) - \chi_1^{\mathrm{sp}}(ab, xx_1) + \chi_1^{\mathrm{sp}}(aa_1, xx_1) = -\frac{(q+1)^2}{q^{j+1}}$$

$$\chi_{(-1)}^{\mathrm{sp}}(ab, xy) - \chi_{(-1)}^{\mathrm{sp}}(aa_1, xy) - \chi_{(-1)}^{\mathrm{sp}}(ab, xx_1) + \chi_{(-1)}^{\mathrm{sp}}(aa_1, xx_1) = \frac{(q+1)^2}{(-q)^{j+1}}.$$

For ab, xy in X_1, we have

$$\chi_1^{\mathrm{sp}}(ab, yx) = -\chi_1^{\mathrm{sp}}(ab, xy)$$
$$\chi_{(-1)}^{\mathrm{sp}}(ab, yx) = \chi_1^{\mathrm{sp}}(ab, xy).$$

CHAPTER 37

The weight of pseudofunctions

In the sequel, we will study the consequences of Proposition 33.3, which describes under which conditions an endpoints series is a coboundary, when applied to the endpoints series which appear in Proposition 36.3 and Proposition 36.5. To this aim, we will need a better understanding of the weight map of pseudofunctions.

37.1. Weight and natural operations

First, we relate the weight construction to the natural operations on pseudofunctions of Section 20.2.

LEMMA 37.1. *Let $k \geq 0$, H be a k-pseudofunction, ab be in X_{k+1} and a_1 and b_1 be the neighbours of a and b on $[ab]$. If k is even, we have*

$$\omega_{k+1}(H^{>})(ab) = \omega_k(H)(ab_1) \text{ and } \omega_{k+1}(H^{>\vee})(ab) = \omega_k(H^{\vee})(a_1 b).$$

If k is odd, we have

$$\omega_{k+1}(H^{>})(ab) = \omega_k(H)(a_1 b) \text{ and } \omega_{k+1}(H^{>\vee})(ab) = \omega_k(H^{\vee})(ab_1).$$

The weight map ω_k was introduced in Definition 36.1.

PROOF. If $k = 0$, and H is the 0-pseudofunction associated to the function u on X (see Section 20.1), we have

$$\omega_1(H^{>})(ab) = \Delta H^{>}_{ab}(b, a) = u(a)(\mathbf{1}_b(b) - \mathbf{1}_b(a)) = u(a) = \omega_0(H)(a)$$

and

$$\begin{aligned}\omega_1(H^{>\vee})(ab) = \Delta H^{>\vee}_{ab}(b, a) &= \Delta H^{>}_{ba}(b, a) \\ &= u(b)(\mathbf{1}_a(b) - \mathbf{1}_a(a)) = -u(b) = \omega_0(H^{\vee})(b).\end{aligned}$$

Assume now $k \geq 1$ and let us write as usual $a_0 = a, a_1, \ldots, a_{k+1} = b$ for the geodesic parametrization on the segment $[ab]$.

If k is even, $k = 2\ell$, $\ell \geq 1$, we have

$$\begin{aligned}\omega_{k+1}(H^{>})(ab) &= \Delta H^{>}_{a_\ell a_{\ell+1}}(b,a) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}} \Delta H^{>}_{cc_-}(b,a_{2i-1}) \\ &= \Delta H_{a_\ell a_{\ell+1}}(b_1,a) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}} \Delta H_{cc_-}(b_1,a_{2i}).\end{aligned}$$

As $b_1 = a_{2\ell}$, the last term of the sum vanishes and, as required, we get $\omega_{k+1}(H^{>})(ab) = \omega_k(H)(a)$. Besides,

$$\begin{aligned}\omega_{k+1}(H^{>\vee})(ab) &= \Delta H^{>}_{a_{\ell+1} a_\ell}(b,a) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}} \Delta H^{>}_{c_- c}(b,a_{2i-1}) \\ &= \Delta H_{a_{\ell+1} a_\ell}(b,a_1) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}} \Delta H_{c_- c}(b,a_{2i-1}).\end{aligned}$$

Now, on one hand,

$$\Delta H_{a_{\ell+1}a_\ell}(b,a_1) + \sum_{\substack{c\sim a_{\ell+1}\\ c\notin\{a_\ell,a_{\ell+2}\}}} \Delta H_{a_{\ell+1}c}(b,a_1) = \Delta H^{\vee}_{a_{\ell+1}a_{\ell+2}}(b,a_1),$$

whereas, on the other hand, for $2 \leq i \leq \ell$, by setting $d = c_-$ in the sums below, we get

$$\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}} \Delta H_{c_- c}(b,a_{2i-1}) = \sum_{\substack{d\in X\\ [a_{\ell+i}d]\cap[ab]=\{a_{\ell+i}\}\\ d(d,a_{\ell+i})=i-1}} \Delta H^{\vee}_{dd_-}(b,a_{2i-1}).$$

We obtain indeed $\omega_{k+1}(H^{>\vee})(ab) = \omega_k(H^{\vee})(a_1 b)$.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, we have

$$\begin{aligned}\omega_{k+1}(H^{>})(ab) &= \Delta H^{>}_{a_{\ell+1}a_{\ell+2}}(b,a) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}} \Delta H^{>}_{cc_-}(b,a_{2i}) \\ &= \Delta H_{a_{\ell+1}a_{\ell+2}}(b,a_1) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}} \Delta H_{cc_-}(b,a_{2i}) \\ &= \omega_k(H)(a_1 b)\end{aligned}$$

and in the same way,

$$\omega_{k+1}(H^{>\vee})(ab) = \Delta H^{>\vee}_{a_{\ell+1}a_{\ell+2}}(b,a) + \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}} \Delta H^{>\vee}_{cc_-}(b,a_{2i}).$$

First, we have

$$\Delta H^{>\vee}_{a_{\ell+1}a_{\ell+2}}(b,a) = \Delta H_{a_{\ell+1}a_\ell}(b_1,a) + \sum_{\substack{c\sim a_{\ell+1}\\ c\notin\{a_\ell,a_{\ell+2}\}}} \Delta H_{a_{\ell+1}c}(b_1,a_1);$$

second, for $1 \leq i \leq \ell$, we have

$$\sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}} \Delta H^{>\vee}_{cc_-}(b,a_{2i}) = \sum_{\substack{d\in X\\ [a_{\ell+i+1}d]\cap[ab]=\{a_{\ell+i+1}\}\\ d(d,a_{\ell+i+1})=i+1}} \Delta H_{d_-d}(b_1,a_{2i+1});$$

in particular, for $i = \ell$, this vanishes as $b_1 = a_k = a_{2\ell+1}$. We get $\omega_{k+1}(H^{>\vee})(ab) = \omega_k(H^{\vee})(ab_1)$. □

37.2. Injectivity properties of the weight

We will show that pseudofunctions are determined by their weight. We start by stating a converse to Lemma 37.1.

LEMMA 37.2. *Let $k \geq 1$ and H be a k-pseudofunction. Assume there exists a function v on X_{k-1} such that, for any ab in X_k, one has $\omega_k(H)(ab) = v(a_1 b)$. Then, if k is even, there exists a $(k-1)$-pseudofunction G with $H = G^{>}$ and $v = \omega_{k-1}(G)$. If k is odd, there exists a $(k-1)$-pseudofunction G with $H = G^{>\vee}$ and $v = \omega_{k-1}(G^{\vee})$*

PROOF. If $k = 1$, for $a \sim b$ in X_1, we have $\omega_1(H)(ab) = \Delta H_{ab}(b,a) = v(b)$, which means that $H_{ab} = v(b)\mathbf{1}_b = -v(b)\mathbf{1}_a$ in

$\overline{V}^0(ab)$. We get $H = G^{>\vee}$ where G is the 0-pseudofunction associated with $-v$ and hence $v = -\omega_0(G) = \omega_0(G^\vee)$ (see Section 20.1 and Section 20.2).

Suppose k is even, $k = 2\ell$, $\ell \geq 1$. Then, saying that there exists a $(k-1)$-pseudofunction G with $H = G^>$ is saying that, for every $x \sim y$ in X and every a, a' in $S^\ell(x)$ such that $d(a, a') = 2$ and $y \notin [ax] \cup [a'x]$, we have $\Delta H_{xy}(a, a') = 0$. Indeed, assume the latter holds. Then, pick b in $S^\ell(x)$ with $y \in [bx]$. By assumption, we have $d(a, b) = d(a', b) = k$. Let a_1 be the neighbour of a on $[ax]$, which is also the neighbour of a' on $[a'x]$ since $d(a, a') = 2$, and write $a_1, a_2, \ldots, a_k = b$ for the geodesic parametrization of the segment $[a_1 b]$. Then, by Definition 36.1 and the asusmption, we have

$$\Delta H_{xy}(b, a) = v(a_1 b) - \sum_{i=1}^{\ell-1} \sum_{\substack{c \in X \\ [a_{\ell+i}c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} \Delta H_{cc_-}(b, a_{2i}) = \Delta H_{xy}(b, a').$$

Hence $\Delta H_{xy}(a, a') = \Delta H_{xy}(b, a') - \Delta H_{xy}(b, a) = 0$ and therefore, there exists a $(k-1)$-pseudofunction G with $H = G^>$. We have $v = \omega_{k-1}(G)$ by Lemma 37.1.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we proceed in the same way. We first show that there exists a $(k-1)$-pseudofunction G with $H = G^{>\vee}$. Indeed this amounts to saying that, for every $x \sim y$ in X and every a, a' in $S^\ell(xy)$ with $d(a, a') = 2$ and $x \in [ay]$, one has $\Delta H_{xy}(a, a') = 0$, which is warranted by Definition 36.1 and the assumption. Then, Lemma 37.1 ensures that $v = \omega_{k-1}(G^\vee)$. □

From this, we can deduce that the weight determines the pseudofunction.

Corollary 37.3. *Let $k \geq 0$ and H be a k-pseudofunction. If $\omega_k(H) = 0$, then $H = 0$.*

In general, for $k \geq 3$, the weight map does not map $\mathcal{H}_k$ onto the space of Γ-invariant functions on X_k.

Proof. For $k = 0, 1$, the proof is immediate. The general case follows by Lemma 37.2 and a straightforward induction argument. □

37.3. Weights and cohomology

Now, we describe under which condition a weight function is a coboundary.

PROPOSITION 37.4. *Let $k \geq 1$ and H be in $\mathcal{H}_k$. Then the following are equivalent:*
(i) *The weight function $\omega_k(H)$ is a coboundary.*
(ii) *There exists G in $\mathcal{H}_{k-1}$ such that $H = G^{\vee>} - G^{>\vee}$.*

This statement is closely related to Theorem 9.1. The equivalence of the different notions of cohomology among elements of $\mathcal{V}_k$ is established in Corollary 32.7.

Formally, Proposition 37.4 will not be used later. Nevertheless, its statement and its proof serve as a model for those of Proposition 37.12 and Proposition 39.3 below. The latter result will play a crucial role in applying the cohomology criterion of Proposition 33.3 to endpoints series as in Proposition 36.3 or Proposition 36.5.

FIRST PART OF THE PROOF. The direction *(ii)⇒(i)* is easy. Indeed, assume $H = G^{\vee>} - G^{>\vee}$ for some G in $\mathcal{H}_{k-1}$. Then, by Lemma 37.1, for ab in X_k we have,

$$\begin{aligned} \omega_k(H)(ab) &= \omega_{k-1}(G^\vee)(a_1 b) - \omega_{k-1}(G^\vee)(ab_1) \qquad \text{if } k \text{ is even} \\ &= \omega_{k-1}(G^\vee)(ab_1) - \omega_{k-1}(G^\vee)(a_1 b) \qquad \text{if } k \text{ is odd.} \end{aligned}$$

The conclusion follows. □

Here is the difficulty for proving the converse statement. Assuming that *(i)* holds, we know from Corollary 32.7 that there exists a function v on X_{k-1} such that, for ab in X_k, one has $\omega_k(H)(ab) = v(ab_1) - v(a_1 b)$. But we don't know whether v is the weight function of some $(k-1)$-pseudofunction. It turns out that this is the case, but this requires some work to be proved.

We will need to introduce a new object. For $k \geq 1$, we define a complete k-pseudofunction as a family $(H_{xy})_{xy \in X_1}$ such that, for xy in X_1, ik k is even, $k = 2\ell$, $\ell \geq 1$, H_{xy} is an element of $V^\ell(x)$; if k is odd, $k = 2\ell + 1$, $\ell \geq 0$, H_{xy} is an element of $V^\ell(xy)$. Thus, the definition is the same as the one of a k-pseudofunction, except that we don't kill the constant functions in the spaces $V^\ell(x)$ and $V^\ell(xy)$.

Starting from a complete k-pseudofunction, one can obtain a k-pseudofunction by killing the constant part. In particular, if $\widehat{\mathcal{H}}_k$ is the space of Γ-invariant complete k-pseudofunctions, we have a natural map $\widehat{\mathcal{H}}_k \to \mathcal{H}_k$.

LEMMA 37.5. *Let $k \geq 1$. The natural map $\widehat{\mathcal{H}}_k \to \mathcal{H}_k$ is surjective.*

Since the stabilizers of the elements of X in Γ are finite, the proof is a direct consequence of the following classical phenomenon in group theory:

LEMMA 37.6. *Let V be a real vector space, equipped with an action of a finite group G. Let W be a G-invariant subspace. Then the natural map $V \to V/W$ maps the space V^G of G-invariant elements of V onto $(V/W)^G$.*

The advantage of dealing with complete pseudofunctions is that for them, we can replace the weight by a function that is defined on smaller segments. Indeed, for $k \geq 1$ and H a complete k-pseudofunction, we define the pseudoweight $\rho_k(H)$ of H as follows. If k is even, $k = 2\ell$, $\ell \geq 1$, for ab in $X_{\ell+1}$, we set

$$\rho_k(H)(ab) = \sum_{\substack{c \in X \\ d(b,c)=\ell \\ b_1 \in [bc]}} H_{cc_-}(b) - \sum_{i=1}^{\ell-1} \sum_{\substack{c \in X \\ d(b_1,c)=i \\ b,b_2 \notin [b_1 c]}} H_{cc_-}(a_i) - H_{b_1 b}(a). \tag{37.1}$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for ab in $X_{\ell+1}$, we set

$$\rho_k(H)(ab) = \sum_{\substack{c \in X \\ d(b,c)=\ell+1 \\ b_1 \in [bc]}} H_{cc_-}(b) - \sum_{i=1}^{\ell} \sum_{\substack{c \in X \\ d(b_1,c)=i \\ b,b_2 \notin [b_1 c]}} H_{cc_-}(a_{i-1}) - H_{b_1 b}(a). \tag{37.2}$$

From Definition 36.1, we directly get

LEMMA 37.7. *Let $k \geq 1$ and H be in $\widetilde{H}_k$. The pseudoweight $\rho_k(H)$ is cohomologous to the weight $\omega_k(H)$.*

By abuse of language, we have denoted by $\omega_k(H)$ the weight of the image of H in $\mathcal{H}_k$.

We define the natural operations $H \mapsto H^>$ and $H \mapsto H^\vee$ for complete pseudofunctions as for pseudofunctions (see Section 20.2).

As in Section 9.4, for $k \geq 1$, we say that a Γ-invariant function w on X_k is split if we can find Γ-invariant functions u and v on X_{k-1} such that, for xy in X_k, one has

$$w(xy) = u(xy_1) + v(x_1 y).$$

We have a criterion for a complete k-pseudofunction to be obtained through complete $(k-1)$-pseudofunctions.

LEMMA 37.8. *Let $k \geq 2$ and H be in $\widehat{\mathcal{H}}_k$.*

If k is odd, there exists F, G in $\widehat{\mathcal{H}}_{k-1}$ with $H = F^> + G^{>\vee}$.

If k is even, $k = 2\ell$, $\ell \geq 1$, then the following are equivalent:

(i) *there exists F, G in $\widehat{\mathcal{H}}_{k-1}$ with $H = F^> + G^{>\vee}$.*

(ii) *the function $xy \mapsto H_{y_1 y}(x)$ is split on $X_{\ell+1}$.*

(iii) *the pseudoweight $\rho_k(H)$ is split on $X_{\ell+1}$.*

PROOF. Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$, and recall from Proposition 4.6 that, for xy in X_1, we have $V^\ell(xy) = J^\ell_{xy}V^\ell(x) + J^\ell_{yx}V^\ell(y)$. Choose a system of representatives $S \subset X_1$. for the action of Γ on X_1. Then, for xy in S, H_{xy} is a $(\Gamma_x \cap \Gamma_y)$-invariant element of $V^\ell(xy)$. By Lemma 37.6, there exists $(\Gamma_x \cap \Gamma_y)$-invariant elements f_{xy} in $V^\ell(x)$ and g_{yx} in $V^\ell(y)$ with

$$H_{xy} = J^\ell_{xy} f_{xy} + J^\ell_{yx} g_{yx}.$$

As f_{xy} and g_{yx} are $\Gamma_x \cap \Gamma_y$-invariant, there exists unique F, G in $\widehat{\mathcal{H}}_{k-1}$ such that $F_{xy} = f_{xy}$ and $G_{yx} = g_{yx}$ for xy in S. By construction, we have $H = F^> + G^{>\vee}$.

Assume now k is even, $k = 2\ell$, $\ell \geq 1$ and first note that, in view of the definition of the pseudoweight $\rho_k(H)$ in (37.1), the function $xy \mapsto \rho_k(H)(xy) + H_{y_1y}(x)$ is split on $X_{\ell+1}$, so that *(ii)* is equivalent to *(iii)*.

Let us prove *(i)⇒(ii)*. Suppose we may write $H = F^> + G^{>\vee}$ with F, G in $\widehat{\mathcal{H}}_{k-1}$. Then, a direct computation gives, for xy in $X_{\ell+1}$,

$$\begin{aligned} H_{y_1y}(x) &= F^>_{y_1y}(x) + \sum_{\substack{z\sim y_1 \\ z\neq y}} G^>_{y_1z}(x) \\ &= F_{y_1y}(x_1) + \sum_{\substack{z\sim y_1 \\ z\notin\{y,y_2\}}} G_{y_1z}(x_1) + G_{y_1y_2}(x) \\ &= F_{y_1y}(x_1) + \sum_{\substack{z\sim y_1 \\ z\neq y_2}} G_{y_1z}(x_1) - G_{y_1y}(x_1) + G_{y_1y_2}(x) \end{aligned}$$

(the last step only being necessary when $k = 2$). As required, the function $xy \mapsto H_{y_1y}(x)$ is split.

Conversely, we now prove *(ii)⇒(i)*. Thus, assume we may find Γ-invariant functions v and w on X_ℓ such that, for xy in $X_{\ell+1}$, one has

$$H_{y_1y}(x) = v(xy_1) + w(x_1y).$$

We define F and G in $\widehat{\mathcal{H}}_{k-1}$ as follows: for xy in X_ℓ, we set

$$\begin{aligned} F_{yy_1}(x) &= H_{yy_1}(x) & F_{y_1y}(x) &= w(xy) \\ G_{yy_1}(x) &= v(xy) & G_{y_1y}(x) &= 0. \end{aligned}$$

A direct computation then shows that $H = F^> + G^{>\vee}$. □

SECOND PART OF THE PROOF OF PROPOSITION 37.4. We prove *(i)⇒(ii)* by induction on $k \geq 1$. Let H be in $\mathcal{H}_k$ and assume that $\omega_k(H)$ is a coboundary. By Corollary 32.7, we may find a Γ-invariant function v on X_{k-1} such that, for any xy in X_k, one has

$$\omega_k(H)(xy) = v(xy_1) - v(x_1y).$$

If $k = 1$, we let G be the 0-pseudofunction associated with the function $-v$ on $X_0 = X$ (see Section 20.1). Then, Definition 36.1 and Lemma 37.1 imply that we have $\omega_1(H - G^{\vee>} + G^{>\vee}) = 0$, hence $H = G^{\vee>} - G^{>\vee}$ by Corollary 37.3.

If $k = 2$, we let G be the 1-pseudofunction associated with the function $xy \mapsto v(yx)$ on X_1 (again, see Section 20.1). As above, we get $H = G^{\vee>} - G^{>\vee}$.

Suppose $k \geq 3$ and the result holds for $k-1$; let us show that it also holds for k.

If k is odd, by Lemma 37.5 and Lemma 37.8, we may find J, K in $\mathcal{H}_{k-1}$ with

$$H = J^{>} + K^{>\vee} = (J + K^{\vee})^{>} - K^{\vee>} + K^{>\vee} = L^{>} - K^{\vee>} + K^{>\vee}, \tag{37.3}$$

where $L = J + K^{\vee}$. Then the first part of the proof and the assumption ensure that the weight $\omega_k(L^{>})$ is a coboundary. By Lemma 37.1, the weight $\omega_{k-1}(L)$ is a coboundary. Thus, by the induction assumption, we may find M in $\mathcal{H}_{k-2}$ with $L = M^{\vee>} - M^{>\vee}$. By (37.3), we get

$$\begin{aligned} H = L^{>} - K^{\vee>} + K^{>\vee} &= M^{\vee>>} - M^{>\vee>} - K^{\vee>} + K^{>\vee} \\ &= (M^{>} + K)^{>\vee} - (M^{>} + K)^{\vee>} \end{aligned}$$

(where we have used Lemma 20.6). The conclusion follows.

If k is even, $k = 2\ell$, $\ell \geq 1$, we will proceed in the same way, by showing that the assumption of Lemma 37.8 is satisfied. Indeed, by Lemma 37.5, we may assume that H is a Γ-invariant complete k-pseudofunction (which we still denote by H by abuse of language). Then, by the assumption and Lemma 37.7, the pseudoweight $\rho_k(H)$ is a coboundary. By Corollary 32.7, this means that we may find a Γ-invariant function v on X_ℓ such that, for any ab in $X_{\ell+1}$, one has

$$\rho_k(H)(ab) = v(ab_1) - v(a_1 b).$$

In particular, $\rho_k(H)$ is split and, by Lemma 37.8, there exists J, K in $\mathcal{H}_{k-1}$ with $H = J^{>} + K^{>\vee}$. We conclude as in the odd case. □

37.4. Sequences of pseudofunctions

We will now prove a statement for sequences of weights that may be seen as a generalization of Proposition 37.4. The proof will rely on some improvements of the techniques used above. It will also serve as a model for the proof of Proposition 39.3 below, which will be a further generalization that will play a crucial role in translating the conclusion of Proposition 33.3 in the language of pseudofunctions.

We start with a definition that is inspired by the language of Proposition 33.3.

DEFINITION 37.9. Let $k \geq 1$. We say that a finitely supported sequence $(w_j)_{j\geq 1}$ in $\mathscr{V}_k$ is cohomologically trivial if there exists a finitely supported sequence $(v_j)_{j\geq 1}$ in $\mathscr{V}_{k-1}$ such that, for $j \geq 1$ and ab in X_k, one has

$$w_j(ab) = v_j(ab_1) - v_{j-1}(a_1 b).$$

This notion is invariant under some shifts.

LEMMA 37.10. *Let $h \geq k \geq 1$ and $(w_j)_{j\geq 1}$ be a finitely supported sequence in $\mathscr{V}_k$. For $j \geq 1$ and ab in X_h, set $w'_j(ab) = w_j(aa_k)$. Assume the sequence $(w'_j)_{j\geq 1}$ is cohomologically trivial in $\mathscr{V}_h$. Then, the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in $\mathscr{V}_k$.*

The proof will use the easy

LEMMA 37.11. *Let $k \geq 1$ and u, v be Γ-invariant functions on X_k. Assume that, for any ab in X_{k+1}, we have*

$$u(ab_1) = v(a_1 b).$$

Then, there exists a Γ-invariant function w on X_{k-1} such that, for every ab on X_k, one has

$$u(ab) = w(a_1 b) \text{ and } v(ab) = w(ab_1).$$

PROOF OF LEMMA 37.10. It suffices to prove the statement when $h = k+1$, the general case following by an easy induction. If $h = k+1$, we can find a finitely supported sequence $(v_j)_{j\geq 0}$ in $\mathscr{V}_k$ such that, for $j \geq 0$ and ab in X_{k+1}, we have

$$w_{j+1}(ab_1) - v_{j+1}(ab_1) = -v_j(a_1 b).$$

By Lemma 37.11, there exists a function u_j on X_{k-1} such that, for ab in X_{k-1}, we have

$$w_{j+1}(ab) - v_{j+1}(ab) = -u_j(a_1 b) \text{ and } v_j(ab) = u_j(ab_1).$$

Thus, for $j \geq 1$, we get

$$w_j(ab) = u_j(ab_1) - u_{j-1}(a_1 b)$$

as required. □

The objective of the remainder of the Chapter is to show

PROPOSITION 37.12. *Let $k \geq 1$ and $(H_j)_{j\geq 1}$ be a finitely supported sequence of elements of $\mathcal{H}_k$. Assume that the sequence $(\omega(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{V}_k$. Then, there exists a finitely supported sequence $(G_j)_{j\geq 0}$ of elements of $\mathcal{H}_{k-1}$ such that, for $j \geq 1$, one has*

$$\begin{aligned} H_j &= G_j^{>\vee} - G_{j-1}^{\vee>} && \text{if } k \text{ is even} \\ &= G_j^{\vee>} - G_{j-1}^{>\vee} && \text{if } k \text{ is odd.} \end{aligned}$$

Note that the converse is also true by Lemma 37.1.

37.5. Splitting sequences

The proof of Proposition 37.12 will follow the same lines as the one of Proposition 37.4. In particular, we will show that the assumption implies that we can apply the criterion of Lemma 37.8 to the pseudofunctions H_j, $j \geq 1$. This is achieved in the following

LEMMA 37.13. *Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence of elements of $\mathcal{H}_k$. Assume that the sequence $(\omega(H_j))_{j\geq 0}$ is cohomologically trivial in $\mathscr{V}_k$. Then, for every $j \geq 1$, there exists F_j and G_j in $\mathcal{H}_{k-1}$ with*

$$H_j = F_j^{>} + G_j^{>\vee}.$$

We introduce new notation for the proof. We keep the language of Section 37.3. For $k \geq 4$ an even integer, $k = 2\ell$, $\ell \geq 2$, and H a complete k-pseudofunction, we define a family of functions on segments of different sizes.

For ab in $X_{\ell+1}$, we set

$$\rho_k^0(H)(ab) = -H_{b_1 b}(a).$$

For $1 \leq i \leq \ell - 2$ and ab in $X_{\ell-i+1}$, we set

$$\rho_k^i(H)(ab) = - \sum_{\substack{c\in X \\ [b_1 c]\cap[ab]=\{b_1\} \\ d(c,b_1)=i}} H_{cc_-}(a).$$

And lastly, for $i = \ell - 1$ and ab in X_2, we set

$$\rho_k^{\ell-1}(H)(ab) = \sum_{\substack{c\in X \\ b_1\in[bc] \\ d(c,b)=\ell}} H_{cc_-}(b) - \sum_{\substack{c\in X \\ [b_1 c]\cap[ab]=\{b_1\} \\ d(c,b_1)=\ell-1}} H_{cc_-}(a).$$

Thus, Definition 36.1 can be rewritten as, for ab in X_k,

$$\omega_k(H)(ab) = \sum_{i=0}^{\ell-1} \rho_k^i(H)(a_{2i} a_{\ell+i+1}). \tag{37.4}$$

If $k = 2$, we set $\rho_2^0(H) = \omega_2(H)$, so that (37.4) still holds.

PROOF OF LEMMA 37.13. Note that, for $k = 2$, the statement directly follows from Definition 36.1, Lemma 37.5 and Lemma 37.8. Assume $k \geq 4$ and let $(v_j)_{j\geq 0}$ be a finitely supported sequence in $\mathscr{V}_{k-1}$ such that, for ab in X_k and $j \geq 1$, we have

$$\omega(H_j)(ab) = v_j(ab_1) - v_{j-1}(a_1 b).$$

By Lemma 37.5, we can assume that $(H_j)_{j\geq 1}$ is a finitely supported sequence of elements of $\widehat{\mathcal{H}}_k$. Then, by (37.4), we get

$$v_j(ab_1) - v_{j-1}(a_1b) = \sum_{i=0}^{\ell-1} \rho_k^i(H_j)(a_{2i}a_{\ell+i+1})$$

$$= \sum_{i=0}^{\ell-1} \rho_k^i(H_{i+j})(a_ia_{\ell+1}) - \sum_{i=1}^{\ell-1}(\rho_k^i(H_{i+j})(a_ia_{\ell+1}) - \rho_k^i(H_j)(a_{2i}a_{\ell+i+1}))$$

$$= \sum_{i=0}^{\ell-1} \rho_k^i(H_{i+j})(a_ia_{\ell+1}) - v'_j(ab_1) + v'_{j-1}(a_1b),$$

where, for $j \geq 0$ and ab in X_{k-1}, we have set

$$v'_j(ab) = \sum_{i=1}^{\ell-1}\sum_{h=0}^{i-1} \rho_k^i(H_{i+j-h})(a_{h+i}a_{h+\ell+1}).$$

By Lemma 37.10, there exists a finitely supported sequence $(v''_j)_{j\geq 0}$ of Γ-invariant functions on $X_{\ell+1}$ such that, for any $j \geq 1$ and ab in $X_{\ell+1}$, one has

$$\sum_{i=0}^{\ell-1} \rho_k^i(H_{i+j})(a_ia_{\ell+1}) = v''_j(ab_1) - v''_{j-1}(a_1b).$$

In particular, the function $\rho_k^0(H_j)$ is split on $X_{\ell+1}$ and the conclusion follows from Lemma 37.8. □

We can now conclude by using an induction argument.

Proof of Proposition 37.12. As for Proposition 37.4, we prove this statement by induction on $k \geq 1$.

Assume $k = 1$. For $j \geq 0$, we let G_j be the 0-pseudofunction associated with the function v_j on X_0 (see Section 20.1). Then, Definition 36.1, Lemma 37.1 and Corollary 37.3 imply that, for $j \geq 1$, we have $H_j = G_{j-1}^{>\vee} - G_j^{\vee>}$.

Assume $k = 2$. For $j \geq 0$, we let G_j be the 1-pseudofunction associated with the function $xy \mapsto v_j(yx)$ on X_1 (see again Section 20.1). Then, as above, for $j \geq 1$, we get $H_j = G_{j-1}^{\vee>} - G_j^{>\vee}$.

Suppose now $k \geq 3$ and the statement holds for $k-1$. If k is odd, we know from Lemma 37.5 and Lemma 37.8 that we may find sequences $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}$ such that, for $j \geq 1$, we have

$$H_j = J_j^{>} + K_j^{>\vee} = (J_j + K_{j+1}^{\vee})^{>} + K_j^{>\vee} - K_{j+1}^{\vee>}.$$

As $(H_j)_{j\geq 1}$ is finitely supported, we may assume that $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ also are.

For $j \geq 1$, we set $L_j = J_j + K_{j+1}^{\vee}$. Then, by Lemma 37.1, for ab in X_k, we have

$$\begin{aligned}\omega_{k-1}(L_j)(ab_1) &= \omega_k(L_j^{>})(ab)\\ &= \omega_k(H_j)(ab) - \omega_k(K_j^{>\vee})(ab) + \omega_k(K_{j+1}^{\vee>})(ab)\\ &= \omega_k(H_j)(ab) - \omega_{k-1}(K_j^{\vee})(a_1b) + \omega_{k-1}(K_{j+1}^{\vee})(ab_1).\end{aligned}$$

Therefore, by Lemma 37.10, the sequence $(\omega_{k-1}(L_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{V}_{k-1}$. By the induction assumption, there exists a finitely supported sequence $(M_j)_{j\geq 0}$ of elements of $\mathcal{H}_{k-2}$ such that, for $j \geq 1$, one has

$$L_j = M_j^{>\vee} - M_{j-1}^{\vee>}.$$

By using Lemma 20.6, we get

$$H_j = L_j^{>} + K_j^{>\vee} - K_{j+1}^{\vee>} = (M_j^{>} - K_{j+1})^{\vee>} - (M_{j-1}^{>} - K_j)^{>\vee}$$

and we are done.

Suppose now k is even. By Lemma 37.13, there exist sequences $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}$, which we may assume to be fintely supported, such that, for $j \geq 1$, we have

$$H_j = J_j^{>} + K_j^{>\vee} = (J_{j+1}^{\vee} + K_j)^{>\vee} + J_j^{>} - J_{j+1}^{\vee>\vee}.$$

We now set $L_j = J_{j+1} + K_j^{\vee}$, so that Lemma 37.1 gives, for ab in X_k,

$$\begin{aligned}\omega_{k-1}(L_j)(ab_1) &= \omega_k(L_j^{\vee>\vee})(ab)\\ &= \omega_k(H_j)(ab) - \omega_k(J_j^{>})(ab) + \omega_k(J_{j+1}^{\vee>\vee})(ab)\\ &= \omega_k(H_j)(ab) - \omega_{k-1}(J_j)(a_1b) + \omega_{k-1}(J_{j+1})(ab_1).\end{aligned}$$

By Lemma 37.10, the sequence $(\omega_{k-1}(L_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{V}_{k-1}$. By the induction assumption, there exists a finitely supported sequence $(M_j)_{j\geq 0}$ of elements of $\mathcal{H}_{k-2}$ such that, for $j \geq 1$, one has

$$L_j = M_j^{\vee>} - M_{j-1}^{>\vee}.$$

By using again Lemma 20.6, we obtain

$$H_j = L_j^{\vee>\vee} + J_j^{>} - J_{j+1}^{\vee>\vee} = (M_j^{\vee>\vee} - J_{j+1}^{\vee})^{>\vee} - (M_{j-1}^{\vee>\vee} - J_j^{\vee})^{\vee>}$$

as required. □

Part 11

Cohomological triviality for tensors

CHAPTER 38

Simplification schemes

Our objective in the next two Chapters is to show Proposition 39.3, which is a generalization of Proposition 37.12 for functions of two sets of variables. Later, it will be used to check the consequences of Proposition 33.3 when it is applied to the objects appearing in the Plancherel formula in Proposition 35.16.

In the present Chapter, as a preliminary, we study certain linear equations on functions on two variables. As analogous equations will also appear later on tensor products of spaces of pseudofunctions, we regroup those two studies in a common abstract formalism, which we call the language of simplication schemes.

38.1. Definition and examples

We now introduce precisely this language and we relate it to our two examples.

Definition 38.1. A simplification scheme is a family

$$(V_-, V, V_+, L, R, L_+, R_+)$$

where V_-, V and V_+ are real vector spaces and L and R are injective linear maps $V_- \to V$ and L_+ and R_+ are injective linear maps $V \to V_+$ with the following property: we have $L_+R = R_+L$ and, for every v and w in V, if

$$L_+v = R_+w,$$

there exists u in V with $v = Ru$ and $w = Lu$.

We set this definition in order to encompass the following two examples.

Example 38.2. For $k \geq 0$, we set $V_- = \mathcal{H}_{k-1}$, $V = \mathcal{H}_k$ and $V_+ = \mathcal{H}_{k+1}$. Then, for G in V_- and H in V, we set

$$LG = G^{>} \qquad RG = G^{\vee >}$$
$$L_+H = H^{>} \qquad R_+H = H^{>\vee}.$$

This defines a simplification scheme as follows from Lemma 20.6 and Proposition 20.8. The same construction works for pseudokernels instead of pseudofunctions, by Lemma 6.8 and Proposition 6.9.

EXAMPLE 38.3. For $k \geq 1$, we set $V_- = \mathscr{V}_{k-1}$, $V = \mathscr{V}_k$ and $V_+ = \mathscr{V}_{k+1}$. Then, for f in V_- and ab in X_k, we set

$$Lf(ab) = f(a_1 b) \text{ and } Rf(ab) = f(ab_1).$$

In the same way, for g in V_+ and ab in X_{k+1}, we set

$$L_+ g(ab) = g(a_1 b) \text{ and } R_+ g(ab) = g(ab_1).$$

This defines a simplification scheme by Lemma 37.11.

We extend this definition for $k = 0$ in the following way. The space $\mathscr{V}_0$ is the space of Γ-invariant functions on $X_0 = X$. We let $\mathscr{V}_{-1}$ be the space of Γ-invariant functions on X which are constant on neighbours: this space is the line of constant functions if Γ is not bipartite; it has dimension 2 else. If f is in $\mathscr{V}_{-1}$, we set Lf to be f, viewed as an element of $\mathscr{V}_0$, and Rf to be the opposite of f (see Section 20.1), that is, for $a \sim b$ in X, $Rf(a) = f(b)$. If g is in $\mathscr{V}_0$, for $a \sim b$ in X, we set

$$L_+ g(ab) = g(b) \text{ and } R_+ g(ab) = g(a).$$

Again, one easily checks that this defines a simplification scheme.

To avoid heavy notation, in the sequel, we will only write (V_-, V, V_+) to mean a simplification scheme and we will use the letters L and R to design the associated linear maps for every simplification scheme. We will also simply write L instead of L_+ and R instead of R_+.

38.2. Tensor products

In the rest of the Chapter, we will describe how the simplification rule of simplification schemes behaves in tensor products. Here, we start by introducing precisely our notation for tensor products.

Let V and W be vector spaces. We write $V \otimes W$ for the algebraic tensor product of V with W. If $V = W$, we write $\otimes^2 V$ for $V \otimes V$. If X is another vector space and $\varphi : V \times W \to X$ is a bilinear map, we still write $\varphi : V \otimes W \to X$ for the linear map such that $\varphi(v \otimes w) = \varphi(v, w)$, for v in V and w in W. If V' and W' are other vector spaces and $\chi : V \to V'$ and $\psi : W \to W'$ are linear maps, we denote by $u \mapsto \chi u \psi$ the natural associated linear map $V \otimes W \to V' \otimes W'$, so that, for v in V and w in W, one has $\chi(v \otimes w)\psi = (\chi v) \otimes (\psi w)$.

The following is standard:

LEMMA 38.4. *Let V, V' and W be vector spaces and $\varphi : V \to V'$ be a linear map. Then the linear map $u \mapsto \varphi u, V \otimes W \to V' \otimes W$ has kernel $(\ker \varphi) \otimes W$ and range $(\varphi V) \otimes W$.*

As in Chapter 33, for $h, k \geq 1$, we write $\mathscr{V}_k$ for the space of Γ-invariant functions on X_k and $\mathscr{W}_{h,k}$ for the space of $(\Gamma \times \Gamma)$-invariant functions on $X_h \times X_k$. We identify $\mathscr{W}_{h,k}$ with the tensor product $\mathscr{V}_h \times \mathscr{V}_k$ in the standard way. More precisely, for v in $\mathscr{V}_h$ and w in $\mathscr{V}_k$, we consider $v \otimes w$ as the function on $X_h \times X_k$ defined by

$$(v \otimes w)(ab, xy) = v(ab)w(xy), \quad ab, xy \in X_k.$$

38.3. Tensor products of simplification schemes

In the abstract framework of simplification schemes, we establish the following result that will allow us to solve functional equations with two variables.

PROPOSITION 38.5. *Let (V_-, V, V_+) and (W_-, W, W_+) be simplification schemes. Take g, h in $V \otimes W_+$ and j, k in $V_+ \otimes W$. Assume we have*

$$Lg + Rh + jL + kR = 0.$$

Then, there exist a in $V_- \otimes W_+$, b in $V_+ \otimes W_-$ and c, d, e, f in $V \otimes W$ such that

$$g = Ra + cL + dR \qquad h = -La + eL + fR$$
$$j = bR - Lc - Re \qquad k = -Lb - Ld - Rf.$$

We will split the proof into several steps.

LEMMA 38.6. *Let (V_-, V, V_+) and (W_-, W, W_+) be simplification schemes. Asssume f is in $V_- \otimes W_+$ and g, h are in $V \otimes W$ and we have*

$$Lf = gL + hR.$$

Then, there exist a in $V \otimes W_-$ and b, c in $V_- \otimes W$ with

$$f = bL + cR$$
$$g = aR + Lb$$
$$h = -aL + Lc.$$

PROOF. The relation $Lf = gL + hR$ implies that $gL + hR$ has trivial image in $(V/LV_-) \otimes W_+$. Therefore, by Definition 38.1 and Lemma 38.4, there exists a in $V \otimes W_-$ such that $g - aR$ and $h + aL$ both belong to $LV_- \otimes W$. In other words, we can find b, c in $V_- \otimes W$ with

$$g = aR + Lb \text{ and } h = -aL + Lc.$$

We get

$$Lf = gL + hR = LbL + LcR$$

and the conclusion follows. □

LEMMA 38.7. *Let (V_-, V, V_+) and (W_-, W, W_+) be simplification schemes and f, g, h, j be in $V \otimes W$. Assume we have*

$$LfL + RgL + LhR + RjR = 0.$$

Then, there exist a, d in $V_- \otimes W$ and b, c in $V \otimes W_-$ such that

$$\begin{aligned} f &= Ra + bR & g &= -La + cR \\ h &= Rd - bL & j &= -Ld - cL. \end{aligned}$$

PROOF. We write the starting equation as

$$(Lf + Rg)L + (Lh + Rj)R = 0.$$

By Definition 38.1 and Lemma 38.4, there exists k in $V_+ \otimes W_-$ such that

$$Lg + Rg = kR \text{ and } Lh + Rj = -kL.$$

By Lemma 38.6, we can find u, x in $V_- \otimes W$ and v, w, y, z in $V \otimes W_-$ with

$$\begin{aligned} k &= Lv + Rw & -k &= Ly + Rz \\ f &= Ru + vR & h &= Rx + yL \\ g &= -Lu + wR & j &= -Lx + zL. \end{aligned}$$

By looking at the two equations above which involve k, we get

$$Lv + Rw + Ly + Rz = 0.$$

By Definition 38.1 and Lemma 38.4, we know that there exists l in $V_- \otimes W_-$ with

$$v + y = Rl \text{ and } w + z = -Ll,$$

which yields

$$\begin{aligned} f &= Ru - yR + RlR & h &= Rx + yL \\ g &= -Lu - zR - LlR & j &= -Lx + zL. \end{aligned}$$

The result folllows with $a = u + lR$, $b = -y$, $c = -z$ and $d = x$. □

PROOF OF PROPOSITION 38.5. The assumption implies that $Lg + Rh$ has trivial image in $V_+ \otimes (W_+/(LW + RW))$. Therefore, by Definition 38.1 and Lemma 38.4, there exists l in $V_- \otimes W_+$ such that $g - Rl$ and $h + Ll$ both belong to $V \otimes (LW + RW)$. We choose g_0, g_1, h_0, h_1 in $V \otimes W$ with

$$g = Rl + g_0L + g_1R \text{ and } h = -Ll + h_0L + h_1R.$$

In the same way, we can find m in $V_+ \otimes W_-$ and j_0, j_1, k_0, k_1 in $V \otimes W$ satisfying

$$j = mR + Lj_0 + Rj_1 \text{ and } k = -mL + Lk_0 + Rk_1.$$

The assumption now reads as

$$Lg_0L + Lg_1R + Rh_0L + Rh_1R + Lj_0L + Rj_1L + Lk_0R + Rk_1R = 0,$$

which we rewrite as

$$L(g_0 + j_0)L + R(h_0 + j_1)L + L(g_1 + k_0)R + R(h_1 + k_1)R = 0.$$

We can therefore apply Lemma 38.7. This tells us that we may find u, x in $V_- \otimes W$ and v, w in $V \otimes W_-$ with

$$\begin{aligned} g_0 + j_0 &= Ru + vR & \qquad h_0 + j_1 &= -Lu + wR \\ g_1 + k_0 &= Rx - vL & \qquad h_1 + k_1 &= -Lx - wL. \end{aligned}$$

We get

$$\begin{aligned} j = mR + Lj_0 + Rj_1 &= mR - Lg_0 + LRu + LvR - Rh_0 - RLu + RwR \\ &= mR - Lg_0 + LvR - Rh_0 + RwR \end{aligned}$$

and in the same way,

$$\begin{aligned} k = -mL + Lk_0 + Rk_1 &= -mL - Lg_1 + LRx - LvL - Rh_1 - RLx - RwL \\ &= -mL - Lg_1 - LvL - Rh_1 - RwL. \end{aligned}$$

Since $g = Rl + g_0L + g_1R$ and $h = -Ll + h_0L + h_1R$, the result follows with

$$\begin{aligned} a &= l & \qquad c &= g_0 & \qquad d &= g_1 \\ b &= m + Lv + Rw & \qquad e &= h_0 & \qquad f &= h_1. \end{aligned}$$

□

38.4. The case of pseudofunctions

We now translate Proposition 38.5 for our concrete examples and we add the description of the boundary cases. In case of pseudofunctions, as in Example 38.2, we get

COROLLARY 38.8. *Let $h, k \geq -1$, G, H be in $\mathcal{H}_h \otimes \mathcal{H}_{k+1}$ and J, K be in $\mathcal{H}_{h+1} \otimes \mathcal{H}_k$. Assume we have*

$$^{>}G + {}^{\vee >}H + J^{>} + K^{>\vee} = 0.$$

Then, if h and k are both ≥ 0, there exist A in $\mathcal{H}_{h-1} \otimes \mathcal{H}_{k+1}$, B in $\mathcal{H}_{h+1} \otimes \mathcal{H}_{k-1}$ and C, D, E, F in $\mathcal{H}_h \otimes \mathcal{H}_k$ such that

$$G = {}^{>\vee}A + C^{>} + D^{>\vee} \qquad H = -{}^{>}A + E^{>} + F^{>\vee}$$
$$J = B^{\vee>} - {}^{>}C - {}^{\vee>}E \qquad K = -B^{>} - {}^{>}D - {}^{\vee>}F.$$

If $h \geq 0$ and $k = -1$, there exist A in $\mathcal{H}_{h-1} \otimes \mathcal{H}_0$ and B, C in $\mathcal{H}_h \otimes \mathcal{H}_{-1}$ with

$$K - J = {}^{>}B + {}^{\vee>}C$$
$$G = {}^{\vee>}A + B^{>}$$
$$H = -{}^{>}A + C^{>}.$$

If $h = k = -1$, there exists A in $\mathcal{H}_{-1} \otimes \mathcal{H}_{-1}$ with

$$G - H = A^{>} \text{ and } K - J = {}^{>}A.$$

PROOF. In case $h, k \geq 0$, this is Proposition 38.5. In case $h \geq 0$ and $k = -1$, the equation reads as

$${}^{>}G + {}^{\vee>}H = (K - J)^{>}$$

and the conclusion follows from Lemma 38.6. Finally, if $h = k = -1$, we have

$${}^{>}(G - H) = (K - J)^{>}$$

and the conclusion is obvious. □

We also include the translation of Lemma 38.6 which we shall need later.

COROLLARY 38.9. *Let $h, k \geq 0$. Asssume F is in $\mathcal{H}_{h-1} \otimes \mathcal{H}_{k+1}$ and G, H are in $\mathcal{H}_h \otimes \mathcal{H}_k$ and we have*

$${}^{>}F = G^{>} + H^{>\vee}.$$

Then, there exist A in $\mathcal{H}_h \otimes \mathcal{H}_{k-1}$ and B, C in $\mathcal{H}_{h-1} \otimes \mathcal{H}_k$ with

$$F = B^{>} + C^{>\vee}$$
$$G = A^{\vee>} + {}^{>}B$$
$$H = -A^{>} + {}^{>}C.$$

38.5. The case of functions on segments

In the case of Example 38.3, we get

COROLLARY 38.10. *Let $h, k \geq 0$, g, h be in $\mathscr{W}_{h,k+1}$ and j, k be in $\mathscr{W}_{h+1,k}$. Assume that, for pq in X_{h+1} and xy in X_{k+1}, we have*

$$g(p_1 q, xy) + h(pq_1, xy) + j(pq, x_1 y) + k(pq, xy_1) = 0.$$

Then, if h and k are both ≥ 1, there exist a in $\mathscr{W}_{h-1,k+1}$, b in $\mathscr{W}_{h+1,k-1}$ and c, d, e, f in $\mathscr{W}_{h,k}$ such that, for pq in X_h and xy in X_{k+1},

$$g(pq, xy) = a(pq_1, xy) + c(pq, x_1y) + d(pq, xy_1)$$
$$h(pq, xy) = -a(p_1q, xy) + e(pq, x_1y) + f(pq, xy_1)$$

and, for pq in X_{h+1} and xy in X_k,

$$j(pq, xy) = b(pq, xy_1) - c(p_1q, xy) - e(pq_1, xy)$$
$$k(pq, xy) = -b(pq, x_1y) - d(p_1q, xy) - f(pq_1, xy).$$

If $h \geq 1$ and $k = 0$, there exist a in $\mathscr{W}_{h-1,1}$, b in $\mathscr{W}_{h+1,-1}$ and c, d, e, f in $\mathscr{W}_{h,0}$ such that, for pq in X_h and X_1,

$$g(pq, xy) = a(pq_1, xy) + c(pq, y) + d(pq, x)$$
$$h(pq, xy) = -a(p_1q, xy) + e(pq, y) + f(pq, x)$$

and, for pq in X_{h+1} and xy in X_1,

$$j(pq, x) = b(pq, y) - c(p_1q, x) - e(pq_1, x)$$
$$k(pq, x) = -b(pq, x) - d(p_1q, x) - f(pq_1, x).$$

If $h = k = 0$, there exist a in $\mathscr{W}_{-1,1}$, b in $\mathscr{W}_{1,-1}$ and c, d, e, f in $\mathscr{W}_{0,0}$ such that, for pq, xy in X_1,

$$g(p, xy) = a(q, xy) + c(p, y) + d(p, x)$$
$$h(pq, xy) = -a(p, xy) + e(p, y) + f(p, x)$$
$$j(pq, x) = b(pq, y) - c(q, x) - e(p, x)$$
$$k(pq, x) = -b(pq, x) - d(q, x) - f(p, x).$$

We also state the results in degenerated cases.

COROLLARY 38.11. *Let $k \geq 0$, g, h be in $\mathscr{W}_{k,0}$ and j be in $\mathscr{W}_{k+1,-1}$. Assume that, for pq in X_{k+1} and x in X, we have*

$$g(p_1q, x) + h(pq_1, x) + j(pq, x) = 0.$$

Then, if $k \geq 1$, there exist a in $\mathscr{W}_{k-1,0}$ and b, c in $\mathscr{W}_{k,-1}$ such that, for pq in X_k and x in X,

$$g(pq, x) = a(pq_1, x) + b(pq, x)$$
$$h(pq, x) = -a(p_1q, x) + c(pq, x)$$

and, for pq in X_{k+1} and x in X,

$$j(pq, x) = -b(p_1q, x) - c(pq_1, x).$$

If $k = 0$, there exist a in $\mathscr{W}_{-1,0}$ and b, c in $\mathscr{W}_{0,-1}$ such that, for pq in X_1 and x in X,

$$\begin{aligned} g(p, x) &= a(q, x) + b(pq, x) \\ h(pq, x) &= -a(p, x) + c(pq, x) \\ j(pq, x) &= -b(q, x) - c(p, x). \end{aligned}$$

PROOF. This is a direct consequence of Lemma 38.6. □

CHAPTER 39

Tensors products of pseudofunctions

In this Chapter, we state and prove Proposition 39.3, which will be our main tool for translating the result of Proposition 33.3 in the language of pseudofunctions.

39.1. Sequences of tensors

We define the double weight of tensors, which is obtained directly from the weight construction for pseudofunctions. We introduce a notion of cohomological triviality for sequences of elements of $\mathscr{W}_{h,k}$, $h, k \geq 0$, that is inspired by the language of Proposition 33.3. Then, we state an analogue of Proposition 37.12.

For $k \geq 0$ and v in $\mathscr{V}_k$, we set $v^\vee$ to be the function $ab \mapsto v(ba)$ on X_k. Let still ω_k be the weight of pseudofunctions from Definition 36.1.

Definition 39.1. For $h, k \geq 0$, and H in $\mathcal{H}_h \otimes \mathcal{H}_k$, we define the double weight $\varpi_{h,k}(H)$ as the element $\omega_h H \omega_k^\vee$ of $\mathscr{W}_{h,k}$. In other words, for any J in $\mathcal{H}_h$, K in $\mathcal{H}_k$, ab in X_h and xy in X_k, one has

$$\varpi_{h,k}(J \otimes K)(ab, xy) = \omega_h(J)(ab)\omega_k(K)(yx).$$

When $h = k$, we write ϖ_k for $\varpi_{k,k}$.

Definition 39.2. Let $h, k \geq 1$. We say that a finitely supported sequence $(w_j)_{j\geq 1}$ in $\mathscr{W}_{h,k}$ is cohomologically trivial if there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $\mathscr{W}_{h,k-1}$ and $(v_j)_{j\geq 0}$ in $\mathscr{W}_{h-1,k}$ such that, for any $j \geq 1$ and ab in X_h and xy in X_k, one has

$$w_j(ab, xy) = u_j(ab, x_1 y) - u_{j-1}(ab, xy_1) + v_j(ab_1, xy) - v_{j-1}(a_1 b, xy).$$

The following statement is a tensor analogue of Proposition 37.12. Its proof will last until the end of the Chapter.

Proposition 39.3. *Let $k \geq 1$ and $(H_j)_{j\geq 1}$ be a finitely supported sequence of elements of $\otimes^2 \mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Then, there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ such that,*

for $j \geq 1$, one has

$$\begin{aligned} H_j &= F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1} && \text{if } k \text{ is even} \\ &= F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1} && \text{if } k \text{ is odd.} \end{aligned}$$

Note that the converse is also true by Lemma 37.1.

39.2. Shortening the dependence

We will prove Proposition 39.3 by following the same lines as for proving Proposition 37.12. In particular, we will need the following analogue of Lemma 37.10:

PROPOSITION 39.4. *Let $h' \geq h \geq 1$, $k' \geq k \geq 1$ and $(w_j)_{j\geq 1}$ be a finitely supported sequence in $\mathscr{W}_{h,k}$. For $j \geq 1$, ab in $X_{h'}$ and xy in $X_{k'}$, set $w'_j(ab, xy) = w_j(aa_h, y_k y)$. Assume that the sequence $(w'_j)_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{h',k'}$. Then, the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{h,k}$.*

This result will follow from several applications of Corollary 38.10 and Corollary 38.11. We summarize them in the technical

LEMMA 39.5. *Let $h \geq 1$, $k \geq 0$ and $(u_j)_{j\geq 0}$ be a finitely supported sequence in $\mathscr{W}_{h,k}$. Assume that there exist a finitely supported sequence $(u'_j)_{j\geq 1}$ in $\mathscr{W}_{h,k-1}$ and finitely supported sequences $(\alpha_j)_{j\geq 1}$, $(\beta_j)_{j\geq 1}$, $(\gamma_j)_{j\geq 1}$ and $(\delta_j)_{j\geq 1}$ in $\mathscr{W}_{h-1,k}$ such that, for any $j \geq 1$, ab in X_h and xy in X_k, one has*

$$\begin{aligned} u_j(ab, xy) &= u'_j(ab, xy_1) + \alpha_j(a_1 b, xy) + \beta_j(ab_1, xy) \\ u_{j-1}(ab, xy) &= u'_j(ab, x_1 y) + \gamma_j(a_1 b, xy) + \delta_j(ab_1, xy). \end{aligned} \tag{39.1}$$

Then, there exist finitely supported sequences $(\varphi_j)_{j\geq 1}$ in $\mathscr{W}_{h-1,k}$ and $(\psi_j)_{j\geq 1}$ in $\mathscr{W}_{h-2,k}$ such that, for any $j \geq 2$, ab in X_{h-1} and xy in X_{k+1}, one has

$$\begin{aligned} &\alpha_j(ab, x_1 y) - \gamma_j(ab, xy_1) + \beta_{j-1}(ab, x_1 y) - \delta_{j-1}(ab, xy_1) = \\ &\qquad \varphi_j(ab, x_1 y) - \varphi_{j-1}(ab, xy_1) + \psi_j(ab_1, x_1 y) - \psi_{j-1}(a_1 b, x_1 y). \end{aligned}$$

Note that, when $h = 1$ or $k = 0$, there is a slight abuse of notation in (39.1) which should be understood by means of the language of Example 38.3.

PROOF. We fix $h \geq 1$ and we prove the result by induction on $k \geq 0$.

For $k = 0$, (39.1) says that, for any $j \geq 1$, ab in X_h and xy in X_1, one has

$$u_j(ab, x) = u'_j(ab, x) + \alpha_j(a_1b, x) + \beta_j(ab_1, x)$$
$$u_{j-1}(ab, x) = u'_j(ab, y) + \gamma_j(a_1b, x) + \delta_j(ab_1, x).$$

If $j \geq 2$, we get

$$u'_{j-1}(ab, x) + \alpha_{j-1}(a_1b, x) + \beta_{j-1}(ab_1, x) \\ = u'_j(ab, y) + \gamma_j(a_1b, x) + \delta_j(ab_1, x).$$

Thus, Corollary 38.11 says that we may find v_j in $\mathscr{W}_{h-2,0}$ and α'_j and β'_j in $\mathscr{W}_{h-1,-1}$ such that, for ab in X_{h-1} and x in X, one has

$$\alpha_{j-1}(ab, x) - \gamma_j(ab, x) = v_j(ab_1, x) + \alpha'_j(ab, x)$$
$$\beta_{j-1}(ab, x) - \delta_j(ab, x) = -v_j(a_1b, x) + \beta'_j(ab, x).$$

If j is large, we assume $v_j = 0$ and $\alpha'_j = \beta'_j = 0$. For $j \geq 2$, ab in X_{h-1} and xy in X_1, we get

$$\begin{aligned}(39.2)\quad \alpha_j(ab, y) - \gamma_j(ab, x) + \beta_{j-1}(ab, y) - \delta_{j-1}(ab, x) = \\ \gamma_{j+1}(ab, y) + v_{j+1}(ab_1, y) + \alpha'_{j+1}(ab, y) - \gamma_j(ab, x) \\ + \delta_j(ab, y) - v_j(a_1b, y) + \beta'_j(ab, y) - \delta_{j-1}(ab, x).\end{aligned}$$

We define a finitely supported sequence $(\varepsilon_j)_{j\geq 1}$ in $\mathscr{W}_{h-1,-1}$ as follows. For $j \geq 1$, ab in X_{h-1} and x in X, we set

$$\varepsilon_j(ab, x) = -\sum_{\substack{i\geq j+1 \\ i-j \text{ even}}} (\alpha'_{i+1}(ab, x) + \beta'_i(ab, x)) \\ - \sum_{\substack{i\geq j+1 \\ i-j \text{ odd}}} (\alpha'_{i+1}(ab, y) + \beta'_i(ab, y)),$$

where y is any neighbour of x. Thus, for $j \geq 2$, ab in X_{h-1} and xy in X_1, we get

$$\alpha'_{j+1}(ab, y) + \beta'_j(ab, y) = \varepsilon_j(ab, y) - \varepsilon_{j-1}(ab, x),$$

hence, from (39.2),

$$\begin{aligned}\alpha_j(ab, y) - \gamma_j(ab, x) + \beta_{j-1}(ab, y) - \delta_{j-1}(ab, x) = \\ \gamma_{j+1}(ab, y) - \gamma_j(ab, x) + \delta_j(ab, y) - \delta_{j-1}(ab, x) \\ + v_{j+1}(ab_1, y) - v_j(a_1b, y) + \varepsilon_j(ab, y) - \varepsilon_{j-1}(ab, x)\end{aligned}$$

and the conclusion follows by setting, for $j \geq 1$ and x in X,

$$\begin{aligned} \varphi_j(ab,x) &= \gamma_{j+1}(ab,x) + \delta_j(ab,x) + \varepsilon_j(ab,x), && ab \in X_{h-1}, \\ \psi_j(ab,x) &= v_{j+1}(ab,x), && ab \in X_{h-2}. \end{aligned}$$

We now deal with the case where $k \geq 1$. To avoid using the same abuse of notation as in the statement, we separate the cases $k = 1$ and $k \geq 2$.

For $k = 1$, (39.1) says that, for any $j \geq 1$, ab in X_h and xy in X_1, one has

$$\begin{aligned} u_j(ab,xy) &= u'_j(ab,x) + \alpha_j(a_1b,xy) + \beta_j(ab_1,xy) \\ u_{j-1}(ab,xy) &= u'_j(ab,y) + \gamma_j(a_1b,xy) + \delta_j(ab_1,xy). \end{aligned}$$

If $j \geq 2$, we get

$$\begin{aligned} u'_{j-1}(ab,x) + \alpha_{j-1}(a_1b,xy) + \beta_{j-1}(ab_1,xy) \\ = u'_j(ab,y) + \gamma_j(a_1b,xy) + \delta_j(ab_1,xy). \end{aligned}$$

Thus, Corollary 38.10 says that we may find u''_j in $\mathscr{W}_{h,-1}$, v_j in $\mathscr{W}_{h-2,1}$ and $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$ in $\mathscr{W}_{h-1,0}$ such that, for ab in X_{h-1} and xy in X_1, one has

$$\begin{aligned} \alpha_{j-1}(ab,xy) - \gamma_j(ab,xy) &= v_j(ab_1,xy) + \alpha'_j(ab,y) - \gamma'_j(ab,x) \\ \beta_{j-1}(ab,xy) - \delta_j(ab,xy) &= -v_j(a_1b,xy) + \beta'_j(ab,y) - \delta'_j(ab,x) \end{aligned} \tag{39.3}$$

and, for ab in X_h and xy in X_1, one has

$$\begin{aligned} u'_j(ab,x) &= u''_j(ab,x) + \alpha'_j(a_1b,x) + \beta'_j(ab_1,x) \\ u'_{j-1}(ab,x) &= u''_j(ab,y) + \gamma'_j(a_1b,x) + \delta'_j(ab_1,x). \end{aligned}$$

If j is large, we assume $u''_j = 0$, $v_j = 0$ and $\alpha'_j = \beta'_j = \gamma'_j = \delta'_j = 0$. Then, in view of the first case, we may find finitely supported sequences $(\varphi'_j)_{j\geq 1}$ in $\mathscr{W}_{h-1,0}$ and $(\psi'_j)_{j\geq 1}$ in $\mathscr{W}_{h-2,0}$ such that, for any $j \geq 2$, ab in X_{h-1} and xy in X_1, one has

$$\begin{aligned} \alpha'_j(ab,y) - \gamma'_j(ab,x) + \beta'_{j-1}(ab,y) - \delta'_{j-1}(ab,x) = \\ \varphi'_j(ab,y) - \varphi'_{j-1}(ab,x) + \psi'_j(ab_1,y) - \psi'_{j-1}(a_1b,y). \end{aligned}$$

Besides, for $j \geq 2$, ab in X_{h-1} and xy in X_2, we get, from (39.3),

$$\begin{aligned} \alpha_j(ab,x_1y) - \gamma_j(ab,xy_1) + \beta_{j-1}(ab,x_1y) - \delta_{j-1}(ab,xy_1) = \\ \gamma_{j+1}(ab,x_1y) + v_{j+1}(ab_1,x_1y) + \alpha'_{j+1}(ab,y) - \gamma'_{j+1}(ab,x_1) - \gamma_j(ab,xy_1) \\ + \delta_j(ab,x_1y) - v_j(a_1b,x_1y) + \beta'_j(ab,y) - \delta'_j(ab,x_1) - \delta_{j-1}(ab,xy_1). \end{aligned}$$

The conclusion follows by setting, for $j \geq 1$ and xy in X_1,

$$\varphi_j(ab, xy) = \gamma_{j+1}(ab, xy) + \delta_j(ab, xy) + \varphi'_{j+1}(ab, y), \quad ab \in X_{h-1},$$
$$\psi_j(ab, xy) = v_{j+1}(ab, xy) + \psi'_{j+1}(ab, y), \quad ab \in X_{h-2}.$$

Now, we assume that $k \geq 2$ and the result holds for $k-1$. Let us show that it also holds for k. This is analogous to the case above. Indeed, (39.1) says that, for any $j \geq 1$, ab in X_h and xy in X_k, one has

$$u_j(ab, xy) = u'_j(ab, xy_1) + \alpha_j(a_1b, xy) + \beta_j(ab_1, xy)$$
$$u_{j-1}(ab, xy) = u'_j(ab, x_1y) + \gamma_j(a_1b, xy) + \delta_j(ab_1, xy).$$

If $j \geq 2$, we get

$$u'_{j-1}(ab, xy_1) + \alpha_{j-1}(a_1b, xy) + \beta_{j-1}(ab_1, xy) = u'_j(ab, x_1y) + \gamma_j(a_1b, xy) + \delta_j(ab_1, xy).$$

Thus, Corollary 38.11 says that we may find u''_j in $\mathscr{W}_{h,k-2}$, v_j in $\mathscr{W}_{h-2,k}$ and $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$ in $\mathscr{W}_{h-1,k-1}$ such that, for ab in X_{h-1} and xy in X_k, one has

(39.4)
$$\alpha_{j-1}(ab, xy) - \gamma_j(ab, xy) = v_j(ab_1, xy) + \alpha'_j(ab, x_1y) - \gamma'_j(ab, xy_1)$$
$$\beta_{j-1}(ab, xy) - \delta_j(ab, xy) = -v_j(a_1b, xy) + \beta'_j(ab, x_1y) - \delta'_j(ab, xy_1)$$

and, for ab in X_h and xy in X_{k-1}, one has

$$u'_j(ab, xy) = u''_j(ab, xy_1) + \alpha'_j(a_1b, xy) + \beta'_j(ab_1, xy)$$
$$u'_{j-1}(ab, xy) = u''_j(ab, x_1y) + \gamma'_j(a_1b, xy) + \delta'_j(ab_1, xy).$$

If j is large, we assume $u''_j = 0$, $v_j = 0$ and $\alpha'_j = \beta'_j = \gamma'_j = \delta'_j = 0$. Then, in view of the induction assumption, we may find finitely supported sequences $(\varphi'_j)_{j\geq 1}$ in $\mathscr{W}_{h-1,k-1}$ and $(\psi'_j)_{j\geq 1}$ in $\mathscr{W}_{h-2,k-1}$ such that, for any $j \geq 2$, ab in X_{h-1} and xy in X_k, one has

$$\alpha'_j(ab, x_1y) - \gamma'_j(ab, xy_1) + \beta'_{j-1}(ab, x_1y) - \delta'_{j-1}(ab, xy_1) = \varphi'_j(ab, x_1y) - \varphi'_{j-1}(ab, xy_1) + \psi'_j(ab_1, x_1y) - \psi'_{j-1}(a_1b, x_1y).$$

Besides, for $j \geq 2$, ab in X_{h-1} and xy in X_{k+1}, we get, from (39.4),

$$\alpha_j(ab, x_1y) - \gamma_j(ab, xy_1) + \beta_{j-1}(ab, x_1y) - \delta_{j-1}(ab, xy_1) =$$
$$\gamma_{j+1}(ab, x_1y) + v_{j+1}(ab_1, x_1y) + \alpha'_{j+1}(a_1b, x_2y) - \gamma'_{j+1}(ab, x_1y_1) - \gamma_j(ab, xy_1)$$
$$+ \delta_j(ab, x_1y) - v_j(a_1b, x_1y) + \beta'_j(ab, x_2y) - \delta'_j(ab, x_1y_1) - \delta_{j-1}(ab, xy_1).$$

The conclusion follows by setting, for $j \geq 1$ and xy in X_k,

$$\varphi_j(ab, xy) = \gamma_{j+1}(ab, xy) + \delta_j(ab, xy) + \varphi'_{j+1}(ab, x_1y), \quad ab \in X_{h-1},$$
$$\psi_j(ab, xy) = v_{j+1}(ab, xy) + \psi'_{j+1}(ab, x_1y), \quad ab \in X_{h-2}.$$

□

Proof of Proposition 39.4. We first deal with the case where $h' = h + 1$ and $k' = k$. Then, by Definition 39.2 and the assumption, there exists finitely supported sequences $(u_j)_{j\geq 0}$ in $\mathscr{W}_{h+1,k-1}$ and $(v_j)_{j\geq 0}$ in $\mathscr{W}_{h,k}$ such that, for any $j \geq 1$, ab in X_{h+1} and xy in X_k, one has

$$w_j(ab_1, xy) = u_j(ab, x_1y) - u_{j-1}(ab, xy_1) + v_j(ab_1, xy) - v_{j-1}(a_1b, xy).$$

Then, Corollary 38.10 says that we may find α_j in $\mathscr{W}_{h-1,k}$, β_j in $\mathscr{W}_{h+1,k-2}$ and $\gamma_j, \delta_j, \varepsilon_j, \zeta_j$ in $\mathscr{W}_{h,k-1}$ such that, for ab in X_h and xy in X_k,

$$v_{j-1}(ab, xy) = \alpha_j(ab_1, xy) + \gamma_j(ab, x_1y) + \delta_j(ab, xy_1) \tag{39.5}$$
$$w_j(ab, xy) - v_j(ab, xy) = -\alpha_j(a_1b, xy) + \varepsilon_j(ab, x_1y) + \zeta_j(ab, xy_1)$$

and, for ab in X_{h+1} and xy in X_{k-1},

$$-u_j(ab, xy) = \beta_j(ab, xy_1) - \gamma_j(a_1b, xy) - \varepsilon_j(ab_1, xy)$$
$$u_{j-1}(ab, xy) = -\beta_j(ab, x_1y) - \delta_j(a_1b, xy) - \zeta_j(ab_1, xy).$$

By Lemma 39.4, the latter tells us that we may find finitely supported sequences $(\varphi_j)_{j\geq 1}$ in $\mathscr{W}_{h,k-1}$ and $(\psi_j)_{j\geq 1}$ in $\mathscr{W}_{h-1,k-1}$ such that, for any $j \geq 2$, ab in X_h and xy in X_k, one has

$$\gamma_j(ab, x_1y) + \delta_j(ab, xy_1) + \varepsilon_{j-1}(ab, x_1y) + \zeta_{j-1}(ab, xy_1) = \tag{39.6}$$
$$\varphi_j(ab, x_1y) - \varphi_{j-1}(ab, xy_1) + \psi_j(ab_1, x_1y) - \psi_{j-1}(a_1b, x_1y).$$

Besides, by using (39.5), we get, for $j \geq 1$,

$$w_j(ab, xy) = \alpha_{j+1}(ab_1, xy) + \gamma_{j+1}(ab, x_1y) + \delta_{j+1}(ab, xy_1) - \alpha_j(a_1b, xy) + \varepsilon_j(ab, x_1y) + \zeta_j(ab, xy_1).$$

Thanks to (39.6), this gives

$$w_j(ab, xy) = \alpha_{j+1}(ab_1, xy) - \alpha_j(a_1b, xy) + \varphi_{j+1}(ab, x_1y) - \varphi_j(ab, xy_1) + \psi_{j+1}(ab_1, x_1y) - \psi_j(a_1b, x_1y)$$

as required.

Now, we can obtain the case where $h' = h$ and $k' = k + 1$ by symmetry. Indeed, in view of Definition 39.2, a sequence $(w_j)_{j\geq 1}$ in $\mathscr{W}_{h,k}$ is cohomologically trivial if and only if the sequence of functions in $\mathscr{W}_{k,h}$

$$(ab, xy) \mapsto w_j(yx, ba), \quad j \geq 1,$$

is cohomologically trivial. The general case follows by an easy induction. □

39.3. Splitting sequences

We pursue the proof of Proposition 39.3 by proving an intermediate result, which will play the role of Lemma 37.13 in the proof of Proposition 37.12.

To state it, we introduce a notation. For $i \geq 0$ an integer, we set $r(i) = 0$ if $i = 0$, $r(i) = 1$ if $i = 1$ and $r(i) = 2(i-1)$ if $i \geq 2$. As in Chapter 20.2, if H is a pseudofunction we write H^+ for $H^{>\vee}$. The purpose of this Section is to establish the following

LEMMA 39.6. *Let $k \geq 4$ be an even integer, $k = 2\ell$, $\ell \geq 2$. Suppose, for any $0 \leq i \leq \ell + 1$, we are given a finitely supported sequence $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)}$. For $j \geq 1$, we set*

$$H_j = H_{0,j}^{+^k} + {}^{+^2}H_{1,j}{}^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i-1)}} + {}^{+^{k-1}}H_{\ell,j}{}^{+^2} + {}^{+^k}H_{\ell+1,j},$$

which is an element of $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Then, for any $0 \leq i \leq h$, there exist finitely supported sequences $(D_{i,j})_{j\geq 1}$ and $(E_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)-1} \otimes \mathcal{H}_{r(i)}$ and $(F_{i,j})_{j\geq 1}$ and $(G_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)-1}$ such that, for $j \geq 1$, one has

$$H_{i,j} = {}^{>}D_{i,j} + {}^{\vee>}E_{i,j} + F_{i,j}^{>} + G_{i,j}^{>\vee}.$$

To prove this, we will aim at applying the tensor version of the criterion of Lemma 37.8. To this purpose, we introduce the notion of a split element of $\mathscr{W}_{h,k}$ for $h, k \geq 0$ (see Section 9.4 and Section 37.3 for the notion of a split elements of $\mathscr{V}_k$, $k \geq 0$). In the language of Example 38.3, we say that an element u in $\mathscr{W}_{h,k}$ is split if it belongs to $L\mathscr{W}_{h-1,k} + R\mathscr{W}_{h-1,k} + \mathscr{W}_{h,k-1}L + \mathscr{W}_{h,k-1}R$. More concretely, for example, if $h, k \geq 1$, this means that there exists v_0, v_1 in $\mathscr{W}_{h-1,k}$ and w_0, w_1 in $\mathscr{W}_{h,k-1}$ such that, for ab in X_h and xy in X_k,

$$u(ab, xy) = v_0(ab_1, xy) + v_1(a_1b, xy) + w_0(ab, xy_1) + w_1(ab, x_1y).$$

If $h = 0$ and $k \geq 1$, this means that there exists v in $\mathscr{W}_{-1,k}$ and w_0, w_1 in $\mathscr{W}_{0,k-1}$ such that, for ab in X_h and xy in X_k,

$$u(a, xy) = v(a, xy) + w_0(a, xy_1) + w_1(a, x_1y).$$

To study triangular families of tensors as in Lemma 39.6, we shall use

LEMMA 39.7. *Let $k \geq 1$ and, for $0 \leq i \leq k$, let $(w_{i,j})_{j\geq 1}$ be a finitely supported sequence in $\mathscr{W}_{k-i,i}$. For $j \geq 1$ and ab, xy in X_k, we set*

$$w_j(ab, xy) = \sum_{i=0}^{k} w_{i,j}(aa_{k-i}, y_i y).$$

Assume that the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Then, for every $j \geq 1$ and $0 \leq i \leq k$, the function $w_{i,j}$ is split in $\mathscr{W}_{k-i,i}$.

PROOF. Let $0 \leq h \leq k+1$ be the least integer such that, for all $h \leq i \leq k$, for all $j \geq 1$, $w_{i,j} = 0$. We will show the statement by induction on h. For $h = 0$, there is nothing to prove.

Assume $h = 1$. By assumption, the sequence of functions in $\mathscr{W}_{k,k}$,

$$(ab, xy) \mapsto w_{0,j}(ab, y), \quad j \geq 1,$$

is cohomologically trivial. By Propositon 39.4, the sequence $(w_{0,j})_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{k,0}$. In other words, by Definition 39.2, there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $\mathscr{W}_{k,-1}$ and $(v_j)_{j\geq 0}$ in $\mathscr{W}_{k-1,0}$ such that, for any $j \geq 1$, ab in X_k and xy in X_1, one has

$$w_{0,j}(ab, y) = u_j(ab, y) - u_{j-1}(ab, x) + v_j(ab_1, y) - v_{j-1}(a_1 b, y).$$

Thus $w_{0,j}$ is split in $\mathscr{W}_{k,0}$ as required.

Assume $h \geq 2$ and the result is true for $h-1$. By assumption, the sequence of functions in $\mathscr{W}_{k,k}$,

$$(ab, xy) \mapsto \sum_{i=0}^{h-1} w_{i,j}(aa_{k-i}, y_i y), \quad j \geq 1,$$

is cohomologically trivial. By Definition 39.2 and Propositon 39.4, there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $\mathscr{W}_{k,h-2}$ and $(v_j)_{j\geq 0}$ in $\mathscr{W}_{k-1,h-1}$ such that, for any $j \geq 1$, ab in X_k and xy in X_{h-1}, one has

$$\sum_{i=0}^{h-1} w_{i,j}(aa_{k-i}, y_i y) = \tag{39.7}$$
$$u_j(ab, x_1 y) - u_{j-1}(ab, xy_1) + v_j(ab_1, xy) - v_{j-1}(a_1 b, xy).$$

We temporarily consider each $w_{i,j}$, $0 \leq i \leq h-1$, as a function from X_{k-i} towards $\mathscr{V}_i$. From (39.7), we get, for ab in X_k,

$$w_{h-1,j}(aa_{k-h+1}) - v_j(ab_1) + v_{j-1}(a_1 b) \in L\mathscr{V}_{h-2} + R\mathscr{V}_{h-2}$$

(where we have used the notation of Example 38.3). Thus, Lemma 37.10 and Lemma 38.4 say that there exist a finitely supported sequence

$(v'_j)_{j\geq 0}$ in $\mathscr{W}_{k-h,h-1}$ such that for any $j \geq 1$ and ab in X_{k-h+1}, one has

$$w_{h-1,j}(ab) - v'_j(ab_1) + v'_{j-1}(a_1b) \in L\mathscr{V}_{h-2} + R\mathscr{V}_{h-2}.$$

In other words, there exist finitely supported sequences $(\alpha_j)_{j\geq 1}$ and $(\beta_j)_{j\geq 1}$ in $\mathscr{W}_{k-h+1,h-2}$ such that, for $j \geq 1$, ab in X_{k-h+1} and xy in X_{h-1}, one has

$$w_{h-1,j}(ab, xy) = v'_j(ab_1, xy) - v'_{j-1}(a_1b, xy) + \alpha_j(ab, xy_1) + \beta_j(ab, x_1y)$$

and the function $w_{h-1,j}$ is split in $\mathscr{W}_{k-h+1,h-1}$.

Besides, we set $\gamma_j = \alpha_{j+1} + \beta_j$ and we rewrite the latter as, for $j \geq 1$, ab in X_{k-h+1} and xy in X_{h-1},

$$w_{h-1,j}(ab, xy) = v'_j(ab_1, xy) - v'_{j-1}(a_1b, xy) + \alpha_j(ab, xy_1) - \alpha_{j+1}(ab, x_1y) + \gamma_j(ab, x_1y).$$

From (39.7), we get, for $j \geq 1$ and ab, xy in X_k,

$$\begin{aligned}\sum_{i=0}^{h-2} w_{i,j}(aa_{k-i}, y_iy) + \gamma_j(aa_{k-h+1}, y_{h-2}y) &= u_j(ab, y_{h-2}y)\\ &\quad - u_{j-1}(ab, y_{h-1}y_1) + v_j(ab_1, y_{h-1}y) - v_{j-1}(a_1b, y_{h-1}y)\\ &\quad - v'_j(aa_{k-h}, y_{h-1}y) + v'_{j-1}(a_1a_{k-h+1}, y_{h-1}y) - \alpha_j(aa_{k-h+1}, y_{h-1}y_1)\\ &\quad + \alpha_{j+1}(aa_{k-h+1}, y_{h-2}y),\end{aligned}$$

so that the sequence in $\mathscr{W}_{k,k}$,

$$(ab, xy) \mapsto \sum_{i=0}^{h-2} w_{i,j}(aa_{k-i}, y_iy) + \gamma_j(aa_{k-h+1}, y_{h-2}y), \quad j \geq 1,$$

is cohomologically trivial. From the induction assumption, it follows that the functions $w_{h-2,j} + \gamma_j R$ and $w_{i,j}$, $0 \leq i \leq h-3$ are split. The conclusion follows. □

To compute the weights of large orthogonal extensions, we shall use

LEMMA 39.8. *Let $k \geq 0$ and H be in $\mathcal{H}_k$. Fix $h \geq 0$ even.*
If k is even, for ab in X_{h+k}, we have

$$\omega_{h+k}\left(H^{+^h}\right)(ab) = \omega_k(H)(ab_h).$$

If k is odd, for ab in X_{h+k+1}, we have

$$\omega_{h+k+1}\left(H^{+^{h+1}}\right)(ab) = \omega_k(H^\vee)(ab_{h+1}).$$

PROOF. This is a direct consequence of Lemma 37.1. □

We now define the tensor squares of the objects introduced in Section 37.5. There, given $k \geq 2$ even, $k = 2\ell$, $\ell \geq 1$, and a complete k-pseudofunction H, we have defined functions $\rho_k^i(H)$ on $X_{\ell-i+1}$ for any $0 \leq i \leq \ell - 1$.

Now, for $k, m \geq 2$ even, $k = 2\ell$, $\ell \geq 1$, $m = 2n$, $n \geq 1$, $0 \leq h \leq \ell-1$ and $0 \leq i \leq n-1$, we let $\sigma_{k,m}^{h,i} : \widehat{\mathcal{H}}_k \otimes \widehat{\mathcal{H}}_m \to \mathscr{W}_{\ell-h+1,n-i+1}$ be the linear map such that, for H in $\widehat{\mathcal{H}}_k$, J in $\widehat{\mathcal{H}}_m$, ab in $X_{\ell-h+1}$ and xy in X_{n-i+1}, one has

$$\sigma_{k,m}^{h,i}(H \otimes J)(ab, xy) = \rho_k^h(H)(ab)\rho_m^i(J)(yx).$$

From (37.4) and Definition 39.1, we get, for H in $\widehat{\mathcal{H}}_k \otimes \widehat{\mathcal{H}}_m$, ab in X_k and xy in X_m,

$$\varpi_{k,m}(H)(ab, xy) = \sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}} \sigma_{k,m}^{h,i}(H)(a_{2h}a_{\ell+h+1}, y_{n+i+1}y_{2i}). \tag{39.8}$$

As in the proof of Lemma 37.13, this yields

LEMMA 39.9. *Let $k, m \geq 2$ be even, $k = 2\ell$, $\ell \geq 1$, $m = 2n$, $n \geq 1$. Let $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\widehat{\mathcal{H}}_k \otimes \widehat{\mathcal{H}}_m$. For $j \geq 1$, ab in X_k and xy in X_m, set*

$$w_j(ab, xy) = \sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}} \sigma_{k,m}^{h,i}(H_{h+i+j})(a_h a_{\ell+1}, y_{n+1}y_i) - \varpi_{k,m}(H)(ab, xy).$$

Then, the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{k,m}$.

PROOF. This is a direct computation. Indeed, by (39.8), for $j \geq 1$, we have, for ab in X_k and xy in X_m,

$$\sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}} \sigma_{k,m}^{h,i}(H_{h+i+j})(a_h a_{\ell+1}, y_{n+1}y_i) - \varpi_{k,m}(H)(ab, xy) =$$

$$\sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}} \sigma_{k,m}^{h,i}(H_{h+i+j})(a_h a_{\ell+1}, y_{\ell+1}y_i) - \sigma_{k,m}^{h,i}(H_j)(a_{2h}a_{\ell+h+1}, y_{n+i+1}y_{2i}).$$

For h, i as above, we write

$$\sigma_{k,m}^{h,i}(H_{h+i+j})(a_h a_{\ell+1}, y_{n+1}y_i) - \sigma_{k,m}^{h,i}(H_j)(a_{2h}a_{\ell+h+1}, y_{n+i+1}y_{2i}) =$$

$$\sum_{p=0}^{h-1} \sigma_{k,m}^{h,i}(H_{h+i+j-p})(a_{h+p}a_{\ell+p+1}, y_{n+1}y_i) - \sigma_{k,m}^{h,i}(H_{h+i+j-p-1})(a_{h+p+1}a_{\ell+p+2}, y_{n+1}y_i)$$

$$+\sum_{q=0}^{i-1} \sigma_{k,m}^{h,i}(H_{i+j-q})(a_{2h}a_{\ell+h+1}, y_{n+q+1}y_{i+q}) - \sigma_{k,m}^{h,i}(H_{i+j-q-1})(a_{2h}a_{\ell+h+1}, y_{n+q+2}y_{i+q+1}).$$

The conclusion follows by Definition 39.2. □

We will need an adapted version of Lemma 39.9 to deal with the boundary terms appearing in Lemma 39.6.

For $k \geq 2$, $k = 2\ell$, $\ell \geq 1$, $0 \leq i \leq \ell - 1$, and $m \in \{0, 1\}$, we let $\theta^i_{k,m} : \widehat{\mathcal{H}}_k \otimes \mathcal{H}_m \to \mathscr{W}_{\ell+1-i,m}$ and $\theta^i_{m,k} : \mathcal{H}_m \otimes \widehat{\mathcal{H}}_k \to \mathscr{W}_{m,\ell+1-i}$ be the linear maps defined by, for H in $\widehat{\mathcal{H}}_k$, J in $\mathcal{H}_m$, ab in $X_{\ell+1-i}$,

$$\theta^i_{k,m}(H \otimes J)(ab, xy) = \rho^i_k(H)(ab)\omega_m(J)(xy) = \theta^i_{m,k}(J \otimes H)(xy, ab).$$

If H is in $\widehat{\mathcal{H}}_k \otimes \mathcal{H}_m$, from (37.4), we now get, for ab in X_k and xy in X_m,

$$\varpi_{k,m}(H)(ab, xy) = \sum_{i=0}^{\ell-1} \theta^i_{k,m}(a_{2i}a_{\ell+i+1}, xy),$$

which, as above, yields

LEMMA 39.10. *Let $k \geq 2$ be even, $k = 2\ell$, $\ell \geq 1$, and m be in $\{0, 1\}$. Let $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\widehat{\mathcal{H}}_k \otimes \mathcal{H}_m$. There exists a finitely supported sequence $(v_j)_{j\geq 0}$ in $\mathscr{W}_{k-1,m}$ such that, for any $j \geq 1$ and ab in X_k and xy in X_m, one has*

$$\sum_{i=0}^{\ell-1} \theta^i_{k,m}(H_{i+j})(a_i a_{\ell+1}, xy) - \varpi_{k,m}(H)(ab, xy) = v_j(ab_1, xy) - v_{j-1}(a_1 b, xy).$$

Finally, we will need to complete the information given by Lemma 37.8 in case $k = 0, 1$.

LEMMA 39.11. *Let H be in $\mathcal{H}_0$. Then $\omega_0(H)$ is split in $\mathscr{V}_0$ if and only if there exists G in $\mathcal{H}_{-1}$ with $H = G^>$.*

Let H be in $\mathcal{H}_1$. Then $\omega_1(H)$ is split in $\mathscr{V}_1$ if and only if there exists F, G in $\mathcal{H}_{-1}$ with $H = F^> + G^{>\vee}$.

PROOF. Recall the conventions on pseudofunctions with low degree from Section 20.1 and Section 20.2. If H in $\mathcal{H}_0$, then H is the 0-pseudofunction associated with the function $\omega_0(H)$ on X_0. Then saying that H is in $\mathcal{H}^>_{-1}$ is saying that $\omega_0(H)$ is constant on neighbours.

If H is in $\mathcal{H}_1$ and $H = F^> + G^{>\vee}$ for F, G in $\mathcal{H}_0$, then $\omega_1(H)$ is split by Lemma 37.1. Conversely, let u, v be in $\mathscr{V}_0$ with $\omega_1(H)(ab) = u(a) + v(b)$, $ab \in X_1$. Then, still by Lemma 37.1, we have $H = F^> - G^{>\vee}$, where F and G are the 0-pseudofunctions associated with u and v. □

We can now conclude the

Proof of Lemma 39.6. By Lemma 39.8, for $j \geq 1$ and ab, xy in X_k, we have

$$\varpi_k(H_j)(ab,xy) = \varpi_{k,0}(H_{0,j})(ab,y) + \varpi_{k-2,1}(H_{1,j}^{\vee})(ab_2, y_1 y) + \sum_{i=2}^{\ell-1} \varpi_{2(\ell-i),2(i-1)}(H_{i,j})(ab_{2i}, x_{2(\ell-i+1)}y) + \varpi_{1,k-2}({}^{\vee}H_{\ell,j})(aa_1, x_2 y) + \varpi_{0,k}(H_{\ell+1,j})(a,xy).$$

By Lemma 37.5, we can assume that, for $j \geq 1$, we have

$$\begin{aligned} H_{0,j} &\in \widehat{\mathcal{H}}_k \otimes \mathcal{H}_0 \\ H_{1,j} &\in \widehat{\mathcal{H}}_{k-2} \otimes \mathcal{H}_1 \\ H_{i,j} &\in \widehat{\mathcal{H}}_{k-2i} \otimes \widehat{\mathcal{H}}_{k-2(i-1)} \qquad 2 \leq i \leq \ell-1 \\ H_{\ell,j} &\in \mathcal{H}_1 \otimes \widehat{\mathcal{H}}_{k-2} \\ H_{\ell+1,j} &\in \mathcal{H}_0 \otimes \widehat{\mathcal{H}}_k. \end{aligned}$$

By the assumption, Proposition 39.4, Lemma 39.7 and Lemma 39.10, we can find finitely supported sequences $(w_{i,j})_{j\geq 1}$ in $\mathscr{W}_{\ell+1-i,i}$, $0 \leq i \leq \ell+1$, such that, on one hand, the sequence in $\mathscr{W}_{\ell+1,\ell+1}$,

$$(ab,xy) \mapsto \sum_{i=0}^{\ell+1} w_j(ab_i, x_{\ell+1-i}y), \quad j \geq 1,$$

is cohomologically trivial and, on the other hand, for $j \geq 1$, the functions

$$\begin{aligned} \theta^0_{k,0}(H_{0,j}) - w_{0,j} \qquad & i = 0 \\ \theta^0_{k-2,1}(H^{\vee}_{1,j}) - w_{1,j} \qquad & i = 1 \\ \sigma^0_{2(\ell-i),2(i-1)}(H_{i,j}) - w_{i,j} \qquad & 2 \leq i \leq \ell-1 \\ \theta^0_{1,k-2}({}^{\vee}H_{\ell,j}) - w_{\ell,j} \qquad & i = \ell \\ \theta^0_{0,k}(H_{\ell+1,j}) - w_{\ell+1,j} \qquad & i = \ell+1 \end{aligned} \tag{39.9}$$

are split in $\mathscr{W}_{\ell+1-i,i}$ for all $0 \leq i \leq \ell+1$. Therefore, by Lemma 39.7, for any $0 \leq i \leq \ell+1$, the function $w_{i,j}$ is split in $\mathscr{W}_{\ell+1-i,i}$. The conclusion follows by Lemma 37.8, Lemma 39.11 and (39.9). □

39.4. Triangular sequences

In this final Section, we will use the previous constructions to prove Proposition 39.3. We will split it into several steps, which will eventually allow us to reduce the question to the study of sequences of the form appearing in Lemma 39.6.

In the first step, we show

LEMMA 39.12. *Let $k \geq 4$ be even, $k = 2\ell$, $\ell \geq 2$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\otimes^2 \mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Then, there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$, and, for $0 \leq i \leq \ell - 1$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{2(i+1)}$ such that, for $j \geq 1$, one has*

$$H_j = \sum_{i=0}^{\ell-1} {}^{+2i}H_{i,j}{}^{+2(\ell-i-1)} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.$$

In the course of the proof, we shall need some easy properties of the natural operations on pseudofunctions.

LEMMA 39.13. *Let $h \geq 0$ and $k \geq -1$.*
If k is odd, for any H in $\mathcal{H}_k$, we have

$$(H^{>\vee})^{+2h} = (H^{+2h})^{>\vee} \text{ and } (H^{\vee>})^{+2h} = (H^{+2h})^{\vee>}.$$

If $k = 0$, for any H in $\mathcal{H}_0$, we have

$$(H^{>\vee})^{+2h+1} = (H^{+2h+1})^{>\vee} \text{ and } (H^{\vee>})^{+2h+1} = (H^{+2h+1})^{\vee>}.$$

PROOF. In both cases, the first equality is obvious since by definition $H^+ = H^{>\vee}$.

Assume first k is odd. Then, using Lemma 20.6 and the fact that $H^{\vee\vee} = H$, we get

$$H^{\vee>++} = H^{\vee>>\vee>\vee} = H^{>>>\vee} = H^{>\vee>>} = H^{>\vee>\vee\vee>} = H^{++\vee>}.$$

The conclusion follows by a straightforward induction.

Assume now $k = 0$. Then, by definition, we have $H^\vee = -H$. Using again Lemma 20.6, we get

$$H^{\vee>+} = -H^{>>\vee} = -H^{\vee>>} = H^{>>} = H^{>\vee\vee>} = H^{+\vee>}.$$

The conclusion follows by the first case. □

PROOF OF LEMMA 39.12. We will actually prove by induction on $0 \leq h \leq \ell - 1$ that there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$, and, for $0 \leq i \leq h$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{k-2(h-i)}$ such that, for $j \geq 1$, one has

$$H_j = \sum_{i=0}^{h} {}^{+2i}H_{i,j}{}^{+2(h-i)} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}. \tag{39.10}$$

For $h = 0$, there is nothing to prove. Assume $0 \leq h \leq \ell - 2$ and (39.10) holds for h. Let us show that it also holds for $h + 1$.

We write $k' = 2(k-h+1)$ and we will apply Lemma 39.6 in $\otimes^2 \mathcal{H}_{k'}$. Indeed, for $j \geq 1$, we set

$$H'_j = \sum_{i=0}^{h} {}^{+2i}H_{i,j}{}^{+2(h-i)}.$$

Then, Lemma 37.1, (39.10) and the assumption imply that the sequence $(\varpi_k(H'_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Therefore, if we now set,

$$H''_j = {}^{+k-2h+2}(H'_j)^{+k-2h+2} = \sum_{i=0}^{h} {}^{+k-2(h-i-1)}H_{i,j}{}^{+k-2(i-1)},$$

then, by Lemma 39.8, the sequence $(\varpi_{k'}(H''_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{k'}$. Thus, the assumption of Lemma 39.6 is satisfied. Hence, for $0 \leq i \leq h$, we may find finitely supported sequences $(J_{i,j})_{j\geq 1}$ and $(K_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i}\otimes\mathcal{H}_{k-2(h-i)-1}$ and $(L_{i,j})_{j\geq 1}$ and $(M_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i-1}\otimes \mathcal{H}_{k-2(h-i)}$ such that, for $j \geq 1$, one has

$$\begin{aligned} H_{i,j} &= J^{>}_{i,j} + K^{>\vee}_{i,j} + {}^{>}L_{i,j} + {}^{\vee>}M_{i,j} \\ &= N^{\vee>\vee}_{i,j} + {}^{\vee>\vee}P_{i,j} + J^{>}_{i,j} - J^{\vee>\vee}_{i,j+1} + {}^{>}L_{i,j} - {}^{\vee>\vee}L_{i,j+1}, \end{aligned}$$

where $N_{i,j} = K^{\vee}_{i,j} + J_{i,j+1}$ and $P_{i,j} = {}^{\vee}M_{i,j} + L_{i,j+1}$.

Note that, since $h \leq \ell - 2$, for $0 \leq i \leq h$, we have

$$k - 2i - 1 \geq k - 2h - 1 \geq 3.$$

Therefore, by Lemma 37.8 and Lemma 38.4, we can find finitely supported sequences $(U_{i,j})_{j\geq 1}$ and $(V_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{k-2(h-i)-2}$ and $(W_{i,j})_{j\geq 1}$ and $(X_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i-2} \otimes \mathcal{H}_{k-2(h-i)}$ with, for $j \geq 1$,

$$\begin{aligned} N_{i,j} &= U^{>}_{i,j} + V^{>\vee}_{i,j} = Y^{>}_{i,j} + V^{>\vee}_{i,j} - V^{\vee>}_{i,j+1} \\ P_{i,j} &= {}^{>}W_{i,j} + {}^{\vee>}X_{i,j} = {}^{>}Z_{i,j} + {}^{\vee>}X_{i,j} - {}^{>\vee}X_{i,j+1}, \end{aligned} \tag{39.11}$$

where we have written $Y_{i,j} = U_{i,j} + V^{\vee}_{i,j+1}$ and $Z_{i,j} = W_{i,j} + {}^{\vee}X_{i,j+1}$. For $j \geq 0$, we set $A_{i,j} = -J^{\vee}_{i,j+1} - V^{\vee>\vee}_{i,j+1}$ and $B_{i,j} = -{}^{\vee}L_{i,j+1} - {}^{\vee>\vee}X_{i,j+1}$ and we get, by using Lemma 20.6, for $j \geq 1$,

$$H_{i,j} = Y^{++}_{i,j} + {}^{++}Z_{i,j} + A^{>\vee}_{i,j} - A^{\vee>}_{i,j-1} + {}^{\vee>}B_{i,j} - {}^{>\vee}B_{i,j-1}.$$

Plugging this into (39.10) and using Lemma 39.13 yields, for $j \geq 1$,

$$\begin{aligned} H_j = Y^{+2(h+1)}_{0,j} + \sum_{i=1}^{h} {}^{+2i}(Z_{i-1,j} + Y_{i,j})^{+2(h+1-i)} + {}^{+2(h+1)}Z_{h,j} \\ + C^{>\vee}_{j} - C^{\vee>}_{j-1} + {}^{\vee>}D_j - {}^{>\vee}D_{j-1}. \end{aligned}$$

with, for $j \geq 0$,

$$C_j = \sum_{i=0}^{h} {}^{+2i}A_{i,j}{}^{+2(h-i)} + F_j \text{ and } D_j = \sum_{i=0}^{h} {}^{+2i}B_{i,j}{}^{+2(h-i)} + G_j.$$

Thus, (39.10) holds also for $h+1$ and we are done. □

By using the same method, we can go one step further, but we have to take into account the fact that $\mathcal{H}_1$ is in general not equal to $\mathcal{H}_0^{>} + \mathcal{H}_0^{>\vee}$.

COROLLARY 39.14. *Let $k \geq 4$ be even, $k = 2\ell$, $\ell \geq 2$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Then, there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$, $(H_{0,j})_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_1$, $(H_{\ell,j})_{j\geq 1}$ in $\mathcal{H}_1 \otimes \mathcal{H}_k$ and, for $1 \leq i \leq \ell - 1$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{2i}$ such that, for $j \geq 1$, one has*

(39.12)

$$H_j = H_{0,j}^{+k-1} + \sum_{i=1}^{\ell-1} {}^{+2i}H_{i,j}{}^{+k-2i} + {}^{+k-1}H_{\ell,j} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.$$

PROOF. By Lemma 39.12, we get, for $j \geq 1$,

$$H_j = \sum_{i=0}^{\ell-1} {}^{+2i}H_{i,j}{}^{+k-2(i+1)} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}, \tag{39.13}$$

where $(F_j)_{j\geq 0}$ is a sequence in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ is a sequence in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$, and, for $0 \leq i \leq \ell - 1$, $(H_{i,j})_{j\geq 1}$ is a sequence in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{2(i+1)}$.

We apply the same procedure as in the proof of Lemma 39.12 by using Lemma 39.6 in $\otimes^2\mathcal{H}_{k+4}$. We keep the same notation. The only difference is that for $j \geq 1$ and $i = 0$, the tensor $N_{0,j}$ belongs to $\mathcal{H}_k \otimes \mathcal{H}_1$ and that, for $i = \ell - 1$, the tensor $P_{\ell-1,j}$ belongs to $\mathcal{H}_1 \otimes \mathcal{H}_k$. Therefore, we can not apply Lemma 37.8 in order to split them as in (39.11). Thus, the elements $Y_{i,j}$ and $V_{i,j}$ are only defined for $1 \leq i \leq \ell - 1$ and the $Z_{i,j}$ and $X_{i,j}$ are only defined for $0 \leq i \leq \ell - 2$.

We now get, for $j \geq 1$,

$$H_{0,j} = N_{0,j}^{\vee>\vee} + {}^{++}Z_{0,j} + J_{0,j}^{>} - J_{0,j+1}^{\vee>\vee} + {}^{\vee>}B_{0,j} - {}^{>\vee}B_{0,j-1}$$
$$H_{\ell-1,j} = Y_{\ell-1,j}^{++} + {}^{\vee>\vee}P_{\ell-1,j} + A_{\ell-1,j}^{>\vee} - A_{\ell-1,j-1}^{\vee>} + {}^{>}L_{\ell-1,j} - {}^{\vee>\vee}L_{\ell-1,j+1}$$

and, for $1 \leq i \leq \ell - 2$,

$$H_{i,j} = Y_{i,j}^{++} + {}^{++}Z_{i,j} + A_{i,j}^{>\vee} - A_{i,j-1}^{\vee>} + {}^{\vee>}B_{i,j} - {}^{>\vee}B_{i,j-1}.$$

Using this in (39.13) and applying Lemma 39.13 gives, for $j \geq 1$,

$$H_j = N_{0,j}^{\vee +^{k-1}} + \sum_{i=1}^{\ell-1} {}^{+^{2i}}(Z_{i-1,j} + Y_{i,j})^{+^{k-2i}} + {}^{+^{k-1}\vee}P_{\ell-1,j} + C_j^{>\vee} - C_{j-1}^{\vee>} + {}^{\vee>}D_j - {}^{>\vee}D_{j-1}.$$

with, for $j \geq 0$,

$$C_j = -J_{0,j+1}^{\vee +^{k-2}} + \sum_{i=1}^{\ell-1} {}^{+^{2i}}A_{i,j}{}^{+^{k-2(i+1)}} + F_j$$

$$\text{and } D_j = \sum_{i=0}^{\ell-2} {}^{+^{2i}}B_{i,j}{}^{+^{k-2(i+1)}} - {}^{+^{k-2}\vee}L_{\ell-1,j+1} + G_j.$$

Thus, (39.12) holds as required. □

In the same way, we can get a last step. Recall that we write $r(0) = 0$, $r(1) = 1$ and $r(i) = 2(i-1)$ for $i \geq 2$.

Corollary 39.15. *Let $k \geq 4$ be even, $k = 2\ell$, $\ell \geq 2$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$. Then, there exists finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ and, for $0 \leq i \leq \ell+1$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)}$ such that, for $j \geq 1$, one has*

(39.14)

$$H_j = H_{0,j}^{+^k} + {}^{+^2}H_{1,j}{}^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i-1)}} + {}^{+^{k-1}}H_{\ell,j}{}^{+^2} + {}^{+^k}H_{\ell+1,j} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.$$

Proof. We start with the decomposition (39.12) given by Corollary 39.15. Again, we apply the induction procedure of the proof of Lemma 39.12 which now relies on the use of Lemma 39.6 in $\otimes^2\mathcal{H}_{k+2}$. We keep the same notation, so that we first write, for $j \geq 1$ and $0 \leq i \leq \ell$,

$$H_{i,j} = N_{i,j}^{\vee>\vee} + {}^{\vee>\vee}P_{i,j} + J_{i,j}^{>} - J_{i,j+1}^{\vee>\vee} + {}^{>}L_{i,j} - {}^{\vee>\vee}L_{i,j+1},$$

where

$$\begin{array}{ll} N_{0,j}, J_{0,j} \in \mathcal{H}_k \otimes \mathcal{H}_0 & P_{0,j}, L_{0,j} \in \mathcal{H}_{k-1} \otimes \mathcal{H}_1 \\ N_{i,j}, J_{i,j} \in \mathcal{H}_{k-2i} \otimes \mathcal{H}_{2i-1} & P_{i,j}, L_{i,j} \in \mathcal{H}_{k-2i-1} \otimes \mathcal{H}_{2i} \quad 1 \leq i \leq \ell-1 \\ N_{\ell,j}, J_{\ell,j} \in \mathcal{H}_1 \otimes \mathcal{H}_{k-1} & P_{\ell,j}, L_{\ell,j} \in \mathcal{H}_0 \otimes \mathcal{H}_k. \end{array}$$

Now, the tensors $Y_{i,j}$ and $V_{i,j}$ may be defined for $2 \leq i \leq \ell$ and the tensors $Z_{i,j}$ and $X_{i,j}$ may be defined defined for $0 \leq i \leq \ell - 2$.

If $\ell \geq 3$, we get, for $\epsilon \in \{0, 1\}$,

$$H_{\epsilon,j} = N_{\epsilon,j}^{\vee>\vee} + {}^{++}Z_{\epsilon,j} + J_{\epsilon,j}^{>} - J_{\epsilon,j+1}^{\vee>\vee} + {}^{\vee>}B_{\epsilon,j} - {}^{>\vee}B_{\epsilon,j-1}$$
$$H_{\ell-\epsilon,j} = Y_{\ell-\epsilon,j}^{++} + {}^{\vee>\vee}P_{\ell-\epsilon,j} + A_{\ell-\epsilon,j}^{>\vee} - A_{\ell-\epsilon,j-1}^{\vee>} + {}^{>}L_{i,j} - {}^{\vee>\vee}L_{i,j+1}$$

and, for $2 \leq i \leq \ell - 2$,

$$H_{i,j} = Y_{i,j}^{++} + {}^{++}Z_{i,j} + A_{i,j}^{>\vee} - A_{i,j-1}^{\vee>} + {}^{\vee>}B_{i,j} - {}^{>\vee}B_{i,j-1}.$$

If $\ell = 2$, we have

$$H_{0,j} = N_{0,j}^{\vee>\vee} + {}^{++}Z_{0,j} + J_{0,j}^{>} - J_{0,j+1}^{\vee>\vee} + {}^{\vee>}B_{0,j} - {}^{>\vee}B_{0,j-1}$$
$$H_{1,j} = N_{1,j}^{\vee>\vee} + {}^{\vee>\vee}P_{1,j} + J_{1,j}^{>} - J_{1,j+1}^{\vee>\vee} + {}^{>}L_{1,j} - {}^{\vee>\vee}L_{1,j+1}$$
$$H_{2,j} = Y_{2,j}^{++} + {}^{\vee>\vee}P_{2,j} + A_{2,j}^{>\vee} - A_{2,j-1}^{\vee>} + {}^{>}L_{2,j} - {}^{\vee>\vee}L_{2,j+1}$$

In both cases, using this in (39.12) and applying Lemma 39.13 gives, for $j \geq 1$,

$$H_j = N_{0,j}^{\vee+^k} + {}^{+^2}(Z_{0,j} + N_{1,j}^{\vee})^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}(Z_{i-1,j} + Y_{i,j})^{+^{k-2(i-1)}}$$
$$+ {}^{+^{k-1}}({}^{\vee}P_{\ell-1,j} + Y_{\ell,j})^{+^2} + {}^{+^k\vee}P_{\ell,j} + C_j^{>\vee} - C_{j-1}^{\vee>}$$
$$+ {}^{\vee>}D_j - {}^{>\vee}D_{j-1}.$$

with, for $j \geq 0$,

$$C_j = -J_{0,j+1}^{\vee+^{k-1}} - {}^{+^2}J_{1,j+1}{}^{\vee+^{k-2}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}A_{i,j}{}^{+^{k-2i}} + {}^{+^{k-1}}A_{\ell,j} + F_j \text{ and}$$

$$D_j = B_{0,j}^{+^{k-1}} + \sum_{i=1}^{\ell-2} {}^{+^{2i}}B_{i,j}{}^{+^{k-2i}} - {}^{+^{k-2}\vee}L_{\ell-1,j+1}{}^{+^2} - {}^{+^{k-1}\vee}L_{\ell,j+1} + G_j.$$

Thus, we have established a decomposition of the form in (39.14) as required. □

We can now conclude.

Proof of Proposition 39.3. As for Proposition 37.12, this relies on an induction argument.

For $k = 1, 2$, the statement is a direct consequence of the definition of the objects and of Lemma 37.1.

We now prove that if $k \geq 4$ is even, $k = 2\ell$, $\ell \geq 2$, and if the statement holds for $k - 2$, it also holds for k. We apply Corollary

39.15. Therefore, we may write, for $j \geq 1$,

$$H_j = H_{0,j}^{+^k} + {}^{+^2}H_{1,j}{}^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i-1)}} + {}^{+^{k-1}}H_{\ell,j}{}^{+^2} + {}^{+^k}H_{\ell+1,j} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1},$$

where the $(F_j)_{j\geq 0}$ are in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, the $(G_j)_{j\geq 0}$ are in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ and, for $0 \leq i \leq \ell+1$, the $(H_{i,j})_{j\geq 1}$ are in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)}$ and all these sequences are finitely supported.

Since the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$, by Lemma 37.1, the assumption of Lemma 39.6 is satisfied. In particular, the conclusion of this result for $i = 0$ says that we may find finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{-1}$ and $(Q_j)_{j\geq 1}$ and $(R_j)_{j\geq 1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_0$ such that, for $j \geq 1$,

$$H_{0,j} = P_j^{>} + {}^{>}Q_j + {}^{\vee>}R_j = P_j^{>} + {}^{\vee>\vee}S_j + {}^{>}Q_j - {}^{\vee>\vee}Q_{j+1},$$

where $S_j = Q_{j+1} + {}^{\vee}R_j$. As $k \geq 4$, by Lemma 37.8 and Lemma 38.4, we can find finitely supported sequences $(T_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ in $\mathcal{H}_{k-2} \otimes \mathcal{H}_0$ such that, for $j \geq 1$,

$$S_j = {}^{>}T_j + {}^{\vee>}U_j = {}^{>}V_j + {}^{\vee>}U_j - {}^{>\vee}U_{j+1},$$

with $V_j = T_j + {}^{\vee}U_{j+1}$. We get, for $j \geq 1$,

$$H_{0,j} = P_j^{>} + {}^{++}V_j + {}^{\vee>}W_j - {}^{>\vee}W_{j-1},$$

where, for $j \geq 0$, $W_j = -{}^{\vee}Q_{j+1} - {}^{\vee>\vee}U_{j+1}$. Now, for $j \geq 0$, we set

$$X_j = \sum_{\substack{i\geq j+1 \\ i-j \text{ even}}} P_i - \sum_{\substack{i\geq j+1 \\ i-j \text{ odd}}} P_i^{\vee},$$

so that, for $j \geq 1$, we get $P_j = -X_j - X_{j-1}^{\vee}$, hence

$$H_{0,j} = {}^{++}V_j + {}^{\vee>}W_j - {}^{>\vee}W_{j-1} + X_j^{>\vee} - X_{j-1}^{\vee>}.$$

Reasoning in the same way from the conclusion of Lemma 39.6 for $i = \ell+1$, we obtain finitely supported sequences $(A_j)_{j\geq 1}$ in $\mathcal{H}_0 \otimes \mathcal{H}_{k-2}$, $(B_j)_{j\geq 0}$ in $\mathcal{H}_0 \otimes \mathcal{H}_{k-1}$ and $(C_j)_{j\geq 0}$ in $\mathcal{H}_{-1} \otimes \mathcal{H}_k$ such that, for $j \geq 1$,

$$H_{\ell-1,j} = A_j^{++} + B_j^{>\vee} - B_{j-1}^{\vee>} + {}^{\vee>}C_j - {}^{>\vee}C_{j-1}.$$

We set

$$J_j = V_j^{+^{k-2}} + H_{1,j}^{+^{k-3}} + \sum_{i=2}^{\ell-1} {}^{+^{2(i-1)}}H_{i,j}{}^{+^{k-2i}} + {}^{+^{k-3}}H_{\ell,j} + {}^{+^{k-2}}A_j,$$

which is an element of $\otimes^2\mathcal{H}_{k-2}$. By using Lemma 39.13, we can write

$$H_j = {}^{++}J_j{}^{++} + K_j^{>\vee} - K_{j-1}^{\vee>} + {}^{\vee>}L_j - {}^{>\vee}L_{j-1},$$

with

$$K_j = F_j + X_j^{+^k} + {}^{+^k}B_j \text{ and } L_j = G_j + W_j^{+^k} + {}^{+^k}C_j.$$

Therefore, as the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$, by Lemma 37.1, Propositon 39.4 and Lemma 39.8, the sequence $(\varpi_{k-2}(J_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{k-2}$. By induction, we may find finitely supported sequences $(M_j)_{j\geq 0}$ in $\mathcal{H}_{k-2}\otimes\mathcal{H}_{k-3}$ and $(N_j)_{j\geq 0}$ in $\mathcal{H}_{k-3}\otimes\mathcal{H}_{k-2}$ such that, for $j\geq 1$,

$$J_j = M_j^{>\vee} - M_{j-1}^{\vee>} + {}^{\vee>}N_j - {}^{>\vee}N_{j-1}.$$

By using Lemma 39.13, we get

$$\begin{aligned} H_j = (K_j + {}^{++}M_j{}^{++})^{>\vee} - (K_{j-1} + {}^{++}M_{j-1}{}^{++})^{\vee>} \\ + {}^{\vee>}(L_j + {}^{++}N_j{}^{++}) - {}^{>\vee}(L_{j-1} + {}^{++}N_{j-1}{}^{++}) \end{aligned}$$

and the conclusion follows.

So far, we have proved the Proposition for $k=1$ and for $k\geq 2$ even. It remains to prove it for $k\geq 3$ odd. For such a k, by Lemma 37.5 and Lemma 37.8, we can find finitely supported sequences $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $(L_j)_{j\geq 1}$ and $(M_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$ such that, for $j\geq 1$,

$$\begin{aligned} H_j &= J_j^{>} + K_j^{>\vee} + {}^{>}L_j + {}^{\vee>}M_j \\ &= N_j^{>} + {}^{>}P_j + K_j^{>\vee} - K_{j+1}^{\vee>} + {}^{\vee>}M_j - {}^{>\vee}M_{j+1}, \end{aligned}$$

where $N_j = J_j + K_{j+1}^{\vee}$ and $P_j = L_j + {}^{\vee}M_{j+1}$. Applying again Lemma 37.8, we can find finitely supported sequences $(U_j)_{j\geq 1}$, $(V_j)_{j\geq 1}$, $(W_j)_{j\geq 1}$ and $(X_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$ with, for $j\geq 1$,

$$\begin{aligned} N_j &= {}^{>}U_j + {}^{\vee>}V_j = {}^{>}Y_j + {}^{\vee>}V_j - {}^{>\vee}V_{j+1} \\ P_j &= W_j^{>} + X_j^{>\vee} = Z_j^{>} + X_j^{>\vee} - X_{j+1}^{\vee>}, \end{aligned}$$

where $Y_j = {}^{>}U_j + {}^{\vee}V_{j+1}$ and $Z_j = W_j + X_{j+1}^{\vee}$. We set $A_j = Y_j + Z_j$, $B_j = K_{j+1} + {}^{>}X_{j+1}$ and $C_j = M_{j+1} + V_{j+1}^{>}$ and we get

$$H_j = {}^{>}A_j{}^{>} + B_{j-1}^{>\vee} - B_j^{\vee>} + {}^{\vee>}C_{j-1} - {}^{>\vee}C_j.$$

As the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_k$, Lemma 37.1 and Proposition 39.4 say that the sequence $(\varpi_{k-1}(A_j))_{j\geq 1}$ is cohomologically trivial in $\mathscr{W}_{k-1}$. Therefore, as the Proposition is true for $k-1$, we can find finitely supported sequences $(D_j)_{j\geq 0}$ and $(E_j)_{j\geq 0}$ in in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-2}$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-2}$ such that, for $j\geq 1$,

$$A_j = D_j^{>\vee} - D_{j-1}^{\vee>} + {}^{\vee>}E_j - {}^{>\vee}E_{j-1}.$$

By using Lemma 20.6, we get

$$H_j = ({}^{>}D_j{}^{>} - B_j)^{\vee >} - ({}^{>}D_{j-1}{}^{>} - B_{j-1})^{>\vee} + {}^{>\vee}({}^{>}E_j{}^{>} - C_j) \\ - {}^{\vee >}({}^{>}E_{j-1}{}^{>} - C_{j-1})$$

and the conclusion follows. □

Part 12

Ultraweight and spectral obstructions

CHAPTER 40

The ultraweight map

We now come back to the study of the Plancherel formula of Proposition 35.16, which, as we have seen in the proof of Corollary 35.17, should be thought of as a universal tool for defining the spectral theory of non-negative Γ-invariant bilinear forms on $\overline{\mathcal{D}}(\partial X)$.

In the present Chapter, for $k \geq -1$, we will use this Plancherel formula to construct a linear map Ω_k, from the space $(\otimes^2\mathcal{H}_k)[t]$ of polynomial functions with values in $\otimes^2\mathcal{H}_k$, towards the space of (cohomology classes of) Hölder continuous functions on $\Gamma\backslash\mathscr{S}$. We will call Ω_k the ultraweight map. Later, in Chapter 42 we will use the ultraweight map to describe the bilinear forms on $\mathcal{H}_k^2[t]$ that are obtained from Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ through the spectral transform of Chapter 24.

40.1. Plancherel formula and the ultraweight map

We start by defining a new object that implicitely appears in the proof of Corollary 35.17.

The following definition is inspired by Proposition 35.16 and the latter proof.

Definition 40.1. Let $k \geq -1$ and H be in $(\otimes^2\mathcal{H}_k)[t]$, that is, H is a polynomial function with values in $\otimes^2\mathcal{H}_k$. We define the ultraweight $\Omega_k(H)$ as the Hölder continuous function on $\Gamma\backslash\mathscr{S}$ given by, for σ in $\mathscr{S}$,

$$\Omega_k(H)(\sigma) = \frac{2}{q+1}\int_{\mathcal{I}_q} \Phi_t\left(^{>\infty}H(t)^{>\infty}\right)(\sigma)\mathrm{d}\mu_q(t)$$
$$+\frac{q-1}{2(q+1)}\Phi_1^{\mathrm{sp}}\left(^{>\infty}H(1)^{>\infty}\right)(\sigma)+\frac{q-1}{2(q+1)}\Phi_{(-1)}^{\mathrm{sp}}\left(^{>\infty}H(-1)^{>\infty}\right)(\sigma).$$

See Definition 35.8 for the spectral fundamental bilinear maps Φ_t, $t \in I_q$, and Definition 35.13 for the special spectral fundamental bilinear maps Φ_1^{sp} and $\Phi_{(-1)}^{\mathrm{sp}}$. The fact that the formula makes sense and defines a Hölder continuous function on $\Gamma\backslash\mathscr{S}$ is a consequence of Lemma 35.9.

As for the fundamental bilinear maps, we will think to the ultraweight map as taking its values in the space of cohomology classes of Hölder continuous functions on $\Gamma\backslash\mathscr{S}$. In this sense, our purpose will now be to describe the null space of the ultraweight map: for $k \geq 1$, we let Θ_k be the space of those H in $(\otimes^2\mathcal{H}_k)[t]$ such that $\Omega_k(H)$ is a coboundary in the space of Hölder continuous functions on $\Gamma\backslash\mathscr{S}$ (in the sense of Section 32.1). We call the elements of Θ_k the k-coboundary polynomial tensors. The purpose of the remainder of the Chapter is to use the previously introduced tools to describe the space Θ_k (up to a finite-dimensional subspace).

40.2. The twist operator

We first construct an operator that preserves the ultraweight in case Γ is bipartite.

If V is a vector space and A is an algebra acting on V, the algebra $A \otimes A$ acts on $V \otimes V$ in a natural way: for v, w in V and a, b in A, we have $(a \otimes b)(v \otimes w) = (av) \otimes (bw)$. In our situation, we fix $k \geq 1$. Then, the space $\mathcal{H}_k$ is equipped with an action of the algebra A_1 of Γ-invariant functions on X_1. Indeed, if H is in $\mathcal{H}_k$ and u is in A_1, we let uH be the k-pseudofunction defined by

$$(uH)_{xy} = u(xy)H_{xy}, \quad xy \in X_1.$$

The algebra $A_1 \otimes A_1$ may be identified naturally with the algebra of $(\Gamma \times \Gamma)$-invariant functions on $X_1 \times X_1$, so that every such function defines an endomorphism of $\otimes^2\mathcal{H}_k$.

Let δ and ε be as in Section 34.2. We have a characterization of bipartite actions:

LEMMA 40.2. *The action of Γ on X is bipartite if and only if the function $\varepsilon(-1)^\delta$ on $X_1 \times X_1$ is $(\Gamma \times \Gamma)$-invariant.*

PROOF. Recall from Section 15.1 and Section 20.1 that saying that Γ is bipartite amounts to saying that the function $(a, x) \mapsto (-1)^{d(a,x)}$ is $(\Gamma \times \Gamma)$-invariant on $X \times X$. Now, in view of the definitions, for ab and xy in X_1, we have $(-1)^{d(a,x)} = \varepsilon(ab, xy)(-1)^{\delta(ab,xy)}$. The conclusion follows. □

Let the twist operation $\wr$ on pseudofunctions be defined as in Section 20.6. From the definitions, one directly gets

LEMMA 40.3. *Let $h, k \geq 1$ and H be in $\mathcal{H}_h \otimes \mathcal{H}_k$. We have ${}^\wr H^\wr = \varepsilon(-1)^\delta H$.*

By abuse of notation, for any $h, k \geq -1$, we will write $\varepsilon(-1)^{\delta} H$ instead of ${}^{\wr}H^{\wr}$ for H in $\mathcal{H}_h \otimes \mathcal{H}_k$.

We can relate multiplication by $\varepsilon(-1)^{\delta}$ and the natural operations.

LEMMA 40.4. *Assume Γ is bipartite. Let $h, k \geq -1$ and H be in $\mathcal{H}_h \otimes \mathcal{H}_k$. We have*

$${}^{\vee}(\varepsilon(-1)^{\delta} H) = (-1)^{k} \varepsilon(-1)^{\delta} ({}^{\vee}H) \text{ and } (\varepsilon(-1)^{\delta} H)^{\vee} = (-1)^{h} \varepsilon(-1)^{\delta} (H^{\vee})$$

as well as

$${}^{>}(\varepsilon(-1)^{\delta} H) = \varepsilon(-1)^{\delta} ({}^{>}H) \text{ and } (\varepsilon(-1)^{\delta} H)^{>} = \varepsilon(-1)^{\delta} (H^{>}).$$

PROOF. This can be obtained by a direct computation or by applying Lemma 20.22 and Lemma 40.3. □

Multiplication by $\varepsilon(-1)^{\delta}$ also behaves well with respect to the double weight map of Definition 39.1.

LEMMA 40.5. *Assume Γ is bipartite. Let $k \geq 1$, H be in $\otimes^2 \mathcal{H}_k$ and ab, xy be in X_k. We have*

$$\varpi_k(\varepsilon(-1)^{\delta} H)(ab, xy) = (-1)^{d(a,y)} \varpi_k(H)(ab, xy).$$

PROOF. It suffices to prove the claim when H is of the form $J \otimes K$ for J, K in $\mathcal{H}_k$.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. Then, by Definition 36.1 and Definition 39.1, we have

$$\begin{aligned}
\varpi_k(\varepsilon(-1)^{\delta} H)(ab, xy) &= (-1)^{d(a_\ell, y_\ell)} \Delta J_{a_\ell a_{\ell+1}}(b, a) \Delta K_{y_\ell y_{\ell+1}}(x, y) \\
&+ \sum_{1 \leq j \leq \ell-1} \sum_{\substack{z \in X \\ [y_{\ell+j} z] \cap [xy] = \{y_{\ell+j}\} \\ d(z, y_{\ell+j}) = j}} (-1)^{d(a_\ell, z)} \Delta J_{a_\ell a_{\ell+1}}(b, a) \Delta K_{z z_-}(x, y_{2i}) \\
&+ \sum_{1 \leq i \leq \ell-1} \sum_{\substack{c \in X \\ [a_{\ell+i} c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} (-1)^{d(c, y_\ell)} \Delta J_{c c_-}(b, a_{2i}) \Delta K_{y_\ell y_{\ell+1}}(x, y) \\
&+ \sum_{1 \leq i, j \leq \ell-1} \sum_{\substack{c, z \in X \\ [a_{\ell+i} c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i \\ [y_{\ell+j} z] \cap [xy] = \{y_{\ell+j}\} \\ d(z, y_{\ell+j}) = j}} (-1)^{d(c, z)} \Delta J_{c c_-}(b, a_{2i}) \Delta K_{z z_-}(x, y_{2i}).
\end{aligned}$$

For $1 \leq i, j \leq \ell - 1$ and c, z as above, we have

$$d(a_\ell, c) = 2i \text{ and } d(y_\ell, z) = 2j,$$

hence

$$(-1)^{d(a_\ell,z)} = (-1)^{d(c,y_\ell)} = (-1)^{d(c,z)} = (-1)^{d(a_\ell,y_\ell)} = (-1)^{d(a,y)}$$

and the conclusion follows.

The proof is the same in the odd case. □

Assume Γ is bipartite. In view of Lemma 40.2, we can construct an operator as follows. For $k \geq 1$ and H in $(\otimes^2 \mathcal{H}_k)[t]$, we define the twist $\widetilde{H}$ of H as the polynomial tensor defined by

$$\widetilde{H}(t) = \varepsilon(-1)^\delta H(-t).$$

This operator preserves the ultraweight.

LEMMA 40.6. *Assume Γ is bipartite. Let $k \geq 1$ and H be in $(\otimes^2 \mathcal{H}_k)[t]$. We have*

$$\Omega_k(\widetilde{H}) = \Omega_k(H).$$

This statement is a direct consequence of the parity properties of the polynomial functions A_j and B_j, $j \geq 0$, of Section 34.1.

PROOF. For $j \geq 0$, the definitions of A_j and B_j give $A_j(-t) = (-1)^j A_j(t)$ and $B_j(t) = (-1)^{j+1} B_j(-t)$, hence $\chi_{-t} = \varepsilon(-1)^\delta \chi_t$. Also, we have $\chi_{-1}^{\mathrm{sp}} = \varepsilon(-1)^\delta \chi_1^{\mathrm{sp}}$. The conclusion now follows from the definition of Ω_k and the fact that the measure μ_q on $\mathbb{R}$ is symmetric. □

40.3. Trivial coboundary polynomial tensors

We now give an a priori list of polynomial tensors which are annihilated by the ultraweight map.

DEFINITION 40.7. (k even) Let $k \geq 2$ be an even integer. Then, we define the space of k-trivial coboundary polynomial tensors $\Theta_k^0 \subset (\otimes^2 \mathcal{H}_k)[t]$ as the subspace of $(\otimes^2 \mathcal{H}_k)[t]$ spanned by the following polynomial tensors:

$$\begin{aligned} J \otimes K - K \otimes J \qquad & J, K \in \mathcal{H}_k[t] \\ H \qquad & H \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-})[t] \\ {}^{>}H^{>} \qquad & H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-})[t] \end{aligned}$$

as well as

$$\begin{aligned} (q+1)t\,{}^{>}H^{>} + (q-1)\,{}^{>}H^{\vee>} - 2\,{}^{>}H^{\vee>\vee} \\ H \in ((\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,+}) \oplus (\mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,-}))[t] \end{aligned}$$

and

$$(q+1)tH^{>>}+(q-1)H^{>\vee>}-H^{\vee>\vee>}-H^{>\vee>\vee} \qquad H\in(\mathcal{H}_k\otimes\mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H}-H,\quad H\in(\otimes^2\mathcal{H}_k)[t].$$

DEFINITION 40.8. (k odd) Let $k\geq 1$ be an odd integer. Then, we define the space of k-trivial coboundary polynomial tensors $\Theta_k^0\subset(\otimes^2\mathcal{H}_k)[t]$ as the subspace of $(\otimes^2\mathcal{H}_k)[t]$ spanned by the following polynomial tensors:

$$\begin{aligned} J\otimes K-K\otimes J &\qquad J,K\in\mathcal{H}_k[t]\\ H &\qquad H\in(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})[t]\\ {}^{>}H^{>} &\qquad H\in(\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-})[t]\end{aligned}$$

as well as

$$(q+1)t\,{}^{>}H^{>}+(q-1)\,{}^{>}H^{>\vee}-2\,{}^{>}H^{\vee>\vee} \qquad H\in((\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,+})\oplus(\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,-}))[t]$$

and

$$(q+1)tH^{>>}+(q-1)H^{\vee>>}-H^{\vee>\vee>}-H^{>\vee>\vee} \qquad H\in(\mathcal{H}_k\otimes\mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H}-H,\quad H\in(\otimes^2\mathcal{H}_k)[t].$$

Note that the two definitions are identical except in the fifth case. These uncomfortable definitions are justified by the following

LEMMA 40.9. *Let $k\geq 1$ and H be a trivial coboundary polynomial tensor in $(\otimes^2\mathcal{H}_k)[t]$. Then, the Hölder continuous function $\Omega_k(H)$ is a coboundary.*

This is a consequence of the properties established in Lemma 35.10 and Lemma 35.15. To prove this precisely, we will need

LEMMA 40.10. *Let $0<\alpha<1$, $I\subset\mathbb{R}$ be a closed interval and $\varphi: I\to\mathscr{H}_\alpha$ be a continuous function such that, for any t in I, $\varphi(t)$ is a coboundary. Then the Hölder continuous function $\int_I\varphi(t)\mathrm{d}t$ is a coboundary.*

PROOF. Indeed, it follows from Livšic Theorem, Proposition 32.1, that the space of coboundaries is a closed subspace of $\mathscr{H}_\alpha$. □

In the context of the Plancherel formula for the fundamental bilinear maps, we will use the previous under the following form.

COROLLARY 40.11. *Let $h \geq 0$ and $k \geq -1$ be integers. For every H, J in $\mathcal{H}_k$, the ultraweight $\Omega_k(t^h H \otimes J)$ of the polynomial tensor $t^h H \otimes J$ is cohomologous to $\Phi(P^h H^{>\infty}, J^{>\infty})$.*

See Definition 35.2 for the global fundamental bilinear map Φ.

PROOF. Indeed, it follows from Lemma 35.9, Lemma 35.10 and Lemma 40.10 that the function

$$\int_{\mathcal{I}_q} \left(t^h \Phi_t \left(H^{>\infty}, J^{>\infty} \right) - \Phi_t \left(P^h H^{>\infty}, J^{>\infty} \right) \right) \mathrm{d}\mu_q(t)$$

is a coboundary. Then, the claim follows from Lemma 35.15 and Proposition 35.16. □

We note that this yields

COROLLARY 40.12. *Let $k \geq -1$ and H be a polynomial tensor in $(\otimes^2 \mathcal{H}_k)[t]$. Then the ultraweight $\Omega_k(H)$ is cohomologous to a smooth function on $\Gamma \backslash \mathscr{S}$.*

REMARK 40.13. In particular, if θ is a T-invariant distribution on $\Gamma \backslash \mathscr{S}$, for H in $(\otimes^2 \mathcal{H}_k)[t]$, we can define $\langle \theta, \Omega_k(H) \rangle$ by means of the convention of Remark 32.8.

PROOF OF LEMMA 40.9. We will only deal with the case where k is even, the odd case being analogous. We will check that in each of the first five cases of Definition 40.7, the ultraweight is a coboundary, whereas in the sixth case, the ultraweight is actually 0 by Lemma 40.6.

Let $h \geq 0$ be an integer.

Let J, K be in $\mathcal{H}_k$. By Corollary 40.11, $\Omega_k(t^h J \otimes K - t^h K \otimes J)$ is cohomologous to $\Phi(P^h J^{>\infty}, K^{>\infty}) - \Phi(P^h K^{>\infty}, J^{>\infty})$, which is a coboundary by Lemma 35.3.

In particular, if J is in $\mathcal{H}_{k,+}$ and K is in $\mathcal{H}_{k,-}$, then, as above $\Omega_k(t^h J \otimes K)$ is cohomologous to $\Phi(P^h J^{>\infty}, K^{>\infty})$. As $J^\vee = qJ$ and $K^\vee = -K$ and P commutes to R and S, we get, by Lemma 35.3,

$$\begin{aligned} \Phi(P^h J^{>\infty}, K^{>\infty}) &= \frac{1}{q} \Phi(P^h J^{\vee > \infty}, K^{>\infty}) = \frac{1}{q} \Phi(P^h R J^{>\infty}, K^{>\infty}) \\ = \frac{1}{q} \Phi(R P^h J^{>\infty}, K^{>\infty}) &\equiv \frac{1}{q} \Phi(P^h J^{>\infty}, R K^{>\infty}) = \frac{1}{q} \Phi(P^h J^{>\infty}, K^{\vee > \infty}) \\ &= -\frac{1}{q} \Phi(P^h J^{>\infty}, K^{>\infty}) \end{aligned}$$

(where we have written $\equiv$ for the cohomology equivalence relation). This gives $\Omega_k(t^h J \otimes K) \equiv 0$ as required.

In the same way, if J is in $\mathcal{H}_{k-1,+}$ and K is in $\mathcal{H}_{k-1,-}$, since $\Omega_k(t^h J^> \otimes K^>) = \Omega_{k-1}(t^h J \otimes K)$, we get $\Omega_k(t^h J^> \otimes K^>) \equiv 0$.

Let now J, K be both in $\mathcal{H}_{k-1,+}$ or both in $\mathcal{H}_{k-1,-}$. Still by Corollary 40.11, we have

$$\begin{aligned}\Omega_k(t^h((q+1)tJ^> \otimes K^> + (q-1)J^> \otimes K^{\vee>} - 2J^> \otimes K^{\vee>\vee}))\\ \equiv (q+1)\Phi(P^{h+1}J^{>\infty}, K^{>\infty}) + (q-1)\Phi(P^h J^{>\infty}, SK^{>\infty})\\ - 2\Phi(P^h J^{>\infty}, RSK^{>\infty}).\end{aligned}$$

As $(q+1)P = RS + SR - (q-1)S$, using Lemma 35.3, we obtain

$$\begin{aligned}\Omega_k(t^h((q+1)tJ^> \otimes K^> + (q-1)J^> \otimes K^{\vee>} - 2J^> \otimes K^{\vee>\vee}))\\ \equiv \Phi(P^h RSJ^{>\infty}, K^{>\infty}) - \Phi(P^h J^{>\infty}, RSK^{>\infty})\end{aligned}$$

If J, K are in $\mathcal{H}_{k-1,+}$, we have $SJ^{>\infty} = J^{>\infty}$ and $SK^{>\infty} = K^{>\infty}$ and the latter is a coboundary. If they are in $\mathcal{H}_{k-1,-}$, we have $SJ^{>\infty} = -J^{>\infty}$ and $SK^{>\infty} = -K^{>\infty}$ and the same holds. Thus, in both cases, we get

$$\Omega_k(t^h((q+1)tJ^> \otimes K^> + (q-1)J^> \otimes K^{\vee>} - 2J^> \otimes K^{\vee>\vee})) \equiv 0$$

as required.

Finally, we take J in $\mathcal{H}_k$ and K in $\mathcal{H}_{k-2}$. By Corollary 40.11, we have

$$\begin{aligned}\Omega_k(t^h((q+1)tJ \otimes K^{>>} + (q-1)J \otimes K^{>\vee>} - J \otimes K^{\vee>\vee>} - J \otimes K^{>\vee>\vee}))\\ \equiv \Phi((q+1)P^{h+1}J^{>\infty}, K^{>\infty})\\ + \Phi(P^h J^{>\infty}, (q-1)SK^{>\infty} - SRK^{>\infty} - RSK^{>\infty}).\end{aligned}$$

As $(q+1)P = RS + SR - (q-1)S$, the conclusion follows from Lemma 35.3. □

Recall that, for $k \geq 1$, we let $\Theta_k \subset (\otimes^2 \mathcal{H}_k)[t]$ be the set of those polynomial tensors H in $(\otimes^2 \mathcal{H}_k)[t]$ such that the Hölder continuous function $\Omega_k(H)$ on $\Gamma\backslash\mathscr{S}$ is a coboundary. We have just shown that we have $\Theta_k^0 \subset \Theta_k$. The next statement says that the reverse inclusion is true up to a finite-dimensional subspace.

PROPOSITION 40.14. *There exists an integer $n \geq 0$ such that, for any $k \geq 2$, we have $\Theta_k \subset \Theta_k^0 + (\otimes^2 \mathcal{H}_k)_{k+n}[t]$, that is, for every H in $(\otimes^2 \mathcal{H}_k)[t]$, if $\Omega_k(H)$ is a coboundary, we may find a trivial coboundary tensor J in Θ_k^0 such that $H - J$ has degree $\leq k+n$ in t.*

The proof of this statement will rely on Proposition 33.3. It will last until the end of the next Chapter.

40.4. Endpoints series formulas for the ultraweight

In order to be able to use Proposition 33.3 to analyse the vanishing of the ultraweight in cohomology, we translate the formulas of Chapter 36 in the language of the ultraweight.

Let V be a vector space. Recall that a tensor in $\otimes^2 V$ is said to be symmetric if it is invariant by the natural involution of $\otimes^2 V$ that maps $v_1 \otimes v_2$ to $v_2 \otimes v_1$ for v_1, v_2 in V. The space of symmetric tensors is denoted by $\mathrm{S}^2 V \subset \otimes^2 V$.

We still let $(C_j)_{j\geq 0}$ be the family of orthogonal polynomials of Section 34.6. From Proposition 36.3 and Proposition 36.4, we get

PROPOSITION 40.15. *Let $k \geq 1$ and H be a symmetric polynomial tensor in $(\mathrm{S}^2\mathcal{H}_k)[t]$. For $j \geq 0$, let H_j be the element of $\mathrm{S}^2\mathcal{H}_k$ defined by*

$$H_j = \int_{\mathcal{I}_q} C_j(t) H(t) \mathrm{d}\mu_q(t).$$

Assume k is even. Then, if H is in $(\mathrm{S}^2\mathcal{H}_{k,+})[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\sigma \mapsto \frac{1}{2}\kappa_{0,k}(H_0) + \frac{1}{2}\sum_{j=1}^{k-2} q^{-(j-1)} \kappa_{j,k}(H_j)(\sigma_0 \sigma_{j+k}) - \sum_{j=k-1}^{\infty} q^{-(j-1)} \varpi_k(H_j)(\sigma_0\sigma_k, \sigma_j \sigma_{j+k}).$$

If H is in $(\mathrm{S}^2\mathcal{H}_{k,-})[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\sigma \mapsto \frac{1}{2q}\kappa_{0,k}\left(qH_0 + (q-1)\sum_{\substack{h>0 \\ h \text{ even}}} H_h\right) - \frac{1}{2}\sum_{j=1}^{k-2} q^{-(j+1)}\kappa_{j,k}\left(q^2 H_j + (q^2-1)\sum_{\substack{h>j \\ j-h \text{ even}}} H_h\right)(\sigma_0\sigma_{j+k}) + \sum_{j=k-1}^{\infty} q^{-(j+1)}\varpi_k\left(q^2 H_j + (q^2-1)\sum_{\substack{h>j \\ j-h \text{ even}}} H_h\right)(\sigma_0\sigma_k, \sigma_j\sigma_{j+k}).$$

Assume k is odd. Then, if H is in $(\mathrm{S}^2\mathcal{H}_{k,+})[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\sigma \mapsto \frac{1}{q+1}\kappa_{0,k}\left((q+1)H_0 + (q-1)\sum_{h>0}(-1)^h H_h\right) + \frac{2}{q+1}\sum_{j=1}^{k-2} q^{-j}\kappa_{j,k}\left(qH_j + (q-1)\sum_{h>j}(-1)^{j-h}H_h\right)(\sigma_0\sigma_{j+k}) - \frac{1}{q+1}\sum_{j=k-1}^{\infty} q^{-j}\varpi_k\left(qH_j + (q-1)\sum_{h>j}(-1)^{j-h}H_h\right)(\sigma_0\sigma_k, \sigma_j\sigma_{j+k}).$$

If H is in $(\mathrm{S}^2\mathcal{H}_{k,-})[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\sigma \mapsto \frac{1}{q+1}\kappa_{0,k}\left((q+1)H_0 + (q-1)\sum_{h>0} H_h\right) - \frac{2}{q+1}\sum_{j=1}^{k-2} q^{-j}\kappa_{j,k}\left(qH_j + (q-1)\sum_{h>j} H_h\right)(\sigma_0\sigma_{j+k}) + \frac{1}{q+1}\sum_{j=k-1}^{\infty} q^{-j}\varpi_k\left(qH_j + (q-1)\sum_{h>j} H_h\right)(\sigma_0\sigma_k, \sigma_j\sigma_{j+k}).$$

Note that, by Lemma 34.6, we have $H_j = 0$ for j large, so that all the sums above are finite and define smooth functions on $\Gamma\backslash\mathscr{S}$.

Remark 40.16. The formulas above only concern symmetric polynomial tensors in $(\otimes^2\mathcal{H}_{k,+}) \oplus (\otimes^2\mathcal{H}_{k,-})$. Indeed, due to the form of trivial polynomial tensors in Definition 40.7 and Definition 40.8, and to Lemma 40.9, the ultraweight of skew symmetric polynomial tensors and of polynomial tensors in $(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})\oplus(\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+})$ is a coboundary. Therefore, for proving Proposition 40.14, we will only need to deal with symmetric polynomial tensors H such that $H^\vee = {}^\vee H$.

The proof uses the following formula for computing the values of a polynomial function at 1 and (-1) by means of its components in the basis $(C_j)_{j\geq 0}$.

Lemma 40.17. *Let f in $\mathbb{R}[t]$ be a polynomial function. For $j \geq 0$, set $f_j = \int_{\mathcal{I}_q} f(t)C_j(t)\mathrm{d}\mu_q(t)$. Then we have*

$$f(1) = \sum_{j\geq 0} f_j \text{ and } f(-1) = \sum_{j\geq 0}(-1)^j f_j.$$

PROOF. Since by Lemma 34.6, the $(C_j)_{j\geq 0}$ form a basis of $\mathbb{R}[t]$, it suffices to check the formulas when f is one of them.

Now, the definition of these polynomials gives $C_0 = 1$ and, for $j \geq 1$,

$$C_j(1) = B_{j+1}(1) - B_{j-1}(1) = \frac{q^{j+1}-1}{q-1} - \frac{q^{j-1}-1}{q-1} = \frac{q^2-1}{q-1}q^{j-1}$$
$$= (q+1)q^{j-1} = \int_{\mathcal{I}_q} C_j(t)^2 \mathrm{d}\mu_q(t),$$

where the last equality follows from Lemma 34.6. Thus, the first formula holds. The second is obtained in the same way. □

PROOF OF PROPOSITION 40.15. This is a direct computation. We start with the endpoints formulas obtained in Proposition 36.3 and Proposition 36.5, and we apply the construction of the ultraweight in Definition 40.1, the fact that integrals of coboundary are coboundaries which was shown in Lemma 40.10, and Lemma 40.17 above which allows to compute the values at (-1) and 1 of the polynomial tensors. □

CHAPTER 41

Building trivial coboundary tensors

In the present Chapter, we will prove Proposition 40.14. Thus, for $k \geq 1$, we are given a polynomial tensor H in $(\otimes^2 \mathcal{H}_k)[t]$ whose ultraweight $\Omega_k(H)$ is a coboundary and we want to build a trivial coboundary tensor J such that $H - J$ has degree $\leq k + n$ for some fixed n.

Our strategy is to apply Proposition 33.3 to the formulas in Proposition 40.15. Then, we will use Proposition 39.3 to say more about the form of sequences related to the coefficients $H_j = \int_{\mathcal{I}_q} C_j(t) H(t) \mathrm{d}\mu_q(t)$ and relate them to the analogue sequences for trivial coboundary tensors.

41.1. Cohomological equations and eigenvectors

First, we analyze precisely the fact that the sequences of polynomial tensors we will need to study from Proposition 40.15 have values in $(\otimes^2 \mathcal{H}_{k,+}) \oplus (\otimes^2 \mathcal{H}_{k,-})$, $k \geq 1$ (see Remark 40.16).

PROPOSITION 41.1. *Let $k \geq 1$ and $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be finitely supported sequences in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1} \otimes H_k$. For $j \geq 1$, we set*

$$\begin{aligned} H_j &= F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1} && \text{if } k \text{ is even} \\ &= F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1} && \text{if } k \text{ is odd} \end{aligned}$$

and we assume that H_j belongs to $(\otimes^2 \mathcal{H}_{k,+}) \oplus (\otimes^2 \mathcal{H}_{k,-})$, that is, ${}^{\vee}H_j = H_j^{\vee}$.

Then, if k is even, there exist finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, $(Q_j)_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$ such that for $j \geq 1$, one has $P_j^{\vee} = {}^{\vee}P_j$, $Q_j^{\vee} = {}^{\vee}Q_j$ and

$$\begin{aligned} {}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} &= Q_j^{>} - (q-1)\,{}^{>}X_j + {}^{\vee>}(X_j + Y_j) \\ G_j^{\vee} - (q-1)G_j + {}^{\vee}G_{j-1} &= {}^{>}P_j - (q-1)Y_j^{>} + (X_j + Y_j)^{>\vee}. \end{aligned}$$

If k is odd, there exist finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, $(Q_j)_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and $(Y_j)_{j\geq 1}$ and $(Z_j)_{j\geq 1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$ such

that for $j \geq 1$, one has $P_j^\vee = {}^\vee P_j$, $Q_j^\vee = {}^\vee Q_j$ and

$$F_j^\vee + {}^\vee F_{j-1} = Q_j^> + {}^> Y_j - {}^{\vee>} Z_j$$
$${}^\vee G_j + G_{j-1}^\vee = {}^> P_j - Y_j^> - Z_j^{>\vee}.$$

The proof is a straightforward consequence of the results of Section 38.4 on the simplification of tensor equations. We split it according to the parity of k.

LEMMA 41.2. *Let $k \geq 2$ be even, F_0, F_1 be in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and G_0, G_1 be in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$. We set*

$$H = F_1^{>\vee} - F_0^{\vee>} + {}^{\vee>} G_1 - {}^{>\vee} G_0.$$

Assume that we have ${}^\vee H = H^\vee$, that is, H belongs to $(\otimes^2 \mathcal{H}_{k,+}) \oplus (\otimes^2 \mathcal{H}_{k,-})$. Then, there exist P in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, Q in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and X, Y in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$ such that $P^\vee = {}^\vee P$, $Q^\vee = {}^\vee Q$ and

$${}^\vee F_1 - (q-1)F_1 + F_0^\vee = Q^> - (q-1)\,{}^> X + {}^{\vee>}(X+Y)$$
$$G_1^\vee - (q-1)G_1 + {}^\vee G_0 = {}^> P - (q-1)Y^> + (X+Y)^{>\vee}.$$

PROOF. We set

$$J = {}^\vee F_1 - (q-1)F_1 + F_0^\vee$$
$$K = G_1^\vee - (q-1)G_1 + {}^\vee G_0.$$

The relation ${}^\vee H = H^\vee$ gives

$${}^\vee F_1^{>\vee} - {}^\vee F_0^{\vee>} + q\,{}^> G_1 + (q-1)\,{}^{\vee>} G_1 - {}^{\vee>\vee} G_0 = qF_1^> + (q-1)F_1^{>\vee} - F_0^{\vee>\vee} + {}^{\vee>} G_1^\vee - {}^{>\vee} G_0^\vee$$

which we rewrite as

$${}^\vee J^> - J^{>\vee} = {}^> K^\vee - {}^{\vee>} K.$$

By Corollary 38.8, we may find U in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, V in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and W, X, Y, Z in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$, such that

$$K^\vee = {}^{>\vee} U + W^> + X^{>\vee} \qquad K = {}^> U - Y^> - Z^{>\vee}$$
$${}^\vee J = V^{\vee>} + {}^> W + {}^{\vee>} Y \qquad J = V^> - {}^> X - {}^{\vee>} Z. \tag{41.1}$$

By comparing the above two expressions for J and K, we get

$${}^{>\vee} U + W^> + X^{>\vee} = {}^> U^\vee - qZ^> - (Y + (q-1)Z)^{>\vee}$$
$$V^{\vee>} + {}^> W + {}^{\vee>} Y = {}^\vee V^> - q\,{}^> Z - {}^{\vee>}(X + (q-1)Z).$$

By Corollary 38.9, we may find A, E, F in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-2}$ and B, C, D in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-1}$ with

$$\begin{aligned} W + qZ &= A^{\vee >} + {}^{>}B = {}^{>\vee}D + E^{>} \\ X + Y + (q-1)Z &= -A^{>} + {}^{>}C = -{}^{>}D + F^{>} \\ U^{\vee} - {}^{\vee}U &= B^{>} + C^{>\vee} \\ {}^{\vee}V - V^{\vee} &= {}^{>}E + {}^{\vee>}F. \end{aligned} \tag{41.2}$$

From the above, we get in particular

$$(A^{\vee} - E)^{>} = {}^{>}({}^{\vee}D - B) \text{ and } (A + F)^{>} = {}^{>}(C + D).$$

Thus, by Lemma 38.4, we can find L, M in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-2}$ with

$$\begin{aligned} E &= A^{\vee} - {}^{>}L & B &= {}^{\vee}D - L^{>} \\ F &= {}^{>}M - A & C &= M^{>} - D. \end{aligned}$$

Using these relations in (41.2) gives

$$\begin{aligned} W + qZ &= A^{\vee >} + {}^{>\vee}D - {}^{>}L^{>} \\ X + Y + (q-1)Z &= -A^{>} - {}^{>}D + {}^{>}M^{>} \\ U^{\vee} - {}^{\vee}U &= {}^{\vee}D^{>} - D^{>\vee} - L^{>>} + M^{\vee >>} \\ {}^{\vee}V - V^{\vee} &= {}^{>}A^{\vee} - {}^{\vee>}A - {}^{>>}L + {}^{>>\vee}M. \end{aligned} \tag{41.3}$$

The last two relations imply that we may find P in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, Q in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and R, S in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-2}$ such that

$$\begin{aligned} P^{\vee} &= {}^{\vee}P & Q^{\vee} &= {}^{\vee}Q \\ U + D^{>} &= P + R^{>>} & V + {}^{>}A &= Q + {}^{>>}S \\ R^{\vee} - {}^{\vee}R &= -L + M^{\vee} & {}^{\vee}S - S^{\vee} &= -L + {}^{\vee}M. \end{aligned} \tag{41.4}$$

We set $T = R + S - M$. Note that the last two relations give $T^{\vee} = {}^{\vee}T$. To simplify the next expressions, we set

$$\begin{aligned} P_1 &= P + \frac{1}{q-1} T^{>>\vee} \\ Q_1 &= Q + \frac{1}{q-1} {}^{\vee>>}T \\ X_1 &= X + A^{>} - {}^{>}S^{>} \\ Y_1 &= Y + {}^{>}D - {}^{>}R^{>}. \end{aligned}$$

In particular, by (41.3), we get

$$X_1 + Y_1 + (q-1)Z + {}^{>}T^{>} = 0$$

and, by (41.1), (41.2) and (41.4),

$$J = Q^{>} - {}^{>}X_1 - {}^{\vee>}Z = Q_1^{>} - {}^{>}X_1 + \frac{1}{q-1}\,{}^{\vee>}(X_1 + Y_1)$$
$$K = {}^{>}P - Y_1^{>} - Z^{>\vee} = {}^{>}P_1 - Y_1^{>} + \frac{1}{q-1}(X_1 + Y_1)^{>\vee}$$

as required. □

In the odd case, the same technique yields

LEMMA 41.3. *Let $k \geq 1$ be odd, F_0, F_1 be in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and G_0, G_1 be in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$. We set*

$$H = F_1^{\vee>} - F_0^{>\vee} + {}^{>\vee}G_1 - {}^{\vee>}G_0.$$

Assume that we have ${}^{\vee}H = H^{\vee}$, that is, H belongs to $(\otimes^2\mathcal{H}_{k,+}) \oplus (\otimes^2\mathcal{H}_{k,-})$. Then, there exist P in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, Q in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and Y, Z in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$ such that $P^{\vee} = {}^{\vee}P$, $Q^{\vee} = {}^{\vee}Q$ and

$$F_1^{\vee} + {}^{\vee}F_0 = Q^{>} + {}^{>}Y - {}^{\vee>}Z$$
$${}^{\vee}G_1 + G_0^{\vee} = {}^{>}P - Y^{>} - Z^{>\vee}.$$

PROOF. The proof follows the same lines as the previous one. We set

$$J = F_1^{\vee} + {}^{\vee}F_0 \text{ and } K = {}^{\vee}G_1 + G_0^{\vee}.$$

As above, the relation ${}^{\vee}H = H^{\vee}$ gives

$${}^{\vee}J^{>} - J^{>\vee} = {}^{>}K^{\vee} - {}^{\vee>}K.$$

By Corollary 38.8, we get U in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, V in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and W, X, Y, Z in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$, such that

$$K^{\vee} = {}^{>\vee}U + W^{>} + X^{>\vee} \qquad K = {}^{>}U - Y^{>} - Z^{>\vee} \tag{41.5}$$
$${}^{\vee}J = V^{\vee>} + {}^{>}W + {}^{\vee>}Y \qquad J = V^{>} - {}^{>}X - {}^{\vee>}Z.$$

We compare the two expressions for J and K and we get

$${}^{>\vee}U + W^{>} + X^{>\vee} = {}^{>}U^{\vee} - Z^{>} - Y^{>\vee}$$
$$V^{\vee>} + {}^{>}W + {}^{\vee>}Y = {}^{\vee}V^{>} - {}^{>}Z - {}^{\vee>}X.$$

By Corollary 38.9, we may find A, E, F in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-2}$ and B, C, D in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-1}$ with

$$W + Z = A^{\vee>} + {}^{>}B = {}^{>\vee}D + E^{>} \tag{41.6}$$
$$X + Y = -A^{>} + {}^{>}C = -{}^{>}D + F^{>}$$
$$U^{\vee} - {}^{\vee}U = B^{>} + C^{>\vee}$$
$${}^{\vee}V - V^{\vee} = {}^{>}E + {}^{\vee>}F.$$

From the above, we get in particular

$$(A^\vee - E)^> = {}^>({}^\vee D - B) \text{ and } (A + F)^> = {}^>(C + D).$$

Thus, by Lemma 38.4, we can find L, M in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-2}$ with

$$E = A^\vee - {}^>L \qquad B = {}^\vee D - L^>$$
$$F = {}^>M - A \qquad C = M^> - D.$$

Using these relations in (41.6) gives

$$\begin{aligned} W + Z &= A^{\vee>} + {}^{>\vee}D - {}^>L^> \\ X + Y &= -A^> - {}^>D + {}^>M^> \\ U^\vee - {}^\vee U &= {}^\vee D^> - D^{>\vee} - L^{>>} + M^{\vee>>} \\ {}^\vee V - V^\vee &= {}^>A^\vee - {}^{\vee>}A - {}^{>>}L + {}^{>>\vee}M. \end{aligned} \tag{41.7}$$

The last two relations imply that we may find P in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, Q in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and R, S in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-2}$ such that

$$\begin{aligned} P^\vee &= {}^\vee P & Q^\vee &= {}^\vee Q \\ U + D^> &= P + R^{>>} & V + {}^>A &= Q + {}^{>>}S \\ R^\vee - {}^\vee R &= -L + M^\vee & {}^\vee S - S^\vee &= -L + {}^\vee M. \end{aligned} \tag{41.8}$$

We set $T = R + S - M$. Note that the last two relations give $T^\vee = {}^\vee T$. To simplify the next expressions, we set

$$X_1 = X + A^> - {}^>S^> \text{ and } Y_1 = Y + {}^>D - {}^>R^>,$$

so that we get, by (41.7)

$$X_1 + Y_1 + {}^>T^> = 0$$

and, by (41.5), (41.6) and (41.8),

$$\begin{aligned} J &= Q^> - {}^>X_1 - {}^{\vee>}Z = (Q + {}^{>>}T)^> + {}^>Y_1 - {}^{\vee>}Z \\ K &= {}^>P - Y_1^> - Z^{>\vee} \end{aligned}$$

as required. □

Proof of Proposition 41.1. This directly follows from Lemma 41.2 and 41.3 that we apply for each $j \geq 1$. □

41.2. Building trivial coboundary tensors from $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$

We are still aiming at proving Proposition 40.14. We will now use the particular form of the sequences appearing in Proposition 41.1 to relate them to the trivial coboundary polynomial tensors of Definition 40.7 and Definition 40.8. Unfortunately, this will require us to check different cases separately. In the present Section, we show

LEMMA 41.4. *Let $k \geq 1$ and $(K_j)_{j\geq 1}$ and $(L_j)_{j\geq 1}$ be finitely supported sequences in $\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}$ and $\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-}$. Let $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be the unique finitely supported sequences in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ such that, for $j \geq 1$, one has, if k is even,*

$$(41.9)\quad {}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} = K_j^{>} \text{ and } {}^{\vee}G_j - (q-1)G_j + G_{j-1}^{\vee} = L_j^{>}$$

and, if k is odd,

$$(41.10)\qquad F_j^{\vee} + {}^{\vee}F_{j-1} = K_j^{>} \text{ and } G_j^{\vee} + {}^{\vee}G_{j-1} = L_j^{>}.$$

Then, there exist trivial coboundary polynomial tensors H in $(\otimes^2 \mathcal{H}_{k,+})[t]$ and J in $(\otimes^2 \mathcal{H}_{k,-})[t]$ such that, for $j \geq 1$, one has, if k is even,

$$F_j^{>\vee} - F_{j-1}^{\vee>} = q^{-j} H_j$$
$$G_j^{>\vee} - G_{j-1}^{\vee>} = q^{-j} J_j + q^{-j}(1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} J_h$$

and, if k is odd,

$$F_j^{\vee>} - F_{j-1}^{>\vee} = q^{-j} H_j + q^{-j}(1-q^{-1}) \sum_{h>j} (-1)^{j-h} H_h$$
$$G_j^{\vee>} - G_{j-1}^{>\vee} = q^{-j} J_j + q^{-j}(1-q^{-1}) \sum_{h>j} J_h,$$

where

$$H_j = \int_{\mathcal{I}_q} C_j(t) H(t) \mathrm{d}\mu_q(t) \text{ and } J_j = \int_{\mathcal{I}_q} C_j(t) J(t) \mathrm{d}\mu_q(t).$$

Note that the sequences $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ are uniquely defined by (41.9) and (41.10) due to the easy

LEMMA 41.5. *Let V be a vector space and $\theta : V \to V$ be an endomorphism. Then, for any finitely supported sequence $(x_j)_{j\geq 1}$ in V, there exists a unique finitely supported sequence $(y_j)_{j\geq 0}$ in V such that, for $j \geq 1$, one has $y_{j-1} = x_j + \theta y_j$.*

PROOF. Indeed, for $h \geq j \geq 0$, one has necessarily

$$y_j = \sum_{i=j+1}^{h} \theta^{i-j-1} x_i + \theta^{h-j} y_h.$$

Since both sequences are finitely supported, this gives

$$y_j = \sum_{i=j+1}^{\infty} \theta^{i-j-1} x_i,$$

hence the uniqueness. The existence follows from the fact that the latter formula actually defines a finitely supported sequence. □

We shall use the following formulas which can be seen as a consequence of the fact that the polynomial functions $(C_j)_{j\geq 0}$ of Section 34.6 are the spherical transforms of the spheres of the tree X in the language of [**21**].

LEMMA 41.6. *For $j \geq 0$, the polynomial function $(q+1)tC_j(t)$ may be written as*

$$\begin{aligned}
(q+1)tC_0(t) &= C_1(t)\\
(q+1)tC_1(t) &= C_2(t) + (q+1)C_0(t)\\
(q+1)tC_j(t) &= C_{j+1}(t) + qC_{j-1}(t) \qquad\qquad j \geq 2.
\end{aligned}$$

PROOF. This directly follows from the definitions in Section 34.6. The first relation is obvious. The third relation follows from the definition of C_j, $j \geq 1$, and the analogue property of B_j, $j \geq 1$ in Lemma 34.1. Finally, for proving the second relation, we write $(q+1)t = \frac{q}{u} + u$, hence

$$\begin{aligned}
C_2(t) = B_3(t) - B_1(t) = \frac{q^2}{u^2} + q + u^2 - 1 &= (q+1)^2t^2 - (q+1)\\
&= (q+1)tC_1(t) - (q+1)C_0(t)
\end{aligned}$$

as required. □

We now start the proof of the main Lemma, which we split according to the parity of k.

PROOF OF LEMMA 41.4 IN CASE k IS EVEN. We first construct H. We look for H to be of the form

$$H(t) = (q+1)tP(t)^{>>} - P(t)^{>\vee>} - P(t)^{>\vee>\vee},$$

where P is in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+})[t]$. Note that H is then a trivial coboundary polynomial tensor in view of Definition 40.7. For $j \geq 0$, we set $P_j = \int_{\mathcal{I}_q} C_j(t)P(t)\mathrm{d}\mu_q(t)$. By Lemma 41.6, we have

$$\begin{aligned}
H_0 &= P_1^{>>} - P_0^{>\vee>} - P_0^{>\vee>\vee}\\
H_1 &= P_2^{>>} + (q+1)P_0^{>>} - P_1^{>\vee>} - P_1^{>\vee>\vee}\\
H_j &= P_{j+1}^{>>} + qP_{j-1}^{>>} - P_j^{>\vee>} - P_j^{>\vee>\vee}, \qquad\qquad j \geq 2.
\end{aligned}$$

We set

$$\begin{aligned}
M_0 &= q^{-1}P_1^{>} - q^{-1}(q+1)P_0^{>\vee}\\
M_j &= q^{-j-1}P_{j+1}^{>} - q^{-j}P_j^{>\vee} \qquad\qquad j \geq 1,
\end{aligned}$$

so that, for $j \geq 1$, as ${}^{\vee}P_j = P_j^{\vee} = qP_j$, we have

$$q^{-j}H_j = M_j^{>\vee} - M_{j-1}^{\vee>}.$$

To conclude, it suffices to choose P in order to have $M_j = F_j$ for any $j \geq 0$. In view of the definition of the $(F_j)_{j\geq 0}$ in the statement, we compute

$$\begin{aligned}
{}^{\vee}M_1 - (q-1)M_1 + M_0^{\vee} &= q^{-2}P_2^{>} - q^{-1}(q+1)P_0^{>} \\
{}^{\vee}M_j - (q-1)M_j + M_{j-1}^{\vee} &= q^{-j-1}P_{j+1}^{>} - q^{1-j}P_{j-1}^{>} \qquad j \geq 2,
\end{aligned}$$

where we have used again the assumption that ${}^{\vee}P_j = qP_j$, $j \geq 0$. Thus, for the conclusion of the Lemma to hold, it suffices to have

$$q^{-2}P_2 - q^{-1}(q+1)P_0 = K_1 \text{ and } q^{-j-1}P_{j+1} - q^{1-j}P_{j-1} = K_j, \quad j \geq 2.$$

By Lemma 41.5, these equations uniquely define the finitely supported sequence $(P_j)_{j\geq 0}$, hence the polynomial tensor $P(t)$. The conclusion follows.

We now construct J, which we will seek to be of the form

$$J(t) = (q+1)tQ(t)^{>>} + qQ(t)^{>\vee>} - Q(t)^{>\vee>\vee},$$

where Q is in $(\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-})[t]$. As above, J is then trivial in view of Definition 40.7. For $j \geq 0$, we set $Q_j = \int_{\mathcal{I}_q} C_j(t)Q(t)\mathrm{d}\mu_q(t)$. By Lemma 41.6, we have

$$\begin{aligned}
J_0 &= Q_1^{>>} + qQ_0^{>\vee>} - Q_0^{>\vee>\vee} \\
J_1 &= Q_2^{>>} + (q+1)Q_0^{>>} + qQ_1^{>\vee>} - Q_1^{>\vee>\vee} \\
J_j &= Q_{j+1}^{>>} + qQ_{j-1}^{>>} + qQ_j^{>\vee>} - Q_j^{>\vee>\vee}, \qquad j \geq 2.
\end{aligned}$$

Now, we set

$$N_0 = -Q_1^{>} - (1+q^{-1})Q_0^{>\vee} - (1-q^{-2}) \sum_{\substack{h>0 \\ h \text{ even}}} Q_{h+1}^{>} + Q_h^{>\vee}$$

and, for $j \geq 1$,

$$N_j = -q^{-j}(Q_{j+1}^{>} + Q_j^{>\vee}) - q^{-j}(1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} Q_{h+1}^{>} + Q_h^{>\vee}.$$

A direct computation gives, for $j \geq 1$, as ${}^{\vee}Q_j = Q_j^{\vee} = -Q_j$,

$$q^{-j}J_j + q^{-j}(1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} J_h = N_j^{>\vee} - N_{j-1}^{\vee>}.$$

We will choose Q in order to have $N_j = G_j$ for any $j \geq 0$. By using again the assumption that ${}^\vee Q_j = -Q_j$, $j \geq 0$, we get

$$\begin{aligned} {}^\vee N_1 - (q-1)N_1 + N_0^\vee &= q^{-2}Q_2^> - q^{-1}(q+1)Q_0^> \\ {}^\vee N_j - (q-1)N_j + N_{j-1}^\vee &= q^{-j-1}Q_{j+1}^> - q^{1-j}Q_{j-1}^> \qquad j \geq 2. \end{aligned}$$

By Lemma 41.5, the equations

$$q^{-2}Q_2 - q^{-1}(q+1)Q_0 = L_1 \text{ and } q^{-j-1}Q_{j+1} - q^{1-j}Q_{j-1} = L_j, \quad j \geq 2.$$

uniquely define the finitely supported sequence $(Q_j)_{j\geq 0}$, hence the polynomial tensor $Q(t)$ and we are done. □

Proof of Lemma 41.4 in case k is odd. Now, we look for H to be of the form

$$H(t) = ((q+1)t + (q-1))P(t)^{>>} - P(t)^{>\vee>} - P(t)^{>\vee>\vee},$$

where P is in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+})[t]$. By Definition 40.8, this is a trivial polynomial tensor. For $j \geq 0$, we set $P_j = \int_{\mathcal{I}_q} C_j(t)P(t)\mathrm{d}\mu_q(t)$. By Lemma 41.6, we have

$$\begin{aligned} H_0 &= P_1^{>>} + (q-1)P_0^{>>} - P_0^{>\vee>} - P_0^{>\vee>\vee} \\ H_1 &= P_2^{>>} + (q-1)P_1^{>>} + (q+1)P_0^{>>} - P_1^{>\vee>} - P_1^{>\vee>\vee} \\ H_j &= P_{j+1}^{>>} + (q-1)P_j^{>>} + qP_{j-1}^{>>} - P_j^{>\vee>} - P_j^{>\vee>\vee}, \qquad j \geq 2. \end{aligned}$$

A tedious computation shows that, by setting

$$M_0 = -(1+q^{-1})P_0^> + q^{-1}P_1^{>\vee} - q^{-1}(1-q^{-1})\sum_{h>1}(-1)^h(P_h^{>\vee} + P_h^>)$$

and, for $j \geq 1$,

$$M_j = -q^{-j}P_j^> + q^{-j-1}P_{j+1}^{>\vee} - q^{-j-1}(1-q^{-1})\sum_{h>j+1}(-1)^{j-h}(P_h^{>\vee} + P_h^>),$$

we get, for $j \geq 1$,

$$q^{-j}H_j + q^{-j}(1-q^{-1})\sum_{h>j}(-1)^{j-h}H_h = M_j^{\vee>} - M_{j-1}^{>\vee}.$$

As above, we want to get $M_j = F_j$ for any $j \geq 0$, so that we compute

$$\begin{aligned} M_1^\vee + {}^\vee M_0 &= q^{-2}P_2^> - q^{-1}(q+1)P_0^> \\ M_j^\vee + {}^\vee M_{j-1} &= q^{-j-1}P_{j+1}^> - q^{1-j}P_{j-1}^> \qquad j \geq 2. \end{aligned}$$

Using Lemma 41.5, we define the finitely supported sequence $(P_j)_{j\geq 0}$, hence the polynomial tensor $P(t)$, by

$$q^{-2}P_2 - q^{-1}(q+1)P_0 = K_1 \text{ and } q^{-j-1}P_{j+1} - q^{1-j}P_{j-1} = K_j, \quad j \geq 2$$

and we are done.

Finally we search for a J of the form

$$J(t) = ((q+1)t - (q-1))Q(t)^{>>} + Q(t)^{>\vee>} - Q(t)^{>\vee>\vee},$$

where Q is in $(\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-})[t]$. Still by Definition 40.8, the polynomial tensor J is then trivial. For $j \geq 0$, we set $Q_j = \int_{\mathcal{I}_q} C_j(t)Q(t)\mathrm{d}\mu_q(t)$. By Lemma 41.6, we have

$$\begin{aligned}
J_0 &= Q_1^{>>} - (q-1)Q_0^{>>} + Q_0^{>\vee>} - Q_0^{>\vee>\vee} \\
J_1 &= Q_2^{>>} - (q-1)Q_1^{>>} + (q+1)Q_0^{>>} + Q_1^{>\vee>} - Q_1^{>\vee>\vee} \\
J_j &= Q_{j+1}^{>>} - (q-1)Q_j^{>>} + qQ_{j-1}^{>>} + Q_j^{>\vee>} - Q_j^{>\vee>\vee}, \qquad j \geq 2.
\end{aligned}$$

In this case, we set

$$N_0 = (1+q^{-1})Q_0^{>} + q^{-1}Q_1^{>\vee} + q^{-1}(1-q^{-1})\sum_{h>1} Q_h^{>\vee} + Q_h^{>}$$

and, for $j \geq 1$,

$$N_j = q^{-j}Q_j^{>} + q^{-j-1}Q_{j+1}^{>\vee} + q^{-j-1}(1-q^{-1})\sum_{h>j+1} Q_h^{>\vee} + Q_h^{>},$$

and we get, for $j \geq 1$,

$$q^{-j}J_j + q^{-j}(1-q^{-1})\sum_{h>j} J_h = N_j^{\vee>} - N_{j-1}^{>\vee}.$$

In order to get $N_j = G_j$ for any $j \geq 0$, we compute

$$\begin{aligned}
N_1^{\vee} + {}^{\vee}N_0 &= q^{-2}Q_2^{>} - q^{-1}(q+1)Q_0^{>} \\
N_j^{\vee} + {}^{\vee}N_{j-1} &= q^{-j-1}Q_{j+1}^{>} - q^{1-j}Q_{j-1}^{>} \qquad j \geq 2.
\end{aligned}$$

As above we use Lemma 41.5, to define the finitely supported sequence $(Q_j)_{j\geq 0}$, hence the polynomial tensor $Q(t)$, by

$$q^{-2}Q_2 - q^{-1}(q+1)Q_0 = L_1 \text{ and } q^{-j-1}Q_{j+1} - q^{1-j}Q_{j-1} = L_j, \quad j \geq 2.$$

The result follows. □

41.3. Building trivial coboundary tensors from $(\otimes^2\mathcal{H}_{k-1,+}) \oplus (\otimes^2\mathcal{H}_{k-1,-})$

We continue preparing the proof of Proposition 40.14. Now, for $k \geq 1$, still in order to reconstruct the formulas from Proposition 41.1, we will use a second kind of trivial polynomial tensors in $(\otimes^2\mathcal{H}_k)[t]$, namely the ones coming from $(\otimes^2\mathcal{H}_{k-1})[t]$ in Definition 40.7 and Definition 40.8. We split the statements according to the parity of k.

LEMMA 41.7. *Let $k\geq 2$ be even and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ be finitely supported sequences in $\otimes^2\mathcal{H}_{k-1}$. Let $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be the unique finitely supported sequences in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$ such that, for $j\geq 1$, one has,*

$$^{\vee}F_j-(q-1)F_j+F_{j-1}^{\vee}=-(q-1)\,{}^{>}X_j+{}^{\vee>}(X_j+Y_j)$$
$$G_j^{\vee}-(q-1)G_j+{}^{\vee}G_{j-1}=-(q-1)Y_j^{>}+(X_j+Y_j)^{>\vee}.$$

Then, there exist a trivial coboundary polynomial tensor H in $(\otimes^2\mathcal{H}_k)[t]$ such that, for $j\geq 2$, one has,

$$q^j(F_j^{>\vee}-F_{j-1}^{\vee>}+{}^{\vee>}G_j-{}^{>\vee}G_{j-1})=(q+1)({}^{\vee}H_j+H_j^{\vee})$$
$$-(q^2-1)H_j-(1-q^{-2})\sum_{\substack{h>j\\ j-h\text{ even}}}{}^{\vee}H_h^{\vee}-q({}^{\vee}H_h+H_h^{\vee})+q^2H_h,$$

where $H_j=\int_{\mathcal{I}_q}C_j(t)H(t)\mathrm{d}\mu_q(t)$.

LEMMA 41.8. *Let $k\geq 1$ be odd and $(Y_j)_{j\geq 1}$ and $(Z_j)_{j\geq 1}$ be finitely supported sequences in $\otimes^2\mathcal{H}_{k-1}$. Let $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be the unique finitely supported sequences in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$ such that, for $j\geq 1$, one has,*

$$F_j^{\vee}+{}^{\vee}F_{j-1}={}^{>}Y_j-{}^{\vee>}Z_j$$
$${}^{\vee}G_j+G_{j-1}^{\vee}=-Y_j^{>}-Z_j^{>\vee}.$$

Then, there exist a trivial coboundary polynomial tensor H in $(\otimes^2\mathcal{H}_k)[t]$ such that, for $j\geq 2$, one has,

$$q^j(F_j^{\vee>}-F_{j-1}^{>\vee}+{}^{>\vee}G_j-{}^{\vee>}G_{j-1})=$$
$${}^{\vee}H_j+H_j^{\vee}+(1-q^{-1})\sum_{\substack{h>j\\ j-h\text{ even}}}{}^{\vee}H_h+H_h^{\vee}-{}^{\vee}H_{h-1}^{\vee}-H_{h-1},$$

where $H_j=\int_{\mathcal{I}_q}C_j(t)H(t)\mathrm{d}\mu_q(t)$.

The complicated structure of the formulas above comes from the need of applying them to the sequences appearing in Proposition 40.15.

To prove Lemma 41.7 and Lemma 41.8, we will need to use the fact that in Proposition 39.3, the sequences $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ are not uniquely determined by the sequence $(H_j)_{j\geq 1}$.

LEMMA 41.9. *Let $k\geq 0$ and $(B_j)_{j\geq 0}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$.*

If k is odd, for $j\geq 0$, we set

$$F_j={}^{>\vee}B_j-{}^{\vee>}B_{j+1}\text{ and }G_j=B_{j+1}^{>\vee}-B_j^{\vee>}.$$

Then, for $j \geq 1$, we have

$$F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1} = 0$$

and

$$\begin{aligned}{}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} = -{}^{>}((q-1){}^{\vee}B_j + qB_{j+1} - {}^{\vee}B_{j-1}^{\vee}) \\ + {}^{\vee>}({}^{\vee}B_j - B_j^{\vee})\end{aligned}$$

as well as

$$\begin{aligned}G_j^{\vee} - (q-1)G_j + {}^{\vee}G_{j-1} = ((q-1)B_j^{\vee} + qB_{j+1} - {}^{\vee}B_{j-1}^{\vee})^{>} \\ + ({}^{\vee}B_j - B_j^{\vee})^{>\vee}.\end{aligned}$$

If k is even, for $j \geq 0$, we set

$$F_j = {}^{\vee>}B_j - {}^{>\vee}B_{j+1} \text{ and } G_j = B_{j+1}^{\vee>} - B_j^{>\vee}.$$

Then, for $j \geq 1$, we have

$$F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1} = 0$$

and

$$\begin{aligned}F_j^{\vee} + {}^{\vee}F_{j-1} &= {}^{>}(B_{j-1} - {}^{\vee}B_{j+1}^{\vee}) - {}^{\vee>}({}^{\vee}B_j - B_j^{\vee}) \\ {}^{\vee}G_j + G_{j-1}^{\vee} &= -(B_{j-1} - {}^{\vee}B_{j+1}^{\vee})^{>} - ({}^{\vee}B_j - B_j^{\vee})^{>\vee}.\end{aligned}$$

Proof. These are direct computations. □

We will split the proofs of Lemma 41.7 and 41.8 according to whether the considered sequences belong to certain eigenspaces of the natural operators.

Proof of Lemma 41.7 when ${}^{\vee}X_j = X_j^{\vee}$ and ${}^{\vee}Y_j = Y_j^{\vee}$, $j \geq 1$. First assume that $X_j + Y_j = 0$ for $j \geq 1$. By Lemma 41.5, we know that there exists a unique finitely supported sequence $(B_j)_{j\geq 0}$ in $(\otimes^2\mathcal{H}_{k-1,+}) \oplus (\otimes^2\mathcal{H}_{k-1,-})$ such that, for $j \geq 1$, one has

$$-\frac{1}{q-1}B_{j-1} + {}^{\vee}B_j + \frac{q}{q-1}B_{j+1} = X_j.$$

Then, the conclusion directly follows from Lemma 41.9.

Now, assume that $X_j = Y_j$ for $j \geq 1$. We will seek for $H(t)$ to be of the form

$$H(t) = (q+1)t\,{}^{>}A(t)^{>} + (q-1)\,{}^{>}A(t)^{\vee>} - 2\,{}^{>}A(t)^{\vee>\vee}$$

where $A(t)$ is a polynomial tensor in $((\otimes^2\mathcal{H}_{k-1,+})\oplus(\otimes^2\mathcal{H}_{k-1,-}))[t]$. This is a trivial polynomial tensor in view of Definition 40.7. For $j \geq 0$, we set $A_j = \int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t)$ so that Lemma 41.6 gives, for $j \geq 2$,

$$H_j = {}^{>}A_{j+1}^{>} + q\,{}^{>}A_{j-1}^{>} + (q-1)\,{}^{>}A_j^{\vee>} - 2\,{}^{>}A_j^{\vee>\vee}.$$

For $j\geq 2$, we set

$$J_j=(q+1)({}^{\vee}H_j+H_j^{\vee})-(q^2-1)H_j \\ -(1-q^{-2})\sum_{\substack{h>j\\ j-h \text{ even}}} {}^{\vee}H_h^{\vee}-q({}^{\vee}H_h+H_h^{\vee})+q^2H_h.$$

A direct computation gives

$$q^{-j}J_j=F_j^{>\vee}-F_{j-1}^{\vee>}+{}^{\vee>}G_j-{}^{>\vee}G_{j-1},$$

where, for $j\geq 0$,

$$q^jF_j= \\ \frac{q^2-1}{2}\,{}^{>}A_j^{\vee}-(q+1)\,{}^{\vee>}A_j^{\vee}+\frac{(q+1)(q^2+1)}{2q}\,{}^{>}A_{j+1}-\frac{q^2-1}{2q}\,{}^{\vee>}A_{j+1} \\ +\frac{(q-1)(q+1)^2}{2q}\sum_{\substack{h>j\\ j-h \text{ even}}} q\,{}^{>}A_h^{\vee}-{}^{\vee>}A_h^{\vee}+q\,{}^{>}A_{h+1}-{}^{\vee>}A_{h+1}$$

and

$$q^jG_j= \\ \frac{q^2-1}{2}\,{}^{\vee}A_j^{>}-(q+1)\,{}^{\vee}A_j^{>\vee}+\frac{(q+1)(q^2+1)}{2q}A_{j+1}^{>}-\frac{q^2-1}{2q}A_{j+1}^{>\vee} \\ +\frac{(q-1)(q+1)^2}{2q}\sum_{\substack{h>j\\ j-h \text{ even}}} q\,{}^{\vee}A_h^{>}-{}^{\vee}A_h^{>\vee}+qA_{h+1}^{>}-A_{h+1}^{>\vee}.$$

In particular, this gives, for $j\geq 2$,

$$q^j({}^{\vee}F_j-(q-1)F_j+F_{j-1}^{\vee})=\frac{q(q^2-1)}{2}\,{}^{>}A_{j-1}-q(q+1)\,{}^{\vee>}A_{j-1} \\ -\frac{q^2-1}{2q}\,{}^{>}A_{j+1}+\frac{q+1}{q}\,{}^{\vee>}A_{j+1}$$

and the symmetric relation

$$q^j(G_j^{\vee}-(q-1)G_j+{}^{\vee}G_{j-1})=\frac{q(q^2-1)}{2}A_{j-1}^{>}-q(q+1)A_{j-1}^{>\vee} \\ -\frac{q^2-1}{2q}A_{j+1}^{>}+\frac{q+1}{q}A_{j+1}^{>\vee}.$$

Thus, to conclude, it suffices to ensure that, for $j\geq 2$, one has

$$q^jX_j=\frac{q+1}{2q}A_{j+1}-\frac{q(q+1)}{2}A_{j-1},$$

which is possible by Lemma 41.5. □

We manage the odd case in an analogous way.

Proof of Lemma 41.8 when ${}^{\vee}Y_j = Y_j^{\vee}$ and ${}^{\vee}Z_j = Z_j^{\vee}$, $j \geq 1$. If $Z_j = 0$ for any $j \geq 1$, then, by Lemma 41.5, there exists a unique finitely supported sequence $(B_j)_{j\geq 0}$ in $(\otimes^2 \mathcal{H}_{k-1,+}) \oplus (\otimes^2 \mathcal{H}_{k-1,-})$ such that, for $j \geq 1$, one has

$$B_{j-1} - {}^{\vee}B_{j+1}^{\vee} = Y_j$$

and as above, the conclusion directly follows from Lemma 41.9.

We now assume $Y_j = 0$ for any $j \geq 1$ and we will construct $H(t)$ of the form

$$H(t) = (q+1)t\,{}^{>}A(t)^{>} + (q-1)\,{}^{>}A(t)^{>\vee} - 2\,{}^{>}A(t)^{\vee>\vee}$$

where $A(t)$ is a polynomial tensor in $((\otimes^2 \mathcal{H}_{k-1,+}) \oplus (\otimes^2 \mathcal{H}_{k-1,-}))[t]$. This is a trivial polynomial tensor in view of Definition 40.8. As usual, for $j \geq 0$, we set $A_j = \int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t)$. By Lemma 41.6, we get, for $j \geq 2$,

$$H_j = {}^{>}A_{j+1}^{>} + q\,{}^{>}A_{j-1}^{>} + (q-1)\,{}^{>}A_j^{>\vee} - 2\,{}^{>}A_j^{\vee>\vee}.$$

For $j \geq 2$, we set

$$J_j = {}^{\vee}H_j + H_j^{\vee} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} {}^{\vee}H_h + H_h^{\vee} - {}^{\vee}H_{h-1}^{\vee} - H_{h-1}.$$

A direct computation gives

$$q^{-j}J_j = F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1},$$

where, for $j \geq 0$,

$$q^j F_j = -\,{}^{>}A_j + \frac{1}{q}\,{}^{\vee>}A_{j+1}^{\vee} + \frac{q-1}{q^2} \sum_{\substack{h>j \\ j-h \text{ even}}} -\,{}^{>}A_h - {}^{>}A_h^{\vee} + {}^{\vee>}A_{h+1} + {}^{\vee>}A_{h+1}^{\vee}$$

and

$$q^j G_j = -A_j^{>} + \frac{1}{q}\,{}^{\vee}A_{j+1}^{>\vee} + \frac{q-1}{q^2} \sum_{\substack{h>j \\ j-h \text{ even}}} -A_h^{>} - {}^{\vee}A_h^{>} + A_{h+1}^{>\vee} + {}^{\vee}A_{h+1}^{>\vee}.$$

For $j\geq 2$, we obtain

$$q^j(F_j^\vee + {}^\vee F_{j-1}) = -q\,{}^{\vee>}A_{j-1} + \frac{1}{q}\,{}^{\vee>}A_{j+1}$$
$$q^j({}^\vee G_j + G_{j-1}^\vee) = -qA_{j-1}^{>\vee} + \frac{1}{q}A_{j+1}^{>\vee}.$$

By Lemma 41.5, we may choose the $(A_j)_{j\geq 0}$ in order to get, for $j\geq 1$,

$$q^{-(j-1)}A_{j-1} - q^{-(j+1)}A_{j+1} = Z_j$$

and the conclusion follows. □

41.4. Building trivial coboundary tensors from $(\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-})\oplus(\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+})$

We now aim at proving Lemma 41.7 and Lemma 41.8 for sequences in $(\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-})\oplus(\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+})$. We will follow the same method as above, which consists into simplifying the proof by removing cases where one can apply Lemma 41.9. Unfortunately, the formulas in the remaining cases are less easy to handle than before. To compensate this, we show

LEMMA 41.10. *Let $k\geq 1$ be odd and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ be finitely supported sequences in $(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})\oplus(\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+})$. Then, there exist finitely supported sequences $(A_j)_{j\geq 0}$ and $(B_j)_{j\geq 0}$ in $(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})\oplus(\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+})$ such that, for $j\geq 1$, one has*

$$q^j(X_j+Y_j) = (q+1)A_j + \frac{q^2-1}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} A_h + 2q^j\,{}^\vee B_j$$

and

$$q^j(X_j-Y_j) = 2\frac{q(q+1)}{q-1}\,{}^\vee A_{j-1} + \frac{(q+1)^2}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} {}^\vee A_{h-1} + \frac{2q}{q-1}q^{j-1}B_{j-1} + \frac{2}{q-1}q^{j+1}B_{j+1}.$$

PROOF. We set $X_0 = 0 = Y_0$. By Lemma 41.5, there exist unique finitely supported sequences $(U_j)_{j\geq -1}$ and $(V_j)_{j\geq -1}$ in $(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})\oplus(\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+})$ such that, for $j\geq 0$, one has

$$\frac{1}{q}U_{j+1} - qU_{j-1} = \frac{q-1}{2(q+1)}q^j(X_j+Y_j)$$
$$\frac{1}{q}V_{j+1} - qV_{j-1} = \frac{q-1}{2(q+1)}q^j(X_j-Y_j).$$

For $j \geq 0$, we set

$$A_j = \frac{2q}{q-1} U_{j-1} + \frac{2}{q-1} U_{j+1} - 2\,{}^{\vee}V_j$$

and

$$q^j B_j = -2\frac{q(q+1)}{q-1}\,{}^{\vee}U_{j-1} - \frac{(q+1)^2}{q} \sum_{\substack{h>j \\ j-h \text{ even}}} {}^{\vee}U_{h-1} + (q+1)V_j + \frac{q^2-1}{q} \sum_{\substack{h>j \\ j-h \text{ even}}} V_h.$$

Straightforward computations show that the conclusion holds. □

In the even case, we have

LEMMA 41.11. *Let $k \geq 0$ be even and $(Y_j)_{j\geq 1}$ and $(Z_j)_{j\geq 1}$ be finitely supported sequences in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+})$. Then, there exist finitely supported sequences $(A_j)_{j\geq 0}$ and $(B_j)_{j\geq 0}$ in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+})$ such that, for $j \geq 1$, one has*

$$q^{j+1} Y_j = \frac{q-1}{2} A_{j+1} - A_{j+1}^{\vee} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} \frac{q-1}{2} A_{h+1} - A_{h+1}^{\vee} + q^{j+1} B_{j-1} - q^{j+1} B_{j+1}$$

and

$$q^{j+1} Z_j = qA_j + \frac{q^2-1}{2q} \sum_{\substack{h>j \\ j-h \text{ even}}} A_h + q^{j+1}((q-1)B_j - 2B_j^{\vee}).$$

PROOF. We set $Y_0 = 0 = Z_0$. By Lemma 41.5, there exist unique finitely supported sequences $(V_j)_{j\geq -1}$ and $(W_j)_{j\geq -1}$ in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+})$ such that, for $j \geq 0$, one has

$$V_{j+1} - q^2 V_{j-1} = q^{j+1} Y_j \text{ and } W_{j+1} - q^2 W_{j-1} = q^{j+1} Z_j.$$

For $j \geq 0$, we set

$$A_j = (q-1)V_j - 2V_j^{\vee} - qW_{j-1} - W_{j+1}$$

and

$$q^{j+1}B_j = -qV_j - \frac{q^2-1}{2q}\sum_{\substack{h>j\\ j-h \text{ even}}} V_h + \frac{q-1}{2}W_{j+1} - W_{j+1}^{\vee} + (1-q^{-1})\sum_{\substack{h>j\\ j-h \text{ even}}} \frac{q-1}{2}W_{h+1} - W_{h+1}^{\vee}.$$

Again, the result follows by direct computations. □

We use the above decompositions to finish the proofs Lemma 41.7 and Lemma 41.8.

Proof of Lemma 41.7 when ${}^{\vee}X_j = -X_j^{\vee}$ and ${}^{\vee}Y_j = -Y_j^{\vee}$, $j \geq 1$. We split the proof according to the decomposition given by Lemma 41.10.

First, assume that there exists a finitely supported sequence $(B_j)_{j\geq 0}$ in $(\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-})\oplus(\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+})$ such that, for $j \geq 1$, one has

$$X_j = \frac{1}{q-1}B_{j-1} + {}^{\vee}B_j + \frac{q}{q-1}B_{j+1}$$
$$\text{and } Y_j = -\frac{1}{q-1}B_{j-1} + {}^{\vee}B_j - \frac{q}{q-1}B_{j+1}.$$

In that case, the conclusion directly follows from Lemma 41.9.

Thus, by Lemma 41.10, we are reduced to deal with the case when there exists a finitely supported sequence $(A_j)_{j\geq 0}$ in $(\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-})\oplus(\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+})$ such that, for $j \geq 1$, one has

$$q^j(X_j+Y_j) = (q+1)A_j + \frac{q^2-1}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} A_h \tag{41.11}$$
$$q^j(X_j-Y_j) = 2\frac{q(q+1)}{q-1}{}^{\vee}A_{j-1} + \frac{(q+1)^2}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} {}^{\vee}A_{h-1}.$$

Then, we let $A(t)$ be the polynomial tensor defined by

$$\int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t) = A_j, \quad j \geq 0.$$

As this polynomial tensor belongs to $((\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})\oplus(\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+}))[t]$, the polynomial tensor $H(t) = {}^{>}A(t)^{>}$ is trivial in view of Definition

40.7. For $j \geq 0$, we set

$$J_j = (q+1)({}^{\vee}H_j + H_j^{\vee}) - (q^2-1)H_j - (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} {}^{\vee}H_h^{\vee} - q({}^{\vee}H_h + H_h^{\vee}) + q^2 H_h.$$

A direct computation gives

$$q^{-j}J_j = F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1},$$

where, for $j \geq 0$,

$$q^j F_j = (q+1)\,{}^{>}A_j + (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} q\,{}^{>}A_h - \frac{1}{2}\,{}^{\vee>}A_h - \frac{q}{2}\,{}^{>\vee}A_{h-1}$$

$$q^j G_j = (q+1)A_j^{>} + (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} qA_h^{>} - \frac{1}{2}A_h^{>\vee} - \frac{q}{2}A_{h-1}^{\vee>}.$$

Using (41.11) yields the conclusion. □

Proof of Lemma 41.8 when ${}^{\vee}Y_j{}^{\vee} = -qY_j$ and ${}^{\vee}Z_j{}^{\vee} = -qZ_j$, $j \geq 1$. We now split the proof according to the decomposition given by Lemma 41.11.

We first assume there exists a finitely supported sequence $(B_j)_{j\geq 0}$ in $(\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-}) \oplus (\mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,+})$ such that, for $j \geq 1$, one has

$$Y_j = B_{j-1} - B_{j+1} \text{ and } Z_j = (q-1)B_j - 2B_j^{\vee}.$$

Then, the conclusion directly follows from Lemma 41.9.

By Lemma 41.11, it remains to manage the case when there exists a finitely supported sequence $(A_j)_{j\geq 0}$ in $(\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-}) \oplus (\mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,+})$ such that, for $j \geq 1$, one has

(41.12)

$$q^{j+1}Y_j = \frac{q-1}{2}A_{j+1} - A_{j+1}^{\vee} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} \frac{q-1}{2}A_{h+1} - A_{h+1}^{\vee}$$

$$q^{j+1}Z_j = qA_j + \frac{q^2-1}{2q} \sum_{\substack{h>j \\ j-h \text{ even}}} A_h.$$

As above, we let $A(t)$ be the polynomial tensor defined by

$$\int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t) = A_j, \quad j \geq 0.$$

By Definition 40.8, the polynomial tensor $H(t) = {}^{>}A(t)^{>}$ is trivial. For $j \geq 0$, we set

$$J_j = {}^{\vee}H_j + H_j^{\vee} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} {}^{\vee}H_h + H_h^{\vee} - {}^{\vee}H_{h-1}^{\vee} - H_{h-1}.$$

We get

$$q^{-j}J_j = F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1},$$

where, for $j \geq 0$,

$$q^{j+1}F_j = -{}^{>}A_{j+1} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} \frac{1}{2} {}^{\vee>}A_{h-1} - {}^{>}A_h$$

$$q^j G_j = -A_{j+1}^{>} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} \frac{1}{2} A_{h-1}^{>\vee} - A_h^{>}.$$

The conclusion follows by using (41.12). □

41.5. Endpoints equations and trivial coboundary tensors

We will now use the previous constructions to finish the proof of Proposition 40.14. We will need the following description of natural projections in tensor spaces.

LEMMA 41.12. *Let $k \geq -1$ and H be in $\otimes^2\mathcal{H}_k$. Write $H = J+K+L$ with J in $\otimes^2\mathcal{H}_{k,+}$, K in $(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})\oplus(\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+})$ and L in $\otimes^2\mathcal{H}_{k,-}$.*

If k is even, we have

$$J = \frac{1}{(q+1)^2}({}^{\vee}H^{\vee} + {}^{\vee}H + H^{\vee} + H)$$

$$K = \frac{1}{(q+1)^2}(-2\,{}^{\vee}H^{\vee} + (q-1)\,{}^{\vee}H + (q-1)H^{\vee} + 2qH)$$

$$L = \frac{1}{(q+1)^2}({}^{\vee}H^{\vee} - q\,{}^{\vee}H - qH^{\vee} + q^2H).$$

If k is odd, we have

$$J = \frac{1}{4}({}^{\vee}H^{\vee} + {}^{\vee}H + H^{\vee} + H)$$

$$K = \frac{1}{2}(-{}^{\vee}H^{\vee} + H)$$

$$L = \frac{1}{4}({}^{\vee}H^{\vee} - {}^{\vee}H - H^{\vee} + H).$$

PROOF. This is a direct computation. □

PROOF OF PROPOSITION 40.14. Let H be in $(\otimes^2\mathcal{H}_k)[t]$ and assume that the ultraweight $\Omega_k(H)$ is a coboundary. Let n be as in Proposition 33.3. We will show that there exists a trivial coboundary tensor K such that $H - K$ has degree $\leq k + n$.

By Definition 40.7 and Definition 40.8, we may assume that H is a symmetric tensor and that it belongs to $(\otimes^2\mathcal{H}_{k,+}) \oplus (\otimes^2\mathcal{H}_{k,-})$, that is, we have ${}^\vee H = H^\vee$. If Γ is bipartite, we can also assume that we have $\widetilde{H} = H$, that is, H is invariant by the twist operator of Section 40.2. As usual, for $j \geq 0$, we set $H_j = \int_{\mathcal{I}_q} C_j(t)H(t)\mathrm{d}\mu_q(t)$.

Suppose k is even. Then, by Lemma 41.12, the component of H in $(\otimes^2\mathcal{H}_{k,+})[t]$ is $(q+1)^{-1}(H + H^\vee)$ and its component in $(\otimes^2\mathcal{H}_{k,-})[t]$ is $(q+1)^{-1}(qH - H^\vee)$. For $j \geq 0$, we set

$$\begin{aligned} J_j &= -(H_j + H_j^\vee) + (qH_j - H_j^\vee) + (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} qH_h - H_h^\vee \\ &= (q-1)H_j - 2H_j^\vee + (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} qH_h - H_h^\vee. \end{aligned}$$

Assume Γ is not bipartite. Then, by Lemma 33.2, Proposition 33.3 and Proposition 40.15, there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $\mathscr{W}_{k,k-1}$ and $(v_j)_{j\geq 0}$ in $\mathscr{W}_{k-1,k}$ such that, for every $j \geq k+n$ and ab, xy in X_k, we have

$$\begin{aligned} (41.13)\quad & q^{-j}\varpi_k(J_j)(ab, xy) = \\ & u_j(ab, x_1y) + v_j(ab_1, xy) - u_{j-1}(ab, xy_1) - v_{j-1}(a_1b, xy). \end{aligned}$$

Assume Γ is bipartite. As above, by Lemma 33.2, Proposition 33.3 and Proposition 40.15, there exist finitely supported sequences $u = (u_j)_{j\geq 0}$ in $\mathscr{W}_{k,k-1}$ and $v = (v_j)_{j\geq 0}$ in $\mathscr{W}_{k-1,k}$ such that, for every $j \geq k+n$, (41.13) holds for any ab, xy in X_k such that $j + d(a,x)$ is even (recall that k is even). Note that in these relations, for $j \geq 0$, we only use the values of the function u_j on the set

$$\{(ab, xy) \in X_k \times X_{k-1} | j + d(a,x) \text{ is odd}\}.$$

Thus, we can assume that, for every (ab, xy) in $X_k \times X_{k-1}$ we have $u_j(ab, xy) = 0$ if $j + d(a,x)$ is even. In the same way, we also assume that, for every (ab, xy) in $X_{k-1} \times X_k$ we have $v_j(ab, xy) = 0$ if $j + d(a,x)$ is even. Besides, recall that $\widetilde{H} = H$, so that, for $j \geq 0$, we have by the definition of the twist operator in Section 40.2, $\varepsilon(-1)^\delta H_j = (-1)^j H_j$, hence, by Lemma 40.4, $\varepsilon(-1)^\delta J_j = (-1)^j J_j$. By Lemma 40.5, for every $j \geq 0$ and ab, xy in X_k such that $j + d(a,x)$ is odd, we get $\varpi_j(J_j)(ab, xy) = 0$ so that (41.13) also holds for such pairs (ab, xy).

Hence, in both cases, we can assume that (41.13) is valid for any $j \geq k+n$ and any ab, xy in X_k, that is, by Definition 39.2, the sequence $(q^{-j}\varpi_k(J_j))_{j\geq k+n}$ is cohomologically trivial in $\mathscr{W}_k$. By Proposition 39.3, we can find finitely supported sequences $(F_j)_{j\geq k+n-1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and $(G_j)_{j\geq k+n-1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ such that, for $j \geq k+n$, one has

$$q^{-j}J_j = F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.$$

Recall that we have ${}^{\vee}J_j{}^{=}J_j^{\vee}$, $j \geq 0$. Therefore, by Proposition 41.1, we may find finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, $(Q_j)_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$ such that for $j \geq 1$, one has $P_j^{\vee} = {}^{\vee}P_j$, $Q_j^{\vee} = {}^{\vee}Q_j$ and, for $j \geq k+n$,

$$\begin{aligned} {}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} &= Q_j^{>} - (q-1)\,{}^{>}X_j + {}^{\vee>}(X_j + Y_j) \\ G_j^{\vee} - (q-1)G_j + {}^{\vee}G_{j-1} &= {}^{>}P_j - (q-1)Y_j^{>} + (X_j + Y_j)^{>\vee}. \end{aligned}$$

Thanks to Lemma 41.5, we can extend the definition of F_j and G_j to all $j \geq 0$ by using the above relations. Now, Lemma 41.4, Lemma 41.7 and Lemma 41.12 precisely tell us that we may find a trivial coboundary polynomial tensor K such that, for any $j \geq k+n$, we have

$$H_j - K_j \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+}),$$

where $K_j = \int_{\mathcal{I}_q} C_j(t)K(t)\mathrm{d}\mu_q(t)$. By Definition 40.7, the component of $H - K$ on $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+}))[t]$ is a trivial coboundary tensor. The conclusion follows in case k is even.

Suppose now k is odd. We just sketch the proof. For $j \geq 0$, we set

$$J_j = H_j^{\vee} + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} H_h^{\vee} - H_{h-1}.$$

Using Lemma 33.2, Proposition 33.3 and Proposition 40.15, we get (41.13) for all $j \geq k+n$ and ab, xy in X_k when Γ is not bipartite. When Γ is bipartite, this is only true when $d(a,x)+j$ is odd. But as we have assumed that $\widetilde{H} = H$, Lemma 40.4 says that $\varepsilon(-1)^{\delta}J_j = (-1)^{j+1}J_j$, $j \geq 0$. Reasoning as above, we show that we can again assume (41.13) to hold for all $j \geq k+n$ and ab, xy in X_k.

We conclude as in the even case by using Lemma 41.4, Lemma 41.8 and Lemma 41.12. □

CHAPTER 42

Spectral obstructions

In Chapter 24, we have introduced the spectral transform of pseudofunctions. For $k \geq 0$, the spectral transform essentially allows to diagonalize the action of the natural operators on the subspace of $\mathcal{H}_\infty$ spanned by the image of $(\mathcal{H}_k)^{>^\infty}$. This construction leads to the description of the spectral theory of orthogonal extensions of Euclidean fields.

Now, as in Section 35.2, every Γ-invariant symmetric bilinear form of $\overline{\mathcal{D}}(\partial X)$ defines in a natural way a symmetric bilinear form on $\mathcal{H}_\infty$ for which the action of the operators R and S are symmetric. By pulling back this bilinear form under the spectral transform (and the polyextension map of Section 20.3), we get a symmetric bilinear form on $\mathcal{H}_k^2[t]$ for which the polynomial operators of Section 24.1 are symmetric and which vanishes on the range of the map described in Proposition 24.5.

In the present Chapter, we will use Proposition 40.14 to show that the last two properties characterize the range of this map $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{Q}(\mathcal{H}_k^2[t])$ up to a finite dimensional subspace.

42.1. The first step extension

For $k \geq -1$, we introduce a map $\mathcal{H}_k^2 \to \mathcal{H}_{k+1}$ that will allow us to pull-back the result of Proposition 40.14 under the spectral transform.

Definition 42.1. Let $k \geq -1$. We define the first step extension map $I_k : \mathcal{H}_k^2 \to \mathcal{H}_{k+1}$ as follows. For $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}_k^2$, we set

$$\begin{aligned} I_k H &= H_0^{\vee >} - (q-1) H_0^{>} + H_1^{>\vee} && \text{if } k \text{ is even} \\ &= H_0^{\vee > \vee} - (q-1) H_0^{\vee >} + H_1^{>} && \text{if } k \text{ is odd.} \end{aligned}$$

We shall see below in Lemma 42.6 that the first step extension map allows to describe the behaviour of the converse of the spectral transform on constant vectors. Before showing this, we establish some basic properties of this map.

There is a compatibility of first step extension with some other extension maps. A direct computation (using as usual Lemma 20.6) yields

LEMMA 42.2. *Let $k \geq 0$ and H be in $\mathcal{H}^2_{k-1}$. We have*

$$\begin{aligned}(I_{k-1}H)^{>} &= I_k(H^{\vee>}) && \text{if } k \text{ is even}\\ &= I_k(H^{>\vee}) && \text{if } k \text{ is odd.}\end{aligned}$$

Using Proposition 20.8 allows to determine the null space of the map I_k:

LEMMA 42.3. *Let $k \geq 0$ and H be in $\mathcal{H}^2_k$. Then $I_k H = 0$ if and only if there exists G in $\mathcal{H}_{k-1}$ with $H = \begin{pmatrix} G^{\vee>\vee} \\ -qG^{>}\end{pmatrix}$.*

PROOF. Assume k is even. Then, if $I_k H = 0$, by Definition 42.1, we have

$$H_0^{\vee>} - (q-1)H_0^{>} = -H_1^{>\vee},$$

hence, by Lemma 20.6, there exists G in $\mathcal{H}_{k-1}$ with

$$H_0^{\vee} - (q-1)H_0 = G^{\vee>} \text{ and } H_1 = -G^{>}.$$

The first relation amounts to $H_0 = q^{-1}G^{\vee>\vee}$. Therefore, we obtain $H = q^{-1}\begin{pmatrix} G^{\vee>\vee} \\ -qG^{>}\end{pmatrix}$ as required. Conversely, if H may be written in this way, a straightforward computation using Lemma 20.6 shows that $I_k H = 0$.

The proof is analogous in the odd case. □

In the bipartite case, the first step extension behaves well with respect to the operations introduced in Section 20.6. Lemma 20.22 and Definition 42.1 give

LEMMA 42.4. *Assume Γ is bipartite. Let $k \geq -1$ and H be in $\mathcal{H}^2_k$. Then, we have*

$$I_k \begin{pmatrix} H_0^{\wr} \\ -H_1^{\wr}\end{pmatrix} = (-1)^k (I_k H)^{\wr}.$$

Now, by using the first step extension map, we can relate the pull back of the natural bilinear forms of Section 35.2 with the ultraweight.

PROPOSITION 42.5. *Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ and θ be the associated (ι, T)-invariant distribution of $\Gamma\backslash\mathscr{S}$. For $k \geq 0$ and H, J in $\mathcal{H}_k^{(\mathbb{N})}$. We have*

$$p(E_k H, E_k J) = \langle \theta, \Omega_{k+1}(I_k \widehat{H}(t) \otimes I_k \widehat{J}(t))\rangle.$$

The relation between Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ and (ι, T)-invariant distributions of $\Gamma\backslash\mathscr{S}$ was explained in Section 15.3. The polyextension map E_k was introduced in Definition 20.11. The spectral transform $H \mapsto \widehat{H}$ was constructed in Proposition 24.3. The ultraweight Ω_k was introduced in Definition 40.1.

Note that, in the formula above, the ultraweight $\Omega_{k+1}(I_k\widehat{H}(t) \otimes I_k\widehat{J}(t))$ is not a priori a smooth function. Nevertheless, by Corollary 40.12, we know that it is cohomologous to a smooth function, so that the formula makes sense, thanks to the convention introduced in Remark 32.8.

The proof relies on the next lemma which tells us that the first extension map is essentially defined by studying the converse of the spectral transform on constant vectors in $\mathcal{H}_k^2[t]$.

LEMMA 42.6. *Let $k \geq -1$ and H be in $\mathcal{H}_k^2$. Define G in $\mathcal{H}_k^{(\mathbb{N})}$ by setting $G_i = 0$ for $i \geq 2$ and*

$$\begin{aligned} &G_0 = H_0^\vee - (q-1)H_0 \qquad && G_1 = H_1 \qquad && \text{if } k \text{ is even} \\ &G_0 = H_1 - (q-1)H_0^\vee \qquad && G_1 = H_0^\vee \qquad && \text{if } k \text{ is even.} \end{aligned}$$

Then we have

$$\widehat{G}(t) = H \text{ and } E_kG = (I_kH)^{>\infty}.$$

Thanks to this Lemma, we could also have recovered Lemma 42.3 as a consequence of Proposition 24.5.

PROOF. Note first that Definition 20.11 gives in both cases

$$E_kG = (G_0^{>} + G_1^{>\vee})^{>\infty} = (I_kH)^{>\infty}.$$

Assume k is even. By Definition 20.16, we have

$$G = G_0\mathbf{1}_0 + G_1\mathbf{1}_1 = G_0\mathbf{1}_0 + S(G_1\mathbf{1}_0).$$

Using (24.1) and Proposition 24.3, we obtain

$$\widehat{G}(t) = \begin{pmatrix} q^{-1}G_0^\vee \\ 0 \end{pmatrix} + \mathfrak{S}_t \begin{pmatrix} q^{-1}G_1^\vee \\ 0 \end{pmatrix} = \begin{pmatrix} q^{-1}G_0^\vee \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ G_1 \end{pmatrix} = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$$

as required.

Assume k is odd. By Definition 20.17, we have

$$G = G_0\mathbf{1}_0 + G_1\mathbf{1}_1 = G_0\mathbf{1}_0 + R(G_1\mathbf{1}_0).$$

Using (24.2) and Proposition 24.3, we obtain

$$\widehat{G}(t) = \begin{pmatrix} 0 \\ G_0 \end{pmatrix} + \mathfrak{R}_t \begin{pmatrix} 0 \\ G_1 \end{pmatrix} = \begin{pmatrix} 0 \\ G_0 \end{pmatrix} + \begin{pmatrix} G_1^\vee \\ (q-1)G_1 \end{pmatrix} = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$$

as required. □

Proof of Proposition 42.5. We fix A, B in $\mathcal{H}_k^2$ and we let K and L be the elements of $\mathcal{H}_k^{(\mathbb{N})}$ given by Lemma 42.6 so that

$$\widehat{K}(t) = A \qquad E_k K = (I_k A)^{>^\infty}$$
$$\widehat{L}(t) = B \qquad E_k L = (I_k B)^{>^\infty}.$$

Let $a, b \geq 0$ be integers. By Proposition 24.3, the spectral transforms of $P^a K$ and $P^b L$ are given by

$$\widehat{P^a K}(t) = t^a A \text{ and } \widehat{P^b L}(t) = t^b B.$$

By Lemma 20.18, Lemma 35.3, Proposition 35.4 and Corollary 40.11, we obtain

$$\begin{aligned} p(E_k(P^a K), E_k(P^b L)) &= p(P^a E_k K, P^b E_k L) \\ &= \langle \theta, \Phi(P^a E_k K, P^b E_k L) \rangle \\ &= \langle \theta, \Phi(P^{a+b} E_k K, E_k L) \rangle \\ &= \langle \theta, \Phi(P^{a+b}(I_k A)^{>^\infty}, (I_k B)^{>^\infty}) \rangle \\ &= \langle \theta, \Omega_{k+1}(t^{a+b}(I_k A) \otimes (I_k B)) \rangle \\ &= \langle \theta, \Omega_{k+1}(I_k \widehat{P^a K} \otimes I_k \widehat{P^b L}) \rangle. \end{aligned}$$

The conclusion follows when H and J are of the form $P^a K$ and $P^b L$. This is sufficient as, by Proposition 24.3, the elements of this form span the space $\mathcal{H}_k^{(\mathbb{N})}$. □

42.2. Tensors with trivial coboundary first step extension

We aim at translating the result of Proposition 40.14 by means the spectral representation obtained through the spectral transform. To this aim, we now describe the inverse image, under the first extension map, of the space of trivial coboundary tensors of Definition 40.7 and Definition 40.8.

We first define a notational convention which complements the notation of Section 38.2. If V and W are vector spaces, the elements of the tensor product $V^2 \otimes W^2$ will be written as matrices $u = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$ whose coefficients are elements of $V \otimes W$. The natural bilinear map $V^2 \times W^2 \to V^2 \otimes W^2$ will be defined by

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \otimes \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} v_0 \otimes w_0 & v_0 \otimes w_1 \\ v_1 \otimes w_0 & v_1 \otimes w_1 \end{pmatrix}, \quad v_0, v_1 \in V, \quad w_0, w_1 \in V.$$

With this matrix convention, we can describe the action of linear maps as follows. Assume V' and W' are other vector spaces and χ :

$V^2 \to (V')^2$ and $\psi : W^2 \to (W')^2$ are linear maps. We may write them as matrices

$$\chi = \begin{pmatrix} \chi_{00} & \chi_{01} \\ \chi_{10} & \chi_{11} \end{pmatrix} \text{ and } \psi = \begin{pmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{pmatrix}$$

whose coefficients are respectiely linear maps $V \to V'$ and $W \to W'$. Then, the associated linear maps $u \mapsto \chi u, V^2 \otimes W^2 \to (V')^2 \otimes W^2$ and $u \mapsto u\psi, V^2 \otimes W^2 \to V^2 \otimes (W')^2$ are given by the following matrix multiplications: for u in $V^2 \otimes W^2$, we have

$$\chi u = \begin{pmatrix} \chi_{00} & \chi_{01} \\ \chi_{10} & \chi_{11} \end{pmatrix} \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \text{ and } u\psi = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \begin{pmatrix} \psi_{00} & \psi_{10} \\ \psi_{01} & \psi_{11} \end{pmatrix}.$$

The reader should beware that, for this computation rule to hold, the antidiagonal coefficients of the latter matrix have to be exchanged.

Coming back to spaces of pseudofunctions, if Γ is bipartite, we extend the definition of the twist operator of Section 40.2 as follows. For $k \geq 0$ and $H(t) = \begin{pmatrix} H_{00}(t) & H_{01}(t) \\ H_{10}(t) & H_{11}(t) \end{pmatrix}$ an element of $(\otimes^2 \mathcal{H}_k^2)[t]$, we will define the twist $\widetilde{H}$ of H as the element

$$\widetilde{H}(t) = \begin{pmatrix} \widetilde{H}_{00}(t) & -\widetilde{H}_{01}(t) \\ -\widetilde{H}_{10}(t) & \widetilde{H}_{11}(t) \end{pmatrix}.$$

This definition allows to get

LEMMA 42.7. *Assume Γ is bipartite. Let $k \geq 0$ and H be in $(\otimes^2 \mathcal{H}_k^2)[t]$. Then we have $I_k(\widetilde{H})I_k = \widetilde{I_k H I_k}$.*

PROOF. This is a direct consequence the definition of the twist operator in Section 40.2 as well as of Lemma 40.3 and Lemma 42.4. □

Now we introduce our candidates for playing the roles of trivial coboundary tensors in $\otimes^2 \mathcal{H}_k^2[t]$.

DEFINITION 42.8. Let $k \geq 0$. We define the space of trivial spectral obstructions in $(\otimes^2 \mathcal{H}_k^2)[t]$ as the subspace Σ_k^0 spanned by the polynomial tensors

$$\begin{aligned} J \otimes K - K \otimes J \qquad & J, K \in \mathcal{H}_k^2[t] \\ \mathfrak{R}_t H - H \mathfrak{R}_t \qquad & H \in (\otimes^2 \mathcal{H}_k^2)[t] \\ \mathfrak{S}_t H - H \mathfrak{S}_t \qquad & H \in (\otimes^2 \mathcal{H}_k^2)[t] \\ H^{\vee > \vee} - H^{>} \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} \qquad & H \in (\mathcal{H}_k^2 \otimes \mathcal{H}_{k-1}^2)[t] \end{aligned}$$

and, if Γ is bipartite,

$$\widetilde{H} - H, \quad H \in (\otimes^2 \mathcal{H}_k)[t].$$

In this Definition, we have used the notation of Section 24.1 for the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$.

PROPOSITION 42.9. *Let $k \geq 1$ and $H(t)$ be in $(\otimes^2 \mathcal{H}_k^2)[t]$. Then the element $I_k H(t) I_k$ of $(\otimes^2 \mathcal{H}_{k+1})[t]$ belongs to Θ_{k+1}^0 if and only if $H(t)$ belongs to Σ_k^0.*

The spaces of trivial coboundary polynomial tensors Θ_k°, $k \geq 0$, were defined in Definition 40.7 and Definition 40.8.

In the proof, we will need to compute explicitely the values of the double of the first step extension operator on the tensors that appear in Definition 42.8. Note that, when k is even, the operator $\mathfrak{S}_t$ of Section 24.1 does not depend on t. In the formulas below, we simply denote it by $\mathfrak{S}$. In the same way, when k is odd, the operator $\mathfrak{R}_t$ is denoted by $\mathfrak{R}$.

LEMMA 42.10. *Let $k \geq 1$. The action of the double first step extension operator on the generators of the space of trivial spectral obstructions in $(\otimes^2 \mathcal{H}_k^2)[t]$ may be computed by using the following formulas.*

If k is even, for H in $\otimes^2 \mathcal{H}_k^2$, we have

$$I_k(\mathfrak{S}H - H\mathfrak{S})I_k = {}^\vee(I_k H I_k) - (I_k H I_k)^\vee.$$

For $H(t) = \begin{pmatrix} H_{00}(t) & H_{01}(t) \\ H_{10}(t) & H_{11}(t) \end{pmatrix}$ *in* $(\otimes^2 \mathcal{H}_k^2)[t]$ *and* $J(t) = \begin{pmatrix} H_{00}(t) & {}^\vee H_{01}(t) \\ H_{10}(t)^\vee & H_{11}(t) \end{pmatrix}$, *we have*

$$\begin{aligned} I_k(\mathfrak{R}_t J(t) - J(t)\mathfrak{R}_t)I_k &= q\,{}^>(H_{00}(t)^\vee - {}^\vee H_{00}(t))^> \\ &+ q(q+1)t\,{}^> H_{10}(t)^> + q(q-1)\,{}^{\vee>} H_{10}(t)^> - q\,{}^{\vee>\vee} H_{10}(t)^> - q\,{}^{\vee>} H_{10}(t)^{\vee>} \\ &- q(q+1)t\,{}^> H_{01}(t)^> - q(q-1)\,{}^> H_{01}(t)^{>\vee} + q\,{}^> H_{01}(t)^{\vee>\vee} + q\,{}^{>\vee} H_{01}(t)^{>\vee} \\ &+ (q+1)t({}^> H_{11}(t)^{>\vee} - {}^{\vee>} H_{11}(t)^>) - {}^{\vee>}({}^\vee H_{11}(t) - H_{11}(t)^\vee)^{>\vee}. \end{aligned}$$

For $H(t)$ in $\mathcal{H}_{k-1}[t]$, we have

$$\begin{aligned} I_k \begin{pmatrix} H_0(t)^{\vee>\vee} \\ H_1(t)^{\vee>\vee} \end{pmatrix} - I_k \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} H_0(t)^> \\ H_1(t)^> \end{pmatrix} &= H_1(t)^{\vee>\vee>\vee} \\ &- (q-1)H_1(t)^{>>} + H_1(t)^{>\vee>} - (q+1)tH_1(t)^{>>\vee}. \end{aligned}$$

If k is odd, for H in $\otimes^2 \mathcal{H}_k^2$, we have

$$I_k(\mathfrak{R}H - H\mathfrak{R})I_k = {}^\vee(I_k H I_k) - (I_k H I_k)^\vee.$$

For $H(t) = \begin{pmatrix} H_{00}(t) & H_{01}(t) \\ H_{10}(t) & H_{11}(t) \end{pmatrix}$ *in* $(\otimes^2 \mathcal{H}_k^2)[t]$, *we have*

$$I_k(\mathfrak{S}_t J(t) - J(t)\mathfrak{S}_t)I_k = -\,{}^{\vee>}H_{00}(t)^{\vee>\vee} + {}^{\vee>\vee}H_{00}(t)^{>\vee}$$
$$+(q-1)\,{}^{\vee>}H_{00}(t)^{\vee>} - (q-1)\,{}^{>\vee}H_{00}(t)^{>\vee} + (q+1)t({}^{>\vee}H_{00}(t)^{\vee>\vee} - {}^{\vee>\vee}H_{00}(t)^{\vee>})$$
$$+\,{}^{>\vee}H_{10}(t)^{\vee>\vee} - (q-1)\,{}^{>\vee}H_{10}(t)^{\vee>} + {}^{>}H_{10}(t)^{>\vee} - (q+1)t\,{}^{>}H_{10}(t)^{\vee>}$$
$$-\,{}^{\vee>\vee}H_{01}(t)^{\vee>} + (q-1)\,{}^{>\vee}H_{01}(t)^{\vee>} - {}^{\vee>}H_{01}(t)^{>} + (q+1)t\,{}^{>\vee}H_{01}(t)^{\vee>}$$
$$+\,{}^{>}({}^{\vee}H_{11}(t) - H_{11}(t)^{\vee})^{>}$$

For $H(t)$ *in* $\mathcal{H}_{k-1}[t]$, *we have*

$$I_k \begin{pmatrix} H_0(t)^{\vee>\vee} \\ H_1(t)^{\vee>\vee} \end{pmatrix} - I_k \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} H_0(t)^{>} \\ H_1(t)^{>} \end{pmatrix} = H_1(t)^{\vee>\vee>}$$
$$+ H_1(t)^{>\vee>\vee} - (q-1)H_1(t)^{>\vee>} - (q+1)tH_1(t)^{>>}.$$

Proof. These are straightforward computations. □

In case $k \geq 2$ is even, Lemma 37.5 and Lemma 37.8 imply that the first step extension I_k maps $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$. This fact will make the proof of Proposition 42.9 easier, so that we start with this case.

Proof of Proposition 42.9 when k is even. By Lemma 37.5 and Lemma 37.8, we have $I_k\mathcal{H}_k = \mathcal{H}_{k+1}$. By comparing the formulas in Definition 40.8 and the ones in Lemma 42.10 (and by using Lemma 42.7 in case Γ is bipartite), we get $I_k\Sigma_k^0 I_k = \Theta_{k+1}^0$. By Lemma 38.4 and Lemma 42.3, the null space of the map $H(t) \mapsto I_k H(t), \mathcal{H}_k^2[t] \to \mathcal{H}_{k+1}[t]$ is the space of polynomial of the form

$$\begin{pmatrix} G(t)^{\vee>\vee} \\ -qG(t) \end{pmatrix} = \begin{pmatrix} G(t)^{\vee>\vee} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} G(t)^{>} \\ 0 \end{pmatrix}$$

where $G(t)$ is in $\mathcal{H}_{k-1}[t]$. Therefore, in view of Definition 42.8, the null space of the map $H(t) \mapsto I_k H(t) I_k, (\otimes^2 \mathcal{H}_k^2)[t] \to (\otimes^2 \mathcal{H}_{k+1})[t]$ is contained in Σ_k^0. The conclusion follows. □

In case k is odd, we will need to give a set of generators for the space $(I_k(\otimes^2 \mathcal{H}_k^2)[t]I_k) \cap \Theta_{k+1}^0$. This will use

Lemma 42.11. *Let* $k \geq -1$, H *be in* $\mathcal{H}_k$ *and* t *be in* $\mathbb{R}$, $t^2 \neq 1$. *Assume that we have*

$$H^{\vee>\vee>} + H^{>\vee>\vee} = (q+1)tH^{>>} + (q-1)H^{>\vee>} \qquad \text{if } k \text{ is even}$$
$$= (q+1)tH^{>>} + (q-1)H^{\vee>>} \qquad \text{if } k \text{ is odd.}$$

Then $H = 0$.

PROOF. We prove the statement by induction on k.

For $k = -1$, as the $\vee$ operator is -1 on 0-pseudofunctions (see Section 20.2), the assumption reads as $tH + H^\vee = 0$, hence $H = 0$ since $t^2 \neq 1$.

Assume now $k \geq 0$ and the Lemma holds for $k-1$.

If k is even, Proposition 20.8 says that we may find J in $\mathcal{H}_k$ with

$$(q+1)tH^> + (q-1)H^{>\vee} - H^{\vee>\vee} = J^{\vee>} \text{ and } H^{>\vee} = J^>.$$

Applying again Proposition 20.8 to the latter, we get K in $\mathcal{H}_{k-1}$ with

$$J = K^{\vee>} \text{ and } H = K^>.$$

The assumption now reads as

$$K^{>\vee>\vee} + K^{\vee>\vee>} = (q+1)tK^{>>} + (q-1)K^{\vee>>} = 0,$$

and the conclusion follows from the induction assumption.

The proof is analogue in the odd case. □

We can now give a new version of Definition 40.7.

LEMMA 42.12. *Let $k \geq 2$ be an even integer. Then, the space*

$$(I_{k-1}(\otimes^2 \mathcal{H}^2_{k-1})[t]I_{k-1}) \cap \Theta^0_k$$

is spanned by the following polynomial tensors:

$$\begin{aligned} J \otimes K - K \otimes J \quad & J, K \in I_{k-1}\mathcal{H}^2_{k-1}[t] \\ H \quad & H \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-})[t] \cap I_{k-1}(\otimes^2 \mathcal{H}^2_{k-1})[t]I_{k-1} \\ {}^>H^> \quad & H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-})[t] \end{aligned}$$

as well as

$$(q+1)t\,{}^>H^> + (q-1)\,{}^>H^{\vee>} - 2\,{}^>H^{\vee>\vee}$$
$$H \in ((\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,+}) \oplus (\mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,-}))[t]$$

and

$$(q+1)tH^{>>} + (q-1)H^{>\vee>} - H^{\vee>\vee>} - H^{>\vee>\vee}$$
$$H \in ((I_{k-1}\mathcal{H}^2_{k-1}) \otimes \mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H} - H, \quad H \in I_{k-1}(\otimes^2 \mathcal{H}^2_{k-1})[t]I_{k-1}.$$

PROOF. The main difficulty is to show that the component of an element of $(I_{k-1}(\otimes^2 \mathcal{H}^2_{k-1})[t]I_{k-1}) \cap \Theta^0_k$ corresponding to the fifth case of Definition 40.7 can be assumed to belong to $(I_{k-1}\mathcal{H}^2_{k-1} \otimes \mathcal{H}_{k-2})[t]$.

We first claim that, in this fifth case, in full generality, it suffices to assume that the element $H(t)$ belongs to $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}) \oplus (\mathcal{H}_{k,-} \otimes$

$\mathcal{H}_{k-2,-}))[t]$ (a fact that was already implicitely used in Section 41.2). Indeed, fix $H(t)$ in $(\mathcal{H}_k \otimes \mathcal{H}_{k-2})[t]$ and set

$$J(t) = (q+1)tH(t)^{>>} + (q-1)H(t)^{>\vee>} - H(t)^{\vee>\vee>} - H(t)^{>\vee>\vee}.$$

When $H(t)$ belongs to $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,+}))[t]$, we have $H(t)^{\vee} + {}^{\vee}H(t) = (q-1)H(t)$, hence

$$J(t) = (q+1)tH(t)^{>>} + {}^{\vee}H(t)^{>\vee>} - H(t)^{>\vee>\vee}$$

and the latter belongs to $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+}))[t]$ so that the first two cases of Definition 40.7 already warrant that it is a trivial coboundary polynomial tensor. Notice in particular, that, if $H(t)$ is in $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-}))[t]$, then

$$J(t) = (q+1)tH(t)^{>>} + (q-1)H(t)^{>\vee>} - {}^{\vee}H(t)^{>\vee>} - H(t)^{>\vee>\vee},$$

so that $J(t)$ belongs to $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,+}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k,-}))[t]$.

In the same way, if Γ is bipartite, by Lemma 40.4, for $H(t)$ in $(\mathcal{H}_k \otimes \mathcal{H}_{k-2})[t]$, we have

$$\widetilde{J}(t) = -(q+1)t\widetilde{H}(t)^{>>} - (q-1)\widetilde{H}(t)^{>\vee>} + \widetilde{H}(t)^{\vee>\vee>} + \widetilde{H}(t)^{>\vee>\vee}.$$

Thus, we can also assume that $\widetilde{H}(t) = -H(t)$ and hence $\widetilde{J}(t) = J(t)$.

Now, let $G(t)$ be in $(\otimes^2\mathcal{H}_{k-1}^2)[t]$ and assume that $I_{k-1}G(t)I_{k-1}$ is in Θ_k^0. In view of the discussion above, we can assume that the component of $I_{k-1}G(t)I_{k-1}$ corresponding to the fifth case of Definition 40.7 is of the form

$$J(t) = (q+1)tH(t)^{>>} + (q-1)H(t)^{>\vee>} - H(t)^{\vee>\vee>} - H(t)^{>\vee>\vee},$$

for some H in $((\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}) \oplus (\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-}))[t]$, with $\widetilde{H}(t) = -H(t)$ if Γ is bipartite. In particular, we have ${}^{\vee}J = J^{\vee}$ and $\widetilde{J} = \widetilde{J}$. We notice that the space $I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1} \subset (\otimes^2\mathcal{H}_k)[t]$ is invariant under the symmetrization of tensors, the map $F(t) \mapsto {}^{\vee}F(t)^{\vee}$ and the twist operator if Γ is bipartite (the latter by Lemma 42.7). Moreover, these maps commute to each other. Therefore, in view of Definition 40.7, the polynomial tensor $J(t)$ may be written as the sum of a skew-symmetric polynomial tensor and an element of $I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}$. In other words, the symmetrization of $J(t)$ belongs to $I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}$. As $J(t)$ belongs to $(\mathcal{H}_k \otimes I_{k-1}\mathcal{H}_{k-1}^2)[t]$, this tells us that $J(t)$ itself belongs to $I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}$. By applying Lemma 38.4 and Lemma 42.11, we obtain that $H(t)$ belongs to $(I_{k-1}\mathcal{H}_{k-1}^2 \otimes \mathcal{H}_{k-2})[t]$.

Therefore, we have shown that the space $(I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}) \cap \Theta_k^0$ is the intersection of $I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}$ with the subspace of

$(\otimes^2\mathcal{H}_k)[t]$ spanned by the polynomial tensors

$$\begin{aligned} J\otimes K - K\otimes J \qquad & J,K\in\mathcal{H}_k[t]\\ H \qquad & H\in(\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-})[t]\\ {}^{>}H^{>} \qquad & H\in(\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-})[t]\end{aligned}$$

as well as

$$(q+1)t\,{}^{>}H^{>}+(q-1)\,{}^{>}H^{\vee>}-2\,{}^{>}H^{\vee>\vee}$$
$$H\in((\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,+})\oplus(\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,-}))[t]$$

and

$$(q+1)tH^{>>}+(q-1)H^{>\vee>}-H^{\vee>\vee>}-H^{>\vee>\vee}$$
$$H\in(I_{k-1}\mathcal{H}_{k-1}^2\otimes\mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H}-H,\quad H\in I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}.$$

The conclusion follows as $I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2)[t]I_{k-1}$ is stable under the symmetrization operator, the map $F(t)\mapsto{}^{\vee}F(t)^{\vee}$ and the twist operator if Γ is bipartite. □

Proof of Proposition 42.9 when k is odd. This is analogue to the proof in case k is even, by using Lemma 42.12 instead of Definition 40.7. □

42.3. Finiteness of genuine spectral obstructions

We now use Proposition 42.9 to translate Proposition 40.14 into a result that only lives in the spectral world.

Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. For $k\geq 0$, we let $\widehat{p}_k$ be the symmetric bilinear form on $\mathcal{H}_k^2[t]$ such that, for every H,J in $\mathcal{H}_k^{(\mathbb{N})}$, one has

$$\widehat{p}_k(\widehat{H},\widehat{J})=p(E_kH,E_kJ),$$

where E_k is the polyextension map of Definition 20.11 and, as in Section 35.2, we still write p for the symmetric bilinear form on $\mathcal{H}_\infty$ associated with p.

The existence and uniqueness of $\widehat{p}_k$ are warranted by Proposition 24.3.

Now, we will associate to $\widehat{p}_k$ a linear functional on $(\otimes^2\mathcal{H}_k^2)[t]$ by the following construction which may be seen as an abstract form of the spectral theorem.

By construction in Section 35.2 (and by using Lemma 10.11), for H, J in $\mathcal{H}_\infty$, we have

$$p(RH, J) = p(H, RJ) \text{ and } p(SH, J) = p(H, SJ) \tag{42.1}$$

where R and S are the natural operators on ∞-pseudofunctions defined in Section 20.5. This gives $p(PH, J) = p(H, PJ)$, hence, by Proposition 24.3, for $H(t), J(t)$ in $\mathcal{H}^2_k[t]$,

$$\widehat{p}_k(tH(t), J(t)) = \widehat{p}_k(H(t), tJ(t)). \tag{42.2}$$

Let V, W be vector spaces and r, s be inderminates. The product map

$$V[r] \times W[s] \to (V \otimes W)[r, s], (v(r), w(s)) \mapsto v(r) \otimes w(s)$$

defines an isomorphism between the algebraic tensor product $V[r] \otimes W[s]$ and the space of tensor valued polynomial functions with two indeterminates $(V \otimes W)[r, s]$. Under this isomorphism, the subspace of $V[r] \otimes W[s]$ spanned by the tensors of the form

$$(rv(r)) \otimes w(s) - v(r) \otimes (sw(s)), \quad v(r) \in V[r], \quad w(s) \in W[s],$$

may be identified with the space $(r - s)(V \otimes W)[r, s]$. Thus, if $\varphi : V[r] \otimes W[s] \to \mathbb{R}$ is a bilinear form such that, for every $v(r)$ in $V[r]$ and $w(s)$ in $W[s]$, we have

$$\varphi(rv(r), w(s)) = \varphi(v(r), sw(s)),$$

we may consider φ as a linear functional on $(V \otimes W)[r, s]$ which vanishes on the space $(r - s)(V \otimes W)[r, s]$. Now, if U a vector space, we have a natural map $\delta : U[r, s] \to U[t]$ defined by letting r and s take the value t, that is

$$\delta u(t) = u(t, t), \quad u(r, s) \in U[r, s].$$

Elementary algebraic considerations show that the null space of δ is exactly the space $(r - s)U[r, s]$. Therefore, if φ is as above, we may consider φ as a linear functional on $(V \otimes W)[t]$.

By applying this construction to the bilinear form $\widehat{p}_k$, since (42.2) holds, we will now consider $\widehat{p}_k$ as a linear functional on $(\otimes^2 \mathcal{H}^2_k)[t]$. This linear functional may also be computed as follows:

LEMMA 42.13. *Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ and θ be the associated (ι, T)-invariant distribution on $\Gamma\backslash\mathscr{S}$. For $k \geq 0$ and H in $(\otimes^2 \mathcal{H}^2_k)[t]$, we have*

$$\widehat{p}_k(H) = \langle \theta, \Omega_{k+1}(I_k H I_k) \rangle.$$

PROOF. This is a direct consequence of the definition of $\widehat{p}_k$ and of Proposition 42.5. □

We shall now study the range of the map $p \mapsto \widehat{p}_k$.

DEFINITION 42.14. Let $k \geq 1$. We define the space of spectral obstructions as the subspace Σ_k of all H in $(\otimes^2 \mathcal{H}_k^2)[t]$ such that, for any Γ-invariant symmetric bilinear form p on $\overline{\mathcal{D}}(\partial X)$, one has $\widehat{p}_k(H) = 0$.

We have an alternative definition of Σ_k by means of the trivial coboundary tensors of Definition 40.7 and Definition 40.8.

LEMMA 42.15. *Let $k \geq 1$. The space $\Sigma_k \subset (\otimes^2 \mathcal{H}_k^2)[t]$ is the inverse image of the space $\Theta_{k+1} \subset (\otimes^2 \mathcal{H}_{k+1})[t]$ under the map $H \mapsto I_k H I_k$. In particular, we have $\Sigma_k^0 \subset \Sigma_k$.*

In other words, all the polynomial tensors that appear in Definition 42.8 are killed by any linear functional $\widehat{p}_k$ as above.

PROOF. The equivalence between the two definitions of Σ_k is a direct consequence of Proposition 32.1, Corollary 32.7 and Lemma 42.13. The inclusion $\Sigma_k^0 \subset \Sigma_k$ then follows by Lemma 40.9 and Proposition 42.9. □

We can use Proposition 40.14 to give a partial converse to Lemma 42.15.

THEOREM 42.16. *For any $k \geq 1$, the space Σ_k^0 has finite codimension in Σ_k. More precisely, there exists an integer $n \geq 0$ such that, for any $k \geq 1$, we have $\Sigma_k \subset \Sigma_k^0 + (\otimes^2 \mathcal{H}_k^2)_{k+n}[t]$, that is, every spectral obstruction may be written as the sum of a trivial spectral obstruction and a polynomial tensor of degree $\leq k + n$.*

As for Proposition 42.9, the proof of this statement will be easier in the even case. In the odd case, we shall need

LEMMA 42.17. *Let $k \geq -1$ and $(F_j)_{j \geq 0}$ be a finitely supported sequence in $\mathcal{H}_k$. Assume that, for every $j \geq 1$, we have*

$$F_j^{\vee >} = F_{j-1}^{> \vee} \qquad \text{if } k \text{ is even}$$
$$F_j^{> \vee} = F_{j-1}^{\vee >} \qquad \text{if } k \text{ is odd.}$$

Then $F_j = 0$ for any $j \geq 0$.

PROOF. As usual, we prove this statement by induction on k.

If $k = -1$, the assumptions says that, for any $j \geq 1$, we have $F_j + F_{j-1}^{\vee} = 0$. The conclusion follows as the sequence $(F_j)_{j \geq 0}$ is finitely supported.

Suppose now $k \geq 0$ and the result is true for $k - 1$.

Assume k is even. From the assumption and Proposition 20.8, we know that there exists a sequence $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1}$ such that, for any $j \geq 1$, we have

$$F_j^{\vee} = G_{j-1}^{\vee >} \text{ and } F_{j-1} = G_{j-1}^{>}.$$

This gives $G_j^{>\vee} = G_{j-1}^{\vee >}$. Besides, as $(F_j)_{j\geq 0}$ is finitely supported, so is $(G_j)_{j\geq 0}$. The conclusion follows by induction.

The proof in the odd case is analogous. □

Using the latter lemma, we can show that the solutions to certain cohomological equations lie in smaller subspaces.

COROLLARY 42.18. *Let $k \geq -1$ be odd and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $I_k(\otimes^2 \mathcal{H}_k^2)I_k$. Assume that there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_{k+1} \otimes \mathcal{H}_k$ and $(G_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k+1}$ such that, for $j \geq 1$, one has*

$$H_j = F_j^{>\vee} - F_{j-1}^{\vee >} + {}^{\vee >}G_j - {}^{>\vee}G_{j-1}.$$

Then, for all $j \geq 0$, we have

$$F_j \in (I_k \mathcal{H}_k^2) \otimes \mathcal{H}_k \text{ and } G_j \in \mathcal{H}_k \otimes (I_k \mathcal{H}_k^2).$$

PROOF. For $j \geq 1$, we have

$$F_j^{>\vee} - F_{j-1}^{\vee >} = H_j - {}^{\vee >}G_j + {}^{>\vee}G_{j-1} \in (I_k \mathcal{H}_k^2) \otimes \mathcal{H}_{k+1}.$$

By Lemma 38.4 and Lemma 42.17, F_j belongs to $(I_k \mathcal{H}_k^2) \otimes \mathcal{H}_k$ for all $j \geq 0$. The proof of the other case is symmetric. □

PROOF OF THEOREM 42.16. Let n be as in Proposition 40.14. Take H in Σ_k. By Lemma 42.15, $I_k H I_k$ belongs to Θ_{k+1} and Proposition 40.14 says that there exists J in Θ_{k+1}^0 such that $H - J$ has degree $\leq k + n + 1$.

If k is even, by Lemma 37.5 and Lemma 37.8, I_k maps $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$ hence J belongs to $I_k(\otimes^2 \mathcal{H}_k^2)[t]I_k$ and Proposition 42.9 implies that we can find K in Σ_k^0 such that $H - K$ has degree $\leq k + n + 1$.

If k is odd, we claim that we can actually choose J to belong to $I_k(\otimes^2 \mathcal{H}_k^2)[t]I_k$. Indeed, due to Corollary 42.18, all the constructions in the proof of Proposition 40.14, when applied to a tensor H in $\Theta_{k+1}^0 \cap I_k(\otimes^2 \mathcal{H}_k^2)[t]I_k$, provide trivial coboundary tensors also living in $I_k(\otimes^2 \mathcal{H}_k^2)[t]I_k$ (see in particular Section 41.1 and Section 41.2). We then conclude as in the even case by applying Proposition 42.9. □

42.4. The spectral projective limit

In Chapter 4, we have introduced the space $\mathcal{F}_k$ of Γ-invariant k-quadratic fields, $k \geq 2$. This is a finite-dimensional space and the reduction map $p \mapsto p^-$ sends $\mathcal{F}_{k+1}$ onto $\mathcal{F}_k$. The projective limit of the system $(\mathcal{F}_k)_{k\geq 2}$ is naturally identified with the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ of Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. We will now give an alternative construction of a projective system whose limit may be identified with $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$. The advantage of this new system is that it will keep track of the spectral theory of non-negative elements in $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$. The drawback is that it will be constructed through infinite-dimensional spaces.

We first note that we have a natural embedding $(\otimes^2\mathcal{H}^2_k)[t]/\Sigma_k \hookrightarrow (\otimes^2\mathcal{H}^2_{k+1})[t]/\Sigma_{k+1}$.

LEMMA 42.19. *Let $k \geq 1$ and H be in $(\otimes^2\mathcal{H}^2_{k+1})[t]$. Then, if k is even, H is in Σ_k if and only if ${}^{\vee>}H^{>\vee}$ is in Σ_{k+1}. If k is odd, H is in Σ_k if and only if ${}^{>\vee}H^{\vee>}$ is in Σ_{k+1}.*

PROOF. Assume k is even. Then from Definition 40.1, Lemma 42.2 and Lemma 42.15, we get

$$\begin{aligned} H \in \Sigma_k &\Leftrightarrow \Omega_{k+1}(I_k H I_k) \text{ is a coboundary} \\ &\Leftrightarrow \Omega_{k+2}({}^{>}I_k H I_k{}^{>}) \text{ is a coboundary} \\ &\Leftrightarrow \Omega_{k+2}(I_{k+1}\,{}^{\vee>}H^{>\vee} I_{k+1}) \text{ is a coboundary} \\ &\Leftrightarrow {}^{\vee>}H^{>\vee} \in \Sigma_{k+1}. \end{aligned}$$

The proof in the odd case is analogous. □

Thus, for any $k \geq 1$, we have defined an injective map $(\otimes^2\mathcal{H}^2_k)[t]/\Sigma_k \hookrightarrow (\otimes^2\mathcal{H}^2_{k+1})[t]/\Sigma_{k+1}$. We denote by

$$\Pi_k : ((\otimes^2\mathcal{H}^2_{k+1})[t]/\Sigma_{k+1})^* \to ((\otimes^2\mathcal{H}^2_k)[t]/\Sigma_k)^*$$

the dual surjective map.

PROPOSITION 42.20. *The maps*

$$p \mapsto \widehat{p}_k : \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to ((\otimes^2\mathcal{H}^2_k)[t]/\Sigma_k)^*, \quad k \geq 1,$$

define a linear isomorphism between $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and the projective limit of the projective system $(((\otimes^2\mathcal{H}^2_k)[t]/\Sigma_k)^*, \Pi_k)_{k\geq 1}$.

The main ingredient of the proof is the following injectivity property:

LEMMA 42.21. *Let p be in $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and assume that the associated symmetric bilinear form on $\mathcal{H}_\infty$ is zero. Then $p = 0$.*

PROOF. First assume that $\Gamma_{xy} = \{e\}$ for some (x, y) in X_1. In this case, if f, g are in $\overline{\mathcal{D}}(\partial X)$, we can find Γ-invariant ∞-pseudofunctions F, G with $F_{xy} = f$, $G_{xy} = g$ and $F_{ab} = G_{ab} = 0$ for any ab in $X_1 \smallsetminus \Gamma(xy)$. Then, by definition, we have $p(f, g) = p(F, G) = 0$, hence $p = 0$.

If all edges admit non trivial stabilizers, then reasoning as above shows that, for any xy in X_1, for any Γ_{xy}-invariant functions f, g in $\overline{\mathcal{D}}(\partial X)$, we have $p(f, g) = 0$. For xy in X_1, we let U_{xy} be as usual

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}.$$

Then, as p is Γ-invariant and U_{xy} is Γ_{xy}-invariant, for any f in $\overline{\mathcal{D}}(\partial X)$, we have

$$|\Gamma_{xy}| p(\mathbf{1}_{U_{xy}}, f) = \sum_{\gamma \in \Gamma_{xy}} p(\mathbf{1}_{U_{xy}}, \gamma f) = p\left(\mathbf{1}_{U_{xy}}, \sum_{\gamma \in \Gamma_{xy}} \gamma f\right) = 0.$$

Since the functions $\mathbf{1}_{U_{xy}}$ span $\overline{\mathcal{D}}(\partial X)$ when xy runs in X_1, we get $p = 0$ as required. □

PROOF OF PROPOSITION 42.20. Let U be the quotient by the cohomology equivalence relation of the space of all Hölder continuous functions on $\Gamma\backslash\mathscr{S}$ which are cohomologuous to a smooth ι-invariant function. Then, in view of Remark 32.8, we may identify $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ with the dual space of U.

By Lemma 42.15, for $k \geq 1$, the map $H \mapsto \Omega_{k+1}(I_k H I_k)$ induces an embedding of $(\otimes^2 \mathcal{H}_k^2)[t]/\Sigma_k$ into U, whose range we denote by U_k. Then, Lemma 42.2 warrants that we have $U_k \subset U_{k+1}$ and Proposition 35.4 and Lemma 42.21 warrant that we have $U = \bigcup_{k \geq 1} U_k$. The conclusion follows since, by Lemma 42.13, the restriction to U_k of the distribution associated to some element p of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ may be identified with $\widehat{p}_k$. □

Thanks to these constructions, Theorem 42.16 yields the

PROOF OF COROLLARY 1.5. Fix $k \geq 0$. We claim that the space $\mathcal{Q}(\mathcal{A}\mathcal{H}_k)^{R,S}$ may be identified with the dual space of the space $(\otimes^2 \mathcal{H}_k^2)[t]/\Sigma_k^\circ$. Indeed, if φ is some symmetric bilinear form on $\mathcal{A}\mathcal{H}_k$ such that the operators R and S are symmetric with respect to φ, we let $\widehat{\varphi}$ be the bilinear form on $\mathcal{H}_k^2[t]$ such that, for every H, J in $\mathcal{H}_k^{(\mathbb{N})}$, one has

$$\widehat{\varphi}(\widehat{H}, \widehat{J}) = \varphi(E_k H, E_k J),$$

where E_k is the polyextension map of Definition 20.11. As in Section 42.3, the existence and uniqueness of $\widehat{\varphi}$ are warranted by Proposition 24.3.

By Lemma 24.1 and Proposition 24.3, multiplication by t is a symmetric endomorphism of $\mathcal{H}_k^2[t]$ with respect to $\widehat{\varphi}$. Therefore, still as in Section 42.3, we can consider $\widehat{\varphi}$ as an element of the dual space of $(\otimes^2\mathcal{H}_k^2)[t]$. It follows from Corollary 20.14, Proposition 24.3 and Proposition 24.5 as well as Definition 42.8 that the map $\varphi \mapsto \widehat{\varphi}$ establishes a linear isomorphism between $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ and the space of those elements in the dual space of $(\otimes^2\mathcal{H}_k^2)[t]$ which vanish on the space Σ_k° of trivial spectral obstructions.

Besides, Proposition 42.20 shows that this map sends the image of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ in $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ onto the space of those elements in the dual space of $(\otimes^2\mathcal{H}_k^2)[t]$ which vanish on the space $\Sigma_k \supset \Sigma_k^\circ$ of all spectral obstructions. The conclusion follows as, by Theorem 42.16, the quotient space Σ_k/Σ_k° has finite dimension. □

APPENDIX A

Euclidean images

In this appendix, we study the notion of the Euclidean image of a non-negative symmetric bilinear form by a surjective map.

A.1. Definition and first properties

We start by defining the Euclidean image of a non-negative symmetric bilinear form under a surjective linear map. This relies on the

LEMMA A.1. *Let V and W be real vector spaces and let q be a non-negative symmetric bilinear form form on V and $\pi : V \to W$ be a surjective linear map. For any w in W, we set*

$$\Phi(w) = \inf_{\substack{v \in \\ \pi(v)=w}} q(v, v).$$

Then Φ is a non-negative quadratic form on W.

DEFINITION A.2. Let the notation be as above. The polar form of Φ is called the Euclidean image of q by π and denoted by $\pi_\star q$.

REMARK A.3. Assume V is a Hilbert space with scalar product q and π is bounded. Let X be the orthogonal complement of $\ker \pi$ in V. Then π induces a linear isomorphism from X onto W and $\pi_\star q$ is the image by this linear isomorphism of the restriction of q to X.

PROOF OF LEMMA A.1. We will proceed to several reductions in order to be brought back to the case in Remark A.3.

First, we will reduce the proof to the case where Φ is non-zero on non-zero vectors. Let W_0 be the set of w in W such that $\Phi(w) = 0$ and let us show that Φ is W_0-invariant, that is, for any w in W and w_0 in W_0, we have $\Phi(w + w_0) = \Phi(w)$. Indeed, for such w and w_0, for any $\varepsilon > 0$, we can find v and v_0 in V with $\pi(v) = w$, $\pi(v_0) = w_0$, $q(v, v) \leq \Phi(w) + \varepsilon$ and $q(v_0, v_0) \leq \varepsilon$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} q(v + v_0, v + v_0) &= q(v, v) + q(v_0, v_0) + 2q(v, v_0) \\ &\leq \Phi(v) + 2\varepsilon + 2\sqrt{\varepsilon(q(v_0, v_0) + \varepsilon)}. \end{aligned}$$

As ε is arbitrary, this gives $\Phi(w + w_0) \leq \Phi(w)$. By symmetry, we get $\Phi(w + w_0) = \Phi(w)$. In particular, if w is also in W_0, we have $\Phi(w + w_0) = 0$ and W_0 is a subspace of W. For w in W, we have

$$\inf_{\substack{v \in V \\ \pi(v) \in w + W_0}} q(v, v) = \inf_{w_0 \in W_0} \Phi(w + w_0) = \Phi(w).$$

Thus, by replacing W with the quotient space W/W_0, we can assume that we have $W_0 = 0$.

Now, we can also assume that q is positive definite. Indeed, if $V_0 \subset V$ is the null space of q, we have $\pi(V_0) \subset W_0$, hence $\pi(V_0) = 0$. Therefore, we can replace V with the quotient space V/V_0 and assume that q is a scalar product. We equip V with the associated topology.

Let $U \subset V$ be the null space of π. We claim that U is closed with respect to this topology. Indeed, if (v_n) is a sequence in U that converges to v in V, we have, by definition of the topology, $q(v - v_n, v - v_n) = \|v - v_n\|^2 \xrightarrow[n \to \infty]{} 0$, hence $\Phi(\pi(v)) = 0$ and $\pi(v) = 0$.

Let H be the completion of V with respect to the positive definite bilinear form q and let X be orthogonal complement of the closure of U in H. Then, as U is closed in V, the orthogonal projection $H \to X$ induces an embedding of $\theta : W \simeq V/U \hookrightarrow X$ and, for w in W, we have $\Phi(w) = q(\theta w, \theta w)$. The result follows. □

Let us give some elementary properties of Euclidean images, whose proofs are straightforward. This operation behaves well under composition of surjective maps.

LEMMA A.4. *Let V, W, X be real vector spaces and $\pi : V \to W$ and $\theta : W \to X$ be surjective linear maps. If q is a non-negative symmetric bilinear form on V, we have*

$$\theta_\star \pi_\star q = (\theta\pi)_\star q.$$

Also, it satisfies a concavity property.

LEMMA A.5. *Let V, W be real vector spaces and $\pi : V \to W$ be a surjective linear map. If p and q are non-negative symmetric bilinear forms on V, we have*

$$\pi_\star(p + q) \geq \pi_\star p + \pi_\star q.$$

We have an invariance under certain translations:

LEMMA A.6. *Let V, W be real vector spaces and $\pi : V \to W$ be a surjective linear map. Let p be a non-negative symmetric bilinear*

form on V and q be a symmetric bilinear form on W. Then $p + \pi^\star q$ is non-negative if and only if $\pi_* p + q$ is non-negative and we then have

$$\pi_\star(p + \pi^\star q) = \pi_\star p + q.$$

Orthogonal splittings are preserved.

LEMMA A.7. *Let $V_1, \dots, V_d, W_1, \dots, W_d$ be real vector spaces and $\pi_i : V_i \to W_i$ be surjective linear maps. Set $V = \bigoplus_{i=1}^d V_i$ and $W = \bigoplus_{i=1}^d W_i$ and write π for the sum map $V \to W$. Then if q is a non-negative symmetric bilinear form on V such that the $V_1, \dots, V_d$ are q-orthogonal to each other, the $W_1, \dots, W_d$ are $\pi_* q$-orthogonal to each other.*

A.2. Approximation of Euclidean images

In the course of the book, we have used the following approximation property of Euclidean images.

PROPOSITION A.8. *Let H be Hilbert space with scalar product p, W be a finite-dimensional real vector space and $\pi : H \to W$ be a continuous surjective linear map. We assume that q is a continuous non-negative symmetric bilinear form on H. Then we have the following convergence of bilinear forms on W:*

$$\pi_\star(\varepsilon p + q) \xrightarrow[\varepsilon \to 0]{} \pi_\star q.$$

REMARK A.9. Even in finite-dimensional vector spaces, the map $q \mapsto \pi_* q$ has bad continuity properties at the undefinite bilinear forms. For example, if $V = \mathbb{R}^2$ and, for any $n \geq 1$, q_n is the polar form of the quadratic form

$$(x, y) \mapsto (x + (1/n)y)^2,$$

then q_n has a non-zero limit, whereas, for any non-zero linear functional φ of V, one has $\varphi_\star q_n \xrightarrow[n \to \infty]{} 0$. This explains why, in Proposition A.8, we have made some additional assumptions to get a limit.

We will prove Proposition A.8 in several steps. The main idea is to reduce it to the case where π is a linear functional. We start by studying this situation.

If V is a vector space, φ is a non-zero linear functional and q is a non-negative symmetric bilinear form, we shall identify the bilinear form $\varphi_\star q$ and the real number

$$\varphi_\star q(1, 1) = \inf_{\substack{v \in V \\ \langle \varphi, v \rangle = 1}} q(v, v).$$

This number is easy to compute:

LEMMA A.10. *Let V be a real vector space, equipped with a non-negative symmetric bilinear form q, $W \subset V$ be the null space of q and φ be a non-zero linear functional of V.*

If φ is not zero on W, then $\varphi_\star q = 0$.

If φ is zero on W and φ is not continuous with respect to the topology induced by q on V/W, then again $\varphi_\star q = 0$.

Finally, if φ is zero on W and continuous with respect to the topology on V/W, then

$$\varphi_\star q = \frac{1}{\|\varphi\|^2},$$

where $\|\varphi\|$ stands for the norm of φ as a bounded linear functional of the normed space V/W.

PROOF. If $\varphi_{|W} \neq 0$, we can find w in W with $\langle \varphi, w\rangle = 1$ and hence $\varphi_\star q = q(w,w) = 0$ and we are done.

Else, we can assume $W = 0$ and q is a scalar product. If φ is not continuous with respect to the topology induced by q, there exists a sequence (u_n) in V with $q(u_n,u_n) = 1$ and $\langle \varphi, u_n\rangle \xrightarrow[n\to\infty]{} \infty$. We set $v_n = \frac{1}{\langle \varphi, u_n\rangle} u_n$ and we have $\langle \varphi, v_n\rangle = 1$ and $q(v_n,v_n) \xrightarrow[n\to\infty]{} 0$, hence $\varphi_\star q = 0$.

Finally, if φ is continuous, we can assume V to be complete with respect to q. Now, let u be the unique vector of V such that $\langle \varphi, v\rangle = q(u,v)$, $v \in V$, so that $\|\varphi\| = \|u\| = q(u,u)^{\frac{1}{2}}$. Any vector v in V with $\langle \varphi, v\rangle = 1$ may be written as $v = \frac{1}{q(u,u)} u + w$ with $q(u,w) = 0$. In particular, we then have $q(v,v) = \frac{1}{q(u,u)} + q(w,w) \geq \frac{1}{q(u,u)}$ and the result follows. □

The data of the numbers $\varphi_\star q$ allows to recover p.

LEMMA A.11. *Let V be a real-vector space and q be a non-negative symmetric bilinear form on V. For any $v \neq 0$ on V, we have*

$$q(v,v) = \sup_{\substack{\varphi \in V^* \\ \langle \varphi, v\rangle = 1}} \varphi_* q.$$

PROOF. The statement is a direct consequence of Cauchy-Schwarz inequality. Let us be more precise.

By construction, we have $q(v,v) \geq \sup_{\substack{\varphi \in V^* \\ \langle \varphi, v\rangle = 1}} \varphi_* q$ and we only need to prove the reverse inequality.

We fix v in V with $q(v,v) \neq 0$ (if $q(v,v) = 0$, the statement is evident). We consider the linear functional

$$\varphi : w \mapsto \frac{q(v,w)}{q(v,v)}$$

so that $\varphi(v) = 1$. Now, Cauchy-Schwarz inequality gives, for any w in V,

$$\varphi(w) \leq \frac{q(w,w)^{\frac{1}{2}}}{q(v,v)^{\frac{1}{2}}},$$

hence, if $\varphi(w) = 1$, $q(w,w) \geq q(v,v)$. Thus, by definition, we get $\varphi_\star q = q(v,v)$ and the result follows. □

Recall that, if V is a finite-dimensional real vector space, we denote by $\mathcal{Q}(V)$ the space of symmetric bilinear forms on V and by $\mathcal{Q}_+(V) \subset \mathcal{Q}(V)$ the set of non-negative ones. The set $\mathcal{Q}_+(V)$ comes with its natural topology as a closed subset of a finite-dimensional vector space. We have an evident semicontinuity property.

LEMMA A.12. *Let V be a finite-dimensional vector space. For any $\varphi \neq 0$ in V^*, the function $q \mapsto \varphi_\star q$ is upper semicontinuous on $\mathcal{Q}_+(V)$, that is, we have, for q in $\mathcal{Q}_+(V)$,*

$$\varphi_\star q = \limsup_{p \to q} \varphi_\star p.$$

PROOF. Indeed, for any v in V, the function $q \mapsto q(v,v)$ is continuous on $\mathcal{Q}(V)$, hence the function $q \mapsto \varphi_\star q$ is the infimum of a family of continuous functions. □

From this, we can deduce a continuity property:

LEMMA A.13. *Let V be a finite-dimensional vector space and q be in $\mathcal{Q}_+(V)$. Assume (q_n) is a sequence in $\mathcal{Q}_+(V)$ such that $q_n \xrightarrow[n\to\infty]{} q$ with $q_n \geq q$, $n \geq 0$. Then, for any $\varphi \neq 0$ in V^*, we have $\varphi_\star q_n \xrightarrow[n\to\infty]{} \varphi_\star q$.*

PROOF. This is a consequence of semicontinuity and concavity. Indeed, on one hand, by Lemma A.12, we have

$$\limsup \varphi_\star q_n \leq \varphi_\star q. \tag{A.1}$$

On the other hand, for any n, we set $p_n = q_n - q$, so that by assumption, the bilinear form p_n is non-negative and $p_n \xrightarrow[n\to\infty]{} 0$. In particular, again by Lemma A.12, we have $\varphi_\star p_n \xrightarrow[n\to\infty]{} 0$. Now, by Lemma A.5, for any n, we have

$$\varphi_\star q_n \geq \varphi_\star q + \varphi_\star p_n,$$

hence

$$\liminf \varphi_\star q_n \geq \varphi_\star q. \tag{A.2}$$

The result follows from (A.1) and (A.2). □

Next, we give a formula for $\varphi_\star q$.

LEMMA A.14. *Let H be Hilbert space with scalar product p, u be a non-zero vector of H, T be a bounded non-negative self-adjoint operator of H and ν be the spectral measure of u with respect to T. Then if φ is the linear functional $v \mapsto p(u,v)$ and q is the bilinear form $(v,w) \mapsto p(Tv,w)$, we have*

$$\varphi_\star q = \left(\int_0^\infty \frac{\mathrm{d}\nu(t)}{t}\right)^{-1}.$$

In particular, we have $\varphi_\star q = 0$ if and only if $\int_0^\infty \frac{\mathrm{d}\nu(t)}{t} = \infty$.

PROOF. Recall that ν is a compactly supported positive Radon measure on $[0,\infty)$. By the Spectral Theorem, we only need to prove the formula when $H = \mathrm{L}^2([0,\infty),\nu)$, u is the constant function $\mathbf{1}$ and T is the operator $f(t) \mapsto tf(t)$.

Note that if $\nu(0) > 0$, we have $q(\mathbf{1}_0,\mathbf{1}_0) = 0$ and $\varphi(\mathbf{1}_0) > 0$, hence $\varphi_* q = 0$ and the result holds. Therefore, we will now assume that we have $\nu(0) = 0$.

In this case, by definition, we have

$$(\varphi_\star q)^{-1} = \sup_{f \in H \smallsetminus \{0\}} \frac{(\int_0^\infty f \mathrm{d}\nu)^2}{\int_0^\infty t f(t)^2 \mathrm{d}\nu(t)}.$$

We let μ be the Radon measure with $\mathrm{d}\mu(t) = t\mathrm{d}\nu(t)$. The supremum above is finite if and only if the function $t \mapsto \frac{1}{t}$ belongs to $\mathrm{L}^2([0,\infty),\mu)$, that is, the function $t \mapsto \frac{1}{t}$ belongs to $\mathrm{L}^1([0,\infty),\mu)$. When this holds, the supremum is equal to $\int_0^\infty t^{-2}\mathrm{d}\mu(t) = \int_0^\infty t^{-1}\mathrm{d}\nu(t)$. □

From this formula, we can deduce a first case of Proposition A.8.

COROLLARY A.15. *Let H be Hilbert space with scalar product p and φ be a non-zero continuous linear functional on H. Then, if q is a continuous non-negative symmetric bilinear form on H, we have:*

$$\varphi_\star(\varepsilon p + q) \xrightarrow[\varepsilon \to 0]{} \varphi_\star q.$$

PROOF. Let u be the vector in H which represents φ and T be the bounded self-adjoint operator on H which represents q. If ν is the

spectral measure of u with respect to T, by Lemma A.14, for any $\varepsilon \geq 0$, we have

$$\varphi_\star(\varepsilon p + q) = \left(\int_0^\infty \frac{\mathrm{d}\nu(t)}{t+\varepsilon} \right)^{-1}.$$

The conclusion follows from the Monotone Convergence Theorem. □

We are now ready to conclude.

Proof of Proposition A.8. For any $\varepsilon > 0$, we set $r_\varepsilon = \pi_\star(\varepsilon p + q)$. As the family $(r_\varepsilon)_{\varepsilon>0}$ is an non-decreasing family of non-negative forms, it has a limit r as $\varepsilon \to 0$. We need to prove that $r = \pi_* q$. On one hand, by Lemma A.4 and Corollary A.15, for any non-zero linear functional φ on W, we have

$$\varphi_\star r_\varepsilon = (\varphi\pi)_\star(\varepsilon p + q) \xrightarrow[\varepsilon\to 0]{} (\varphi\pi)_\star q = \varphi_\star \pi_\star q.$$

On the other hand, as, for any $\varepsilon > 0$, $r_\varepsilon \geq r$, by Lemma A.13, we have

$$\varphi_\star r_\varepsilon \xrightarrow[\varepsilon\to 0]{} \varphi_\star r.$$

We get $\varphi_\star \pi_\star q = \varphi_\star r$ for any φ, hence $\pi_\star q = r$ by Lemma A.11. □

A.3. Derivative of the image map

For a finite-dimensional vector space V, the map $q \mapsto \pi_* q$ is smooth on the space $\mathcal{Q}_{++}(V)$ of positive definite bilinear forms on V. We we will now prove an infinite-dimensional version of this result.

Fix a Hilbert space H with scalar product p. We recall that a continuous symmetric bilinear form q on H is said to be coercive if there exists $\varepsilon > 0$ with $q(v,v) \geq \varepsilon p(v,v)$, $v \in H$, or equivalently if q is positive definite and defines the same topology as p on H. We denote by $\mathcal{Q}^\infty_{++}(H)$ the space of coercive continuous bilinear symmetric forms on H, which is an open subset for the norm topology of the space $\mathcal{Q}^\infty(H)$ of continuous bilinear symmetric forms.

Proposition A.16. *Let H be a Hilbert space, W be a finite-dimensional vector space and $\pi : H \to W$ be a continuous surjective linear map. We denote by $\pi^{\dagger p} : W \to H$ the adjoint operator of π, that is, the linear map with $(\pi_\star p)(\pi(v), w) = p(v, \pi^{\dagger p}(w))$, $v \in H$, $w \in W$. Then the map $q \mapsto \pi_* q, \mathcal{Q}^\infty_{++}(H) \to \mathcal{Q}_{++}(W)$ is smooth. Its derivative at p is the linear map $q \mapsto (\pi^{\dagger p})^\star q, \mathcal{Q}^\infty(H) \to \mathcal{Q}(W)$.*

Proof. Let q be in $\mathcal{Q}^\infty_{++}(H)$. We will compute the adjoint $\pi^{\dagger q}$ of π with respect to the scalar product q on H. Let T be the unique self-adjoint bounded operator on H such that $q(v_1, v_2) = p(Tv_1, v_2)$ for v_1, v_2 in H. As q is coercive, T is invertible. Let $L \subset H$ be the

kernel of π. The space $\pi^{\dagger p}(W)$ is the orthogonal complement $L^{\perp p}$ of L with respect to p. By construction, the orthogonal complement $L^{\perp q}$ of L with respect to q is $T^{-1}\pi^{\dagger p}(W)$. In particular, the endomorphism $\pi T^{-1}\pi^{\dagger p}$ of W is injective, hence bijective. We denote by U its inverse. We claim that we have $\pi^{\dagger q} = T^{-1}\pi^{\dagger p}U$. Indeed, on one hand, the range of the linear operator $T^{-1}\pi^{\dagger p}U$ is $T^{-1}\pi^{\dagger p}(W) = L^{\perp q}$ and on the other hand we have $\pi T^{-1}\pi^{\dagger p}U = \mathrm{Id}_W$. Therefore, we have $\pi^{\dagger q} = T^{-1}\pi^{\dagger p}U$ and, for w_1, w_2 in W,

$$\pi_* q(w_1, w_2) = \pi^{\dagger q}(w_1, w_2) = q(T^{-1}\pi^{\dagger p}Uw_1, T^{-1}\pi^{\dagger p}Uw_2) = p(\pi^{\dagger p}Uw_1, T^{-1}\pi^{\dagger p}Uw_2). \tag{A.3}$$

As the inverse map is smooth on the space of invertible operators, $\pi_* q$ depends smoothly on q. Let us now compute its derivative $\mathrm{d}_p(\pi_*)$ at p. For this, we will derivate the quantity in (A.3) for T close to the identity operator.

We let $\mathcal{B}(H)$ denote the space of bounded operators on H. The derivative of the inverse map at the identity operator is $\Theta \mapsto -\Theta, \mathcal{B}(H) \to \mathcal{B}(H)$. Hence, by the chain rule, the derivative of the map $T \mapsto (\pi T^{-1}\pi^{\dagger p})^{-1}$ at the identity operator is the map $\Theta \mapsto \pi\Theta\pi^{\dagger p}, \mathcal{B}(H) \to \mathrm{End}(W)$. Let q be in $\mathcal{Q}^\infty(H)$ and Θ be the self-adjoint operator associated to q. We get, from (A.3), for w_1, w_2 in W,

$$\begin{aligned}&\mathrm{d}_p(\pi_*)(q)\\ &= p(\pi^{\dagger p}\pi\Theta\pi^{\dagger p}w_1, \pi^{\dagger p}w_2) - p(\pi^{\dagger p}w_1, \Theta\pi^{\dagger p}w_2) + p(\pi^{\dagger p}w_1, \pi^{\dagger p}\pi\Theta\pi^{\dagger p}w_2).\end{aligned}$$

Now, we claim that the three numbers in the right hand-side of the latter are actually equal to each other. Indeed, by construction the operator $\pi^{\dagger p}\pi$ is the p-orthogonal projection on $\pi^\dagger(W)$, so that

$$p(\pi^{\dagger p}\pi\Theta\pi^{\dagger p}w_1, \pi^{\dagger p}w_2) = p(\Theta\pi^{\dagger p}w_1, \pi^{\dagger p}w_2) = p(\pi^{\dagger p}w_1, \Theta\pi^{\dagger p}w_2) = p(\pi^{\dagger p}w_1, \pi^{\dagger p}\pi\Theta\pi^{\dagger p}w_2)$$

and

$$\mathrm{d}_p(\pi_*)(q) = p(\pi^{\dagger p}w_1, \Theta\pi^{\dagger p}w_2) = q(\pi^{\dagger p}w_1, \pi^{\dagger p}w_2)$$

as required. □

APPENDIX B

Euclidean projective limits

The purpose of this appendix is to define and study the notion of a Euclidean projective limit that has been used throughout the book.

B.1. Non-negative projective systems

We first define non-negative projective systems and the associated Euclidean projective limits.

DEFINITION B.1. A non-negative projective system is a sequence $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ where, for any integer $\ell \geq 0$, X_ℓ is a finite-dimensional real vector space, q_ℓ is a non-negative symmetric bilinear form on X_ℓ and $\pi_\ell : X_{\ell+1} \to X_\ell$ is a surjective map such that $(\pi_\ell)^\star q_\ell \leq q_{\ell+1}$ (that is, the bilinear symmetric form $q_{\ell+1} - (\pi_\ell)^\star q_\ell$ is non-negative).

REMARK B.2. For any $\ell \geq 0$, let $Y_\ell \subset X_\ell$ be the null space of q_ℓ. Then, we have $\pi_\ell Y_{\ell+1} \subset Y_\ell$. Set $\overline{X}_\ell = X_\ell/Y_\ell$ and let $\overline{\pi}_\ell : \overline{X}_{\ell+1} \to \overline{X}_\ell$ be the natural map and $\overline{q}_\ell$ be the bilinear form induced by q_ℓ on $\overline{X}_\ell$. The sequence $(\overline{X}_\ell, \overline{q}_\ell, \overline{\pi}_\ell)_{\ell\geq 0}$ is again a non-negative projective system and the forms $(\overline{q}_\ell)_{\ell\geq 0}$ are all positive definite. We shall use this construction repeatedly.

Recall that the algebraic projective limit X of the projective system (X_ℓ, π_ℓ) is defined as

$$X = \varprojlim_{\ell\geq 0} X_\ell = \left\{ x = (x_\ell)_{\ell\geq 0} \in \prod_{\ell\geq 0} X_\ell \,\middle|\, \forall \ell \geq 0 \quad x_\ell = \pi_\ell(x_{\ell+1}) \right\}.$$

To any non-negative projective system we can associate a natural Hilbert space.

LEMMA B.3. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a non-negative projective system and let X be the algebraic projective limit of the projective system (X_ℓ, π_ℓ). We let $L \subset X$ be the set of those $x = (x_\ell)_{\ell\geq 0}$ in X such that*

$$\sup_{\ell\geq 0} q_\ell(x_\ell, x_\ell) < \infty.$$

Then, L is a vector subspace of X and there exists a unique non-negative symmetric bilinear form q on L with

$$q(x,x) = \sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell), \quad x \in L.$$

The space $H = L/\ker q$, equipped with the positive definite bilinear form induced by q is complete.

DEFINITION B.4. If $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ is a non-negative projective system the Hilbert space H from Lemma B.3 is called the Euclidean projective limit of the system.

REMARK B.5. Note that in general, there is no reason for H not to be reduced to 0. We will address this question in the next sections.

PROOF OF LEMMA B.3. By Minkowski inequality (that is, the triangle inequality for non-negative quadratic forms), for any x, y in L, we have, for $\ell \geq 0$,

$$q_\ell(x_\ell + y_\ell, x_\ell + y_\ell) \leq (q_\ell(x_\ell)^{\frac{1}{2}} + q_\ell(y_\ell)^{\frac{1}{2}})^2,$$

hence $x + y$ belongs to L. Now, we set

$$q(x,y) = \lim_{\ell \to \infty} \frac{1}{2}(q_\ell(x+y, x+y) - q_\ell(x,x) - q_\ell(y,y))$$

One checks that q is a symmetric bilinear form and that, for any x in L,

$$q(x,x) = \sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell).$$

In particular, q is non-negative.

It remains to prove that the space $H = L/\ker q$ is complete for the bilinear form induced by q, which we still denote by q. First, we assume that, for any $\ell \geq 0$, q_ℓ is positive definite. In this case, we have $\ker q = \{0\}$ and $H = L$. We need to prove that any absolutely convergent series in H is convergent. Let us pick a sequence (x_n) in H and assume it is absolutely convergent, that is,

$$\sum_n q(x_n, x_n)^{\frac{1}{2}} < \infty.$$

Set, for any n, $x_n = (x_{\ell,n})_{\ell \geq 0}$. We have, for $\ell \geq 0$,

$$\sum_n q_\ell(x_{\ell,n}, x_{\ell,n})^{\frac{1}{2}} < \infty.$$

Hence the series $\sum_n x_{\ell,n}$ converges in the finite-dimensional Hilbert space (X_ℓ, q_ℓ) towards an element x_ℓ. By uniqueness of the limit, we have $\pi_\ell(x_{\ell+1}) = x_\ell$. Therefore, the element $x = (x_\ell)_{\ell \geq 0}$ in $\prod_{\ell \geq 0} X_\ell$

actually belongs to the algebraic projective limit X. Now, for any ℓ, we have

$$q_\ell(x_\ell, x_\ell) \leq \left(\sum_n q_\ell(x_{\ell,n}, x_{\ell,n})^{\frac{1}{2}} \right)^2 \leq \left(\sum_n q(x_n, x_n)^{\frac{1}{2}} \right)^2,$$

hence x belongs to H. In the same way, one checks that $\sum_n x_n = x$ in H.

In the general case, we let $(\overline{X}_\ell, \overline{q}_\ell, \overline{\pi}_\ell)_{\ell \geq 0}$ be as in Remark B.2 above, that is $\overline{X}_\ell$ is the quotient of X_ℓ by the null space of q_ℓ, $\overline{q}_\ell$ is the induced symmetric bilinear form on $\overline{X}_\ell$ and $\overline{\pi}_\ell : \overline{X}_{\ell+1} \to \overline{X}_\ell$ is the natural map. Then the algebraic projective limit $\overline{X}$ of the projective system $(\overline{X}_\ell, \overline{q}_\ell)_{\ell \geq 0}$ is exactly the quotient of X by the space

$$\ker q = \{x = (x_\ell)_{\ell \geq 0} \in X | \forall \ell \geq 0 \quad x_\ell \in \ker q_\ell\}.$$

We are brought back to the case where all the q_ℓ are positive definite. □

B.2. Straight systems

We now would like to have conditions for the Euclidean projective limit to be large. To this aim, we introduce a new notion.

Definition B.6. Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a non-negative projective system. We shall say that the system is straight if, for every $\ell \geq 0$, we have $(\pi_\ell)_\star q_{\ell+1} = q_\ell$, that is, q_ℓ is the Euclidean image of $q_{\ell+1}$.

See Section A.1 for the notion of the Euclidean image of a non-negative bilinear form by a surjective linear map.

Straight systems have good Euclidean projective limits.

Lemma B.7. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a straight non-negative projective system and let H be its Euclidean projective limit. Then, for any $\ell \geq 0$, the natural map $\rho_\ell : H \to X_\ell / \ker q_\ell$ is onto and one has $(\rho_\ell)_\star q = q_\ell$.*

A subspace $L \subset H$ is dense in H if and only if, for any $\ell \geq 0$, $\rho_\ell(L) = X_\ell / \ker q_\ell$ and $(\rho_\ell)_\star q_{|L} = q_\ell$.

Proof. As noticed in Remark B.2, we can assume that, for any $\ell \geq 0$, the symmetric bilinear form q_ℓ is positive definite. Set $W_0 = X_0$ and $p_0 = q_0$ and, for $\ell \geq 1$, set $W_\ell = \ker \pi_{\ell-1}$ and let p_ℓ be the restriction of q_ℓ to W_ℓ. The system being straight, we have an isomorphism $X_\ell \to W_0 \oplus \cdots \oplus W_\ell$ which sends q_ℓ to $p_0 + \cdots + p_\ell$ and which identifies π_ℓ with the natural map $W_0 \oplus \cdots \oplus W_{\ell+1} \to W_0 \oplus \cdots \oplus W_\ell$. Now we see that H may be defined as Hilbertian direct sum of the Euclidean spaces $(W_\ell, p_\ell)_{\ell \geq 0}$. In particular, the first part of the lemma follows easily.

Let now L be a closed subspace of H such that, for any $\ell \geq 0$, $\rho_\ell(L) = X_\ell$ and $(\rho_\ell)_\star q_{|L} = q_\ell$ and let us prove that $L = H$. Indeed, we can identify ρ_ℓ with the orthogonal projection $H \to X_\ell = W_0 \oplus \cdots \oplus W_\ell$. Now, as $(\rho_\ell)_\star q_{|L} = q_\ell$, there exists a closed subspace Y_ℓ of L such that ρ_ℓ is an isometry from Y_ℓ onto X_ℓ. But as ρ_ℓ is an orthogonal projection of H, this implies that $Y_\ell = X_\ell$, hence $X_\ell \subset L$. As this is true, for any ℓ, we get $L = H$ as required. □

B.3. Straightenable systems

We shall now see how to build a straight system from one that is not.

For $k \geq \ell$, let us write $\pi_{k,\ell}$ for the natural product map

$$\pi_{k,\ell} = \pi_{k-1} \cdots \pi_\ell : X_k \to X_\ell.$$

If the system is not straight, we can try to straighten it. There is a natural formula for doing so.

Lemma B.8. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a non-negative projective system. Assume that, for any ℓ and any x in X_ℓ, one has*

$$\Phi_\ell(x) = \sup_{k \geq \ell} (\pi_{k,\ell})_\star q_k(x,x) = \sup_{k \geq \ell} \inf_{\substack{y \in X_k \\ \pi_{k,\ell}(y) = x}} q_k(y,y) < \infty.$$

Then, for any ℓ, Φ_ℓ is a quadratic form on X_ℓ. Let p_ℓ be its polar form. The family $(X_\ell, p_\ell, \pi_\ell)_{\ell \geq 0}$ is a straight non-negative projective system.

Definition B.9. Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a non-negative projective system. We say that it is straightenable if, for any ℓ and any x in X_ℓ, we have

$$\sup_{k \geq \ell} (\pi_{k,\ell})_\star q_k(x,x) < \infty.$$

In this case, the straight non-negative projective system $(X_\ell, p_\ell, \pi_\ell)_{\ell \geq 0}$ from Lemma B.8 is called the straightening of $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$.

Proof of Lemma B.8. Fix $\ell \geq 0$. For any $k \geq \ell$, let Φ_ℓ^k denote the quadratic form $x \mapsto (\pi_{k,\ell})_\star q_k(x,x)$ on X_ℓ. If $k > \ell$, since $q_k \geq (\pi_{k-1})^\star q_{k-1}$, we have, for any x in X_ℓ

$$\Phi_\ell^k(x) \geq \inf_{\substack{y \in X_k \\ \pi_{k,\ell}(y) = x}} q_{k-1}(\pi_k(y), \pi_k(y)) = \Phi_\ell^{k-1}(x),$$

where the latter inequality holds because of the surjectivity of π_{k-1}. In particular, we have

$$\Phi_\ell^k(x) \xrightarrow[k \to \infty]{} \Phi_\ell(x)$$

and Φ_ℓ is a quadratic form. As in the statement, we let p_ℓ denote its polar form. Since $\Phi_\ell \geq \Phi_\ell^\ell$ and Φ_ℓ^ℓ is the quadratic form associated to q_ℓ, Φ_ℓ is non-negative. Finally, for any x in $X_{\ell+1}$ and any $k \geq \ell+1$, we have

$$\{y \in X_k | \pi_{k,\ell+1}(y) = x\} \subset \{y \in X_k | \pi_{k,\ell}(y) = \pi_\ell(y)\},$$

hence $\Phi_{\ell+1} \geq (\pi_\ell)^\star \Phi_\ell$ and the family $(X_\ell, p_\ell, \pi_\ell)_{\ell \geq 0}$ is a non-negative projective system. It remains to prove that this system is straight, which is the main difficulty of the proof.

To this aim, we need to introduce more notation. For any ℓ, let $W_\ell \subset X_\ell$ be the null space of Φ_ℓ. As Φ_ℓ is the non-decreasing limit of the Φ_ℓ^k, $k \geq \ell$, there exists a smallest $k \geq \ell$ such that W_ℓ is the null space of Φ_ℓ^k. We denote it by $j(\ell)$. We also set $V_\ell = X_\ell / W_\ell$.

Now, fix $\ell \geq 0$ and x in X_ℓ. We claim that there exists y in $X_{\ell+1}$ such that $\pi_\ell(y) = x$ and, for any $k \geq \ell+1$, $\Phi_{\ell+1}^k(y) \leq \Phi_\ell(y)$, which finishes the proof

Indeed, by Lemma A.4, for $k \geq \ell+1$, we have $(\pi_\ell)_\star \Phi_{\ell+1}^k = \Phi_\ell^k \leq \Phi_\ell$. As $X_{\ell+1}$ is finite-dimensional, this implies that the set

$$A_k = \{y \in X_{\ell+1} | \pi_\ell(y) = x \text{ and } \Phi_{\ell+1}^k(y) \leq \Phi_\ell(x)\}$$

is not empty. Note that one has $A_{k+1} \subset A_k$. We let B_k be the image of A_k in $V_{\ell+1}$. Then, if $k \geq j(\ell+1)$, B_k is a compact subset of V_k. As this sequence is non-increasing, we have $\bigcap_{k \geq j(\ell+1)} B_k \neq \emptyset$ and we are done. □

The straightened system and the original one have the same Euclidean projective limit:

LEMMA B.10. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell \geq 0}$ be a straightenable non-negative projective system and let $(X_\ell, p_\ell, \pi_\ell)_{\ell \geq 0}$ be its straightening. For any $x = (x_\ell)_{\ell \geq 0}$ in the algebraic projective limit, we have*

$$\sup_{\ell \geq 0} p_\ell(x_\ell, x_\ell) = \sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell).$$

In particular, both systems have the same Euclidean projective limit.

PROOF. On one-hand, we have, for any $\ell \geq 0$, $q_\ell \leq p_\ell$, hence

$$\sup_{\ell \geq 0} q_\ell(x_\ell, x_\ell) \leq \sup_{\ell \geq 0} p_\ell(x_\ell, x_\ell).$$

On the other hand, for any $k \geq \ell \geq 0$, we have $\pi_{k,\ell}(x_k) = x_\ell$, hence

$$p_\ell(x_\ell, x_\ell) = \sup_{k \geq \ell} (\pi_{k,\ell})_\star q_k(x,x) = \sup_{k \geq \ell} \inf_{\substack{y \in X_k \\ \pi_{k,\ell}(y) = x_\ell}} q_k(y,y) \leq \sup_{k \geq \ell} q_k(x_k, x_k).$$

□

The notion of a straightenable system allows us to characterize the case where the Euclidean projective limit is large enough.

PROPOSITION B.11. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a non-negative projective system with Euclidean projective limit H. The following are equivalent:*
(i) *The system is straightenable.*
(ii) *For any $\ell \geq 0$, the natural map $H \to X_\ell/\ker q_\ell$ is onto.*
(iii) *There exists a Hilbert space K and a family $(\theta_\ell)_{\ell\geq 0}$ where, for any $\ell \geq 0$, θ_ℓ is a surjective continuous linear map $K \to X_\ell/\ker q_\ell$ with $\theta_\ell = \pi_\ell\theta_{\ell+1}$ and $q_\ell(\theta_\ell(v), \theta_\ell(v)) \leq \|v\|^2$ for any v in K.*

PROOF. *(i)⇒(ii)* This follows from Lemmas B.7, B.8 and B.10.

(ii)⇒(iii) This is evident by taking $H = K$.

(iii)⇒(i) Let $\ell \geq 0$ and v be in K. For any $k \geq \ell$, we have $\pi_{k,\ell}(\theta_k(v)) = \theta_\ell(v)$. Hence,

$$\sup_{k\geq\ell} \inf_{\substack{y\in X_k \\ \pi_{k,\ell}(y)=\theta_\ell(v)}} q_k(y,y) \leq \|v\|^2 .$$

As the maps θ_ℓ are surjective, the system is straightenable. □

APPENDIX C

Quadratic duality

We recall here some basic facts about the duality between the spaces of symmetric bilinear forms on a vector space and on its dual space.

C.1. Definition and elementary properties

Let V be a finite-dimensional vector space. As usual, we let V^* denote its dual space and $\mathcal{Q}(V)$ denote the space of symmetric bilinear forms on V.

For φ, ψ in V^*, we let $\varphi\psi \in \mathcal{Q}(V)$ denote the bilinear form

$$(v, w) \mapsto \frac{1}{2}(\varphi(v)\psi(w) + \psi(v)\varphi(w))$$

on V. If $\varphi = \psi$, we write φ^2 for $\varphi\varphi$. In the same way, for v, w in V, we let vw denote the bilinear form

$$(\varphi, \psi) \mapsto \frac{1}{2}(\varphi(v)\psi(w) + \psi(v)\varphi(w))$$

on V^* and, when $v = w$, we write v^2 for vv. In the formalism of multilinear algebra, the map $\varphi \mapsto \varphi^2$ (resp. $v \mapsto v^2$) defines an isomorphism between the spaces S^2V^* (resp. S^2V) and $\mathcal{Q}(V)$ (resp. $\mathcal{Q}(V^*)$).

Any p in $\mathcal{Q}(V)$ defines a linear map $\theta_p : V \to V^*$ such that, for v, w in V, one has $p(v, w) = \langle \theta_p v, w\rangle$. The fact that p is symmetric translates into saying that θ_p is equal to its adjoint operator, that is, $\theta_p = \theta_p^*$ (when V is identified with the dual space of V^*!)

For any p in $\mathcal{Q}(V)$ and q in $\mathcal{Q}(V^*)$, we set

$$\langle p, q\rangle = \mathrm{tr}(\theta_p\theta_q) = \mathrm{tr}(\theta_q\theta_p).$$

Lemma C.1. *Let V be a finite-dimensional vector space. For any p in $\mathcal{Q}(V)$ and v in V, we have $\langle p, v^2\rangle = p(v, v)$. In particular, the pairing $\langle ., .\rangle$ between $\mathcal{Q}(V)$ and $\mathcal{Q}(V^*)$ is non-degenerate. For any φ in V^* and v in V, one has $\langle \varphi^2, v^2\rangle = \varphi(v)^2$ and this property uniquely determines the pairing $\langle ., .\rangle$.*

The pairing $\langle ., .\rangle$ is called the quadratic duality in this book.

Proof. For v in V and $q = v^2$, the linear map $\theta_q : V^* \to V$ reads as $\varphi \mapsto \varphi(v)v$. Thus, for p in $\mathcal{Q}(V)$, $\theta_p\theta_q$ is the endomorphism

$w \mapsto p(v,w)v$ of V whose trace is $p(v,v)$. Non-degeneracy follows. The uniqueness property comes from the fact that the φ^2, $\varphi \in V^*$, span $\mathcal{Q}(V)$ as a vector space. □

The quadratic duality behaves well under linear maps.

LEMMA C.2. *Let V and W be finite-dimensional real vector spaces and $T : V \to W$ be a linear map with adjoint linear map $T^* : W^* \to V^*$. If p is a symmetric bilinear form on W and q is a symmetric bilinear form on V^*, we have*

$$\langle T^\star p, q\rangle = \langle p, (T^*)^\star q\rangle.$$

PROOF. Indeed, one has $\theta_{T^\star p} = T^*\theta_p T$ and $\theta_{(T^*)^\star q} = T\theta_q T^*$, hence

$$\langle T^\star p, q\rangle = \mathrm{tr}(T^*\theta_p T\theta_q) = \mathrm{tr}(\theta_p T\theta_q T^*) = \langle p, (T^*)^\star q\rangle.$$

□

C.2. A Euclidean formula

Let V be a finite-dimensional vector space. We will see how the formalism of the quadratic duality allows to describe the classical $\mathrm{GL}(V)$-invariant Riemannian metric on the set $\mathcal{Q}_{++}(V)$ of positive definite symmetric bilinear forms on V.

If p is in $\mathcal{Q}_{++}(V)$, the map $\theta_p : V \to V^*$ is a linear isomorphism. We define the dual form $\delta_V(p)$ of p as the positive symmetric bilinear form $(\theta_p^{-1})^\star p$ on V^*. The map $\delta_V : \mathcal{Q}_{++}(V) \to \mathcal{Q}_{++}(V^*)$ is a smooth diffeomorphism and we can compute its derivative.

LEMMA C.3. *Let V be a finite-dimensional vector space, p be a scalar product on V and q be in $\mathcal{Q}(V)$. We have*

$$\mathrm{d}_p\delta_V(q) = -(\theta_p^{-1})^\star q.$$

For p, q as in the setting, there exists a unique p-symmetric endomorphism A of V with $q(v,w) = p(Av,w)$ for v, w in V. We say that A is the endomorphism which represents q with respect to p.

PROOF. Let first q be in $\mathcal{Q}_{++}(V)$ and A be the p-symmetric endomorphism which represents q with respect to p. For any φ, ψ in V^*, let $v = \theta_p^{-1}\varphi$ and $w = \theta_p^{-1}\psi$ be the vectors such that $\varphi(u) = p(u,v)$ and $\psi(u) = p(u,w)$ for u in V. We have, by definition, $\theta_q = \theta_p A$, hence

$$\delta_V(q)(\varphi,\psi) = q(A^{-1}v, A^{-1}w) = p(v, A^{-1}w).$$

Therefore, for q in $\mathcal{Q}(V)$, $\mathrm{d}_p\delta_V(q)(\varphi,\psi) = -p(v,Aw)$, which we may write as $\mathrm{d}_p\delta_V(q) = -(\theta_p^{-1})^\star q$. □

The derivative of δ_V allows to define the natural Riemannian metric of $\mathcal{Q}_{++}(V)$ (see [**27**, Chapter VI]).

LEMMA C.4. *Let V be a finite-dimensional vector space, p be a scalar product on V and q, r be in $\mathcal{Q}(V)$. We have*

$$-\langle q, \mathrm{d}_p\delta_V(r)\rangle = \mathrm{tr}(AB),$$

where A and B are the p-symmetric endomorphisms of V which represent q and r with respect to p.

PROOF. We have $\theta_q = \theta_p A$ and $\theta_r = \theta_p B$. By Lemma C.3, we get $\theta_{\mathrm{d}_p\delta_V(r)} = -B\theta_p^{-1}$, hence, by definition, $\langle q, \mathrm{d}_p\delta_V(r)\rangle = -\mathrm{tr}(\theta_p AB\theta_p^{-1}) = -\mathrm{tr}(AB)$. □

C.3. A formula for finite sets

We give a formula for the quadratic duality which was used in the proof of the weight formula in Section 12.2.

Let A be a finite set and V be the space of real-valued functions on A. We identify V with its dual space through the bilinear form $(f, g) \mapsto \sum_{a\in A} f(a)g(a)$. As usual, we set $\overline{V}$ to be the quotient space of V by the line of constant functions and we identify the dual space of V with the space $V_0 = \{f \in V \,|\, \sum_a f(a) = 0\}$.

If p and q are symmetric bilinear forms on $\overline{V}$ and V_0, we set, for a, b in A,

$$\varphi_p(a, b) = -p(\mathbf{1}_a, \mathbf{1}_b)$$
$$K_q(a, b) = q(\mathbf{1}_a - \mathbf{1}_b, \mathbf{1}_a - \mathbf{1}_b),$$

where by abuse of notation, we write $\mathbf{1}_a, \mathbf{1}_b$ for their images in $\overline{V}$.

LEMMA C.5. *Let p and q be symmetric bilinear forms on $\overline{V}$ and V_0. We have*

$$\langle p, q\rangle = \frac{1}{2}\sum_{(a,b)\in A^2} \varphi_p(a, b)K_q(a, b).$$

PROOF. Let $T : V \to \overline{V}$ be the natural quotient map, so that T^* is the inclusion $V_0 \hookrightarrow V$. We define a symmetric bilinear form $\hat{q}$ on V by setting, for f, g in V,

$$\hat{q}(f, g) = -\frac{1}{2}\sum_{(a,b)\in A^2} K_q(a, b)f(a)g(b).$$

Then the restriction of $\hat{q}$ to V_0 is q, that is, $(T^*)^\star\hat{q} = q$. Thus, by Lemma C.2, we have

$$\langle p, q\rangle = \langle p, (T^*)^\star\hat{q}\rangle = \langle T^\star p, \hat{q}\rangle.$$

Through the identification between V and its dual space, the basis $(\mathbf{1}_a)_{a\in A}$ is equal to its dual basis, so that, by definition, we have

$$\langle T^\star p, q\rangle = \sum_{(a,b)\in A^2} p(T\mathbf{1}_a, T\mathbf{1}_b)\hat{q}(\mathbf{1}_a, \mathbf{1}_b)$$

and the result follows. □

APPENDIX D

Algebraic properties of standard representations

In this appendix, we achieve certain abstract constructions that were used in the proof of Proposition 19.3 and Theorem 19.5.

Let G be a discrete group. If A is a set, we shall say that an action of G on A is standard if G has finitely many orbits in A and the stabilizers in G of the elements of A are finite. If H is a (real) Hilbert space, equipped with a unitary action of G, we shall say that the representation of G on H is standard if there exists a standard action of G on a set A and a closed non necessarily unitary G-equivariant embedding of H inside the space $\ell^2(A)$ of square-summable functions on A.

It is a widely used fact in the theory of Von Neumann algebras (see [**31**, Chapter 8]) that the algebra $\mathcal{B}(H)^G$ of bounded G-equivariant endomorphisms of H can then be equipped with a natural trace. This functional allows to introduce algebraic structures on spaces related to H that are analogues of the algebraic structures studied in Appendix C. These constructions are classical, but we will give a presentation of them that is adapted to our purpose.

D.1. Standard trace

We begin by defining the above mentioned trace functional. If A is a set, we write $\langle ., . \rangle$ for the usual scalar product on $\ell^2(A)$. If H is a Hilbert space, we denote by $\mathcal{B}(H)$ the algebra of bounded endomorphisms of H. The following is a classical result from the theory of Von Neumann algebras:

PROPOSITION D.1. *Let G be a discrete group and H be a Hilbert space equipped with a standard unitary representation of G. Then there exists a unique linear functional tr_G on $\mathcal{B}(H)^G$ with the following property: let A be a set with a standard G-action and $I : H \to \ell^2(A)$ be a closed G-equivariant embedding. Then, for every T in $\mathcal{B}(H)^G$ and S in $\mathcal{B}(\ell^2(A))^G$ with $SI = IT$ and $S\ell^2(A) \subset IH$, we have*

$$\mathrm{tr}_G\, T = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \langle \mathbf{1}_a, S\mathbf{1}_a \rangle.$$

The functional tr_G *is a trace, meaning that, if* H, L *are Hilbert spaces equipped with standard unitary representations of* G*, for every bounded* G*-equivariant linear maps* $U : H \to L$ *and* $V : L \to H$*, we have*

$$\mathrm{tr}_G(UV) = \mathrm{tr}_G(VU). \tag{D.1}$$

REMARK D.2. The assumption on S and T in the statement means that, for a direct decompostion of $\ell^2(A)$ as $H \oplus L$, the operator S has a matrix of the form

$$\begin{pmatrix} T & * \\ 0 & 0 \end{pmatrix}.$$

DEFINITION D.3. Let G be a discrete group and H be a standard unitary representation of G. Then the functional tr_G is called the standard trace on $\mathcal{B}(H)^G$.

Assume the set A is equipped with a standard G-action. We start building the standard trace in case $H = \ell^2(A)$. Let T be in $\mathcal{B}(\ell^2(A))^G$. We set

$$\mathrm{tr}_G(T) = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \langle \mathbf{1}_a, T\mathbf{1}_a \rangle. \tag{D.2}$$

LEMMA D.4. *Let* G *be a discrete group,* A *be a set equipped with a standard action of* G *and* S, T *be in* $\mathcal{B}(\ell^2(A))^G$*. For* a, b *in* A*, set* $\varphi(a,b) = \langle \mathbf{1}_a, S\mathbf{1}_b \rangle$ *and* $\psi(a,b) = \langle \mathbf{1}_a, T\mathbf{1}_b \rangle$*. Then we have*

$$\mathrm{tr}_G(ST) = \sum_{(a,b) \in G\backslash A^2} \frac{1}{|G_a \cap G_b|} \varphi(a,b)\psi(b,a), \tag{D.3}$$

where the series is absolutely convergent. In particular, we have $\mathrm{tr}_G(ST) = \mathrm{tr}_G(TS)$*, that is, the functional* tr_G *is a trace on* $\mathcal{B}(\ell^2(A))^G$*.*

PROOF. Indeed, for a, b in A, we have

$$\langle \mathbf{1}_a, ST\mathbf{1}_a \rangle = \sum_{b \in A} \varphi(a,b)\psi(b,a),$$

meaning that the series is absolutely convergent. By the definition of the trace in (D.2), we get

$$\mathrm{tr}_G(ST) = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \sum_{b \in A} \varphi(a,b)\psi(b,a).$$

By using Lemma 10.11, (D.3) follows. □

Let $H \subset \ell^2(A)$ be a closed G-invariant subspace $I : H \to \ell^2(A)$ be the natural injection and $P : \ell^2(A) \to H$ be the orthogonal projection. For T in $\mathcal{B}(H)^G$, set

$$\mathrm{tr}_G(T) = \mathrm{tr}_G(ITP). \tag{D.4}$$

LEMMA D.5. *Let G be a group and A be a set equipped with a standard G-action. Let H be a closed G-invariant subspace of $\ell^2(A)$. Then, for any S in $\mathcal{B}(H)^G$ with $IT = SI$ and $S\ell^2(A) \subset IH$, we have*

$$\mathrm{tr}_G(T) = \mathrm{tr}_G(S).$$

In particular, if $Q : \ell^2(A) \to H$ is a non necessarily orthogonal projection, we have

$$\mathrm{tr}_G(T) = \mathrm{tr}_G(ITQ).$$

PROOF. As the range of S is contained in IH, we have $S = IPS$. Therefore, by Lemma D.4,

$$\mathrm{tr}_G(S) = \mathrm{tr}_G(IPS) = \mathrm{tr}_G(SIP) = \mathrm{tr}_G(ITP) = \mathrm{tr}_G(T).$$

If Q is as in the statement, we can set $S = ITQ$ and the conclusion follows. □

PROOF OF PROPOSITION D.1. Let us prove the uniqueness property of the trace. We will show that the trace tr_G defined on $\mathcal{B}(H)^G$ in (D.4) does not depend on the embedding of H as a subspace of the space of square-integrable functions on some standard action of G.

Let A and B be two sets, both equipped with standard G-actions. Assume that we are given two G-equivariant closed embeddings $I : H \to \ell^2(A)$ and $J : H \to \ell^2(B)$. Set $C = A \sqcup B$ and equip it with the natural action. Identify $\ell^2(C)$ with $\ell^2(A) \oplus \ell^2(B)$. Then, a direct computation shows that the traces on $\mathcal{B}(\ell^2(A))^G$ and $\mathcal{B}(\ell^2(B))^G$ are the same as the ones induced by the closed embeddings of $\ell^2(A)$ and $\ell^2(B)$ inside $\ell^2(C)$. Denote by $P : \ell^2(C) \to \ell^2(A)$ and $Q : \ell^2(C) \to \ell^2(B)$ the orthogonal projections which we consider as endomorphisms of $\ell^2(C)$. Write $K = I \oplus J$ for the diagonal embedding of H in $\ell^2(C)$. We get

$$PK = I \text{ and } QK = J. \tag{D.5}$$

Temporarily write tr_G^A, tr_G^B and tr_G^C for the traces on $\mathcal{B}(H)^G$ associated with the embeddings I, J and K. Let $P' : \ell^2(A) \to H$ and $Q' : \ell^2(B) \to H$ be the orthogonal projections. Note that $P'P$ and $Q'Q$ are projections $\ell^2(C) \to H$. Therefore, for T in $\mathcal{B}(H)^G$, by Lemma D.5, we have

$$\mathrm{tr}_G^C(T) = \mathrm{tr}_G(KTP'P) = \mathrm{tr}_G(KTQ'Q). \tag{D.6}$$

By using (D.5), we get

$$\mathrm{tr}_G(KTP'P) = \mathrm{tr}_G(KTP'P^2) = \mathrm{tr}_G(PKTP'P) = \mathrm{tr}_G(ITP) = \mathrm{tr}_G^A(T)$$

and in the same way, $\mathrm{tr}_G(KTQ'Q) = \mathrm{tr}_G^B(T)$. By (D.6), we get

$$\mathrm{tr}_G^A(T) = \mathrm{tr}_G^C(T) = \mathrm{tr}_G^B(T)$$

as required. By Lemma D.5, we have built a functional on $\mathcal{B}(H)^G$ with the required properties.

To conclude, it remains to prove (D.1). Thus, we let H and L be Hilbert spaces with standard representations of G, and $U : H \to L$ and $V : L \to H$ be bounded G-equivariant linear maps. We set M to be the Hilbert space $H \oplus L$, and S and T to be the endomorphisms defined by the matrices

$$S = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}.$$

By the uniqueness of the trace and by Lemma D.4, we have

$$\mathrm{tr}_G(UV) = \mathrm{tr}_G(ST) = \mathrm{tr}_G(TS) = \mathrm{tr}_G(VU).$$

□

D.2. Trace inequalities

We will use the standard trace to write inequalities in $\mathcal{B}(H)^G$ which are analogues of inequalities for endomorphisms of finite dimensional vector spaces.

We first state a consequence of (D.3).

COROLLARY D.6. *Let G be a group and H be a Hilbert space with scalar product p. Assume H is equipped with a standard unitary representation of G. Then the bilinear form*

$$(S, T) \mapsto \mathrm{tr}_G(S^{\dagger_p} T)$$

is a scalar product on $\mathcal{B}(H)^G$.

PROOF. Let A be a standard G-set. If $H = \ell^2(A)$, the result is clear from (D.3).

In general, let $I : H \to \ell^2(A)$ be a closed embedding and assume first that H is equipped with the restriction of the usual scalar product of $\ell^2(A)$. Then, the adjoint of I is the orthogonal projection $P : \ell^2(A) \to H$. Therefore, for S, T in $\mathcal{B}(H)^G$, we have

$$\mathrm{tr}_G(S^\dagger T) = \mathrm{tr}_G(IS^\dagger TP) = \mathrm{tr}_G(IS^\dagger PITP) = \mathrm{tr}_G((ISP)^\dagger (ITP)),$$

and the result follows from the previous case.

Finally, suppose p is any scalar product on H, which defines the topology of H. We let V be the endomorphism of H which represents p with respect to the scalar product induced by the usual scalar product $\langle .,. \rangle$ of $\ell^2(A)$. As p is positive and defines the topology of H, V is invertible and positive. For T in $\mathcal{B}(H)$ and x, y in H, we have

$$p(Tx, y) = \langle VTx, y \rangle = \langle x, T^\dagger V y \rangle = p(x, V^{-1}T^\dagger V y),$$

hence $T^{\dagger p} = V^{-1}T^\dagger V$. Let W be the square-root of V, that is, the unique self-adjoint endomorphism with respect to the usual scalar product which is non-negative and such that $W^2 = V$. As V commutes with G, so does W by uniqueness. For S, T in $\mathcal{B}(H)^G$, we have

$$\begin{aligned}\mathrm{tr}_G(S^{\dagger p}T) &= \mathrm{tr}_G(V^{-1}S^\dagger VT) = \mathrm{tr}_G(W^{-2}S^\dagger W^2 T) \\ &= \mathrm{tr}_G(W^{-1}S^\dagger W^2 T W^{-1}) = \mathrm{tr}_G((WSW^{-1})^\dagger (WTW^{-1})).\end{aligned}$$

The result follows by the previous case. □

Corollary D.7. *Let G be a group and H be a Hilbert space with a standard unitary representation of G. Let T be a non-negative self-adjoint element of $\mathcal{B}(H)^G$. Then we have*

$$\mathrm{tr}_G(T) \geq 0.$$

Proof. Let S be the square-root of T. Then S is self-adjoint and belongs to $\mathcal{B}(H)^G$. By Corollary D.6, we have

$$\mathrm{tr}_G(T) = \mathrm{tr}_G(S^2) \geq 0.$$

□

If H is a standard unitary representation of G, the trace of the identity endomorphism of H is called the Von Neumann dimension of H and denoted by $\dim_G H$. We get a bound of the norm of the trace functional on $\mathcal{B}(H)^G$.

Corollary D.8. *Let G be a group and H be a Hilbert space with a standard unitary representation of G. For any T in $\mathcal{B}(H)^G$, we have*

$$\mathrm{tr}_G(T) \leq \|T\| \dim_G(H),$$

where $\|T\|$ is the operator norm of T.

Proof. By Corollary D.6 we get, thanks to Cauchy-Schwarz inequality,

$$\mathrm{tr}_G(T)^2 \leq \mathrm{tr}_G(T^\dagger T) \dim_G H.$$

Besides, $T^\dagger T$ is self-adjoint and has norm $\|T\|^2$. Therefore, the endomorphism $\|T\|^2 - T^\dagger T$ is self-adjoint and non-negative. Thus, by Corollary D.7, we get

$$\mathrm{tr}_G(T^\dagger T) \leq \|T\|^2 \dim_G H.$$

The conclusion follows. □

D.3. Standard quadratic pairing

Now that we have introduced the trace, we can use it to define an analogue of the quadratic duality of Appendix C, that is, the natural duality between $\mathcal{Q}(V)$ and $\mathcal{Q}(V^*)$ where V is a finite dimensional vector space.

Let G be a group and H be a Hilbert space, equipped with a standard unitary representation of G. We write $\mathcal{Q}^\infty(H)$ for the space of bounded symmetric bilinear forms on H and $\mathcal{Q}^\infty(H)^G$ for the set of G-invariant such forms. Let H' be the topological dual space of H: then, the dual action of G on H' is standard. Let q be in $\mathcal{B}(H)^G$ and let r be in $\mathcal{B}(H')^G$. There exists G-equivariant bounded linear operators $S : H \to H'$ and $T : H' \to H$ such that, for any v, w in H and φ, ψ in H', one has

$$q(v, w) = (Sv)(w) \text{ and } r(\varphi, \psi) = \varphi(T\psi).$$

We set

$$\langle q, r\rangle_G = \mathrm{tr}_G(ST) = \mathrm{tr}_G(TS).$$

We call $\langle ., .\rangle_G$ the standard quadratic pairing between $\mathcal{Q}^\infty(H)^G$ and $\mathcal{Q}^\infty(H')^G$.

This pairing behaves well under linear maps.

Lemma D.9. *Let G be a group and H and L be standard unitary representations of G. Let $S : H \to L$ be a G-equivariant bounded linear map, q be in $\mathcal{Q}^\infty(L)^G$ and r be in $\mathcal{Q}^\infty(H')^G$. We have*

$$\langle S^\star q, r\rangle_G = \langle q, (S')^\star r\rangle_G$$

where $S' : L' \to H'$ is the adjoint linear map of S.

Proof. Let $T : L \to L'$ and $U : H' \to H$ be the bounded linear maps associated to q and r. We have

$$\begin{aligned}\langle S^\star q, r\rangle_G = \mathrm{tr}_G((S'TS)U) = \mathrm{tr}_G(S'TSU) = \mathrm{tr}_G(TSUS') \\ = \mathrm{tr}_G(T(SUS')) = \langle q, (S')^\star r\rangle_G.\end{aligned}$$

□

The standard pairing allows to state an analogue of Lemma 14.9

PROPOSITION D.10. *Let G be a group, H be a Hilbert space with a standard unitary representation of G and p be in $\mathcal{Q}^\infty(H)^G$. Then p is non-negative if and only if, for every non-negative q in $\mathcal{Q}^\infty(H')^G$, one has $\langle p, q\rangle_G \geq 0$.*

The proof uses a construction of bounded G-invariant bilinear forms.

LEMMA D.11. *Let A be a standard G-set and φ be a finitely supported function on $\Gamma\backslash A^2$. Then, the bilinear form defined on finitely supported functions by $(\rho, \theta) \mapsto \sum_{(a,b)\in A^2} \varphi(a,b)\rho(a)\theta(b)$ is bounded on $\ell^2(A)$.*

PROOF. We first note that, for ρ, θ in $\ell^2(A)$ and (a, b) in A^2, we have, by Cauchy-Schwarz inequality,

$$\sum_{g\in G} |\rho(ga)\theta(gb)| \leq \left(\sum_{g\in G} \rho(ga)^2\right)^{\frac{1}{2}} \left(\sum_{g\in G} \theta(gb)^2\right)^{\frac{1}{2}} \leq |G_a|^{\frac{1}{2}} |G_b|^{\frac{1}{2}} \|\rho\|_2 \|\theta\|_2 .$$

Therefore, if $S \subset \Gamma\backslash A^2$ is the support of φ, by Lemma 13.20, we have

$$\sum_{(a,b)\in A^2} |\varphi(a,b)\rho(a)\theta(b)| = \sum_{(a,b)\in G\backslash A^2} \frac{1}{|G_a \cap G_b|} |\varphi(a,b)| \sum_{g\in G} |\rho(ga)\theta(gb)| \leq |S| \|\varphi\|_\infty \sup_{a\in A} |G_a| \|\rho\|_2 \|\theta\|_2 .$$

The conclusion follows. □

PROOF OF PROPOSITION D.10. First assume that p is non-negative. Let q be a G-invariant coercive bilinear form on H', that is, a scalar product which defines the topology of H'. We write $\langle ., .\rangle$ for the duality between H and H' and $S : H \to H'$ and $T : H' \to H$ for the linear maps associated to p and q. By definition, we have $\langle p, q\rangle_G = \mathrm{tr}_G(ST)$. For x, y in H', we have $q(x, y) = \langle x, Ty\rangle$ and

$$p(Tx, Ty) = \langle STx, Ty\rangle = q(STx, y).$$

Therefore, the G-equivariant endomorphism ST is self-adjoint and non-negative with respect to q on H'. By Corollary D.7, we get $\langle p, q\rangle_G = \mathrm{tr}_G(ST) \geq 0$. Besides, by Corollary D.8, the quadratic pairing is continuous on $\mathcal{Q}^\infty(H)^G \times \mathcal{Q}^\infty(H')^G$ for the norm topology. As

the set of coercive symmetric bilinear forms is dense in the set of non-negative bilinear forms, we still get $\langle p, q\rangle_G \geq 0$ for any non-negative q in $\mathcal{Q}^\infty(H')^G$.

Conversely, suppose now that, for any non-negative q in $\mathcal{Q}^\infty(H')^G$, we have $\langle p, q\rangle_G \geq 0$, and let us show that p is non-negative.

Assume first that H is $\ell^2(A)$, equipped with the usual scalar product, where A is a standard G-set. Let θ be a finitely supported function on A and let us show that $p(\theta, \theta) \geq 0$. For (a, b) in A^2, set

$$\varphi(a, b) = \sum_{g\in G} \theta(ga)\theta(gb). \tag{D.7}$$

By construction, the support of φ has finite image in $G\backslash A^2$. Hence, by Lemma D.11, there exists q in $\mathcal{Q}^\infty(H)^G$ with

$$\varphi(a, b) = q(\mathbf{1}_a, \mathbf{1}_b) \quad a, b \in A.$$

We claim that q is non-negative. Indeed, if ρ is another finitely supported function, we have

$$\begin{aligned} q(\rho, \rho) = \sum_{(a,b)\in A^2} \varphi(a, b)\rho(a)\rho(b) &= \sum_{g\in G} \sum_{(a,b)\in A^2} \theta(ga)\theta(gb)\rho(a)\rho(b) \\ &= \sum_{g\in G} \left(\sum_{a\in A} \theta(ga)\rho(a) \right)^2 \geq 0. \end{aligned}$$

Let $U : \ell^2(A)' \to \ell^2(A)$ be the isomorphism associated to the usual scalar product. By assumption, we have $\langle p, U^\star q\rangle_G \geq 0$. Now, by (D.3), we get

$$\langle p, U^\star q\rangle_G = \sum_{(a,b)\in G\backslash A^2} \frac{1}{|G_a \cap G_b|} p(\mathbf{1}_a, \mathbf{1}_b)\varphi(a, b),$$

hence, thanks to Lemma 13.20 and the definition of φ in (D.7),

$$\langle p, U^\star q\rangle_G = \sum_{(a,b)\in A^2} p(\mathbf{1}_a, \mathbf{1}_b)\theta(a)\theta(b) = p(\theta, \theta).$$

We get $p(\theta, \theta) \geq 0$ as required.

In general, if $I : H \to \ell^2(A)$ is a closed embedding, we let $P : \ell^2(A) \to H$ be the orthogonal projection. Then, the adjoint P' of P is a closed embedding $H' \to \ell^2(A)'$. If q is a non-negative element of $\mathcal{Q}^\infty(\ell^2(A)')^G$, then $(P')^\star q$ is non-negative on H' and, by Lemma D.9, we have

$$\langle P^\star p, q\rangle_G = \langle p, (P')^\star q\rangle_G \geq 0.$$

By the previous case, $P^\star p$ is non-negative. As P is surjective, p is non-negative. □

D.4. Standard scalar product

We finally define an analogue of the $\mathrm{GL}(V)$-invariant Riemannian metric on the space $\mathcal{Q}_{++}(V)$ of positive definite symmetric bilinear forms on V, where V is a finite dimensional real vector space (see [**27**, Chapter VI]).

Let H be a standard unitary representation of G and p be a G-invariant scalar product on H which defines the topology of H. Then p induces a G-equivariant isomorphism from H onto H'. For q in $\mathcal{Q}^\infty(H)^G$, let q' be the element of $\mathcal{Q}^\infty(H')^G$ defined through this isomorphism. For q, r in $\mathcal{Q}^\infty(H)^G$, we set

$$p_G^*(q,r) = \langle q', r\rangle_G.$$

We have formulated the definition in this way in order to emphasize the relation with the standard quadratic pairing. Here comes an equivalent definition.

Lemma D.12. *Let G be a group and H be a standard unitary representation of G with scalar product p. Let q and r in $\mathcal{Q}^\infty(H)^G$ be respectively represented by the self-adjoint endomorphisms S and T. Then we have*

$$p_G^*(q,r) = \mathrm{tr}_G(ST).$$

In particular, the bilinear form p_G^* is symmetric and positive definite on $\mathcal{Q}^\infty(H)^G$.

Definition D.13. Given a group G and a standard unitary representation H with scalar product p as above, the associated scalar product p_G^* on $\mathcal{Q}^\infty(H)^G$ is called the standard scalar product.

Proof of Lemma D.12. For v, w in H, we have

$$q(v,w) = p(Sv,w) \text{ and } r(v,w) = p(Tv,w).$$

Let $U : H \to H'$ be the G-equivariant isomorphism associated to p. The linear map $H' \to H$ associated to q' is SU^{-1} and the linear map $H \to H'$ associated to r is UT. By definition, we get

$$p_G^*(q,r) = \langle q', r\rangle_G = \mathrm{tr}_G(SU^{-1}UT) = \mathrm{tr}_G(ST). \tag{D.8}$$

The remainder of the proof follows from Corollary D.6. □

The construction of the standard scalar product is natural in the following sense:

Lemma D.14. *Let G be a group and H and L be standard unitary representations of G, with scalar products p and q. Let $T : H \to L$ be*

a surjective G-equivariant continuous linear map with $T_\star p = q$. Then, for every r, s in $\mathcal{Q}(L)$, we have

$$p_G^*(T^\star r, T^\star S) = q_G^*(r, s).$$

As in Appendix A, the notation $T_\star p = q$ means that the adjoint operator $T^\dagger$ of T is an isometric embedding $L \to H$.

PROOF. Let A and B be the continuous bounded endomorphisms of L which represent r and s with respect to q. Then, $T^\star r$ and $T^\star s$ are respectively represented by $T^\dagger AT$ and $T^\dagger BT$ with respect to p. By Lemma D.12, we have $q_G^*(r, s) = \mathrm{tr}_G(AB)$ and, as by assumption $TT^\dagger$ is the identity operator of L,

$$\begin{aligned} p_G^*(T^\star r, T^\star S) = \mathrm{tr}_G(T^\dagger ATT^\dagger BT) = \mathrm{tr}_G(T^\dagger ABT) = \mathrm{tr}_G(ABTT^\dagger) \\ = \mathrm{tr}_G(AB). \end{aligned}$$

□

Bounded G-equivariant endomorphisms act as bounded endomorphisms on $\mathcal{Q}^\infty(H)^G$ with respect to the standard scalar product.

LEMMA D.15. *Let G be a group, H be a standard unitary representation of G with scalar product p and T be in $\mathcal{B}(H)^G$. Write $\|T\|$ for the operator norm of T with respect to p. Then, for q in $\mathcal{Q}^\infty(H)^G$, we have*

$$p_G^*(T^\star q, T^\star q) \leq \|T\|^4 p_G^*(q, q).$$

PROOF. Let S be the element of $\mathcal{B}(H)^G$ that represents q with respect to p. For x, y in H, we have

$$T^\star q(x, y) = q(Tx, Ty) = p(STx, Ty) = p(T^\dagger STx, y),$$

hence, by Lemma D.12,

$$p_G^*(T^\star q, T^\star q) = \mathrm{tr}_G(T^\dagger STT^\dagger ST).$$

For x, y in H, we have

$$\begin{aligned} p(T^\dagger STT^\dagger STx, y) = p(T^\dagger STx, T^\dagger STy) \leq \|T\|^2 p(STx, STy) \\ = \|T\|^2 p(T^\dagger S^2 Tx, y), \end{aligned}$$

that is, the symmetric endomorphism $\|T\|^2 T^\dagger S^2 T - T^\dagger STT^\dagger ST$ is nonnegative. Therefore, by Lemma D.7 and the trace property, we get

$$\mathrm{tr}_G(T^\dagger STT^\dagger ST) \leq \|T\|^2 \mathrm{tr}_G(T^\dagger S^2 T) = \|T\|^2 \mathrm{tr}_G(STT^\dagger S).$$

Again, for x, y in H,

$$p(STT^\dagger Sx, y) = p(T^\dagger Sx, T^\dagger Sy) \leq \|T\|^2 p(Sx, Sy) = \|T\|^2 p(S^2 x, y),$$

hence, still by Lemma D.7,

$$\mathrm{tr}_G(STT^\dagger S) \leq \|T\|^2 \,\mathrm{tr}_G(S^2).$$

The result follows from this chain of inequalities. □

COROLLARY D.16. *Let G be a group and H be a standard unitary representation of G with scalar product p. If q is another G-invariant scalar product on H which defines the topology of H, then the norms associated to the standard scalar products p^*_G and q^*_G on $\mathcal{Q}^\infty(H)^G$ are equivalent.*

PROOF. As usual, let V be the positive invertible element of $\mathcal{B}(H)^G$ which represents q with respect to p, and let W be the square-root of V. For r in $\mathcal{Q}^\infty(H)^G$, we have $q^\star_G(W^\star r, W^\star r) = p^\star_G(r,r)$. The conclusion follows from Lemma D.15. □

In particular, the completion of $\mathcal{Q}^\infty(H)^G$ does not depend on the choice of a G-invariant scalar product. The following is a direct consequence of the definition of the standard scalar product.

COROLLARY D.17. *Let G be a group and H be a standard unitary representation of G with scalar product p. Then the standard quadratic pairing is continuous with respect to the topologies on $\mathcal{Q}^\infty(H)^G$ and $\mathcal{Q}^\infty(H')^G$ associated to a standard scalar product. This pairing defines an identification of the completion of $\mathcal{Q}^\infty(H')^G$ with the topological dual space of the completion of $\mathcal{Q}^\infty(H)^G$.*

We will need the following description of the completion of $\mathcal{Q}(\ell^2(A))^G$ with respect to the standard scalar product.

LEMMA D.18. *Let G be a group and A be a standard G-set. We equip $\ell^2(A)$ with the usual scalar product. For any q in $\mathcal{Q}^\infty(\ell^2(A))^G$, let φ_q be the function on A^2*

$$\varphi_q : A^2 \to \mathbb{R}, (a,b) \mapsto q(\mathbf{1}_a, \mathbf{1}_b).$$

We use the map $\Phi : q \mapsto \varphi_q$ to identify $\mathcal{Q}^\infty(\ell^2(A))^G$ with a subspace of the space of symmetric G-invariant functions on A^2. Then, Φ may be extended as an isometry from the completion of $\mathcal{Q}^\infty(\ell^2(A))^G$ with respect to the standard scalar product onto the space of square-summable symmetric functions φ on $G\backslash A^2$, equipped with the scalar product

$$(\varphi, \psi) \mapsto \sum_{(a,b)\in\Gamma\backslash A^2} \frac{1}{|G_a \cap G_b|} \varphi(a,b)\psi(a,b). \tag{D.9}$$

PROOF. Equation (D.3) implies directly that Φ induces an isometric embedding of the completion of $\mathcal{Q}^\infty(\ell^2(A))^G$ with respect to the

standard scalar product inside the space of square-summable symmetric functions φ on $G\backslash A^2$, equipped with the scalar product (D.9). By Lemma D.11, the range of this embedding contains all symmetric functions with finite support in $G\backslash A^2$. The conclusion follows. $\square$

APPENDIX E

Haagerup inequality

In the course of the book, we have used Haagerup inequality from [**24**] to ensure that certain convolution operators were bounded. In this appendix, we show precisely how to adapt the original statement and proof by Haagerup in order to have them fitting in our framework. This adaptation could also be seen as following from [**25**] and [**30**].

We keep the notation of the book. In particular, X is a tree and Γ is a discrete group of automorphisms of X such that $\Gamma\backslash X$ is finite.

E.1. Norms of convolutors

The original Haagerup inequality dominates the norm of the convolution operator on $\ell^2(\Gamma)$ associated to a function f on $\ell^2(\Gamma)$ by a weighted ℓ^2-norm of f. As, in our case, the action of Γ on X is not necessarily transitive, we use convolution operators by Γ-invariant functions on X_*. We shall first define precisely which kind of norms we will use on the space of such functions.

For φ a Γ-invariant function on X^2, we set

$$(\|\varphi\|_2^\Gamma)^2 = \sum_{(x,y)\in\Gamma\backslash X^2} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi(x,y)^2.$$

LEMMA E.1. *There exists $C > 0$ such that, for any Γ invariant function on φ on X_*, we have*

$$\frac{1}{C}\|\varphi\|_2^\Gamma \leq \sup_{x\in X} \|\varphi(x,.)\|_2 \leq C\|\varphi\|_2^\Gamma,$$

where, for x in X, $\|\varphi(x,.)\|_2$ is the norm of the function $y \mapsto \varphi(x,y)$ in $\ell^2(X)$ if this function belongs to this space and ∞ else.

PROOF. Lemma 10.11 gives

$$(\|\varphi\|_2^\Gamma)^2 = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}\|\varphi(x,.)\|_2^2.$$

The conclusion follows from the fact that $\Gamma\backslash X$ is finite. □

E.2. Bounded convolution operators

We recall that X_* stands for the set of (x, y) in X^2 with $x \neq y$ and that, for $k \geq 1$, X_k stands for the set of (x, y) with $d(x, y) = k$. As above, we write $\ell^2_-(X_1)$ for the space of square-integrable skew-symmetric functions on X_1. For a Γ-invariant function φ on X_*, we defined in (11.1) the associated convolution operator P_φ by the following formula: if ψ is a finitely supported skew-symmetric function ψ on X_1, for (x, y) in X_1,

$$P_\varphi\psi(x,y) = \sum_{\substack{(a,b)\in X_1\\ y,b\in[xa]}} \varphi(x,a)\psi(b,a) - \sum_{\substack{(a,b)\in X_1\\ x,b\in[ya]}} \varphi(y,a)\psi(b,a) - \frac{1}{2}(\varphi(x,y)+\varphi(y,x))\psi(x,y).$$

Haagerup inequality states as

PROPOSITION E.2. *Let φ be a Γ-invariant function φ on X_* such that*

$$\sum_{(x,y)\in\Gamma\backslash X_*} \varphi(x,y)^2 d(x,y)^\alpha < \infty$$

for some $\alpha > 2$. Then the convolution operator P_φ is bounded in $\ell^2_-(X_1)$.

We will prove this by following the same lines as in [**24**]. First, we translate [**24**, Lemma 1.3] which is the key observation of the proof. We fix a point o in X that will play the role of an origin and, for any integer $k \geq 1$, we set

$$Y_k = \{(x, y) \in X_1 | \max(d(o, x), d(o, y)) = k\}.$$

We get

LEMMA E.3. *There exists $C > 0$ such that the following holds. Let $j, k, \ell \geq 1$ and φ be a Γ-invariant function on X_*, with support on X_j and ψ be a skew-symmetric function on X_1 with support on Y_k. If $j \leq k + \ell$, $k \leq \ell + j$ and $\ell \leq j + k$, we have*

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2 \leq C \|\varphi\|_2^\Gamma \|\psi\|_2 .$$

In all other cases, we have $P_\varphi\psi = 0$ on Y_ℓ.

PROOF. If $j = 1$, P_φ is just the multiplication operator by the function $(x, y) \mapsto \frac{1}{2}(\varphi(x, y) + \varphi(y, x))$ on $\ell^2_-(X_1)$. The required inequality follows since all the norms on the finite-dimensional space of Γ-invariant functions on X_1 are equivalent.

Therefore, we assume $j \geq 2$. In that case, for any function ψ in $\ell^2_-(X_1)$ and any $x \sim y$ in X, we have

$$P_\varphi\psi(x,y) = \sum_{\substack{a\in S^j(x)\\ y\in[xa]}} \varphi(x,a)\psi(a_1,a) - \sum_{\substack{b\in S^j(y)\\ x\in[yb]}} \varphi(y,b)\psi(b_1,b),$$

where, for a, b as above a_1, b_1 are their neighbour which are closest to $[xy]$. For x in X with $d(x,o) = x \neq o$, we let x_- denote its neighbour on $[ox]$. We must dominate the quantity

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 = 2\sum_{d(x,o)=\ell} (P_\varphi\psi(x_-,x))^2.$$

Now, recall that φ has support in X_j and ψ has support in Y_k. Hence if, for some x with $d(x,o) = \ell$, we have $P_\psi\varphi(x_-,x) \neq 0$, then three possibilities can occur:
— *(i)* there exists a in X with $d(x,a) = j$, $d(a,o) = k$, $x_- \in [ax]$ and $a \notin [ox]$. In that case, we have, by the triangle identity, $j \leq k+\ell$, $k \leq \ell + j$, and $\ell \leq j + k$. Let y be the point such that $[ao]\cap[xo] = [yo]$ and $i = d(y,o)$. As x_- belongs to $[ax]$, we have $y \neq x$ hence $i \leq \ell - 1$. Besides, we have

$$j = d(a,x) = d(a,y) + d(y,x) = (k-i) + (\ell - i) = k + \ell - 2i,$$

so that $2i = k+\ell-j$ and this number must be even. Also, as $i \leq \ell-1$, we have $k \leq j + \ell - 2$ and, as $a \notin [ox]$, we have $i \leq k-1$, hence $\ell \leq j + k - 2$.
— *(ii)* there exists a in $[ox_-]$ with $d(x,a) = j$ and $d(a,o) = k-1$. In that case, we have $\ell = d(x,o) = j + k - 1$.
— *(iii)* there exists b in X with $d(x,b) = j-1$, $d(b,o) = k$ and $x_- \notin [xb]$. In that case, as x_- is not in $[xb]$, we have $x \in [ob]$ hence $k = \ell + j - 1$. The three cases are pictured on Figure 1.

Note in particular that the inequalities over j, k, ℓ imply that no two of those three cases can happen simultaneously.

We now prove the inequality in case *(i)*, that is, we assume that we have $j \leq k+\ell$, $k \leq \ell + j - 2$, $\ell \leq j + k - 2$ and that $j + k + \ell$ is even and we set $i = \frac{1}{2}(k + \ell - j)$. The reasoning above, gives
(E.1)

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 = 2\sum_{d(y,o)=i}\ \sum_{\substack{d(x,y)=\ell-i\\ y\in[xo]}} \left(\sum_{\substack{d(a,y)=k-i\\ [ao]\cap[xo]=[yo]}} \varphi(x,a)\psi(a_-,a)\right)^2.$$

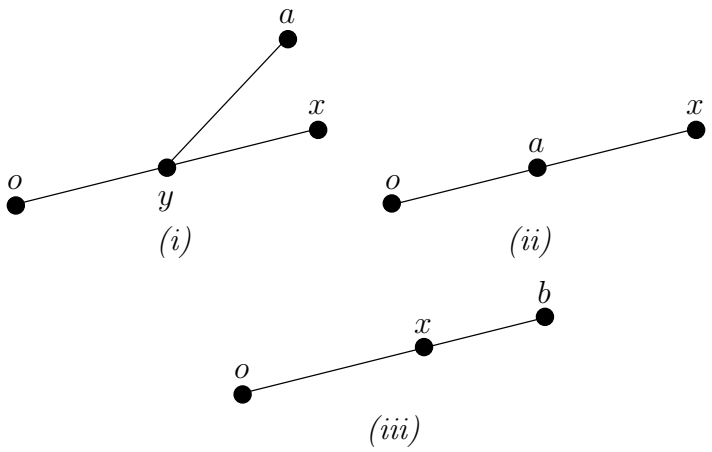

FIGURE 1. The three cases in the proof of Lemma E.3

We define a new Γ-invariant function φ' on X_*. For any (x, y) in X_*, if $d(x, y) = \ell - i$, we set

$$\varphi'(x,y) = \left(\sum_{\substack{d(x,z)=j \\ y\in[xz]}} \psi(x,z)^2 \right)^{\frac{1}{2}} .$$

Else, we set $\varphi'(x, y) = 0$. By (E.1) and Cauchy-Schwarz inequality, we have

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 \leq 2 \sum_{d(y,o)=i} \sum_{\substack{d(a,o)=k \\ y\in[ao]}} \psi(a_-,a)^2 \sum_{\substack{d(x,o)=\ell \\ y\in[xo]}} \varphi'(x,y)^2$$
$$\leq \sup_{y\in X} \|\varphi'(.,y)\|_2^2 \, \|\psi\|_2^2 .$$

Now, let C be as in Lemma E.1, as φ' is Γ-invariant, we have

$$\sup_{y\in X} \|\varphi'(.,y)\|_2 \leq C^2 \sup_{x\in X} \|\varphi'(x,.)\|_2$$

and by the definition of φ',

$$\sup_{x\in X} \|\varphi'(x,.)\|_2 = \sup_{x\in X} \|\varphi(x,.)\|_2 \leq C \, \|\varphi\|_2^\Gamma .$$

The result follows.

In case *(ii)*, we have $\ell = j + k - 1$ and we can write

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 = 2\sum_{d(y,o)=k}\sum_{\substack{d(x,y)=j-1\\ y\in[xo]}}\varphi(x,y_-)^2\psi(y,y_-)^2$$
$$\leq \sup_{y\in X}\|\varphi(.,y)\|_2^2\,\|\psi\|_2^2\,,$$

and we conclude again by Lemma E.1.

Finally, in case *(iii)*, we have $k = \ell + j - 1$ and

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2^2 = 2\sum_{d(x,o)=\ell}\left(\sum_{\substack{d(b,o)=k\\ x\in[bo]}}\varphi(x_-,b)\psi(b_-,b)\right)^2.$$

As in case *(i)*, we define a Γ-invariant function φ' on X_1 by setting, for any (u,v) in X_1,

$$\varphi'(u,v) = \left(\sum_{\substack{d(u,w)=j\\ y\in[xz]}}\psi(u,w)^2\right)^{\frac{1}{2}},$$

so that Cauchy-Schwarz inequality gives

$$\|(P_\varphi\psi)\mathbf{1}_{Y_\ell}\|_2 \leq \sup_{(u,v)\in X_1}\varphi'(u,v)\,\|\psi\|_2\,,$$

and we conclude as above. □

From Lemma E.3, we easily deduce Proposition E.2 as in [**24**, Lemma 1.4, Lemma 1.5].

Proof of Proposition E.2. Pick a Γ-invariant function φ on X_*. Let us for the moment fix $j \geq 1$ and set $\varphi_j = \varphi\mathbf{1}_{X_j}$. Let ψ be a finitely suported skew-symmetric function on X_1. For $k \geq 1$, we write $\psi_k = \psi\mathbf{1}_{Y_k}$, so that $\psi = \sum_{k\geq 1}\psi_k$. By Lemma E.3, we can find $C > 0$ such that, for any such ψ, for $\ell \geq 1$, one has

$$\left\|P_{\varphi_j}\psi\mathbf{1}_{Y_\ell}\right\|_2 \leq \sum_{k\geq 1}\left\|(P_{\varphi_j}\psi_k)\mathbf{1}_{Y_\ell}\right\|_2 \leq C\,\|\varphi_j\|_2^\Gamma\sum_{k=|j-\ell|}^{j+\ell}\|\psi_k\|_2\,.$$

Indeed, for any $k \notin [|j-\ell|, j+\ell]$, Lemma E.3 says that $(P_{\varphi_j}\psi_k)\mathbf{1}_{Y_\ell} = 0$. We can dominate the norm of $P_{\varphi_j}\psi$ by

$$\begin{aligned}
\left\|P_{\varphi_j}\psi\right\|_2^2 &= \sum_{\ell\geq 1}\left\|P_{\varphi_j}\psi\mathbf{1}_{Y_\ell}\right\|_2^2 \\
&\leq C^2(\|\varphi_j\|_2^\Gamma)^2\sum_{\ell\geq 1}\left(\sum_{k=|j-\ell|}^{j+\ell}\|\psi_k\|_2\right)^2 \\
&\leq C^2(\|\varphi_j\|_2^\Gamma)^2\sum_{\ell\geq 1}(2\min(j,\ell)+1)\sum_{k=|j-\ell|}^{j+\ell}\|\psi_k\|_2^2 \\
&\leq C^2 3j(\|\varphi_j\|_2^\Gamma)^2\sum_{k\geq 1}(2\min(j,k)+1)\,\|\psi_k\|_2^2 \\
&\leq C^2 9j^2(\|\varphi_j\|_2^\Gamma)^2\,\|\psi\|_2^2\,,
\end{aligned}$$

where we have used Cauchy-Schwarz inequality.

Now, we have

$$\|P_\varphi\psi\|_2 \leq \sum_{j\geq 1}\left\|P_{\varphi_j}\psi\right\|_2 \leq 3C\,\|\psi\|_2\sum_{j\geq 1} j\,\|\varphi_j\|_2^\Gamma\,.$$

Fix $\alpha > 2$ and set $C' = \sum_{j\geq 1} j^{1-\alpha}$. For any sequence $(x_j)_{j\geq 1}$ of non-negative real numbers, Cauchy-Schwarz inequality gives $(\sum_{j\geq 1} jx_j)^2 \leq C'\sum_{j\geq 1} j^\alpha x_j^2$. Thus, we get

$$\|P_\varphi\psi\|_2^2 \leq 9C^2C'\,\|\psi\|_2^2\sum_{j\geq 1} j^\alpha(\|\varphi_j\|_2^\Gamma)^2$$

and we are done. □

APPENDIX F

Harmonic cocycles

As in the book, we assume that X is a locally finite tree equipped with a proper cofinite action of a group Γ and we use in particular the notations and conventions of Section 2.1. The purpose of this Appendix is to explain how the study of non-negative Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ can be considered as the study of a wide class of unitary representations of Γ, namely the ones admitting a cyclic harmonic first cohomology class.

F.1. Geometric cocycles

We start by recalling the basic definitions of 1-cohomology. We also introduce a geometric version of these definitions. Later, we will show that both versions define the same notion of cohomology.

Let G be a group with a linear representation on a real vector space V. A 1-cocycle of G in V is a map $\sigma : G \to V$ such that, for any g_1, g_2 in G, one has

$$\sigma(g_1 g_2) = \sigma(g_1) + g_1 \sigma(g_2).$$

This cocycle is said to be a coboundary if there exists some v in V such that, for g in G, one has

$$\sigma(g) = gv - v.$$

The space of 1-cocycles is denoted by $\mathrm{Z}^1(G, V)$ and the one of 1-coboundaries by $\mathrm{B}^1(G, V)$. The latter may be identified with the quotient space V/V^G of V by the space of G-invariant elements in V. Two cocycles are said to be cohomologous if their difference is a coboundary. The quotient space $\mathrm{H}^1(G, V) = \mathrm{Z}^1(G, V)/\mathrm{B}^1(G, V)$ is called the first cohomology group of G in V.

Now, assume G is our tree lattice Γ. In this case, the group $\mathrm{H}^1(\Gamma, V)$ may be computed in a different way. Define a geometric cocycle of Γ in V as a Γ-equivariant map $\sigma : X_1 \to V$ which is skew-symmetric, that is,

$$\sigma(y, x) + \sigma(x, y) = 0, \quad (x, y) \in X_1.$$

If $\varphi : X \to V$ is any map, we denote by $\mathrm{d}\varphi : X_1 \to V$ the map defined by

$$\mathrm{d}\varphi(x, y) = \varphi(y) - \varphi(x), \quad (x, y) \in X_1.$$

We shall say that a geometric cocycle σ is a geometric coboundary if there exists a Γ-equivariant map $\varphi : X \to V$ with $\sigma = \mathrm{d}\varphi$. Write $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$ for the space of geometric cocycles and $\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ for the one of geometric coboundaries. Again, two geometric cocycles are said to be cohomologous if their difference is a geometric coboundary.

F.2. Loops and integration

We aim at showing that the quotient space $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ may be identified with the first cohomology group $\mathrm{H}^1(\Gamma, V)$. This identification will rely on an integration procedure of cocycles that we will now introduce.

For x, y in X, we define a path from x to y as a sequence $x_0 = x, x_1, \ldots, x_n = y$ in X with $x_k \sim x_{k-1}$, $1 \leq k \leq n$. As in Section 2.1, we shall say that this path is geodesic if moreover, we have $x_{k-1} \neq x_{k+1}$, $1 \leq k \leq n-1$. By assumption, there exists a unique geodesic path from x to y.

LEMMA F.1. *Let x be in X and $x_0 = x, x_1, \ldots, x_n = x$ be a path from x to itself. Then, n is even and, if $n > 0$ and x does not belong to the set $\{x_1, \ldots, x_{n-1}\}$, we have $x_1 = x_{n-1}$.*

PROOF. We show this statement by induction on n. For $n = 0$, there is nothing to prove. For $n = 1$, there is no path of length 1 from x to itself.

Assume $n \geq 2$ and the statement holds for any $n' < n$. Let $x_0 = x, x_1, \ldots, x_n = x$ be a path from x to itself. Then, by uniqueness, the path is not geodesic, that is, there exists $1 \leq k \leq n-1$ with $x_{k-1} = x_{k+1}$. For $0 \leq j \leq n-2$, we set

$$\begin{aligned} y_j &= x_j && \text{if } j \leq k-1 \\ y_j &= x_{j+2} && \text{if } j \geq k. \end{aligned}$$

Then, $y_0, \ldots, y_{n-2}$ is a path from x to itself. In particular, the induction assumption implies that n is even.

Now, assume x does not belong to the set $\{x_1, \ldots, x_{n-1}\}$. If $n = 2$, we have $x_1 = x_{n-1}$ as required. If $n \geq 4$, then, necessarily, we have $2 \leq k \leq n-2$, $y_1 = x_1$ and $y_{n-3} = x_{n-1}$. As x does not belong to the set $\{y_1, \ldots, y_{n-3}\} \subset \{x_1, \ldots, x_{n-1}\}$, the induction assumption says that we must have $y_1 = y_{n-3}$, hence $x_1 = x_{n-1}$. The conclusion follows. □

Using this result, we can show that the sum of a skew-symmetric map along a path only depends on the endpoints.

COROLLARY F.2. *Let V be a real vector space and $\sigma : X_1 \to V$ be a skew-symmetric map. Then, for any x, y in X, the element of V*

$$\sum_{k=1}^{n} \sigma(x_{k-1}, x_k)$$

does not depend on the choice a path $x_0 = x, x_1, \ldots, x_n = y$ from x to y.

In the sequel, for $\sigma : X_1 \to V$ a skew-symmetric map and x, y in X, we set

$$\sum_{x}^{y} \sigma = \sum_{k=1}^{n} \sigma(x_{k-1}, x_k)$$

where $x_0 = x, x_1, \ldots, x_n = y$ is a path x to y. Note that, for x, y, z in X, we have the chain rule

$$\sum_{x}^{y} \sigma + \sum_{y}^{z} \sigma = \sum_{x}^{z} \sigma. \tag{F.1}$$

PROOF OF COROLLARY F.2. Fix x in X. We claim that, given a path $x_0 = x, x_1, \ldots, x_{2n} = x$ from x to itself, we have $\sum_{k=1}^{2n} \sigma(x_{k-1}, x_k) = 0$. We show this statement by induction on $n \geq 0$.

If $n = 0$, there is nothing to prove. Assume $n \geq 1$ and the statement holds for all $n' < n$. Let $x_0 = x, x_1, \ldots, x_{2n} = x$ be a path from x to itself. If there exits $1 \leq j \leq 2n - 1$ with $x_j = x$, then the sequences $x_0, \ldots, x_j$ and $x_j, \ldots, x_{2n}$ are paths from x to itself and, by the induction assumption, we have

$$\sum_{k=1}^{2n} \sigma(x_{k-1}, x_k) = \sum_{k=1}^{j} \sigma(x_{k-1}, x_k) + \sum_{k=j+1}^{2n} \sigma(x_{k-1}, x_k) = 0.$$

If there exists no such j, by Lemma F.1, we have $x_1 = x_{2n-1}$, hence, the sequence $x_1, \ldots, x_{2n-1}$ is a path from x_1 to itself. By the induction assumption, we get

$$\begin{aligned}\sum_{k=1}^{2n} \sigma(x_{k-1}, x_k) = \sigma(x, x_1) + \sum_{k=2}^{2n-1} \sigma(x_{k-1}, x_k) + \sigma(x_{2n-1}, x) \\ = \sigma(x, x_1) + \sigma(x_1, x) = 0,\end{aligned}$$

as σ is skew-symmetric. The claim follows.

Now assume x, y are in X and $u_0 = x, \dots, u_m = y$ and $v_0 = x, \dots, v_n = y$ are two paths from x to y. Since $u_0, \dots, u_n, v_{m-1}, \dots, v_0$ is a path from x to itself, we get

$$\sum_{k=1}^{m} \sigma(u_{k-1}, u_k) + \sum_{k=1}^{n} \sigma(v_k, v_{k-1}) = 0.$$

As σ is skew-symmetric, this gives

$$\sum_{k=1}^{m} \sigma(u_{k-1}, u_k) = \sum_{k=1}^{n} \sigma(v_{k-1}, v_k)$$

as required. □

F.3. Geometric representation of cohomology

We will use the constructions above to build an isomorphism between the quotient space $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ and the cohomology group $\mathrm{H}^1(\Gamma, V)$. This is an explicit version of [**43**, Proposition II.13].

Note that, for σ in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$, we have the following equivariance property of path summation:

$$\sum_{\gamma x}^{\gamma y} \sigma = \gamma \sum_{x}^{y} \sigma, \quad \gamma \in \Gamma, \quad x, y \in X. \tag{F.2}$$

This together with (F.1) will be instrumental in proving

Proposition F.3. *Let V be a real vector space equipped with a linear action of Γ. Let σ be in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. For x in X and γ in Γ, set*

$$\sigma_x(\gamma) = \sum_{x}^{\gamma x} \sigma.$$

Then σ_x is a 1-cocycle of Γ in V.

If y is another element of X, the cocycles σ_x and σ_y are cohomologous. If θ is a geometric cocycle that is cohomologous to σ, the coycles σ_x and θ_x are cohomologous.

The linear map $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V) \to \mathrm{H}^1(\Gamma, V)$ associated with this construction is an isomorphism.

Proof. Fix σ in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. For x in X and γ_1, γ_2 in Γ, we have, by (F.1) and (F.2),

$$\sigma_x(\gamma_1\gamma_2) = \sum_{x}^{\gamma_1 x} \sigma + \sum_{\gamma_1 x}^{\gamma_1\gamma_2 x} \sigma = \sum_{x}^{\gamma_1 x} \sigma + \gamma_1 \sum_{x}^{\gamma_2 x} \sigma = \sigma_x(\gamma_1) + \gamma_1 \sigma_x(\gamma_2).$$

Thus, σ_x is a 1-cocycle of Γ in V.

Besides, for x, y in X and γ in Γ, still by (F.1) and (F.2), we have

$$\sigma_x(\gamma) = \sum_x^y \sigma + \sum_y^{\gamma y} \sigma + \sum_{\gamma y}^{\gamma x} \sigma = \sum_x^y \sigma + \sigma_y(\gamma) - \gamma \sum_x^y \sigma,$$

hence $\sigma_x - \sigma_y$ is a coboundary.

Assume that σ is a coboundary in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. Then, there exists a Γ-equivariant map $\varphi : X \to V$ with $\sigma = \mathrm{d}\varphi$. For x in X and γ in Γ, we get

$$\sigma_x(\gamma) = \varphi(\gamma x) - \varphi(x) = \gamma\varphi(x) - \varphi(x),$$

hence σ_x is a coboundary in $\mathrm{Z}^1(\Gamma, V)$.

Conversely, fix x in X and suppose σ_x is a coboundary in $\mathrm{Z}^1(\Gamma, V)$. Choose v in V with

$$\sigma_x(\gamma) = \gamma v - v \quad \gamma \in \Gamma.$$

For y in X, we set

$$\varphi(y) = \sum_x^y \sigma + v.$$

We claim that $\varphi : X \to V$ is Γ-equivariant. Indeed, by (F.1) and (F.2), for γ in Γ, we have

$$\varphi(\gamma y) = \sum_x^{\gamma x} \sigma + \sum_{\gamma x}^{\gamma y} \sigma + v = \sigma_x(\gamma) + \gamma \sum_x^y \sigma + v = \gamma v + \gamma \sum_x^y \sigma = \gamma\varphi(y).$$

Again by (F.1), for $y \sim z$ in X, we have $\mathrm{d}\varphi(y, z) = \varphi(z) - \varphi(y) = \sigma(y, z)$, hence σ is a geometric coboundary.

So far, we have shown that the map $\sigma \mapsto \sigma_x$ defines an injective linear map $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V) \to \mathrm{H}^1(\Gamma, V)$ that does not depend on x. To conclude, it remains to prove that this map is surjective. Therefore, we fix θ in $\mathrm{Z}^1(\Gamma, V)$ and we will build σ in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$ such that, for x in X, θ and σ_x are cohomologous.

First, we use a standard trick of finite groups theory to eliminate the difficulties associated with stabilizers of vertices. For x in X, we set

$$v_x = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \theta(\gamma).$$

We notice that, by the cocycle property, for γ in Γ_x,

$$\gamma v_x - v_x = \frac{1}{|\Gamma_x|} \sum_{\eta \in \Gamma_x} (\gamma\theta(\eta) - \theta(\gamma\eta)) = -\theta(\gamma).$$

The latter implies the following property: if γ and η are in Γ and $\gamma x = \eta x$, we have

$$\theta(\gamma) + \gamma v_x = \theta(\eta) + \eta v_x. \tag{F.3}$$

Indeed, we write $\eta = \gamma\zeta$ where ζ belongs to Γ_x and we get, from the cocycle property,

$$\theta(\eta) + \eta v_x - \theta(\gamma) - \gamma v_x = \gamma\theta(\zeta) + \gamma\zeta v_x - \gamma v_x = 0.$$

Now, we fix a system of representatives $S \subset X$ for the action of Γ on X. In other words, we have $X = \Gamma S$ and, for x in S, $S \cap \Gamma x = \{x\}$. To build σ, we will first build a map $\varphi : X \to V$ which will play the role of a primitive of σ, that is, we will have $\sigma = \mathrm{d}\varphi$. More precisely, for x in X, we choose γ in Γ with $\gamma^{-1}x \in S$ and we set

$$\varphi(x) = \theta(\gamma) + \gamma v_{\gamma^{-1}x}.$$

Due to (F.3), this does not depend on the choice of γ. The map $\varphi : X \to V$ is not Γ-equivariant in general. But, for x in X and γ in Γ, we have, by the construction and the cocycle property of θ,

$$\varphi(\gamma x) - \gamma\varphi(x) = \theta(\gamma).$$

Therefore, if we set $\sigma = \mathrm{d}\varphi$, the map $\sigma : X_1 \to V$ is Γ-equivariant. Thus, σ is a geometric cocycle. To conclude, we compute, for x in S and γ in Γ,

$$\sigma_x(\gamma) = \varphi(\gamma x) - \varphi(x) = \theta(\gamma) + \gamma v_x - v_x.$$

Therefore, σ_x and θ are cohomologous as required. □

F.4. Harmonic cohomology classes

In the sequel, we use Proposition F.3 to identify the spaces $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ and $\mathrm{H}^1(\Gamma, V)$. We will now introduce a notion of a harmonic cohomology class that is inspired by Hodge theory and is essentially the same as the one in [**4, 39**].

Let V be a vector space. A skew-symmetric map $\sigma : X_1 \to V$ is said to be harmonic if, for x in X, one has

$$\sum_{y \sim x} \sigma(x, y) = 0.$$

Example F.4. Let V be $\overline{\mathcal{D}}(\partial X)$. For $x \sim y$ in X_1, set $\sigma(x, y) = \mathbf{1}_{U_{xy}}$ where, as usual,

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}.$$

Then, σ is a harmonic skew-symmetric map.

This example is universal in the following sense:

LEMMA F.5. *Let V be a real vector space and let $\sigma : X_1 \to V$ be a harmonic skew-symmetric map. Then, there exists a unique linear map $\rho : \overline{\mathcal{D}}(\partial X) \to V$ such that, for $x \sim y$ in X, one has*

$$\sigma(x, y) = \rho(\mathbf{1}_{U_{xy}}).$$

If V is equipped with an action of Γ and σ is Γ-equivariant (that is, σ is a harmonic geometric cocycle), then ρ is also Γ-equivariant.

PROOF. In case $V = \mathbb{R}$ the existence and uniqueness of ρ are established in Lemma 3.5. The general case can be obtained in the same way. The equivariance property for cocycles directly follows from uniqueness. □

A map $\varphi : X \to V$ will be said to be harmonic if, for x in X, one has

$$\frac{1}{d(x)} \sum_{y \sim x} \varphi(y) = \varphi(x).$$

Let V be a vector space with an action of Γ. We will say that a cohomology class in $\mathrm{H}^1(\Gamma, V)$ is harmonic if it admits a harmonic representative in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. We can describe the obstruction for this representative to be unique.

LEMMA F.6. *Let V be a real vector space equipped with a linear action of Γ. Then, the harmonic representatives of the trivial cohomology class in $\mathrm{H}^1(\Gamma, V)$ are the geometric 1-cocycles of the form $\mathrm{d}\varphi$, where $\varphi : X \to V$ is a harmonic Γ-equivariant map.*

The proof is immediate.

F.5. Unitary representations

We now gather all the previous constructions to describe the space of harmonic cohomology classes of a unitary representation.

If Γ acts on a vector space V, write $\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V)$ for the space of all Γ-equivariant linear maps $\overline{\mathcal{D}}(\partial X) \to V$. In Lemma F.5, we have defined a natural isomorphism between the spaces $\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V)$ and the space of harmonic cocycles in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$.

PROPOSITION F.7. *Let V be a Hilbert space equipped with a unitary action of Γ. The natural map*

$$\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V) \to \mathrm{H}^1(\Gamma, V)$$

is an isomorphism onto the space of harmonic cohomology classes.

The proof uses the following generalization of Appollonius Theorem which gives a strong convexity property of the balls in Hilbert spaces.

LEMMA F.8. *Let V be a Hilbert space, $v_1, \dots, v_n$ be vectors in V and $t_1, \dots, t_n$ be non-negative real numbers with $\sum_{i=1}^n t_i = 1$. We have*

$$\left\| \sum_{1 \leq i \leq n} t_i v_i \right\|^2 + \frac{1}{2} \sum_{1 \leq i,j \leq n} t_i t_j \left\| v_i - v_j \right\|^2 = \sum_{1 \leq i \leq n} t_i \left\| v_i \right\|^2 .$$

The proof is immediate.

PROOF OF PROPOSITION F.7. By Lemma F.5, it only remains to prove that every harmonic cohomology class admits a unique harmonic representative. This, we will show by using the criterion in Lemma F.6.

To this aim, we study harmonic Γ-equivariant maps $\varphi : X \to V$. We claim that any such φ is constant, with values in the space V^Γ of Γ-invariant vectors of V. This will follow from the maximum principle. Indeed, the function $x \mapsto \|\varphi(x)\|$ is Γ-invariant on X. As Γ has finitely many orbits in X, this function reaches its maximum value, that is, the set

$$E = \{x \in X | \, \|\varphi(x)\| = \sup_{y \in X} \|\varphi(y)\|\}$$

is not empty. For x in E, since φ is harmonic, we have

$$\varphi(x) = \frac{1}{d(x)} \sum_{y \sim x} \varphi(y).$$

Note that all the $\varphi(y)$, $y \sim x$, have norm $\leq \|\varphi(x)\|$. By applying Lemma F.8 to the vectors $\varphi(y)$, $y \sim x$, with constants coefficients, we obtain that all these vectors are equal to each other and hence to $\varphi(x)$. Therefore $E = X$ and φ is constant with value some $v \in V$. Since φ is Γ-equivariant, the vector v is Γ-invariant.

In particular, the harmonic cocycles defined in Lemma F.6 are all 0, which amounts to say that a harmonic cohomology class admits a unique harmonic representative. □

REMARK F.9. Given a harmonic geometric cocycle σ let us say that the cohomology class of σ is cyclic if the range of $\sigma : X_1 \to V$ spans a dense subspace of V. This amounts to saying that the linear map $\rho : \overline{\mathcal{D}}(\partial X) \to V$ associated with σ has dense image. Then, the pull back of the scalar product of V under ρ is a Γ-invariant non-negative symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. In that sense, the study of these bilinear forms may be understood as the study of all unitary representations of Γ which admit a cyclic harmonic cohomology class.

F.6. Spectral gap

We now assume that, for any x in X, the number $d(x)$ of neighbours of x is constant with value $q+1$, for some integer $q \geq 2$.

To conclude, we will give a sufficient criterion for all cohomology classes of a unitary representation to be harmonic. Before stating it, we discuss the properties of unitary representations that are far away from the trivial representation.

Let V be a Hilbert space equipped with a unitary action of Γ. We denote by $\mathcal{F}(X,V)^\Gamma$ and $\mathcal{F}(X_1,V)^\Gamma$ the spaces of Γ-equivariant maps $X \to V$ and $X_1 \to V$, which we equip with the natural Hilbert spaces structures defined in Section 1.5. In particular, the operator Q on $\mathcal{F}(X,V)^\Gamma$ defined by

$$Q\varphi(x) = \frac{1}{q+1} \sum_{y \sim x} \varphi(y), \quad \varphi \in \mathcal{F}(X,V)^\Gamma, \quad x \in X,$$

is self-adjoint.

Recall that Γ is said to have almost invariant vectors in V if, for any $\varepsilon > 0$ and any finite subset F of Γ, there exists v in V with $\|v\| = 1$ such that one has

$$\|\gamma v - v\| \leq \varepsilon, \quad \gamma \in F.$$

The next result may be seen as a reformulation of Kesten's criterion for amenability [**32**].

Proposition F.10. *Let V be a Hilbert space equipped with a unitary action of Γ. The following are equivalent*
(i) *the group Γ has almost invariant vectors in V.*
(ii) *the number 1 belongs to the spectrum of Q in $\mathcal{F}(X,V)^\Gamma$.*

If these conditions are not satisfied, we shall say that the representation of Γ in V has a spectral gap.

The main difficulty of the proof of Proposition F.10 is to establish the following technical statement which may be seen as an effective version of the proof of Proposition F.7.

Lemma F.11. *There exists a non decreasing sequence $(\alpha_n)_{n\geq 0}$ of continuous non-negative functions on $[0,\infty)$ such that $\alpha_n(0) = 0$ for any $n \geq 0$ and with the following property. Let V be a Hilbert space equipped with a unitary action of Γ, $\varepsilon > 0$ and φ be in $\mathcal{F}(X,V)^\Gamma$ with $\|\varphi\| = 1$ and $\|Q\varphi - \varphi\| \leq \varepsilon$. Pick x in X with $\|\varphi(x)\| = \sup_{y\in X} \|\varphi(y)\|$. Then, for any y in X with $d(x,y) = n$, one has*

$$\|\varphi(x) - \varphi(y)\| \leq \alpha_n(\varepsilon).$$

PROOF. Set $C = \sup_{x\in X} |\Gamma_x|^{\frac{1}{2}}$ and, for $\varepsilon \geq 0$, define by induction

$$\alpha_0(\varepsilon) = 0 \tag{F.4}$$
$$\alpha_{n+1}(\varepsilon) = \alpha_n(\varepsilon) + 2(qC(\alpha_n(\varepsilon) + C\varepsilon))^{\frac{1}{2}} + C\varepsilon, \quad n \geq 0.$$

The sequence is non decreasing and the functions $(\alpha_n)_{n\geq 0}$ are all continuous with value 0 at 0.

Now, let V, $\varepsilon > 0$ and φ be as in the statement. Set

$$M = \sup_{x\in X} \|\varphi(x)\| .$$

Note that the definition of the norm on $\mathcal{F}(X,V)^\Gamma$ gives

$$1 = \|\varphi\|^2 = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \|\varphi(x)\|^2 ,$$

hence $M \leq C$. Besides, we have

$$\sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \|Q\varphi(x) - \varphi(x)\|^2 \leq \varepsilon^2,$$

hence, for all x in X,

$$\|Q\varphi(x) - \varphi(x)\| \leq C\varepsilon. \tag{F.5}$$

Fix x in X with $\|\varphi(x)\| = M$. We will show by induction on $n \geq 0$ that, for every y in X with $d(x,y) = n$, one has $\|\varphi(x) - \varphi(y)\| \leq \alpha_n(\varepsilon)$. For $n = 0$, there is nothing to prove. Assume the statement holds for $n \geq 0$. Fix y in X with $d(x,y) = n$. By applying the formula in Lemma F.8 to the vectors $\varphi(z)$, $z \sim y$, we obtain

$$\frac{1}{2(q+1)^2} \sum_{w,z\sim y} \|\varphi(w) - \varphi(z)\|^2 \leq M^2 - \|Q\varphi(y)\|^2 .$$

From (F.5) and the induction assumption, we get

$$\|Q\varphi(y)\| \geq \|\varphi(y)\| - C\varepsilon \geq M - (\alpha_n(\varepsilon) + C\varepsilon),$$

hence

$$\frac{1}{2(q+1)^2} \sum_{w,z\sim y} \|\varphi(w) - \varphi(z)\|^2 \leq M^2 - (M - (\alpha_n(\varepsilon) + C\varepsilon))^2 \tag{F.6}$$
$$\leq 2C(\alpha_n(\varepsilon) + C\varepsilon).$$

Fix $z \sim y$. By Cauchy-Schwarz inequality, we have

$$\|\varphi(z) - Q\varphi(y)\| \leq \frac{1}{q+1} \sum_{\substack{w \sim y \\ w \neq z}} \|\varphi(z) - \varphi(w)\|$$

$$\leq \frac{q^{\frac{1}{2}}}{q+1} \left(\sum_{\substack{w \sim y \\ w \neq z}} \|\varphi(z) - \varphi(w)\|^2 \right)^{\frac{1}{2}}.$$

Applying (F.6) yields

$$\|\varphi(z) - Q\varphi(y)\| \leq 2(qC(\alpha_n(\varepsilon) + C\varepsilon))^{\frac{1}{2}},$$

hence, from (F.5),

$$\|\varphi(z) - \varphi(y)\| \leq 2(qC(\alpha_n(\varepsilon) + C\varepsilon))^{\frac{1}{2}} + C\varepsilon.$$

Therefore, by (F.4) and the induction assumption, we get

$$\|\varphi(x) - \varphi(z)\| \leq \alpha_{n+1}(\varepsilon)$$

as required. □

Proof of Proposition F.10. *(i)⇒(ii)* First assume that Γ has almost invariant vectors in V. Fix a system of representatives $S \subset X$ for the action of Γ and set

$$F = \{\gamma \in \Gamma | \exists x, x' \in S \quad x \sim \gamma x'\} \cup \bigcup_{x \in S} \Gamma_x. \tag{F.7}$$

Since the action of Γ on X is proper, the set F is finite.

Fix $0 < \varepsilon < 1$ and a unit vector v in V such that $\|\gamma v - v\| \leq \varepsilon$ for any γ in F. We will use v to build φ in $\mathcal{F}(X, V)^\Gamma$ with $Q\varphi$ close to φ. For x in S, set

$$v_x = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \gamma v.$$

We have $\|v_x - v\| \leq \varepsilon$, hence $\|v_x\| \geq 1 - \varepsilon$. Let φ be the unique element of $\mathcal{F}(X, V)^\Gamma$ such that $\varphi(x) = v_x$ for x in S. By definition, we have

$$\|\varphi\|^2 = \sum_{x \in S} \frac{1}{|\Gamma_x|} \|v_x\|^2,$$

hence $\|\varphi\| \geq (1 - \varepsilon)c$, where $c = \left(\sum_{x \in S} |\Gamma_x|^{-1}\right)^{\frac{1}{2}}$.

We claim that, for every x in S and $y \sim x$, we have $\|\varphi(y) - \varphi(x)\| \leq 3\varepsilon$. Indeed, for such x, y, choose γ in Γ with $z = \gamma^{-1} y \in S$. Then, by (F.7), γ belongs to F and we get

$$\|\varphi(y) - \varphi(x)\| = \|\gamma v_z - v_x\| \leq \|\gamma v_z - \gamma v\| + \|\gamma v - v\| + \|v - v_x\| \leq 3\varepsilon.$$

Since by definition, we have

$$\|Q\varphi - \varphi\|^2 = \sum_{x\in S} \frac{1}{|\Gamma_x|} \left\| \frac{1}{q+1} \sum_{y\sim x} (\varphi(y) - \varphi(x)) \right\|^2 ,$$

we obtain $\|Q\varphi - \varphi\| \leq 3c\varepsilon$ and hence $\|Q\varphi - \varphi\| \leq 3\varepsilon(1-\varepsilon)^{-1} \|\varphi\|$. As ε is arbitrary, 1 is a spectral value of Q in $\mathcal{F}(X,V)^\Gamma$.

(i)⇒(ii) Assume now that 1 is a spectral value of Q in $\mathcal{F}(X,V)^\Gamma$. We need to show that V admits almost invariant vectors. Let $(\alpha_n)_{n\geq 0}$ be as in Lemma F.11 and fix $\varepsilon > 0$ and a finite subset F of Γ. We still take $S \subset X$ to be a set of representatives for the action of Γ and we set

$$n = \sup_{\substack{x\in S \\ \gamma\in F}} d(\gamma x, x).$$

As the function α_n is continuous at 0, we can find $\eta > 0$ such that $\alpha_n(\eta) \leq \varepsilon$. As 1 is a spectral value of the self-adjoint operator Q, we can find φ in $\mathcal{F}(X,V)^\Gamma$ with $\|\varphi\| = 1$ and $\|Q\varphi - \varphi\| \leq \eta$. Choose x in S with $\|\varphi(x)\| = \sup_{y\in X} \|\varphi(y)\|$ and set $v = \varphi(x)$. Since

$$1 = \|\varphi\|^2 = \sum_{y\in S} \frac{1}{|\Gamma_y|} \|\varphi(y)\|^2 ,$$

we have $\|v\| \geq c^{-1}$, where, as above, $c = \left(\sum_{x\in S} |\Gamma_x|^{-1}\right)^{\frac{1}{2}}$. Besides, Lemma F.11 yields, for γ in F,

$$\|\gamma v - v\| = \|\varphi(\gamma x) - \varphi(x)\| \leq \alpha_n(\eta) \leq \varepsilon.$$

The conclusion follows. □

F.7. Spectral gap and harmonic cocycles

We now show that, when the representation has a spectral gap, all cohomology classes are harmonic. The following is mostly a translation from [**4, 39**].

PROPOSITION F.12. *Let V be a Hilbert space equipped with a unitary action of Γ. If Γ has a spectral gap in V, any 1-cohomology class in $\mathrm{H}^1(\Gamma, V)$ is harmonic. In other words, the natural map*

$$\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V) \to \mathrm{H}^1(\Gamma, V)$$

is an isomorphism.

PROOF. Note that the two statements are equivalent by Proposition F.7.

Consider the operator $\mathrm{d} : \mathcal{F}(X,V)^\Gamma \to \mathcal{F}(X_1,V)^\Gamma$. For an element φ in $\mathcal{F}(X,V)^\Gamma$, we get, by using Lemma 10.11,

$$\begin{aligned}\|\mathrm{d}\varphi\|^2 &= \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \|\varphi(y)-\varphi(x)\|^2 \\ &= (q+1)^2 \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \|Q\varphi(x)-\varphi(x)\|^2 = (q+1)^2 \|Q\varphi-\varphi\|^2 .\end{aligned}$$

By Proposition F.10 and the assumption, 1 is not a spectral value of Q. Since Q is self-adjoint in $\mathcal{F}(X,V)^\Gamma$, this tells us that we may find $\varepsilon > 0$ such that, for φ in $\mathcal{F}(X,V)^\Gamma$, we have $\|Q\varphi-\varphi\| \geq \varepsilon \|\varphi\|$. Therefore, we get $\|\mathrm{d}\varphi\| \geq (q+1)\varepsilon \|\varphi\|$ and the operator d has closed range in $\mathcal{F}(X_1,V)^\Gamma$. Note that the orthogonal complement of $\mathrm{d}\mathcal{F}(X,V)^\Gamma$ in $\mathcal{F}(X_1,V)^\Gamma$ is the kernel of the adjoint operator $\mathrm{d}^\dagger$ of d. Thus, every element σ in $\mathcal{F}(X_1,V)^\Gamma$ may be written as

$$\sigma = \mathrm{d}\varphi + \theta, \tag{F.8}$$

with φ in $\mathcal{F}(X,V)^\Gamma$, θ in $\mathcal{F}(X_1,V)^\Gamma$ and $\mathrm{d}^\dagger\theta = 0$.

We claim that, for θ in $\mathcal{F}(X_1,V)^\Gamma$ and x in X, we have

$$\mathrm{d}^\dagger\theta(x) = \sum_{y\sim x} \theta(y,x) - \theta(x,y).$$

Indeed, for φ in $\mathcal{F}(X,V)^\Gamma$, by applying again Lemma 10.11, we obtain

$$\begin{aligned}\langle \mathrm{d}\varphi, \theta\rangle &= \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle \varphi(y)-\varphi(x), \theta(x,y)\rangle \\ &= \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle \varphi(x), \sum_{y\sim x} \theta(y,x) - \theta(x,y)\rangle.\end{aligned}$$

Now, let σ be a geometric coycle of Γ in V, so that σ is a skew-symmetric element of $\mathcal{F}(X_1,V)^\Gamma$. Decompose σ as in (F.8). As σ and $\mathrm{d}\varphi$ are skew-symmetric, so is θ, so that θ is also a geometric cocycle. In particular, for x in X, we get

$$0 = \mathrm{d}^\dagger\theta(x) = \sum_{y\sim x} \theta(y,x) - \theta(x,y) = -2\sum_{y\sim x} \theta(x,y).$$

Hence θ is harmonic as required. □

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