Random walks on reductive groups

Y. Benoist & J.-F. Quint

To Dominique and Clémence

Contents

Introduction

0.1. What is this book about? This book deals with "products of random matrices". Let us describe in concrete terms the questions we will be studying all over this book. Let $d \geq 1$ be a positive integer. We choose a sequence $g_1, \ldots, g_n, \ldots$ of $d \times d$ of invertible matrices with real coefficients. These matrices are chosen independently and according to an identical law μ. We want to study the sequence of product matrices $p_n := g_n \cdots g_1$. In particular, we want to know :

(0.1) *Can one describe the asymptotic behavior of the matrices p_n?*

A naive way to ask this question is to fix a Euclidean norm on the vector space $V = \mathbb{R}^d$, to fix a nonzero vector v on V and a nonzero linear functional f on V and to ask

(0.2) *What is the asymptotic behavior of the norms $\|p_n\|$?*

(0.3) *What is the asymptotic behavior of the coefficients $f(p_n v)$?*

The first aim of this book is to explain the answer to these questions, which was guessed at the very early stage of the theory : under suitable irreducibility and moment assumptions, the real random variables $\log \|p_n\|$ and $\log |f(p_n v)|$ behave very much like a "sum of independent identically distributed (iid) real random variables".

Indeed we will see that, under suitable assumptions, these variables satisfy many properties that are classical for "sums of iid random real numbers" like the Law of Large Numbers (LLN), the Central Limit Theorem (CLT), the Law of Iterated Logarithm (LIL), the Large Deviations Principle (LDP), and the Local Limit Theorem (LLT).

The answer to Questions (0.2) and (0.3) will be obtained by focusing first on the following two related questions :

(0.4) *What is the asymptotic distribution of the vectors $\frac{p_n v}{\|p_n v\|}$?*

(0.5) *What is the asymptotic behavior of the norms $\|p_n v\|$?*

0.2. When did this topic emerge? The theory of "products of random matrices" or more precisely "products of iid random matrices" is sometimes also called "random walks on linear groups". It began in the middle of the 20$^{\text{th}}$ century. It finds its roots in the speculative work of Bellman in [**8**] who guessed that an analog of classical Probability Theory for "sums of random numbers" might be true for the coefficients of products of random matrices. The pioneers of this topic are Kesten, Furstenberg, Guivarc'h,...

At that time, in 1960, Probability Theory was already based on very strong mathematical foundations, and the language of σ-algebras, measure theory and Fourier transform was widely adopted among the specialists interested in probabilistic phenomena. A few textbooks on “sum of random numbers” were already available (like the ones by Kolmogorov [**80**] in USSR, by Lévy [**85**] in France and by Cramér [**36**] in UK,...), and many more were about to appear like the ones by Loève [**86**], Spitzer [**118**], Breiman [**28**], Feller [**44**],...

It took about half a century for the theory of “products of random matrices” to achieve its maturity. The reason may be the following. Even though some of the new characters who happen to play an important role in this new realm, like the “martingales and the Markov chains” and the “ergodic theory of cocycles” were very popular among specialists of this topic, others like the “semisimple algebraic groups” and the “highest weight representations” were less popular, and others like the “spectral theory of transfer operators” and the “asymptotic properties of discrete linear groups” were still waiting to be developed.

This book is also an introduction to all these tools.

The main contributors of the theorems we are going to explain in this book are not only Kesten, Furstenberg, Guivarc’h, but also Kifer, Le Page, Raugi, Margulis, Goldsheid,...

The topic of this book is the same as the nice and very influential book written by Bougerol-Lacroix 30 years ago. We also recommend the surveys by Ledrappier [**83**] and Furman [**48**] on related topics. This theory has had recently nice applications to the study of subgroups of Lie groups (as in [**58**], [**26**] or [**27**, Section 12]). Beyond these applications, we were urged to write this book so that it could serve as a background reference for our joint work in [**14**], [**15**], and [**16**].

Even though our topic is very much related to the almost homonymous topic “random walks on countable groups”, we will not discuss here this aspect of the theory and its ties with the “geometric group theory” and the “growth of groups”.

0.3. Is this topic related to sums of random numbers? Yes. The classical theory of “sums of random numbers” or more precisely “sums of iid random numbers” is sometimes also called ”random walks on $\mathbb{R}^d$”. Let us describe in concrete terms the question studied in this classical theory.

We choose a sequence $t_1, \ldots, t_n, \ldots$ of real numbers. These real numbers are chosen independently and according to an identical law μ. This law μ is a Borel probability measure on the real line $\mathbb{R}$. We denote by A the support of μ. For instance, when $\mu = \frac{1}{2}(\delta_0 + \delta_1)$, the set A is $\{0,1\}$, and we are choosing the t_k to be either 0 or 1 with equal probability and independently of the previous choices of t_j for $j < k$. We want to study the sequence of partial sums $s_n := t_1 + \cdots + t_n$. In particular, we want to know :

(0.6) *What is the asymptotic behavior of s_n?*

We will explain in Section 0.4 various classical answers to this question.

On the one hand, some of these classical answers describe the behavior in law of this sequence. They tell us what we can expect at time n when n is large. These statements only involve the law of the random variable s_n which is nothing else than the n^{th}-convolution power μ^{*n} of μ i.e.

$$\mu^{*n} = \mu * \cdots * \mu.$$

For instance, the Central Limit Theorem (CLT), the Large Deviations Principle (LDP) and the Local Limit Theorem (LLT) are statements in law. An important tool in this point of view is Fourier analysis.

On the other hand, some classical answers describe the behavior of the individual trajectories $s_1, s_2, \ldots, s_n, \ldots$ These statements are true for almost every trajectory. The trajectories are determined by elements of the Bernoulli space

$$B := A^{\mathbb{N}^*} := \{b = (t_1, \ldots, t_n, \ldots) \mid t_n \in A\}$$

of all possible sequences of random choices. Here "almost every" refer to the Bernoulli probability measure

$$\beta := \mu^{\otimes \mathbb{N}^*}$$

on this space B. This space B is also called the space of *forward trajectories*. For instance, the Law of Large Numbers (LLN) and the Law of the Iterated Logarithm (LIL) are statements about almost every trajectory. An important tool in this point of view is the conditional expectation.

The interplay between these two aspects is an important feature of Probability Theory. The Borel-Cantelli lemma sometimes allows one to transfer results in law into almost-sure results. Conversely, the point of view of trajectories gives us a much deeper level of analysis on the probabilistic phenomena that cannot be reached by the sole study of the laws μ^{*n}.

0.4. What classical results should I know? This short book is as self-contained as possible. We will reprove many classical facts from Probability Theory. However we will take for granted basic facts from Linear Agebra, Integration Theory and Functional Analysis. A few results on real reductive algebraic groups, their representations and their discrete subgroups will be quoted without proof.

The reader will more easily appreciate the streamlining of this book if he or she knows classical Probability Theory. Indeed the main objective of this book is to present for "products of iid random matrices" the analogs of the following five classical theorems for "sums of iid random numbers".

In these five classical theorems, we fix a probability measure μ on $\mathbb{R}$ and set $b = (t_1, \ldots, t_n, \ldots) \in B$ and $s_n = t_1 + \cdots + t_n$ for the partial sums. The sequence b is chosen according to the law β, which means that the coordinates t_k are iid random real numbers of law μ.

The first theorem is the Law of Large Numbers due to many authors from Bernoulli up to Kolmogorov. It tells us that, when μ has a finite first moment i.e. when $\int_{\mathbb{R}} |t| \, \mathrm{d}\mu(t) < \infty$, almost every trajectory has a drift which is equal to the average of the law :

$$\lambda := \int_{\mathbb{R}} t \, \mathrm{d}\mu(t). \tag{0.7}$$

THEOREM 0.1. **(LLN)** *Let μ be a Borel probability measure on $\mathbb{R}$ with a finite first moment. Then, for β-almost all b in B, one has*

$$\lim_{n\to\infty} \tfrac{1}{n}\, s_n = \lambda. \tag{0.8}$$

The second theorem is the Central Limit Theorem which is also due to many authors from Laplace up to Lindeberg and Lévy. It tells us that, when μ is non-degenerate i.e. is not a Dirac mass, and when μ has a finite second moment i.e. when $\int_{\mathbb{R}} t^2 \, \mathrm{d}\mu(t) < \infty$, the recentered law of μ^{*n} spreads at speed $\sqrt{n}$, more precisely, it tells us that the renormalized variables $\frac{s_n - n\lambda}{\sqrt{n}}$ converge in law to a Gaussian variable which has the same variance Φ as μ :

$$\Phi := \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\mu(t).$$

THEOREM 0.2. **(CLT)** *Let μ be a non-degenerate Borel probability measure on $\mathbb{R}$ with a finite second moment. Then, for any bounded continuous function ψ on $\mathbb{R}$, one has*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \psi\left(\tfrac{s-n\lambda}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(s) = \int_{\mathbb{R}} \psi(s) \frac{e^{-\frac{s^2}{2\Phi}}}{\sqrt{2\pi\Phi}} \, \mathrm{d}s. \tag{0.9}$$

The third theorem is the Law of the Iterated Logarithm discovered by Khinchin. It tells us that almost all recentered trajectories spread at a slightly higher speed than $\sqrt{n}$. More precisely it tells us that the precise scale at which almost all recentered trajectories fill a bounded interval is $\sqrt{n \log\log n}$.

Theorem 0.3. **(LIL)** *Let μ be a non-degenerate Borel probability measure on $\mathbb{R}$ with a finite second moment. Then, for β-almost all b in B, the set of cluster points of the sequence*

$$\frac{s_n - n\lambda}{\sqrt{2\Phi\, n \log\log n}}$$

is equal to the interval $[-1, 1]$.

The fourth theorem is the Large Deviations Principle due to Cramér. It tells us that when μ has a finite exponential moment i.e. when $\int_{\mathbb{R}} e^{\alpha|t|}\, \mathrm{d}\mu(t) < \infty$, for some $\alpha > 0$, the probability of an excursion away from the average decays exponentially. We will just state below the upper bound in the large deviations principle.

Theorem 0.4. **(LDP)** *Let μ be a Borel probability measure on $\mathbb{R}$ with a finite exponential moment. Then, for any $t_0 > 0$, one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{t \in \mathbb{R} \mid \ |t - n\lambda| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{0.10}$$

The fifth theorem is the Local Limit Theorem due to many authors from de Moivre up to Stone. It tells us that the rate of decay for the probability that the recentered sum $s_n - n\lambda$ belongs to a fixed interval is $1/\sqrt{n}$. For sake of simplicity, we will assume below that μ is aperiodic i.e. μ is not supported by an arithmetic progression $m_0 + t\mathbb{Z}$ with $m_0 \in \mathbb{R}$ and $t > 0$. Indeed the statement is just slightly different when μ is supported by an arithmetic progression.

Theorem 0.5. **(LLT)** *Let μ be an aperiodic Borel probability measure on $\mathbb{R}$ with a finite second moment. Then, for all $a_1 \leq a_2$, one has*

$$\lim_{n\to\infty} \sqrt{n}\ \mu^{*n}(n\lambda + [a_1, a_2]) = \frac{a_2 - a_1}{\sqrt{2\pi\Phi}}\,.$$

0.5. Can you show me nice sample results from this topic? The five main results that we will explain in this book are the analogs of the five classical theorems that we just quoted in the previous section. We will state below special cases of these five results. We will explain in Section 0.8 what kind of generalizations of these special cases is needed for a better answer to Question 0.1.

In these five results, we fix a Borel probability measure μ on the special linear group $G := \mathrm{SL}(d, \mathbb{R})$, we set $V = \mathbb{R}^d$, and we fix a Euclidean norm $\|.\|$ on V. We denote by A the support of μ, and by

Γ_μ the closed subsemigroup of G spanned by A. For $n \geq 1$, we denote by μ^{*n} the n^{th}-convolution power

$$\mu^{*n} := \mu * \cdots * \mu.$$

The forward trajectories are determined by elements of the Bernoulli space

$$B := A^{\mathbb{N}^*} := \{b = (g_1, \ldots, g_n, \ldots) \mid g_n \in A\} \tag{0.11}$$

endowed with the Bernoulli probability measure

$$\beta := \mu^{\otimes \mathbb{N}^*}.$$

As in Section 0.4, the sequence b is chosen according to the law β which means that b is a sequence of iid random matrices g_k chosen with law μ, and we want to understand the asymptotic behavior of the products $p_n := g_n \cdots g_1$. We assume, to simplify this introduction, that

(0.12) - *μ has a finite exponential moment,*
- *Γ_μ is unbounded and acts strongly irreducibly on V.*

In these assumptions, *finite exponential moment* means that one has $\int_G \|g\|^\alpha \, \mathrm{d}\mu(g) < \infty$ for some $\alpha > 0$. Notice that the word *exponential* is natural in this context if one keeps in mind the equality $\|g\|^\alpha = e^{\alpha \log \|g\|}$. In these assumptions, *strongly irreducible* means that no proper finite union of vector subspaces of V is Γ_μ-invariant.

These conditions are satisfied for instance when

$$\mu = \tfrac{1}{2}(\delta_{a_0} + \delta_{a_1}) \ \text{ where } \ a_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \ \text{ and } \ a_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

or, more generally, where

$$a_0 = \begin{pmatrix} 2 & 1 & 0 & . & 0 \\ 1 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \end{pmatrix} \ \text{ and } \ a_1 = \begin{pmatrix} 0 & -1 & 0 & . & 0 \\ 0 & 0 & -1 & . & 0 \\ 0 & 0 & 0 & . & 0 \\ . & . & . & . & -1 \\ 1 & 0 & 0 & . & 0 \end{pmatrix}.$$

In this example, one has $A = \{a_0, a_1\}$ and we are choosing the g_k to be either a_0 or a_1 with equal probability and independently of the previous choices of g_j for $j < k$. The partial products $p_n := g_n \cdots g_1$ can take 2^n values with equal probability. This concrete example is very interesting to keep in mind. Indeed, the whole machinery we are going to explain in this book is necessary to understand the asymptotic behavior of p_n in this case.

We denote by $\lambda_1 = \lambda_{1,\mu}$ the first Lyapunov exponent of μ, i.e.

$$\lambda_1 := \lim_{n\to\infty} \frac{1}{n} \int_G \log \|g\| \,\mathrm{d}\mu^{*n}(g). \tag{0.13}$$

The first result tells us that the variables $\log \|p_n v\|$ satisfy the Law of Large Numbers. It is due to Furstenberg.

Theorem 0.6. **(LLN)** *For all v in $V \smallsetminus \{0\}$, for β-almost all b in B, one has*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|g_n \cdots g_1 v\| = \lambda_1, \quad \text{and one has} \quad \lambda_1 > 0. \tag{0.14}$$

The second result tells us that the variables $\log \|p_n v\|$ satisfy the Central Limit Theorem i.e. that the renormalized variables $\frac{\log \|p_n v\| - n\lambda_1}{\sqrt{n}}$ converge in law to a nondegenerate Gaussian variable.

Theorem 0.7. **(CLT)** *The limit*

$$\Phi := \lim_{n\to\infty} \frac{1}{n} \int_G (\log \|g\| - n\,\lambda_1)^2 \,\mathrm{d}\mu^{*n}(g)$$

exists and is positive $\Phi > 0$. For all v in $V \smallsetminus \{0\}$, for any bounded continuous function ψ on $\mathbb{R}$, one has

$$\lim_{n\to\infty} \int_G \psi\left(\tfrac{\log \|gv\| - n\lambda_1}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) = \int_{\mathbb{R}} \psi(s) \frac{e^{-\frac{s^2}{2\Phi}}}{\sqrt{2\pi\Phi}} \,\mathrm{d}s. \tag{0.15}$$

The third result tells us that the variables $\log \|p_n v\|$ satisfy a law of the iterated logarithm.

Theorem 0.8. **(LIL)** *For all v in $V \smallsetminus \{0\}$, for β-almost all b in B, the set of cluster points of the sequence*

$$\frac{\log \|g_n \cdots g_1 v\| - n\lambda_1}{\sqrt{2\Phi\, n \log\log n}}$$

is equal to the interval $[-1, 1]$.

The fourth result tells us that the variables $\log \|p_n v\|$ satisfy a Large Deviations Principle.

Theorem 0.9. **(LDP)** *For all v in $V \smallsetminus \{0\}$, for any $t_0 > 0$, one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{g \in G \mid\ |\log \|gv\| - n\lambda_1| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{0.16}$$

The fifth result tells us that the variables $\log \|p_n v\|$ satisfy a Local Limit Theorem.

THEOREM 0.10. **(LLT)** *For all $a_1 \leq a_2$, for all v in $V \smallsetminus \{0\}$, one has*

$$\lim_{n\to\infty} \sqrt{n}\, \mu^{*n}(\{g \in G \mid \log \|gv\| - n\lambda_1 \in [a_1, a_2]\,\}) = \frac{a_2 - a_1}{\sqrt{2\pi\Phi}}\,.$$

Theorems 0.7 up to 0.10 are in Le Page's thesis under technical assumptions. Since then, the statements have been extended and simplified by Guivarc'h, Raugi, Goldsheid, Margulis, and the authors.

0.6. How does one prove these nice results? Thanks for your enthusiasm. As for sums of random numbers, we will use tools coming from Probability Theory like the Doob Martingale Theorem, tools coming from Ergodic Theory like the Birkhoff Ergodic Theorem and tools coming from Harmonic Analysis like the Fourier Inversion Theorem.

New tools will be needed. We will be able to understand the asymptotic behavior of the product p_n of iid random matrices, only by first studying the associated Markov chain on the projective space $\mathbb{P}(V)$ whose trajectories, starting from $x = \mathbb{R}v$, are $n \mapsto x_n := p_n x$. We will also study the ergodic properties along these trajectories of the cocycle σ_1 on $\mathbb{P}(V)$ given by

$$\sigma_1(g, x) = \tfrac{\|gv\|}{\|v\|}.$$

Indeed, for a vector v of norm $\|v\| = 1$, the quantity $s_n := \log \|p_n v\|$ that we want to study is nothing else than the sum

$$\log \|p_n v\| = \sum_{k=1}^{n} \sigma_1(g_k, x_{k-1}).$$

These random real variables $t_k := \sigma_1(g_k, x_{k-1})$ whose sum is s_n are not always independent because the point x_{k-1} depends on what happened before. This is why we will need tools from Markov chains.

First we have to understand the statistics of the trajectories x_k i.e. we have to answer to Question (0.4). That is why we will study the invariant probability measures ν of this Markov chain, i.e. the probability measures ν on $\mathbb{P}(V)$ which satisfy $\mu * \nu = \nu$. Those probability measures ν are also called *μ-stationary*. This will allow us to prove the LLN and to give a formula for the drift analog to (0.7) :

$$\lambda_1 = \int_{G\times\mathbb{P}(V)} \sigma_1(g, x)\, \mathrm{d}\mu(g)\, \mathrm{d}\nu(x). \tag{0.17}$$

This formula is due to Furstenberg.

We will see that, when the action of Γ_μ on V is *proximal* the invariant probability measure ν on $\mathbb{P}(V)$ is unique. The assumption *proximal* means that there exists a rank-one matrix which is a limit of matrices

$\lambda_n\gamma_n$ with $\lambda_n > 0$ and γ_n in Γ_μ. In this case Furstenberg's formula (0.17) reflects the fact that, for all starting point x in $\mathbb{P}(V)$, the sequence $(x_n)_{n\geq 1}$ becomes equidistributed according to the law ν, for β-almost all b. When Γ_μ is not proximal, the asymptotic behavior of the sequence $(x_n)_{n\geq 1}$ is described in [**13**].

Second we have to understand the *transfer operator* P and its generalisation the *complex transfer operator* $P_{i\theta}$ with $\theta \in \mathbb{R}$. This operator $P_{i\theta}$ is the bounded operator on $\mathcal{C}^0(\mathbb{P}(V))$, given by, for any φ in $\mathcal{C}^0(\mathbb{P}(V))$ and any x in $\mathbb{P}(V)$,

$$P_{i\theta}\varphi(x) = \int_G e^{i\theta\,\sigma_1(g,x)}\varphi(gx)\,\mathrm{d}\mu(g). \tag{0.18}$$

The CLT 0.7 describes the asymptotic behavior of the probability measures on $\mathbb{R}$

$$\mu_{n,x} := \text{image of } \mu^{*n} \text{ by the map } g \mapsto \log \tfrac{\|gv\|}{\|v\|}.$$

The Fourier transform of these measures is given by the classical and elegant formula with θ in $\mathbb{R}$,

$$\widehat{\mu_{n,x}}(\theta) = P_{i\theta}^n\mathbf{1}(x), \tag{0.19}$$

where $\mathbf{1}$ is the constant function on $\mathbb{P}(V)$ equal to 1. The behavior of the righthand side of this formula will be controlled by the "largest" eigenvalue of $P_{i\theta}$. This formula (0.19) explains how spectral data from the complex transfer operator $P_{i\theta}$ can be used in combination with the Fourier Inversion Theorem to prove not only the CLT but also the LIL, the LDP and the LLT. We will be able to reduce our analysis to the case where the action of Γ_μ on V is proximal. We will see then that this operator $P_{i\theta}$ has a unique "largest" eigenvalue $\lambda_{i\theta}$ when θ is small, and that this eigenvalue $\lambda_{i\theta}$ varies analytically with θ.

0.7. Can you answer your own questions now? You are right, what took us so long to explain are nothing but answers to Questions (0.4) and (0.5). We will deduce answers to Questions (0.2) and (0.3) from these.

Indeed, we will first check that, under assumption (0.12), the random variables $\log\|p_n\|$ satisfy the same LLN, CLT, LIL and LDP as $\log\|p_nv\|$. Technically, this will not be too difficult since these four limit laws involve a renormalization which will erase the difference between $\log\|p_n\|$ and $\log\|p_nv\|$

We will also check that, when moreover Γ_μ is proximal, the random variables $\log|f(p_nv)|$ satisfy the same LLN, CLT, LIL and LDP as $\log\|p_nv\|$. This will be more delicate since we will have to control the excursions of the sequence p_nx near the kernel of f. The key point will be to prove a Hölder regularity result for the stationary measure ν which is due to Guivarc'h.

0.8. Why is this book less simple than these samples? The quantity

$$\kappa_1(g) := \log \|g\|$$

gives us information on the size of a matrix g only "in one direction". It is much more useful in the applications to deal with all the logarithms of singular values $\kappa_j(g) := \log \frac{\|\wedge^j(g)\|}{\|\wedge^{j-1}(g)\|}$ and to introduce the "multinorm"

$$\kappa_V(g) := (\kappa_1(g), \dots, \kappa_d(g)). \tag{0.20}$$

A less naive way to ask our question (0.1) is :

(0.21) *Can one describe the asymptotic behavior of $\kappa_V(p_n)$?*

The answer to this question is Yes! These random variables $\kappa_V(p_n)$ satisfy a LLN with average λ. However they do not exactly satisfy a CLT: the renormalized variable $\frac{\kappa_V(p_n)-n\lambda}{\sqrt{n}}$ converges in law but the limit law is only a "folded Gaussian law" i.e. the "image of a Gaussian law by a homogeneous continuous locally linear map"!

The support of this limit law depends only on λ and the "Zariski closure" G_μ of the semigroup Γ_μ. This Zariski closure G_μ is always a reductive algebraic group with compact center. The "folding" phenomenon occurs already when $d = 4$ and $G_\mu = \mathrm{SO}(2,2)$!

The whole picture becomes much clearer when one adopts the following more intrinsic point of view.

We start with a connected real semisimple algebraic group, call it again G, and a Borel probability measure μ on G. We consider iid random variables $g_n \in G$ of law μ and want, again, to describe the asymptotic behavior of the products $p_n := g_n \cdots g_1$. In this point of view, we forget about the embedding ρ of G in $\mathrm{GL}(V)$ which was responsible for the folding of the Gaussian law. We replace the conditions (0.12) by

(0.22)
- *μ has a finite exponential moment,*
- *the semigroup Γ_μ spanned by A is Zariski dense in G,*

where A is the support of μ.

The projective space $\mathbb{P}(V)$ is replaced by the flag variety $\mathcal{P}$ of G, and the norm is replaced by the *Cartan projection* κ of G. Exactly as in Section 0.6, we will use a cocycle $\sigma(g, \eta)$ on the flag variety $\mathcal{P}$, called the Iwasawa or Busemann cocycle. The Iwasawa cocycle σ takes its values in a real vector space $\mathfrak{a}$ called the *Cartan subspace* whose dimension is the *real rank* r of G. The Cartan projection κ and takes its values in a simplicial cone $\mathfrak{a}^+$ of $\mathfrak{a}$ called the *Weyl chamber*. The precise definitions will be given later. For every η in $\mathcal{P}$, the asymptotic behavior of $\kappa(p_n)$ will be related to the asymptotic behavior of $\sigma(p_n, \eta)$. Our questions now become

(0.23) *What is the asymptotic behavior of $\kappa(p_n)$ and $\sigma(p_n, \eta)$?*

We will see that the random variables $\sigma(p_n, \eta)$ and $\kappa(p_n)$ satisfy a LLN, CLT, LIL and LDP. We will also check the LLT for the random variables $\sigma(p_n, \eta)$.

0.9. Can you state these more general limit theorems? Here are the statements for the Iwasawa cocycle σ. The assumptions on μ are given in (0.22).

THEOREM 0.11. **(LLN)** *There exists a unique μ-stationary probability measure ν on $\mathcal{P}$. The average*

$$\sigma_\mu := \int_{G\times\mathcal{P}} \sigma(g,\eta)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(\eta)$$

belongs to the interior of the Weyl chamber $\mathfrak{a}^+$.

For η in $\mathcal{P}$, for β-almost all b in B, one has

$$\lim_{n\to\infty} \tfrac{1}{n}\sigma(g_n\cdots g_1, \eta) = \sigma_\mu$$

This multidimensional version of Theorem 0.6 is due to Guivarc'h-Raugi and Goldsheid-Margulis. An important new output there is the fact that the Lyapunov vector σ_μ belongs to the interior of the Weyl chamber $\mathfrak{a}^+$.

THEOREM 0.12. **(CLT)** *There exists a Euclidean norm $\|.\|_\mu$ on $\mathfrak{a}$ such that, for all η in $\mathcal{P}$, for any bounded continuous function ψ on $\mathfrak{a}$,*

$$\lim_{n\to\infty} \int_G \psi\left(\tfrac{\sigma(g,\eta)-n\sigma_\mu}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) = (2\pi)^{-r/2}\int_{\mathfrak{a}} \psi(v)e^{-\frac{\|v\|_\mu^2}{2}}\,\mathrm{d}\pi_\mu(v),$$

where $\mathrm{d}\pi_\mu(v) = \mathrm{d}v_1\cdots\mathrm{d}v_r$ *in an orthonormal basis for* $\|.\|_\mu$.

This multidimensional version of Theorem 0.7 is due to Guivarc'h and Goldsheid. An important new output there is the fact that the support of the limit Gaussian law is the whole Cartan subspace $\mathfrak{a}$.

Here are the multidimensional versions of Theorems 0.8, 0.9 and 0.10.

THEOREM 0.13. **(LIL)** *For all η in $\mathcal{P}$, for β-almost all b in B, the set of cluster points of the sequence*

$$\frac{\sigma(g_n\cdots g_1,\eta)-n\sigma_\mu}{\sqrt{2n\log\log n}}$$

is equal to the unit ball K_μ *of* $\|.\|_\mu$.

THEOREM 0.14. **(LDP)** *For any* $t_0 > 0$*, one has*

$$\limsup_{n\to\infty} \sup_{\eta\in\mathcal{P}} \mu^{*n}(\{g \in G \mid \|\sigma(g,\eta) - n\sigma_\mu\| \geq nt_0\})^{\frac{1}{n}} < 1.$$

THEOREM 0.15. **(LLT)** *For all bounded open convex set* C *of* $\mathfrak{a}$*, for all* η *in* $\mathcal{P}$ *belonging to the support of* ν*, one has*

$$\lim_{n\to\infty} (2\pi\, n)^{r/2}\, \mu^{*n}(\{g \in G \mid \sigma(g,\eta) - n\sigma_\mu \in C\}) = \pi_\mu(C)\,.$$

It is remarkable that, in Theorem 0.15, no further "aperiodicity" assumptions have to be made as in Theorem 0.5. This will follow from a general fact for "Zariski dense subgroups of semisimple Lie groups" in [**11**].

We will also prove a version of this local limit theorem where we allow moderate deviation i.e. where we allow the "window" C to be translated by a vector $v_n \in \mathfrak{a}$ as soon as $\|v_n\|$ do not grow faster than $\sqrt{n\log n}$. Indeed this version, which adapts Breuillard's LLT for sums of iid real numbers in [**30**], is the one which is needed in [**15**].

0.10. Are the proofs as simple as for the simple samples? Well, ... at least the proofs of these five theorems follow the same lines as in Section 0.6.

First we study the associated Markov chain on the flag variety $\mathcal{P}$. Since this flag variety is equivariantly embedded in product of projective spaces on which the action of Γ_μ is "proximal", we will be able to use results previously proven for these proximal actions.

Second, we study the spectral properties of the complex transfer operator. This operator $P_{i\theta}$ is defined for any $\theta \in \mathfrak{a}^*$. It is the bounded operator on $\mathcal{C}^0(\mathcal{P})$, given, for any φ in $\mathcal{C}^0(\mathcal{P})$ and η in $\mathcal{P}$, by the following formula similar to (0.18),

$$P_{i\theta}\varphi(\eta) = \int_G e^{i\theta(\sigma(g,\eta))}\varphi(g\eta)\,\mathrm{d}\mu(g).$$

Another consequence of the contraction property of the action on $\mathcal{P}$, will be again the existence of a unique "largest" eigenvalue $\lambda_{i\theta}$ for the operator $P_{i\theta}$ when θ is small, and the fact that this eigenvalue $\lambda_{i\theta}$ varies analytically with θ.

The CLT 0.12 for the Iwasawa cocycle σ describes the asymptotic behavior of the probability measures on $\mathfrak{a}$

$$\mu_{n,\eta} := \text{image of } \mu^{*n} \text{ by the map } g \mapsto \sigma(g,\eta).$$

The Fourier transform of these measures is given by the classical and elegant formula similar to (0.19), with θ in $\mathfrak{a}^*$,

$$\widehat{\mu_{n,\eta}}(\theta) = P^n_{i\theta}\mathbf{1}(\eta). \tag{0.24}$$

Thanks to this formula, we can use, as in Section 0.6, the uniqueness of the "largest" eigenvalue of the complex transfer operator $P_{i\theta}$, in combination with the Fourier Inversion Theorem, to prove the CLT for the Iwasawa cocycle σ.

This intrinsic approach allows us to answer Question (0.5) not only when the action of the semigroup Γ_μ on $\mathbb{R}^d$ is irreducible but also when this action is semisimple, i.e. when every Γ_μ-invariant vector subspace of $\mathbb{R}^d$ admits a Γ_μ-invariant complementary subspace.

0.11. Why is the Iwasawa cocycle so important to you? Both the Cartan projection and the Iwasawa cocycle are important to us. We recall that they are constructed thanks to the Cartan decomposition and the Iwasawa decomposition of a connected real reductive algebraic group

$$G = K \exp \mathfrak{a}^+ K \quad \text{and} \quad G = K \exp \mathfrak{a} N.$$

Here K is a maximal compact subgroup of G, exp is the exponential map of G, $\mathfrak{a}$ is a Cartan subspace of the Lie algebra $\mathfrak{g}$ of G that is orthogonal to the Lie algebra $\mathfrak{k}$ of K with respect to the Killing form, $\mathfrak{a}^+$ is a Weyl chamber in $\mathfrak{a}$, and N is the corresponding unipotent subgroup of G. Let M be the centralizer of $\mathfrak{a}$ in K. With these notations, the *flag variety* is the quotient space

$$\mathcal{P} = G/P \quad \text{where} \quad P = M \exp \mathfrak{a} N$$

is the normalizer of N. This group P is called the *minimal parabolic subgroup* associated to $\mathfrak{a}^+$.

The precise formulas defining κ and σ are, for g in G and η in $\mathcal{P}$,

$$g \in K e^{\kappa(g)} K \quad \text{and} \quad gk \in K e^{\sigma(g,\eta)} N$$

where k in K is chosen so that $k^{-1}\eta$ is N-invariant.

For instance, when $G = \mathrm{GL}(d, \mathbb{R})$, one can take $\mathfrak{a}$ to be the space of diagonal matrices, $\mathfrak{a}^+$ the subset of diagonal matrices with non-increasing coefficients, $K = \mathrm{SO}(d, \mathbb{R})$, and N the group of upper triangular unipotent matrices. In this case the Cartan decomposition is the "polar decomposition", the Cartan projection κ is the multinorm κ_V given by Formula (0.20), and the Iwasawa decomposition is obtained by the "Gram-Schmidt orthonormalisation process".

For g in G, the Cartan projection $\kappa(g)$ is important because it simultaneously controls for all representations ρ of G the norms of the matrices $\rho(g)$. Similarly, for g in G and η in $\mathcal{P}$, the Iwasawa cocycle $\sigma(g,\eta)$ is important because it controls simultaneously the norms of all vectors $\frac{1}{\|v\|}\rho(g)v$ when $\mathbb{R}v$ is a line invariant by the stabilizer of η. More precisely, one has the following fact:

When (V,ρ) is an irreducible algebraic representation of G, one has, for a suitable K-invariant norm on V, the equalities, for all g in G, η in $\mathcal{P}$, and every line $\mathbb{R}v$ in V which is invariant by the stabilizer of η,

$$\log\|\rho(g)\| = \chi(\kappa(g)) \quad\text{and}\quad \log\tfrac{\|\rho(g)v\|}{\|v\|} = \chi(\sigma(g,\eta))$$

where the linear functional $\chi \in \mathfrak{a}^*$ is the "highest weight" of V.

Because of this fact, the five theorems of Section 0.9 are multidimensional extensions of the five theorems of Section 0.5.

0.12. I am allergic to local fields. Is it safe to open this book? In this text we will not only study the asymptotic behavior of product of iid random real matrices, but we will allow the coefficients of these matrices to be in any local field $\mathbb{K}$. We recall that a local field $\mathbb{K}$ is a finite extension of either the field of p-adic numbers $\mathbb{Q}_p$, the field of Laurent series $\mathbb{F}_p((T))$ with coefficients in the finite field $\mathbb{F}_p$, where p is prime number, or the field $\mathbb{Q}_\infty = \mathbb{R}$.

For a first reading, you can assume that $\mathbb{K} = \mathbb{R}$. Except in very few places that we will point out, the proofs are not simpler over $\mathbb{R}$ than they are over any local field $\mathbb{K}$. A reader more familiar with local fields may assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$ since all the difficulties already occurs in these cases.

So you may wonder in the first place why we want to state these results over local fields. The reason is that those extended results give new information of an arithmetic flavor. For instance when the support of the law μ consists of finitely many matrices in $\mathrm{SL}(d,\mathbb{Q})$, the coefficients of the random products p_n are rational numbers. The results over $\mathbb{K} = \mathbb{R}$ give information on the size of these coefficients while the extended results over $\mathbb{K} = \mathbb{Q}_p$ give information on the size of the denominators of these coefficients, and more precisely on the powers of the prime number p which occur in these denominators.

As a by-product of this point of view, we will be see that the five limit theorems we quoted in Section 0.5 can be adapted over any local field $\mathbb{K}$, even in positive characteristic, except that the variance Φ might be equal to 0 (see Section 13.7).

0.13. Why are there so many chapters in this book? Sometimes chapters are related two by two, the first one dealing with general cocycles over semigroup actions, the second one applying these general results to products of random matrices.

In Chapter 1, we recall basic facts on Markov chains.

In Chapter 2, we prove the LLN for cocycles over a semigroup action.

In Chapter 3, we prove the LLN for products of random matrices.

In Chapter 4, we explain how to induce a random walk to a finite index subsemigroup.

In Chapter 5, we check that Zariski dense semigroups in semisimple real Lie groups always contain loxodromic elements.

In Chapter 6, we focus on the Jordan projection of Zariski dense semigroups in semisimple real Lie groups.

In Chapter 7, we recall a few basic facts on reductive algebraic groups over local fields, their algebraic representations, their flag varieties, their Iwasawa cocycle and their Cartan projection.

In Chapter 8, we study the Zariski dense semigroups in algebraic reductive $\mathcal{S}$-adic Lie groups.

In Chapter 9, we reformulate the LLN for products of random matrices in the intrinsic language of Chapter 7.

In Chapter 10, we study the spectral properties of the complex transfer operator for a cocycle over a contracting semigroup action.

In Chapter 11, we prove the CLT, LIL and LDP for a cocycle over a contracting semigroup action.

In Chapter 12, we deduce the CLT, LIL and LDP for the Iwasawa cocycle and the Cartan projection.

In Chapter 13, we give a short proof of the Hölder regularity of the stationary measure on the flag variety. We apply it to prove the LLN, CLT, LIL and LDP for the coefficients and for the spectral radius.

In Chapter 14, we study more deeply the spectral properties of the complex transfer operator.

In Chapter 15, we prove the LLT for a cocycle over a contracting semigroup action.

In Chapter 16, we deduce the LLT for the Iwasawa cocycle. We apply it to prove the LLT for the Cartan projection, and for the norm of vectors.

In Appendix 1, we recall basic facts on Martingales and their applications to the LLN for "sums of random numbers".

In appendix 2, we recall basic facts on bounded operators in Banach spaces, their spectrum and their essential spectrum. These facts are used in the proof of the Local Limit Theorem.

In Appendix 3, we quote our sources.

0.14. Whom do you thank? Institutions, referees, colleagues, students, friends, and families who financed us, teased us, helped us, read us, encouraged us, and supported us.

Part 1

Law of Large Numbers

1. Stationary measures

In this preliminary chapter, we first state general properties of a Markov operator P on a Borel space X. We study the P-invariant probability measures ν on X, and we prove the ergodicity of the associated *forward* dynamical system when ν is ergodic.

We focus then on the Markov-Feller operators, and in particular on the Markov-Feller operator P_μ associated to a random walk. For this operator P_μ and for the P_μ-invariant probability measures ν, which are also called *μ-stationary*, we explain the construction of the *backward* dynamical system and prove its ergodicity, when ν is ergodic.

In the next chapters, this space X will be a projective space or a flag variety and the Markov-Feller operator P will be the operator P_μ associated to a probability measure μ on the group G of automorphisms of X.

1.1. Markov operators.

We begin by general facts about Markov operators P and the probability measures ν they preserve (Lemma 1.3). We will give various equivalent definitions for the ergodicity of ν (Proposition 1.8). A key tool in order to prove the equivalence of these definitions is the *adjoint Markov operator* P^* (Lemma 1.4).

1.1.1. *Markov chains on standard Borel spaces.* Let $(X, \mathcal{X})$ be a *standard Borel space*. By a *Markov chain* on X, we mean a Borel map $x \mapsto P_x$ from X to the space of Borel probability measures on X. This space X will be sometimes called the *state space* of the Markov chain. For any bounded Borel function φ on X and any x in X, we set

$$P\varphi(x) = \textstyle\int_X \varphi \, \mathrm{d}P_x$$

and we say P is the *Markov operator* associated to the Markov chain. A function φ is said to be *P-invariant* if $P\varphi = \varphi$.

Let us recall the construction of the *Markov probability measures* $\mathbb{P}_x$ associated to P on the space Ω of *forward trajectories*. We set $\Omega = X^{\mathbb{N}}$ and we equip it with the product σ-algebra $\mathcal{B} = \mathcal{X}^{\otimes \mathbb{N}}$. An element ω in Ω will be written as a sequence $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$. For any x in X, there exists a unique Borel probability measure $\mathbb{P}_x$ on Ω such that, for any bounded Borel functions $\varphi_0, \ldots, \varphi_n$ on X, one has

$$\textstyle\int_\Omega \varphi_0(\omega_0) \cdots \varphi_n(\omega_n) \, \mathrm{d}\mathbb{P}_x(\omega) = (\varphi_0 P(\ldots(\varphi_{n-1} P(\varphi_n))\ldots))(x).$$

In other words, $\mathbb{P}_x$ is implicitely defined by $\mathbb{P}_x = \delta_x \otimes (\int_X \mathbb{P}_y \, \mathrm{d}P_x(y))$. We say $\mathbb{P}_x$ is the Markov measure associated to P and x (see Neveu's book [**91**, Chap. 3] for more details).

A probability measure ν on $(X, \mathcal{X})$ is said to be *P-invariant* if for every bounded Borel function φ on X, one has $\nu(P\varphi) = \nu(\varphi)$.

1.1.2. *Measure preserving Markov operators.* Let now $(X, \mathcal{X}, \nu)$ be a probability space and P an operator on the Banach space $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$ of (equivalence classes of) bounded measurable complex-valued functions on X. The operator P is called a *contraction* if $\|P\| \leq 1$. The operator P is called *non-negative*, if for every non-negative function $\varphi \in \mathrm{L}^\infty(X, \nu)$, the image $P\varphi$ is also non-negative. The operator P is called a *measure preserving Markov operator* on $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$ if it is a non-negative contraction such that $P\mathbf{1} = \mathbf{1}$ and, for every function $\varphi \in \mathrm{L}^\infty(X, \nu)$, one has $\int_X P\varphi \, \mathrm{d}\nu = \int_X \varphi \, \mathrm{d}\nu$.

If $(X, \mathcal{X})$ is a standard Borel space, P a Markov chain on $(X, \mathcal{X})$ and ν is a P-invariant probability measure, then P defines a measure preserving Markov operator on $(X, \mathcal{X}, \nu)$. Conversely if, $(X, \mathcal{X}, \nu)$ is a Lebesgue probability space, then every measure preserving Markov operator on $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$ comes from a Markov chain on a set of full measure in X.

Let us again assume $(X, \mathcal{X}, \nu)$ is any probability space and P is a general measure preserving Markov operator on $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$. We shall prove that P may be extended, for any $1 \leq p < \infty$, as a continuous contraction on the space $\mathrm{L}^p(X, \mathcal{X}, \nu)$ of functions φ for which $|\varphi|^p$ is integrable. This will follow from an elementary extension of Jensen's inequality:

Lemma 1.1. *Let P be a measure preserving Markov operator on $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$ and $\theta : \mathbb{C} \to \mathbb{R}$ be a convex function. Then, for any φ in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$, one has*

$$\theta(P\varphi) \leq P(\theta(\varphi)).$$

Proof. Pick φ in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$. By standard arguments about convex functions, there exists a sequence τ_n of affine functions $\mathbb{C} \to \mathbb{R}$ such that, for every z in $\mathbb{C}$, one has $\theta(z) = \sup_n \tau_n(z)$. Now, using successively the fact that P is non-negative and the equality $P\mathbf{1} = \mathbf{1}$, we get, for ν-almost every x in X, for any n in $\mathbb{N}$,

$$P\theta(\varphi)(x) \geq P\tau_n(\varphi)(x) = \tau_n(P\varphi(x)).$$

Thus $P\theta(\varphi)(x) \geq \theta(P\varphi(x))$ and we are done. $\square$

Corollary 1.2. *Let P be a measure preserving Markov operator on $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$. Then, for every $1 \leq p < \infty$, the operator P extends as a continuous contraction on $\mathrm{L}^p(X, \mathcal{X}, \nu)$.*

PROOF. By Lemma 1.1, for any φ in $\mathrm{L}^\infty(X,\mathcal{X},\nu)$, one has $|P\varphi|^p \leq P\,|\varphi|^p$, hence, since P is measure preserving,

$$\|P\varphi\|_p = (\textstyle\int_X |P\varphi|^p \,\mathrm{d}\nu)^{1/p} \leq (\int_X P\,|\varphi|^p \,\mathrm{d}\nu)^{1/p} = \|\varphi\|_p\,,$$

which completes the proof. □

A $\mathcal{X}$-measurable subset $E \subset X$ is called ν-almost P-invariant if its characteristic functions $\mathbf{1}_E$ is P-invariant as an element of $\mathrm{L}^\infty(X,\mathcal{X},\nu)$.

The following lemma tells us that every P-invariant function is a limit of linear combinations of P-invariant subsets.

LEMMA 1.3. *Let P be a measure preserving Markov operator on $\mathrm{L}^\infty(X,\mathcal{X},\nu)$. Then, for any $1 \leq p \leq \infty$, the vector subspace generated by the characteristic functions of ν-almost everywhere P-invariant subsets is dense in the space $\mathrm{L}^p(X,\mathcal{X},\nu)^P$ of P-invariant functions.*

PROOF OF LEMMA 1.3. It suffices to prove the result for functions with real values. Let φ be a real function in $\mathrm{L}^1(X,\mathcal{X},\nu)^P$. First note that the function $\varphi_+ := \max(\varphi, 0)$ is also P-invariant. Indeed, since P is non-negative, we have

$$P\varphi_+ \geq \max(P\varphi, 0) = \varphi_+.$$

Combining this inequality with the equality $\int_X P\varphi_+ \,\mathrm{d}\nu = \int_X \varphi_+ \,\mathrm{d}\nu$, we get $P\varphi_+ = \varphi_+$ in $\mathrm{L}^1(X,\mathcal{X},\nu)$. Now, we claim that the characteristic function $\mathbf{1}_{\{\varphi>0\}}$ is also P-invariant. Indeed, this function is the limit in $\mathrm{L}^1(X,\mathcal{X},\nu)$ of the functions $\min(1, n\,\varphi_+)$ and, by Corollary 1.2, P is continuous in $\mathrm{L}^1(X,\mathcal{X},\nu)$. As a consequence, for $a < b$, the characteristic function $\mathbf{1}_{\{a<\varphi\leq b\}}$ is also P-invariant. The result follows, since every real φ in $\mathrm{L}^p(X,\mathcal{X},\nu)$ is the limit in $\mathrm{L}^p(X,\mathcal{X},\nu)$

$$\varphi = \lim_{n\to\infty} \textstyle\sum_{-n^2\leq k\leq n^2} \frac{k}{n}\, \mathbf{1}_{\{k/n<\varphi\leq(k+1)/n\}}.$$ □

In the following lemma, we define the *adjoint operator* P^* of P and we check that P and P^* have the same invariant functions:

LEMMA 1.4. *Let P be a measure preserving Markov operator on $\mathrm{L}^\infty(X,\mathcal{X},\nu)$.*
a) Then there exists a unique measure preserving Markov operator P^* on $\mathrm{L}^\infty(X,\mathcal{X},\nu)$, called the adjoint operator of P, such that, for every $\varphi, \varphi' \in \mathrm{L}^\infty(X,\mathcal{X},\nu)$, one has

$$\textstyle\int_X P\varphi\,\varphi' \,\mathrm{d}\nu = \int_X \varphi\, P^*\varphi' \,\mathrm{d}\nu. \tag{1.1}$$

b) A function φ in $\mathrm{L}^1(X,\mathcal{X},\nu)$ is P-invariant if and only if it is P^*-invariant.

Proof. *a*) By Lemma 1.1.2, P extends as a continuous operator of $\mathrm{L}^1(X, \mathcal{X}, \nu)$. Let P^* be the adjoint operator to P on $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$, viewed as the dual space of $\mathrm{L}^1(X, \mathcal{X}, \nu)$, so that (1.1) holds and let us check that P^* is a measure preserving Markov operator.

Since P is a contraction, so is P^*. Since P is non-negative, for any $\varphi, \varphi' \geq 0$ in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$, one has

$$\int_X \varphi\, P^*\varphi' \,\mathrm{d}\nu = \int_X P\varphi\, \varphi' \,\mathrm{d}\nu \geq 0,$$

so that $P^*\varphi' \geq 0$ and P^* is non-negative.

Finally, since P is measure preserving, for any φ in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$, one has

$$\int_X \varphi \,\mathrm{d}\nu = \int_X P\varphi \,\mathrm{d}\nu = \int_X \varphi(P^*\mathbf{1}) \,\mathrm{d}\nu,$$

that is, $P^*\mathbf{1} = \mathbf{1}$. In the same way,

$$\int_X P^*\varphi \,\mathrm{d}\nu = \int_X \varphi(P\mathbf{1}) \,\mathrm{d}\nu = \int_X \varphi \,\mathrm{d}\nu,$$

that is, P^* is measure preserving, which was to be shown.

b) We first check the direct implication when φ is a characteristic function $\varphi = \mathbf{1}_E$ where E be a ν-almost surely P-invariant measurable subset of X. According to (1.1) with $\varphi = \varphi' = \mathbf{1}_E$ and to the bounds $0 \leq P^*\mathbf{1}_E \leq 1$ the function $P^*\mathbf{1}_E$ is equal to 1 on E. Since $\int_X P^*\mathbf{1}_E \,\mathrm{d}\nu = \nu(E)$, we get $P^*\mathbf{1}_E = \mathbf{1}_E$. Now, by Corollary 1.2, P^* acts continuously on $\mathrm{L}^1(X, \mathcal{X}, \nu)$ and, by Lemma 1.3, the characteristic functions of ν-almost surely P-invariant measurable subsets span a dense subset of $\mathrm{L}^1(X, \mathcal{X}, \nu)^P$, so that if φ is P-invariant in $\mathrm{L}^1(X, \mathcal{X}, \nu)$, one has $P^*\varphi = \varphi$. This proves the direct implication. The converse implication follows since $P^{**} = P$. □

Remark 1.5. The definition of the adjoint operator of a Markov operator depends on the measure. For example, let $X = \{0, 1\}^{\mathbb{N}}$ be the set of sequences of 0's and 1's, equipped with the natural σ-algebra, and P be the Markov operator associated to the shift map, that is, for every x in X, P_x is the Dirac mass at Tx, where $(Tx)_k = x_{k+1}$. Fix $0 < p < 1$ and let ν be the Bernoulli measure with parameter p, that is $\nu = (p\delta_0 + (1-p)\delta_1)^{\otimes \mathbb{N}}$. Then, one checks that ν is P-invariant and, for any φ in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$, for ν-almost any x in X, one has $P^*\varphi(x) = p\varphi(0x) + (1-p)\varphi(1x)$, which depends on p.

1.1.3. *Ergodicity of Markov operators.* We now let again $(X, \mathcal{X})$ be a standard Borel space, P be a Markov chain on $(X, \mathcal{X})$ and ν be a P-invariant probability measure. We shall give equivalent definitions for ergodicity. First let us describe the functions which are ν-almost surely P-invariant.

Lemma 1.6. *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator on X and ν be a P-invariant probability measure. Then, every ν-almost surely P-invariant bounded Borel function φ is equal ν-almost everywhere to a P-invariant bounded Borel function ψ.*

Proof. Let φ be a bounded Borel function such that one has $P\varphi = \varphi$ in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$. For x in X, we set

$$\varphi_\infty(x) = \liminf_{n\to\infty} P^n\varphi(x).$$

By Fatou's lemma, we have $P\varphi_\infty \leq \varphi_\infty$. We set, for any x in X,

$$\psi(x) = \lim_{n\to\infty} P^n\varphi_\infty(x).$$

By the monotone convergence theorem, we have $P\psi = \psi$.

Now, since φ is P-invariant in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$, there exists a Borel subset E of X with $\nu(E) = 1$ such that, for any x in E, for any $n \geq 0$, one has $P^n\varphi(x) = \varphi(x)$, hence $\varphi_\infty(x) = \varphi(x)$. In particular, φ_∞ is P-invariant in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$ and there exists a Borel subset F of X with $\nu(F) = 1$ such that, for any x in F, for any $n \geq 0$, one has $P^n\varphi_\infty(x) = \varphi_\infty(x)$, hence $\psi(x) = \varphi_\infty(x)$. We are done, since $\psi = \varphi$ on $E \cap F$. $\square$

Remark 1.7. Here is a subtle point in the definition of ν-almost P-invariant subsets : there may exist ν-almost P-invariant subsets E of X which are not ν-almost everywhere equal to an invariant subset. For example, let X be a triple $\{a, b, c\}$ and P be the Markov operator such that

$$P_a = \tfrac{1}{2}(\delta_b + \delta_c)\ ,\quad P_b = \delta_b \text{ and } \ P_c = \delta_c.$$

The measure $\nu := \frac{1}{2}(\delta_b + \delta_c)$ is P-invariant and the set $E := \{b\}$ is ν-almost P-invariant. Indeed, the characteristic function $\varphi := \mathbf{1}_E$ is ν-almost everywhere equal to the ν-almost P-invariant function $\psi := \frac{1}{2}\mathbf{1}_{\{a\}} + \mathbf{1}_{\{b\}}$. One cannot choose ψ to be a characteristic function since the only P-invariant subsets of X are $\emptyset$ and X.

We can now give five equivalent definitions for ergodicity:

Proposition 1.8. *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator on X and ν be a P-invariant Borel probability measure. The following are equivalent:*
(i) every P-invariant bounded Borel function is constant ν-almost everywhere.
(ii) every P-invariant element in $\mathrm{L}^1(X, \mathcal{X}, \nu)$ is constant.
(iii) every P-invariant element in $\mathrm{L}^\infty(X, \mathcal{X}, \nu)$ is constant.
(iv) every ν-almost P-invariant Borel subset of X has measure 0 or 1.

(v) ν is extremal in the convex set of P-invariant Borel probability measures.
In this case ν is said to be P-ergodic.

PROOF. The implications (ii)⇒(iii)⇒(iv) are immediate and their converse (iv)⇒(ii) follows from Lemma 1.3. The implication (i)⇒(iii) is a consequence of Lemma 1.6 and its converse (iii)⇒(i) is immediate.

Let us prove (ii)⇒(v). Let P^* be the adjoint of P with respect to ν as in Lemma 1.4. If ν is equal to a convex combination $t\nu_1 + (1-t)\nu_2$ where ν_1 and ν_2 are P-invariant Borel probability measures and $0 < t < 1$, for $i = 1, 2$, ν_i is absolutely continuous with respect to ν and hence can be written as $\varphi_i \nu$, where φ_i belongs to $\mathrm{L}^1(X, \mathcal{X}, \nu)$ and has integral 1. Since ν_i is P-invariant, one has $P^* \varphi_i = \varphi_i$. Again by Lemma 1.4.*b*), one has $P\varphi_i = \varphi_i$, hence by assumption, $\varphi_i = 1$ ν-almost everywhere, that is $\nu_i = \nu$, which was to be shown.

Finally, let us prove (v)⇒(iv). If $E \in \mathcal{X}$ is a ν-almost P-invariant subset of X, by Lemma 1.4.*b*), one has $P^* \mathbf{1}_E = \mathbf{1}_E$, hence the Borel measures $\nu_{|E}$ and $\nu_{|E^c}$ are P-invariant. Since ν is extremal, we get $\nu(E) = 0$ or $\nu(E^c) = 0$ as required. □

1.2. Ergodicity and the forward dynamical system.

In this section we introduce the dynamical system on the space of forward trajectories of a Markov chain, and we interpret the P-ergodicity of a measure as an ergodicity property of this dynamical system.

Let P be a Markov chain on a standard Borel space $(X, \mathcal{X})$. The *forward dynamical system* $(\Omega, \mathcal{B}, T)$ is the dynamical system on the space of forward trajectories given by

$$T : \Omega \to \Omega \; ; \; (\omega_0, \omega_1, \ldots) \mapsto (\omega_1, \omega_2, \ldots)$$

For any Borel probability measure ν on X we set $\mathbb{P}_\nu$ for the probability measure on $(\Omega, \mathcal{B})$

$$\mathbb{P}_\nu := \int_X \mathbb{P}_x \, \mathrm{d}\nu(x)$$

and $\mathbb{E}_\nu$ for the corresponding expectation operator.

The following proposition interprets the P-invariance and the P-ergodicity of ν as an invariance property and an ergodicity property of the measured forward dynamical system $(\Omega, \mathcal{B}, T, \mathbb{P}_\nu)$.

PROPOSITION 1.9. *Let ν be a Borel probability measure on X.*
a) Then ν is P-invariant if and only if $\mathbb{P}_\nu$ is T-invariant.
b) In this case, ν is P-ergodic if and only if $\mathbb{P}_\nu$ is T-ergodic.

Proof of Proposition 1.9. We denote by $\mathcal{X}_0 \subset \mathcal{B}$ the sub-σ-algebra generated by ω_0. More generally, we denote by $\mathcal{X}_n \subset \mathcal{B}$ the sub-σ-algebra generated by $\omega_0, \ldots, \omega_n$. By construction of the measures $\mathbb{P}_x$, $x \in X$, and $\mathbb{P}_\nu$, for any bounded Borel function ψ on Ω, the conditional expectation of ψ is given by the formula, for $\mathbb{P}_\nu$-almost all ω in Ω,

$$\mathbb{E}_\nu(\psi \mid \mathcal{X}_n)(\omega) = \int_\Omega \psi(\omega_0, \ldots, \omega_{n-1}, \omega'_0, \omega'_1, \ldots)\, \mathrm{d}\mathbb{P}_{\omega_n}(\omega'). \tag{1.2}$$

Hence, in particular,

$$\mathbb{E}_\nu(\psi \circ T^n \mid \mathcal{X}_n) = \mathbb{E}_\nu(\psi \mid \mathcal{X}_0) \circ T^n. \tag{1.3}$$

a) If ψ is a bounded Borel function on Ω, we let φ denote the bounded Borel function on X given by, for every x in X,

$$\varphi(x) = \int_\Omega \psi(\omega)\, \mathrm{d}\mathbb{P}_x(\omega).$$

In other words, $\varphi(x)$ is the expected value of the function ψ for the trajectories of the Markov chain starting at x. The map $\psi \mapsto \varphi$ is onto and, we have, for ν-almost any ω in Ω,

$$\mathbb{E}_\nu(\psi \mid \mathcal{X}_0)(\omega) = \varphi(\omega_0) \text{ and } \mathbb{E}_\nu(\psi \circ T \mid \mathcal{X}_0)(\omega) = P\varphi(\omega_0).$$

Thus, we get

$$\mathbb{E}_\nu(\psi) = \nu(\varphi) \text{ and } \mathbb{E}_\nu(\psi \circ T) = \nu(P\varphi),$$

whence the result.

b) We assume first that ν is P-ergodic and we want to prove that any T-invariant bounded Borel function ψ on Ω is constant. We still set, for any x in X, $\varphi(x) = \int_\Omega \psi(\omega)\, \mathrm{d}\mathbb{P}_x(\omega)$. We get

$$P\varphi(x) = \int_X \int_\Omega \psi(\omega)\, \mathrm{d}\mathbb{P}_y(\omega)\, \mathrm{d}P_x(y) = \int_\Omega \psi(T\omega)\, \mathrm{d}\mathbb{P}_x(\omega) = \varphi(x).$$

Thus, φ is constant ν-almost everywhere and we may assume that $\varphi = 0$. Now, since the σ-algebra $\mathcal{B}$ is spanned by the increasing union of the σ-algebras $\mathcal{X}_n$, $n \geq 0$, ψ is the limit in $\mathrm{L}^1(\Omega, \mathbb{P}_\nu)$ of the functions $\mathbb{E}_\nu(\psi \mid \mathcal{X}_n)$. One computes

$$\mathbb{E}_\nu(\psi \mid \mathcal{X}_n) = \mathbb{E}_\nu(\psi \circ T^n \mid \mathcal{X}_n) = \mathbb{E}_\nu(\psi \mid \mathcal{X}_0) \circ T^n = 0.$$

Hence $\psi = 0$ as required.

Conversely, we assume that $\mathbb{P}_\nu$ is T-ergodic and we want to prove that any P-invariant bounded Borel function φ on X is constant ν-almost everywhere. Indeed, let us set, for any $n \geq 0$ and ω in Ω,

$$\psi_n(\omega) = \varphi(\omega_n).$$

By construction, for any $n \geq 1$, for $\mathbb{P}_\nu$-almost any ω, one has

$$\mathbb{E}_\nu(\psi_n \mid \mathcal{X}_{n-1})(\omega) = P\varphi(\omega_{n-1}) = \varphi(\omega_{n-1}) = \psi_{n-1}(\omega),$$

that is, the sequence ψ_n is a uniformly bounded martingale. By Doob's martingale convergence theorem 1.3, it converges almost everywhere to a function ψ in $\mathrm{L}^\infty(\Omega, \mathbb{P}_\nu)$. By construction, one has, for $\mathbb{P}_\nu$-almost every ω,

$$\psi(T\omega) = \lim_{n\to\infty} \varphi(\omega_{n+1}) = \psi(\omega)$$

and ψ is constant $\mathbb{P}_\nu$-almost everywhere. Since, for $\mathbb{P}_\nu$-almost every ω, one has

$$\varphi(\omega_0) = \psi_0(\omega) = \mathbb{E}_\nu(\psi|\mathcal{X}_0)(\omega),$$

the function φ is constant ν-almost everywhere, as required. □

1.3. Markov-Feller operators.

We define Markov-Feller operators: they are the analogues, in the theory of Markov operators, of continuous transformations in the theory of classical dynamical systems.

When X is a compact space, a *Markov-Feller operator* on X is a nonnegative operator P on the space of continuous functions on X such that $P\mathbf{1} = \mathbf{1}$. In other terms, a Markov-Feller operator is a Markov chain on X such that the map $x \mapsto P_x$ is continuous, when the space $\mathcal{P}(X)$ of Borel probability measures of X is equipped with the weak-$*$ topology.

The following lemma reduces the study of P-invariant measures to the study of those that are ergodic.

LEMMA 1.10. *Let P be a Markov-Feller chain on a compact metric space X. Then there exists P-invariant Borel probability measures on X. In the dual space of $\mathcal{C}^0(X)$, equipped with the weak-$*$ topology, the set of P-invariant Borel probability measures is the closed convex hull of the set of ergodic ones.*

PROOF. Since X is a compact space, the space $\mathcal{M}(X)$ of complex *Borel measures* on X is the dual space of the space $\mathcal{C}^0(X)$ of continuous functions on X. We endow it with the weak-$*$ topology. The subset $\mathcal{P}(X)$ of Borel probability measures on X is then a compact subset of X.

We use Markov-Kakutani's argument: we start from any point x in X and consider the sequence of probability measures on X

$$\nu_n : \varphi \mapsto \tfrac{1}{n}(\varphi(x) + P\varphi(x) + \cdots + P^{n-1}\varphi(x)).$$

Since the set $\mathcal{P}(X)$ is compact, ν_n admits a cluster point ν_∞ in the weak-$*$ topology. Passing to the limit in the equalities, with φ in $\mathcal{C}^0(X)$,

$$\nu_n(P\varphi) - \nu_n(\varphi) = \tfrac{1}{n}(P^n\varphi(x) - \varphi(x)),$$

one gets

$$\nu_\infty(P\varphi) = \nu_\infty(\varphi).$$

Hence the probability measure ν_∞ is P-invariant.

Finally, by Proposition 1.8, a P-invariant Borel probability measure is P-ergodic if and only if it is extremal. The last part of the lemma now follows from Krein-Millman Theorem. $\square$

A Markov-Feller operator P is said to be *uniquely ergodic* if it admits a unique P-invariant Borel probability measure. As a corollary of the proof of the previous lemma, we get a nice interpretation of unique ergodicity.

COROLLARY 1.11. *Let P be a Markov-Feller operator on the compact metric space X. The following are equivalent:*
(i) *P is uniquely ergodic.*
(ii) *there exists a Borel probability measure ν on X such that, for any continuous function φ, one has*

$$\tfrac{1}{n}\sum_{k=0}^{n-1} P^k\varphi \xrightarrow[n\to\infty]{} \int_X \varphi \,\mathrm{d}\nu$$

uniformly.

PROOF. *(ii)⇒(i)* Let ν' be a P-invariant Borel probability measure on X. By the dominated convergence theorem, we have, for any continuous function φ,

$$\int_X \varphi \,\mathrm{d}\nu' = \int_X (\tfrac{1}{n}\sum_{k=0}^{n-1} P^k\varphi)\,\mathrm{d}\nu' \xrightarrow[n\to\infty]{} \int_X \varphi\,\mathrm{d}\nu.$$

(i)⇒(ii) Let x_n be a sequence in X. Reasoning as in the proof of Lemma 1.10, we get that any limit point of the sequence of measures $\nu_n := \frac{1}{n}\sum_{k=0}^{n-1}(P^*)^k\delta_{x_n}$ is P-invariant. Hence this sequence ν_n converges to ν $\square$

1.4. Stationary measures and the forward dynamical system.

> In this section, we give an alternative construction of the forward dynamical system associated to the action of a probability measure μ on a compact space X.

We recall that a *semigroup* is a set G endowed with an associative multiplication law $G \times G \to G$ and containing a neutral element. For instance, for any set X, the set $\mathcal{F}(X, X)$ of maps from X to X is a semigroup for the composition of applications. A *morphism* of semigroups $\rho : G \to H$ is a map sending the neutral element of G to the neutral element of H and such that, for any g, g' in G, $\rho(gg') = \rho(g)\rho(g')$. An *action* of G on a space X is a morphism from G to $\mathcal{F}(X, X)$.

A *topological semigroup* is a semigroup G endowed with a topology such that the multiplication is continuous. For instance when X is a compact space, the semigroup $\mathcal{C}^0(X,X)$ of continuous transformations of X endowed with the topology of uniform convergence is a topological semigroup. A *continuous action* of G on X is a continuous morphism of semigroups $G \to \mathcal{C}^0(X,X)$.

Let G be a second countable locally compact semigroup and X be a compact metrizable topological space on which G acts continuously. We denote by $\mathcal{G}$ the Borel σ-algebra of G and by $\mathcal{X}$ the Borel σ-algebra of X.

Let μ be a Borel probability measure on G, we denote by Γ_μ the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$. For any Borel probability measure ν on X, let $\mu * \nu$ denote the probability measure on X which is the image of the product measure $\mu \otimes \nu$ on $G \times X$ under the action map, that is

$$\mu * \nu = \textstyle\int_G g_* \nu \, \mathrm{d}\mu(g).$$

The Borel probability measure ν is said to be *μ-stationary* if

$$\mu * \nu = \nu.$$

If it is the case, it is said to be *μ-ergodic* if it cannot be written as a proper convex combination of two different μ-stationary Borel probability measures.

For instance any Γ_μ-invariant probability measure is μ-stationary. The converse is not true in general but Lemma 1.12 tells us that it is true when X is finite.

Lemma 1.12. *When X is a finite set, any μ-stationary probability measure ν on X is Γ_μ-invariant.*

Proof. We can assume that G is finite, equal to Γ_μ and that ν is ergodic. Let $S_\mu \subset G$ be the support of μ and $S_\nu \subset X$ be the support of ν. Stationarity of ν means that

$$\nu(\{x\}) = \textstyle\sum_{g \in S_\mu} \mu(\{g\}) \nu(g^{-1}\{x\}) \tag{1.4}$$

for every x in X. In particular one has the equality $S_\mu S_\nu = S_\nu$. Hence by replacing X with S_ν, we can also assume, with no loss of generality, that $X = S_\nu$ and that $S_\mu X = X$. Let X_0 be the set of points x in X such that $\nu(\{x\})$ is minimal.

Equality (1.4) implies that, for all x in X_0 and g in S_μ, one has

$$\nu(\{x\}) = \nu(g^{-1}\{x\}).$$

This means that ν is Γ_μ-invariant. $\square$

We now introduce the one-sided *Bernoulli shift* $(B, \mathcal{B}, \beta, T)$ with *alphabet* $(G, \mathcal{G}, \mu)$, that is $B = G^{\mathbb{N}^*}$ where $\mathbb{N}^*$ is the set of positive integers, $\mathcal{B}$ is the product σ-algebra $\mathcal{G}^{\otimes \mathbb{N}^*}$, β is the product measure $\mu^{\otimes \mathbb{N}^*}$, and T is the shift map given, by

$$Tb = (b_2, \ldots, b_{n+1}, \ldots) \text{ for } b = (b_1, \ldots, b_n, \ldots) \in B.$$

We now construct the *forward dynamical system* on $B \times X$. We equip $B \times X$ with the σ-algebra $\mathcal{B} \otimes \mathcal{X}$ of Borel subsets and we introduce the skew-product transformation

$$T^X : (b, x) \mapsto (Tb, b_1 x).$$

We identify the σ-algebra $\mathcal{X}$ of Borel subsets of X with the sub-σ-algebra of Borel subsets of $B \times X$ which do not depend on the first coordinate.

For any x in X, set

$$P_{\mu,x} = \mu * \delta_x.$$

One easily check that this defines a Markov-Feller operator P_μ on X.

We explain now how the forward dynamical system on $B \times X$ is related to the forward dynamical system (Ω, T) of the Markov operator $P = P_\mu$ that we introduced in Section 1.2. For any x in X, the associated Markov measure $\mathbb{P}_{\mu,x}$ on Ω is the image of the measure $\beta = \mu^{\otimes \mathbb{N}^*}$ on $B = G^{\mathbb{N}^*}$ under the map

$$(b_k)_{k \geq 1} \mapsto (b_k \cdots b_1 x)_{k \geq 0}. \tag{1.5}$$

If ν is a Borel probability measure on X, then ν is μ-stationary if and only if it is P_μ-invariant and, in this case, the measure $\mathbb{P}_\nu$ on Ω is the image of $\beta \otimes \nu$ under the map

$$(b, x) \mapsto (b_k \cdots b_1 x)_{k \geq 0},$$

which intertwines the maps T^X and T. By Proposition 1.8, ν is μ-ergodic if and only if it is P_μ-ergodic.

Remark 1.13. In general, the map $(b, x) \mapsto (b_k \cdots b_1 x)_{k \geq 0}$ is not a Borel isomorphism between $B \times X$ and Ω since non-trivial elements of G may have fixed points in X. Nevertheless, we have the following analogue of Proposition 1.9.

Proposition 1.14. *Let ν be a Borel probability measure on X.*
a) Then ν is μ-stationary if and only if $\beta \otimes \nu$ is T^X-invariant.
b) In this case, ν is μ-ergodic if and only if $\beta \otimes \nu$ is T^X-ergodic.

Proof. It follows the same lines as for the proof of Proposition 1.9. □

REMARK 1.15. There may exist a T^X-invariant Borel probability measure on $B \times X$ whose image by the projection on the first factor is equal to β but which is not of the form $\beta \otimes \nu$ for some μ-stationary Borel probability measure ν on X. For example, let G be the free group on two generators g and h, X be the Gromov boundary of G, i.e. the set of reduced one-sided infinite words in $g^{\pm}$ and $h^{\pm}$ and μ be the probability measure $\mu = \frac{1}{2}(\delta_g + \delta_h)$. For β-almost every b in B, b is a reduced word, that is, b may be seen as an element x_b of X. By construction, one has $x_{Tb} = b_1 x_b$. Hence, the image of β by the graph map $b \mapsto (b, x_b)$ on $B \times X$ is T^X-invariant. It is clearly not a product measure. In fact, this image measure is an example of the measures invariant by the backward dynamical system that we will construct below.

LEMMA 1.16. *Given μ, there exists a μ-stationary Borel probability measure on the compact space X.*

PROOF. This is a special case of Lemma 1.10. □

1.5. The limit measures and the backward dynamical system.

> For every μ-stationary probability measure on X, we construct in this section an equivariant measurable family of probability measures ν_b on X indexed by the Bernoulli shift and called the limit measures. We will use this family in order to construct the dynamical system of backward trajectories.

We keep the notations of section 1.4. In particular, G is a second countable locally compact semigroup, μ is a Borel probability measure on G, $(B, \mathcal{B}, \beta, T)$ is the associated one-sided Bernoulli shift, the semigroup G acts continuously on the compact metrizable topological space X and ν is a μ-stationary Borel probability measure on X.

Here is the construction of the *limit measures*.

LEMMA 1.17. *There exists a Borel map $b \mapsto \nu_b$ from B to $\mathcal{P}(X)$ such that, for β-almost any b in B, one has $(b_1 \cdots b_n)_* \nu \xrightarrow[n\to\infty]{} \nu_b$.*

REMARK 1.18. In this lemma, the compactness assumption on X can be removed (see [**14**, Lemma 3.2]).

PROOF. The main tool is Doob martingale theorem. Let, for any n in $\mathbb{N}$, $\mathcal{B}_n$ be the sub-σ-algebra of $\mathcal{B}$ spanned by the coordinate functions with indices p, $1 \leq p \leq n$. If ν is a μ-stationary Borel probability measure on X, one checks that, for any bounded Borel function φ on X, the sequence of functions

$$f_n : b \mapsto \int_X \varphi(b_1 \cdots b_n x)\, \mathrm{d}\nu(x)$$

on B is a uniformly bounded martingale with respect to the filtration $(\mathcal{B}_n)_{n\in\mathbb{N}}$: for β-almost all b in B and all $n \geq 0$, one has

$$\mathbb{E}(f_{n+1} \mid \mathcal{B}_n)(b) = f_n(b).$$

By applying Doob martingale convergence theorem (Theorem 1.3) to a countable dense subset D of functions $\varphi \in \mathcal{C}^0(X)$, we deduce that, for b in a subset $B' \subset B$ with $\beta(B') = 1$, for all φ in D, the limit

$$\nu_b(\varphi) := \lim_{n\to\infty} (b_1 \cdots b_n)_*\nu\,(\varphi)$$

exists. Hence, by approximation, this limit exists for all φ in $\mathcal{C}^0(X)$, i.e. the limit $\nu_b = \lim\limits_{n\to\infty} (b_1 \cdots b_n)_*\nu$ exists for all b in B'. □

The following lemma tells us that the stationary measure ν can be recovered from its limit measures ν_b by a simple averaging, and that these limit measures satisfy a nice equivariant property.

Lemma 1.19. *One has $\nu = \int_B \nu_b \, \mathrm{d}\beta(b)$ and, for β-almost any b in B, one has $\nu_b = (b_1)_*\nu_{Tb}$.*

Proof. Let φ belong to $\mathcal{C}^0(X)$. As ν is μ-stationary, for any n in $\mathbb{N}$, one has

$$\int_X \varphi \, \mathrm{d}\nu = \int_B f_n(b)\, \mathrm{d}\beta(b).$$

Passing to the limit, the first equality follows by the dominated convergence theorem.

The second assertion follows directly from the definition of ν_b. □

Remark 1.20. Conversely, according to [**14**, Lemma 3.2], if $b \mapsto \nu_b$ is a Borel map from B to $\mathcal{P}(X)$ such that for β-almost any b in B, one has $\nu_b = (b_1)_*\nu_{Tb}$, then the Borel probability measure $\nu := \int_B \nu_b \, \mathrm{d}\beta(b)$ on X is μ-stationary and, for β-almost any b in B, ν_b is equal to the limit probability measure $\lim_{n\to\infty}(b_1 \cdots b_n)_*\nu$.

We will also need an enhanced version of Lemma 1.17.

Lemma 1.21. *For any m in $\mathbb{N}$, for $\beta \otimes \mu^{*m}$-almost any (b, g) in $B \times G$, one has $(b_1 \cdots b_n g)_*\nu \xrightarrow[n\to\infty]{} \nu_b$.*

Proof. Let φ be in $\mathcal{C}^0(X)$ and set Φ to be the function on G

$$\Phi : h \mapsto \int_X \varphi(hx)\, \mathrm{d}\nu(x).$$

Since ν is μ-stationary, one has the equality, for n in $\mathbb{N}$ and h in G,

$$\int_G \Phi(hg)\,\mathrm{d}\mu^{*m}(g) = \Phi(h). \tag{1.6}$$

For g in G, we set f_n^g to be the function on B

$$f_n^g : b \mapsto \Phi(b_1 \cdots b_n g).$$

By Lemma 1.17, since $\mathcal{C}^0(X)$ is separable, it suffices to check that, for μ^{*m}-almost any g in G, the sequence of functions $f_n^g(b) - f_n(b)$ on B converges for β-almost all b towards 0. For any n in $\mathbb{N}$, using (1.6), we compute the integral

$$\begin{aligned} I_n &= \int_G \int_B |f_n^g(b) - f_n(b)|^2 \,\mathrm{d}\beta(b)\,\mathrm{d}\mu^{*m}(g) \\ &= \int_G \int_G |\Phi(hg) - \Phi(h)|^2 \,d\mu^{*m}(g)\,d\mu^{*n}(h) = J_{n+m} - J_n, \end{aligned}$$

where $J_n := \int_G \Phi(h)^2 \,\mathrm{d}\mu^{*n}(h)$. Since J_n is bounded by $\|\varphi\|_\infty$, one gets $\sum_{n=0}^\infty I_n < \infty$, and, for $\beta \otimes \mu^{*m}$-almost any (b, g) in $B \times G$,

$$\sum_{n=0}^{\infty} |f_n^g(b) - f_n(b)|^2 < \infty,$$

hence $f_n^g(b) - f_n(b)$ goes to zero as $n \to \infty$, whence the result. □

In order to appreciate the strength of the previous lemmas, we deduce the following corollary which is a reformulation of the classical Choquet-Deny Theorem in [**33**]. We recall that Γ_μ is the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$.

Corollary 1.22. *When G is abelian, every μ-stationary probability measure ν on X is Γ_μ-invariant.*

Proof. Since G is abelian, by Lemmas 1.17 and 1.21, for μ-almost every g in G and β-almost every b in B, one has the equality $\nu_b = g_*\nu_b$. Hence, averaging this equality over B and using Lemma 1.19, one gets the equality $\nu = g_*\nu$ for μ-almost every g in G. Now, the result follows, since the stabilizer of ν in G is a closed subsemigroup containing the support of μ. □

We now construct, when G is a group, the *backward dynamical system* on $B \times X$, or dynamical system of *backward trajectories*. We recall that $(B, \mathcal{B}, \beta, T)$ is the one-sided Bernoulli shift with alphabet $(G, \mathcal{G}, \mu)$. We equip the product space $B^X := B \times X$ with the σ-algebra $\mathcal{B}^X := \mathcal{B} \otimes \mathcal{X}$ of Borel subsets and we introduce the *skew-product* transformation

$$T^{\vee X} : (b, x) \mapsto (Tb, b_1^{-1}x)$$

and the Borel probability measure β^X on B^X given by

$$\beta^X := \int_B \delta_b \otimes \nu_b \,\mathrm{d}\beta(b).$$

The following proposition is an analog of Proposition 1.9. It interprets the P-ergodicity of ν as the ergodicity of the backward dynamical system $(B^X, \mathcal{B}^X, T^X, \beta^X)$.

Proposition 1.23. *Let G be a second countable locally compact group acting continuously on a compact metrizable topological space X, and ν be a μ-stationary Borel probability measure on X.*
a) Then the probability measure β^X on B^X is $T^{\vee X}$-invariant.
b) The measure β^X is $T^{\vee X}$-ergodic if and only if ν is μ-ergodic.

Proof. *a*) This follows from the following calculation which uses Lemma 1.19

$$\begin{aligned}\textstyle\int_{B^X} \varphi(T^{\vee X}(b,x))\,\mathrm{d}\beta^X(b,x) &= \textstyle\int_B \int_X \varphi(Tb, b_1^{-1}x)\,\mathrm{d}\nu_b(x)\,\mathrm{d}\beta(b)\\ &= \textstyle\int_B \int_X \varphi(Tb, x)\,\mathrm{d}\nu_{Tb}(x)\,\mathrm{d}\beta(b)\\ &= \textstyle\int_B \int_X \varphi(b, x)\,\mathrm{d}\nu_b(x)\,\mathrm{d}\beta(b)\\ &= \textstyle\int_{B^X} \varphi(b,x)\,\mathrm{d}\beta^X(b,x),\end{aligned}$$

where $\varphi : B^X \to \mathbb{R}_+$ is a $(\mathcal{B} \otimes \mathcal{X})$-measurable function.

b) First, assume β^X is $T^{\vee X}$-ergodic and let ν be equal to a convex combination $t\nu_1 + (1-t)\nu_2$ of μ-stationary probability measures with $0 < t < 1$. We get, for β-almost any b in B,

$$\nu_b = t\nu_{1,b} + (1-t)\nu_{2,b},$$

hence

$$\beta^X = t\beta_1^X + (1-t)\beta_2^X,$$

where, for $i = 1, 2$, β_i^X is constructed from ν_i. Since β^X is $T^{\vee X}$-ergodic, we have $\beta_1^X = \beta_2^X = \beta^X$ and therefore, by projecting on X, $\nu = \nu_1 = \nu_2$. By Proposition 1.8, ν is μ-ergodic.

Conversely, assume now ν is μ-ergodic and let us prove that β^X is $T^{\vee X}$-ergodic. This can be seen as an immediate consequence of the ergodicity of the forward dynamical system thanks to the ideas that will be introduced in Section 1.6 below. But we can also give a direct, more computational proof.

Let θ be a $T^{\vee X}$-invariant bounded Borel function on B^X. We want to prove that this function θ is β^X-almost surely constant. Let φ be any bounded Borel function on X and set

$$\rho(\varphi) = \int_{B^X} \varphi(x)\theta(b,x)\,\mathrm{d}\beta^X(b,x).$$

We first claim that the complex measure ρ on X is μ-stationary. This follow from the following calculation, with φ as above,

$$\begin{aligned}\int_G\int_X \varphi(gx)\,\mathrm{d}\rho(x)\,\mathrm{d}\mu(g) &= \int_G\int_B\int_X \varphi(gx)\theta(b',x)\,\mathrm{d}\nu_{b'}(x)\,\mathrm{d}\beta(b')\,\mathrm{d}\mu(g)\\ &= \int_B\int_X \varphi(b_1x)\theta(Tb,x)\,\mathrm{d}\nu_{Tb}(x)\,\mathrm{d}\beta(b)\\ &= \int_B\int_X \varphi(y)\theta(b,y)\,\mathrm{d}\nu_b(y)\,\mathrm{d}\beta(b) = \int_X \varphi\,\mathrm{d}\rho.\end{aligned}$$

We prove now that the measure ρ is absolutely continuous with respet to ν. Indeed, if φ is a non-negative Borel function on X such that $\int_X \varphi\,\mathrm{d}\nu = 0$, we have, for β-almost any b in B, $\int_X \varphi\,\mathrm{d}\nu_b = 0$ hence $\varphi = 0$ on a set of ν_b-full measure and $\int_X \varphi\,\mathrm{d}\rho = 0$. That is, ρ is absolutely continuous with respect to ν.

By Proposition 1.8, as ν is μ-ergodic, ρ is a multiple of ν. It remains to prove the implication

$$\rho = 0 \Rightarrow \theta = 0.$$

Assume that $\rho = 0$. Let $n \geq 0$ and φ, ψ be bounded Borel functions on X and on G^n respectively. We calculate

$$\begin{aligned}&\int_{B^X} \psi(b_1,\dots,b_n)\varphi(b_n^{-1}\cdots b_1^{-1}x)\theta(b,x)\,\mathrm{d}\beta^X(b,x)\\ &= \int_{B^X} \psi(b_1,\dots,b_n)\varphi(b_n^{-1}\cdots b_1^{-1}x)\theta(T^nb,b_n^{-1}\cdots b_1^{-1}x)\,\mathrm{d}\beta^X(b,x)\\ &= \int_B\int_X \psi(b_1,\dots,b_n)\varphi(y)\theta(T^nb,y)\,\mathrm{d}((b_n^{-1}\cdots b_1^{-1})_*\nu_b)(y)\,\mathrm{d}\beta(b)\\ &= \int_{G^n}\int_B\int_X \psi(b_1,\dots,b_n)\varphi(y)\theta(b',y)\,\mathrm{d}\nu_{b'}(y)\,\mathrm{d}\beta(b')\,\mathrm{d}\mu^{\otimes n}(b_1,\cdots,b_n)\\ &= \mu^{\otimes n}(\psi)\,\rho(\varphi) \;=\; 0.\end{aligned}$$

Since the map

$$G^n \times X \to G^n \times X, (g_1,\dots,g_n,x) \mapsto (g_1,\dots,g_n,g_n^{-1}\cdots g_1^{-1}x)$$

is a homeomorphism, we get, for any bounded Borel function ψ on $G^n \times X$,

$$\int_{B^X} \psi(g_1,\dots,g_n,x)\theta(b,x)\,\mathrm{d}\beta^X(b,x) = 0.$$

This proves that $\theta = 0$, β^X-almost everywhere. $\square$

1.6. The two-sided fibered dynamical system.

We explain in this section how the forward and the backward dynamical systems are related. Indeed, both occur as factors of the space of biinfinite trajectories either equipped with the shift transformation or its inverse.

We keep the notations of Proposition 1.23. We denote by $(\widetilde{B},\widetilde{\mathcal{B}},\widetilde{\beta},\widetilde{T})$ the *two-sided Bernoulli shift* with alphabet $(G,\mathcal{G},\mu)$, that is, $\widetilde{B}$ is the product space $G^{\mathbb{Z}}$, $\widetilde{\mathcal{B}}$ is the product σ-algebra $\mathcal{G}^{\otimes\mathbb{Z}}$, $\widetilde{\beta}$ is the product measure $\mu^{\otimes\mathbb{Z}}$, and $\widetilde{T}$ is the shift map given, by

$$\widetilde{T}b = (\ldots, b_{n+1}, \ldots) \text{ for } b = (\ldots, b_n, \ldots) \in \widetilde{B}$$

This dynamical system is invertible and the probability measure $\widetilde{\beta}$ is $\widetilde{T}$-invariant.

For $\widetilde{\beta}$-almost every b in $\widetilde{B}$, we denote $b_+ := (b_1, b_2, \ldots) \in B$ and $b_- := (b_0, b_{-1}, b_{-2}, \ldots) \in B$. The map $b \mapsto b_+$ realizes the two-sided Bernoulli shift $(\widetilde{B}, \widetilde{\beta}, \widetilde{T})$ as the natural invertible extension of the one-sided Bernoulli shift (B, β, T). Similarly, the map $b \mapsto b_-$ realizes the inverse $(\widetilde{B}, \widetilde{\beta}, \widetilde{T}^{-1})$ of the two-sided Bernoulli shift as the natural invertible extension of the one-sided Bernoulli shift (B, β, T).

We now construct the *two-sided fibered dynamical system* on the space $\widetilde{B} \times X$ that we heuristically consider as the space of biinfinite trajectories. We endow this space with the σ-algebra $\widetilde{\mathcal{B}} \otimes \mathcal{X}$ of Borel subsets and we introduce the skew-product transformation

$$\widetilde{T}^X : (b, x) \mapsto (\widetilde{T}b, b_1 x)$$

and the Borel probability measure $\widetilde{\beta}^X$ on $B \times X$ defined by

$$\widetilde{\beta}^X := \textstyle\int_{\widetilde{B}} \delta_b \otimes \nu_{b_-} \, \mathrm{d}\widetilde{\beta}(b)$$

This dynamical system is invertible and the probability measure $\widetilde{\beta}^X$ is $\widetilde{T}$-invariant.

The map $(b, x) \mapsto (b_+, x)$ realizes the two-sided dynamical system $(\widetilde{B}^X, \widetilde{\beta}^X, \widetilde{T}^X)$ as the natural invertible extension of the forward dynamical system $(B^X, \beta \otimes \nu, T^X)$. Similarly, the map $(b, x) \mapsto (b_-, x)$ realizes the inverse $(\widetilde{B}^X, \widetilde{\beta}^X, (\widetilde{T}^X)^{-1})$ of the two-sided dynamical system as the backward dynamical system $(B^X, \beta^X, T^{\vee X})$. Since the natural invertible extension of an ergodic probability preserving dynamical system is also ergodic, and since the inverse of an ergodic transformation is also ergodic, this discussion gives a direct proof of the equivalences

$$\beta \otimes \nu \text{ is } T^X\text{-ergodic} \;\Leftrightarrow\; \widetilde{\beta}^X \text{ is } \widetilde{T}^X\text{-ergodic} \;\Leftrightarrow\; \beta^X \text{ is } T^{\vee X}\text{-ergodic.}$$

and explains how Propositions 1.9 and 1.23 are related.

1.7. Proximal stationary measures.

In this section, we introduce the property of μ-proximality for stationary measures. This proximality property will be satisfied by the stationary measures on projective spaces in Section 3.2 and by the stationary measures on the flag varieties in Section 9.1.

Let G be a second countable locally compact semigroup acting continuously on a compact metrizable topological space X, Say that a μ-stationary Borel probability measure ν on X is *μ-proximal* if, for β-almost any b in B, the Borel probability measure ν_b is a Dirac mass. An important example of a proximal stationary probability measure will be given in Proposition 9.1.

More generally, given a morphism $s : G \to F$ onto a finite group F, we define a *fibration over F* of X as a G-equivariant continuous map $X \to F$. We say that X is *fibered over F* if it is equipped with such a fibration. In this case, we say that ν is *μ-proximal over F* if, for β-almost any b in B, the Borel probability measure ν_b is a uniform average of $|F|$ Dirac masses and its image in F is the normalized counting measure on F. This definition will be used in Section 4.3, and an important example of such a situation will be given in Proposition 9.2.

We will apply the following lemma to the embedding of a flag variety in a product of projective spaces in order to prove Proposition 9.1.

LEMMA 1.24. *Let $X, X_1, \ldots, X_k$ be compact metrizable topological spaces, all of them equipped with a continuous action of a second countable locally compact semigroup G and, let $\pi : X \to X_1 \times \ldots \times X_k$ be a continuous injective G-equivariant map. Suppose, for any $1 \leq i \leq k$, there exists a unique μ-stationary Borel probability measure ν_i on X_i and ν_i is μ-proximal. Then, there exists a unique μ-stationary Borel probability measure on X and it is μ-proximal.*

PROOF. For any $1 \leq i \leq k$, since the probability measures ν_i is μ-proximal, there exists a Borel map $\xi_i : B \to X_i$ such that, for β-almost any b in B, one has $(\nu_i)_b = \delta_{\xi_i(b)}$. Set $\pi_i : X \to X_i$ to be the projection map on the factor X_i and set $\xi = (\xi_1, \ldots, \xi_k)$. Let ν be a μ-stationary Borel probability measure on X. Since, for any $1 \leq i \leq k$, the Borel probability measure $(\pi_i)_*\nu$ is μ-stationary, by uniqueness, one has $(\pi_i)_*\nu = \nu_i$ and, for β-almost any b in B, $(\pi_i)_*\nu_b = \delta_{\xi_i(b)}$, so that $\pi_*\nu_b = \delta_{\xi(b)}$. Hence ν is μ-proximal, and, for β-almost any b in B, one has $\xi(b) \in \pi(X)$ and $\pi_*\nu = \xi_*\beta$, whence the result. $\square$

2. Law of Large Numbers

The main goal of this Chapter is to prove a Law of Large Numbers for a general real valued cocycle with a unique average (Theorem 2.9).

In order to do this, we first reduce this statement to a Law of Large Numbers for a function with a unique average using Proposition 2.2. Then we prove the Law of Large Numbers for a function with a unique average (Corollary 2.8).

We will apply this Law of Large Numbers to the norm cocycle in Section 3.6 and to the Iwasawa cocycle in Section 9.4.

2.1. Birkhoff averages for functions on $G \times X$.

The aim of this section is Proposition 2.2 which reduces the proof of a Law of Large Numbers for a function σ on $G \times X$ to a Law of Large Numbers for a function φ on X called the drift function. This function φ is the expected value of σ.

As in Chapter 1, G is a second countable locally compact semigroup, μ is a Borel probability measure on G, $(B, \mathcal{B}, \beta, T)$ is the associated one-sided Bernoulli shift and the group G acts continuously on the compact metrizable topological space X.

The following Lemma is an application of Birkhoff Ergodic Theorem. Its conclusion will be our guideline towards more precise results.

Lemma 2.1. *Let ν be a μ-stationary μ-ergodic Borel probability measure on X and $\sigma : G \times X \to \mathbb{R}$ be a measurable function. Assume that*

$$\int_{G\times X} |\sigma| \, \mathrm{d}(\mu \otimes \nu) < \infty, \quad \text{and set} \quad \sigma_\mu := \int_{G\times X} \sigma \, \mathrm{d}(\mu \otimes \nu).$$

Then, one has

$$\frac{1}{n} \sum_{k=1}^{n} \sigma(b_k, b_{k-1} \cdots b_1 x) \xrightarrow[n\to\infty]{} \sigma_\mu, \tag{2.1}$$

$\beta \otimes \nu$-almost anywhere and in $\mathrm{L}^1(B \times X, \beta \otimes \nu)$.

Proof. We will use the forward dynamical system. For b in B and x in X, set $\varphi(b, x) = \sigma(b_1, x)$. Then φ is $\beta \otimes \nu$-integrable and, for b in B, x in X and $n \geq 1$, the left-hand side of (2.1) is equal to the Birkhoff average

$$\tfrac{1}{n}(\varphi(b, x) + \ldots + \varphi((T^X)^{n-1}(b, x))).$$

According to Proposition 1.9, $\beta \otimes \nu$ is T^X-ergodic, hence by Birkhoff theorem, this Birkhoff average converges $(\beta \otimes \nu)$-almost anywhere and in $\mathrm{L}^1(B \times X, \beta \otimes \nu)$ towards the spatial average $(\beta \otimes \nu)(\varphi) = (\mu \otimes \nu)(\sigma)$. □

We want to describe conditions under which the convergence of the Birkhoff averages (2.1) is uniform in x. The following proposition reduces this question to the Birkhoff averages of a function on X. Its proof relies on the classical Law of Large Numbers proven in Appendix 1.

PROPOSITION 2.2. *Let $\sigma : G \times X \to \mathbb{R}$ be a continuous function and*

$$\sigma_{\sup} : G \to \mathbb{R}\ ;\ g \mapsto \sigma_{\sup}(g) := \sup_{x\in X} |\sigma(g,x)|.$$

Assume that $\int_G \sigma_{\sup}(g)\,\mathrm{d}\mu(g) < \infty$ and introduce the drift function

$$\varphi : X \mapsto \mathbb{R}\ ;\ x \mapsto \varphi(x) := \int_G \sigma(g,x)\,\mathrm{d}\mu(g).$$

Then, for every x in X, for β-almost every b in B, one has

$$\tfrac{1}{n}\sum_{k=1}^{n} \left(\sigma(b_k, b_{k-1}\cdots b_1 x) - \varphi(b_{k-1}\cdots b_1 x)\right) \xrightarrow[n\to\infty]{} 0.$$

Moreover this sequence converges also in $\mathrm{L}^1(B,\beta)$ uniformly for $x \in X$.

PROOF. This is a direct application of the Law of Large Numbers, Theorem 1.6. Let $\varphi_n : B \to \mathbb{R}$ be the integrable function given by

$$\varphi_n(b) = \sigma(b_n, b_{n-1}\cdots b_1 x)$$

and $\mathcal{B}_n$ be the sub-σ-algebra of $\mathcal{B}$ generated by $b_1,\ldots,b_n$. One has the equality, for β-almost every b in B,

$$\mathbb{E}(\varphi_n \mid \mathcal{B}_{n-1}) = \varphi(b_{n-1}\cdots b_1 x).$$

Hence we only have to check that Condition (1.1) is satisfied. Since the coordinates b_n are independent and identically distributed, one has the bound, for $t > 0$,

$$\begin{aligned}\beta(\{|\varphi_n| \geq t\} \mid \mathcal{B}_{n-1}) &\leq \beta(\{\sigma_{\sup}(b_n) \geq t\} \mid \mathcal{B}_{n-1})\\ &= \beta(\{\sigma_{\sup}(b_n) \geq t\}) \leq \beta(\{\sigma_{\sup}(b_1) \geq t\}).\end{aligned}$$

This proves (1.1) with domination by the function $\psi : B \to \mathbb{R}; b \mapsto \sigma_{\sup}(b_1)$.

We note that this function ψ does not depend on x and that the L^1-convergence is therefore uniform in x. □

2.2. Breiman Law of Large Numbers.

In this section we prove the Law of Large Numbers for functions over a Markov chain.

Let $(X,\mathcal{X})$ be a standard Borel space, P be a Markov chain on X and, for x in X, set $\mathbb{P}_x$ for the Markov probability measure on the space Ω of trajectories.

The following technical lemma compares the Birkhoff averages of a function φ along the trajectories of a Markov chain with the Birkhoff averages of $P\varphi$.

LEMMA 2.3. (Breiman [**29**]) *Let φ be a bounded Borel function on X. For every x in X, for $\mathbb{P}_x$-almost every ω in Ω, one has*

$$\tfrac{1}{n}\sum_{k=0}^{n-1}\varphi(\omega_k) - \tfrac{1}{n}\sum_{k=0}^{n-1}P\varphi(\omega_k) \xrightarrow[n\to\infty]{} 0.$$

PROOF. The main ingredient of the proof is Corollary 1.8. For any integer $n \geq 1$, we introduce the functions

$$\varphi_n : \Omega \to \mathbb{R}\ ;\ \omega \mapsto \varphi(\omega_n) - P\varphi(\omega_{n-1}),$$

and the sub-σ-algebras $\mathcal{B}_n$ generated by $\omega_0, \ldots, \omega_n$. This sequence of functions on Ω is bounded by $2\sup_X |\varphi|$ and, by construction, one has

$$\mathbb{E}_x(\varphi_n|\mathcal{B}_{n-1}) = 0.$$

Therefore, by Corollary 1.8, the sequence $\frac{1}{n}\sum_{k=1}^{n}\varphi_k$ goes to 0 $\mathbb{P}_x$-almost everywhere. □

When P is a Markov-Feller chain, one can reformulate Lemma 2.3 using the so-called *empirical measures* :

COROLLARY 2.4. *Let X be a compact metrizable topological space and P be a Markov-Feller operator on X. Then, for any x in X, for $\mathbb{P}_x$-almost any ω in Ω, any weak limit of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{\omega_k}$ is P-invariant.*

In particular, using the weak compactness of the space of probability measures on X, we retrieve the Law of Large Numbers for functions over a Markov chain which is due to Breiman in [**29**]:

We say that a function $\varphi \in \mathcal{C}^0(X)$ has *a unique average* if

(2.2) there exists a constant ℓ_φ such that, for any P-invariant probability measure ν on X, one has $\nu(\varphi) = \ell_\varphi$.

REMARK 2.5. A function φ has a unique average ℓ_φ, if and only if one can write $\varphi - \ell_\varphi$ as a uniform limit of a sequence $P\psi_n - \psi_n$ with ψ_n in $\mathcal{C}^0(X)$. This follows from Hahn-Banach Theorem and Riesz representation Theorem.

In Chapter 10, we will find out conditions on a Markov operator P which ensure that the image of the operator $P-1$ is closed so that every function φ with a unique average ℓ_φ can be written as $\varphi = P\psi - \psi + \ell_\varphi$, with ψ in $\mathcal{C}^0(X)$.

COROLLARY 2.6. *Let X be a compact metrizable topological space and P be a Markov-Feller operator on X. Let φ be a continuous function on X with a unique average ℓ_φ. Then for any x in X, for $\mathbb{P}_x$-almost any ω in Ω, one has*

$$\tfrac{1}{n}\sum_{k=0}^{n-1}\varphi(\omega_k) \xrightarrow[n\to\infty]{} \ell_\varphi.$$

This sequence converges also in $\mathrm{L}^1(\Omega, \mathbb{P}_x)$*, uniformly for* $x \in X$*, i.e.*

$$\lim_{n\to\infty} \int_\Omega |\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1} \varphi(\omega_k) - \ell_\phi| \,\mathrm{d}\mathbb{P}_x(\omega) = 0 \text{ uniformly for } x \in X.$$

Proof. For $x \in X$ and $\varphi \in \mathcal{C}^0(X)$, we introduce for $n, \ell \geq 1$ the bounded functions Ψ_n and $\Psi_{\ell,n}$ on Ω given by, for $\omega \in \Omega$,

$$\Psi_n(\omega) = \varphi(\omega_n) \text{ and } \Psi_{\ell,n}(\omega) = (P_\mu^\ell \varphi)(\omega_n).$$

We will again use the sub-σ-algebras $\mathcal{B}_n$ generated by $\omega_0, \ldots, \omega_n$. These functions satisfy the equality, for $\mathbb{P}_x$-almost every ω in Ω, and $\ell \leq k$,

$$\mathbb{E}_x(\Psi_k | \mathcal{B}_{k-\ell})(\omega) = (P_\mu^\ell \varphi)(\omega_{k-\ell}) = \Psi_{\ell,k-\ell}(\omega).$$

On the one hand, by Theorem 1.6 (using the fact that φ is uniformly bounded to kill the boundary terms), for every $\ell \geq 1$, one has the convergence, for $\mathbb{P}_x$-almost all ω in Ω,

$$\tfrac{1}{n}\textstyle\sum_{k=1}^{n} (\Psi_k(\omega) - \Psi_{\ell,k}(\omega)) \xrightarrow[n\to\infty]{} 0.$$

This sequence converges also in $\mathrm{L}^1(\Omega, \mathbb{P}_x)$ uniformly for $x \in X$. Hence one has also the convergence, for $\mathbb{P}_x$-almost all ω in Ω,

$$\tfrac{1}{n}\textstyle\sum_{k=1}^{n} (\Psi_k(\omega) - \tfrac{1}{\ell}\sum_{j=1}^{\ell} \Psi_{j,k}(\omega)) \xrightarrow[n\to\infty]{} 0. \tag{2.3}$$

This sequence converges also in $\mathrm{L}^1(\Omega, \mathbb{P}_x)$ uniformly for $x \in X$.

On the other hand, since the function φ has a unique average ℓ_φ, one has the uniform convergence

$$\tfrac{1}{\ell}\textstyle\sum_{j=1}^{\ell} P_\mu^j \varphi \xrightarrow[\ell\to\infty]{} \ell_\varphi$$

in $\mathcal{C}^0(X)$. Hence one has also the convergence

$$\tfrac{1}{\ell}\textstyle\sum_{j=1}^{\ell} \Psi_{j,k}(\omega) \xrightarrow[\ell\to\infty]{} \ell_\varphi \tag{2.4}$$

in $\mathrm{L}^\infty(\Omega, \mathbb{P}_x)$ uniformly in $k \geq 1$ and in $x \in X$.

Combining (2.3) and (2.4) one gets the convergence, for $\mathbb{P}_x$-almost all ω in Ω,

$$\tfrac{1}{n}\textstyle\sum_{k=1}^{n} \Psi_k(\omega) \xrightarrow[n\to\infty]{} \ell_\varphi. \tag{2.5}$$

This sequence converges also in $\mathrm{L}^1(\Omega, \mathbb{P}_x)$ uniformly for $x \in X$. □

Note that Condition (2.2) is automatically satisfied when P is *uniquely ergodic* . Hence one has the following :

Corollary 2.7. *Let X be a compact metrizable topological space, P be a uniquely ergodic Markov-Feller operator on X and ν be the unique P-invariant probability measure on X. Let φ be a continuous* *function on X. Then for any x in X, for $\mathbb{P}_x$-almost any ω in Ω, one has*

$$\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1}\varphi(\omega_k)\xrightarrow[n\to\infty]{}\nu(\varphi).$$

This sequence converges also in $\mathrm{L}^1(\Omega,\mathbb{P}_x)$*, uniformly for $x\in X$.*

2.3. Law of Large Numbers for cocycles.

In this section we deduce from Breiman Law of Large Numbers a Law of Large Numbers for a cocycle.

2.3.1. *Random walks on X.* We come back to the notations of section 1.4. In particular, G is a second countable locally compact semigroup, μ is a Borel probability measure on G, $(B,\mathcal{B},\beta,T)$ is the associated one-sided Bernoulli shift, the group G acts continuously on a compact metrizable topological space X and ν is a μ-stationary Borel probability measure on X. We will apply the results of Section 2.2 to the Markov chain on X given by $x\mapsto P_x=\mu*\delta_x$.

This will give the following Law of Large Numbers for a function over a random walk

Corollary 2.8. *Let G be a locally compact semigroup, X be a compact metrizable G-space, and μ be a Borel probability measure on G. Then, for any x in X, for β-almost every b in B, for any continuous function $\varphi\in\mathcal{C}^0(X)$ with a unique average ℓ_φ, one has*

$$\tfrac{1}{n}\textstyle\sum_{k=1}^{n}\varphi(b_k\cdots b_1x)\xrightarrow[n\to\infty]{}\ell_\varphi.$$

This sequence converges also in $\mathrm{L}^1(B,\beta)$*, uniformly for $x\in X$.*

Proof. We use the forward dynamical system on $B\times X$. This corollary is almost a special case of Corollary 2.7, if we take into account the formula for $\mathbb{P}_{\mu,x}$ given in (1.5). $\square$

2.3.2. *Cocycles.* The Law of Large Numbers will be proved for a class of cocycles called *cocycles with a unique average* that we define now.

Let E be a finite dimensional real vector space. A continuous function $\sigma: G\times X\to E$ is said to be a *cocycle* if one has

$$\sigma(gg',x)=\sigma(g,g'x)+\sigma(g',x)\quad\text{for any } g,g'\in G,\ x\in X. \tag{2.6}$$

In particular, one has $\sigma(e,x)=0$, for any x in X. Two cocycles σ and σ' are said to be *cohomologous* if there exists a continuous function $\varphi: X\to E$ with

$$\sigma(g,x)+\varphi(x)=\sigma'(g,x)+\varphi(gx)\quad(g\in G, x\in X).$$

A cocycle that is cohomologous to 0 is called a *coboundary.*

For a cocyle σ we introduce the functions *sup-norm* $\sigma_{\sup}$. It is given by, for g in G,

$$\sigma_{\sup}(g) = \sup_{x\in X} \|\sigma(g,x)\|\,, \tag{2.7}$$

The cocycle is said to be $(\mu\otimes\nu)$-integrable if one has

$$\int_{G\times X} \|\sigma(g,x)\| \,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) < \infty.$$

For instance, a cocycle with $\sigma_{\sup} \in \mathrm{L}^1(G,\mu)$ is $(\mu\otimes\nu)$-integrable for any μ-stationary probability measure ν.

When σ is $(\mu\otimes\nu)$-integrable, the vector

$$\sigma_\mu(\nu) := \int_{G\times X} \sigma(g,x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) \in E$$

is then called the *average* of the cocycle.

The cocycle σ is said to *have a unique average* if

(2.8) the average $\sigma_\mu = \sigma_\mu(\nu)$ does not depend on the choice of ν.

A cocycle σ with a unique average is said to be *centered* if $\sigma_\mu = 0$.

Let us introduce a trick which reduces the study of cocycles with a unique average to the study of those which are centered. Replace G by $G' := G\times\mathbb{Z}$ where $\mathbb{Z}$ acts trivially on X, replace μ by $\mu' := \mu\otimes\delta_1$ so that any μ-stationary probability measure is also μ'-stationary, and replace σ by the cocycle

$$\sigma' : G'\times X \to E \;\text{ given by }\; \sigma'((g,n),x) = \sigma(g,x) - n\sigma_\mu. \tag{2.9}$$

2.3.3. *Law of Large Cocycles.* Here is the Law of Large Numbers for cocycles.

THEOREM 2.9. *Let G be a locally compact semigroup, X a compact metrizable G-space, E a finite dimensional real vector space and μ a Borel probability measure on G. Let $\sigma : G\times X\to E$ be a continuous cocycle with $\int_G \sigma_{\sup}(g)\,\mathrm{d}\mu(g) < \infty$ and with a unique average σ_μ. Then, for any x in X, for β-almost every b in B, one has*

$$\tfrac{1}{n}\sigma(b_n\cdots b_1, x) \xrightarrow[n\to\infty]{} \sigma_\mu. \tag{2.10}$$

This sequence converges also in $\mathrm{L}^1(B,\beta,E)$ uniformly for $x\in X$.

In particular, uniformly for $x\in X$, one has

$$\tfrac{1}{n}\int_G \sigma(g,x)\,\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \sigma_\mu.$$

Note that the assumption (2.8) is automatically satisfied when there exists a unique μ-stationary Borel probability measure ν on X.

Proof. Just combine Proposition 2.2 and Corollary 2.8 applied to the drift function $\varphi \in \mathcal{C}^0(X)$ which is given by $\varphi(x) = \int_G \sigma(g,x)\,\mathrm{d}\mu(g)$, for all x in X. This function has a unique average $\ell_\varphi := \sigma_\mu$. $\square$

2.3.4. *Invariance property.* When working on linear groups that are not connected, we will encounter cocycles which enjoy equivariance properties under the action of a finite group. The following lemma tells us that such equivariance properties imply invariance properties of the associated average.

Lemma 2.10. *We keep the notations and assumptions of Theorem 2.9. Besides, we let F be a finite group which acts linearly on E and which acts continuously on the right on X. We assume that the F-action and the G-action on X commute and that*

(2.11) *the cocycles $(g,x) \mapsto \sigma(g,xf)$ and $(g,x) \mapsto f^{-1}\sigma(g,x)$ are cohomologous for all f in F.*

Then the vector $\sigma_\mu \in E$ is F-invariant.

Remark 2.11. Assumption (2.11) is satisfied when those two cocycles are equal, i.e. when

$$f\,\sigma(g,xf) = \sigma(g,x) \ \text{ for all } f \text{ in } F,\ g \text{ in } G \text{ and } x \text{ in } X.$$

Proof of Lemma 2.10. Let ν be a stationary probability measure on X, f be an element of F and $\varphi_f : X \to E$ be a continuous function such that

$$f^{-1}\sigma(g,.) = \sigma(g,.f) - \varphi_f \circ g + \varphi_f$$

for any g in G. Since the F-action commutes with the G-action, the probability measure $f_*\nu$ is also μ-stationary, hence as σ has a unique average, we have

$$\begin{aligned}\sigma_\mu &= \textstyle\int_{G\times X} \sigma(g,xf)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x)\\ &= \textstyle\int_{G\times X} (f^{-1}\sigma(g,x) + \varphi_f(gx) - \varphi_f(x))\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x)\\ &= f^{-1}(\sigma_\mu) + \textstyle\int_X (P_\mu\varphi_f - \varphi_f)\,\mathrm{d}\nu = f^{-1}(\sigma_\mu),\end{aligned}$$

that is, σ_μ is F-invariant. $\square$

2.4. Convergence of the covariance 2-tensors.

In this section we deduce from Breiman Law of Large Numbers a convergence result for the covariance 2-tensors which will be useful for the Central Limit Theorem. This convergence is true for a particular class of cocycles that we call special cocycles.

2.4.1. *Special cocycles.* Let $\sigma : G\times X \to E$ be a continuous cocycle. When the function $\sigma_{\sup}$ is μ-integrable, we define the *drift* of σ as the continuous function $X \to E; x \mapsto \int_G \sigma(g,x)\,\mathrm{d}\mu(g)$. One says that σ has *constant drift* if the drift is a constant function:

$$\int_G \sigma(g,x)\,\mathrm{d}\mu(g) = \sigma_\mu. \tag{2.12}$$

One says that σ has *zero drift* if the drift is a null function.

A continuous cocycle $\sigma : G \times X \to E$ is said to be *special* if it is the sum

$$\sigma(g,x) = \sigma_0(g,x) + \psi(x) - \psi(gx) \tag{2.13}$$

of a cocycle $\sigma_0(g,x)$ with constant drift and of a coboundary $\psi(x) - \psi(gx)$ given by a continuous function $\psi : X \to E$. A special cocycle always has a unique average: for any μ-stationary probability measure ν on X, one has

$$\int_{G\times X} \sigma(g,x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) = \sigma_\mu. \tag{2.14}$$

As we will see in Remark 2.15, there exist non special cocycles. However, one has the following easy lemma

Lemma 2.12. *Let G be a locally compact semigroup, X be a compact metrizable G-space, E be a finite dimensional real vector space, and μ be a Borel probability measure on G such that there exists a unique μ-stationary Borel probability measure ν on X. Let $\sigma : G \times X \to E$ be a special cocycle. Then the decomposition* (2.13) *is unique provided $\nu(\psi) = 0$.*

Proof. Let ψ be as in (2.13) with $\nu(\psi) = 0$. Since ν is the unique μ-stationary probability measure on X, by Corollary 1.11, one has the uniform convergence on X, $\frac{1}{n}\sum_{k=0}^{n-1} P_\mu^k\psi \xrightarrow[n\to\infty]{} \nu(\psi)$. One gets

$$\psi(x) = \lim_{n\to\infty} \tfrac{1}{n}\sum_{k=0}^{n-1} \int_G (\sigma(g,x) - k\sigma_\mu)\,\mathrm{d}\mu^{*k}(g)$$

for all $x \in X$. $\square$

2.4.2. *Covariance tensor.* We will now study the covariance 2-tensors of a cocycle. Let us introduce some terminology. We let S^2E denote the symmetric square of E, that is, the subspace of $\bigotimes^2 E$ spanned by the elements $v^2 =: v \otimes v$, $v \in E$. We identify S^2E with the space of symmetric bilinear functionals on the dual space E^* of E, through the linear map which, for any v in E, sends v^2 to the bilinear functional $(\varphi,\psi) \mapsto \varphi(v)\psi(v)$ on E^*.

Given Φ in S^2E, we define the *linear span* of Φ as being the smallest vector supspace $E_\Phi \subset E$ such that Φ belongs to S^2E_Φ: in other words, the space $E_\Phi^\perp \subset E^*$ is the kernel of Φ as a bilinear functional on E^*.

We say Φ is *non-negative*, which we write $\Phi \geq 0$, if it is non-negative as a bilinear functional on E^*. In this case, Φ induces a Euclidean scalar product on E_Φ and we call the unit ball $K_\Phi \subset E_\Phi$ of this scalar product the *unit ball* of Φ. One has

$$K_\Phi = \{v \in E \mid v^2 \leq \Phi\}. \tag{2.15}$$

THEOREM 2.13. *Let G be a locally compact semigroup, X be a compact metrizable G-space, E be a finite dimensional real vector space and μ be a Borel probability measure on G such that there exists a unique μ-stationary Borel probability measure ν on X. Let $\sigma : G \times X \to E$ be a special cocycle, i.e. σ satisfies* (2.13)*. Assume $\int_G \sigma_{\sup}(g)^2 \, \mathrm{d}\mu(g) < \infty$ and introduce the covariance 2-tensor*

$$\Phi_\mu := \textstyle\int_{G\times X} (\sigma_0(g,x) - \sigma_\mu)^2 \, \mathrm{d}\mu(g) \, \mathrm{d}\nu(x) \in \mathrm{S}^2 E. \tag{2.16}$$

Then one has the convergence in $\mathrm{S}^2 E$

$$\tfrac{1}{n} \textstyle\int_G (\sigma(g,x) - n\sigma_\mu)^2 \, \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \Phi_\mu. \tag{2.17}$$

This convergence is uniform for x in X.

REMARK 2.14. Choose an identification of E with $\mathbb{R}^d$. Then the covariance 2-tensor on the left-hand side of (2.17) is nothing but the *covariance matrix* of the random variable $\frac{\sigma}{\sqrt{n}}$ on $(G \times X, \mu^{*n} \otimes \delta_x)$. Similarly the limit Φ_μ of these covariance 2-tensors is nothing but the *covariance matrix* of the random variable σ_0 on $(G \times X, \mu \otimes \nu)$. This 2-tensor Φ_μ is non-negative. The linear span E_{Φ_μ} of Φ_μ is the smallest vector subspace E_μ of E such that

$$\sigma_0(g,x) \in \sigma_\mu + E_\mu \text{ for all } g \text{ in } \operatorname{Supp}\mu \text{ and } x \text{ in } \operatorname{Supp}\nu$$

REMARK 2.15. The conclusion of Theorem 2.13 is not correct if one does not assume the cocycle σ to be special. Here is an example where the random walk is deterministic. We choose $X = \mathbb{R}/\mathbb{Z}$, $G = \mathbb{Z}$, $\mu = \delta_1$ and the action of μ on X is a translation by an irrational number α. The unique μ-stationary probability measure on X is the Lebesgue probability measure $\mathrm{d}x$. We let $\sigma(1,x)$ be a continuous function φ with 0 integral and $x = 0$, so that for $n \geq 0$, $\sigma(n,x)$ is the Birkhoff sum

$$S_n\varphi(0) := \sum_{k=0}^{n-1} \varphi(k\alpha).$$

We claim that one can choose φ in such a way that the left-hand side $\frac{1}{n} S_n\varphi(x)^2$ of (2.17) is not bounded, so that the theorem does not hold.

Indeed assume that, for any φ with $\int_X \varphi(x) \, \mathrm{d}x = 0$, one has

$$\sup_n \tfrac{1}{\sqrt{n}} |S_n\varphi(0)| < \infty.$$

Then, by Banach-Steinhaus Theorem, there would exist $C > 0$ such that, for any such φ, one has

$$\sup_n \tfrac{1}{\sqrt{n}} |S_n\varphi(0)| \leq C \left\|\varphi\right\|_\infty .$$

Choose a sequence $k_\ell \to \infty$ such that $\exp(2i\pi k_\ell \alpha) \xrightarrow[\ell\to\infty]{} 1$ and write $\exp(2i\pi k_\ell\alpha) = \exp(2i\pi\varepsilon_\ell)$ with $\varepsilon_\ell \xrightarrow[\ell\to\infty]{} 0$. Set $n_\ell = \left[\frac{1}{2\varepsilon_\ell}\right]$. We have then $\exp(2i\pi k_\ell n_\ell\alpha) \xrightarrow[\ell\to\infty]{} -1$. Let φ_ℓ be the function $x \mapsto \exp(2i\pi k_\ell x)$. We have

$$\tfrac{1}{\sqrt{n_\ell}} \left|S_{n_\ell}\varphi_\ell(0)\right| = \tfrac{1}{\sqrt{n_\ell}} \left|\tfrac{\exp(2i\pi k_\ell n_\ell \alpha)-1}{\exp(2i\pi k_\ell\alpha)-1}\right| \sim \tfrac{\sqrt{2}}{\pi\sqrt{\varepsilon_\ell}} \to \infty,$$

hence a contradiction. Thus, one can find a function φ such that the conclusion of the Theorem 2.13 does not hold for the associated cocycle σ.

Remark 2.16. The 2-tensor Φ_μ will play a crucial role in the Central Limit Theorem and its unit ball $K_\mu := K_{\Phi_\mu}$ will play a crucial role in the law of the iterated logarithm in Theorem 11.1.

Proof of Theorem 2.13. Using the trick (2.9), we may assume that the average σ_μ is 0.

The integral $M_n(x) := \int_G \sigma(g,x)^2 \,\mathrm{d}\mu^{*n}(g)$ is the sum of three terms $M_n(x) = M_{0,n}(x) + M_{1,n}(x) + M_{2,n}(x)$ where

$$\begin{aligned}
M_{0,n}(x) &= \textstyle\int_G \sigma_0(g,x)^2 \,\mathrm{d}\mu^{*n}(g),\\
M_{1,n}(x) &= \textstyle\int_G 2\,\sigma_0(g,x)(\psi(x) - \psi(gx)) \,\mathrm{d}\mu^{*n}(g),\\
M_{2,n}(x) &= \textstyle\int_G (\psi(x) - \psi(gx))^2 \,\mathrm{d}\mu^{*n}(g),
\end{aligned}$$

where σ_0 and ψ are as in (2.13).

We compute the first term. Since $\sigma_\mu = 0$, the "zero drift" condition (2.12) implies that, for every $m, n \geq 1$, one has

$$M_{0,m+n} = P_\mu^m M_{0,n} + M_{0,m}.$$

Hence $M_{0,n}$ is the Birkhoff sum

$$M_{0,n} = \textstyle\sum_{k=0}^{n-1} P_\mu^k M_{0,1}.$$

Since ν is the unique μ-stationary probability on the compact space X, by Corollary 1.11, one has the convergence in S^2E, uniformly for $x \in X$,

$$\tfrac{1}{n} M_{0,n}(x) \xrightarrow[n\to\infty]{} \nu(M_{0,1}) = \Phi_\mu. \tag{2.18}$$

We now compute the second term. According to Theorem 2.9, one has the convergence

$$\tfrac{1}{n}\sigma(b_n\cdots b_1,x)\xrightarrow[n\to\infty]{}\sigma_\mu=0$$

in $\mathrm{L}^1(B,\mathcal{B},E)$ uniformly for $x\in X$. Hence one has the convergence, uniformly for $x\in X$,

$$\tfrac{1}{n}|M_{1,n}(x)|\leq\tfrac{2}{n}\,\|\psi\|_\infty\int_G\|\sigma_0(g,x)\|\,\mathrm{d}\mu^{*n}(g)\xrightarrow[n\to\infty]{}0. \tag{2.19}$$

The last term is the easiest one to control:

$$\tfrac{1}{n}|M_{2,n}(x)|\leq\tfrac{4}{n}\,\|\psi\|_\infty^2\xrightarrow[n\to\infty]{}0. \tag{2.20}$$

The convergence (2.17) follows from (2.18), (2.19) and (2.20). □

Again, in the study of non-connected groups, we will need the following invariance property analogous to Lemma 2.10.

Lemma 2.17. *We keep the notations and assumptions of Theorem 2.13. Let F be a finite group which acts linearly on E and which acts continuously on the right on X. We assume that the F-action and the G-action on X commute and that the cocycles $(g,x)\mapsto\sigma(g,xf)$ and $(g,x)\mapsto f^{-1}\sigma(g,x)$ are cohomologous for all f in F. Then the 2-tensor $\Phi_\mu\in\mathrm{S}^2E$ is F-invariant.*

Proof. By Lemma 2.12, we have $f^{-1}\sigma_0(g,.)=\sigma_0(g,.f)$ for any g in G and f in F. The proof is then analogue to the one of Lemma 2.10, by using (2.16). □

2.5. Divergence of Birkhoff sums.

The aim of this section is to prove Lemma 2.18 which tells us that when Birkhoff sums of a real function diverge, they diverge with linear speed.

This lemma 2.18 will be a key ingredient in the proof of the positivity of the first Lyapunov exponent in Theorem 3.31, in the proof of the regularity of the Lyapunov vector in Theorem 9.9, and hence in the proof of the simplicity of the Lyapunov exponents in Corollary 9.15.

Lemma 2.18 (Divergence of Birkhoff sums). *Let $(X,\mathcal{X},\chi)$ be a probability space, equipped with an ergodic measure-preserving map T, let φ be in $\mathrm{L}^1(X,\mathcal{X},\chi)$ and, for any n in $\mathbb{N}$, let $\varphi_n=\varphi+\ldots+\varphi\circ T^{n-1}$ be the n-th Birkhoff sum of φ. Then, one has the equivalences*

$$\lim_{n\to\infty}\varphi_n(x)=+\infty \text{ for } \chi\text{-almost all } x \text{ in } X \iff \int_X\varphi\,\mathrm{d}\chi>0,$$

$$\lim_{n\to\infty}|\varphi_n(x)|=+\infty \text{ for } \chi\text{-almost all } x \text{ in } X \iff \int_X\varphi\,\mathrm{d}\chi\neq 0.$$

Here is the interpretation of this last equivalence: one introduces the fibered dynamical system on $X \times \mathbb{R}$ given by $(x,t) \mapsto (Tx, t+\varphi(x))$ which preserves the infinite volume measure $\chi \otimes \mathrm{d}t$; this dynamical system is conservative if and only if the function φ has zero average.

PROOF. Suppose first $\int_X \varphi \, \mathrm{d}\chi > 0$. Then, by Birkhoff theorem, one has, χ-almost everywhere, $\varphi_n \xrightarrow[n\to\infty]{} +\infty$.

Similarly, when $\int_X \varphi \, \mathrm{d}\chi < 0$, one has $\varphi_n \xrightarrow[n\to\infty]{} -\infty$.

Suppose now $\int_X \varphi \, \mathrm{d}\chi = 0$ and let us prove that, for χ-almost any x in X, there exists arbitrarily large n such that $|\varphi_n(x)| \leq 1$. Suppose this is not the case, that is, for some $p \geq 1$, the set

$$A = \{x \in X \mid \forall n \geq p \;\; |\varphi_n(x)| > 1\}$$

has positive measure.

Let us first explain roughly the idea of the proof. By definition of A, the intervals of length 1 centered at $\varphi_m(x)$, for m integer such that $T^m x$ sits in A, are disjoints. We will see that by Birkhoff Theorem this gives too many intervals since the sequence $\varphi_m(x)$ grows sublinearly.

Here is the precise proof. By Birkhoff theorem, for χ-almost any x in X, one has

$$\tfrac{1}{n}\,\varphi_n(x) \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \tfrac{1}{n}\,|\{m \in [0, n-1] \mid T^m x \in A\}| \xrightarrow[n\to\infty]{} \chi(A).$$

Pick such an x and fix $q \geq p$ such that, for any $n \geq q$, one has

$$|\varphi_n(x)| \leq \tfrac{n}{4p}\,\chi(A) \quad \text{and} \quad |\{m \in [0, n-1] \mid T^m x \in A\}| \geq \tfrac{3n}{4}\,\chi(A).$$

Then, for $n \geq q$, the set

$$E_n = \{m \in [q, n-1] \mid T^m x \in A\}$$

admits at least $\frac{3n}{4}\chi(A) - q$ elements. For each m in E_n, we consider the intervals

$$I_m := [\varphi_m(x) - \tfrac{1}{2}, \varphi_m(x) + \tfrac{1}{2}].$$

On the one hand, for m, m' in E_n with $m' \geq m + p$, as $T^m x$ belongs to A, one has

$$|\varphi_{m'}(x) - \varphi_m(x)| = |\varphi_{m'-m}(T^m x)| > 1,$$

hence the intervals I_m and $I_{m'}$ are disjoint, so that one has

$$\lambda\left(\cup_{m\in E_n} I_m\right) \geq \tfrac{1}{p}\textstyle\sum_{m\in E_n} \lambda\left(I_m\right) \geq \tfrac{1}{p}(\tfrac{3n}{4}\chi(A) - q),$$

where λ denotes Lebesgue measure. On the other hand, for $q \leq m \leq n-1$, the interval I_m is included in $[-\frac{n}{4p}\chi(A) - \frac{1}{2}, \frac{n}{4p}\chi(A) + \frac{1}{2}]$, so that

$$\lambda\left(\textstyle\bigcup_{m\in E_n} I_m\right) \leq \tfrac{1}{2p}\chi(A)n + 1.$$

Thus, for any $n \geq q$, one has

$$\tfrac{1}{p}(\tfrac{3n}{4}\chi(A) - q) \leq \tfrac{n}{2p}\chi(A) + 1,$$

which is absurd, whence the result. □

3. Linear random walks

The aim of this chapter is to prove the Law of Large Numbers for the norm a product of random matrices when the representation is irreducible (Theorem 3.28) and to prove the positivity of the first Lyapunov exponent when moreover this representation is unimodular, unbounded and strongly irreducible (Theorem 3.31). To do this, we have to understand the stationary measures on the projective space for such irreducible actions. We will begin by the simplest case: when the representation is strongly irreducible and proximal. In this case, we check that there exists a unique μ-stationary measure on the projective space. It is called the Furstenberg measure.

3.1. Linear groups.

In this section, we study semigroups Γ of matrices over a local field. When Γ is irreducible, we define its proximal dimension. When moreover Γ is proximal, i.e. when the proximal dimension is 1, we define its limit set.

Let $\mathbb{K}$ be a *local field*. We recall that this means that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, or a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ for p a prime number, or the field of Laurent series $\mathbb{F}_q((T))$ with coefficients in the finite field $\mathbb{F}_q$, where q is a prime power. Let V be a finite dimensional $\mathbb{K}$-vector space and $d = \dim_{\mathbb{K}} V$.

If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, let $|.|$ be the usual modulus on $\mathbb{K}$ and q be the number e. Fix a scalar product on V and let $\|.\|$ denote the associated norm.

If $\mathbb{K}$ is non-archimedean, let $\mathcal{O}$ be its valuation ring, ϖ be a *uniformizing element* of $\mathbb{K}$, that is, a generator of the maximal ideal of $\mathcal{O}$, and let q be the cardinal of the finite field $\mathcal{O}/\varpi\mathcal{O}$. Equip $\mathbb{K}$ with the absolute value $|.|$ such that $|\varpi| = \frac{1}{q}$. Fix a ultrametric norm $\|.\|$ on V.

We denote by $\mathbb{P}(V) := \{\text{lines in } V\}$ the projective space of V and $\mathbb{G}_r(V) := \{r\text{-planes in } V\}$ the Grassmann variety of V when $0 \leq r \leq d$.

We endow the ring of endomorphisms $\mathrm{End}(V)$ with the norm given by $\|f\| := \max\limits_{v \neq 0} \dfrac{\|f(v)\|}{\|v\|}$, for every endomorphism f of V.

Recall that a nonzero endomorphism f of V is said to be *proximal* if f admits a unique eigenvalue with maximal absolute value and if the multiplicity of this eigenvalue in the characteristic polynomial of

f is 1. In this case, this eigenvalue and this eigenspace are defined over $\mathbb{K}$. Note that this amounts to saying that the action of f on $\mathbb{P}(V) \smallsetminus \mathbb{P}\left(\mathrm{Ker} f^d\right)$ admits *attracting fixed point*, i.e. a point admitting a compact neighborhood b^+ such that, uniformly for x in b^+, the powers $f^n(x)$ converge to this point. This point is sometimes denoted $V_f^+ \in \mathbb{P}(V)$ and sometimes x_f^+. This line V_f^+ is the eigenspace of f whose eigenvalue has maximal absolute value. We let $V_f^< \subset V$ denote the unique f-stable hyperplane with $V_f^+ \not\subset V_f^<$. The action of the adjoint map f^* of f on the dual space V^* to V is also proximal and one has

$$(V^*)_{f^*}^+ = (V_f^<)^\perp \text{ and } (V^*)_{f^*}^< = (V_f^+)^\perp.$$

Let Γ be a subsemigroup of $\mathrm{GL}(V)$. Say that the action of Γ on V is *irreducible*, or that Γ is *irreducible*, if every Γ-stable subspace of V either equals V or $\{0\}$. Say it is *strongly irreducible*, or that Γ is *strongly irreducible*, if, for any finite set $V_1, \ldots, V_l$ of subspaces of V, if the set $V_1 \cup \ldots \cup V_l$ is Γ-stable, then either there exists $1 \leq i \leq l$ with $V_i = V$ or $V_1 = \ldots = V_l = \{0\}$.

Let $r := r_\Gamma$ be the *proximal dimension* of Γ, i.e. the smallest integer $r \geq 1$ for which there exists an endomorphism π in $\mathrm{End}(V)$ of rank r such that

$$\pi = \lim_{n\to\infty} \lambda_n \, g_n \text{ with } \lambda_n \text{ in } \mathbb{K} \text{ and } g_n \text{ in } \Gamma.$$

Say Γ is *proximal* if $r_\Gamma = 1$. For instance, when Γ contains a proximal element, the semigroup Γ is proximal.

The following lemma tells us that, when Γ is irreducible, the converse is also true.

LEMMA 3.1. *Let Γ be an irreducible proximal subsemigroup of $\mathrm{GL}(V)$. Then Γ contains a proximal element.*

Moreover, for any proper subspace W of V, there exists a proximal element g of Γ with $V_g^+ \not\subset W$.

PROOF. Let π in $\mathrm{End}(V)$ be a rank one endomorphism such that $\pi = \lim_{n\to\infty} \lambda_n \, g_n$ with λ_n in $\mathbb{K}$ and g_n in G. As Γ is irreducible, there exists h, h' in Γ with $h(\mathrm{Im}\pi) \not\subset W$ and $h'h(\mathrm{Im}\pi) \not\subset \mathrm{Ker}\pi$. Then $h\pi h'$ is a multiple of a rank one projector whose image is not included in W. Note that

$$h\pi h' = \lim_{n\to\infty} \lambda_n \, hg_nh'.$$

We claim that the element hg_nh' is proximal, for n large, and $V_{hg_nh'}^+ \not\subset W$. Indeed, if b is a compact neighborhood of $\mathbb{P}(h(\mathrm{Im}\pi))$ in $\mathbb{P}(V)$ which intersects neither $\mathbb{P}(W)$ nor $\mathbb{P}(h'^{-1}(\mathrm{Ker}\pi))$, then, for n large, $hg_nh'(b)$ is contained in the interior of b and the restriction of hg_nh' to b is a

$\frac{1}{2}$-contraction, thus, hg_nh' admits an attracting fixed point in $\mathbb{P}(V)$, which belongs to b. □

The following lemma 3.2 introduces the limit set in $\mathbb{P}(V)$ of an irreducible proximal subsemigroup. This lemma is also useful when the representation is not proximal. Indeed, it introduces the limit set in the Grassmann variety of V on which one controls the norms of the image vectors. This limit set will be used in the proof of the Law of Large Numbers for the norm.

LEMMA 3.2. *Let Γ be an irreducible subsemigroup of* $\mathrm{GL}(V)$ *and let $r = r_\Gamma$ be its proximal dimension. Let $\Lambda_\Gamma^r \subset \mathbb{G}_r(V)$ be the set of r-dimensional subspaces W of V which are images of elements $\pi \in$* $\mathrm{End}(V)$ *which belong to the closure $\overline{\mathbb{K}\Gamma}$.*
a) Then Λ_Γ^r is a minimal Γ-invariant subset of $\mathbb{G}_r(V)$. It is called the limit set of Γ in $\mathbb{G}_r(V)$.
b) There exists $C > 0$ such that, for every g in Γ, W in Λ_Γ^r, and v, v' nonzero in W, one has

$$\frac{\|gv'\|}{\|v'\|} \leq C\,\frac{\|gv\|}{\|v\|}. \tag{3.1}$$

c) When $r = 1$, Λ_Γ^1 is the unique minimal Γ-invariant subset of $\mathbb{P}(V)$, and is called the limit set of Γ in $\mathbb{P}(V)$.

We recall that a Γ-invariant subset is said to be *minimal* if it is closed and all its Γ-orbits are dense.

Point b) means that, on the limit r-subspaces $W \in \Lambda_\Gamma^r$, the elements of Γ almost act by similarities. In case $\mathbb{K} = \mathbb{R}$, the constant C can be chosen to be $C = 1$ for a suitable choice of norms.

REMARK 3.3. In case $\mathbb{K} = \mathbb{R}$, the constant C can be chosen to be $C = 1$ for a suitable choice of norm (see Lemmas 5.23 and 5.33).

REMARK 3.4. When $r > 1$, the Γ-invariant subset $\Lambda_\Gamma^r \subset \mathbb{G}_r(V)$ may not be the only one which is minimal. Indeed, there may exist uncountably many minimal subsets in $\mathbb{G}_r(V)$. For example, let $\Gamma = \mathrm{SO}(d-1,1)$ act on $V = \wedge^2\mathbb{R}^d$ with $d > 6$. One has then $r = d-2$. We denote by $e_{i,j} := e_i \wedge e_j$, with $1 \leq i < j \leq d$, the standard basis of V. For instance when $d = 7$, $r = 5$ and the quadratic form is $x_1x_7 + x_2^2 + \cdots + x_6^2$, the subspace

$$W := \langle e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{1,6}\rangle$$

belongs to Λ_Γ^r while, for $t > 1$, the subspaces

$$W_t := \langle e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{2,3} + te_{4,5}\rangle$$

are in distinct compact orbits of Γ in $\mathbb{G}_r(V)$.

Proof of Lemma 3.2. a) Fix $W = \operatorname{Im}\pi$ and $W' = \operatorname{Im}\pi'$ in Λ^r_Γ. We want to prove that W is in the closure of the Γ-orbit of W'. Since Γ is irreducible, one can find g in Γ such that the product $\pi g \pi'$ is nonzero. By definition of r, the product $\pi g \pi'$ has rank r. Write $\pi = \lim\limits_{n\to\infty} \lambda_n g_n$ with $\lambda_n \in \mathbb{K}$, $g_n \in \Gamma$. Then one has, as required,

$$W = \lim_{n\to\infty} g_n g W'.$$

b) First, note that, for any $\varepsilon > 0$, there exists $\alpha > 0$ such that, for any $x \in \mathbb{P}(V)$ and π in $\overline{\mathbb{K}\Gamma}$ with rank r, if $d(x, \mathbb{P}(\operatorname{Ker}\pi)) \geq \varepsilon$, one has

$$\|\pi w\| \geq \alpha \, \|\pi\| \, \|w\| \, .$$

Indeed, if this were not the case, one could find a sequence of elements of $\overline{\mathbb{K}\Gamma}$ with rank r but with a nonzero cluster point of rank $< r$.

Using the compactness of the Grassmann varieties, we pick $\varepsilon > 0$ such that, for any U in $\mathbb{G}_{n-r}(V)$ and U' in $\mathbb{G}_{n-r+1}(V)$, there exists x in $\mathbb{P}(U')$ with $d(x, \mathbb{P}(U)) \geq \varepsilon$, and we let α be as above. For g in Γ, $W = \operatorname{Im}\pi$ in Λ^r_Γ and $v \neq 0$ in W, we can find w in V such that $\pi w = v$ and $d(\mathbb{K}w, \mathbb{P}(\operatorname{Ker}\pi)) \geq \varepsilon$. We get

$$\alpha \, \|\pi\| \, \|w\| \leq \|v\| \leq \|\pi\| \, \|w\|$$
$$\alpha \, \|g\pi\| \, \|w\| \leq \|gv\| \leq \|g\pi\| \, \|w\|$$

hence

$$\alpha \tfrac{\|g\pi\|}{\|\pi\|} \leq \tfrac{\|gv\|}{\|v\|} \leq \tfrac{1}{\alpha} \tfrac{\|g\pi\|}{\|\pi\|}$$

and (3.1) follows immediately.

c) Same proof as in a). Assume $r = 1$. Fix $W = \operatorname{Im}\pi$ in Λ^1_Γ and x in $\mathbb{P}(V)$. We want to prove that W is in the closure of the Γ-orbit of x. Since Γ is irreducible, one can find g in Γ such that gx is not in $\operatorname{Ker}\pi$. Write $\pi = \lim\limits_{n\to\infty} \lambda_n g_n$ with $\lambda_n \in \mathbb{K}$, $g_n \in \Gamma$. Then one has, $W = \lim\limits_{n\to\infty} g_n g x$ as required. □

3.2. Stationary measures on $\mathbb{P}(V)$ for V strongly irreducible.

We study now the stationary measures ν on the projective space for strongly irreducible actions. We construct the Furstenberg boundary map. In particular, when the action is proximal, ν is unique and its limit measures ν_b are Dirac masses.

We keep the notations of Section 3.1. For a Borel probability measure μ on $\mathrm{GL}(V)$, we let Γ_μ denote the smallest closed subsemigroup of $\mathrm{GL}(V)$ such that $\mu(\Gamma_\mu) = 1$. We also keep the notations of Chapter 1 with $G = \mathrm{GL}(V)$. In particular, $(B, \mathcal{B}, \beta)$ is the one-sided Bernoulli space with alphabet $(G, \mathcal{G}, \mu)$.

The following lemma tells us that the proximal dimension is reached by almost every trajectory and it constructs the so-called Furstenberg *boundary map*.

LEMMA 3.5. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that Γ_μ is strongly irreducible. Let $r = r_{\Gamma_\mu}$. Then*
a) There exists a Borel map $\xi : B \to \mathbb{G}_r(V)$ such that for β-almost any b in B, every nonzero limit point f in $\mathrm{End}(V)$ *of a sequence $\lambda_n b_1 \cdots b_n$ with λ_n in $\mathbb{K}$ has rank r and admits $\xi(b)$ as its image.*
b) Let ν be a μ-stationary Borel probability measure on $\mathbb{P}(V)$. Then, for β-almost any b in B, $\xi(b)$ is the smallest vector subspace $V_b \subset V$ such that the limit measure ν_b is supported by $\mathbb{P}(V_b)$.

We shall use the strong irreducibility assumption under the following form:

LEMMA 3.6. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$, *$r_0 > 0$, ν be a μ-stationary Borel probability measure on $\mathbb{G}_{r_0}(V)$ and W be a proper nontrivial subspace of V.*
a) If Γ_μ is irreducible, then one has $\nu(\mathbb{G}_{r_0}(W)) \neq 1$.
b) If Γ_μ is strongly irreducible, then one has $\nu(\mathbb{G}_{r_0}(W)) = 0$.

PROOF. *a*) Let W_0 be the intersection of all the subspaces W of V such that $\nu(\mathbb{G}_{r_0}(W)) = 1$, that is, such that $\mathbb{G}_{r_0}(W)$ contains the support of ν. We still have $\nu(\mathbb{G}_{r_0}(W_0)) = 1$. The equality

$$\nu(\mathbb{G}_{r_0}(W_0)) = \textstyle\int_G \nu(\mathbb{G}_{r_0}(g^{-1}W_0))\, \mathrm{d}\mu(g)$$

tells us that, for μ-almost any g in $\mathrm{GL}(V)$, one has

$$\nu(\mathbb{G}_{r_0}(g^{-1}W_0)) = 1,$$

and hence $W_0 = g^{-1}W_0$. We get $\Gamma_\mu W_0 = W_0$. Now, since W_0 is nonzero and V is irreducible, we get $W_0 = V$ as required.

b) Let $r \geq r_0$ be the smallest positive integer such that there exists a nontrivial subspace W of V with dimension r such that $\nu(\mathbb{G}_{r_0}(W)) \neq 0$. As, for any $W_1 \neq W_2$ in $\mathbb{G}_r(V)$, one has $\nu(\mathbb{G}_{r_0}(W_1 \cap W_2)) = 0$, for any countable family $(W_i)_{i\in\mathbb{N}}$ of elements of $\mathbb{G}_r(V)$, one has

$$\textstyle\sum_{i\in\mathbb{N}} \nu(\mathbb{G}_{r_0}(W_i)) = \nu(\bigcup_{i\in\mathbb{N}} \mathbb{G}_{r_0}(W_i)) \leq 1.$$

Hence, for any $m > 0$, the set of W in $\mathbb{G}_r(V)$ with $\nu(\mathbb{G}_{r_0}(W)) \geq m$ is finite. Let

$$m := \sup_{W\in\mathbb{G}_r(V)} \nu(\mathbb{G}_{r_0}(W))$$

and let M be the non-empty finite set

$$M := \{W \in \mathbb{G}_r(V) \mid \nu(\mathbb{G}_{r_0}(W)) = m\}.$$

Again, for any W in M, the equality

$$\nu(\mathbb{G}_{r_0}(W)) = \int_G \nu(\mathbb{G}_{r_0}(g^{-1}W))\,\mathrm{d}\mu(g),$$

tells us that, for μ-almost any g in G, $g^{-1}W$ belongs to M. Hence, the finite union $\bigcup_{W\in M} W$ is Γ_μ-stable and, since Γ_μ is strongly irreducible, r is the dimension of V, which completes the proof. $\square$

Note that every endomorphism f of V induces a continuous map $\mathbb{P}(V) \smallsetminus \mathbb{P}(\operatorname{Ker} f) \to \mathbb{P}(V)$.

Proof of Lemma 3.5. A crucial feature of the proof consists in dealing simultaneously with the statements *a*) and *b*). Let ν be a μ-stationary Borel probability measure on $\mathbb{P}(V)$. Such a measure does exist by Lemma 1.10. By Lemma 1.21, for β-almost any b in B, for any integer $m \geq 0$, for μ^{*m}-almost any g in G, one has

$$(b_1 \cdots b_n g)_*\nu \xrightarrow[n\to\infty]{} \nu_b.$$

We set $\xi(b)$ to be the smallest vector subspace of V such that

$$\nu_b(\mathbb{P}(\xi(b))) = 1.$$

Let f be a nonzero limit point in the space of endomorphisms of V of a sequence $\lambda_n b_1 \cdots b_n$ with λ_n in $\mathbb{K}$. By Lemma 3.6, one has $\nu(\mathbb{P}(\operatorname{Ker} fg)) = 0$ for any g in $\mathrm{GL}(V)$. Hence, for any m in $\mathbb{N}$, for μ^{*m}-almost any g in $\mathrm{GL}(V)$, one has $(fg)_*\nu = \nu_b$. Thus, by continuity, one gets

$$(fg)_*\nu = \nu_b, \text{ for any } g \text{ in } \Gamma_\mu. \tag{3.2}$$

In particular, one has

$$f_*\nu = \nu_b.$$

On the one hand, this gives $\xi(b) \subset \operatorname{Im} f$. On the other hand, one gets $\nu(f^{-1}\xi(b)) = 1$, hence, by Lemma 3.6, $f^{-1}\xi(b) = V$ and $\xi(b) \supset \operatorname{Im} f$. This proves the equality $\xi(b) = \operatorname{Im} f$. This proves simultaneously that the image $\operatorname{Im} f$ does not depend on the choice of the limit point f and that the space $\xi(b)$ does not depend on the choice of the stationary measure ν.

It only remains to check that $\dim \xi(b) = r$. Let π be a rank r endomorphism of V which is a limit $\pi = \lim_{n\to\infty} \lambda_n g_n$ with λ_n in $\mathbb{K}$ and g_n in Γ_μ. Since Γ_μ is irreducible, we can choose π in such a way that $f\pi \neq 0$. By Lemma 3.6, $\nu(\operatorname{Ker}\pi) = 0$. Hence, applying Equation (3.2) to $g = g_n$ and passing to the limit, one gets

$$(f\pi)_*\nu = \nu_b.$$

This proves that $\xi(b) = \mathrm{Im}\,(f\pi)$ and $\dim \xi(b) \leq r$. By definition of r, this inequality has to be an equality. $\square$

The following Proposition 3.7 is just a restatement of Lemma 3.5 when Γ_μ is proximal. In this case the Furstenberg boundary map ξ takes its values in the projective space.

Proposition 3.7. *Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that Γ_μ is proximal and strongly irreducible. Then there exists a unique μ-stationary Borel probability measure ν on $\mathbb{P}(V)$.*

This probability ν is μ-proximal, i.e. there exists a Borel map

$$\xi : B \to \mathbb{P}(V)$$

such that, for β-almost any b in B, ν_b is the Dirac mass at $\xi(b) \in \mathbb{P}(V)$. In particular, one has $\nu = \xi_*\beta$.

For β-almost any b in B, every nonzero limit point f in $\mathrm{End}(V)$ of a sequence $\lambda_n b_1 \cdots b_n$ with λ_n in $\mathbb{K}$ has rank one and admits the line $\xi(b)$ as its image.

Proof of proposition 3.7. Thanks to Lemma 3.5, it only remains to check the uniqueness of the μ-stationary probability measure ν on $\mathbb{P}(V)$. Since $r_{\Gamma_\mu} = 1$, according to Lemma 3.5, for β-almost any b in B, the corresponding limit measure ν_b is a Dirac mass at the point $\xi(b)$. Hence by Lemma 1.19, one has $\nu = \int_B \delta_{\xi(b)}\, \mathrm{d}\beta(b)$. $\square$

Applying Lemma 3.5 to the dual representation, one gets :

Corollary 3.8. *Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that Γ_μ is strongly irreducible. Let $r = r_{\Gamma_\mu}$*
a) For β-almost any b in B, there exists $V_b \in \mathbb{G}_{d-r}(V)$ such that every nonzero limit point f in $\mathrm{End}(V)$ of a sequence $\lambda_n b_n \cdots b_1$ with λ_n in $\mathbb{K}$ has rank r and admits V_b as its kernel.
b) For every x in $\mathbb{P}(V)$, one has $\beta(\{b \in B \mid x \subset V_b\}) = 0$.

Proof. a) For $g \in \mathrm{GL}(V)$ we denote by $g^* \in \mathrm{GL}(V^*)$ the adjoint operator of g. The adjoint subsemigroup $\Gamma_\mu^* \subset \mathrm{GL}(V^*)$ is also strongly irreducible and one has

$$r_{\Gamma_\mu} = r_{\Gamma_\mu^*}.$$

Hence we can apply Lemma 3.5 to the image measure μ^* of μ by the adjoint map. This tells us that, for β-almost any b in B and any λ_n in $\mathbb{K}$, any nonzero limit value of $\lambda_n b_1^* \cdots b_n^*$ is a rank r operator in $\mathrm{End}(V^*)$ whose image $\xi^*(b) \in \mathbb{G}_r(V^*)$ does not depend on the limit value. Let $V_b \subset V$ be the vector subspace

$$V_b := (\xi^*(b))^\perp.$$

Any limit value of $\lambda_n b_n \cdots b_1$ is a rank r operator in $\mathrm{End}(V)$ whose kernel is V_b.

b) Note that, by construction, for β-almost any b in B, one has

$$\xi^*(Tb) = (b_1^*)^{-1}\xi^*(b),$$

so that, by Remark 1.20, the Borel probability measure ν^* on $\mathbb{G}_r(V^*)$, image of β by the map ξ^*, is μ^*-stationary. The result now follows from Lemma 3.6 applied to ν^*. □

Remark 3.9. The assumption that Γ_μ is proximal is crucial in Proposition 3.7. For instance, if one chooses μ in such a way such that Γ_μ is a connected compact subgroup of $\mathrm{GL}(V)$ which acts irreducibly on V but which does not act transitively on $\mathbb{P}(V)$, then there are infinitely many stationary measures on $\mathbb{P}(V)$, since every Γ_μ-orbit carries one. One can give similar examples with a non-compact Γ_μ by using the group constructed in Remark 3.4.

Remark 3.10. The assumption that Γ_μ is strongly irreducible is also crucial in Proposition 3.7. One cannot weaken it by just assuming Γ_μ to be irreducible. For example, if G is the group of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}$ with $a \neq 0$ in $\mathbb{R}$, which acts on $\mathbb{R}^2$, we let μ be a compactly supported Borel probability measure on G such that $\Gamma_\mu = G$. In this case, one checks that, since a centered random walk on $\mathbb{R}$ is recurrent, for β-almost every b in B, the set of cluster points of the sequence $\mathbb{R}b_1 \cdots b_n \in \mathbb{P}(\mathrm{End}(\mathbb{R}^2))$ contains both rank 1 and rank 2 matrices.

An analogue example can be constructed with a semisimple group G (see Section 12.9 for details).

We will see in Section 3.3 how to take into account Remark 3.10 and how to adapt the main results of Section 3.2 to general irreducible actions.

3.3. Virtually invariant subspaces.

In this section, we introduce purely algebraic tools to reduce the study of irreducible representations to the study of strongly irreducible representations.

Let Γ be a subsemigroup of $\mathrm{GL}(V)$. We say that a subspace W of V is *virtually invariant* by Γ if the set $\Gamma W = \{gW | g \in \Gamma\}$ is finite. We say that a nonzero virtually invariant subspace W is *strongly irreducible* if it does not contain any proper nontrivial virtually invariant subspace. Note that, since V is finite dimensional, there always exists a strongly irreducible virtually invariant subspace W in V. Note that this definition of strong irreducibility extends the one given in Section 3.1.

Lemma 3.11. *Let* Γ *be a subsemigroup of* $\mathrm{GL}(V)$.
a) If W *is a virtually invariant subspace, so is* gW *for any* g *in* Γ.
b) If moreover W *is strongly irreducible, so is* gW *for any* g *in* Γ.
c) If W_1 *and* W_2 *are virtually invariant subspaces, so are* $W_1 + W_2$ *and* $W_1 \cap W_2$.

Proof. *a*) follows from the fact that $\Gamma gW \subset \Gamma W$ (and even $\Gamma gW = \Gamma W$ since the latter set is finite).

b) is immediate if Γ is a group. In general, this follows from the fact that any finite subsemigroup of a group is a group. More precisely, there exists $m > n$ such that $g^m W = g^n W$. Hence, setting $h = g^{m-n-1}$, one gets $hgW = W$. Now, if $U \subset gW$ is virtually invariant, then, by *a*), $hU \subset W$ is also virtually invariant and we get $hU = W$, hence $U = gW$, which was to be shown.

c) follows from the identites $g(W_1 + W_2) = (gW_1) + (gW_2)$ and $g(W_1 \cap W_2) = (gW_1) \cap (gW_2)$, for g in Γ. □

The following lemma decomposes any irreducible representation as a sum of strongly irreducible subspaces:

Lemma 3.12. *Let* Γ *be an irreducible subsemigroup of* $\mathrm{GL}(V)$ *and let* $W_1, \ldots, W_\ell$ *be a minimal family of virtually invariant and strongly irreducible subspaces of* V *such that* V *is spanned by* $W_1, \ldots, W_\ell$. *Then one has* $V = W_1 \oplus \cdots \oplus W_\ell$.

Proof. By minimality, we have $W_1 \cap (W_2 + \cdots + W_\ell) \neq W_1$. By Lemma 3.11, $W_1 \cap (W_2 + \cdots + W_\ell)$ is a virtually invariant subspace. Thus, we get $W_1 \cap (W_2 + \cdots + W_\ell) = \{0\}$ and the result follows. □

Note that such a family W_i always exists. Note also that one cannot always expect such a family W_i to be invariant under the action of Γ. This is why we introduce the following definition.

If Γ is an irreducible subsemigroup of $\mathrm{GL}(V)$, we shall say that a family $(V_i)_{i \in I}$ of subspaces of V is a *transitive strongly irreducible* Γ-*family* if, for any i, V_i is virtually invariant and strongly irreducible and if the family is Γ-invariant and transitively permuted by Γ. In other words, it is of the form ΓW, where W is a virtually invariant and strongly irreducible subspace of V. Such a family necessarily spans V since $\bigcup_{g \in \Gamma} gW$ spans a Γ-invariant subspace of V and Γ acts irreducibly on V. Since V admits virtually invariant and strongly irreducible subspaces, it also admits transitive strongly irreducible Γ-families.

EXAMPLE 3.13. If Γ is a finite group, the V_i have dimension 1. If Γ is strongly irreducible, one has $V_i = V$.

LEMMA 3.14. *Let* Γ *be an irreducible subsemigroup of* $\mathrm{GL}(V)$*,* W *be a nonzero virtually invariant and strongly irreducible subspace of* V *and* $\Gamma_W = \{g \in \Gamma \mid gW = W\}$*. Then, the dimension of* W *and the proximal dimension of* Γ_W *in* W *do not depend on* W*.*

We call this proximal dimension r the *virtual proximal dimension* of Γ and we say Γ is *virtually proximal* if $r = 1$.

PROOF. Let $(V_i)_{i\in I}$ be a transitive strongly irreducible Γ-family in V. We claim that the semigroups $\Gamma_i := \Gamma_{V_i}$ all have the same proximal dimension in the spaces V_i. Indeed, let i, j be in I and g, h be in Γ with $gV_i = V_j$ and $hV_j = V_i$. We get $g\Gamma_i h \subset \Gamma_j$, hence the proximal dimension of Γ_j is bounded above by the proximal dimension of Γ_i. By reversing the roles of i and j, we get equality.

Now, by Lemma 3.12, one can find a subset J of I such that one has $V = \bigoplus_{i\in J} V_i$. We let p_i denote the projection on V_i in this decomposition.

Let W be a virtually invariant and strongly irreducible nonzero subspace of V. As W is nonzero, there exists $i \in J$ with $p_i(W) \neq \{0\}$. We claim that p_i induces an isomorphism between W and V_j. Indeed, since the set $\Gamma W \times \prod_{j\in J} \Gamma V_j$ is finite, if $\Delta = \Gamma_W \cap \bigcap_{j\in J} \Gamma_j$, there exists a finite subset F of Γ such that $\Gamma = F\Delta$. Hence, since the spaces $p_i(W)$ and $W \cap \operatorname{Ker} p_i$ are Δ-invariant, they are virtually invariant. Since $p_i(W)$ is a nonzero subspace of V_i, we get $p_i(W) = V_i$. Since $W \cap \operatorname{Ker} p_i$ is a proper subspace of W, we get $W \cap \operatorname{Ker} p_i = \{0\}$, which was to be shown. In particular, W and V_i have the same dimension.

Let now g_n be a sequence in Γ_W and λ_n be a sequence in $\mathbb{K}$ such that $\lambda_n g_n$ converges in the space of endomorphisms of W towards a map π with rank the proximal dimension r of Γ_W in W. Since the set $\Gamma_W(V_j)_{j\in J}$ is finite, one can find a finite subset $F' \subset \Gamma_W$ such that $\Gamma_W = F'\Delta$. Thus, for any n in $\mathbb{N}$, there exists f_n in F' with $f_n g_n V_j = V_j$ for any j in J. In other words, after having replaced g_n by $f_n g_n$ and taken a subsequence, one can assume $g_n \in \Gamma_{V_j}$ for any n, for any j in J. In particular $p_i g_n = g_n p_i$. Since p_i induces an isomorphism between W and V_i, the sequence $\lambda_n g_n$ converges in the space of endomorphisms of V_i towards a rank r map and the proximal dimension of Γ_i in V_i is bounded by r. The result follows by exchanging the roles of the Γ-families $(V_i)_{i\in I}$ and ΓW. $\square$

3.4. Stationary measures on $\mathbb{P}(V)$.

We will now use the language of Section 3.3 to extend the study of stationary measures on projective spaces to irreducible actions which are not strongly irreducible. An alternative approach will be explained in Chapter 4.

Here is the extension of Lemma 3.5 which constructs the Furstenberg boundary map.

LEMMA 3.15. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that the semigroup Γ_μ is irreducible. Let r be the virtual proximal dimension of Γ_μ. Let $(V_i)_{i\in I}$ be a transitive strongly irreducible Γ_μ-family. Then*
a) There exist Borel maps $\xi_{V_i} : B \to \mathbb{G}_r(V_i)$, for $i \in I$, such that, for any i, j in I, for β-almost any b in B, every nonzero limit point f in $\mathrm{Hom}(V_j, V_i)$ *of a sequence $\lambda_n b_1 \cdots b_n \,|_{V_j}$ with λ_n in $\mathbb{K}$ has rank r and admits $\xi_{V_i}(b)$ as its image.*
b) Let ν be a μ-stationary Borel probability measure on $\cup_{i\in I}\mathbb{P}(V_i)$. Then, for β-almost any b in B, $\xi_{V_i}(b)$ is the smallest vector subspace $V_{i,b} \subset V_i$ such that the limit measure ν_b is supported by $\cup_{i\in I}\mathbb{P}(V_{i,b})$.

REMARK 3.16. By construction these maps ξ_{V_i} satisfy the following equivariance property. For all i, j in I and β-almost all b in B such that $b_1 V_j = V_i$, one has

$$\xi_{V_i}(b) = b_1\, \xi_{V_j}(Tb).$$

Here is the extension of Lemma 3.6.

LEMMA 3.17. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that Γ_μ is irreducible. Let W be a virtually invariant and strongly irreducible subspace of V for Γ_μ. Let $r_0 > 0$ and ν be a μ-stationary Borel probability measure on $\mathbb{G}_{r_0}(V)$. Then, for any proper nontrivial subspace U of W, one has $\nu(\mathbb{G}_{r_0}(U)) = 0$.*

PROOF OF LEMMA 3.17 . Same proof as for Lemma 3.6. □

PROOF OF LEMMA 3.15. We copy the proof of Lemma 3.5 taking into account the subspaces V_i which are permuted by Γ. We simultaneously prove the two statements. Let ν be a μ-stationary Borel probability measure on X. We set ν_i for the restriction of ν to $\mathbb{P}(V_i)$ and, for β-almost all b in B, we set $\nu_{i,b}$ for the restriction of ν_b to $\mathbb{P}(V_i)$. By Lemma 1.21, for β-almost any b in B, for any integer $m \geq 0$, for μ^{*m}-almost any g in G , one has $(b_1 \cdots b_n g)_* \nu \xrightarrow[n\to\infty]{} \nu_b$. We set $\xi_{V_i}(b)$ to be the smallest vector subspace of V_i such that $\nu_b(\mathbb{P}(\xi_{V_i}(b))) = 1$.

Let i, j, k in I and g in $\mathrm{GL}(V)$ be such that $gV_k = V_j$. Let $f \in \mathrm{Hom}(V_j, V_i)$ be a nonzero limit point of a sequence $\lambda_n b_1 \cdots b_n|_{V_j}$

with λ_n in $\mathbb{K}$. By lemma 3.17, one has $\nu(\mathbb{P}(\mathrm{Ker}_{V_k} fg)) = 0$. Hence, for any m in $\mathbb{N}$, for μ^{*m}-almost any g in $\mathrm{GL}(V)$ such that $gV_k = V_j$, one has $(fg)_*\nu_k = \nu_{i,b}$. Thus, by continuity, one gets

$$(fg)_*\nu_k = \nu_{i,b}, \text{ for any } g \text{ in } \Gamma_\mu \text{ such that } gV_k = V_j. \tag{3.3}$$

In particular, one has

$$f_*\nu_j = \nu_{i,b}.$$

Hence, using again Lemma 3.17, one has the equality

$$\xi_{V_i}(b) = f(V_j).$$

This simultaneously proves that the image $f(V_j)$ does not depend on the limit point f and that the space $\xi_{V_i}(b)$ does not depend on the choice of the stationary measure ν.

It remains only to check that $\dim \xi_{V_i}(b) = r$. Let $\pi \in \mathrm{End}(V_j)$ be a rank r element which is a limit $\pi = \lim\limits_{n\to\infty} \lambda_n g_n|_{V_j}$ with λ_n in $\mathbb{K}$ and g_n in Γ_μ, $g_nV_j = V_j$. Since the stabilizer of V_j in Γ_μ is irreducible in V_j, we can choose π in such a way that $f\pi \neq 0$. By Lemma 3.6, $\nu(\mathrm{Ker}_{V_j}\pi) = 0$. Hence, applying Equation (3.3) to $g = g_n$ and passing to the limit, one gets

$$(f\pi)_*\nu_j = \nu_{i,b}.$$

This proves that $\xi_{V_i}(b) = f\pi(V_j)$ and $\dim \xi_{V_i}(b) \leq r$. By definition of r, this inequality has to be an equality. $\square$

Focusing on virtually proximal representations, one obtains the following extension of Proposition 3.7.

PROPOSITION 3.18. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that the semigroup Γ_μ is irreducible and virtually proximal. Let $(V_i)_{i\in I}$ be a transitive strongly irreducible Γ-family. Then there exists a unique μ-stationary Borel probability measure ν on $\cup_{i\in I}\mathbb{P}V_i$.*

This probability ν is μ-proximal over I i.e. for each i in I, there exists a Borel map

$$\xi_i : B \to \mathbb{P}(V_i)$$

such that, for β-almost any b in B, ν_b is the average $\frac{1}{|I|}\sum_{i\in I}\delta_{\xi_i(b)}$. In particular, one has $\nu_{|\mathbb{P}(V_i)} = (\xi_i)_*\beta$.

For $i, j \in I$, for β-almost any b in B, every nonzero limit point f of a sequence $\lambda_n(b_1\cdots b_n)_{|V_j} \in \mathrm{Hom}(V_j, V_i)$ with λ_n in $\mathbb{K}$ has rank one and admits the line $\xi_i(b)$ as its image.

REMARK 3.19. In case $\mathbb{K} = \mathbb{R}$, one can prove that every ergodic stationary measure on $\mathbb{P}(V)$ is of the form described in Lemma 3.15, i.e. is supported by $\cup_{i\in I}\mathbb{P}(V_i)$ for some transitive strongly irreducible

Γ_μ-family (this is explained in [**13**]). In case $\mathbb{K}$ is non-archimedean, a counter-example is constructed in Section 12.9.

PROOF. Thanks to Lemma 3.15, it only remains to check the uniqueness of the μ-stationary measure ν on $\cup_{i\in I}\mathbb{P}(V)$. Note first that the semigroup Γ acts on the finite set I and hence, by the *maximum principle* the image of ν on I is Γ-invariant.

Since $r_{\Gamma_\mu} = 1$, according to Lemma 3.15, for β-almost any b in B, the corresponding limit measure ν_b is given by the formula

$$\nu_b = \tfrac{1}{|I|} \textstyle\sum_{i\in I} \delta_{\xi_i(b)}. \tag{3.4}$$

Hence ν is unique since by Lemma 1.19, one has $\nu = \int_B \nu_b \,\mathrm{d}\beta(b)$. $\square$

Applying Lemma 3.15 to the dual representation, one obtains the following extension of Corollary 3.8.

COROLLARY 3.20. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that the semigroup Γ_μ is irreducible. Let r be the virtual proximal dimension of Γ_μ, and W be a virtually invariant and strongly irreducible subspace of V. Then*
a) For β-almost any b in B, there exists $W_b \in \mathbb{G}_{d-r}(W)$ such that every nonzero limit point f in $\mathrm{Hom}(W,V)$ *of a sequence $\lambda_n b_n \cdots b_1|_W$ with λ_n in $\mathbb{K}$ has rank r and admits W_b as its kernel.*
b) For every x in $\mathbb{P}(V)$, one has $\beta(\{b\in B \mid x\subset W_b\}) = 0$.

PROOF. a) For $g \in \mathrm{GL}(V)$ we denote by $g^* \in \mathrm{GL}(V^*)$ the adjoint operator of g. The adjoint subsemigroup $\Gamma_\mu^* \subset \mathrm{GL}(V^*)$ is also irreducible with virtual proximal dimension r. Let U be a virtually invariant and strongly irreducible subspace of V^* such that the restriction to U of the natural map $i^* : V^* \to W^*$ is nonzero. Since the image of U in W^* is virtually invariant, i^* maps U onto W^* isomorphically. Let $\xi_U^* : B \to \mathbb{G}_r(U)$ be the map constructed in Lemma 3.15. For b in B, we set

$$W_b = (i^*\xi_U(b))^\perp,$$

which is a codimension r subspace of W and we claim that the Corollary holds for this choice of the map $b \mapsto W_b$.

Indeed, let b be in B such that the conclusion of Lemma 3.15 holds for b and the transitive strongly irreducible Γ_μ-family $\Gamma_\mu U$. Let λ_k be a sequence in $\mathbb{K}$ and n_k be a sequence of positive integers such that the sequence $\lambda_k(b_{n_k}\cdots b_1)_{|W}$ admits a nonzero limit point π in $\mathrm{Hom}(W,V)$. After maybe extracting a subsequence, one can assume there exists subspaces W' of V and U' of V^* such that, for any k, one has

$$b_{n_k}\cdots b_1 W = W' \ \text{ and } \ b_1^*\cdots b_{n_k}^* U' = U.$$

In particular, i^* induces an isomorphism between U' and $(W')^*$. Now, by construction and by Lemma 3.15, the restriction of $\lambda_k b_1^* \cdots b_{n_k}^*$ to U' converges towards a rank r element ϖ of $\mathrm{Hom}(U', U)$ with image $\xi_U(b)$ and we get $\pi^*_{|U'} = i^*\varpi$, so that π has rank r and kernel W_b, which was to be shown.

b) First note that, by definition, if $x \not\subset W$, one has

$$\beta(\{b \in B \mid x \subset W_b\}) = 0,$$

so that we can assume $x \subset W$. We keep the notations of a) and we set $X = x^\perp \cap U$, which is a proper subspace of U. For β-almost any b in B, one has the equivalence

$$x \subset W_b \iff \xi_U^*(b) \subset X.$$

Let $(V_i^*)_{i\in I}$ be the transitive strongly irreducible Γ_μ^*-family $\Gamma_\mu^* U$ and, for β-almost any b in B, for i in I, let $V_{i,b}^*$ be the subspace constructed in Lemma 3.15. We set $\nu^*(b) = \frac{1}{|I|}\sum_{i\in I} \delta_{V_{i,b}^*}$ which is a Borel probability measure on $\mathbb{G}_r(V^*)$. By construction, for β-almost any b in B, one has $\nu_{Tb}^* = (b_1^*)^{-1}\nu_b^*$ so that, by Remark 1.20, the Borel probability measure $\nu^* = \int_B \nu_b^* \,\mathrm{d}\beta(b)$ is μ^*-stationary. The conclusion now follows from Lemma 3.6 since one has

$$\beta(\{b \in B \mid x \subset W_b\}) \;=\; |I|\, \nu^*(\mathbb{G}_r(X)). \qquad \square$$

3.5. Norms of vectors and norms of matrices.

In this section we prove that for almost every trajectory b, the size of all the columns of the matrix $b_n \cdots b_1$ are comparable.

PROPOSITION 3.21. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that Γ_μ is strongly irreducible. For any nonzero vector v in V, for β-almost any b in B, there exists $\varepsilon > 0$ such that, for any $n \in \mathbb{N}$, one has*

$$\|b_n \cdots b_1 v\| \geq \varepsilon \, \|b_n \cdots b_1\| \, \|v\| \,. \tag{3.5}$$

REMARK 3.22. In Proposition 3.21, one cannot replace the assumption "Γ_μ *is strongly irreducible*" by "Γ_μ *is irreducible*". Indeed it may exist two virtually invariant and strongly irreducible subspaces V_i and V_j of V such that, for β-almost every b in B, one has

$$\sup_{n\geq 1} \frac{\|b_n \cdots b_1|_{V_i}\|}{\|b_n \cdots b_1|_{V_j}\|} = \infty.$$

An example of such a situation will be constructed in Section 12.9.

If we only assume that "Γ_μ *is irreducible*", we have to replace Inequality (3.5) by Inequality (3.6). This is the content of the following proposition.

PROPOSITION 3.23. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that Γ_μ is irreducible. Let $(V_i)_{i\in I}$ be a transitive strongly irreducible Γ_μ-family. For any i in I, v nonzero in V_i, for β-almost any b in B, there exists $\varepsilon > 0$ such that, for any $n \in \mathbb{N}$, one has*

$$\|b_n \cdots b_1 v\| \geq \varepsilon \|b_n \cdots b_1|_{V_i}\| \, \|v\|. \tag{3.6}$$

To estimate norms of random products, we shall use the following

LEMMA 3.24. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of* $\mathrm{GL}(V)$ *and $f \in \mathrm{End}(V)$ be a nonzero limit of a sequence $\lambda_n g_n$ with λ_n in $\mathbb{K}$.*
a) Then, for any compact subset M of $\mathbb{P}(V) \smallsetminus \mathbb{P}(\mathrm{Ker}\, f)$, there exists a real number $\varepsilon > 0$ such that, for any $n \in \mathbb{N}$ and any v in V with $\mathbb{R}v \in M$, one has $\|g_n v\| \geq \varepsilon \, \|g_n\| \, \|v\|$.
b) If f is non invertible, one has $\frac{\|g_n\|^d}{|\det g_n|} \xrightarrow[n\to\infty]{} \infty$.
c) More precisely if f has rank $r < d$, one has $\frac{\|g_n\|^{r+1}}{\|\wedge^{r+1} g_n\|} \xrightarrow[n\to\infty]{} \infty$.

PROOF OF LEMMA 3.24. These statements are proved by contradiction. After a renormalization, we may assume that the sequence g_n converges towards f. In particular, one has $\|g_n\| \xrightarrow[n\to\infty]{} \|f\| \neq 0$.

a) If there exists a sequence of nonzero vectors v_n with $\mathbb{K}v_n$ in M such that the ratio $\frac{\|g_n v_n\|}{\|g_n\|\|v_n\|}$ goes to 0, then one can assume that v_n converges to a nonzero vector v_∞. The line $\mathbb{K}v_\infty$ is also in M and the limit ratio $\frac{\|f v_\infty\|}{\|f\|\|v_\infty\|}$ is nonzero.

b) If $\frac{\|g_n\|^d}{|\det g_n|}$ is bounded, then f is invertible.

c) If $\frac{\|g_n\|^{r+1}}{\|\wedge^{r+1} g_n\|}$ is bounded, then $\wedge^{r+1} f$ is nonzero. □

PROOF OF PROPOSITION 3.23 . For any x in $\mathbb{P}(V_i)$, one has, by Corollary 3.20, $\beta(\{b \in B \mid x \subset V_{i,b}\}) = 0$, so that our statement follows from Lemma 3.24.*a*. □

The following corollary tells us that the random walk on $V \smallsetminus \{0\}$ is *transient*.

COROLLARY 3.25. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that Γ_μ is irreducible. If, for some virtually invariant and strongly irreducible subspace W of V, the image in* $\mathrm{PGL}(W)$ *of the stabilizer $\Gamma_{\mu,W}$ of W in Γ_μ is not bounded, then, for any nonzero vector v in V, for β-almost any b in B, one has*

$$\lim_{n\to\infty} \|b_n \cdots b_1 v\| = \infty. \tag{3.7}$$

Note that, if Γ is an irreducible subsemigroup of $\mathrm{GL}(V)$, then the virtual proximal dimension of Γ equals the dimension of some (equivalently any) virtually invariant and strongly irreducible subspace W if and only if, for some (equivalently any) such subspace W, the image in $\mathrm{PGL}(W)$ of the stabilizer Γ_W of W is bounded.

PROOF. Let r be the virtual proximal dimension of Γ_μ. Let $(V_i)_{i\in I}$ be a transitive strongly irreducible Γ_μ-family. All these spaces V_i have the same dimension, call it d_0. Since the image of Γ_μ in $\mathrm{PGL}(V)$ is unbounded, one has $r < d_0$.

It is enough to prove (3.7) for v in one V_i. According to (3.6), for β-almost all b in B, the sequence

$$\tfrac{\|b_n \cdots b_1|_{V_i}\|}{\|b_n \cdots b_1 v\|}$$

is bounded above. Since $r < d_0$, according to Lemma 3.15 and Lemma 3.24.b, for β-almost all b in B, one has

$$\lim_{n\to\infty} \|b_n \cdots b_1|_{V_i}\| = \infty.$$

This proves (3.7). $\square$

REMARK 3.26. Here is a slight improvement of Proposition 3.21, which we will not use in this book, in which the convergence in v is uniform. This statement has a similar proof (See [**14**, Cor. 5.5]) :

Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that Γ_μ is strongly irreducible. For any $\alpha < 1$ there exists $\varepsilon > 0$ such that for any nonzero vector v in V, one has

$$\beta(\{b \in B \mid \|b_n \cdots b_1 v\| \geq \varepsilon \|b_n \cdots b_1\| \, \|v\| \text{ for all } n \geq 1\}) > \alpha. \tag{3.8}$$

3.6. Law of Large Numbers on $\mathbb{P}(V)$.

We now introduce the *norm cocycle* on the projective space which, roughly speaking, controls the growth of the norm of a matrix and we prove the Law of Large Numbers for this cocycle.

We want to describe the behavior of the norm of the product of random elements of the group $G := \mathrm{GL}(V)$ that are independent and identically distributed with law μ. For any g in G, we set

$$N(g) := \max\left(\|g\|, \|g^{-1}\|\right), \tag{3.9}$$

and for x in the space $X := \mathbb{P}(V)$,

$$\sigma(g, x) := \log \tfrac{\|gv\|}{\|v\|} \tag{3.10}$$

where v is a nonzero element of the line x. The map $\sigma : G \times X \to \mathbb{R}$ is a continuous cocycle which we will call the *norm cocycle*. The function $\sigma_{\sup} : G \to \mathbb{R}$ introduced in (2.7) is given here by

$$\sigma_{\sup}(g) = \log N(g).$$

We will say that a Borel probability measure μ on $\mathrm{GL}(V)$ has a *finite first moment* if one has $\int_G \log N(g)\,\mathrm{d}\mu(g) < \infty$ (which does not depend on the choice of the norm). In this case the sequence of real numbers $(\int_G \log \|g\|\,\mathrm{d}\mu^{*n}(g))$ is subadditive. We set

$$\lambda_{1,\mu} = \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{GL}(V)} \log \|g\|\,\mathrm{d}\mu^{*n}(g)$$

and we call it the *first Lyapunov exponent* of μ. ¿From Kingman's subadditive ergodic theorem we get the following very general fact:

LEMMA 3.27. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *with a finite first moment. Then, for β-almost any b in B, one has*

$$\frac{1}{n}\log\|b_n \cdots b_1\| \xrightarrow[n\to\infty]{} \lambda_{1,\mu} \text{ and}$$
$$\frac{1}{n}\log\|b_1 \cdots b_n\| \xrightarrow[n\to\infty]{} \lambda_{1,\mu}$$

and these sequences also converge in $\mathrm{L}^1(B,\beta)$.

PROOF. For any $n \geq 1$ set, for b in B,

$$f_n(b) = \log\|b_n \cdots b_1\|.$$

Then f_n is integrable. Besides, for any m, n, one has $f_{n+m} \leq f_n + f_m \circ T^n$ (where as usual T is the shift map on B). By results Kingman's subadditive ergodic theorem (see for example [**119**]), $\frac{1}{n} f_n$ converges almost everywhere and in $\mathrm{L}^1(B,\beta)$ towards $\lim_{n\to\infty} \frac{1}{n}\int_B f_n\,\mathrm{d}\beta$.

Besides, since, for every g in EndV, one has $\|g\| = \|{}^t g\|$ (where ${}^t g$ denotes the adjoint map of g, acting on the dual space V^*), we get

$$\lambda_{1,\mu} = \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{GL}(V)} \log\|{}^t g\|\,\mathrm{d}\mu^{*n}(g)$$

and hence, for β-almost any b in B,

$$\frac{1}{n}\log\|b_1 \cdots b_n\| = \frac{1}{n}\log\|{}^t b_n \cdots {}^t b_1\| \xrightarrow[n\to\infty]{} \lambda_{1,\mu}$$

and the sequence also converges in $\mathrm{L}^1(B,\beta)$. $\square$

We will show that, when Γ_μ is irreducible, the first Lyapunov exponent $\lambda_{1,\mu}$ may be given an alternate definition. The following Theorem

3.28.b is the Law of Large Numbers for the norm cocycle. The L^1-convergence in this Law of Large Numbers is useful in order to check that all the definitions of the Lyapunov exponent are equivalent.

THEOREM 3.28 (Law of Large Numbers for $\|gv\|$). *Let μ be a Borel probability measure on $G = \mathrm{GL}(V)$ with Γ_μ irreducible and with a finite first moment i.e. such that $\int_G \log N(g)\, \mathrm{d}\mu(g) < \infty$. Let ν be a μ-stationary Borel probability measure on $X = \mathbb{P}(V)$.*
a) Then the cocycle σ is $(\mu \otimes \nu)$-integrable i.e. $\int_{G\times X} |\sigma|\, \mathrm{d}(\mu\otimes\nu) < \infty$ and its average is equal to the first Lyapunov exponent of μ

$$\lambda_{1,\mu} = \int_{G\times X} \sigma\, \mathrm{d}(\mu\otimes\nu).$$

In particular, it does not depend on ν. Indeed, for β-almost any b in B, one has

$$\tfrac{1}{n}\log\|b_n\cdots b_1\| \xrightarrow[n\to\infty]{} \lambda_{1,\mu}.$$

Moreover this sequence converges also in $\mathrm{L}^1(B,\beta)$.
b) For any x in $\mathbb{P}(V)$, for β-almost any b in B, one has

$$\tfrac{1}{n}\sigma(b_n\cdots b_1, x) \xrightarrow[n\to\infty]{} \lambda_{1,\mu}.$$

This sequence converges also in $\mathrm{L}^1(B,\beta)$ uniformly for $x\in\mathbb{P}(V)$.
c) One has,

$$\tfrac{1}{n}\int_G \log\|g\|\, \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \lambda_{1,\mu}.$$

d) Uniformly for x in $\mathbb{P}(V)$, one has,

$$\tfrac{1}{n}\int_G \sigma(g,x)\, \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \lambda_{1,\mu}.$$

In Theorem 3.28, one does not assume Γ_μ to be proximal, hence the μ-stationary measure ν on X may not be unique.

PROOF OF THEOREM 3.28. *a*) For any g in $\mathrm{GL}(V)$ and x in $\mathbb{P}(V)$, one has

$$|\sigma(g,x)| \le \log N(g), \tag{3.11}$$

thus σ is $\mu\otimes\nu$ integrable and its average $\sigma_\mu(\nu) := \int_{G\times X}\sigma\, \mathrm{d}(\mu\otimes\nu)$ is well-defined. We want to prove that this average does not depend on ν. We may assume that ν is ergodic.

We will use the forward dynamical system on $B\times X$. By Proposition 1.9, the Borel probability measure $\beta\otimes\nu$ is invariant and ergodic under the transformation $T^X : B\times X \to B\times X, (b,x)\mapsto (Tb, b_1x)$. The function $(b,x)\mapsto \varphi(b,x) := \sigma(b_1,x)$ on $B\times X$ is $\beta\otimes\nu$-integrable. By definition, for any (b, x) in $B \times X$, any $v \neq 0$ in x and any n in $\mathbb{N}$, the n-th Birkhoff sum of φ is given by

$$\varphi_n(b, x) = \sigma(b_n \cdots b_1, x) = \log \|b_n \cdots b_1 v\| - \log \|v\|.$$

By Birkhoff theorem, for $\beta \otimes \nu$-almost any (b, x) in $B \times \mathbb{P}(V)$, one has,

$$\tfrac{1}{n}\varphi_n(b, x) \xrightarrow[n\to\infty]{} \sigma_\mu(\nu).$$

In particular,

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \|b_n \cdots b_1\| \geq \sigma_\mu(\nu).$$

Since, by Lemma 3.6, for any proper subspace W of V, one has $\nu(\mathbb{P}(W)) < 1$, one can find a basis $(v_i)_{1\leq i\leq d}$ of V such that, for β-almost all b in B, for all i, one has

$$\tfrac{1}{n} \log \|b_n \cdots b_1 v_i\| \xrightarrow[n\to\infty]{} \sigma_\mu(\nu).$$

Since all the norms of the finite dimensional vector space $\mathrm{End}(V)$ are comparable, there exists $\varepsilon > 0$ such that, for any g in $\mathrm{GL}(V)$, one has

$$\max_{1\leq i\leq d} \|g v_i\| \geq \varepsilon \|g\| .$$

As a consequence, for β-almost all b in B, one has

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \|b_n \cdots b_1\| \leq \sigma_\mu(\nu)$$

and hence

$$\tfrac{1}{n} \log \|b_n \cdots b_1\| \xrightarrow[n\to\infty]{} \sigma_\mu(\nu). \tag{3.12}$$

In particular, $\sigma_\mu(\nu)$ does not depend on ν and is equal to $\lambda_{1,\mu}$ by Lemma 3.27.

Still by Lemma 3.27, the sequence (3.12) of integrable functions converges also in $\mathrm{L}^1(B, \beta)$. Let us also prove it directly in this case. It is enough to check that this sequence is uniformly integrable. This follows from the fact that these functions are bounded by the functions

$$\Psi_n(b) := \tfrac{1}{n} \textstyle\sum_{i=1}^n \log N(b_i),$$

and that the sequence Ψ_n is uniformly integrable since, by the Law of Large Numbers (Theorem 1.5), it converges in $\mathrm{L}^1(B, \beta)$

$$\Psi_n(b) \xrightarrow[n\to\infty]{} \textstyle\int_G \log N(g)\, \mathrm{d}\mu(g).$$

b) This follows from *a*) and Theorem 2.9.

c) Again, this follows from Lemma 3.27, but can be established directly, since, from the convergence in $L^1(B, \beta)$ proven in *a*), one gets

$$\tfrac{1}{n} \textstyle\int_G \log \|g\|\, \mathrm{d}\mu^{*n}(g) = \tfrac{1}{n} \int_B \log \|b_n \cdots b_1\|\, \mathrm{d}\beta(b) \xrightarrow[n\to\infty]{} \lambda_{1,\mu}.$$

d) By *b*), one gets

$$\frac{1}{n}\int_G \sigma(g,x)\,\mathrm{d}\mu^{*n}(g) = \frac{1}{n}\int_B \sigma(b_n\cdots b_1,x)\,\mathrm{d}\beta(b) \xrightarrow[n\to\infty]{} \lambda_{1,\mu},$$

uniformly for x in $\mathbb{P}(V)$, which was to be shown. □

REMARK 3.29. In the general context of Theorem 2.9, for every g, g' in G one still has

$$\sigma_{\sup}(gg') \leq \sigma_{\sup}(g) + \sigma_{\sup}(g').$$

Hence, as in the proof of Lemma 3.27, by Kingman's subadditive ergodic Theorem [**119**], one knows that there exists a real constant κ_μ such that, for β-almost every b in B,

$$\tfrac{1}{n}\sigma_{\sup}(b_n\cdots b_1) \xrightarrow[n\to\infty]{} \kappa_\mu.$$

By construction one has the inequality

$$\sigma_\mu \leq \kappa_\mu.$$

We have just shown that, in the context of Theorem 3.28, this inequality is indeed an equality. However, in the general context of Theorem 2.9 this inequality is not always an equality. To get an example, one can choose G to be $\mathrm{SL}(V)$, μ to be a Borel probability measure on G such that Γ_μ is strongly irreducible and $X = \mathbb{P}(V)$, as in Theorem 3.28, but one replaces the cocycle σ by its opposite. Then, by Theorem 3.31 below, σ_μ is negative whereas κ_μ is non-negative.

3.7. Positivity of the first Lyapunov exponent.

In this section we use the method of Guivarc'h and Raugi to prove the positivity of the first Lyapunov exponent, which is originally due to Furstenberg. This method relies on the linear speed of divergence of Birkhoff sums (Lemma 2.18).

We keep the notations of Section 3.6. For any g in G, set

$$\delta(g) := \tfrac{1}{d}\log|\det g|\,, \tag{3.13}$$

where d is the dimension of V.

We will need the following elementary lemma.

LEMMA 3.30. *For any g in $\mathrm{GL}(V)$, one has*

$$|\det g| \leq \|g\|^d \ \textit{and}\ |\delta(g)| \leq \log N(g).$$

PROOF. Equip V with a Haar measure λ. For any $r > 0$, let $B(r) \subset V$ be the closed ball with radius r and center 0. If $\mathbb{K}$ is archimedean, we have $\lambda(B(r)) = r^d\lambda(B(1))$. If $\mathbb{K}$ is non-archimedean, we have $\lambda(B(qr)) = q^d \lambda(B(r))$, where q is the cardinality of the residual field of $\mathbb{K}$. In both cases, one has

$$0 < R := \sup_{r>0} r^{-d} \lambda(B(r)) < \infty.$$

For any g in $\mathrm{GL}(V)$ and $r > 0$, we have $gB(r) \subset B(\|g\|\, r)$ hence

$$|\det g|\, \lambda(B(r)) = \lambda(gB(r)) \subset \lambda(B(\|g\|\, r)) \leq r^d \|g\|^d\, R,$$

whence the first inequality. The second follows by applying the first one to g and g^{-1}. □

Note that, as the determinant is a morphism $G \to \mathbb{K}^*$, the random sequence $\delta(b_n \cdots b_1)$ is a sum of independent and identically distributed elements of $\mathbb{R}$. When the function $\log N$ is μ-integrable, the function δ is also μ-integrable, and, by the classical Law of Large Numbers, for β-almost all b in B, one has

$$\tfrac{1}{n}\delta(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \delta_\mu \quad \text{where} \quad \delta_\mu := \textstyle\int_G \delta \, \mathrm{d}\mu. \tag{3.14}$$

In the following theorem, we keep the notations of Theorem 3.28.

THEOREM 3.31 (Positivity of the first Lyapunov exponent). *Let μ be a Borel probability measure on $G = \mathrm{GL}(V)$ with a finite first moment, i.e. $\int_G \log N(g)\, \mathrm{d}\mu(g) < \infty$. Assume that Γ_μ is strongly irreducible and that the image of Γ_μ in $\mathrm{PGL}(V)$ is not bounded.*
Then the first Lyapunov exponent $\lambda_{1,\mu}$ satisfies

$$\lambda_{1,\mu} > \delta_\mu.$$

When μ is supported by $\mathrm{SL}(V)$, one can restate Theorem 3.31 as :

COROLLARY 3.32. *Let μ be a Borel probability measure on $\mathrm{SL}(V)$ with a finite first moment. If Γ_μ is strongly irreducible and unbounded, then the first Lyapunov exponent is positive : $\lambda_{1,\mu} > 0$.*

REMARK 3.33. There are various proofs for the positivity of the first Lyapunov exponent relying on the spectral gap of an operator acting on a Hilbert space. For instance the original proof of Furstenberg is based on Kesten's amenability criterion in [**76**]. See also [**123**] or [**115**]. Here we will follow an argument due to Guivarc'h and Raugi which does not rely on a spectral gap.

REMARK 3.34. In Theorem 3.31, one cannot replace the assumption "Γ_μ *is strongly irreducible*" by "Γ_μ *is irreducible*". This can be seen on the example of Remark 3.10. In this example, the group G consists of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}$ with $a \neq 0$ in $\mathbb{R}$, the Borel probability measure μ on G is compactly supported and satisfies

$\Gamma_\mu = G$. In this case, the first Lyapunov exponent of μ on $\mathbb{R}^2$ is $\lambda_{1,\mu} = 0$ (See Proposition 4.9).

We will prove the following slightly more general theorem, without the strong irreducibility assumption. In this theorem, the assumptions are similar to the assumptions in Corollary 3.25.

THEOREM 3.35. *Let μ be a Borel probability measure on $G = \mathrm{GL}(V)$ such that Γ_μ is irreducible and $\int_G \log N(g)\,\mathrm{d}\mu(g) < \infty$. If, for some virtually invariant and strongly irreducible subspace W of V, the image of $\Gamma_{\mu,W}$ in $\mathrm{PGL}(W)$ is not bounded, then one has $\lambda_{1,\mu} > \delta_\mu$.*

One could first prove Theorem 3.31 and deduce the more general Theorem 3.35 by using the measure induced by μ on a finite index subgroup as in Section 4.3 below. Instead, we will give a direct proof:

PROOF OF THEOREM 3.35. The key step is Lemma 2.18.

Let $(V_i)_{i\in I}$ be a transitive strongly irreducible Γ_μ-family in V and let d_1 be the dimension of these subspaces. For i in I, equip V_i with an alternate d_1-form ω_i.

First, let us give a formula for the computation of determinants. Let $\Delta \subset \mathrm{GL}(V)$ be the subgroup spanned by Γ_μ and $\Lambda \subset \Delta$ be the finite index normal subgroup of those g in Δ such that $gV_i \subset V_i$ for any i in I. We set $F = \Delta/\Lambda$ and we let Δ (and F) act on I in the natural way, that is, for any g in Δ and i in I, we set $gi = j$, where j is such that $gV_i = V_j$. For g in Δ and i in I, let $D_i(g)$ be the determinant of g, viewed as a linear map from (V_i, ω_i) to (V_{gi}, ω_{gi}), and

$$\delta_i(g) = \frac{1}{d_1} \log |D_i(g)|.$$

We claim that, for any g in Δ, one has the equality

$$\delta(g) = \tfrac{1}{|I|} \textstyle\sum_{i\in I} \delta_i(g). \tag{3.15}$$

In order to prove this equality, we fix a minimal subset $J \subset I$ such that V is spanned by $(V_i)_{i\in J}$. Then, by Lemma 3.12, one has $V = \bigoplus_{i\in J} V_i$. In particular, $|J| = \frac{d}{d_1}$ and, for any g in Λ and f in F, one has

$$\det{}_V(g) = \textstyle\prod_{i\in fJ} D_i(g),$$

hence

$$\det{}_V(g)^{|F|} = \textstyle\prod_{f\in F} \prod_{i\in fJ} D_i(g) = (\prod_{i\in I} D_i(g))^p,$$

where $p = |J|\frac{|F|}{|I|} = \frac{d}{d_1}\frac{|F|}{|I|}$. Now, the map $\Delta \to \mathbb{K}^*$,

$$g \mapsto (\textstyle\prod_{i\in I} D_i(g))^p \det{}_V(g)^{-|F|}$$

is a group morphism. Since it is trivial on the finite index subgroup Λ, it takes values in the group of roots of 1 in $\mathbb{K}^*$. In particular, taking absolute values, we get Equality (3.15).

For β-almost any b in B, for any i in I, we let $V_{i,b} \subset V_i$ be as in Corollary 3.20 so that any nonzero cluster point in $\mathrm{Hom}(V_i, V)$ of a sequence $\lambda_n b_n \cdots b_1|_{V_i}$ with λ_n in $\mathbb{K}$ has kernel $V_{i,b}$. Since the virtual proximal dimension of Γ_μ is $< d_1$, one has $V_{i,b} \neq \{0\}$, hence, by Lemma 3.24.b),

$$\log(\|b_n \cdots b_1|_{V_i}\|) - \delta_i(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \infty. \tag{3.16}$$

Let us fix an ergodic μ-stationary Borel probability measure ν on $\bigcup_{i\in I} \mathbb{P}(V_i)$. Such a measure does exist by Lemma 1.10. By Proposition 3.23, for β-almost any b in B, for ν-almost any x in $\mathbb{P}(V)$, there exists $\varepsilon > 0$ such that, for $v \neq 0$ in x, one has

$$\|b_n \cdots b_1 v\| \geq \varepsilon \left\|b_n \cdots b_1|_{V_{i(x)}}\right\| \|v\| \quad \text{for all } n \geq 1, \tag{3.17}$$

where $i(x) \in I$ is such that $x \in \mathbb{P}\left(V_{i(x)}\right)$. From (3.16) and (3.17), we get

$$\sigma(b_n \cdots b_1, x) - \delta_{i(x)}(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \infty. \tag{3.18}$$

We use again the forward dynamical system on $B \times X$. By Proposition 1.9, the Borel probability measure $\beta \otimes \nu$ is invariant and ergodic under the transformation

$$T^X : B \times X \to B \times X, (b, x) \mapsto (Tb, b_1 x).$$

Set, for b in B and x in $\bigcup_{i\in I} \mathbb{P}(V_i)$,

$$\varphi(b, x) = \sigma(b_1, x) - \delta_{i(x)}(b_1).$$

Then, (3.18) reads as

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ (T^X)^k \xrightarrow[n\to\infty]{} \infty,$$

$\beta \otimes \nu$-almost everywhere. By Lemma 2.18, we get

$$\int_{B\times\mathbb{P}(V)} \varphi \,\mathrm{d}(\beta \otimes \nu) > 0.$$

We claim we have $\int_{B\times\mathbb{P}(V)} \varphi \,\mathrm{d}(\beta \otimes \nu) = \lambda_{1,\mu} - \delta_\mu$, which finishes the proof. Indeed, on one hand, by Theorem 3.28, we have

$$\int_{B\times\mathbb{P}(V)} \sigma(b_1, x) \,\mathrm{d}(\beta \otimes \nu) = \lambda_{1,\mu}.$$

On the other hand, since, by Proposition 3.18, for any i in I, $\nu(\mathbb{P}(V_i)) = \frac{1}{|I|}$, we get

$$\int_{B\times\bigcup_{i\in I}\mathbb{P}(V_i)} \delta_{i(x)}(b_1)\,\mathrm{d}(\beta\otimes\nu) = \tfrac{1}{|I|}\sum_{i\in I}\int_G \delta_i(g)\,\mathrm{d}\mu(g) = \delta_\mu,$$

where the last equality follows from (3.14) and (3.15). $\square$

3.8. Proximal and non-proximal representations.

In this section we explain a method which allows to control norms of matrices thanks to norms in proximal irreducible representations.

This purely algebraic method will not be used before Section 13.5.

LEMMA 3.36. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let Γ be a strongly irreducible sub-semigroup of $\mathrm{GL}(V)$. Let $r \geq 1$ be the proximal dimension of Γ in V, and let $V_r \subset \wedge^r V$ be the subspace spanned by the lines $\wedge^r\pi(V)$, where π is a rank r element of $\overline{\mathbb{K}\Gamma}$. Then,*
a) V_r admits a largest proper Γ-invariant subspace U_r.
b) The action of Γ on the quotient $V_r' := V_r/U_r$ is proximal and strongly irreducible.
c) Moreover, there exists $C \geq 1$ such that, for any g in Γ, one has

$$C^{-1}\|g\|^r \leq \|\wedge^r g\|_{V_r'} \leq \|g\|^r. \tag{3.19}$$

REMARK 3.37. In case $\mathbb{K}$ has characteristic 0, the action of Γ on $\wedge^r V$ is semisimple and $V_r' = V_r$.

In case $\mathbb{K} = \mathbb{R}$, the constant C can be chosen to be $C = 1$ for a suitable choice of norms.

PROOF OF LEMMA 3.36. *a*) We will prove that V_r contains a largest proper Γ-invariant subspace and that this space is equal to

$U_r := \cap_\pi \mathrm{Ker}_{V_r}(\wedge^r\pi)$, where π runs among all rank r elements of $\overline{\mathbb{K}\Gamma}$.

This space U_r is clearly Γ-invariant. We have to check that the only Γ-invariant subspace U of V_r which is not included in U_r is $U = V_r$. Let π be a rank r element of $\overline{\mathbb{K}\Gamma}$ such that U is not included in $\mathrm{Ker}(\wedge^r\pi)$. The endomorphism $\wedge^r\pi$ is proximal and one has

$$\wedge^r\pi(U) \subset U\,.$$

As $\wedge^r\pi$ has rank one, one has

$$\mathrm{Im}(\wedge^r\pi) \subset U\,.$$

Let π' be any rank r element of $\overline{\mathbb{K}\Gamma}$. Since Γ is irreducible in V, there exists f in Γ such that $\pi' f\pi \neq 0$. As $\pi' f\pi$ also belongs to $\overline{\mathbb{K}\Gamma}$, we get $\mathrm{rk}(\pi' f\pi) = r$ and, since $\wedge^r(\pi' f)$ preserves U, one has

$$\mathrm{Im}(\wedge^r\pi') = \mathrm{Im}(\wedge^r(\pi' f\pi)) \subset U\,.$$

Since this holds for any π', by definition of V_r, we get $U = V_r$, which was to be shown.

b) The above argument proves also that, for any rank r element π of $\overline{\mathbb{K}\Gamma}$, one has

$$\text{(3.20)} \qquad \operatorname{Im}(\wedge^r\pi) = \wedge^r\pi(V_r) \text{ and } \operatorname{Im}(\wedge^r\pi) \not\subset U_r\,.$$

In particular, the action of Γ on the quotient space $V'_r := V_r/U_r$ is proximal.

Let us prove now that the action of Γ on V'_r is strongly irreducible. Let $U_{(1)}, \dots, U_{(\ell)}$ be subspaces of V_r, all of them containing U_r, such that Γ preserves $U_{(1)} \cup \cdots \cup U_{(\ell)}$. Since V'_r is Γ-irreducible, the spaces $U_{(1)}, \dots, U_{(\ell)}$ span V_r. Let $\Delta \subset \Gamma$ be the sub-semigroup

$$\Delta := \{g \in \Gamma \mid gU_{(i)} = U_{(i)} \text{ for all } 1 \leq i \leq \ell\}.$$

There exists a finite subset $F \subset \Gamma$ such that

$$\Gamma = \Delta F = F\Delta.$$

In particular, since Γ is strongly irreducible in V, so is Δ. Besides, Δ also has proximal dimension r and, since $\overline{\mathbb{K}\Gamma} = \overline{\mathbb{K}\Delta}F$, V_r is also spanned by the lines $\operatorname{Im}(\wedge^r\pi)$ for rank r elements π of $\overline{\mathbb{K}\Delta}$. By applying the first part of the proof to Δ, since the Δ-invariant subspaces $U_{(i)}$ span V_r, one of them is equal to V_r. Therefore, V'_r is strongly irreducible.

c) We want to prove the bounds (3.19). First, for g in $\mathrm{GL}(V)$, one has $\| \wedge^r g\| \leq \|g\|^r$. As for g in Γ, we have $(\wedge^r g)V_r = V_r$ and $(\wedge^r g)U_r = U_r$, we get

$$\| \wedge^r g\|_{V'_r} \leq \|g\|^r\,.$$

Assume now there exists a sequence (g_n) in Γ with

$$\|g_n\|^{-r}\| \wedge^r g_n\|_{V'_r} \to 0$$

and let us reach a contradiction. If $\mathbb{K}$ is $\mathbb{R}$, set $\lambda_n = \|g_n\|^{-1}$. In general, pick λ_n in $\mathbb{K}$ such that $\sup_n |\log(|\lambda_n|\|g_n\|)| < \infty$. After extracting a subsequence, we may assume $\lambda_n g_n \to \pi$, where π is a nonzero element of $\overline{\mathbb{K}\Gamma}$. In particular, π has rank $\geq r$ and we have $\lambda_n^r \wedge^r g_n \to \wedge^r\pi$. Thus, since $\|\lambda_n^r \wedge^r g_n\|_{V'_r} \to 0$, we get $\| \wedge^r \pi\|_{V'_r} = 0$, that is,

$$\wedge^r\pi(V_r) \subset U_r.$$

We argue now as in *a)*. Let π' be a rank r element of $\overline{\mathbb{K}\Gamma}$. Since Γ is irreducible in V, there exists f in Γ such that $\pi' f\pi \neq 0$. Since $\pi' f\pi$ has rank at least r, it has rank exactly r and, since $\wedge^r(\pi' f)$ preserves U_r, one has

$$\operatorname{Im}(\wedge^r\pi') = \operatorname{Im}(\wedge^r(\pi' f\pi)) \subset U_r.$$

This contradicts (3.20). $\square$

Here is an application of Lemma 3.36. We use the notations of Lemma 3.2.

Lemma 3.38. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that the semigroup* $\Gamma := \Gamma_\mu$ *is strongly irreducible. Let $r \geq 1$ be the proximal dimension of Γ in V and Λ_Γ^r be the limit set of Γ in the Grassmann variety $\mathbb{G}_r(V)$. Then there exists a unique μ-stationary Borel probability measure ν_r on Λ_Γ^r.*

Remark 3.39. When $r > 1$, the measure ν_r may not be the only μ-stationary measure on the Grassmannian $\mathbb{G}_r(V)$. Indeed, there may exist uncountably many ergodic μ-stationary probability measures on $\mathbb{G}_r(V)$. See Remark 3.4 for an example.

Proof. According to Lemma 3.36, there exists a strongly irreducible and proximal representation $\rho' : \Gamma \to \mathrm{GL}(V'_r)$, in a $\mathbb{K}$-vectorspace V'_r and a Γ-equivariant embedding $i'_r : \Lambda_\Gamma^r \to \mathbb{P}(V'_r)$. Since, by Proposition 3.7, the μ-stationary probability measure on $\mathbb{P}(V'_r)$ is unique, then the μ-stationary probability measure on Λ_Γ^r is also unique. □

Remark 3.40. One can reinterpret this unique μ-stationary probability measure ν_r on the limit set Λ_Γ^r thanks to the Furstenberg boundary map $\xi : B \to \mathbb{G}_r(V)$ introduced in Lemma 3.5. Indeed ν_r is equal to the image $\nu_r = \xi_*(\beta)$ of the Bernoulli probability measure β on B by the Furstenberg boundary map ξ.

4. Finite index subsemigroups

This chapter contains general results relating the random walks on a semigroup and the induced random walks on its finite index subsemigroups.

4.1. Expected Birkhoff sum at the first return time.

We begin by a general result from ergodic theory, relating averages of an ergodic dynamical system with averages for an induced dynamical system.

Let $(X, \mathcal{X}, \chi)$ be a probability space, equipped with a measure preserving map T, and φ be a $\mathcal{X}$-measurable function on X. Let $A \subset X$ be a $\mathcal{X}$-measurable subset such that,

$$\chi(\cup_{q=0}^{\infty} T^{-q}(A)) = 1. \tag{4.1}$$

For χ-almost any x in X, we introduce the *first return time*

$$t_A(x) = \min\{n \geq 1 | T^n x \in A\},$$

which is almost surely finite, and the corresponding Birkhoff sum

$$\varphi_A(x) = \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{t_A(x)-1}x).$$

LEMMA 4.1. *Let $(X, \mathcal{X}, \chi)$ be a probability space, equipped with a measure preserving transformation T. Let A be an element of $\mathcal{X}$ satisfying (4.1). Then, for any integrable function φ on X, φ_A is integrable on A and one has*

$$\int_A \varphi_A \,\mathrm{d}\chi = \int_X \varphi \,\mathrm{d}\chi. \tag{4.2}$$

REMARK 4.2. In case $\varphi = \mathbf{1}$, this is just Kac formula $\int_A t_A \,\mathrm{d}\chi = 1$.
When T is ergodic, the condition (4.1) is equivalent to $\chi(A) > 0$.

PROOF. We first give a short proof of Lemma 4.1 in case T is invertible. We write $A = \cup_{n\geq 1} A_n$ where $A_n := A \cap t_A^{-1}(n)$. Up to negligeable sets, one can write X as the disjoint union

$$X = \cup_{0\leq k<n} T^k(A_n).$$

It suffices to prove Formula (4.2) when φ is the characteristic function of some $\mathcal{X}$-measurable set $B \subset X$ and we can also suppose that

$$B \subset T^k(A_n),$$

for some integers $0 \leq k < n$. In this case, Formula (4.2) follows from the T-invariance of χ. $\square$

PROOF. We give now another proof of Lemma 4.1 in case T is ergodic. This proof is based on a double application of Birkhoff ergodic theorem. One for the transformation T of X and one for the first return map $R : x \mapsto T^{t_A(x)}x$ which is a transformation of A. The transformation R is then ergodic too. We can also assume $\varphi > 0$. We write, for χ-almost all x in X and $n \geq 1$,

$$t_{n,A}(x) := t_A(x) + \ldots + t_A(R^{n-1}x).$$

Hence the following sum is both a Birkhoff sum for T and R,

$$S_n(x) := \varphi_A(x) + \ldots + \varphi_A(R^{n-1}x) = \varphi(x) + \ldots \varphi(T^{t_{n,A}(x)-1}x).$$

Then by a double application of Birkhoff ergodic theorem, one has, for χ-almost all x in A,

$$\frac{\int_A \varphi_A \,\mathrm{d}\chi}{\int_X \varphi \,\mathrm{d}\chi} = \chi(A) \frac{\lim_{n\to\infty} \frac{1}{n} S_n(x)}{\lim_{n\to\infty} \frac{1}{t_{A,n}(x)} S_n(x)} = \chi(A) \lim_{n\to\infty} \frac{t_{n,A}(x)}{n}.$$

In particular, this ratio does not depend on φ, hence, computed with the characteristic function $\varphi = \mathbf{1}_A$, is equal to 1. This proves (4.2). $\square$

Proof. We end with a tricky and elementary proof, with no further assumptions. It suffices to prove this formula when φ is the characteristic function of some $\mathcal{X}$-measurable set $B \subset X$ and we can also suppose that

$$B \subset t_A^{-1}(n),$$

for some integer $n \geq 1$. In this case, the function $\varphi_A \mathbf{1}_A$ is the characteristic function of the set C which is a disjoint union

$$C = \bigcup_{\ell \geq 0} C_\ell \ \text{ where } \ C_\ell = A \cap T^{-\ell} B \cap t_A^{-1}(\ell + n)$$

and we have to prove that $\chi(C) = \chi(B)$. By construction, the sets $D_m^\ell := T^{-(m-\ell)} C_\ell$ are disjoint, when ℓ varies between 0 and m, and one has

$$\bigcup_{\ell=0}^m D_m^\ell = T^{-m} B \cap \Big(\bigcup_{q=0}^m T^{-q} A \Big).$$

Therefore one has

$$\chi\left(\bigcup_{\ell=0}^m C_\ell\right) = \sum_{\ell=0}^m \chi\left(C_\ell\right) = \sum_{\ell=0}^m \chi\left(D_m^\ell\right) = \chi\left(\bigcup_{\ell=0}^m D_m^\ell\right) \tag{4.3}$$

and, using (4.1), one has

$$\chi\left(T^{-m} B \smallsetminus \bigcup_{\ell=0}^m D_m^\ell\right) \leq \chi\Big(X \smallsetminus \bigcup_{q=0}^m T^{-q} A\Big) \xrightarrow[m\to\infty]{} 0. \tag{4.4}$$

Now, combining (4.3) and (4.4), one gets as required

$$\chi(C) = \lim_{m\to\infty} \chi\left(\bigcup_{\ell=0}^m C_\ell\right) = \lim_{m\to\infty} \chi\left(\bigcup_{\ell=0}^m D_\ell\right) = \lim_{m\to\infty} \chi(T^{-m}B) = \chi(B). \qquad \square$$

4.2. The first return in a finite index subsemigroup.

A probability measure μ on a semigroup induces, on each closed finite index subsemigroup, a new probability measure: the law of the first return of the random walk in this finite index subsemigroup. We check that the left random walk and the right random walk on a semigroup induce the same law on such a finite index subsemigroup.

We check also that the return time has an exponential moment, and apply this fact to control the moments of the induced probability measure in terms of the moments of μ.

We will say that a subsemigroup H in a semigroup G is a *finite index subsemigroup*, if H is the stabilizer in G of a point f_0 in a finite set F on which G acts transitively by permutations. We will denote by

$$s : G \to F \simeq G/H; g \mapsto g f_0$$

the quotient map. We will say that H is a finite index *normal subsemigroup* if H is the kernel of a morphism $s : G \to F$ onto a finite group F.

Let G be a second countable locally compact topological semigroup with Borel σ-algebra $\mathcal{G}$. Let H be a closed finite index subsemigroup of G. Denote by $\mathrm{d}f$ the normalized counting measure on the finite set $F = G/H$.

If μ is a Borel probability measure on G, we let, as usual, $(B, \mathcal{B}, \beta, T)$ be the one-sided Bernoulli shift with alphabet $(G, \mathcal{G}, \mu)$. We set Γ_μ to be the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$.

For β-almost any b in B, define integers $t_s(b)$ and $u_s(b)$ by

$$t_s(b) := \min\{n \geq 1 \mid b_n \cdots b_1 \in H\} \ ,$$
$$u_s(b) := \min\{n \geq 1 \mid b_1 \cdots b_n \in H\} \ .$$

The following lemma tells us that the left random walk and the right random walk on G induce the same law on H.

Lemma 4.3. *Let μ be a Borel probability measure on G. Then the image measure μ_H on H of μ by the map $B \to H, b \mapsto b_{t_s(b)} \cdots b_1$ equals the image measure μ'_H of μ by the map $B \to H, b \mapsto b_1 \cdots b_{u_s(b)}$.*

This measure μ_H is called the *measure induced by μ on H*.

Proof. For any $n \geq 1$, let S_n be the set of $(g_1, \ldots, g_n)$ in G^n with $g_n \cdots g_1 \in H$ and, for any $1 \leq m \leq n-1$, $g_m \cdots g_1 \notin H$.

Similarly, let U_n be the set of $(g_1, \ldots, g_n)$ in G^n with $g_1 \cdots g_n \in H$ and, for any $1 \leq m \leq n-1$, $g_1 \cdots g_m \notin H$. One has

$$t_s^{-1}(n) = S_n \times B \ \text{ and } \ u_s^{-1}(n) = U_n \times B.$$

Since the semigroup G acts by permutation on the finite set F, for any two elements g, g' in G with $g'g$ in H, one has the equivalence $g \in H \Leftrightarrow g' \in H$. In particular, the set U_n is also the set of $(g_1, \ldots, g_n)$ in G^n with $g_1 \cdots g_n \in H$ and, for any $1 \leq m \leq n-1$, $g_{m+1} \cdots g_n \notin H$. This proves that the map $\Phi : (g_1, \ldots, g_n) \mapsto (g_n, \ldots, g_1)$ exchanges the sets S_n and U_n. As this map Φ preserves the restriction of the measure $\mu^{\otimes n}$, the result follows. $\square$

The following lemma tells us that the expected value of the return time in H is given by the index of H.

Lemma 4.4 (Expected return time). *Let G be a second countable locally compact topological semigroup, H be a closed finite index subsemigroup of G and $F = G/H$. Let μ be a Borel probability measure on G such that Γ_μ acts transitively on F. Set (B, β, T) to be the one-sided Bernoulli shift with alphabet (G, μ).*
a) One has $\int_B t_s(b)\, \mathrm{d}\beta(b) = |F|$.
b) Let $\varphi : B \to \mathbb{R}$ be a β-integrable function. Then the function

$$\psi : B \to \mathbb{R}; b \mapsto \varphi(b) + \cdots + \varphi(T^{t_s(b)-1} b) \tag{4.5}$$

is β-integrable and one has

$$\int_B \psi \, \mathrm{d}\beta = |F| \int_B \varphi \, \mathrm{d}\beta$$

Proof. Since a) is a consequence of b) with $\varphi = 1$, we only have to prove b). Let $\mathrm{d}f$ be the normalized counting probability measure on F. We use again the forward dynamical system. Indeed, we just apply Lemma 4.1 to the measure preserving transformation T^s of $(B \times F, \beta \otimes \mathrm{d}f)$ given by

$$T^s(b, f) = (Tb, b_1 f), \text{ for all } (b, f) \text{ in } B \times F,$$

to the function $\Phi : B \times F \to \mathbb{R}; (b, f) \mapsto \varphi(b)$ and to the subset $A = B \times \{e\}$.

Note that, since Γ_μ acts transitively on F, this transformation T^s is ergodic by Proposition 1.14. □

The following lemma tells us that the return time in H has a finite exponential moment.

Lemma 4.5 (Exponential moment for the return time). *Let G be a second countable locally compact topological semigroup, H be a closed finite index subsemigroup of G and $F = G/H$. Let μ be a Borel probability measure on G. Set (B, β, T) to be the one-sided Bernoulli shift with alphabet (G, μ).*
a) There exists $t_0 > 0$ such that $\int_B e^{t_0 \, t_s(b)} \, \mathrm{d}\beta(b) < \infty$.
b) Assume that a function $\varphi : G \to \mathbb{R}$ has a finite exponential moment, i.e. there exists $t_0 > 0$ such that $\int_G e^{t_0 \varphi(g)} \, \mathrm{d}\mu(g) < \infty$. Then the function

$$\psi : B \to \mathbb{R} \; ; \; b \mapsto \varphi(b_1) + \cdots + \varphi(b_{t_s(b)})$$

also has a finite exponential moment, i.e. there exists $t > 0$ such that $\int_B e^{t\psi(b)} \, \mathrm{d}\beta(b) < \infty$.

Proof. a) The semigroup H is the stabilizer in G of a point on a finite set on which the semigroup G acts. By replacing H by the kernel of this action, we can assume that H is normal in G. By replacing G by Γ_μ, we can also assume that Γ_μ acts transitively on F. In this case, by Lemma 1.12, the normalized counting measure $\mathrm{d}f$ is the unique μ-stationary probability measure on F. In particular (for example by Corollary 1.11), for any g in G, one has

$$\tfrac{1}{n} \sum_{k=1}^{n} \mu^{*k}(gH) \xrightarrow[n \to \infty]{} \tfrac{1}{|F|}$$

and there exist $n_0 \geq 1$ and $p_0 > 0$ such that, for any g in G, one has

$$\tfrac{1}{n_0} \sum_{k=1}^{n_0} \mu^{*k}(gH) \geq p_0.$$

Now, using the Markov property, one gets, for all $k \geq 1$,

$$\beta(\{b \in B \mid t_s(b) \geq kn_0\}) \leq (1-p_0)^k.$$

Hence t_s has a finite exponential moment.

b) The finite integral $I_t := \int_B e^{t\psi}\,\mathrm{d}\beta$ can be decomposed as $I_t = \sum_{n\geq 1} I_{t,n}$ where

$$I_{t,n} = \int_{\{t_s=n\}} e^{t\psi(b)}\,\mathrm{d}\beta(b).$$

Using Cauchy-Schwartz inequality and the independence of the coordinates b_i, one computes

$$\begin{aligned} I_{t,n} &\leq \beta(\{t_s=n\})^{\frac{1}{2}} \left(\int_B e^{2t(\varphi(b_1)+\cdots+\varphi(b_n))}\,\mathrm{d}\beta(b)\right)^{\frac{1}{2}} \\ &\leq \beta(\{t_s=n\})^{\frac{1}{2}} \left(\int_G e^{2t\varphi(g)}\,\mathrm{d}\mu(g)\right)^{\frac{n}{2}}. \end{aligned}$$

Since, by *a*), the sequence $\beta(\{t_s=n\})$ decays exponentially and since, by Lebesgue convergence theorem, one has $\lim_{t\to 0}\int_G e^{2t\varphi}\,\mathrm{d}\mu = 1$, one gets that, for t small enough, the sequence $I_{t,n}$ decays also exponentially and hence the exponential moment I_t is finite. □

As a corollary of these two lemmas we prove that, when a probability measure μ on a linear group G has a finite first moment (resp. a finite exponential moment), so has the induced measure μ_H on a finite index subgroup H. We will again use the notation $N(.)$ from (3.9).

Corollary 4.6 (Moments and finite index subgroups). *Let G be a closed subgroup of $\mathrm{GL}(d,\mathbb{K})$, H be a closed finite index subgroup of G, $F = G/H$, and μ be a Borel probability measure on G.*
a) Assume μ has a finite first moment, i.e. $\int_G \log N(g)\,\mathrm{d}\mu(g) < \infty$. Then μ_H also has a finite first moment, i.e. $\int_H \log N(h)\,\mathrm{d}\mu_H(h) < \infty$.
b) Assume μ has a finite exponential moment, i.e. there exists $t_0 > 0$ such that $\int_G N(g)^{t_0}\,\mathrm{d}\mu(g) < \infty$. Then μ_H also has a finite exponential moment, i.e. there exists $t > 0$ such that $\int_H N(h)^t\,\mathrm{d}\mu_H(h) < \infty$.

Proof. *a*) After replacing G by Γ_μ, the proof is an application of Lemma 4.4 with the function $\varphi(b) = \log N(b_1)$ on the one-sided Bernoulli shift (B,β,T) whith alphabet (G,μ). Indeed, one has

$$\begin{aligned} \int_H \log N(h)\,\mathrm{d}\mu_H(h) &= \int_B \log N(b_{t_s(b)}\cdots b_1)\,\mathrm{d}\beta(b) \\ &\leq \int_B \log N(b_1) + \cdots + \log N(b_{t_s(b)})\,\mathrm{d}\beta(b) \\ &= |F| \int_B \log N(b_1)\,\mathrm{d}\beta(b) = |F| \int_G \log N(g)\,\mathrm{d}\mu(g)\,. \end{aligned}$$

This proves that μ_H has a finite first moment.

b) The proof is similar, applying Lemma 4.5 with the function $\varphi(g) = \log N(g)$. One gets for t small enough,

$$\int_H N(h)^t \, \mathrm{d}\mu_H(h) \le \int_B N(b_1)^t \cdots N(b_{t_{s(b)}})^t \, \mathrm{d}\beta(b) < \infty\,.$$

This proves that μ_H has a finite exponential moment. □

4.3. Stationary measures for finite extensions.

In this section we prove that the μ-stationary measures are also μ_H-stationary for the probability measure induced by μ on a finite index subsemigroup H. We give then a few applications of this fact.

Let G be a second countable locally compact topological semigroup, H be a closed finite index subsemigroup of G and $F = G/H$. Let μ be a Borel probability measure on G, Γ_μ be the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$ and μ_H be the induced measure on H.

Let Y be a metrizable compact G-space. We let G act on $F = G/H$ by the natural left action and on $X := F \times Y$ by the product action.

The following lemma will be used in Section 9.1.

Lemma 4.7. *Let ν be a μ-stationary Borel probability measure on Y.*
a) This probability measure ν is also μ_H-stationary. The probability measure $\mathrm{d}f \otimes \nu$ on $X := F \times Y$ is also μ-stationary, and, for β-almost any b in B, one has $(\mathrm{d}f \otimes \nu)_b = \mathrm{d}f \otimes \nu_b$.
b) The probability measure ν is μ-proximal if and only if it is μ_H-proximal. In this case, $\mathrm{d}f \otimes \nu$ is μ-proximal over F.
c) If ν is the unique μ_H-stationary Borel probability measure on Y, then ν is also the unique μ-stationary Borel probability measure on Y.
d) If moreover Γ_μ acts transitively on F, the Borel probability measure $\mathrm{d}f \otimes \nu$ is the unique μ-stationary Borel probability measure on X.

Proof. *a*) Pick a non-negative continuous function φ on Y and let us prove that the integral $I := \int_H \int_Y \varphi(hy) \, \mathrm{d}\nu(y) \, \mathrm{d}\mu_H(h)$ is equal to $\int_Y \varphi \, \mathrm{d}\nu$. Indeed, using lemma 4.3 and the fact that ν is μ-stationary, one computes:

$$\begin{aligned} I &= \textstyle\int_{B\times Y} \varphi(b_1 \cdots b_{u_s(b)} y) \, \mathrm{d}\nu(y) \, \mathrm{d}\beta(b) \\ &= \lim_{m\to\infty} \textstyle\sum_{n=1}^m \int_{B\times Y} \varphi(b_1 \cdots b_n y) \, \mathrm{d}\nu(y) \mathbf{1}_{\{u_s(b)=n\}} \, \mathrm{d}\beta(b) \\ &= \lim_{m\to\infty} \textstyle\sum_{n=1}^m \int_{B\times Y} \varphi(b_1 \cdots b_m y) \, \mathrm{d}\nu(y) \mathbf{1}_{\{u_s(b)=n\}} \, \mathrm{d}\beta(b) \\ &= \lim_{m\to\infty} \textstyle\int_{B\times Y} \varphi(b_1 \cdots b_m y) \, \mathrm{d}\nu(y) \mathbf{1}_{\{u_s(b)\le m\}} \, \mathrm{d}\beta(b). \end{aligned}$$

Now, again, as ν is μ-stationary, one has, for any $m \geq 1$,

$$\textstyle\int_{B\times Y} \varphi(b_1 \cdots b_m y)\,\mathrm{d}\nu(y)\,\mathrm{d}\beta(b) = \int_Y \varphi\,\mathrm{d}\nu$$

while

$$\textstyle\int_{B\times Y} \varphi(b_1 \cdots b_m y)\mathbf{1}_{\{u_s(b)>m\}}\,\mathrm{d}\nu(y)\,\mathrm{d}\beta(b) \leq \|\varphi\|_\infty\, \beta(\{u_s(b) > m\})$$

goes to 0 when m goes to ∞. This proves that $I = \int_Y \varphi\,\mathrm{d}\nu$ as required. The last statement is easy.

b) If ν is μ_H-proximal, set, for β-almost any b in B, $u_0(b) = 0$ and, for any $p \geq 1$,

$$u_p(b) = u(b) + u(T^{u(b)}b) + \ldots + u(T^{u_{p-1}(b)}b),$$

so that the $u_p(b)$, $p \in \mathbb{N}$, are the successive times when the right random walk $e, b_1, b_1b_2, \ldots, b_1b_2\ldots b_n, \ldots$ visits H. Then, by definition, $(b_1 \cdots b_{u_p(b)})_*\nu$ converges to a Dirac mass, so that ν_b is a Dirac mass. The proof of the converse is similar.

c) In particular, if there exists a unique μ_H-stationary Borel probability measure ν on Y, then ν is *a fortiori* the unique μ-stationary Borel probability measure on Y. The last statement follows from *a*)

d) If Γ_μ acts transitively on F, $\mathrm{d}f$ is the unique μ-stationary probability measure on F. Hence, the image in F of any μ-stationary Borel probability measure $\widetilde{\nu}$ on $F \times Y$ necessarily equals $\mathrm{d}f$. Let f_0 be a point in F whose stabilizer in G is H, the restriction of such a measure to $\{f_0\} \times Y$ is μ_H-stationary, hence equals $\frac{1}{|F|}\delta_{f_0} \otimes \nu$.

When H is normal in G, this argument applies to every point of F and hence one has $\widetilde{\nu} = \mathrm{d}f \otimes \nu$.

In general, the proof is slightly longer. We will use the forward dynamical system. By Proposition 1.9, the product measure $\chi := \beta \otimes \widetilde{\nu}$ on $B \times X$ is invariant under the map

$$T^X : (b, X) \mapsto (Tb, b_1 x).$$

Let φ be a continuous function on X. By Lemma 4.1 applied to the transformation T^X, the function φ and the subset $A := B \times \{f_0\} \times Y$, we get the equality

$$\int_X \varphi(x)\,\mathrm{d}\widetilde{\nu}(x) = \frac{1}{|F|}\int_{B\times Y} \sum_{k=0}^{t_s(b)-1} \varphi(b_k \cdots b_1(f_0, y))\,\mathrm{d}\beta(b)\,\mathrm{d}\nu(y).$$

Therefore the μ-stationary Borel probability measure $\widetilde{\nu}$ on $F \times X$ is unique. Hence it is equal to $\mathrm{d}f \otimes \nu$. $\square$

REMARK 4.8. A bounded Borel function Φ on G is said to be μ-harmonic if, for any g in G,

$$\Phi(g) = \int_G \Phi(gh)\,\mathrm{d}\mu(h).$$

By using the same argument, one proves that *the restriction to H of a μ-harmonic function on G is μ_H-harmonic.*

4.4. Cocycles and finite extensions.

We compare the averages of a cocycle σ for the μ-action and for the μ_H-action.

PROPOSITION 4.9. *Let G be a second countable locally compact topological semigroup, H be a closed normal finite index subsemigroup of H and $F = G/H$. Let μ be a Borel probability measure on G such that Γ_μ maps onto F, μ_H be the induced probability measure on H, X be a compact second-countable G-space and ν be a μ-stationary Borel probability measure on X. Let $\sigma : G \times X \to E$ be a $\mu \otimes \nu$-integrable Borel cocycle Then σ is also $\mu_H \otimes \nu$-integrable and the averages*

$$\sigma_{\mu_H} := \textstyle\int_{H\times X} \sigma\,\mathrm{d}(\mu_H \otimes \nu) \;\text{ and }\; \sigma_\mu := \textstyle\int_{G\times X} \sigma\,\mathrm{d}(\mu \otimes \nu)$$

satisfy the equality $\sigma_{\mu_H} = |F|\,\sigma_\mu$.

PROOF. We will again use the forward dynamical system. By Proposition 1.9, the product measure $\chi := \beta \otimes \mathrm{d}f \otimes \nu$ on on $B \times F \times X$ is invariant under the map

$$T^{F,X} : (b, f, x) \mapsto (Tb, s(b_1)f, b_1 x).$$

The function

$$\varphi : B \times F \times X \to E, (b, x) \mapsto \sigma(b_1, x)$$

is $\beta \otimes \mathrm{d}f \otimes \nu$-integrable, and, by definition, one has the equality

$$\sigma_\mu = \textstyle\int_{B\times F\times X} \sigma(b_1, x)\,\mathrm{d}\beta(b)\,\mathrm{d}f\,\mathrm{d}\nu(x).$$

By Lemma 4.3, one has the equality

$$\sigma_{\mu_H} = \textstyle\int_{B\times X} \sigma(b_{t_s(b)} \cdots b_1, x)\,\mathrm{d}\beta(b)\,\mathrm{d}\nu(x).$$

By Lemma 4.1 applied to the transformation $T^{F,X}$, the function φ and the subset $A := B \times \{e\} \times X$, we know that these two right-hand sides are equal up to a factor $|F|$. Note that the condition (4.1) is satisfied since Γ_μ maps onto F (same argument as for Lemma 4.5). Hence, one has the equality $\sigma_{\mu_H} = |F|\,\sigma_\mu$. $\square$

Part 2

Reductive groups

5. Loxodromic elements

The aim of this chapter is to prove the existence of so-called "loxodromic" elements in Zariski dense semigroups of semisimple real Lie groups (Theorem 5.36). This result will be used in Chapter 9 to prove the regularity of the Lyapunov vector in the Law of Large Numbers.

We will focus mainly in this chapter on real Lie groups since this result does not extend to other local fields.

5.1. Basics on Zariski topology.

> We begin by recalling the very basic facts about Zariski topology that will be used in this book.

We will define Zariski topology on algebraic varieties and recall some of its elementary properties. The reader may find more about this topic in any introductory book on algebraic geometry, such as [**114**].

Let k be a field an V be a finite dimensional k-vector space. By a polynomial function on V, we mean a function from V to k which may be expressed as a polynomial function in the coordinates of a basis of V. We let $k[V]$ denote the algebra of polynomial functions on V.

Definition 5.1. Let k be a field. An *algebraic subvariety* Z in a finite dimensional k-vector space V is the set of zeroes of a family of polynomial functions. The *Zariski topology* on V is the topology whose closed subsets are the algebraic subvarieties.

In other words, a subset Z of V is an algebraic subvariety, or equivalently is Zariski closed, if there exists a set $\mathcal{F}$ of polynomial functions such that

$$Z = \{v \in V | \forall f \in \mathcal{F} \; f(v) = 0\}.$$

Proof. We need to check that this definition makes sense, that is, that the algebraic subvarieties are indeed the closed subsets of a topology. This is straightforward.

First, note that $\emptyset$ and V are algebraic varieties since they are respectively the zero sets of the constant functions 1 and 0.

Now, let $Z_1, \ldots, Z_r$ be algebraic subvarieties of V and $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be sets of polynomial functions such that, for $1 \le i \le r$,

$$Z_i = \{v \in V | \forall f \in \mathcal{F}_i \; f(v) = 0\}.$$

We let $\mathcal{F}$ be the set of functions which may be written as $f_1 \cdots f_r$ with $f_i \in \mathcal{F}_i$, for $1 \le i \le r$. We immediately get

$$Z_1 \cup \cdots \cup Z_r = \{v \in V | \forall f \in \mathcal{F} \; f(v) = 0\},$$

that is, $Z_1 \cup \cdots \cup Z_r$ is an algebraic subvariety.

Finally, let $(Z_i)_{i\in I}$ be a family of algebraic subvarieties and, for any i, let still $\mathcal{F}_i$ be a set of polynomial functions such that

$$Z_i = \{v \in V | \forall f \in \mathcal{F}_i \ f(v) = 0\}.$$

We now set $\mathcal{F} = \bigcup_{i\in I} \mathcal{F}_i$ and we get

$$\bigcap_{i\in I} Z_i = \{v \in V | \forall f \in \mathcal{F} \ f(v) = 0\},$$

that is, $\bigcap_{i\in I} Z_i$ is an algebraic subvariety. □

We can now speak of a *Zariski open* subset, a *Zariski closed* subset, a *Zariski connected* subset or a *Zariski dense* subset.

For instance the group $\mathrm{SL}(V)$ is a Zariski closed subset of the vector space $\mathrm{End}(V)$. The group $\mathrm{GL}(V)$ is a Zariski open subset of $\mathrm{End}(V)$. By definition, an *algebraic subgroup* of $\mathrm{GL}(V)$ is a subgroup of $\mathrm{GL}(V)$ which is Zariski closed in $\mathrm{GL}(V)$.

If Z is a subset of V, we let $I(Z)$ denote the set of polynomial functions of V which vanish identically on V. This is an ideal of the k-algebra $k[V]$.

Lemma 5.2. *Let Z be a subset of V. Then the Zariski closure of Z is the set*

$$\{v \in V | \forall f \in I(Z) \ f(v) = 0\}.$$

In particular, if Z is an algebraic subvariety, this set is equal to Z.

Proof. This is immediate. □

Remark 5.3. It follows from Hilbert's basis Theorem that the algebra $k[V]$ is noetherian. In particular, the ideal $I(Z)$ is always finitely generated, which means that any algebraic subvariety may be defined by a *finite* set of polynomial equations.

We shall soon see other consequences of the Noetherian property of $k[V]$ for the Zariski topology.

One easily checks that the points of V are closed subsets for the Zariski topology. But this topology is not Hausdorff as soon as k is infinite. More precisely, in this case it satisfies a property which can be considered as a strong converse of the Hausdorff property.

Let us say that a topological space X is irreducible if any two non empty open subsets of X have non empty intersection or equivalently if X may not be written as the union of two proper closed subsets.

Lemma 5.4. *Assume k is infinite. Then the Zariski topology on V is irreducible.*

In other words, any non empty Zariski open subset of V is Zariski dense.

Proof. Let Z_1 and Z_2 be proper Zariski closed subsets of V. As Z_1 is proper, $I(Z_1)$ contains a non-zero function f_1. In the same way, $I(Z_2)$ contains a non zero function f_2. Now, since k is infinite, the choice of a basis of V induces an isomorphism from the algebra $k[V]$ onto the abstract algebra $k[t_1, \dots, t_d]$, where d is the dimension of V (this can easily be shown by induction on d). In particular, the algebra $k[V]$ is an integral domain and the function $f = f_1 f_2$ is non zero. Since f belongs to $I(Z_1 \cup Z_2)$, we have $Z_1 \cup Z_2 \neq V$ and we are done. $\square$

Example 5.5. Let W_1 and W_2 be two distinct proper hyperplanes of V. Then the space $Z = W_1 \cup W_2$ is not irreducible for the Zariski topology.

Remark 5.6. If X is an irreducible topological space, so is every open subset of X. In particular, the algebraic group $\mathrm{GL}(V)$ is irreducible for the Zariski topology.

As we saw in the proof of Lemma 5.4 above, irreducibility follows from the integrity of the ring of functions. Let us see how the Noetherian property translates.

We say that a topological space X is *Noetherian* if any non-increasing sequence of closed subsets of X is eventually stationary.

Lemma 5.7. *The Zariski topology on V is Noetherian.*

Proof. This is straightforward: assume (Z_n) is a a non-increasing sequence of algebraic subvarieties of V. Since $k[V]$ is Noetherian, there exists n_0 such that, for any $n \geq n_0$, one has $I(Z_n) = I(Z_{n_0})$. By Lemma 5.2, we get $Z_n = Z_{n_0}$ for $n \geq n_0$. $\square$

Remark 5.8. If X is a Noetherian topological space, so is every subset of X for the induced topology.

We can now state the main result of this section. Its proof diretly follows from the Noetherian property.

Lemma 5.9. *Let k be a field, $V = k^d$ and X be a subset of V. There exists a decomposition*

$$X = X_1 \cup \ldots \cup X_\ell$$

where the X_i are Zariski closed in F, are Zariski irreducible and are not included in one another. This decomposition is unique up to permutations.

These closed irreducible subsets are called the *irreducible components* of X.

Proof. This is a general feature of Noetherian topological spaces.

Let X be such a space and let us prove that X may be written as a finite union of irreducible closed subspaces. We proceed by contradiction and we assume that such a decomposition does not exist. Since in particular, X is not irreducible, we may write X as a union $X' \cup X''$ where X' and X'' are proper closed subsets. Since X may not be written as the union of finitely many closed irreducible subsets, so is the case for at least one among X' and X''. Call X_1 this proper closed subset of X. By iterating the process, we construct a decrasing sequence (X_n) of closed subsets of X. This is a contradiction.

Now that the existence of such a decomposition is proved, write X as $X_1 \cup \cdots \cup X_\ell$ where the X_i are closed irreducible subsets and ℓ is minimal. In particular, for any $1 \leq i \neq j \leq \ell$, we have $X_i \not\subset X_j$. Besides, if Y is a closed irreducible subset of X, we have

$$Y = \bigcup_{i=1}^{\ell} Y \cap X_i,$$

hence, by irreducibility, $Y \subset X_i$ for some $1 \leq i \leq \ell$. The result follows. $\square$

5.2. Zariski dense semigroups in SL(d,$\mathbb{R}$).

We now start the study of Zariski dense subgroups of semisimple real Lie groups. To be very concrete, we will first state and prove our main result for the group $G = \mathrm{SL}(d, \mathbb{R})$.

Let $V = \mathbb{R}^d$ and $e_1, \ldots, e_d$ be its standard basis. Let $G = \mathrm{SL}(d, \mathbb{R})$ and $\mathfrak{g} := \{y \in \mathrm{End}(\mathbb{R}^d) \mid \mathrm{tr}(y) = 0\}$ be its Lie algebra. We introduce the *Cartan subspace* of $\mathfrak{g}$,

$$\mathfrak{a} := \{x = \mathrm{diag}(x_1, \ldots, x_d) \;/\; x_i \in \mathbb{R} \;,\; x_1 + \cdots + x_d = 0\},$$

i.e. the Lie subalgebra of diagonal matrices, and the *Weyl chamber*

$$\mathfrak{a}^+ = \{x \in \mathfrak{a} \;/\; x_1 \geq \cdots \geq x_d\}.$$

The *Jordan projection* $\lambda : G \to \mathfrak{a}^+$ is defined by, for every g in G,

$$\lambda(g) = \mathrm{diag}(\log \lambda_1(g), \ldots, \log \lambda_d(g)),$$

where the d-tuple $(\lambda_1(g), \ldots, \lambda_d(g))$ is the sequence of moduli of the eigenvalues of g in $\mathbb{C}$ in non-increasing order and repeated according to their multiplicities. The largest one $\lambda_1(g)$ is the *spectral radius* of g.

Definition 5.10. An element g of $\mathrm{SL}(d,\mathbb{R})$ is said to be *loxodromic* if $\lambda(g)$ belongs to the interior of $\mathfrak{a}^+$, or, equivalently, if the moduli of the eigenvalues of g are distinct:

$$\lambda_1(g) > \cdots > \lambda_d(g).$$

Equivalently this means that the eigenvalues of g^2 are distinct and positive.

Proposition 5.11. *Let Γ be a Zariski dense subsemigroup of $\mathrm{SL}(d,\mathbb{R})$. Then the set Γ_{lox} of loxodromic elements of Γ is also Zariski dense.*

Remark 5.12. In particular, Γ contains at least one loxodromic element. It is easy to see that Γ contains elements g whose eigenvalues are distinct. Indeed the discriminant D of the characteristic polynomial of g is a nonzero polynomial function on $G = \mathrm{SL}(d,\mathbb{R})$, hence it is nonzero on Γ. What proposition 5.11 tells us is that Γ contains many elements whose eigenvalues are distinct and positive.

Remark 5.13. One cannot replace in this proposition the field $\mathbb{R}$ by $\mathbb{C}$. For example, the unitary group $\Gamma = \mathrm{U}(d) \subset G = \mathrm{SL}(d,\mathbb{C})$ is Zariski dense but all the eigenvalues of the elements of Γ have modulus 1.

One can neither replace $\mathbb{R}$ by the field $\mathbb{Q}_p$. For example, the compact open subgroup of matrices whose coefficients are p-adic integers $\Gamma = \mathrm{SL}(d,\mathbb{Z}_p) \subset \mathrm{SL}(d,\mathbb{Q}_p)$ is also Zariski dense and all the eigenvalues of the elements of Γ have also modulus 1.

Remark 5.14. One may wonder why, in Proposition 5.11, we are dealing with subsemigroups Γ instead of subgroups Γ. There are two reasons. First, what occurs naturally when dealing with a random walk on G is the semigroup spanned by the support of the law. Second, even if we want to deal only with subgroups Γ, the key point of the proof will still involve semigroups.

5.3. Zariski closure of semigroups.

We begin by very general lemmas on the Zariski closure of subsemigroups.

Lemma 5.15. *Let k be a field and Γ be a subsemigroup of $\mathrm{GL}(d,k)$. Then the Zariski closure G of Γ in $\mathrm{GL}(d,k)$ is a group.*

Remark 5.16. We will often use this lemma under the equivalent formulation :

Let k be a field, $g \in \mathrm{GL}(d,k)$ and $n_0 \geq 0$. Then the sequence $(g^n)_{n \geq n_0}$ is Zariski dense in the group $<g>$ spanned by g.

Proof. Let $V = k^d$, let $k[\mathrm{End}V]$ be the algebra of k-valued polynomial functions on $\mathrm{End}(V)$, let

$$I := I(\Gamma) = \{P \in k[\mathrm{End}V] \;/\; \forall g \in \Gamma \;,\; P(g) = 0\}$$

so that, by Lemma 5.2, G is the set of zeroes of the ideal I, that is

$$G = \{g \in \mathrm{End}(V) \mid \forall P \in I \;,\; P(g) = 0\}\,.$$

For $m \geq 0$, let $I^m = \{P \in I \;/\; \mathrm{d}^o P \leq m\}$ where $\mathrm{d}^o P$ is the total degree of the polynomial P in d^2 variables.

We first prove the easy implication: $g, h \in G \Longrightarrow gh \in G$. Fix P in I. For g in Γ, the polynomial function $h \to P(gh)$ is null on Γ an hence also on G. Hence, for h in G, the polynomial function $g \to P(gh)$ is null on Γ and hence also on G. This proves that for any g, h in G, one has $P(gh) = 0$ and the product gh also belongs to G.

It remains to prove the implication: $g \in G \Longrightarrow g^{-1} \in G$. Fix g in G and denote by T_g the automorphism of $k[\mathrm{End}(V)]$ defined by

$$T_g(P)(h) = P(gh) \text{ for all } P \text{ in } k[\mathrm{End}(V)] \text{ and } h \text{ in } \mathrm{End}(V).$$

One has the inclusion

$$T_g(I^m) \subset I^m$$

since g belongs to G. Since I^m is finite dimensional, this inclusion is an equality:

$$T_g(I^m) = I^m.$$

Hence one has $T_g^{-1}(I) = I$. One writes then, for all P in I,

$$P(g^{-1}) = (T_g^{-2}(P))(g) = 0.$$

This proves that g^{-1} belongs to G. □

The second lemma focuses on real linear groups.

Lemma 5.17. *Every compact subsemigroup H of $\mathrm{GL}(d, \mathbb{R})$ is a subgroup.*

Proof. This fact is a general property of compact subsemigroups in topological groups. Indeed let h be an element of H. We want to prove that its inverse h^{-1} also belongs to H. Since H is compact, the sequence $(h^n)_{n\geq 1}$ has a cluster point k in H. Let U be a neighborhood of e in H. One can find another neighborhood V of e such that $V\,V^{-1} \subset U$. Let $m < n$ be positive integers such that both h^m and h^n belong to Vk. The element h^{n-m-1} belongs to Uh^{-1} Hence h^{-1} is also a cluster point of the sequence $(h^n)_{n\geq 1}$ and hence belongs to H. □

Lemma 5.18. *Every compact subgroup H of $\mathrm{GL}(d, \mathbb{R})$ preserves a positive definite quadratic form q_0 on $\mathbb{R}^d$.*

The proof uses the Haar measure. We recall that every locally compact group H supports a left H-invariant Radon measure $\mathrm{d}h$ called the *Haar measure* (see [**90**]). This measure is unique up to normalization. When H is compact, this measure is finite and is also left H-invariant. In this case, one can normalize $\mathrm{d}h$ so that it is a probability measure.

Proof. Let q be a positive definite quadratic form on $\mathbb{R}^d$, let $\mathrm{d}h$ be the Haar probability measure on H and let q_0 the average of the translates of q: this quadratic form q_0 is defined by

$$q_0(v) = \int_H q(hv)\,\mathrm{d}h \quad \text{for all } v \text{ in } \mathbb{R}^d.$$

By construction q_0 is positive definite and H-invariant as required. $\square$

With similar arguments, one can prove the following fact that we will not use in the sequel but that clarifies our approach.

Lemma 5.19. *Every compact subgroup H of* $\mathrm{GL}(d,\mathbb{R})$ *is Zariski closed.*

Remark 5.20. The field of real numbers $k = \mathbb{R}$ cannot be replaced here by the field of p-adic numbers $k = \mathbb{Q}_p$ or the field of complex numbers $k = \mathbb{C}$. For instance the compact group $\mathrm{SL}(d,\mathbb{Z}_p)$ is Zariski dense in $\mathrm{SL}(d,\mathbb{Q}_p)$. Similarly the unitary group $\mathrm{U}(d)$ is compact and Zariski dense in the complex group $\mathrm{GL}(d,\mathbb{C})$. However, this group $\mathrm{U}(d)$ is Zariski closed in the group $\mathrm{GL}(d,\mathbb{C})$ seen as an algebraic real Lie group.

Proof. Fix an element g of $\mathrm{End}(\mathbb{R}^d)$ which does not belong to H. We need to find a polynomial function P null on H such that $P(g) \neq 0$.

Let φ be a real valued continuous fonction on $\mathrm{End}(\mathbb{R}^d)$ that is equal to 0 on H and equal to 1 on the class $Hg = \{hg \mid h \in H\}$. Stone-Weierstrass Theorem ensures that there exists a polynomial function Q on $\mathrm{End}(\mathbb{R}^d)$ that is near φ on the compact set $H \cup Hg$. For instance we may require

$$Q(h) \leq \tfrac{1}{3} \quad \text{and} \quad Q(hg) \geq \tfrac{2}{3} \text{ for all } h \text{ in } H.$$

Let Q_0 be the average of the translates of Q: it is defined by

$$Q_0(g) = \int_H Q(hg)\,\mathrm{d}h \quad \text{for all } g \text{ in } \mathrm{End}(\mathbb{R}^d).$$

This polynomial function Q_0 is equal to a constant $C \leq \frac{1}{3}$ on H and is larger than $\frac{2}{3}$ on Hg. Hence the difference $P = Q_0 - C$ fulfills our requirements. $\square$

To finish this section, let us prove that, for algebraic groups, the irreducible components from Lemma 5.9 are Zariski connected components.

LEMMA 5.21. *Let k be a field and $V = k^d$.*
a) The Zariski connected component H_e of a subgroup H of $\mathrm{GL}(V)$ *is a finite index normal subgroup of H which is Zariski irreducible.*
b) A Zariski connected subsemigroup Γ of $\mathrm{GL}(V)$ *is Zariski irreducible.*

PROOF. *a*) The group H acts by conjugation on its irreducible components $(H_i)_{1 \le i \le \ell}$. The set

$$H_0 := \{h \in H \mid hH_i = H_i \text{ for all } i \le \ell\}$$

is a Zariski closed, finite index normal subgroup of H whose translates $H_0 h$ are included in the irreducible components H_i. Since they are Zariski irreducible the H_i's are equal to translates $H_0 h_i$ of H_0. The Zariski connected component H_e of H is then equal to H_0.

b) By Lemma 5.15 the Zariski closure H of Γ is a group. This group H is still Zariski connected. By point *a*) this group H is Zariski irreducible, and Γ is also Zariski irreducible. □

COROLLARY 5.22. *If k is infinite, the group* $\mathrm{SL}(V)$ *is irreducible.*

PROOF. We assume $d \ge 2$ since otherwise the result is trivial. Fix a basis of V and let U be the group of matrices of the form

$$\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{d-2} \end{pmatrix}$$

with t in k. This is an algebraic subgroup of $\mathrm{GL}(V)$ and the algebra of functions on U which are restrictions of polynomial functions on $\mathrm{End} V$ is isomorphic to $k[t]$. In particular, since this algebra is an integral domain, by arguing as in the proof of Lemma 5.4, one proves that U is Zariski connected. Let H be the Zariski connected component of e in $\mathrm{SL}(V)$. We have $U \subset H$. Since H is normal in $\mathrm{GL}(V)$, we have $gUg^{-1} \subset H$ for any g in $\mathrm{GL}(V)$. As these subgroups span $\mathrm{SL}(V)$, $\mathrm{SL}(V)$ is connected, hence irreducible by Lemma 5.21. □

The reader should not mistake the *Zariski irreducible subsemigroups* of $\mathrm{GL}(V)$ we just discussed for the *irreducible semigroups* of $\mathrm{GL}(V)$ that we introduced in Chapter 3, that is the semigroups in $\mathrm{GL}(V)$ whose action on V is irreducible.

5.4. Proximality and Zariski closure.

In this section, we check that two irreducible real linear semigroups with the same Zariski closure have equal proximal dimensions.

The following Lemma 5.23 gives also an easily checkable criterion to detect the existence of proximal elements in an irreducible real linear semigroup.

LEMMA 5.23. *Let $V = \mathbb{R}^d$, let Γ be an irreducible subsemigroup of* $\mathrm{GL}(V)$ *and let G be the Zariski closure of Γ in* $\mathrm{GL}(V)$*. Then the proximal dimensions are equal*

$$r_\Gamma = r_G\,.$$

In particular if G is proximal in V, then Γ contains a proximal element.

We recall that, according to Lemma 3.1, an irreducible semigroup $\Gamma \subset \mathrm{GL}(V)$ contains a proximal element if and only if Γ is proximal, i.e. if and only if its proximal dimension r_Γ is equal to 1.

PROOF. By definition of the proximal dimension, one has the inequality $r_G \leq r_\Gamma$. Assume by contradiction that one has the strict inequality $r_G < r_\Gamma$. By definition of the proximal dimension r_Γ, there exists an element $\pi \in \mathrm{End}(V)$ of rank r_Γ that belongs to the closure $\overline{\mathbb{R}\Gamma}$. Let $W = \mathrm{Im}\,\pi \subset V$ be its image and $W' = \mathrm{Ker}\,\pi \subset V$ be its kernel. Using the fact that Γ is irreducible and replacing if necessary π by a product $g\pi$ with g in Γ, we can assume that $\pi^2 \neq 0$. By minimality, the rank of π and π^2 are equal, hence one has the decomposition

$$V = W \oplus W'.$$

¿From now on, using this decomposition, we will consider $\mathrm{End}(W)$ as a subalgebra of $\mathrm{End}(V)$. One has then the equality

$$\mathrm{End}(W) = \pi \mathrm{End}(V)\pi\,.$$

Let H' and H be the subsemigroups of $\mathrm{End}(W)$:

$$H' := \pi\overline{\mathbb{R}\Gamma}\pi \ \text{ and } \ H := \{h \in H' \mid \det_W h = \pm 1\}.$$

Note that, by minimality of r_Γ, every nonzero element of H' belongs to $\mathrm{GL}(W)$, and hence is a scalar multiple of an element of H.

We claim that the semigroup H is bounded. Indeed, if this were not the case, there would exist a sequence $(h_n)_{n\geq 1}$ in H' with $\|h_n\| = 1$ and with $\det_W(h_n) \underset{n\to\infty}{\longrightarrow} 0$. But then, every cluster point τ of the sequence h_n would be a nonzero element of H' which is not invertible on W. A contradiction.

Hence H is a compact subsemigroup of $\mathrm{GL}(W)$. According to Lemma 5.18, there exists a H-invariant positive definite quadratic form q_0 on W. In particular, H' is included in the set $\mathrm{Sim}(q_0)$ of similarities of q_0. Since this set is Zariski closed and since Γ is Zariski dense in G, one has the inclusion

$$\pi G\pi \subset \mathrm{Sim}(q_0).$$

As a consequence one gets the inclusion

$$\pi\overline{\mathbb{R}G}\pi \subset \mathrm{Sim}(q_0). \tag{5.1}$$

Let $\tau \in \mathrm{End}(V)$ be an element of rank r_G that belongs to $\overline{\mathbb{R}\,G}$. Since Γ is irreducible in V, there exists g_1, g_2 in Γ such that, the following element of $\overline{\mathbb{R}G}$ is nonzero :

$$\pi g_1\tau g_2\pi \neq 0.$$

Since $r_G < r_\Gamma$, it does not belong to $\mathrm{GL}(W)$. This contradicts (5.1). $\square$

Remark 5.24. In the last argument, instead of using the existence of q_0 given by Lemma 5.18, we could have applied directly the more powerful Lemma 5.19.

Now we could end the proof of Proposition 5.11, by applying Lemma 5.23 to a suitable irreducible representation of $\mathrm{SL}(d,\mathbb{R})$ as in [**97**], but we will instead use a technic involving simultaneously finitely many irreducible representations. This technic will be useful throughout this book.

5.5. Simultaneous proximality.

According to Lemma 3.1, every irreducible proximal subsemigroup Γ of $\mathrm{GL}(V)$ contains at least one proximal element. We will need a version of this lemma that involves simultaneously finitely many representations.

Lemma 5.25. *Let $\mathbb{K}$ be a local field, let Γ be a semigroup and, for all positive integers $i \leq s$, let $\rho_i : \Gamma \to \mathrm{GL}(V_i)$ be representations of Γ in finite dimensional $\mathbb{K}$-vector spaces V_i that are strongly irreducible and proximal. Then there exists g in Γ such that, for all $i \leq s$, the element $\rho_i(g)$ is proximal.*

Moreover, for any nonzero endomorphism $q_i \in \mathrm{End}(V_i)$, one can choose such a g in Γ such that $q_i(V^+_{i,g}) \not\subset V^<_{i,g}$.

Here the notations $V^+_{i,g}$ and $V^<_{i,g}$ are shorthands for the attracting line of $\rho_i(g)$ and for its invariant complementary hyperplane. They were defined in Section 3.1.

Proof. Let $V := \oplus_{i\leq s} V_i$. We can assume that Γ is included in $\mathrm{GL}(V)$ and that the representations ρ_i are the restrictions to V_i. Replacing if necessary Γ by a finite index subgroup, we can also assume, thanks to Lemma 5.21 and to the strong irreducibility of V, that Γ is Zariski connected. For $i = 1, \dots, s$, let $(\gamma_{i,p})_{p\geq 1}$ be a sequence of elements of Γ and $(\lambda_{i,p})_{p\geq 1}$ be a sequence of scalars such that the limit in $\mathrm{End}(V_i)$

$$\pi_i := \lim_{p\to\infty} \lambda_{i,p}\, \rho_i(\gamma_{i,p})$$

exists and is a rank one operator. Set, for $p \geq 1$,

$$g_p := h_0\, \gamma_{1,p}\, h_1\, \gamma_{2,p}\, h_2 \cdots \gamma_{s,p}\, h_s \in \Gamma$$

where the elements $h_0, \dots, h_s \in \Gamma$ will be chosen later. We will find our element g among these g_p. Indeed, there exists a sequence $S \subset \mathbb{N}$ and sequences $(\lambda_{i,j,p})_{p\in S}$ of scalars, for $i, j \leq s$, such that the limit in $\mathrm{End}(V_i)$

$$\pi_{i,j} := \lim_{p\in S} \lambda_{i,j,p}\, \rho_i(\gamma_{j,p})$$

exists and is nonzero and such that $\lambda_{i,i,p} = \lambda_{i,p}$. By assumption, for $i \leq s$, the limits $\pi_{i,i}$ are rank one operators. Hence, for any $i \leq s$, the following operators

$$\tau_i := \rho_i(h_0)\, \pi_{i,1}\, \rho_i(h_1)\, \pi_{i,2}\, \rho_i(h_2) \cdots \pi_{i,s}\, \rho_i(h_s)\,.$$

have rank at most one.

Since the representations V_i are irreducible, for any $i \leq s$, one can choose elements $h_0, \dots, h_s$ in Γ in such a way that,

$$\mathrm{Im}\, \tau_i \not\subset \mathrm{Ker}\, \tau_i \ \text{ and } \ q_i(\mathrm{Im}\, \tau_i) \not\subset \mathrm{Ker}\, \tau_i. \tag{5.2}$$

Since the semigroup Γ is Zariski connected, by Lemma 5.21, this group is also Zariski irreducible, and one can choose the elements $h_0, \dots, h_s$ in Γ such that (5.2) is valid simultaneously for all $i \leq s$. Now setting $\lambda'_{i,p} = \prod_{j\leq s} \lambda_{i,j,p}$ for any $i \leq s$ and p in S one gets

$$\lambda'_{i,p}\, \rho_i(g_p) \xrightarrow[p\to\infty]{} \tau_i \quad \text{in} \quad \mathrm{End}(V_i).$$

Reasoning as in the proof of Lemma 3.1, for $p \in S$ large enough, we deduce that, for any $i \leq s$, the element $\gamma := g_p$ acts proximally in V_i and satisfies $q_i(V^+_{i,\gamma}) \not\subset V^<_{i,\gamma}$. □

The following corollary tells us that many elements of Γ are simultaneously proximal in all the V_i's.

Corollary 5.26. *Let $\mathbb{K}$ be a local field and for $i \leq s$, let V_i be a finite dimensional $\mathbb{K}$-vector space and $q_i \in \mathrm{End}(V_i)$ be a nonzero endomorphism. Let $\Gamma \subset \prod_{i\leq s} \mathrm{GL}(V_i)$ be a Zariski connected subsemigroup* *such that, for all $i \le s$, Γ is irreducible and proximal in V_i. Then the set*

$$\Gamma' := \{g \text{ in } \Gamma \mid \text{for all } i \le s,\ g \text{ is proximal in } V_i \text{ and } q_i(V^+_{i,g}) \not\subset V^<_{i,g}\}$$

is Zariski dense in Γ.

Proof. Denote by $\rho_i : G \to \mathrm{GL}(V_i)$ the restriction map. According to Lemma 5.25, there exists at least one element γ_0 in Γ'. For any $i \le s$, there exists a sequence, $(\lambda_{i,p})_{p\ge 1}$ of scalars such that the limit in $\mathrm{End}(V_i)$

$$\pi_i := \lim_{p\to\infty} \lambda_{i,p}\, \rho_i(\gamma_0^p)$$

exists and is a rank-one endomorhism of V_i. Since the representations V_i are irreducible, for all $i \le s$ the set

$$\Gamma_{(i)} := \{\gamma \in \Gamma \mid \pi_i \rho_i(\gamma)\pi_i \neq 0\}$$

is a non empty Zariski open subset of Γ. Since the semigroup Γ is Zariski connected, by Lemma 5.21, this group is also Zariski irreducible and the intersection $\Gamma'' := \cap_{i\le s}\Gamma_{(i)}$ is also a non empty Zariski open subset of Γ. Reasoning as in the proof of Lemma 3.1, we deduce that, for any element γ in Γ'', for n large, the element $\gamma_0^n\gamma\gamma_0^n$ belongs to Γ'. Since, by Lemma 5.15, the Zariski closure of a semigroup is always a group, for every integer $n \in \mathbb{Z}$ the element $\gamma_0^n\gamma\gamma_0^n$ belongs to the Zariski closure of Γ'. In particular the element γ belongs to the Zariski closure of Γ'. This proves that Γ' is Zariski dense in Γ. □

5.6. Loxodromic and proximal elements.

We explain now that being loxodromic can be interpreted as being proximal in suitable representations.

Lemma 5.27. *Let $G = \mathrm{SL}(d,\mathbb{R})$. An element g of G is loxodromic if and only if, for all $1 \le i < d$, the element $\wedge^i g$ is proximal in $\wedge^i\mathbb{R}^d$.*

We recall that a basis of the exterior product $\wedge^i\mathbb{R}^d$ is given by the elements $e_E = e_{j_1} \wedge \cdots \wedge e_{j_i}$ where $E = \{j_1, \ldots, j_i\}$ runs among the subsets of $\{1, \ldots, s\}$ with cardinality i. We recall also that the endomorphism $\wedge^i g$ is given by,

$$\wedge^i g\,(v_1 \wedge \cdots \wedge v_i) = (gv_1) \wedge \cdots \wedge (gv_i)\ ,$$

for all vectors v_j in $\mathbb{R}^d$.

Proof. Indeed, for $1 \le i < d$, the moduli of the eigenvalues of $\wedge^i g$ are the product $\mu_E = \prod_{j\in E} \lambda_j(g)$ where E runs among the subsets of $\{1, \ldots, s\}$ with cardinality i. This product is maximal when $E = \{1, \ldots, i\}$. The element $\wedge^i g$ is proximal in $\wedge^i\mathbb{R}^d$ if and only if no other subset E' achieves this maximum. This is the case if and only if one has the strict inequality $\lambda_i(g) > \lambda_{i+1}(g)$. □

We can now prove the existence of loxodromic elements in Zariski dense subsemigroups Γ of $\mathrm{SL}(d, \mathbb{R})$

Proof of Proposition 5.11. For $1 \leq i < d$, the action of the group $G = \mathrm{SL}(d, \mathbb{R})$ on $\wedge^i \mathbb{R}^d$ is proximal. By Lemma 5.23, since Γ is Zariski dense in G, the action of Γ on $\wedge^i \mathbb{R}^d$ is also proximal. By Lemma 5.25, there exists an element g in Γ such that, for all $i < d$, the element $\wedge^i g$ is proximal. By Lemma 5.27, such an element g is loxodromic in G. By Corollary 5.26, these loxodromic elements are Zariski dense in G. □

Our aim now is to extend Proposition 5.11 to semisimple real Lie groups.

5.7. Semisimple real Lie groups.

We recall without proof basic definitions and basic facts on semisimple real Lie groups (see [**64**]). We use the language of algebraic groups and root systems which is very convenient to deal with semisimple Lie groups.

We gather here more notations than what is needed to prove the existence of loxodromic elements. In particular, we will discuss the Cartan projection, the Iwasawa cocycle, the Jordan projection and the parabolic subgroups. We expect that this section will help the reader to feel more confortable when we will need to introduce similar notions in the context of $\mathcal{S}$-adic Lie groups in Chapter 7.

5.7.1. *Algebraic groups and maximal compact subgroups.* Let G be an *algebraic* real Lie group. Pedantically, this means that G is the group of real points $G = \mathbf{G}(\mathbb{R})$ of an algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. In this chapter and the next one, we will abusively think of G as a Zariski closed subgroup of a group $\mathrm{SL}(d, \mathbb{R})$ for some $d \geq 1$. For instance $\mathrm{GL}(d, \mathbb{R})$ is an algebraic real Lie group since it can be seen as the stabilizer in $\mathrm{SL}(d+1, \mathbb{R})$ of the decomposition $\mathbb{R}^{d+1} = \mathbb{R}^d \oplus \mathbb{R}$. An *algebraic morphism* $\varphi : G \to H$ between two algebraic real Lie groups is a map which is both a group morphism and a polynomial map.

We say that G is a *semisimple* algebraic Lie group, if it does not contain an infinite abelian normal subgroup. We say that G is a *connected* algebraic Lie group if it is Zariski connected.

We will assume in this chapter that G is a semisimple connected algebraic Lie group. Important examples are $G = \mathrm{SL}(d, \mathbb{R})$, $\mathrm{SL}(d, \mathbb{C})$,

$\mathrm{SL}(d,\mathbb{H})$, $\mathrm{SO}(p,q)$, $\mathrm{Sp}(d,\mathbb{R})$, $\mathrm{SU}(p,q)$,.... The full list, up to finite covers and finite products, can be seen in Helgason's book [**64**].

The group G contains a maximal compact subgroup K and all such subgroups are conjugate. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ be the Lie algebra of K. We introduce the *Killing form* on $\mathfrak{g}$ given by

$$\mathrm{Killing}(x,y) = \mathrm{tr}(\mathrm{ad}x\ \mathrm{ad}y).$$

Let $\mathfrak{s}$ be the orthogonal subspace of $\mathfrak{k}$ for the Killing form. This Killing form is negative definite on $\mathfrak{k}$, is positive definite on $\mathfrak{s}$ and one has the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}. \tag{5.3}$$

5.7.2. *Cartan subspaces and restricted roots.* For x in $\mathfrak{g}$, we denote by $\mathrm{ad}x$ the endomorphism of $\mathfrak{g}$ given by $\mathrm{ad}x(y) = [x,y]$ for all y in $\mathfrak{g}$. An element x of $\mathfrak{g}$ is said to be *hyperbolic* if $\mathrm{ad}\,x)$ is diagonalizable over $\mathbb{R}$. A *Cartan subspace* of $\mathfrak{g}$ is a commutative subalgebra $\mathfrak{a}$ whose elements are hyperbolic and which is maximal for these properties. All Cartan subspaces are conjugate under G and a maximal commutative algebra in $\mathfrak{s}$ is a Cartan subspace. Let us choose such a Cartan subspace $\mathfrak{a} \subset \mathfrak{s}$. We denote by A the connected algebraic subgroup of G with Lie algebra $\mathfrak{a}$. It does exist (see [**22**]). By definition, the *real rank* of G is the dimension of $\mathfrak{a}$. Endowed with the Killing form, the space $\mathfrak{a}$ and its dual space $\mathfrak{a}^*$ are Euclidean.

For every algebraic character α of the algebraic group A, we still denote by α its differential (in the following chapters, this differential will also be denoted by α^ω, see Section 7.2). It belongs to the dual space $\mathfrak{a}^\star$. Let us diagonalize $\mathfrak{g}$ under the adjoint action of A or $\mathfrak{a}$. One denotes by Σ the set of *restricted roots*, i.e. the set of nontrivial weights for this action:

$$\Sigma = \{\alpha \in \mathfrak{a}^* \smallsetminus \{0\} \mid \mathfrak{g}^\alpha \neq \{0\}\} \ \text{ where}$$

$$\mathfrak{g}^\alpha := \{y \in \mathfrak{g} \mathrel{/} \forall x \in \mathfrak{a}\ ,\ \mathrm{ad}x(y) = \alpha(x)y\}$$

is the root space associated to α. This finite set Σ is a *root system* in the Euclidean space $\mathfrak{a}^*$. Note that it is not always a *reduced root system.* One has the decomposition

$$\mathfrak{g} = \mathfrak{z} \oplus (\oplus_{\alpha\in\Sigma}\, \mathfrak{g}^\alpha),$$

where $\mathfrak{z}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.

The group G is said to be *split* if one has $\mathfrak{z} = \mathfrak{a}$. This happens if and only if all the root spaces $\mathfrak{g}^\alpha$ are 1-dimensional.

5.7.3. *Simple restricted roots and Weyl chambers.* Let $\Sigma^+ \subset \Sigma$ be a choice of *positive roots* and $\Pi \subset \Sigma^+$ be the subset of *simple roots.* This subset Π is a basis of $\mathfrak{a}^\star$. Let

$$\mathfrak{u} := \oplus_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha \quad \text{and let} \quad \mathfrak{p} = \mathfrak{z} \oplus \mathfrak{u}$$

be the *minimal parabolic subalgebra* associated to Σ^+. Its normalizer is the *minimal parabolic subgroup* $P := N_G(\mathfrak{p})$ associated to Σ^+. The Lie algebra of P is equal to $\mathfrak{p}$. Let

$$\mathfrak{a}^+ := \{x \in \mathfrak{a} \ / \ \forall\alpha \in \Sigma^+ \ , \ \alpha(x) \geq 0\}$$

be the corresponding *Weyl chamber* in $\mathfrak{a}$.

5.7.4. *Cartan projection.* One has the *Cartan decomposition*

$$G = K\exp(\mathfrak{a}^+)K.$$

For g in G one denote by $\kappa(g) \in \mathfrak{a}^+$ the *Cartan projection* of g, that is the unique element of $\mathfrak{a}^+$ such that

$$g \in K\exp(\kappa(g))K.$$

Remark 5.28. Here is the geometric interpretation of the Cartan projection. The quotient G/K endowed with the G-invariant Riemannian metric given by the restriction of the Killing form to $\mathfrak{s}$ is the so-called *Riemannian symmetric space associated to G*. Let m_0 be the point of G/K whose stabilizer is K. In this space G/K the maximal flat totally geodesic subspaces are exactly the translates $g\exp(\mathfrak{a})m_0$ with g in G. They are called *apartments.* The subsets $g\exp(\mathfrak{a}^+)m_0$ are called *chambers* with vertex gm_0. The Cartan decomposition tells us that any two points of G/K belong simultaneously to at least one apartment. More precisely, it tells us that, when k varies in K, the chambers $k\exp(\mathfrak{a}^+)m_0$ form a covering of G/K. When G has real rank 1, it just tells us that any two points of G/K can be joined by a geodesic. The distance on G/K is also given by the formula

$$d(gm_0, hm_0) = \|\kappa(h^{-1}g)\| \ .$$

The fact that the right-hand side defines a distance follows from the definitions and the following inequality which will be proved in Corollary 5.34

$$\|\kappa(g_1g_2)\| \leq \|\kappa(g_1)\| + \|\kappa(g_2)\| \ , \ \text{ for all } g_1,\, g_2 \text{ in } G. \tag{5.4}$$

5.7.5. *Iwasawa cocycle.* Let Z be the centralizer of $\mathfrak{a}$ in G and $M := Z \cap K$. We denote by U the connected algebraic subgroup of G with Lie algebra $\mathfrak{u}$. It does exist and is a maximal unipotent subgroup of G One has the *Iwasawa decomposition*

$$G = K\exp(\mathfrak{a})U \ .$$

More precisely, the product map $K \times (\exp \mathfrak{a}) \times U \to G$ is a homeomorphism. Note that $\exp(\mathfrak{a})$ is equal to the *analytical connected component* A_e of A. One also has the equality $P = M\exp(\mathfrak{a})U$. Let

$$\mathcal{P} = G/P$$

be the *flag variety* of G and, for any g in G and η in $\mathcal{P}$, if $\eta = kP$ for some k in K, let $\sigma(g, \eta)$ be the unique element of $\mathfrak{a}$ such that

$$gk \in K \exp(\sigma(g, \eta))\, U.$$

LEMMA 5.29. *The continuous map $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is a cocycle.*

This cocycle is called the *Iwasawa cocycle* by group theoretists and the *Busemann cocycle* by geometers.

PROOF. For g, g' in G and $\eta = kP$ in $\mathcal{P}$ with k in K, let $k' \in K$ and $x, x' \in \mathfrak{a}$ be such that

$$g'k \in k' \exp(x')U \quad \text{and} \quad gk' \in K \exp(x)U.$$

We have $\sigma(g', \eta) = x'$ and $\sigma(g, g'\eta) = x$ and

$$gg'k \in gk' \exp(x')U \subset K \exp(x)U \exp(x')U = K \exp(x + x')U,$$

hence $\sigma(gg', \eta) = x + x'$ and σ satisfies the cocycle property (2.6). $\square$

REMARK 5.30. Here is the geometric interpretation of the Iwasawa cocycle. Let G/K be the associated Riemannian symmetric space and m_0 the point of G/K whose stabilizer is K. We fix x in $\mathfrak{a}^+$ of norm 1. For $\eta = kP \in \mathcal{P}$, we introduce the geodesic ray on G/K given by $m_t := k \exp(tx)m_0$. The geometric interpretation of the Iwasawa cocycle comes from the equality

$$< x, \sigma(g, \eta) >= \lim_{t\to\infty} d(g^{-1}m_0, m_t) - d(m_0, m_t). \tag{5.5}$$

The right hand side of this equality is the Buseman cocycle (see for instance [**6**, Sect. II.2] or [**19**, Sect. 2.4] in the context of hyperbolic groups). When x belongs to the interior of $\mathfrak{a}^+$, this equality (5.5) follows from the definitions and the following stronger equality which will be proved in Corollary 5.34

$$\sigma(g, \eta) = \lim_{t\to\infty} \kappa(gke^{tx}) - tx. \tag{5.6}$$

5.7.6. *Jordan projection.* An element g of G is said to be *semisimple* if it is diagonalizable over $\mathbb{C}$. It is said to be *elliptic* if it is semisimple with eigenvalues of modulus one. It is said to be *hyperbolic* if it is semisimple with positive real eigenvalues. It is said to be *unipotent* if all its eigenvalues are equal to 1. These notions do not depend on the algebraic embedding of G as a linear group.

For every g in G, one has a unique decomposition, called the *Jordan decomposition of* g, as a product $g = g_e g_h g_u$ of commuting elements, where g_e is elliptic g_h is hyperbolic and g_u is unipotent. A striking property, valid more generally for any algebraic real Lie group, is that those three components g_e, g_h g_u still belong to G. Another useful property is the following fact. Let $\varphi : G \to H$ be an algebraic morphism between two algebraic real Lie groups. Then the image $\varphi(g)$ of a semisimple (resp. elliptic, hyperbolic, or unipotent) element g of G is also a semisimple (resp. elliptic, hyperbolic, or unipotent) element of H. In particular, the Jordan decomposition does not depend on the representation of G as a group of matrices.

We recall that G is assumed here to be a connected semisimple real algebraic Lie group. The hyperbolic component g_h of g is then conjugated under G to an element $\exp(\lambda(g))$ with $\lambda(g) \in \mathfrak{a}^+$. This element $\lambda(g)$ is uniquely determined and the map $\lambda : G \to \mathfrak{a}^+$ is called the *Jordan projection*.

Remark 5.31. The geometric interpretation of the Jordan projection comes from the equality, for all g in G, m in G/K

$$\|\lambda(g)\| = \lim_{n\to\infty} \tfrac{1}{n} d(g^n m, m) \tag{5.7}$$

The right hand side of this equality does not depend on m and is called the *stable length* of g. This equality (5.7) follows from the definitions and the following equality which will be proved in Corollary 5.34

$$\lambda(g) = \lim_{n\to\infty} \tfrac{1}{n}\kappa(g^n). \tag{5.8}$$

Another useful formula, that we will not use, is

$$\|\lambda(g)\| = \min_{m\in G/K} d(gm, m),$$

Moreover, when g is hyperbolic, there exists at least one g-invariant chamber in G/K and the action of g on such a chamber is nothing but a translation by the element $\lambda(g)$.

In order to illustrate all these notions, we describe now their meaning for the two examples $G = \mathrm{SL}(d, \mathbb{R})$ and $G = \mathrm{SO}(p, q)$.

5.7.7. *Example:* $G = \mathrm{SL}(d, \mathbb{R})$. Let $V = \mathbb{R}^d$, let $e_1, \ldots, e_d$ be its standard basis, and let $G = \mathrm{SL}(d, \mathbb{R})$. The Lie algebra $\mathfrak{g}$ of G is the space of matrices with zero trace

$$\mathfrak{g} = \{f \in \mathrm{End}(\mathbb{R}^d) \mid \mathrm{tr}(f) = 0\}.$$

One can choose the maximal compact subgroup K to be the subgroup of orthogonal matrices $K = \mathrm{SO}(d)$. As in Section 5.2, one can choose the Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$ to be the subspace of diagonal matrices

$$\mathfrak{a} = \{x = \mathrm{diag}(x_1, \dots, x_d) \;/\; x_1 + \cdots x_d = 0\}.$$

Hence the real rank of G is $d-1$. One can choose the Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{g}$ to be the set of elements of $\mathfrak{a}$ with decreasing coefficients

$$\mathfrak{a}^+ = \{x \in \mathfrak{a} \;/\; x_1 \geq \cdots \geq x_d\}.$$

The group A is then

$$A = \{a = \mathrm{diag}(a_1, \dots, a_d) \;/\; a_i \neq 0 \;,\; a_1 \cdots a_d = 1\}.$$

The set Σ of restricted root is

$$\Sigma = \{\varepsilon_i - \varepsilon_j \;,\; i \neq j \;,\; 1 \leq i, j \leq d\}$$

where $\varepsilon_i \in \mathfrak{a}^\star$ is given by $\varepsilon_i(x) = x_i$. For $i \neq j$, the root spaces $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ are 1-dimensional and are spanned by the elementary matrices $E_{i,j} = e_j^\star \otimes e_i$. The centralizer of $\mathfrak{a}$ is $\mathfrak{z} = \mathfrak{a}$. Hence the group G is split. The set of positive roots of $\mathfrak{g}$ may be chosen to be

$$\Sigma^+ = \{\varepsilon_i - \varepsilon_j \;,\; 1 \leq i < j \leq d\},$$

and the set of simple roots is then

$$\Pi = \{\varepsilon_i - \varepsilon_{i+1} \;,\; 1 \leq i < d\}.$$

The minimal parabolic subgroup P and its unipotent radical U are

$$P = \left\{ \begin{pmatrix} * & * & * \\ & \ddots & * \\ 0 & & * \end{pmatrix} \in G \right\} \quad , \quad U = \left\{ \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{pmatrix} \right\} \quad .$$

The group P is the stabilizer in G of the maximal flag

$$V_1 \subset \ldots \subset V_d$$

where V_i is the vector subspace of $\mathbb{R}^d$ spanned by $e_1, \dots, e_i$. Hence the flag variety $\mathcal{P}$ of G is the set of all maximal flags of V.

For g in G, the *Cartan decomposition of* g is nothing but the *polar decomposition* of g. It expresses g as a product $g = k_1 e^{\kappa(g)} k_2$ with k_1, k_2 in K and $\kappa(g)$ in $\mathfrak{a}^+$. This element

$$\kappa(g) = \mathrm{diag}(\log \kappa_1(g), \dots, \log \kappa_d(g))$$

is the *Cartan projection* of g. Here one has $\kappa_1(g) = \|g\|$, where $\|g\|$ is the norm of g as an endomorphism of the Euclidean space $\mathbb{R}^d$ (see Section 3.1). For $i \geq 1$, $\kappa_i(g)$ is the i^{th}-*singular value* of g, i.e.

$$\kappa_i(g) = \frac{\| \wedge^i g\|}{\| \wedge^{i-1} g\|}.$$

Here, again, $\| \wedge^i g\|$ is the norm of $\wedge^i g$ as an endomorphism of the Euclidean space $\wedge^i \mathbb{R}^d$. The Euclidean norm on $\wedge^i \mathbb{R}^d$ is the standard one, i.e. it is the one for which the vectors $e_{\ell_1}\wedge\cdots\wedge e_{\ell_i}$, for $\ell_1<\ldots<\ell_i$, form an orthonormal basis of $\wedge^i\mathbb{R}^d$.

5.7.8. *Example:* $G=\mathrm{SO}(p,q)$. Let $1\leq p\leq q$ with $d=p+q\geq 3$ and let $S_{p,q}$ be the symmetric matrix of size d,

$$S_{p,q}=\left\{\begin{pmatrix} 0 & 0 & J_p\\ 0 & I_{q-p} & 0\\ J_p & 0 & 0\end{pmatrix}\right\}\quad\text{where}\quad J_p=\left\{\begin{pmatrix} 0 & 0 & 1\\ 0 & \iddots & 0\\ 1 & 0 & 0\end{pmatrix}\right\}$$

is the antidiagonal matrix of size p and I_{q-p} is the identity matrix of size $q-p$. The group $G=\mathrm{SO}(p,q)$ is the group

$$G=\{g\in\mathrm{SL}(d,\mathbb{R})\mid g\,S_{p,q}\,{}^tg=S_{p,q}\}.$$

Its Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g}=\{f\in\mathrm{End}(\mathbb{R}^d)\mid f\,S_{p,q}+S_{p,q}\,{}^tf=0\}.$$

One can choose the maximal compact subgroup K to be the subgroup of orthogonal matrices $K=\mathrm{SO}(d)\cap G\simeq S(O(p)\times O(q))$. One can choose the Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$ to be the subspace of diagonal matrices

$$\mathfrak{a}=\{x=\mathrm{diag}(x_1,\ldots,x_p,0,\ldots,0,-x_p,\ldots,-x_1)\}.$$

Hence the real rank of G is p. One can choose the Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{g}$ to be the set of elements of $\mathfrak{a}$ with decreasing coefficients

$$\mathfrak{a}^+=\{x\in\mathfrak{a}\ /\ x_1\geq\cdots\geq x_p\geq 0\}.$$

The group A is then

$$A=\{a=\mathrm{diag}(a_1,\ldots,a_p,1,\ldots,1,a_p^{-1},\ldots,a_1^{-1})\ /\ a_i\neq 0\}.$$

The set Σ of restricted root is

$$\Sigma=\{\pm\varepsilon_i\ ,1\leq i\leq p\}\cup\{\pm\varepsilon_i\pm\varepsilon_j\ ,1\leq i<j\leq p\}\ \text{ when }\ p>q\,,$$
$$\Sigma=\{\pm\varepsilon_i\pm\varepsilon_j\ ,1\leq i<j\leq p\}\ \text{ when }\ p=q\,,$$

where $\varepsilon_i\in\mathfrak{a}^\star$ is given by $\varepsilon_i(x)=x_i$. For $i\neq j$, the root spaces $\mathfrak{g}_{\pm\varepsilon_i\pm\varepsilon_j}$ are 1-dimensional but the root spaces $\mathfrak{g}_{\pm\varepsilon_i}$ have dimension $q-p$. The centralizer of $\mathfrak{a}$ is $\mathfrak{z}=\mathfrak{a}\oplus\mathfrak{m}$ where $\mathfrak{m}=\mathfrak{so}(q-p)$ is the Lie algebra of antisymmetric matrices of size $q-p$. Hence the group G is split if an only if $q=p$ or $p+1$. The set of positive roots of $\mathfrak{g}$ may be chosen to be

$$\Sigma^+=\{\varepsilon_i\ ,1\leq i\leq p\}\cup\{\varepsilon_i\pm\varepsilon_j\ ,\ 1\leq i<j\leq p\},\ \text{ when }\ p>q\,,$$
$$\Sigma^+=\{\varepsilon_i\pm\varepsilon_j\ ,1\leq i<j\leq p\},\ \text{ when }\ p=q\,,$$

and the set of simple roots is then

$$\Pi=\{\varepsilon_i-\varepsilon_{i+1}\ ,\ 1\leq i<p\}\cup\{\varepsilon_p\},\ \text{ when }\ p>q\,,$$
$$\Pi=\{\varepsilon_i-\varepsilon_{i+1}\ ,1\leq i<p\}\cup\{\varepsilon_{p-1}+\varepsilon_p\},\ \text{ when }\ p=q\,,$$

The minimal parabolic subgroup P is the stabilizer in G of the maximal isotropic flag

$$V_1 \subset \ldots \subset V_p$$

where V_i is still the vector subspace of $\mathbb{R}^d$ spanned by $e_1, \ldots, e_i$. Hence the flag variety $\mathcal{P}$ of G is the set of all maximal isotropic flags of V.

5.8. Representations of G.

For $G = \mathrm{SL}(d, \mathbb{R})$, the representations $\wedge^i V$ in Section 5.6 played a crucial role in the proof of Proposition 5.11. For a semisimple real Lie group G, they will be replaced by the representations V_α that we will introduce below.

Let G be a connected algebraic semisimple real Lie group. We keep the notations of Section 5.7.

Let (V, ρ) be an algebraic representation of G in a finite dimensional real vector space V. This means that $\rho : G \to \mathrm{GL}(V)$ is an algebraic morphism. For every character χ of $\mathfrak{a}$, we set

$$V^\chi := \{v \in V \ / \ \forall x \in \mathfrak{a} \ , \ \rho(x)v = \chi(x)v\}$$

to be the corresponding eigenspace. Let

$$\Sigma(\rho) := \{\chi \ / \ V^\chi \neq 0\}$$

be the set of *restricted weights* of V. Most of the time, we will just say *weights* of V. Since the group $\rho(A)$ is commutative and its elements are diagonalizable over $\mathbb{R}$, one has

$$V = \bigoplus\nolimits_{\chi \in \Sigma(\rho)} V^\chi \ .$$

We endow $\Sigma(\rho)$ with the partial order:

$$\chi_1 \leq \chi_2 \iff \chi_2 - \chi_1 \ \text{ is a sum of positive roots.} \tag{5.9}$$

We assume ρ to be irreducible. The set $\Sigma(\rho)$ has then a largest element χ called the *highest restricted weight* of V. The corresponding eigenspace is the space

$$V^U := \{v \in V \mid Uv = v\}.$$

The representation ρ is proximal if and only if $\dim V^U = 1$. This is always the case when G is split.

The dimension $r_{V,G} := \dim V^U$ is the proximal dimension of G in V. The map $g \mapsto gV^U$ factors as a map from the flag variety to the Grassmann variety

$$\begin{array}{rcl} \mathcal{P} & \to & \mathbb{G}_{r_{V,G}}(V) \\ \eta = gP & \mapsto & V_\eta := gV^U. \end{array} \tag{5.10}$$

Lemma 5.32. *Let G be a connected algebraic semisimple real Lie group. For every α in Π, there exists a proximal irreducible algebraic representation (ρ_α, V_α) of G whose highest weight χ_α is a multiple of the fundamental weight ϖ_α associated to α.*

These weights $(\chi_\alpha)_{\alpha\in\Pi}$ form a basis of the dual space $\mathfrak{a}^*$.

Moreover, the product of the maps given by (5.10)

$$\mathcal{P} \;\to\; \textstyle\prod_{\alpha\in\Pi}\mathbb{P}(V_\alpha)$$

is an embedding of the flag variety in this product of projective spaces.

This condition on χ_α means that χ_α is orthogonal to β for every simple root $\beta \neq \alpha$. It implies that the restricted weights of ρ_α are χ_α, $\chi_\alpha - \alpha$ and weights of the form $\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta\, \beta$ with n_β non-negative integers.

Proof. See [**121**]. □

5.9. Interpretation with representations of G.

In this section, we give an interpretation of the Cartan projection, the Iwasawa cocycle and the Jordan projection in terms of representations of G.

We keep the notations of Sections 5.7 and 5.8, and we relate now κ, σ and λ to the representations of G. The Cartan projection controls the norm of the image matrices in all representations, the Jordan projection controls their spectral radii and the Iwasawa cocycle controls the growth of the highest weight vectors.

The following Lemma should be seen as a dictionary wich translates the language of the geometry of G into the language of the representations of G and vice-versa.

Lemma 5.33. *Let G be a connected algebraic semisimple real Lie group and (V, ρ) be an irreducible representation of G with highest weight χ.*

a) There exists a good norm on V i.e. a K-invariant Euclidean norm such that, for all a in A, $\rho(a)$ is a symmetric endomorphism.

b) For such a good norm, one has, for all g in G, η in $\mathcal{P}$ and v in V_η,

i) $\chi(\kappa(g)) = \log(\|\rho(g)\|)$,

ii) $\chi(\lambda(g)) = \log(\lambda_1(\rho(g)))$,

iii) $\chi(\sigma(g,\eta)) = \log \dfrac{\|\rho(g)v\|}{\|v\|}$.

Proof. *a*) The group G is the group $\mathbf{G}_\mathbb{R}$ of real point of an algebraic group. We let $\mathbf{G}_\mathbb{C}$ be the corresponding group of complex points so that we get a representation $\mathbf{G}_\mathbb{C} \to \mathrm{GL}(V_\mathbb{C})$ where $V_\mathbb{C} = \mathbb{C}\otimes_\mathbb{R} V$.

Using the decomposition (5.3), one introduces the Lie subalgebra $\mathfrak{g}' := \mathfrak{k} + i\mathfrak{s} \subset \mathfrak{g}_{\mathbb{C}}$. Since the Killing form is negative definite on $\mathfrak{g}'$, this Lie algebra $\mathfrak{g}'$ is the Lie algebra of a compact subgroup G' of $\mathbf{G}_{\mathbb{C}}$ (see [**64**, Chap.V §2] for more details). As in Lemma 5.18, we choose a hermitian scalar product on $V_{\mathbb{C}}$ that is G'-invariant. Then, the Euclidean norm that it induces on V is good. Indeed this norm is clearly K-invariant, and the element $\rho(x)$ for x in $\mathfrak{a} \subset \mathfrak{s}$ are symmetric since, by construction, they are both real and hermitian.

b) For x in $\mathfrak{a}^+$, the eigenvalues of $\rho(e^x)$ are exactly the real numbers $e^{\chi'(x)}$ where χ' runs among the weights of V. Since χ is the largest weight for the order (5.9), one always has $\chi(x) \geq \chi'(x)$. Hence one has

$$\log \lambda_1(\rho(e^x)) = \log \|\rho(e^x)\| = \chi(x).$$

This proves that, for any g in G, one has

$$\log(\|\rho(g)\|) = \log \|\rho(e^{\kappa(g)})\| = \chi(\kappa(g)) \;\text{ and}$$

$$\log(\lambda_1(\rho(g))) = \log \lambda_1(\rho(e^{\lambda(g)})) = \chi(\lambda(g)) \;.$$

In the same way, for x in $\mathfrak{a}$ and v_0 in V^U, one has

$$\log \frac{\|\rho(e^x)v_0\|}{\|v_0\|} = \chi(x).$$

Hence, when $\eta = kP$ with k in K, one writes $v = \rho(k)v_0$ and $gk \in Ke^xU$ with $x = \sigma(g,\eta)$, and one computes

$$\log \frac{\|\rho(g)v\|}{\|v\|} = \log \frac{\|\rho(e^{\sigma(g,\eta)})v_0\|}{\|v_0\|} = \chi(\sigma(g,\eta)),$$

as required. □

As a corollary, we get a proof of Formulas (5.4), (5.6) and (5.8) relating Cartan projection, Iwasawa cocycle and Jordan projection, that we used in Section 5.7 to understand the geometric interpretation of these notions.

Corollary 5.34. *Let G be a connected algebraic semisimple real Lie group.*
a) *One has the inequality, for all g_1, g_2 in G,*

$$\|\kappa(g_1g_2)\| \leq \|\kappa(g_1)\| + \|\kappa(g_2)\|.$$

b) *One has the equality, for all g in G, $\eta = kP \in \mathcal{P}$ with k in K, and x in the interior of $\mathfrak{a}^+$ of norm* 1

$$\sigma(g,\eta) = \lim_{t\to\infty} \kappa(gk\exp(tx)) - tx.$$

c) One has the equality, for all g in G,

$$\lambda(g) = \lim_{n\to\infty} \tfrac{1}{n}\kappa(g^n).$$

We fix once for all a family of representations $(\rho_\alpha, V_\alpha)_{\alpha\in\Pi}$ of G as in Lemma 5.32, and we equip each of them with a good norm.

PROOF. We recall from Lemma 5.32 that the family of highest weights $(\chi_\alpha)_{\alpha\in\Pi}$ is a basis of the dual space $\mathfrak{a}^*$.

a) For all α in Π, one has the inequality

$$\|\rho_\alpha(g_1g_2)\| \leq \|\rho_\alpha(g_1)\|\,\|\rho_\alpha(g_2)\|.$$

Hence using Lemma 5.33, one has the inequality

$$\chi_\alpha(\kappa(g_1g_2)) \leq \chi_\alpha(\kappa(g_1) + \kappa(g_2)).$$

Since the vectors χ_α are multiples of the fundamental weights, for any x in $\mathfrak{a}^+$, the dual linear form on $\mathfrak{a}$, $y \mapsto \langle x, y\rangle$ belongs to the convex cone of $\mathfrak{a}^*$ spanned by the vectors χ_α. One deduces

$$\|\kappa(g_1g_2)\|^2 \leq \langle \kappa(g_1g_2), \kappa(g_1) + \kappa(g_2)\rangle$$

and hence

$$\|\kappa(g_1g_2)\| \leq \|\kappa(g_1) + \kappa(g_2)\| \leq \|\kappa(g_1)\| + \|\kappa(g_2)\|\ .$$

b) We can assume that $k = e$. According to Lemma 5.32, we only have to check that the image by χ_α of this equality is true, i.e., using Lemma 5.33, we only have to check the equality

$$\log\frac{\|gv_\alpha^+\|}{\|v_\alpha^+\|} = \lim_{t\to\infty}\frac{\|\rho_\alpha(ge^{tx})\|}{\|\rho_\alpha(e^{tx})\|} \tag{5.11}$$

where $v_\alpha^+ \in V_\alpha^U$ is a highest weight vector of V_α. Let π_α be the orthogonal projection on the line V_α^U. Since V is endowed with a good norm, arguing as in *a*), one obtains the equality

$$\pi_\alpha = \lim_{t\to\infty}\frac{\rho_\alpha(e^{tx})}{\|\rho_\alpha(e^{tx})\|}.$$

Formula (5.11) follows then from the simple equality

$$\|\rho_\alpha(g)\pi_\alpha\| = \frac{\|gv_\alpha^+\|}{\|v_\alpha^+\|}.$$

c) As in *b*), using Lemmas 5.32 and 5.33, we only have to check the equality

$$\log\lambda_1(\rho_\alpha(g)) = \lim_{n\to\infty}\tfrac{1}{n}\log\|\rho_\alpha(g)^n\|.$$

which is nothing but the spectral radius formula. $\square$

5.10. Zariski dense semigroups in semisimple Lie groups.

We can now extend Proposition 5.11 to any semisimple real Lie group G, i.e. we can prove the existence of loxodromic elements in any Zariski dense subsemigroup of G.

DEFINITION 5.35. An element g of G is said to be *loxodromic* if $\lambda(g)$ belongs to the interior of $\mathfrak{a}^+$.

THEOREM 5.36. *Let G be a connected algebraic semisimple real Lie group and Γ be a Zariski dense subsemigroup of G. Then the set Γ_{lox} of loxodromic elements of Γ is still Zariski dense.*

The proof uses the following Lemma which generalizes Lemma 5.27.

LEMMA 5.37. *Let G be a connected algebraic semisimple real Lie group. An element g of G is loxodromic if and only if, for all α in Π, the element $\rho_\alpha(g)$ is proximal in V_α.*

PROOF. Recall from Section 5.8 that the weights of $\mathfrak{a}$ in V_α are χ_α, $\chi_\alpha - \alpha$ and other weights of the form $\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta \beta$, where, for β in Π, n_β belongs to $\mathbb{N}$.

In particular, for any x in $\mathfrak{a}^+$, one has the equivalence : the endomorphism $\rho_\alpha(e^x)$ is a proximal endomorphism of V_α if and only if $\alpha(x) > 0$. □

PROOF OF THEOREM 5.36. For α in Π, the action of the group G on the representation (V_α, ρ_α) is proximal. By Lemma 5.23, since Γ is Zariski dense in G, the action of Γ on V_α is also proximal. By Lemma 5.25, there exists an element g in Γ such that, for all α in Π, the element $\rho_\alpha(g)$ is proximal. By Lemma 5.37, such an element g is loxodromic in G. By Corollary 5.26, these loxodromic elements are Zariski dense in G. □

We finish this section by the following two lemmas on loxodromic elements.

The first lemma will be useful in Section 6.7.

LEMMA 5.38. *In a connected algebraic semisimple real Lie group G, every loxodromic element g is semisimple.*

PROOF. Recall that the *Jordan decomposition* of g is the decomposition of g as a product of commuting elements $g = g_e g_h g_u$, where g_e is elliptic, g_h is hyperbolic and g_u is unipotent. After conjugation, we can assume that the component g_h is equal to $\exp(\lambda(g))$. The component g_u can be written as $g_u = \exp(y)$ where y is a nilpotent element of $\mathfrak{g}$ which commutes with $\lambda(g)$. Since the Jordan projection $\lambda(g)$ belongs to the interior of the Weyl chamber $\mathfrak{a}^+$, its centralizer is equal to $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{a}$. Since $\mathfrak{z}$ does not contain nonzero nilpotent element, one has $y = 0$ and the element g is semisimple. $\square$

The second lemma characterizes the loxodromic elements in terms of their action on the flag variety.

LEMMA 5.39. *Let G be a connected algebraic semisimple real Lie group. An element g of G is loxodromic if and only if it has an attracting fixed point ξ_g^+ on the flag variety $\mathcal{P}$ of G.*

Attracting fixed point means that this point ξ_g^+ admits a compact neighborhood b^+ such that, uniformly for ξ in b^+, the powers $g^n(\xi)$ converge to ξ_g^+.

PROOF. If the element g is loxodromic, after conjugation one can assume that $g = me^x$ where $x = \lambda(g)$ belongs to the interior of $\mathfrak{a}^+$ and where m belongs to the centralizer M of $\mathfrak{a}$ in K. The adjoint action of g on $\mathfrak{g}/\mathfrak{p}$ is contracting, hence the base point of $\mathcal{P}$ is an attracting fixed point of g.

Conversely, assume that g has an attracting fixed point in $\mathcal{P}$. After conjugation, one can assume that this point is the base point of $\mathcal{P}$ so that g belongs to the minimal parabolic subgroup P of G, and that the adjoint action of g on $\mathfrak{g}/\mathfrak{p}$ is contracting. The three components g_e, g_h and g_u of the Jordan decomposition of g belong also to P. For each α in Π, the adjoint action of g on the space $(\mathfrak{g}_{-\alpha} \oplus \mathfrak{p})/\mathfrak{p}$ is contracting hence one has $\alpha(\lambda(g)) > 0$. This proves that g is loxodromic. $\square$

6. The Jordan projection of semigroups

We gather in this chapter two key results on Zariski dense subsemigroups of semisimple real Lie groups: the convexity and non-degeneracy of the limit cone (Theorem 6.2) and the density of the group spanned by the Jordan projections (Theorem 6.4). These results will be used to prove the non-degeneracy of the Gaussian law (Proposition 12.19) in the Central Limit Theorem 12.17 and the aperiodicity condition (Proposition 16.1) in the Local Limit Theorem 16.6.

We will focus mainly in this chapter on real Lie groups since these results do not extend to other local fields.

6.1. Convexity and density.

We first state the two main results of this chapter.

We recall a few notations from Section 5.7. We fix a connected algebraic semisimple real Lie group G, a Cartan subspace $\mathfrak{a}$ of its Lie algebra $\mathfrak{g}$ and a Weyl chamber $\mathfrak{a}^+$. We denote by $\lambda : G \to \mathfrak{a}^+$ the Jordan projection and we recall from Definition 5.35 that an element g of G is loxodromic if $\lambda(g)$ belongs to the interior of $\mathfrak{a}^+$.

We recall that, when $G = \mathrm{SL}(d, \mathbb{R})$, the Cartan subspace $\mathfrak{a}$ can be chosen to be the space of diagonal matrices with zero trace, the Weyl chamber $\mathfrak{a}^+$ to be the cone of matrices in $\mathfrak{a}$ with nonincreasing coefficients. For g in G, the coefficients of the Jordan projection $\lambda(g)$ are then the logarithms of the moduli of the eigenvalues of g.

Let Γ be a Zariski dense subsemigroup of G. We saw in Chapter 5 that the set Γ_{lox} of loxodromic elements of Γ is still Zariski dense in G. The following two theorems give useful informations on the image of Γ_{lox} by the Jordan projection.

DEFINITION 6.1. The *limit cone* of Γ is the smallest closed cone L_Γ in $\mathfrak{a}^+$ containing $\lambda(\Gamma_{lox})$.

In other words, L_Γ is the closure of the union of the half-lines spanned by the Jordan projections of the loxodromic elements of Γ:

$$L_\Gamma := \overline{\bigcup_{g\in\Gamma_{lox}} \mathbb{R}^+\lambda(g)}.$$

In this definition, the word cone does not presuppose that L_Γ is convex. The fact that this cone is indeed convex is part of our first main theorem.

THEOREM 6.2. *Let G be a connected algebraic semisimple real Lie group and Γ be a Zariski dense subsemigroup of G. Then the limit cone L_Γ is convex with non-empty interior.*

REMARK 6.3. let us quote without proof a few more properties of L_Γ.
(i) The limit cone L_Γ contains also $\lambda(\Gamma)$.
(ii) The limit cone L_Γ is the asymptotic cone of the image of Γ by the Cartan projection, i.e.

$$L_\Gamma = \{x \in \mathfrak{a}^+ \mid \exists g_n \in \Gamma\,,\, \exists t_n \searrow 0 \ \lim_{n\to\infty} t_n\, \kappa(g_n) = x\}.$$

(iii) For any closed convex cone with non empty interior L of $\mathfrak{a}^+$, there exists a Zariski dense subsemigroup Γ of G such that $L_\Gamma = L$.
(iv) The convexity of L_Γ is also true over non-archimedean fields.

These properties will not be used in this book. See [**10**] for more details.

The fact that L_Γ is convex will be proved in Section 6.4. The fact that L_Γ has non-empty interior will then be a consequence of our second main theorem.

THEOREM 6.4. *Let G be a connected algebraic semisimple real Lie group and Γ be a Zariski dense subsemigroup of G. Then the subgroup of $\mathfrak{a}$ spanned by the elements $\lambda(gh) - \lambda(g) - \lambda(h)$, for g, h and gh in Γ_{lox}, is dense in $\mathfrak{a}$.*

The proof of Theorem 6.4 will be given in Section 6.8.

6.2. Products of proximal elements.

In this section we relate the spectral radius of the product of two *transversally proximal* matrices with the product of their spectral radii. This will be the key ingredient in the proof of the convexity of the limit cone in Section 6.4.

We first recall some notations from Section 3.1. Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. For any proximal element g in $\mathrm{End}(V)$, we recall that V_g^+ is the attracting g-invariant line and that $V_g^<$ is the unique g-invariant complementary hyperplane. We choose a nonzero vector $v_g^+ \in V_g^+$ and a linear functional $\varphi_g^< \in V^*$ whose kernel is $V_g^<$ and such that $\varphi_g^<(v_g^+) = 1$. We introduce the rank-one projection $\pi_g := \varphi_g^< \otimes v_g^+$. It is given by $\pi_g(v) = \varphi_g^<(v)v_g^+$, for all v in V. Its image is V_g^+ and its kernel is $V_g^<$. This rank-one projection π_g can be obtained as the limit

$$\pi_g := \lim_{n\to\infty} \frac{g^n}{\mathrm{tr}(g^n)}. \tag{6.1}$$

Indeed, since g is proximal, when n goes to infinity the norm of g^n, the spectral radius of g^n and the absolute value of the trace of g^n are equivalent:

$$\|g^n\| \sim \lambda_1(g)^n \sim |\mathrm{tr}(g^n)|\ .$$

Here the symbol $a_n \sim b_n$ means that the ratio a_n/b_n converges to 1. Note then that the limit operator in the right-hand side of (6.1) has image V_g^+, kernel $V_g^<$ and trace equal to 1. Hence this operator is equal to π_g.

These projections π_g are very useful to approximate the spectral radius of a product. Indeed, one has the following lemma. We write $m \wedge n$ for the minimum of m and n.

LEMMA 6.5. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let g, h be two proximal elements of* $\mathrm{End}(V)$ *and let f_1, f_2 be two elements of* $\mathrm{End}(V)$.

Then one has the limit

$$\lim_{m\wedge n\to\infty} \frac{\lambda_1(g^m f_1 h^n f_2)}{\lambda_1(g)^m \lambda_1(h)^n} \;=\; |\mathrm{tr}(\pi_g f_1 \pi_h f_2)|\;.$$

In particular, when $\mathrm{tr}(\pi_g f_1 \pi_h f_2) \neq 0$, this limit is nonzero.

PROOF. An easy but crucial point in the proof is the equality

$$\lambda_1(\sigma) = |\mathrm{tr}(\sigma)|$$

which is valid as soon as σ is a rank-one endomorphism of V.

Using Formula (6.1) for both g and h and the fact that the spectral radius of a matrix depends continuously on the matrix, one computes the limits for $m \wedge n \to \infty$,

$$\begin{aligned}\lim_{m\wedge n\to\infty} \frac{\lambda_1(g^m f_1 h^n f_2)}{\lambda_1(g)^m \lambda_1(h)^n} &= \lim_{m\wedge n\to\infty} \lambda_1\left(\frac{g^m}{\mathrm{tr}(g^m)} f_1 \frac{h^n}{\mathrm{tr}(h^n)} f_2\right)\\ &= \lambda_1(\pi_g f_1 \pi_h f_2) = |\mathrm{tr}(\pi_g f_1 \pi_h f_2)|,\end{aligned}$$

as required. □

DEFINITION 6.6. Two proximal elements g, h of $\mathrm{End}(V)$ are called *transversally proximal* if $\mathrm{tr}(\pi_g \pi_h) \neq 0$.

Geometrically this transversality condition means that

$$V_g^+ \not\subset V_h^< \;\text{ and }\; V_h^+ \not\subset V_g^<\;,$$

and the quantity

$$B_1(V_g^+, V_g^<, V_h^+, V_h^<) := \mathrm{tr}(\pi_g \pi_h)$$

is the *cross-ratio* of this quadruple. Indeed, one has the formula

$$B_1(V_g^+, V_g^<, V_h^+, V_h^<) = \frac{\varphi_g^<(v_h^+)\,\varphi_h^<(v_g^+)}{\varphi_g^<(v_g^+)\,\varphi_h^<(v_h^+)}\;. \tag{6.2}$$

This equation (6.2) follows from the formula $\pi_g \pi_h = \varphi_g^<(v_h^+)\,\varphi_h^< \otimes v_g^+$.

A special case of Lemma 6.5 is the following corollary.

COROLLARY 6.7. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let g, h be two proximal elements of* $\mathrm{End}(V)$. *Then one has the limit*

$$\lim_{m\wedge n\to\infty} \frac{\lambda_1(g^m h^n)}{\lambda_1(g)^m \lambda_1(h)^n} \;=\; |\mathrm{tr}(\pi_g \pi_h)|\;.$$

In particular, when g, h are transversally proximal this limit is nonzero.

PROOF. This follows from Lemma 6.5 with $f_1 = f_2 = 1$. □

6.3. Products of loxodromic elements.

Using the dictionary introduced in Section 5.9, we translate now the results of Section 6.2: we relate the Jordan projection of the product of two *transversally loxodromic* elements with the sum of their Jordan projections.

We first recall some notations from Section 5.8. We fix a connected algebraic semisimple real Lie group G, a Cartan subspace $\mathfrak{a}$ of its Lie algebra $\mathfrak{g}$, a Weyl chamber $\mathfrak{a}^+$ and the corresponding set Π of simple restricted roots. For every α in Π, we denote by (V_α, ρ_α) the irreducible proximal representation of G introduced in Lemma 5.32, whose highest weight χ_α is a multiple of the corresponding fundamental weight.

For g loxodromic in G, we will write $V^+_{\alpha,g}$, $V^<_{\alpha,g}$, and $\pi_{\alpha,g}$ as shorthands for $(V_\alpha)^+_{\rho_\alpha(g)}$, $(V_\alpha)^<_{\rho_\alpha(g)}$, and $\pi_{\rho_\alpha(g)}$.

Definition 6.8. Two elements g, h of G are called *transversally loxodromic* if , for every α in Π, the elements $\rho_\alpha(g)$, $\rho_\alpha(h)$ are transversally proximal.

For instance, when g is loxodromic, the duplicate elements g, g are transversally loxodromic.

Remark 6.9. This definition does not depend on the choice of the family ρ_α. Indeed, using Lemma 5.39, one can check that two loxodromic elements g, h are transversally loxodromic if and only if the G-orbit of the pair (ξ^+_g, ξ^+_h) of attracting points is the open orbit in $\mathcal{P} \times \mathcal{P}$.

It is in general not true that the Jordan projection $\lambda(gh)$ of the product of two elements g and h is equal to the sum $\lambda(g) + \lambda(h)$ of their Jordan projections. The following Lemma 6.10 and its Corollary 6.11 tell us that under suitable transversality assumptions this fact is asymptotically true up to a converging error term.

Lemma 6.10. *Let G be a connected algebraic semisimple real Lie group. Let g, h be two loxodromic elements of G. Then there exists a non-empty Zariski open subset $G_{g,h}$ of G^2 such that, for every $f = (f_1, f_2)$ in $G_{g,h}$ the following limit*

$$\lim_{m\wedge n\to\infty} \lambda(g^m f_1 h^n f_2) - m\lambda(g) - n\lambda(h) \tag{6.3}$$

exists in $\mathfrak{a}$.

Proof. We define $G_{g,h}$ to be

$$G_{g,h} := \{f = (f_1, f_2) \in G^2 \mid \operatorname{tr}(\pi_{\alpha,g}\, \rho_\alpha(f_1)\, \pi_{\alpha,h}\, \rho_\alpha(f_2)) \neq 0, \text{ for } \alpha \in \Pi\}. \tag{6.4}$$

The transversality condition means exactly that the pair $(1,1)$ belongs to the Zariski open set $G_{g,h}$.

Since the linear functionals $(\chi_\alpha)_{\alpha\in\Pi}$ form a basis of the dual space $\mathfrak{a}^*$, we can define, for f in $G_{g,h}$ an element $\nu_f(g,h)$ in $\mathfrak{a}$ by the equalities

$$\chi_\alpha(\nu_f(g,h)) = \log|\mathrm{tr}(\pi_{\alpha,g}\,\rho_\alpha(f_1)\,\pi_{\alpha,h}\,\rho_\alpha(f_2))| \ \text{ for } \alpha\in\Pi. \tag{6.5}$$

We will check that the limit (6.3) is equal to this vector $\nu_f(g,h)$.

Equivalently, we will prove, for every α in Π, the convergence

$$\chi_\alpha(\lambda(g^m f_1 h^n f_2) - \lambda(g^m) - \lambda(h^n)) \xrightarrow[m\wedge n\to\infty]{} \chi_\alpha(\nu_f(g,h))$$

But, by Lemma 5.33, the left-hand side is equal to

$$\log \frac{\lambda_1(\rho_\alpha(g^m f_1 h^n f_2)}{\lambda_1(\rho_\alpha(g))^m \lambda_1(\rho_\alpha(h))^n}$$

By Lemma 6.5, it converges to $\log|\mathrm{tr}(\pi_{\alpha,g}\,\rho_\alpha(f_1)\,\pi_{\alpha,h}\,\rho_\alpha(f_2))|$. □

Corollary 6.11. *Let G be a connected algebraic semisimple real Lie group, let g, h be two transversally loxodromic elements of G and let $\nu(g,h)$ be the element of $\mathfrak{a}$ defined by*

$$\chi_\alpha(\nu(g,h)) = \log|\mathrm{tr}(\pi_{\alpha,g}\pi_{\alpha,h})| \ \text{ for all } \alpha \text{ in } \Pi. \tag{6.6}$$

Then one has the equality

$$\nu(g,h) = \lim_{m\wedge n\to\infty} \lambda(g^m h^n) - m\lambda(g) - n\lambda(h). \tag{6.7}$$

Remark 6.12. Conversely, if for two loxodromic elements g, h in G, the limit (6.7) exists then the pair (g,h) is transversally loxodromic. This fact follows from the proof. This fact tells us also that Definition 6.8 does not depend on the choices of ρ_α.

Proof. This follows from Lemma 6.10 and its proof with $f_1 = f_2 = 1$. □

The element $\nu(g,h)$ will be called the *multicross-ratio* of g and h.

6.4. Convexity of the limit cone.

Using the results of Section 6.3, we prove now the convexity of the limit cone of a Zariski dense semigroup Γ.

Proof of Theorem 6.2. We first prove the convexity of the cone L_Γ. Since this cone L_Γ is closed, it is enough to prove the following:

For any g, h in Γ_{lox}, the sum $\lambda(g)+\lambda(h)$ belongs to L_Γ.

Since the set $G_{g,h}$ introduced in (6.4) is a non-empty Zariski open set, the intersection

$$\Gamma_{g,h} := \Gamma^2 \cap G_{g,h}$$

is non-empty. Let $f = (f_1, f_2)$ be an element of $\Gamma_{g,h}$. According to Lemma 6.10, the Jordan projection $\lambda(g^n f_1 h^n f_2)$ remains at bounded distance from $n\lambda(g) + n\lambda(h)$. In particular, for n large enough, the product $g^n f_1 h^n f_2$ is loxodromic and the sum

$$\lambda(g) + \lambda(h) = \lim_{n\to\infty} \tfrac{1}{n}\lambda(g^n f_1 h^n f_2)$$

belongs to L_Γ as required.

The fact that L_Γ has non-empty interior will follow from Theorem 6.4. □

6.5. The group Δ_Γ.

We explain in this Section how to prove the density Theorem 6.4 thanks to the group Δ_Γ of multicross-ratios.

DEFINITION 6.13. The group Δ_Γ of *multicross-ratios* of Γ is the subgroup of $\mathfrak{a}$ spanned by the multicross-ratios $\nu(g, h)$ where the pair (g, h) runs among the pairs of transversally loxodromic elements of Γ.

Here is the main result of this chapter.

PROPOSITION 6.14. *Let G be a connected algebraic semisimple real Lie group and Γ be a Zariski dense subsemigroup of G. The group Δ_Γ is dense in $\mathfrak{a}$.*

This proposition will be proved in Section 6.8.

PROOF OF PROPOSITION 6.14 $\Longrightarrow$ THEOREM 6.4. Let Δ'_Γ be the subgroup of $\mathfrak{a}$ spanned by the differences $\lambda(gh) - \lambda(g) - \lambda(h)$ for g, h and gh loxodromic elements of Γ. We will prove the inclusion between the closures

$$\overline{\Delta_\Gamma} \subset \overline{\Delta'_\Gamma}.$$

Let g_0, h_0 be two transversally loxodromic elements of Γ. According to Corollary 6.11, the multi crossratio $\nu(g_0, h_0)$ is given by the limit

$$\nu(g_0, h_0) = \lim_{n\to\infty} \lambda(g_0^n h_0^n) - \lambda(g_0^n) - \lambda(h_0^n),$$

and, for n large, the element $g_0^n h_0^n$ is also loxodromic. Hence $\nu(g_0, h_0)$ belongs to $\overline{\Delta'_\Gamma}$ and Δ_Γ is included in $\overline{\Delta'_\Gamma}$. □

Our aim now is to prove Proposition 6.14.

6.6. Asymptotic expansion of cross-ratios.

The proof of Proposition 6.14 will rely on an estimation of suitable cross-ratios associated to transversally proximal elements. This estimation will be valid only under a stronger transversality condition involving the second leading eigenspaces.

For a sequence $S \subset \mathbb{N}$ and sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ of nonzero real numbers, we write $a_m \underset{m\in S}{\asymp} b_m$ if there exist real numbers $c, d > 0$ such that, for m large enough in S, $c\,|a_m| \leq |b_m| \leq d\,|a_m|$, and we write $a_m = o(b_m)$ if the ratio a_m/b_m converges to 0

Let $\mathbb{K}$ be a local field and g be a proximal element of $\mathrm{End}(\mathbb{K}^d)$. We denote by $V_g^{<+} \subset V_g^{<}$ the subspace of V_g that is the sum of the generalized eigenspaces with eigenvalues of modulus $\lambda_2(g)$. We denote by τ_g the projection on $V_g^{<+}$ whose kernel is g-invariant.

The following lemma will allow us to construct, in a given proximal and strongly irreducible semigroup Γ, pairs of transversally proximal elements (g, h) such that the cross-ratio $\mathrm{tr}(\pi_g\pi_h)$ is close to 1 but not 1.

LEMMA 6.15. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let g, h be two transversally proximal elements of* $\mathrm{End}(V)$.
a) Then, for m, n large enough the product $g^m h^n$ is proximal and one has the convergence

$$\lim_{n\to\infty} \mathrm{tr}(\pi_g \pi_{g^m h^n}) = c_m(g, h) := \frac{\mathrm{tr}(\pi_g g^m \pi_h)}{\mathrm{tr}(g^m \pi_h)}.$$

b) If moreover g is semisimple and $\tau_g(V_h^+) \not\subset V_h^<$, there exists a sequence S_g in $\mathbb{N}$ such that one has

$$\log|c_m(g, h)| \underset{m\in S_g}{\asymp} \frac{\lambda_2(g)^m}{\lambda_1(g)^m}\ . \tag{6.8}$$

REMARK 6.16. The real number $c_m(g, h)$ is also a cross-ratio. Indeed one has the equality

$$c_m(g, h) = B_1(V_g^+, V_g^<, g^m V_h^+, V_h^<).$$

DEFINITION 6.17. A transversally proximal pair (g, h) satisfying the extra condition $\tau_g(V_h^+) \not\subset V_h^<$ will be called *strongly transversally proximal.*

Proof. *a*) Choose m large enough so that $\mathrm{tr}(g^m\pi_h) \neq 0$. One has the equality

$$\lim_{n\to\infty} \frac{g^m h^n}{\mathrm{tr}(g^m h^n)} = \frac{g^m\pi_h}{\mathrm{tr}(g^m\pi_h)}.$$

Hence since the map $f \mapsto \pi_f$ is continuous on the set of proximal endomorphisms, one also has the equality

$$\lim_{n\to\infty} \pi_{g^m h^n} = \frac{g^m\pi_h}{\mathrm{tr}(g^m\pi_h)}.$$

Our claim follows by applying the map $f \mapsto \mathrm{tr}(\pi_g\, f)$ to both sides.

b) Using this formula, one has the asymptotic

$$\log|c_m(g,h)| \underset{m\to\infty}{\sim} c_m(g,h) - 1 = \frac{\mathrm{tr}((\pi_g - 1)g^m\pi_h)}{\mathrm{tr}(g^m\pi_h)}\ .$$

We have already computed the denominator. One has

$$\mathrm{tr}(g^m\pi_h) \quad = \quad \varphi_h^<(g^m v_h^+)\ .$$

We compute now the numerator. We set $w_0 := \tau_g(v_h^+)$, so that one has

$$\mathrm{tr}((1-\pi_g)g^m\pi_h) = \varphi_h^<((1-\pi_g)g^m v_h^+) \underset{m\to\infty}{=} \varphi_h^<(g^m\tau_g v_h^+) + o(\lambda_2(g)^m)\ .$$

Since g is semisimple, there exist a sequence $S_g \subset \mathbb{N}$ depending only on g, and elements t_m in $\mathbb{K}$ with $|t_m| = \lambda_2(g)^m$ such that

$$t_m^{-1} g^m \tau_g \underset{m\in S_g}{\longrightarrow} \tau_g.$$

Since neither v_g^+ nor $\tau_g v_h^+$ belong to $V_h^<$, one has

$$|\varphi_h^<(g^m v_h^+)| \underset{m\to\infty}{\asymp} \lambda_1(g)^m \quad \text{and} \quad |\varphi_h^<(g^m\tau_g v_h^+)| \underset{m\in S_g}{\asymp} \lambda_2(g)^m\ .$$

Putting all this together, one gets (6.8). $\square$

6.7. Strongly transversally loxodromic elements.

Using the dictionary introduced in Section 5.9, we translate now the results of Sections 5.5 and 6.6 into the language of the geometry of G.

Let G be a connected algebraic semisimple real Lie group.

Definition 6.18. Two elements g, h of G are called *strongly transversally loxodromic* if, for every α in Π, the elements $\rho_\alpha(g)$, $\rho_\alpha(h)$ are strongly transversally proximal.

We recall that $S_g \subset \mathbb{N}$ is the sequence introduced in Lemma 6.15.

COROLLARY 6.19. *Let G be a connected algebraic semisimple real Lie group and g, h be two transversally loxodromic elements of G.*
a) For m large enough, the following limit exists

$$\tau_m(g,h) = \lim_{n\to\infty} \nu(g, g^m h^n) \in \mathfrak{a}.$$

b) Moreover, if g, h are strongly transversally loxodromic, one has, for all α in Π,

$$|\chi_\alpha(\tau_m(g,h))| \underset{m\in S_g}{\asymp} e^{-m\,\alpha(\lambda(g))} \tag{6.9}$$

PROOF. *a*) According to Corollary 6.11, for all α in Π, one has

$$\chi_\alpha(\nu(g, g^m h^n)) = \log|\mathrm{tr}(\pi_{\alpha,g}\pi_{\alpha,g^m h^n})|.$$

Hence by Lemma 6.15, one has, for m large enough,

$$\lim_{n\to\infty} \chi_\alpha(\nu(g, g^m h^n)) = \log|c_m(\rho_\alpha(g), \rho_\alpha(h))|.$$

This proves that the limit $\tau_m(g,h)$ exists and satisfies, for all α in Π,

$$\chi_\alpha(\tau_m(g,h)) = \log|c_m(\rho_\alpha(g), \rho_\alpha(h))|. \tag{6.10}$$

b) According to Lemma 5.38, the loxodromic element g is semisimple. This tells us that all the proximal endomorphisms $\rho_\alpha(g)$ are semisimple. Using Equation (6.10) and Lemma 6.15, one gets the asymptotics:

$$|\chi_\alpha(\tau_m(g,h))| \underset{m\in S_g}{\asymp} \frac{\lambda_2(\rho_\alpha(g))^m}{\lambda_1(\rho_\alpha(g))^m}.$$

Now, using the description of the restricted weights of the representations V_α from Lemma 5.32 and using Lemma 5.33, one gets the equalities

$$\lambda_1(\rho_\alpha(g)) = e^{\chi_\alpha(\lambda(g))} \text{ and } \lambda_2(\rho_\alpha(g)) = e^{(\chi_\alpha-\alpha)(\lambda(g))}.$$

This proves (6.9). □

The following lemma tells us that, in a Zariski dense semigroup, there are many pairs (g,h) of strongly transversally loxodromic elements.

LEMMA 6.20. *Let G be a connected algebraic semisimple real Lie group, Γ be a Zariski dense subsemigroup of G, and g be a loxodromic element of Γ. Then the set*

$$\Gamma_g := \{h \in \Gamma_{lox} \mid g \text{ and } h \text{ are strongly transversally loxodromic}\}$$

is Zariski dense in G.

Proof. This set Γ_g is the set of elements h such that, for all α in Π, $\rho_\alpha(h)$ is proximal in V_α with $\pi_{\alpha,g}(V^+_{\alpha,h}) \not\subset V^<_{\alpha,h}$ and $\tau_{\rho_\alpha(g)}(V^+_{\alpha,h}) \not\subset V^<_{\alpha,h}$. According to Corollary 5.26, this set is Zariski dense. $\square$

6.8. Density of the group of multicross-ratios.

We are now ready to prove Proposition 6.14

At the very beginning of this proof, we will need a loxodromic element in Γ with extra properties. This element will be given by the following lemma.

Lemma 6.21. *Let G be a connected algebraic semisimple real Lie group and Γ be a Zariski dense subsemigroup of G. Then there exists a loxodromic element g of Γ such that the real numbers $\alpha(\lambda(g))$ for $\alpha \in \Pi$ are distinct.*

Remark 6.22. Note that Lemma 6.21 is a special case of Theorem 6.2 which tells us that the limit cone L_Γ is convex and is not included in a proper subspace of $\mathfrak{a}$. However we need to give a proof of Lemma 6.21 since we have not finished yet the proof of Theorem 6.2. What we will check in the proof of Lemma 6.21 is that the cone L_Γ is not included in a proper "rational" subspace of $\mathfrak{a}$, by noticing that such an inclusion will contradicts the Zariski density of Γ_{lox}.

Proof of Lemma 6.21. By Theorem 5.36, Γ_{lox} is Zariski dense in G. By Lemma 5.21, G is Zariski irreducible. Hence it is enough to prove that, for every two restricted roots α_1 and α_2, there exists a non-empty Zariski open set U_{α_1,α_2} of G such that,

$$\alpha_1(\lambda(g)) \neq \alpha_2(\lambda(g))\ ,\ \text{for all loxodromic element } g \text{ in } U_{\alpha_1,\alpha_2}.$$

Since both α_1 and α_2 belong to the $\mathbb{Q}$-span of the linear functionals χ_α, there exists even integers $(p_\alpha)_{\alpha\in\Pi}$ not all zero, such that $\sum_{\alpha\in\Pi} p_\alpha\,\chi_\alpha$ is a multiple of $\alpha_1 - \alpha_2$. Now, for any g in G let us introduce the multiplicity $m_1(g)$ of the eigenvalue 1 in the characteristic polynomial of the matrix $\bigotimes_{\alpha\in\Pi}\rho_\alpha(g)^{\otimes p_\alpha}$, with the convention that, for a matrix A, a negative tensor power like $A^{\otimes -k}$ means $(A^{-1})^{\otimes k}$. Let $m_{1,min}$ be the minimal value of those integers $m_1(g)$ when g runs in G. The set

$$U_{\alpha_1,\alpha_2} = \{g \in G \mid m_1(g) = m_{1,min}\}$$

is the Zariski open subset of G we were looking for.

Indeed, let g be a loxodromic element satisfying $\alpha_1(\lambda(g)) = \alpha_2(\lambda(g))$. We want to see that g does not belong to U_{α_1,α_2}. One has the equality

$$\textstyle\sum_{\alpha\in\Pi} p_\alpha\,\chi_\alpha(\lambda(g)) = 0.$$

According to Lemma 5.33, this means that

$$\prod_{\alpha\in\Pi} \lambda_1(\rho_\alpha(g)^{p_\alpha}) = 1.$$

Since the local field is $\mathbb{R}$ and since the p_α are even integers, the leading eigenvalues of $\rho_\alpha(g)$ are real numbers and this relation between their moduli is a relation between the leading eigenvalues themselves. This proves that g does not belong to U_{α_1,α_2} as required. □

Proof of Proposition 6.14. Assume by contradiction, that there exists a nonzero linear functional φ in $\mathfrak{a}^*$ such that $\varphi(\Delta_\Gamma) \subset \mathbb{Z}$. Write

$$\varphi = \sum_{\alpha\in\Pi} \varphi_\alpha \, \chi_\alpha \ \text{ with } \ \varphi_\alpha \in \mathbb{R}.$$

Choose, using Lemma 6.21 a loxodromic element g of Γ such that the positive real numbers $\alpha(\lambda(g))$, for $\alpha \in \Pi$, are distinct. Choose then α in Π with $\varphi_\alpha \neq 0$ for which $\alpha(\lambda(g))$ is minimal. Choose, using Lemma 6.20, an element h in Γ_{lox} such that g, h are strongly transversally loxodromic. According to Corollary 6.19, for m large, the element $\tau_m(g,h)$ belongs to $\overline{\Delta_\Gamma}$, and one has

$$|\varphi(\tau_m(g,h))| \underset{m\in S_g}{\asymp} e^{-m\,\alpha(\lambda(g))}.$$

This contradicts the fact that $\varphi(\overline{\Delta_\Gamma}) \subset \mathbb{Z}$. □

This finishes also the proof of Theorems 6.2 and 6.4.

7. Reductive groups and their representations

In order to study random walks on reductive groups over local fields, we collect in this chapter a few notations and facts about these groups: the definition of the flag variety, the Cartan projection and the Iwasawa cocycle. Those extend the notations and facts for semisimple real Lie groups that we collected in Section 5.7. Even though these notations and facts look at a first glance rather heavy, they will allow us to express the asymptotic behavior of random walks on G in an intrinsic way i.e. in a way which does not depend on an embedding of G into a linear group. To prove these intrinsic results, we will only use a special kind of irreducible representations of G, the so-called *proximal* representations. We will later be able to deduce from the intrinsic results the asymptotic behavior of the random walk in *any* linear representation of G.

7.1. Reductive groups.

We first introduce the main definitions and notations for reductive groups over local fields.

Let still $\mathbb{K}$ be a local field and keep the notations from chapter 3. Let $\mathbf{G}$ be a reductive $\mathbb{K}$-group i.e. a reductive algebraic group defined over $\mathbb{K}$ and set $G = \mathbf{G}(\mathbb{K})$. Equip G with its natural locally compact topology.

Choose a maximal $\mathbb{K}$-split torus $\mathbf{A}$ of $\mathbf{G}$, a maximal unipotent $\mathbb{K}$-subgroup $\mathbf{U}$ of $\mathbf{G}$ that is normalized by $\mathbf{A}$ and let $\mathbf{P} = \mathbf{N}_{\mathbf{G}}(\mathbf{U})$ be the normalizer of $\mathbf{U}$ in $\mathbf{G}$. Let Σ be the root system of the pair $(\mathbf{G}, \mathbf{A})$, that is, the set of non-trivial weights of the adjoint representation of $\mathbf{A}$ in the Lie algebra of $\mathbf{G}$, $\Sigma^+ \subset \Sigma$ be the set of positive roots associated to the choice of $\mathbf{P}$ and Π be the set of simple roots of Σ^+. Let $\mathbf{Z}$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$. Let A, Z, U and P be the groups of $\mathbb{K}$-points of $\mathbf{A}$, $\mathbf{Z}$, $\mathbf{U}$ and $\mathbf{P}$ (see [**22**] for more details).

Let $\mathfrak{a}$ be the dual vector space to the real vector space of continuous homomorphisms $A \to \mathbb{R}$. Since any continuous morphism $A \to \mathbb{R}$ extends in a unique way to a morphism $Z \to \mathbb{R}$, there exists a unique morphism $\omega : Z \to \mathfrak{a}$ whose restriction to A is the natural morphism $A \to \mathfrak{a}$ (see [**122**, Lemma 4.11.4]).

Let $\mathrm{X}(\mathbf{A})$ be the character group of $\mathbf{A}$. For any character χ of $\mathrm{X}(\mathbf{A})$, we let χ^ω be the unique linear functional on $\mathfrak{a}$, such that, for any a in A,

$$|\chi(a)| = e^{\chi^\omega(\omega(a))}.$$

The set Σ^ω is a root system in $\mathfrak{a}^*$ and Π^ω is a basis of this root system. We set $\mathfrak{a}^+$ for the closed Weyl chamber of Π^ω,

$$\mathfrak{a}^+ := \{x \in \mathfrak{a} \mid \forall \alpha \in \Sigma^+ \ \alpha^\omega(x) \geq 0\},$$

and

$$\mathfrak{a}^{++} := \{x \in \mathfrak{a} \mid \forall \alpha \in \Sigma^+ \ \alpha^\omega(x) > 0\}$$

for the open Weyl chamber.

We set W for the Weyl group of Σ^ω and $\iota : \mathfrak{a}^+ \to \mathfrak{a}^+$ for the associated opposition involution, that is $-\iota$ is the unique element of W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$.

Remark 7.1. When $\mathbb{K} = \mathbb{R}$, these notations have been introduced in a simpler way in Section 5.7 : the vector space $\mathfrak{a}$ is the Lie algebra of A, and for every algebraic character χ of A, the linear functional χ^ω on $\mathfrak{a}$ is the differential of χ.

7.2. Iwasawa cocycle for reductive groups.

The two main outputs of this section are the Cartan projection κ which is a multidimensional avatar of the norm and the Iwasawa cocycle σ which is a multidimensional avatar of the norm cocycle. The main asymptotic laws in this book will describe the behavior of κ and σ.

7.2.1. *Iwasawa cocycle for connected reductive groups.* We define first the Iwasawa cocycle and the Cartan projection for connected groups since it is slightly easier in this case.

Let $\mathbf{G}_c$ be the connected component of $\mathbf{G}$, $\mathbf{Z}_c := \mathbf{Z} \cap \mathbf{G}_c$ and $\mathbf{P}_c := \mathbf{P} \cap \mathbf{G}_c$, which is a minimal parabolic $\mathbb{K}$-subgroup of $\mathbf{G}_c$. Let G_c, Z_c and $P_c = Z_c U$ be their groups of $\mathbb{K}$-points and

$$Z_c^+ := \{z \in Z_c \mid \omega(z) \in \mathfrak{a}^+\}.$$

Let K_c be a *good maximal compact* subgroup of G_c with respect to the torus A.

When $\mathbb{K}$ is archimedean this means the Lie algebras of A and K are orthogonal for the Killing form as is explained in Section 5.7.

When $\mathbb{K}$ is non-archimedean this notion is introduced in [**32**], where the existence of such a group is also established.

In both cases, for such a group K_c, one has the Cartan decomposition

$$G_c = K_c Z_c^+ K_c$$

(see [**32**] in the non-archimedean case). For any g in G_c, let $\kappa(g)$ be the unique element of $\mathfrak{a}^+$ such that

$$g \in K_c\, \omega^{-1}(\kappa(g))\, K_c.$$

The map

$$\kappa : G_c \to \mathfrak{a}^+$$

is called the *Cartan projection.* For all g in G, one has

$$\kappa(g^{-1}) = \iota(\kappa(g)).$$

Besides, one has the Iwasawa decomposition

$$G_c = K_c Z_c U.$$

Let

$$\mathcal{P}_c = G_c / P_c$$

be the *flag variety* of G_c and, for any g in G_c and η in $\mathcal{P}_c$, if $\eta = kP_c$ for some k in K, let $\sigma(g, \eta)$ be the unique element of $\mathfrak{a}$ such that

$$gk \in K_c\, \omega^{-1}(\sigma(g, \eta))\, U.$$

The following lemma is a straightforward generalization of Lemma 5.29.

Lemma 7.2. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. The map $\sigma : G_c \times \mathcal{P}_c \to \mathfrak{a}$ is a continuous cocycle.*

This cocycle is still called the *Iwasawa cocycle* or the *Busemann cocycle.*

Proof. The proof is the same as for Lemma 5.29. Indeed, for g, g' in G and η in $\mathcal{P}_c$, if $\eta = kP_c$ with k in K_c, let k' in K_c and z, z' in Z be such that

$$g'k \in k'z'U \quad \text{and} \quad gk' \in K_c zU.$$

We have $\sigma(g', \eta) = \omega(z')$ and $\sigma(g, g'\eta) = \omega(z)$ and

$$gg'k \in gk'z'U \subset K_c zUz'U = K_c(zz')U,$$

hence $\sigma(gg', \eta) = \omega(zz')$ and σ satisfies the cocycle property (2.6).

This cocycle σ is continuous. Indeed, in case $\mathbb{K}$ is non-archimedean, since K_c is open, the cocycle σ is locally constant. In case $\mathbb{K}$ is archimedean, the continuity has been checked in Lemma 5.29. $\square$

7.2.2. *Iwasawa cocycle for general reductive groups in the archimedean case.* We now extend the definition of the Iwasawa cocycle to non-connected groups. For technical reasons, the definition is easier in the archimedean case, that is when $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, which we temporarily assume. We set $F = G/G_c$.

Let K be the normalizer of K_c in G. As *the maximal compact subgroups of* G_c *are all conjugated*, we have

$$G = G_c K \;\text{ and }\; K \cap G_c = K_c$$

(see [**64**, Sections 6.1 and 6.2]). Hence the natural map

$$K/K_c \to F$$

is an isomorphism and we get the non-connected Cartan decomposition

$$G = KZ_c^+K_c.$$

For g in G, we let again $\kappa(g)$ be the unique element of $\mathfrak{a}^+$ such that

$$g \in K\,\omega^{-1}(\kappa(g))\,K_c = K\,\exp(\kappa(g))\,K_c.$$

We still say κ is the *Cartan projection* of G.

In the same way, we have $G = KP_c = KZ_cU$. We let

$$\mathcal{P} = G/P_c$$

be the *flag variety* of G and, for any g in G and η in $\mathcal{P}$, if $\eta = kP_c$ with k in K, we let $\sigma(g, \eta)$ be the unique element of $\mathfrak{a}$ such that

$$gk \in K\,\omega^{-1}(\sigma(g, \eta))\,U = K\,\exp(\sigma(g, \eta))\,U.$$

As in Lemma 7.2, one checks that the map σ is a continuous cocycle, which we still call the *Iwasawa cocycle*.

Let us now study the equivariance properties of this Iwasawa cocycle under the group $F = G/G_c$. First note that, since the minimal parabolic $\mathbb{K}$-subgroups of $\mathbf{G}_c$ are all conjugated (see [**22**]) by an element of G_c, we have $G = G_c P$ and the natural map

$$P/P_c \to F$$

is an isomorphism. Now, since the connected component P_c is normal in P, the group P/P_c acts on the right on G/P_c and this action may be read as an action of F. This action is right equivariant with respect to the natural map $G/P_c \to G/G_c = F$. Besides, since $P_c = Z_c U$ and U is equal to the commutator group $[A, U]$, the morphism $\omega : Z_c \to \mathfrak{a}$ extends in a unique way as a morphism $P_c \to \mathfrak{a}$, which we still denote by ω. By definition of ω, there exists a unique linear action of $F = P/P_c$ on $\mathfrak{a}$ which makes ω an F-equivariant morphism. Since P normalizes U, the action of F on $\mathfrak{a}$ preserves $\mathfrak{a}^+$.

The following lemma tells us that the Iwasawa cocycle is F-equivariant.

Lemma 7.3. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. Assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For any g in G, η in $\mathcal{P}$ and f in F, one has*

$$\sigma(g, \eta f) = f^{-1}\sigma(g, \eta). \tag{7.1}$$

Proof. Indeed, assume $\eta = kP_c$ with k in K. Since $\mathbb{K}$ is archimedean, we have

$$P = (K \cap P)P_c,$$

and we can find a representant for f which belongs to $K \cap P$; we still denote it by f. We get $\eta f = kfP_c$. By definition, we have

$$gk \in K\,\omega^{-1}(\sigma(g, \eta))\,U,$$

hence

$$gkf \in K\,\omega^{-1}(\sigma(g, \eta))\,Uf = K\,f^{-1}\,\omega^{-1}(\sigma(g, \eta))\,fU,$$

which completes the proof. $\square$

7.2.3. *Iwasawa cocycle for general reductive groups over an arbitrary field.* We now drop the assumption that $\mathbb{K}$ is archimedean. We will extend the previous construction. The only new difficulty is that the maximal compact subgroups of G_c are in general not all conjugated in G_c but may be conjugated in G. When this happens, this prevents the existence of a maximal compact subgroup G that would map onto G/G_c. We will overcome it by using a suitable section τ of the quotient map $s : G \to F = G/G_c \simeq P/P_c$. We choose a map

$$\tau : F \to P\ ;\ f \mapsto \tau_f$$

which is a section for the natural projection, that is, for any f in F, one has $\tau_f \in P \cap s^{-1}(f)$. We also assume that $\tau(e) = e$. We introduce the subset of G

$$K := \tau(F)K_c.$$

This set K may not be a subgroup, but it is still suitable for constructing the Cartan projection and the Iwasawa cocycle.

We define again the *Cartan decomposition* of G in an analogue way: for any g in G, we let $\kappa(g)$ be the unique element of $\mathfrak{a}^+$ such that

$$g \in K\,\omega^{-1}(\kappa(g))\,K_c. \tag{7.2}$$

For η in $\mathcal{P}$, we can write

$$\eta = k\,P_c, \quad \text{with } k \text{ in } K.$$

For g in G and η in $\mathcal{P}$, we let $\sigma(g,\eta)$ be the unique element of $\mathfrak{a}$ such that

$$g\,k \in K\,\omega^{-1}(\sigma(g,\eta))\,U. \tag{7.3}$$

This function σ is well defined since k is unique up to the right multiplication by an element of $K_c \cap P_c$.

Lemma 7.4. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. This map $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is a continuous cocycle.*

The proof is the same as for Lemma 7.2. We still call σ the *Iwasawa cocycle*

Remark 7.5. In case $\mathbb{K}$ is archimedean, we can choose K to be a maximal subgroup of G, we have $P = (K \cap P)P_c$, so that we can assume τ to take values in $K \cap P$. We retrieve the construction from the previous paragraph.

The finite group $F = P/P_c$ is still acting on the right on the flag variety $\mathcal{P} = G/P_c$ of G. With this definition of σ, we lost the property of equivariance (7.1) under the action of the group F. However, we still get

Lemma 7.6. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. For any f in F, the cocycles*

$$(g,\eta) \mapsto f^{-1}\sigma(g,\eta) \ \text{ and } \ (g,\eta) \mapsto \sigma(g,\eta f)$$

are cohomologous.

Proof. For η in $\mathcal{P}$, write $\eta = kP_c$ with k in K and let $\varphi_f(\eta)$ be the unique element of $\mathfrak{a}$ such that

$$k\,\tau_f \in K\,\omega^{-1}(\varphi_f(\eta))\,U. \tag{7.4}$$

Now, if g belongs to G, let k' and k'' be in K such that

$$g\,k \in k'\,\omega^{-1}(\sigma(g,\eta))\,U \text{ and} \tag{7.5}$$

$$k\,\tau_f \in k''\,\omega^{-1}(\varphi_f(\eta))\,U. \tag{7.6}$$

On the one hand, since $g\eta = k'\,P_c$, we have, using (7.4),

$$k'\,\tau_f \in K\,\omega^{-1}(\varphi_f(g\eta))\,U,$$

hence, using (7.5) and the fact that τ_f normalizes P_c,

$$g\,k\,\tau_f \in K\,\omega^{-1}(\varphi_f(g\eta) + f^{-1}\sigma(g,\eta))\,U.$$

On the other hand, by (7.6), $\eta f = k''\,P_c$, hence, by the definition (7.3) of σ, we have

$$g\,k'' \in K\,\omega^{-1}(\sigma(g,\eta f))\,U.$$

Therefore, using again (7.6),

$$g\,k\,\tau_f \in K\,\omega^{-1}(\sigma(g,\eta f) + \varphi_f(\eta))\,U.$$

Thus, we get

$$\varphi_f(g\eta) + f^{-1}\sigma(g,\eta) = \sigma(g,\eta f) + \varphi_f(\eta),$$

which completes the proof. $\square$

7.3. Jordan decomposition.

We introduce now the Jordan projection λ which is a multidimensional avatar of the spectral radius.

Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$.

We already discussed the case when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ in Section 5.7. Let us recall it. In this case, every element g of G has a unique decomposition, called the *Jordan decomposition*, as a product of commuting elements $g = g_e g_h g_u$, where g_e is semisimple with eigenvalues of modulus one, g_h is semisimple with positive eigenvalues and g_u is unipotent. The component g_h is conjugated to an element z_g of Z_c^+ and we let

$$\lambda(g) := \omega(z_g) \in \mathfrak{a}^+.$$

When $\mathbb{K}$ is a non-archimedean local field, we fix a uniformizing element $\varpi \in \mathbb{K}$. Every element g of G has a power g^{n_0} with $n_0 \geq 1$, which admits a *Jordan decomposition*, i.e. a decompositon as a product of commuting elements $g^{n_0} = g_e g_h g_u$, where g_e is semisimple with eigenvalues of modulus one, g_h is semisimple with eigenvalues in $\varpi^{\mathbb{Z}}$ and g_u is unipotent. The component g_h is conjugated to an element z of $A^+ := A \cap Z_c^+$ and we let

$$\lambda(g) := \tfrac{1}{n_0}\,\omega(z_g) \in \mathfrak{a}^+.$$

Remark 7.7. This map does not depend on the choices that we made, and one still have the following formula:

$$\lambda(g) = \lim_{n\to\infty} \tfrac{1}{n}\kappa(g). \tag{7.7}$$

Proof. This will follow from Lemmas 7.8, 7.15, and 7.17, and from the spectral radius formula. For more details, see [**10**]. $\square$

The following lemma tells us that $\lambda(g)$ encodes the moduli of all the eigenvalues of g in all the representations of G

Lemma 7.8. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. Let (ρ, V) be an algebraic representation of G. Then, for g in G, the moduli of the eigenvalues of $\rho(g)$ are the numbers $e^{\chi^\omega(\lambda(g))}$, where χ runs among the weights of A in V.*

In particular, if (ρ, V) is an irreducible representation of G_c with highest weight χ, the spectral radius of $\rho(g)$ is equal to $e^{\chi^\omega(\lambda(g))}$.

Proof. By definition of the Jordan projection, it is enough to prove this assertion when g admits a Jordan decomposition $g = g_e g_h g_u$. Then, since all the eigenvalues of $\rho(g_u)$ are equal to one, and since all the eigenvalues of $\rho(g_e)$ have modulus one, one can assume $g = g_h$. In this case, g is conjugated to an element of A^+ and one can also assume $g \in A^+$. Now, the eigenvalues of $\rho(g)$ in V are the numbers $\chi(g)$ and the result follows. $\square$

7.4. Representations of reductive groups.

In the next section, we will explain how to analyze the behavior of the Iwasawa cocycle of G thanks to suitable representations of G endowed with *good norms*.

We construct these representations and their norms in this section, extending the construction of Section 5.8.

Let (ρ, V) be an *algebraic* representation of G. This means that V is a finite dimensional $\mathbb{K}$-vector space and ρ is the restriction to G of a $\mathbb{K}$-rational representation $(\rho, \mathbf{V})$ of $\mathbf{G}$. For any character χ of A, we let V^χ be the associated weight space in V, that is,

$$V^\chi = \{v \in V \mid \forall a \in A\ \rho(a)v = \chi(a)v\}$$

and, for v in V, we set v^χ for its A-equivariant projection on V^χ.

7.4.1. *Good norms for connected groups.* Assume $\mathbf{G}$ is connected i.e. $\mathbf{G} = \mathbf{G}_c$.

In case $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, a norm $\|.\|$ on V is said to be *good* or (ρ, A, K_c) good, if it is Euclidean and if the elements of $\rho(K_c)$ are $\|.\|$-unitary and those of $\rho(A)$ are $\|.\|$-symmetric.

In case $\mathbb{K}$ is non-archimedean, a norm $\|.\|$ on V is said to be (ρ, A, K_c) good, if it is ultrametric, $\rho(K_c)$-invariant, if, for any v in V, one has $\|v\| = \max_\chi \|v^\chi\|$ and if, for any character χ of A, any v in V^χ and z in Z, one has

$$\|\rho(z)v\| = e^{\chi^\omega(\omega(z))} \|v\| .$$

The following lemma tells us that, for connected groups, good norms always exist.

Lemma 7.9. *Let G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group $\mathbf{G}$. For any algebraic representation (ρ, V) of G, such a good norm on V always exists.*

Proof. In the archimedean case, we gave the proof in Lemma 5.33. In the non-archimedean case, this is proved in [**99**, §6].

□

Remark 7.10. In case $\mathbb{K}$ is archimedean and $\mathbf{G}$ non-connected Lemma 7.9 is still true. However, when $\mathbb{K}$ is non-archimedean and $\mathbf{G}$ non-connected Lemma 7.9 is not always true.

7.4.2. *Good norms in induced representations.* Our aim now is to state a lemma which will play the role of Lemma 7.9 for non-connected groups $\mathbf{G}$. This will be Lemma 7.13 below.

First, let us recall some general facts from representation theory.

Let Γ be a group and Δ be a subgroup of Γ. Given a representation ρ of Δ in V, the *induced representation* $\mathrm{Ind}_\Delta^\Gamma(\rho)$ is the space W of maps $\varphi : \Gamma \to V$ such that, for any g in Γ, h in Δ, one has

$$\varphi(gh) = \rho(h)^{-1}\varphi(g),$$

equipped with the natural action of Γ, that is,

$$g\varphi(g') = \varphi(g^{-1}g') \text{ for any } g, g' \text{ in } \Gamma \text{ and } \varphi \text{ in } W.$$

For any f in Γ/Δ, define $V_f \subset W$ as the space of φ in W with $\varphi_{|f'\Delta} = 0$ for $f' \neq f$ in Γ/Δ. Then V_f is $f\Delta f^{-1}$-invariant and one has

$$W = \bigoplus\nolimits_{f\in\Gamma/\Delta} V_f.$$

For v in W, we let v_f be its component in V_f for this decomposition.

In all the sequel, we identify V and V_e through the map that sends some v in V to the function φ such that $\varphi(h) = \rho(h^{-1})v$ for h in Δ and $\varphi(g) = 0$ for g in $\Gamma \smallsetminus \Delta$.

Even if V is irreducible, the induced representation is not necessarily irreducible. For instance, when V is trivial, the induced representation W is the regular representation of Γ on Γ/Λ. However, we have the following Lemma 7.11 which will allow us to project induced representations in irreducible quotients. This technical lemma will be used in the proof of Theorem 9.9.

Lemma 7.11. *Let Γ be a group and Δ be a finite index subgroup of Γ. If V is a vector space and ρ an irreducible representation of Δ in V, for any proper Γ-invariant subspace U of $W = \mathrm{Ind}_\Delta^\Gamma(\rho)$, for any f in Γ/Δ, one has $V_f \cap U = \{0\}$.*

Remark 7.12. Assume W/U is Γ-irreducible and V is Δ-strongly irreducible. Then the image of $(V_f)_{f\in\Gamma/\Delta}$ in W/U is a transitive strongly irreducible Γ-family.

Proof. As W is spanned by the V_f, there exists f in Γ/Δ with $V_f \not\subset U$. Since V_f is $f\Delta f^{-1}$-irreducible, we have $V_f \cap U = \{0\}$. Since U is Γ-invariant, we have $V_{f'} \cap U = \{0\}$ for any f' in Γ/Δ, which was to be shown. □

Let us come back to the context of reductive groups. Given an algebraic representation ρ of G_c in V, the induced representation $\mathrm{Ind}_{G_c}^G(\rho)$ in W is an algebraic representation of G. We will only define the *good norms* for these induced representations.

In case $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, a norm on W is (ρ, A, K_c)-*good* if it is Euclidean and K-invariant, if the sum $W = \bigoplus_{f\in F} V_f$ is orthogonal and if the elements of A act as symmetric endomorphisms on W.

In case $\mathbb{K}$ is non-archimedean, a norm on W is (ρ, A, K_c, τ)-*good* if it is ultrametric, if, for any v in W, $\|v\| = \max_{f\in F}\|v_f\|$ and if the restriction of the norm to V is (ρ, A, K_c)-good and if , for any f in F, the element τ_f induces an isometry $V \to V_f$.

The following lemma tells us that such good norms do exist.

Lemma 7.13. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. For any algebraic representation (ρ, V) of G_c, the induced representation $\mathrm{Ind}_{G_c}^G(\rho)$ always admits such a good norm.*

Proof. In case $\mathbb{K}$ is archimedean, the proof mimics the connected case. In case $\mathbb{K}$ is non-archimedean, we fix a (ρ, A, K_c)-good norm on V, which exists by Lemma 7.9. Now, for f in F, we equip V_f with the image of this norm by τ_f, and we set $\|v\| = \max\limits_{f\in F}\|v_f\|$. □

7.4.3. *Highest weight.* Let (ρ, V) be an algebraic representation of G_c.

Let χ be a *parabolic weight* of A in V i.e. χ is a weight of A in the space
$$V^U := \{v \in V \mid Uv = v\}.$$

We write $V^{U,\chi}$ for the corresponding *weight space*

$$V^{U,\chi} := V^U \cap V^\chi.$$

One has

$$P_c V^{U,\chi} \subset V^{U,\chi}.$$

If (ρ, V) is an irreducible representation of G_c, it admits a unique parabolic weight which is also the largest weight and is traditionally called the *highest weight* . If (ρ, V) extends as a representation of G, the set of parabolic weights is stable under the natural action of F. Moreover, if (ρ, V) is an irreducible representation of G, all the parabolic weights of V belong to the same F-orbit and the parabolic weights are exactly the maximal weights for the order (5.9).

Set $W = \mathrm{Ind}_{G_c}^G(\rho)$. Let χ be a parabolic weight of (ρ, V) and $r = r_\chi = \dim V^{U,\chi}$. The map $g \mapsto gV^{U,\chi}$ factors as a map

$$\begin{aligned} \mathcal{P} &\to \textstyle\bigcup_{f\in F} \mathbb{G}_r(V_f) \\ \eta = gP_c &\mapsto V_{\chi,\eta} := gV^{U,\chi}. \end{aligned} \tag{7.8}$$

If V is G_c-irreducible, we write V_η for $V_{\chi,\eta}$.

7.4.4. *Proximal representations.* Let (ρ, V) be an irreducible algebraic representation of G. The representation (ρ, V) is said to be *proximal* if there exists a parabolic weight χ of A in V whose corresponding weight space is a line: $\dim V^{U,\chi} = 1$. In this case, the other parabolic weight spaces $V^{f\chi}$ also are one-dimensional.

Remark 7.14. A strongly irreducible algebraic representation (ρ, V) of G is proximal if and only if there exists g in G such that $\rho(g)$ is a proximal endomorphism of V.

7.4.5. *Construction of representations.* We quote now a lemma which constructs a few proximal representations of G_c. Recall that we already quoted this construction for real Lie groups in Lemma 5.32.

Lemma 7.15. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. For every α in Π, there exists a proximal irreducible algebraic representation $(\rho_{c,\alpha}, V_{c,\alpha})$ of G_c with a highest weight χ_α such that χ_α^ω is a multiple of the fundamental weight ϖ_α^ω associated to α^ω.*

Moreover any product $\chi = \prod_{\alpha\in\Pi} \chi_\alpha^{n_\alpha}$ with $n_\alpha \geq 0$ is also the highest weight of a proximal irreducible representation of G.

Proof. As for Lemma 5.32, we refer to [**121**]. $\square$

This condition on χ_α means that χ_α^ω is orthogonal to β^ω for every simple root $\beta \neq \alpha$ and also for every character of G_c.

The others weights of A in $V_{c,\alpha}$ are $\chi_\alpha - \alpha$ and weights of the form $\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta \beta$, where, for β in Π, n_β belongs to $\mathbb{N}$. In particular, for any z in Z_c^+, the endomorphism $\rho_{c,\alpha}(z)$ is a proximal endomorphism of $V_{c,\alpha}$ if and only if $\alpha^\omega(\omega(z)) > 0$.

Definition 7.16. We fix once for all such a family of representations $(\rho_{c,\alpha}, V_{c,\alpha})$ of G_c, for α in Π, and we let (ρ_α, V_α) be the induced representation $\mathrm{Ind}_{G_c}^G(\rho_{c,\alpha})$, which we equip with a $(\rho_\alpha, A, K_c, \tau)$-good norm.

7.5. Representations and Iwasawa cocycle.

We relate now κ, σ and λ to norm behavior in representations: the Cartan projection controls the norm of the image matrices in all representations, the Iwasawa cocycle controls the growth of highest weight vectors, and the Jordan projection controls the spectral radius.

We first state these properties as a lemma when $\mathbf{G}$ is connected. This lemma explains why the Cartan projection, the Iwasawa cocycle and the Jordan projection, can be seen as mutidimensional avatars of the norm, the norm cocycle and the spectral radius.

Lemma 7.17. *Let G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group $\mathbf{G}$. Let (ρ, V) be an irreducible algebraic representation of G equipped with a (ρ, A, K_c)-good norm and χ be the highest weight of A in V. Then, one has, for any g in G,*

$$\|\rho(g)\| = e^{\chi^\omega(\kappa(g))}, \tag{7.9}$$

for any η in $\mathcal{P}$ and v in V_η,

$$\|\rho(g)v\| = e^{\chi^\omega(\sigma(g,\eta))}\,\|v\|\,, \tag{7.10}$$

and

$$\lambda_1(\rho(g)) = e^{\chi^\omega(\lambda(g))}. \tag{7.11}$$

As we will see, this lemma is an application of the definitions of the Cartan projection, the Iwasawa cocycle, the Jordan projection and the good norms.

Here is the extension of Lemma 7.17 to non-connected groups $\mathbf{G}$. We let $s : G \to G_c$ be the natural morphism.

Lemma 7.18. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group $\mathbf{G}$. Let (ρ, V) be an algebraic irreducible representation of G_c, χ be the highest weight of A in V and $W = \mathrm{Ind}_{G_c}^G(V)$. Equip W with* *a (ρ, A, K_c, τ)-good norm. For any g in G, one has $\rho(g)V = V_{s(g)}$ and the norm of g as a linear operator between these G_c-submodules is*

$$\|\rho(g)_{|V}\| = e^{\chi^\omega(\kappa(g))}, \tag{7.12}$$

For η in $\mathcal{P}$ and v in the space V_η, one has

$$\|gv\| = e^{\chi^\omega(\sigma(g,\eta))}\,\|v\|\,, \tag{7.13}$$

and, introducing the sum V' of the images g^nV for $n \geq 0$,

$$\lambda_1(\rho(g)|_{V'}) = e^{\chi^\omega(\lambda(g))}. \tag{7.14}$$

Remark 7.19. These formulae are the main reason, and also the main tool, for us to study the behavior of the Iwasawa cocycle and the Cartan projection of a large product of random elements.

Proof. First, we prove (7.12). Write

$$g \in KzK_c \ \text{ with } z \text{ in } Z.$$

By Definition (7.2), one has

$$\omega(z) = \kappa(g) \in \mathfrak{a}^+.$$

By construction, we have

$$\|\rho(g)_{|V}\| = \|\rho(z)_{|V}\|$$

and the result follows since χ is the highest weight of A in V.

Now, we prove (7.13). Write

$$\eta = k\,P_c \ \text{ with } k \text{ in } K \text{ and}$$

$$g\,k = k'zu \ \text{ with } u \text{ in } U,\ k' \text{ in } K,\ z \text{ in } Z_c.$$

By definition (7.3), one has

$$\omega(z) = \sigma(g, \eta).$$

Setting $w = k^{-1}v$, so that w is in V^χ and $\|w\| = \|v\|$, one has

$$gv = gkw = k'zuw = k'zw$$

and

$$\|gv\| = \|zw\| = e^{\chi^\omega(\omega(z)))}\,\|w\| = e^{\chi^\omega(\sigma(g,\eta))}\,\|v\|\,.$$

The proof of (7.14) is similar. □

Equip once for all $\mathfrak{a}$ with a Euclidean norm $\|.\|$ which is invariant by the Weyl group W and by F. In order to control the size of elements in $\mathfrak{a}$, we just have to control the image of these elements by sufficiently many linear functionals on $\mathfrak{a}$. The following corollary gives examples of application of this technique similar to those in Corollary 5.34.

Corollary 7.20. *Let G be the group of $\mathbb{K}$-points of a reductive $\mathbb{K}$-group.*
a) For every g in G and η in $\mathcal{P}_c$, one has

$$\sigma(g,\eta) \in \mathrm{Conv}(W\kappa(g)), \tag{7.15}$$

in particular, one has

$$\|\sigma(g,\eta)\| \leq \|\kappa(g)\|. \tag{7.16}$$

b) For every g_1, g_2 in G, one has

$$\|\kappa(g_1g_2)\| \leq \|\kappa(g_1\tau_2) + \kappa(g_2)\|, \tag{7.17}$$

where $\tau_2 = \tau_{s(g_2)} \in F$. In particular, one has

$$\|\kappa(g_1g_2)\| \leq \|\kappa(g_1\tau_2)\| + \|\kappa(g_2)\|. \tag{7.18}$$

c) Moreover, there exists $C > 0$ such that, for every g, g_1, g_2 in G_c,

$$\|\kappa(g_1gg_2) - \kappa(g)\| \leq C(\|\kappa(g_1)\| + \|\kappa(g_2)\|) \tag{7.19}$$

and, for every g, g_1, g_2 in G,

$$\|\kappa(g_1gg_2) - f_2^{-1}\kappa(g)\| \leq C(\|\kappa(g_1)\| + \|\kappa(g_2)\| + 1), \tag{7.20}$$

where $f_2 = s(g_2)$. Moreover, for any g in G and f in F,

$$\left\|\kappa(g\,\tau_f) - f^{-1}\kappa(g)\right\| \leq C. \tag{7.21}$$

Proof of Corollary 7.20. *a*) See [**81**] for a more precise statement when G is a real Lie group. Here is a short proof. We may assume that G is semisimple. Besides, since we have, by construction, for any g in G and η in $\mathcal{P}_c$, $\kappa(\tau_{s(g)}^{-1}g) = \kappa(g)$ and $\sigma(\tau_{s(g)}^{-1}g,\eta) = \sigma(g,\eta)$, we may assume that $\mathbf{G}$ is connected.

For p in $\mathfrak{a}^+$, we introduce the set

$$C_p := \{q \in \mathfrak{a} \mid \chi^\omega_\alpha(wq) \leq \chi^\omega_\alpha(p) \text{ for all } w \text{ in } W,\ \alpha \text{ in } \Pi\}.$$

First step: We check that

$$\mathrm{Conv}(Wp) = C_p. \tag{7.22}$$

Since C_p is convex and W-invariant, in order to prove the inclusion $\mathrm{Conv}(Wp) \subset C_p$, we only have to check that p belongs to C_p. Since p is dominant i.e. belongs to $\mathfrak{a}^+$, for every w in W, $p - wp$ is a positive linear combination of simple roots and hence $\chi^\omega_\alpha(wp) \leq \chi^\omega_\alpha(p)$ for all α in Π.

In order to prove the inclusion $\mathrm{Conv}(Wp) \supset C_p$, by Krein-Milman Theorem, it suffices to prove that any extremal point q of C_p belongs to Wp. Since C_p is W-invariant, we may assume that q is dominant and we want to prove that $q = p$. If this is not the case, there exists $\alpha \in \Pi$

such that $\chi^\omega_\alpha(q) < \chi^\omega_\alpha(p)$, but then, for ε small enough, the interval $q + [-\varepsilon, \varepsilon]\alpha^\omega$ is included in C_p, whence a contradiction.

Second step: We have the equivalence, for g, g' in G,

$$\kappa(g') \in \mathrm{Conv}(W\kappa(g)) \iff \|\rho(g')\| \leq \|\rho(g)\| \text{ for all } \rho. \tag{7.23}$$

In the right-hand side of this equivalence, "for all ρ" means for all ireducible algebraic representation (ρ, V) of G endowed with a (ρ, A, K_c) good norm. This equivalence follows from the first step and Equality (7.9) applied to all the representations (ρ_α, V_α) introduced in 7.4.5.

Third step: Let (ρ, V) be an irreducible algebraic representation of G endowed with a (ρ, A, K_c) good norm. For all z in Z and u in U, one has

$$\|\rho(z)\| \leq \|\rho(zu)\|. \tag{7.24}$$

Indeed, let χ be a weight of A in V such that $\chi^\omega(\omega(z))$ is maximal. Since the norm is (ρ, A, K_c)-good, we have $\|\rho(z)\| = e^{\chi^\omega(\omega(z))}$. Now, if $v \neq 0$ is a vector in V_χ, we have

$$\rho(u)v \in v + \textstyle\bigoplus_{\chi' \neq \chi} V^{\chi'}.$$

Again, since the norm is (ρ, A, K_c)-good, we get

$$\|\rho(zu)v\| \geq \|\rho(z)v\| = e^{\chi^\omega(\omega(z))}\,\|v\| = \|\rho(z)\|\,\|v\|\,,$$

and we are done.

Fourth step: We prove (7.15). Write $\eta = k_0 P_c$ with k_0 in K_c, $g = k_1 z^+ k_2$ with k_1, k_2 in K_c and z^+ in Z^+, so that $\kappa(g) = \omega(z^+)$. Write $gk_0 = kzu$ with k in K_c, z in Z and u in U, so that $\sigma(g, \eta) = \omega(z)$. According to Inequality (7.24), one has, for any ρ,

$$\|\rho(z)\| \leq \|\rho(z^+)\|. \tag{7.25}$$

Now (7.15) follows from (7.23) and (7.25).

b) Let (ρ, V) be an irreducible representation of G_c with highest weight χ and equip the induced representation $\mathrm{Ind}_{G_c}^G(\rho) = \bigoplus_{f \in F} V_f$ with a (ρ, A, K_c, τ)-good norm. We have, setting $f_2 = s(g_2)$,

$$e^{\chi^\omega(\kappa(g_1 g_2))} = \|\rho(g_1 g_2)|_V\| \leq \|\rho(g_1)|_{V_{f_2}}\|\|\rho(g_2)|_V\| = \|\rho(g_1)|_{V_{f_2}}\|e^{\chi^\omega(\kappa(g_2))}.$$

Now, since τ_2 induces an isometry between V and V_{f_2},

$$\|\rho(g_1)|_{V_{f_2}}\| = \|\rho(g_1\tau_2)|_V\| = e^{\chi^\omega(\kappa(g_1\tau_2))}.$$

Applying this property to the representations (ρ_α, V_α), $\alpha \in \Pi$, and using (7.22) one gets

$$\kappa(g_1 g_2) \in \mathrm{Conv}(W(\kappa(g_1\tau_2) + \kappa(g_2))).$$

This implies (7.17) and (7.18).

c) Again, if (ρ, V) is an irreducible representation of G_c with highest weight χ, equipped with a (ρ, A, K_c)-good norm, for g, g_1 and g_2 in G_c, we have

$$\|\rho(g_1^{-1})\|^{-1}\|\rho(g_2^{-1})\|^{-1} \leq \|\rho(g_1 g g_2)\|/\|\rho(g)\| \leq \|\rho(g_1)\|\,\|\rho(g_2)\|,$$

hence

$$-\chi^\omega(\iota(\kappa(g_1)+\kappa(g_2))) \leq \chi^\omega(\kappa(g_1 g g_2)-\kappa(g)) \leq \chi^\omega(\kappa(g_1)+\kappa(g_2)),$$

which gives (7.19), since the dual space of $\mathfrak{a}$ is spanned by finitely many highest weights of representations. Now, (7.19) and (7.20) are proved in the same way by using the good norms in $W = \mathrm{Ind}_{G_c}^G(\rho)$ and the fact that the finite set $\tau(F)$ has bounded image in $\mathrm{GL}(W)$. The bound (7.21) follows immediately. $\square$

7.6. Partial flag varieties.

When $\mathbb{K} \neq \mathbb{R}$, we need to introduce also the partial flag varieties associated to subsets $\Theta \subset \Pi$. When $\mathbb{K}$ is $\mathbb{R}$, the subset $\Theta = \Pi$ is the only one which will be useful in this text.

For $\Theta \subset \Pi$, let $\mathbf{A}_\Theta$ be the intersection of the kernels of the elements of $\Pi \smallsetminus \Theta$ in $\mathbf{A}$ and $\mathbf{Z}_{\Theta,c}$ be the centralizer of $\mathbf{A}_\Theta$ in $\mathbf{G}_c$. Set $\mathbf{P}_{\Theta,c} = \mathbf{Z}_{\Theta,c}\mathbf{U}$. For instance, one has

$$\mathbf{A}_\Pi = \mathbf{A},\ \mathbf{A}_\emptyset = \mathbb{K}\text{-split center of } \mathbf{G}_c,\ \mathbf{P}_{\Pi,c} = \mathbf{P}_c,\ \mathbf{P}_\emptyset = \mathbf{G}_c.$$

The $\mathbb{K}$-groups $\mathbf{P}_{\Theta,c}$, $\Theta \subset \Pi$, are exactly the $\mathbb{K}$-subgroups of $\mathbf{G}_c$ which contain $\mathbf{P}_c$. Set $P_{\Theta,c} = \mathbf{P}_{\Theta,c}(\mathbb{K})$, and introduce the *partial flag variety* of G and G_c

$$\mathcal{P}_\Theta := G/P_{\Theta,c} \ \text{ and } \ \mathcal{P}_{\Theta,c} = G_c/P_{\Theta,c}.$$

Those partial flag varieties will be better understood tanks to the representations (ρ_α, V_α) in Definition 7.16. For any $\alpha \in \Theta$, one has $\rho_\alpha(P_{\Theta,c})(V_{c,\alpha})^{\chi_\alpha} \subset (V_{c,\alpha})^{\chi_\alpha}$ and the map

$$\begin{aligned} \mathcal{P}_\Theta &\to \textstyle\bigcup_{f\in F} \mathbb{P}(V_{\alpha,f}) \\ \eta = gP_{\Theta,c} &\mapsto V_{\alpha,\eta} := \rho_\alpha(g)(V_{c,\alpha})^{\chi_\alpha} \end{aligned}$$

is well defined. The product map

$$\mathcal{P}_\Theta \to \textstyle\prod_{\alpha\in\Theta}(\bigcup_{f\in F}\mathbb{P}(V_{\alpha,f})) \tag{7.26}$$

is a G-equivariant embedding. Set, $\Theta^c := \Pi \smallsetminus \Theta$,

$$\begin{aligned} \mathfrak{a}_\Theta &= \{x \in \mathfrak{a} \mid \forall \alpha \in \Theta^c \quad \alpha^\omega(x) = 0\}, \\ \mathfrak{a}_\Theta^+ &= \mathfrak{a}_\Theta \cap \mathfrak{a}^+ \ \text{ and} \\ \mathfrak{a}_\Theta^{++} &= \{x \in \mathfrak{a}_\Theta^+ \mid \forall \alpha \in \Theta\ ,\ \ \alpha^\omega(x) > 0\}. \end{aligned}$$

We let $W_\Theta \subset \mathrm{GL}(\mathfrak{a})$ be the subgroup of the Weyl group of Σ^ω spanned by the reflections associated to the elements of $\Pi \smallsetminus \Theta$. Then, $\mathfrak{a}_\Theta$ is the space of fixed points of W_Θ in $\mathfrak{a}$. For instance, $\mathcal{P}_\Pi = \mathcal{P}$, $\mathfrak{a}_\Pi = \mathfrak{a}$ and $W_\Pi = W$, while $\mathcal{P}_\emptyset = F$, $\mathfrak{a}_\emptyset$ is the subspace of $\mathfrak{a}$ spanned by the image of the center of $\mathbf{G}_c$ by ω and $W_\emptyset = \{1\}$. One let $p_\Theta : \mathfrak{a} \to \mathfrak{a}_\Theta$ denote the unique W_Θ-equivariant projection.

LEMMA 7.21. *The image $p_\Theta \circ \sigma : G \times \mathcal{P} \to \mathfrak{a}_\Theta$ of the Iwasawa cocycle σ by p_Θ factors as a cocycle*

$$\sigma_\Theta : G \times \mathcal{P}_\Theta \to \mathfrak{a}_\Theta. \tag{7.27}$$

We call this cocycle the *partial Iwasawa cocycle.*

PROOF. In case $\mathbf{G}$ is connected, this is proved for example in [**100**, lemme 6.1]. In general, from the connected case, we get, for any g in G and z in $Z_{\Theta,c}$,

$$p_\Theta(\sigma(g, \xi_\Pi)) = p_\Theta(\sigma(\tau_{s(g)}^{-1} g, \xi_\Pi)) = p_\Theta(\sigma(\tau_{s(g)}^{-1} g, z\xi_\Pi)) = p_\Theta(\sigma(g, z\xi_\Pi))$$

and, by the cocycle property, the same holds for any η in $\mathcal{P}$. □

Assume that the subset $\Theta \subset \Pi$ is stable by F. On the one hand the right action of F on $\mathcal{P}$ factors as an action on $\mathcal{P}_\Theta$. On the other hand, the subspace $\mathfrak{a}_\Theta$ is F-invariant and the projection p_Θ commutes with F. One has the following generalisation of Lemma 7.6.

LEMMA 7.22. *Assume Θ is F-invariant. For any f in F, the cocycles $(g, \eta) \mapsto f^{-1}\sigma_\Theta(g, \eta)$ and $(g, \eta) \mapsto \sigma_\Theta(g, \eta f)$ are cohomologous.*

PROOF. This follows from the proof of Lemma 7.6. Keeping the notations of this proof, we just have to notice that the function $p_\Theta \circ \varphi_f$ descends to $\mathcal{P}_\Theta$ and hence gives the required coboundary. □

Still assume that the set Θ is F-stable. Let $\mathbf{P}_\Theta \subset \mathbf{G}$ be the normalizer of $\mathbf{P}_{\Theta,c}$ and P_Θ be its group of $\mathbb{K}$-points. Since $P_\Theta \cap G_c = P_{\Theta,c}$ and $P \subset P_\Theta$, the natural map

$$P_\Theta / P_{\Theta,c} \to F$$

is an isomorphism. Since Θ is F-stable, for every g in G, $gP_{\Theta,c}g^{-1}$ is conjugated in G_c to P_Θ, that is, we have $G = G_c P_\Theta$ and the natural map

$$\mathcal{P}_{\Theta,c} = G_c / P_{\Theta,c} \to G / P_\Theta$$

is an isomorphism. To summarize, G acts in a natural way on $\mathcal{P}_{\Theta,c}$ and we have a G-equivariant identification

$$\mathcal{P}_\Theta \simeq \mathcal{P}_{\Theta,c} \times F. \tag{7.28}$$

Under this identification, the action of F o $\mathcal{P}_\Theta$ reads as its right action on the second factor.

For $G = \mathrm{SL}(d, \mathbb{K})$, one can describe concretely the parabolic subgroups P_Θ and their unipotent radical U_Θ. Choosing for instance Θ^c with only one simple root, that is, with the notations of Section 5.7.7, choosing $\Theta^c = \{\varepsilon_{i+1} - \varepsilon_i\}$ for some $1 \le i < d$, one has, in terms of block matrices with blocks of size i and $d-i$,

$$P_\Theta = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad , \quad U_\Theta = \left\{ \begin{pmatrix} \mathbf{1} & * \\ 0 & \mathbf{1} \end{pmatrix} \right\} \quad .$$

Note that another value for Θ would give different numbers and sizes of block matrices.

7.7. Algebraic reductive $\mathcal{S}$-adic Lie groups.

In this section we introduce the class of locally compact groups that we will work with. This class contains both the reductive algebraic real Lie groups and the reductive algebraic p-adic Lie groups.

We now let $\mathcal{S}$ be a finite set of local fields. For any s in $\mathcal{S}$, we will sometimes denote by $\mathbb{K}_s$ the local field s. These local fields are two by two non isomorphic.

Definition 7.23. An *algebraic $\mathcal{S}$-adic Lie group G* is a subgroup of a product $G \subset \prod_{s\in\mathcal{S}} G_s$ such that,
- for s in $\mathcal{S}$, the group G_s is the group of $\mathbb{K}_s$-points of a $\mathbb{K}_s$-group $\mathbf{G}_s$,
- G contains the finite index subgroup $G_c := \prod_{s\in\mathcal{S}} G_{s,c}$, and,
- for s in $\mathcal{S}$, the projection map $G \to G_s$ is onto.

We denote by F the finite group $F = G/G_c$. We say that G is *connected* if $G = G_c$. We say that G is *reductive* if the $\mathbb{K}_s$-groups $\mathbf{G}_s$ are reductive.

The *real factor* $G_\mathbb{R}$ of G will mean the group G_s for $\mathbb{K}_s = \mathbb{R}$.

We keep the notations of Sections 7.4 and 7.6, adding a subscript s to each of them: thus, $\mathcal{P}_s$ is the flag manifold of G_s, $\mathfrak{a}_s$ a Cartan space for G_s, Π_s a set of simple restricted roots, etc. We set $P_c = \prod_{s\in\mathcal{S}} P_{s,c}$, $\mathfrak{a} = \prod_{s\in\mathcal{S}} \mathfrak{a}_s$. We define the flag variety of G as $\mathcal{P} := G/P_c$. It is an open and compact G-orbit in the product of the flag varieties $\prod_{s\in\mathcal{S}} \mathcal{P}_s$.

We define the *Cartan projection* of G

$$\kappa : G \to \mathfrak{a}$$

as the map obtained by taking the product of the Cartan projections $\kappa_s : G_s \to \mathfrak{a}_s$ of G_s, $s \in \mathcal{S}$.

We define the *Iwasawa cocycle* of G

$$\sigma : G \times \mathcal{P} \to \mathfrak{a}$$

as the cocycle obtained by taking he product of the Iwasawa cocycles $\sigma_s : G_s \times \mathcal{P}_s \to \mathfrak{a}_s$ of G_s, $s \in \mathcal{S}$.

We define the *Jordan projection* of G

$$\lambda : G \to \mathfrak{a}$$

as the map obtained by taking the product of the Jordan projections $\lambda_s : G_s \to \mathfrak{a}_s$ of G_s, $s \in \mathcal{S}$.

When Θ is an F-invariant subset of the set $\Pi := \bigsqcup_{s\in\mathcal{S}} \Pi_s$, we set $P_{\Theta,c} = \prod_{s\in\mathcal{S}} P_{\Theta_s,s,c}$, $\mathcal{P}_\Theta = G/P_{\Theta,c}$, $\mathcal{P}_{\Theta,c} = G/P_\Theta$, $\mathfrak{a}_\Theta = \prod_{s\in\mathcal{S}} \mathfrak{a}_{\Theta_s}$ where, for any s in $\mathcal{S}$, $\Theta_s = \Pi_s \cap \Theta$. We set $p_\Theta : \mathfrak{a} \to \mathfrak{a}_\Theta$ to be the projection and the *partial Iwasawa cocycle*

$$\sigma_\Theta : G \times \mathcal{P}_\Theta \to \mathfrak{a}_\Theta. \tag{7.29}$$

to be the cocycle which is the product of the cocycles $\sigma_{\Theta_s} : G_s \times \mathcal{P}_{\Theta_s} \to \mathfrak{a}_{\Theta_s}$, $s \in \mathcal{S}$.

As a shorthand, we will say that a representation (ρ, V) of G in a $\mathbb{K}_s$-vector space is *algebraic* if it factors as an algebraic representation of the quotient group G_s. We will say that this representation is *proximal* if it is proximal as a representation of G_s, and so on...

8. Zariski dense subsemigroups

This is the third chapter which is devoted to Zariski dense subsemigroups. While Chapters 5 and 6 were dealing with algebraic reductive real Lie groups, the present chapter is dealing with algebraic reductive $\mathcal{S}$-adic Lie groups. We freely use the language of Section 7.

8.1. Zariski dense subsemigroups.

In this section we introduce the set Θ_Γ of simple roots associated to a Zariski dense subsemigroup Γ of G.

Let G be a reductive algebraic $\mathcal{S}$-adic Lie group. As a shorthand, we will say that a subsemigroup Γ of G is *Zariski dense in* G if Γ is not included in a proper algebraic $\mathcal{S}$-adic Lie subgroup H of G. Equivalently, Γ is Zariski dense in G if, for each s in $\mathcal{S}$, the projection Γ_s of Γ on the reductive algebraic $\mathbb{K}_s$-algebraic group G_s is Zariski dense, and if one has the equality $G = \Gamma G_c$. In this case, we set

$$\Theta_\Gamma := \{\alpha \in \Pi \mid \alpha^\omega(\kappa(\Gamma)) \text{ is not bounded}\}. \tag{8.1}$$

By Theorem 5.36, this set Θ_Γ always contains the set $\Pi_\mathbb{R}$ of simple roots of the real Lie group $G_\mathbb{R}$ In particular, one has

(8.2) *When G is a reductive algebraic real Lie group, this set Θ_Γ is equal to Π and the partial flag variety $\mathcal{P}_{\Theta_\Gamma}$ is equal to the full flag variety $\mathcal{P}$.*

LEMMA 8.1. *Let Γ be a Zariski dense subsemigroup of G. Then one has the equality*

$$\Theta_\Gamma = \Theta_{\Gamma\cap G_c}. \tag{8.3}$$

Moreover the set Θ_Γ is F-stable.

PROOF. The first assertion follows from Corollary 7.20.c).

Pick f in F and g in Γ such that $s(g) = f$. Again using Corollary 7.20.c), one has $\sup_{\gamma\in\Gamma} \|\kappa(\gamma g) - f^{-1}\kappa(\gamma)\| < \infty$. The second assertion follows. $\square$

Note that, by the spectral radius formula (7.7), for g in Γ, one has $\lambda(g) \in \mathfrak{a}_{\Theta_\Gamma}$.

8.2. Loxodromic elements in semigroups.

In this section, we give a few properties of the set Θ_Γ.

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. For $\Theta \subset \Pi$, we say that *an element g of G_c is Θ-proximal* if, for every α in Θ, $\rho_\alpha(g)$ is a proximal endomorphism of V_α (where the ρ_α are as in section 7.4.5). This amounts to say that the action of g on $\mathcal{P}_{\Theta,c}$ admits an attracting fixed point $\xi^+_{\Theta,g}$. For any α in Θ, the line $V_{\alpha,\xi^+_{\Theta,g}} \subset V_\alpha$ is then the eigenspace associated to the dominant eigenvalue of $\rho_\alpha(g)$.

According to Lemma 7.8, an element

$$g \text{ is } \Theta\text{-proximal if and only if } \alpha^\omega(\lambda(g)) > 0 \text{ for any } \alpha \text{ in } \Theta$$

and one then has

$$\sigma_\Theta(g, \xi^+_{\Theta,g}) = p_\Theta(\lambda(g)).$$

Let Γ be a Zariski dense subsemigroup of G. Note that the set Θ_Γ is also the set of simple roots α for which $\rho_\alpha(\Gamma)$ is proximal.

The following lemma proves the existence of elements in Γ which are simultaneously proximal in these representations ρ_α. It is an extension of Lemma 5.25 where we allow simultaneously representations of Γ over different local fields.

LEMMA 8.2. *Let G be a connected algebraic reductive $\mathcal{S}$-adic Lie group and Γ be a Zariski dense subsemigroup of G.*
a) Then, the semigroup Γ contains Θ_Γ-proximal elements.
b) More precisely, the set of Θ_Γ-proximal elements of Γ is Zariski dense in G.

The proof uses the following

LEMMA 8.3. *Let G be a connected algebraic reductive $\mathcal{S}$-adic Lie group and Γ be a Zariski dense subsemigroup of G. For $i = 1, \dots, s$, let (ρ_i, V_i) be an algebraic irreducible representation of G, v_i be a nonzero vector of V_i and W_i be a proper subspace of V_i. Then there exists g in Γ such that $gv_i \notin W_i$ for all $1 \le i \le s$.*

PROOF OF LEMMA 8.3. In case G is an algebraic group over a fixed local field, this follows from Zariski connectedness of G. In general, the main new difficulty is that the representations may be defined over different fields.

We may assume that Γ is closed. Then, we can choose a Zariski dense probability measure μ on G such that $\Gamma = \Gamma_\mu$.

By Lemma 3.6, for $1 \le i \le s$, if ν is a μ-stationary Borel probability measure on $\mathbb{P}(V_i)$, we have

$$\nu(\mathbb{P}(W_i)) = 0.$$

Let x_i be the image of v_i in $\mathbb{P}(V_i)$. Since every limit point of the sequence of probability measures

$$\tfrac{1}{n} \textstyle\sum_{k=1}^n \mu^{*k} * \delta_{x_i}$$

is μ-stationary, we get

$$\tfrac{1}{n} \textstyle\sum_{k=1}^n \mu^{*k}\{g \in G \mid gv_i \in W_i\} \xrightarrow[n\to\infty]{} 0.$$

Pick n large enough so that each of these terms is $< \frac{1}{s}$. We get

$$\tfrac{1}{n} \textstyle\sum_{k=1}^n \mu^{*k}\{g \in G \mid \forall 1 \le i \le s \quad gv_i \notin W_i\} > 0$$

and we are done. □

PROOF OF LEMMA 8.2. This is Lemma 5.25 when G is an algebraic Lie group over a local field. The proof in general is very similar.

a) We denote by $\alpha_1, \dots, \alpha_s$ the elements of Θ_Γ. For $i = 1, \dots s$, let $\gamma_{i,p}$ be a sequence of elements of Γ with $\alpha_i^\omega(\kappa(\gamma_{i,p})) \xrightarrow[p\to\infty]{} \infty$, and set, for $p \ge 1$,

$$g_p := \gamma_{1,p}\, h_1 \gamma_{2,p}\, h_2 \cdots \gamma_{s,p}\, h_s$$

where the elements $h_1, \dots, h_s \in \Gamma$ will be chosen later. There exists a sequence $S \subset \mathbb{N}$ such that, for any α in Θ_Γ and $i = 1, \dots, s$, there exists a sequence, $(\lambda_{i,p,\alpha})_{p\in S}$ of scalars such that the limit in $\mathrm{End}(V_\alpha)$

$$\pi_{\alpha,i} := \lim_{p\in S} \lambda_{i,p,\alpha} \rho_\alpha(\gamma_{i,p})$$

exists and is nonzero. By assumption, for $i = 1, \dots, s$, the limits $\pi_{\alpha_i,i}$ are rank one operators. Hence, for any α in Θ_Γ, the following operators

$$\tau_\alpha := \pi_{\alpha,1}\,\rho_\alpha(h_1)\,\pi_{\alpha,2}\,\rho_\alpha(h_2)\cdots\pi_{\alpha,s}\,\rho_\alpha(h_s)\,.$$

have rank at most one.

By Lemma 8.3, one can choose the elements $h_1, \dots, h_s$ in Γ in such a way that, for any $\alpha \in \Theta_\Gamma$, $\operatorname{Im}\tau_\alpha \not\subset \operatorname{Ker}\tau_\alpha$ and hence τ_α has rank one. Now, for any $\alpha \in \Theta_\Gamma$, there exists a sequence $(\lambda_{p,\alpha})_{p\in S}$ of scalars with

$$\lambda_{p,\alpha}\rho_\alpha(g_p) \xrightarrow[p\to\infty]{} \tau_\alpha \quad \text{in} \quad \operatorname{End}(V_\alpha).$$

Reasoning as in the proof of Lemma 3.1, for $p \in S$ large enough, we deduce that the element $\gamma := g_p$ acts proximally in V_α, for any α in Θ_Γ.

b) We want to prove now that the set

$$\Gamma_{prox} := \{\gamma \in \Gamma \mid \gamma \text{ is } \Theta_\Gamma\text{-proximal}\}$$

is Zariski dense in G. Let $\gamma_0 \in \Gamma$ be a Θ_Γ-proximal element. For any α in Θ_Γ, there exists a sequence, $(\lambda_{p,\alpha})_{p\in\mathbb{N}}$ of scalars such that the limit in $\operatorname{End}(V_\alpha)$

$$\pi_\alpha := \lim_{p\to\infty} \lambda_{p,\alpha}\rho_\alpha(\gamma_0^p)$$

exists and is a rank-one endomorhism of V_α. Since V_α is irreducible and G is Zariski connected, the set

$$\Gamma' := \{\gamma \in \Gamma \mid \pi_\alpha\rho_\alpha(\gamma)\pi_\alpha \neq 0 \ \text{ for all } \alpha \text{ in } \Theta_\Gamma\}$$

is Zariski dense in Γ. For any element γ in Γ', for n large, the element $\gamma_0^n\gamma\gamma_0^n$ belongs to Γ_{prox}. Since the Zariski closure of a semigroup is always a group, the element γ belongs to the Zariski closure of Γ_{prox}. This proves that Γ_{prox} is Zariski dense in G. $\square$

By reasoning as in the proof of Lemma 8.3, one gets:

Lemma 8.4. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, Γ be a Zariski dense subsemigroup of G and f be an element of $F = G/G_c$. For $i = 1, \dots, s$, let (ρ_i, V_i) be an algebraic irreducible representations of G, U_i be an irreducible G_c-submodule of V_i, v_i be a nonzero vector of U_i and W_i be a proper subspace of fU_i. Then there exists g in Γ such that $gG_c = f$ and $gv_i \notin W_i$ for $1 \le i \le s$.*

Proof. Assume that Γ is closed and let μ be a Borel probability measure on G with $\Gamma = \Gamma_\mu$. Note that, since Γ maps onto F, the only μ-stationary Borel probability measure on F is the normalized counting measure, so that one has

$$\tfrac{1}{n}\textstyle\sum_{k=1}^{n} \mu^{*k}\{g \in G | gG_c = f\} \xrightarrow[n\to\infty]{} \tfrac{1}{|F|}.$$

Then one argues as in the proof of Lemma 8.3, replacing the use of Lemma 3.6 by the use of Lemma 3.17. ☐

8.3. The limit set of Γ.

In this section, we define the limit set of a Zariski dense subsemigroup of a reductive algebraic $\mathcal{S}$-adic Lie group

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and Γ be a Zariski dense subsemigroup of G.

Define the limit set $\Lambda_{\Gamma,c}$ of Γ in $\mathcal{P}_{\Theta_\Gamma,c}$ as the closure in this flag variety of the set of attracting fixed points of Θ_Γ-proximal elements of $\Gamma \cap G_c$.

LEMMA 8.5. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and Γ be a Zariski dense subsemigroup of G.*
a) One has $\Gamma\Lambda_{\Gamma,c} = \Lambda_{\Gamma,c}$.
b) For any η in $\mathcal{P}_{\Theta_\Gamma,c}$, one has $\Lambda_{\Gamma,c} \subset \overline{\Gamma\eta}$.

In other words, $\Lambda_{\Gamma,c}$ is the unique Γ-minimal closed invariant subset of $\mathcal{P}_{\Theta_\Gamma,c}$.

PROOF. Let g be a Θ_Γ-proximal element of $\Gamma \cap G_c$.

a) Let h be an element of Γ. Let us prove that $h\xi^+_{\Theta_\Gamma,g}$ belongs to $\Lambda_{\Gamma,c}$. If Γ is a group, this is trivial since then the element hgh^{-1} belongs to $\Gamma \cap G_c$, is Θ_Γ proximal and its attracting fixed point is

$$\xi^+_{\Theta_\Gamma,hgh^{-1}} = h\,\xi^+_{\Theta_\Gamma,g}.$$

Since Γ is only assumed to be a semigroup, the argument will be longer. Set $f = s(h)^{-1}$. For any α in Θ_Γ, let $W_\alpha = \mathrm{Ind}_{G_c}^G V_\alpha$. Then, since Θ_Γ is F-stable, g acts as a proximal endomorphism of fV_α. We denote by $V^{f,+}_{\alpha,g} \subset f\,V_\alpha$ its dominant eigenline and by $V^{f,<}_{\alpha,g} \subset f\,V_\alpha$ the g-invariant complementary subspace of $V^{f,+}_{\alpha,g}$. The line $V^{f,+}_{\alpha,g}$ is the image of $\xi^+_{\Theta_\Gamma,g}$ by the unique G_c-equivariant map $\mathcal{P}_{\Theta_\Gamma,c} \to \mathbb{P}(fV_\alpha)$. By Lemma 8.4 applied to G-irreducible quotients of the spaces W_α, there exists h' in Γ such that $s(h') = f$ and, for any α in Θ_Γ,

$$h'hV^{f,+}_{\alpha,g} \not\subset V^{f,<}_{\alpha,g}.$$

Reasoning again as in the proof of Lemma 3.1, one sees that, for large n, the element $\rho_\alpha(hg^nh')$ is a proximal endomorphism of V_α and that its dominant eigenline converges to $hV^{f,+}_{\alpha,g}$. By uniqueness of the G_c-equivariant map $\mathcal{P}_{\Theta_\Gamma,c} \to \mathbb{P}(fV_\alpha)$, we get

$$hV^{f,+}_{\alpha,g} = V_{\alpha,f\xi^+_{\Theta_\Gamma,g}}.$$

Therefore, if n is large enough, the element hg^nh' of Γ is Θ_Γ-proximal and we have

$$\xi^+_{\Theta_\Gamma,hg^nh'} \xrightarrow[n\to\infty]{} h\xi^+_{\Theta_\Gamma,g}.$$

In particular, $h\xi^+_{\Theta_\Gamma,g}$ belongs to $\Lambda_{\Gamma,c}$ as required.

b) Now, let η be in $\mathcal{P}_{\Theta,c}$ and let us prove $\xi^+_{\Theta_\Gamma,g}$ belongs to $\overline{\Gamma\eta}$. By Lemma 8.3, there exists h in $\Gamma\cap G_c$ such that, for any α in Θ_Γ, one has $\rho_\alpha(h)V_{\alpha,\eta}\notin V^<_{\alpha,\rho_\alpha(g)}$ and hence

$$\rho_\alpha(g^nh)V_{\alpha,\eta} \xrightarrow[n\to\infty]{} V^+_{\alpha,\rho_\alpha(g)} = V_{\alpha,\xi^+_{\Theta_\Gamma,g}}.$$

We get $g^nh\eta \xrightarrow[n\to\infty]{} \xi^+_{\Theta_\Gamma,g}$ and we are done. □

Corollary 8.6. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F = G/G_c$ and Γ be a Zariski dense subsemigroup of G. Then the set*

$$\Lambda_\Gamma := \Lambda_{\Gamma,c}\times F \subset \mathcal{P}_{\Theta_\Gamma,c}\times F \simeq \mathcal{P}_{\Theta_\Gamma}$$

is the unique Γ-minimal closed invariant subset in $\mathcal{P}_{\Theta_\Gamma}$.

This set Λ_Γ is called the *limit set* of Γ in $\mathcal{P}_{\Theta_\mu}$.

Proof. By definition, one has $\Lambda_{\Gamma\cap G_c,c} = \Lambda_{\Gamma,c}$, hence by Lemma 8.5, the action of $\Gamma\cap G_c$ on $\Lambda_{\Gamma,c}$ is also minimal. Our claim follows. □

8.4. The Jordan projection of Γ.

In this section, we give an extension of the result of Section 6.1 which will be used to determine the support of the covariance 2-tensor for random walks on algebraic reductive $\mathcal{S}$-adic Lie groups.

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. For any s in $\mathcal{S}$, we set $\mathfrak{b}_s$ to be the orthogonal in $\mathfrak{a}_s$ of the subspace of $\mathfrak{a}_s^*$ spanned by the algebraic characters of the center of $G_{s,c}$. We set $\mathfrak{b}_\mathbb{R}$ to be this subspace $\mathfrak{b}_s$ when the local field is $\mathbb{K}_s = \mathbb{R}$.

Let Γ be a Zariski dense subsemigroup of G. We define the *limit cone* of Γ is the smallest closed cone L_Γ in $\mathfrak{a}^+$ containing the elements $\lambda(g)$ where g runs among the Θ_Γ-proximal elements of Γ (see Lemma 8.2).

The following proposition extends Theorem 6.2. It will be used in the determination of the support of the Gaussian law in the TCL in Proposition 12.19

Proposition 8.7. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and Γ be a Zariski dense subsemigroup of G. Then the limit cone L_Γ is a convex cone whose intersection with $\mathfrak{b}_\mathbb{R}$ has non-empty interior.*

Proof. The proof is similar to the proof of Theorem 6.2. □

The following proposition extends Theorem 6.4. It will be used in the determination of the essential image of the Iwasawa cocycle in Proposition 16.1.

PROPOSITION 8.8. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and Γ be a Zariski dense subsemigroup of G. Then the closed subgroup of $\mathfrak{a}$ spanned by the elements $\lambda(gh) - \lambda(g) - \lambda(h)$, for g, h and gh Θ_Γ-proximal elements of Γ contains $\mathfrak{b}_{\mathbb{R}}$.*

PROOF. The proof is similar to the proof of Theorem 6.4. □

9. Random walks on reductive groups

The main result of this chapter is the Law of Large Numbers for the Iwasawa cocycle and for the Cartan projection together with the regularity of the corresponding Lyapunov vector (Theorem 9.9). These results will be obtained as translations of the results of Chapter 3 in the intrinsic language of reductive algebraic $\mathcal{S}$-adic Lie groups introduced in Chapter 7. We keep the notations of this Chapter 7.

9.1. Stationary measures on flag varieties.

We first translate the results of Section 3.2 in the language of reductive groups.

When G is a reductive algebraic $\mathcal{S}$-adic Lie group and μ is a Borel probability measure on G, we define Γ_μ to be the subsemigroup of G spanned by the support of μ and set $\Theta_\mu = \Theta_{\Gamma_\mu}$. We say that μ *is Zariski dense in G* if the semigroup Γ_μ is Zariski dense in G.

The first proposition deals with connected groups. It tells us that the partial flag variety $\mathcal{P}_{\Theta_\mu}$ supports a unique μ-stationary measure. This proposition is similar to Lemma 3.6 and Proposition 3.7.

PROPOSITION 9.1. *Let $\mathbb{K}$ be a local field and G be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group $\mathbf{G}$. Let μ be a Zariski dense Borel probability measure on G.*
a) Then there exists a unique μ-stationary Borel probability measure on the flag variety $\mathcal{P}_{\Theta_\mu}$. This probability ν is μ-proximal.
b) Let $\mathbf{M}$ be a homogeneous space of $\mathbf{G}$ and ν be a μ-stationary Borel probability measure on $\mathbf{M}(\mathbb{K})$. For any proper subvariety $\mathbf{N}$ of $\mathbf{M}$, one has $\nu(\mathbf{N}(\mathbb{K})) = 0$.

PROOF. *a*) For any α in Θ_μ, $\rho_\alpha(\Gamma_\mu)$ is a proximal strongly irreducible subsemigroup of $\mathrm{GL}(V_\alpha)$. Hence, by Proposition 3.7, there exists a unique μ-stationary Borel probability measure on $\mathbb{P}(V_\alpha)$ and this measure is μ-proximal. Therefore, as $\mathcal{P}_{\Theta_\mu}$ embeds G-equivariantly in the product $\prod_{\alpha\in\Theta_\mu}\mathbb{P}(V_\alpha)$, according to lemma 1.24, there exists a unique μ-stationary Borel probability measure ν on $\mathcal{P}_{\Theta_\mu}$ and it is μ-proximal.

b) Consider now the set $\mathcal{N}$ of irreducible subvarieties $\mathbf{N}$ of $\mathbf{M}$ such that $\nu(\mathbf{N}(\mathbb{K}))>0$ and that the dimension of $\mathbf{N}$ is minimal for this property. Then, for any $\mathbf{N}_1\neq\mathbf{N}_2$ in $\mathcal{N}$, one has $\mathbf{N}_1\cap\mathbf{N}_2\notin\mathcal{N}$, so that, reasoning as in the proof of Lemma 3.6, one proves that, if $\mathcal{N}_\nu$ is the set of elements $\mathbf{N}$ of $\mathcal{N}$ such that $\nu(\mathbf{N}(\mathbb{K}))$ is maximal, then $\mathcal{N}_\nu$ is non-empty, finite and Γ_μ-invariant. Thus, the $\mathbb{K}$-points of the subvariety $\bigcup_{\mathbf{N}\in\mathcal{N}_\nu}\mathbf{N}$ form a Zariski closed Γ_μ-invariant subset of $\mathbf{M}(\mathbb{K})$, so that, Γ_μ being Zariski dense, one has $\mathcal{N}_\nu=\{\mathbf{M}\}$, whence the result. $\square$

We now extend the study of the stationary probability measures on flag varieties to the context of algebraic reductive $\mathcal{S}$-adic Lie groups.

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. When μ is a Borel probability measure on G, we let, as in Section 4.2, μ_{G_c} be the Borel probability measure induced by μ on G_c. One has $\Gamma_{\mu_{G_c}}=\Gamma_\mu\cap G_c$ and we set $\Theta_\mu:=\Theta_{\Gamma_\mu}$. Note that, by (8.3), one has $\Theta_\mu=\Theta_{\mu_{G_c}}$. We still denote by $\mathrm{d}f$ the normalized counting measure on $F=G/G_c$.

The second proposition extends Proposition 9.1 to non-connected groups. It tells us that the partial flag variety $\mathcal{P}_{\Theta_\mu}$ still supports a unique μ-stationary measure ν.

PROPOSITION 9.2. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G.*
a) *There exists a unique μ-stationary Borel probability measure ν_c on $\mathcal{P}_{\Theta_\mu,c}$ and ν_c is μ-proximal.*
b) *There exists also a unique μ-stationary Borel probability measure ν on $\mathcal{P}_{\Theta_\mu}$ and ν is μ-proximal over F. More precisely, through the identification $\mathcal{P}_{\Theta_\mu}\simeq F\times\mathcal{P}_{\Theta_\mu,c}$ as in* (7.28), *the measure ν reads as $\mathrm{d}f\otimes\nu_c$.*

PROOF. *a*) and *b*). From Proposition 9.1, we know that there exists a unique μ_{G_c}-stationary Borel probability measure ν_c on $\mathcal{P}_{\Theta_\mu,c}$ and ν_c is μ_G-proximal. Hence our claims follow from Lemma 4.7. $\square$

The support of ν depends only on Γ_μ. Indeed the following lemma tells us that it is equal to the limit set of Γ_μ in $\mathcal{P}_{\Theta_\mu}$. This Lemma will be used in the proof of Proposition 12.19.

LEMMA 9.3. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F=G/G_c$, μ be a Zariski dense Borel probability measure on G and ν be the μ-stationary Borel probability measure on $\mathcal{P}_{\Theta_\mu}$. Then the support of ν is Λ_{Γ_μ}.*

PROOF OF LEMMA 9.3. On the one hand, by Lemma 1.10, every closed Γ_μ-invariant subset of $\mathcal{P}_{\Theta_\mu}$ supports a μ-stationary probability measure. On the other hand, by Proposition 9.2, ν is the unique μ-stationary probability measure on $\mathcal{P}_{\Theta_\mu}$. This proves our claim. This also gives another proof of the uniqueness of the minimal Γ_μ-invariant subset of $\mathcal{P}_{\Theta_\mu}$ (see Corollary 8.6). $\square$

9.2. Stationary measures on Grassmann varieties.

In this section, we draw a link between the stationary measure on the flag variety $\mathcal{P}_{\Theta_\mu}$ and the boundary map in Lemma 3.5.

Assume that $\mathbf{G}$ is a connected $\mathbb{K}$-group, where $\mathbb{K}$ is a local field. Let μ be a Zariski dense Borel probability measure on the group $G := \mathbf{G}(\mathbb{K})$. According to Proposition 9.2, the unique μ-stationary probability measure ν on $\mathcal{P}_{\Theta_\mu}$ is μ-proximal. This means that there exists a Borel map $\xi : B \to \mathcal{P}_{\Theta_\mu}$ (where, as usual $(B, \beta) = (G, \mu)^{\mathbb{N}^*}$), also called the Furstenberg *boundary map*, such that, for β-almost all b in B, the limit measure ν_b is the Dirac mass $\nu_b = \delta_{\xi(b)}$.

Let (ρ, V) be an irreducible algebraic representation of G with highest weight χ. We set V^μ to be the sum of weight spaces V^ρ of A in V such that $\chi - \rho$ is a sum of elements of $\Pi \smallsetminus \Theta_\mu$ and $r = \dim V^\mu$. By definition, one has $P_{\Theta_\mu} V^\mu \subset V^\mu$, so that the map

$$G \to \mathbb{G}_r(V)\ ;\ g \mapsto gV^\mu$$

factors as a G-equivariant map

$$\mathcal{P}_{\Theta_\mu} \to \mathbb{G}_r(V), \eta \to V^\mu_\eta.$$

Hence the boundary map can be seen as a map $\xi : B \to \mathbb{G}_r(V)$.

REMARK 9.4. We claim that, for β-almost any b in B,

$\xi(b)$ is the space constructed in Lemma 3.5.

PROOF. It suffices to prove that, for β-almost any b in B, any nonzero limit point in the space of endomorphisms of V of a sequence of the form $\lambda_n \rho(b_1 \cdots b_n)$ with λ_n in $\mathbb{K}$, has image $\xi(b)$.

By Lemma 8.2, for any α in Θ_μ, the semigroup $\rho_\alpha(\Gamma_\mu)$ is proximal, so that, by Proposition 3.7, for β-almost any b in B, the nonzero limit points in $\mathrm{End}(V_\alpha)$ of a sequence $\lambda_n \rho_\alpha(b_1 \cdots b_n)$ with λ_n in $\mathbb{K}$ have rank one. Writing, for any n, $b_1 \cdots b_n = k_n z_n l_n$, with k_n, l_n in K, z_n in Z^+ and $\omega(z_n) = \kappa(b_1 \cdots b_n)$, this implies that the nonzero limit points of $\lambda_n \rho_\alpha(z_n)$ as $n \to \infty$ have rank one. This proves that

$$\lim_{n \to \infty} \alpha^\omega(\kappa(b_1 \cdots b_n)) = \infty, \ \text{ for } \alpha \text{ in } \Theta_\mu.$$

Besides, by definition,

$$\alpha^\omega(\kappa(b_1\cdots b_n)) \text{ remains bounded for } \alpha \text{ in } \Pi \smallsetminus \Theta_\mu.$$

Hence, every nonzero limit point in $\mathrm{End}(V)$ of a sequence $\lambda_n\rho(z_n)$ with λ_n in $\mathbb{K}$ has rank r and its image equals V^μ. Therefore, every nonzero limit point of a sequence $\lambda_n\rho(b_1\cdots b_n)$ has rank r and its image equals $\xi(b)$. $\square$

REMARK 9.5. Recall that *there may exist more than one μ-stationary Borel probability measure on $\mathbb{G}_r(V)$*. Indeed, there may exist uncountably many compact G-orbits in $\mathbb{G}_r(V)$. An example is given in Remark 3.4 where $G = \mathrm{SO}(n,1)$ is acting on $V = \wedge^2\mathbb{R}^{n+1}$ with $n \geq 6$. In this example, one has $r = n-1$.

However, *there exists a unique μ-stationary Borel probability measure on the Γ_μ-minimal set $\Lambda^r_{\rho(\Gamma_\mu)}$* introduced in Lemma 3.2. Indeed, this follows from Proposition 9.1, since, by Remark 9.4, the image of the map $\eta \mapsto V^\mu_\eta$ contains $\wedge^r_{\rho(\Gamma_\mu)}$.

9.3. Moments and exponential moments.

We define two integrability conditions which will be useful assumptions to get asymptotic laws for products of random elements of G.

The first integrability condition will be used in the Law of Large Numbers (Theorem 9.9)

LEMMA 9.6 (First moment). *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. Let μ be a Borel probability measure on G. The following statements are equivalent :*
(i) $\int_G \|\kappa(g)\|\,\mathrm{d}\mu(g) < \infty$
(ii) For any algebraic representation (ρ, V) of G, one has,

$$\int_G \log N(\rho(g))\,\mathrm{d}\mu(g) < \infty. \tag{9.1}$$

(iii) There exists a finite family of algebraic representations (ρ_i, V_i) of G such that $\bigcap_i \mathrm{Ker}\,\rho_i$ is finite and (9.1) *holds for each (ρ_i, V_i).*

In this case, we say that *μ has finite first moment.*

PROOF. *(i)* $\Longrightarrow$ *(ii)* First, assume ρ to be irreducible. Let V' be a G_c-irreducible submodule of V, so that V is a quotient of the induced representation $W' = \mathrm{Ind}_{G_c}^G(V')$. We equip the latter with a good norm and it now suffices to prove the claim in W'. Let χ be the highest weight of A in V'. By Lemma 7.18 and Corollary 7.20.c), one has

$$\int_G |\log\|\rho(g)\||\,\mathrm{d}\mu(g) \leq \int_G \max_{f\in F} |\chi^\omega(\kappa(g\tau_f))|\,\mathrm{d}\mu(g) < \infty.$$

As this also holds for the dual representation, this gives (9.1).

Now, assume ρ is any representation and let (ρ_i, V_i) be the irreducible subquotients of a Jordan-Hölder filtration of (ρ, V).

In case ρ is defined over a field $\mathbb{K}$ with characteristic 0, we have $V = \bigoplus_i V_i$ as a representation of G. Hence, there exists $C > 0$ such that, for any g in G,

$$\|\rho(g)\| \leq C \max_i \|\rho_i(g)\| \tag{9.2}$$

and (9.1) follows from the irreducible case.

In case ρ is defined over a field $\mathbb{K}$ of positive characteristic, (9.1) also follows from the irreducible case, since, as we will see, (9.2) still holds.

It remains to check that (9.2) still holds. Since $\mathbf{A}$ is a $\mathbb{K}$-split torus, as A-modules, we have $V \simeq \bigoplus_i V_i$ and (9.2) holds when g belongs to A. As A is cocompact in Z, it also holds when g belongs to Z, up to changing the constant C. Now, as K_c is a compact group, we can assume all the norms to be K_c-invariant and, as $G_c = K_c Z K_c$, (9.2) holds for any g in G_c. Finally, since G_c has finite index in G, up to again changing the constant C, (9.2) holds for any g in G and we are done.

$(iii) \implies (i)$ One uses again Lemma 7.18 and the fact that the sum of the highest weights of the G_c-irreducible subquotients of the V_i is in the interior of the dual cone of $\mathfrak{a}^+$, which follows from the finiteness of the kernel. $\square$

Later on, in Theorem 12.17, we will need the following stronger integrability condition.

LEMMA 9.7 (Exponential moment). *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. Let μ be a Borel probability measure on G. The following statements are equivalent:*
(i) *There exists $t_0 > 0$ such that*

$$\int_G e^{t_0 \|\kappa(g)\|} \, \mathrm{d}\mu(g) < \infty \tag{9.3}$$

(ii) *For any algebraic representation (ρ, V) of G, there exists $t_0 > 0$ such that*

$$\int_G N(\rho(g))^{t_0} \, \mathrm{d}\mu(g) < \infty. \tag{9.4}$$

(iii) *There exists a finite family of algebraic representations (ρ_i, V_i) of G such that $\bigcap_i \operatorname{Ker} \rho_i$ is finite and $t_0 > 0$ such that (9.4) holds for each (ρ_i, V_i).*

In this case, we say that μ *has a finite exponential moment.*

Proof. $(i) \Longrightarrow (ii)$ By reasoning as in the proof of Lemma 9.6, we can assume ρ to be irreducible. Let still V' and W' be as in this proof and χ be the highest weight of A in V'. Again by Lemma 7.17 and Corollary 7.20, one has

$$\int_G \|\rho(g)\|^{t_0}\, \mathrm{d}\mu(g) \;\leq\; \int_G \max_{f\in F} e^{t_0\chi^\omega(\kappa(g\,\tau_f))}\, \mathrm{d}\mu(g) < \infty,$$

for t_0 small enough. Applying also this bound to the dual representation of $(\rho, \mathbf{V})$, one deduces (9.4).

$(iii) \Longrightarrow (i)$ Again, one argues as in the proof of Lemma 9.6. $\square$

The following lemma tells us that these two integrability conditions (9.1) and (9.4) are automatically transmitted to the induced measure on G_c. Note that this would not be the case for a "compact support condition".

Lemma 9.8. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, μ be a Zariski dense Borel probability measure on G and μ_{G_c} be the measure induced by μ on G_c.*

If μ has finite first moment then μ_{G_c} also has finite first moment.

If μ has a finite exponential moment then μ_{G_c} also has a finite exponential moment.

Proof. This follows from Corollary 4.6, Lemmas 9.6 and 9.7. $\square$

9.4. Law of Large Numbers on G.

We now translate Theorem 3.28 in the language of reductive groups.

We denote by $\mathrm{L}^1(B, \beta, \mathfrak{a})$ the space of $\mathfrak{a}$-valued β-integrable functions on the one-sided Bernoulli space (B, β) with alphabet (G, μ).

Theorem 9.9 (Law of Large Numbers on G). *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with finite first moment. Let ν be a μ-stationary Borel probability measure on the flag variety $\mathcal{P}$.*
a) Then the Iwasawa cocycle $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is integrable i.e. one has $\int_{G\times\mathcal{P}} \|\sigma\|\, \mathrm{d}\mu\, \mathrm{d}\nu < \infty$. Its average

$$\sigma_\mu := \textstyle\int_{G\times\mathcal{P}} \sigma\, \mathrm{d}\mu\, \mathrm{d}\nu \in \mathfrak{a}\,.$$

is called the Lyapunov vector of μ. It is F-invariant and does not depend on ν. Indeed, for β-almost any b in B, one has

$$\tfrac{1}{n}\kappa(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \sigma_\mu.$$

Moreover this sequence also converges in $\mathrm{L}^1(B,\beta,\mathfrak{a})$.
b) For any η *in* $\mathcal{P}$, *for* β*-almost any* b *in* B, *one has*

$$\tfrac{1}{n}\sigma(b_n\cdots b_1,\eta)\xrightarrow[n\to\infty]{}\sigma_\mu.$$

This sequence converges also in $\mathrm{L}^1(B,\beta,\mathfrak{a})$, *uniformly for* η *in* $\mathcal{P}$.
c) Uniformly for η *in* $\mathcal{P}$, *one has*

$$\tfrac{1}{n}\int_G \sigma(g,\eta)\,\mathrm{d}\mu^{*n}(g)\xrightarrow[n\to\infty]{}\sigma_\mu.$$

d) For any η *in* $\mathcal{P}_c$, *for* β*-almost any* b *in* B, *there exists* $M>0$, *such that for any* $n\in\mathbb{N}$, *one has*

$$\|\sigma(b_n\cdots b_1,\eta)-\kappa(b_n\cdots b_1)\|\leq M.$$

e) (Regularity of σ_μ) *The Lyapunov vector* σ_μ *belongs to* $\mathfrak{a}^{++}_{\Theta_\mu}$.
f) In particular, when G *is a real Lie group, the Lyapunov vector belongs to the open Weyl chamber :* $\sigma_\mu\in\mathfrak{a}^{++}$.

Remark 9.10. When G is a real Lie group, the μ-stationary probability measure ν on $\mathcal{P}$ is unique since $\Theta_\mu=\Pi$. In general, this is not always the case, but, as a consequence of *b*), the Lyapunov vector σ_μ does not depend on the choice of ν.

Proof. We will use the same technique as in the proof of Corollary 7.20: we just have to control the image of these sequences by sufficiently many linear functionals on $\mathfrak{a}$.

By (7.16), the cocycle σ is integrable on $G\times\mathcal{P}$. We set $\sigma_\mu=\int_{G\times\mathcal{P}}\sigma\,\mathrm{d}(\mu\otimes\nu)$.

Let (ρ,V) be a proximal irreducible algebraic representation of G_c with highest weight χ. For instance (ρ,V) may be one of the representations introduced in Lemma 7.15, or (ρ,V) may be a scalar representation associated to an algebraic character of G_c. Equip $\mathrm{Ind}_{G_c}^G(\rho)$ with a good norm and let W be an irreducible quotient of this induced representation. Let $\pi:\mathrm{Ind}_{G_c}^G(\rho)\to W$ be the quotient map and θ be the representation of G in W. By Lemma 7.11, for any f in F, the map π is injective on V_f. Therefore, we have

$$\sup_{g\in G}\left|\log\tfrac{\|\rho(g)\|}{\|\theta(g)\|}\right|<\infty.$$

By Lemma 7.18 and Corollary 7.20, we get

$$\sup_{g\in G}\left|\max_{f\in F}\chi^\omega(f\kappa(g))-\log(\|\theta(g)\|)\right|<\infty. \tag{9.5}$$

Recall from(7.8) that, for any η in $\mathcal{P}$, V_η is a line in V_f with $f=\eta\,G_c$. We let W_η be the image of V_η in W. The image measure of ν by the map $\mathcal{P}\to\mathbb{P}(V);\eta\mapsto W_\eta$ is a μ-stationary probability measure on $\mathbb{P}(W)$.

If U is a line in W and g is in $\mathrm{GL}(W)$, we set

$$\sigma_W(g,U) = \log \tfrac{\|gu\|}{\|u\|},$$

where u is a nonzero element of U. For any η in $\mathcal{P}$, we set

$$\varphi(\eta) = \log \tfrac{\|\pi v\|}{\|v\|},$$

where v is a nonzero element of V_η. Then φ is a continuous function $\mathcal{P} \to \mathbb{R}$. Since the projection π is G-equivariant, we get from Lemma 7.18, for any g in G,

$$\chi^\omega(\sigma(g,\eta)) + \varphi(g\eta) = \sigma_W(\theta(g), W_\eta) + \varphi(\eta). \tag{9.6}$$

In particular, since ν is μ-stationary, we have

$$\textstyle\int_G \int_{\mathcal{P}} \sigma_W(\theta(g), W_\eta)\,\mathrm{d}\nu(\eta)\,\mathrm{d}\mu(g) = \chi^\omega(\sigma_\mu).$$

Therefore, by Theorem 3.28, for β-almost any b in B, we have

$$\tfrac{1}{n}\log(\|\theta(b_n \cdots b_1)\|) \xrightarrow[n\to\infty]{} \chi^\omega(\sigma_\mu),$$

hence, by (9.5),

$$\tfrac{1}{n}\max\nolimits_{f\in F} \chi^\omega(f\kappa(b_n \cdots b_1)) \xrightarrow[n\to\infty]{} \chi^\omega(\sigma_\mu).$$

In particular, since the set of highest weights of proximal representations of G_c spans $\mathfrak{a}^*$, σ_μ is F-invariant. Besides, this convergence also takes place in $\mathrm{L}^1(B,\beta)$.

Now, by Theorem 3.28.b) and (9.6), for any η in $\mathcal{P}$, for β-almost any b in B, we have

$$\tfrac{1}{n}\chi^\omega(\sigma(b_n \cdots b_1, \eta)) \xrightarrow[n\to\infty]{} \chi^\omega(\sigma_\mu)$$

and this sequence also converges in $\mathrm{L}^1(B,\beta)$, that is we get *b*). Besides, again by Lemma 7.18, for η in $\mathcal{P}_c$, we have

$$\begin{aligned}\chi^\omega(\sigma_\mu) = \lim_{n\to\infty} \tfrac{1}{n}\sigma_W(\theta(b_n \cdots b_1), W_\eta) &= \liminf_{n\to\infty} \tfrac{1}{n}\chi^\omega(\kappa(b_n \cdots b_1)) \\ &\leq \limsup_{n\to\infty} \tfrac{1}{n}\chi^\omega(\kappa(b_n \cdots b_1)) \\ &\leq \lim_{n\to\infty} \tfrac{1}{n} \max_{f\in F} \chi^\omega(f\kappa(b_n \cdots b_1)) \\ &= \chi^\omega(\sigma_\mu).\end{aligned}$$

Therefore, we have

$$\tfrac{1}{n}\chi^\omega(\kappa(b_n \cdots b_1)) \xrightarrow[n\to\infty]{} \chi^\omega(\sigma_\mu),$$

and this convergence also holds in $\mathrm{L}^1(B,\beta)$, that is, *a*) is proved.

c) directly follows from *b*).

d) By Proposition 3.23, for any η in $\mathcal{P}_c$, for β-almost any b in B, the sequence

$$\log \|\theta(b_n \cdots b_1)_{|W_e}\| - \sigma_W(\theta(b_n \cdots b_1), W_\eta)$$

is bounded. Now, this sequence is equal, up to a uniform constant, to the sequence

$$\chi^\omega(\kappa(b_n \cdots b_1)) - \chi^\omega(\sigma(b_n \cdots b_1, \eta))$$

and *d*) follows.

e) We want to prove that σ_μ belongs to $\mathfrak{a}_{\Theta_\mu}$ and that $\alpha^\omega(\sigma_\mu) > 0$ for any α in Θ_μ.

According to Lemma 9.8, the induced probability measure μ_{G_c} on G_c also has finite first moment. By Lemma 4.7, ν is also μ_{G_c} stationary. By Proposition 4.9, one has $\sigma_\mu = \frac{1}{|F|}\sigma_{\mu_{G_c}}$. Hence, we may assume that $G = G_c$.

First if α belongs to $\Pi \smallsetminus \Theta_\mu$, since $\sup_{\Gamma_\mu}(\alpha^\omega \circ \kappa) < \infty$, one has, for β-almost any b in B,

$$\alpha^\omega(\sigma_\mu) = \lim_{n\to\infty} \tfrac{1}{n}\alpha^\omega(\kappa(b_n \cdots b_1)) = 0,$$

hence $\sigma_\mu \in \mathfrak{a}_{\Theta_\mu}$.

Now, fix α in Θ_μ. By Proposition 3.7, for β-almost all b in B, any nonzero limit point in $\mathrm{End}(V_\alpha)$ of a sequence

$$\lambda_n \rho_\alpha(b_n \cdots b_1),$$

with $\lambda_n \in \mathbb{K}$ has rank one. Thus, choosing z_n in Z^+ with $b_n \cdots b_1 \in K z_n K$, every nonzero limit point of a sequence $\lambda_n \rho_\alpha(z_n)$, has rank one. As, for any v in the weight space $V^{\chi_\alpha - \alpha}$ and for any n in $\mathbb{N}$, one has

$$\|\rho_\alpha(z_n)v\| = e^{-\alpha^\omega(\omega(z_n))}\, \|\rho_\alpha(z_n)\|\ \|v\|\,,$$

this necessarily implies that $\alpha^\omega(\omega(z_n)) \xrightarrow[n\to\infty]{} \infty$, that is,

$$\alpha^\omega(\kappa(b_n \cdots b_1)) \xrightarrow[n\to\infty]{} \infty.$$

Hence by *c*), for ν-almost any η in $\mathcal{P}$,

$$\alpha^\omega(\sigma(b_n \cdots b_1, \eta)) \xrightarrow[n\to\infty]{} \infty.$$

Now, using Lemma 2.18 as in the proof of Theorem 3.31, this implies $\alpha^\omega(\sigma_\mu) > 0$, whence the result.

f) This follows from *e*). Indeed, since G is a real Lie group, the set Θ_μ is equal to Π. $\square$

9.5. Simplicity of the Lyapunov exponents.

We give in this section concrete consequences of the regularity of the Lyapunov vector. For instance, we prove the simplicity of the first Lyapunov exponent for proximal representations.

The following corollary relates the Lyapunov vectors of μ and $\mu^\vee$.

COROLLARY 9.11. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with finite first moment. Let $\mu^\vee$ be the image of μ by the map $g \mapsto g^{-1}$. Then the Lyapunov vector of $\mu^\vee$ is equal to the image of the Lyapunov vector of μ by the opposition involution: $\sigma_{\mu^\vee} = \iota(\sigma_\mu)$.*

PROOF. One computes using twice Theorem 9.9 and using the equality $\kappa(g^{-1}) = \iota(\kappa(g))$

$$\begin{aligned}\sigma_{\mu^\vee} &= \lim_{n\to\infty} \tfrac{1}{n} \int_B \kappa(b_n^{-1} \cdots b_1^{-1})\, \mathrm{d}\beta(b)\\ &= \lim_{n\to\infty} \tfrac{1}{n} \int_B \iota(\kappa(b_1 \cdots b_n))\, \mathrm{d}\beta(b) = \iota(\sigma_\mu)\end{aligned}$$

as required. □

Recall that, in Section 3.6, when V is a finite dimensional $\mathbb{K}$-vector space, and μ is a Borel probability measure on $\mathrm{GL}(V)$, we defined its first Lyapunov exponent as the limit

$$\lambda_{1,\mu} = \lim_{n\to\infty} \tfrac{1}{n} \int_{\mathrm{GL}(V)} \log \|g\|\, \mathrm{d}\mu^{*n}(g).$$

As a consequence of Theorem 9.9 and Lemma 7.17, one gets the following reformulation of Theorem 3.28 in which we compute the first Lyapunov exponent by means of the Lyapunov vector.

COROLLARY 9.12. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with a finite first moment. Let (ρ, V) be an algebraic representation of G and let $\rho_*\mu$ be the image of μ on $\mathrm{GL}(V)$ under the map ρ. We have*

$$\lambda_{1,\rho_*\mu} = \max_\chi \chi^\omega(\sigma_\mu), \tag{9.7}$$

where χ runs among the weights of A in V. In particular, if (ρ, V) is irreducible and χ is a maximal weight, we have

$$\lambda_{1,\rho_*\mu} = \chi^\omega(\sigma_\mu). \tag{9.8}$$

REMARK 9.13. In case V is strongly irreducible, it has a unique highest weight χ. In general the maximal (or parabolic) weights of V form a F-orbit. Since, by Theorem 9.9, the Lyapunov vector σ_μ

is F-invariant, the limit $\chi^\omega(\sigma_\mu)$ does not depend on the choice of the maximal weight.

PROOF. The formula follows from an analogue formula for elements of G.

Fix a norm on V such that the decomposition of V into weight spaces for the action of A is good. For any a in A, we have

$$\|\rho(a)\| = \max_{\chi} |\chi(a)|.$$

Since A is cocompact in Z and the set of weights of A in V is finite, there exists $C \geq 0$ such that, for any z in Z, we have

$$|\log \|\rho(z)\| - \max_{\chi} \chi^\omega(\omega(z))| \leq C.$$

As K is compact, up to enlarging C, this gives for any g in G,

$$|\log \|\rho(g)\| - \max_{\chi} \chi^\omega(\kappa(g))| \leq 2C.$$

Hence, by Lemma 3.27, for β-almost any b in B,

$$\frac{1}{n} \max_{\chi} \chi^\omega(\kappa(b_n \cdots b_1)) \xrightarrow[n\to\infty]{} \lambda_{1,\rho_*\mu}. \tag{9.9}$$

Now, by Theorem 9.9, we have, for β-almost any b in B,

$$\frac{1}{n}\kappa(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \sigma_\mu. \tag{9.10}$$

¿From (9.9) and (9.10), we get (9.7). Since σ_μ belongs to $\mathfrak{a}^+$, Equation (9.7) still holds when χ runs among the set of maximal weights. As recalled in Remark 9.13, when ρ is irreducible, this set is an F-orbit and (9.8) follows since σ_μ is F-invariant. □

Let us relate the Lyapunov vector to the other Lyapunov exponents of probability measures. Let d be the dimension of V. For $1 \leq k \leq d$ we define inductively the k-th Lyapunov exponent $\lambda_{k,\mu}$ of μ by the formula

$$\lambda_{1,\mu} + \cdots + \lambda_{k,\mu} = \lim_{n\to\infty} \tfrac{1}{n} \int_{\mathrm{GL}(V)} \log \| \wedge^k g\| \,\mathrm{d}\mu^{*n}(g),$$

where the existence of the limit follows from subadditivity. Note that this definition does not depend on the choice of the norms on the exterior powers.

LEMMA 9.14. *Let μ be a Borel probability measure on* $\mathrm{GL}(V)$*. The sequence of its Lyapunov exponents is non-increasing, that is, we have*

$$\lambda_{1,\mu} \geq \cdots \geq \lambda_{d,\mu}.$$

To prove this result, we need to introduce in general the singular values of an element of $\mathrm{GL}(V)$ which, in the real case, were defined in Section 5.7.7. Since the definition of the Lyapunov exponents does not depend on the choice of the norms, we chose some that are particularly convenient.

If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, we equip V with a Euclidean or Hermitian scalar product. We equip each of the $\wedge^k V$, $1 \leq k \leq d$, with the associated scalar product.

If $\mathbb{K}$ is non archimedean, we equip V with the sup norm given by a basis and each of the $\wedge^k V$, $1 \leq k \leq d$, with the sup norm coming from the associated basis.

In both cases, let $K \subset \mathrm{GL}(V)$ be the group of isometries of the norm. The Cartan decomposition of $\mathrm{GL}(V)$ allows to write any g in $\mathrm{GL}(V)$ as a product kal where k and l belong to K and the matrice a is diagonal, with entries $a_1, \ldots, a_d$ such that

$$|a_1| \geq \cdots \geq |a_d|.$$

The real numbers $\kappa_k(g) = |a_k|$, $1 \leq k \leq d$, only depend on g and on the norm and are called the singular values of g. By construction, for $1 \leq k \leq d$, we have

$$\| \wedge^k g\| = \kappa_1(g) \cdots \kappa_k(g). \tag{9.11}$$

Proof of Lemma 9.14. The Lemma relies on an analogue formula for the norms of the $\wedge^k g$, $1 \leq k \leq d$, for g in $\mathrm{GL}(V)$. Indeed, for such a g, by (9.11), for $1 \leq k \leq d-1$, we have

$$\| \wedge^{k-1} g\| \| \wedge^{k+1} g\| \leq \| \wedge^k g\|^2.$$

By the definition of the Lyapunov exponents, this gives

$$(\lambda_{1,\mu} + \cdots + \lambda_{k-1,\mu}) + (\lambda_{1,\mu} + \cdots + \lambda_{k+1,\mu}) \leq 2(\lambda_{1,\mu} + \cdots + \lambda_{k,\mu})$$

which in turn amounts to $\lambda_{k,\mu} \geq \lambda_{k+1,\mu}$. □

The following corollary of Theorem 9.9 explains on a concrete case the meaning of the regularity of the Lyapunov vector.

Corollary 9.15 (Simplicity of the Lyapunov exponents). *Let $V = \mathbb{K}^d$ and μ be a Borel probability measure on $G = \mathrm{GL}(V)$ with a finite first moment, i.e. $\int_G \log N(g)\, \mathrm{d}\mu(g) < \infty$, and such that Γ_μ is strongly irreducible in V.*
a) *If Γ_μ is proximal in V, the two first Lyapunov exponents satisfy $\lambda_{1,\mu} > \lambda_{2,\mu}$.*
b) *More precisely, one always has $\lambda_{1,\mu} = \cdots = \lambda_{r,\mu} > \lambda_{r+1,\mu}$ where r is*

the proximal dimension of Γ_μ.
c) *If* $\mathbb{K} = \mathbb{R}$ *and* Γ_μ *is Zariski dense in* $\mathrm{SL}(V)$ *or* $\mathrm{GL}(V)$*, then one has*

$$\lambda_{1,\mu} > \lambda_{2,\mu} > \cdots > \lambda_{d,\mu}.$$

To rely the proximal dimension of Γ_μ with the objects that have been defined for abstract reductive groups, we will use the

LEMMA 9.16. *Let* $V = \mathbb{K}^d$ *and* Γ *be a strongly irreducible sub-semigroup of* $\mathrm{GL}(V)$ *with proximal dimension* r.
a) *There exists* $c_0 > 0$ *such that, for any* g *in* Γ*, one has* $\kappa_r(g) \geq c_0\kappa_1(g)$ *and one has* $\sup_{g\in\Gamma} \kappa_r(g)/\kappa_{r+1}(g) = \infty$.
b) *Let* G *be the Zariski closure of* Γ *in* $\mathrm{GL}(V)$*, let* χ *be the highest weight of* G *in* V *and set* X *to be the set of weights* χ' *of* A *in* V *which are of the form* $\chi' = \chi - \sum_{\alpha\in\Theta_\Gamma^c} n_\alpha\alpha$*, where the* n_α *are nonnegative integers. Then we have*

$$r = \sum_{\chi'\in X} \dim V^{\chi'}.$$

Recall that the Zariski closure of an irreducible sub-semigroup of $\mathrm{GL}(V)$ is a reductive group.

PROOF. *a)* Assume that, for some $2 \leq k \leq d$, we have a sequence (g_n) of elements of Γ with $\sup_n \kappa_1(g_n)/\kappa_k(g_n) = \infty$. Let (λ_n) be a sequence of elements of $\mathbb{K}^*$ with $|\lambda_n| = \kappa_1(g_n)^{-1}$. After extracting a subsequence, we can assume that $\lambda_n g_n$ converges to a non zero endomorphism π. By assumption, since $\lambda_n\kappa_k(g_n) \xrightarrow[n\to\infty]{} 0$, π has rank $< k$, hence $k > r$. The existence of c_0 follows.

Conversely, let π be a rank r element of $\overline{\mathbb{K}\Gamma}$. Write $\pi = \lim_{n\to\infty} \lambda_n g_n$, $g_n \in \Gamma$, $\lambda_n \in \mathbb{K}$. As π is non zero, we have $\liminf_{n\to\infty} \lambda_n\kappa_1(g_n) > 0$. As π has rank r, we have $\lambda_n\kappa_{r+1}(g_n) \xrightarrow[n\to\infty]{} 0$. The result follows.

b) By reasoning as in the proof of Corollary 9.12, one sees that there exists $C \leq 0$ such that, for any g in G, the sets

$$\{\log \kappa_k(g) | 1 \leq k \leq d\}$$

and

$$\{(\chi')^\omega(\kappa(g)) | \chi' \text{ is a weight of } A \text{ in } V\}$$

are equal up to C (that is, more precisely, the Hasudorff distance between these two finite sets of real numbers is $\leq C$). The result follows from *a)* and this remark. $\square$

PROOF OF COROLLARY 9.15. *a)* and *b)* Let χ_0 be the highest weight of G in V. By Corollary 9.12, for $1 \leq k \leq d$, one has

$$\lambda_{1,\mu} + \cdots + \lambda_{k,\mu} = \max_\chi \chi^\omega(\sigma_\mu),$$

where χ runs among the set X_k of weights of A in $\wedge^k V$. In particular, let k be the largest integer such that $\lambda_{1,\mu} = \lambda_{k,\mu}$. Then k is the dimension of the space

$$\bigoplus_{\substack{\chi \in X_1 \\ \chi(\sigma_\mu) = \chi_0(\sigma_\mu)}} V_\chi .$$

As, by Theorem 9.9, σ_μ belongs to $\mathfrak{a}^{++}_{\Theta_\mu}$, for any χ in X_1, one has $\chi(\sigma_\mu) = \chi_0(\sigma_\mu)$ if and only if $\chi_0 - \chi$ is a linear combination of elements of Θ^c_μ. We get $k = r$ by Lemma 9.16.*b)* and we are done.

c) Assume for instance that Γ_μ is Zariski-dense in $\mathrm{GL}(V)$. Since $\mathbb{K} = \mathbb{R}$, by (8.2),one has

$$\mathfrak{a}^{++}_{\Theta_\mu} = \mathfrak{a}^{++} = \{\mathrm{diag}(x_1, \dots, x_d) \mid x_1 > x_2 > \cdots > x_d\}.$$

Our claims follow then from Theorem 9.9 and Corollary 9.12 applied to the representations $\wedge^k V$. $\square$

Part 3

Central Limit Theorem

10. Transfer operators over contracting actions

We come back to the abstract framework of Chapter 2, studying the actions on a compact space X of a locally compact semigroup G endowed with a probability measure μ. and studying the behavior of the cocycles over this action. When this action is μ-contracting (Definition 10.1) and under suitable integrability conditions, we introduce the corresponding complex transfer operators P_θ. We study the spectral properties of P_θ when the parameter θ is small enough (Lemmas 10.17 and 10.18). We will use them in Chapter 11 to prove various limit laws for random walks on groups satisfying some exponential moment conditions.

10.1. Contracting actions.

We define in this section the μ-contracting actions and we prove that they admit a unique μ-stationary probability measure.

We still let G be a second countable locally compact semigroup, $s : G \to F$ be a continuous morphism onto a finite group F, and μ be a Borel probability measure on G. We shall say that μ *spans* F if the image in F of the support of μ spans F. We shall say that μ is *aperiodic* in F if it spans F and if, for any non-trivial morphism from F to a cyclic group, the image of μ is not a Dirac mass.

Let X be a compact metric G-space which is fibered over F (see Section 1.7), and let $x \mapsto f_x$ be the G-equivariant fibration. For any g in G, we define the *Lipschitz constant* $\mathrm{Lip}(g)$ of g by

$$\mathrm{Lip}(g) = \sup_{f_x = f_{x'}} \frac{d(gx, gx')}{d(x, x')}$$

where the supremum is taken over the pairs x, x' in X with $f_x = f_{x'}$ and $x \neq x'$.

Definition 10.1. Let X be a compact metric G-space which is fibered over F and $\gamma_0 > 0$. We shall say that the action of G on X is (μ, γ_0)*-contracting over* F if one has

$$\textstyle\int_G \mathrm{Lip}(g)^{\gamma_0}\, \mathrm{d}\mu(g) < \infty \tag{10.1}$$

and, for some $n \geq 1$,

$$\sup_{f_x = f_{x'}} \int_G \frac{d(gx, gx')^{\gamma_0}}{d(x, x')^{\gamma_0}}\, \mathrm{d}\mu^{*n}(g) < 1. \tag{10.2}$$

We will say that *the action is* μ*-contracting over* F or, in short, that *the* G*-space* X *is* μ*-contracting over* F if this action is (μ, γ_0) contracting over F for some $\gamma_0 > 0$. In this case, the action is also (μ, γ)-contracting for any $0 < \gamma \leq \gamma_0$.

When F is trivial, we just say that the action is *μ-contracting.*

In other words, the action is *μ-contracting over F* when the action of G on fibers of the G-equivariant fibration tends to contract on average. Note that, if the definition holds, there exist $0 \leq \delta < 1$ and $C_0 > 0$ such that, for any n in $\mathbb{N}$ and x, x' in X, with $f_x = f_{x'}$ one has

$$\int_G d(gx, gx')^{\gamma_0} \,\mathrm{d}\mu^{*n}(g) \leq C_0\, \delta^n d(x, x')^{\gamma_0}. \tag{10.3}$$

We will often only use the definition under the form (10.3) but we will also sometimes need the moment condition (10.1).

Example 10.2. The main example we will study in this book is the action of an algebraic reductive $\mathcal{S}$-adic Lie group G on a projective space or a flag variety. In this case F is the group G/G_c (see Chapter 12).

Example 10.3. Here is a trivial example. Let X be a compact metric space, and, for x in X, let c_x be the constant map on X given by $c_x : y \mapsto x$. Let G be the semigroup of transformations of the compact space X which are either the identity e or a constant map c_x, and μ be a probability measure on X, viewed as a subset of G. In this case, the limit theorems 11.1 and 15.1 that we will prove follow from the classical limit theorems for random walks on $\mathbb{R}^d$.

Example 10.4. Another enlightening example to keep in mind while reading this text is the following. Let X be the compact space $X = \{0, 1\}^{\mathbb{N}}$ endowed with the distance $d(x, y) = 2^{-\min\{k \geq 0 | x_k \neq y_k\}}$. Let s_i, $i = 0, 1$, be the two *prefix maps* of X defined, for $x = (x_1, x_2, \ldots) \in X$, by $s_i(x) = (i, x_1, x_2, \cdots)$. Let G be the discrete free semigroup spanned by s_0 and s_1, and $\mu := \frac{1}{2}(\delta_{s_0} + \delta_{s_1})$. This action of G on X is μ-contracting (here the group F is trivial). In this case, the spectral properties of the complex perturbations of the Markov operator P_μ that we will discuss in this chapter also follow from [**93**].

The following lemma tells us roughly that, for a μ-contracting action, the behavior of the random trajectories does not depend on the starting point except for an exponentially small error term.

Lemma 10.5 (Exponential convergence of orbits). *Let G be a second countable locally compact semigroup and $s : G \to F$ be a continuous morphism onto a finite group F. Let μ be a Borel probability measure on G such that μ spans F. Let X be a compact metric G-space which is fibered over F and μ-contracting over F.*

a) There exist $\gamma > 0$ and $C > 0$ such that, for every x, x' in X with $f_x = f_{x'}$, for every $n \geq 1$, one has

$$(10.4) \qquad \mu^{*n}(\{g \in G \mid d(gx, gx') \geq e^{-\gamma n} d(x, x')\}) \;\leq\; C\, e^{-\gamma n}.$$

b) There exists $\gamma > 0$ such that, for every x, x' in X with $f_x = f_{x'}$, for β-almost every b in B, for all but finitely many $n \geq 1$, one has

$$(10.5) \qquad d(b_n \cdots b_1 x, b_n \cdots b_1 x') \;\leq\; e^{-\gamma n} d(x, x').$$

c) There exists a unique μ-stationary Borel probability measure ν on X. This μ-stationary measure ν is μ-proximal over F.

PROOF. a) Inequality (10.4) is a direct consequence of Equation (10.3) with $C = C_0$ and γ small enough so that $0 < \gamma \leq \frac{|\log \delta|}{1+\gamma_0}$.

b) This follows from Equation (10.4) and Borel Cantelli Lemma.

c) For x, x' in X, set $d_0(x, x') = d(x, x')\, \mathbf{1}_{f_x = f_{x'}}$. Let ν and ν' be two μ-stationary measures on X. Using Lemma 1.17 and Lebesgue convergence theorem, one gets from *b)*,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \textstyle\int_{X\times X} d_0(b_1 \cdots b_n x, b_1 \cdots b_n x')\, \mathrm{d}\nu(x)\, \mathrm{d}\nu'(x') \\ &= \textstyle\int_{X\times X} d_0(x, x')\, \mathrm{d}\nu_b(x)\, \mathrm{d}\nu'_b(x'). \end{aligned}$$

Hence for $(\nu_b \otimes \nu'_b)$-almost all (x, x') in $X \times X$, one has $d_0(x, x') = 0$. This proves that the restriction of the limit measures ν_b and ν'_b to each fiber is a multiple of the same Dirac mass. Since μ spans F, the images of ν and ν' in F are F-invariant. The same is true for the images of the limit measures ν_b and ν'_b. Hence for β-almost every b in B and f in F, there exists $\xi_{b,f} \in X$ in the fiber over f such that

$$\nu_b = \nu'_b = \tfrac{1}{|F|} \textstyle\sum_{f\in F} \delta_{\xi_{b,f}}.$$

This proves that $\nu = \nu'$ and that ν is μ-proximal over F. □

10.2. The transfer operator for finite groups.

We describe in this section a few basic spectral properties for the transfer operator P of a random walk on a finite group.

Let μ be a probability measure on a finite group F. Let $P = P_\mu$ be the averaging operator on $\mathbb{C}^F = \mathcal{C}^0(F)$ given, for $\varphi : F \to \mathbb{C}$ and $f \in F$, by

$$(10.6) \qquad P\varphi(f) = \textstyle\int_F \varphi(hf)\, \mathrm{d}\mu(h) = \sum_{h\in F} \mu(h)\varphi(hf).$$

As for any Markov-Feller operator, the norm of P in $\mathcal{C}^0(F)$ is at most 1, hence its eigenvalues have modulus at most 1.

The following lemma describe the eigenvalues of modulus 1 of the averaging operator P.

LEMMA 10.6. *Let μ be a probability measure on a finite group F whose support spans F.*
a) There exists a smallest normal subgroup F_μ of F such that the quotient group F/F_μ is cyclic and the image of μ in F/F_μ is a Dirac mass at some generator f_μ of this group.

Let $p_\mu := |F/F_\mu|$.
b) The eigenvalues ζ of modulus 1 of the operator P in $\mathbb{C}^F$ are the p_μ^{th}-roots of 1. These eigenvalues are simple and the associated eigenline is spanned by the character χ_ζ of F/F_μ for which $\chi_\zeta(f_\mu F_\mu) = \zeta$.
c) The probability measure μ^{*p_μ} is aperiodic in F_μ.

In particular, when μ is aperiodic in F, the only eigenvalue of modulus 1 of the transfer operator P is 1, and the corresponding eigenfunctions are constant.

PROOF. a) We first check the existence of F_μ. Let Ξ be the set of characters of F which are constant on the support of μ. This set Ξ is a subgroup of the group of characters of F. In particular this group Ξ is abelian. We define now F_μ to be the intersection of the kernels of the elements of Ξ. This subgroup F_μ is normal in F and the quotient F/F_μ is also an abelian group and is the dual group of Ξ. As the elements of Ξ are constant on the support of μ, the image of μ in F/F_μ is a Dirac mass at some element f_μ of F/F_μ. As the support of μ spans F, f_μ spans F/F_μ, which is therefore cyclic. Clearly, this group F_μ is the smallest one with those properties.

b) Let φ be a nonzero element of $\mathbb{C}^F$ and ζ be a complex number of modulus 1 with $P\varphi = \zeta\varphi$. We want to prove that ζ is a p_μ^{th}-root of unity. We have the inequality

$$P\,|\varphi| \geq |P\varphi| = |\varphi|\,.$$

Let M be the set of f in F with $|\varphi(f)| = \max_F |\varphi|$. By the maximum principle, for any f in F with $\mu(f) > 0$, we have $fM \subset M$, hence, as the support of μ spans F, we have $M = F$, that is $|\varphi|$ is a constant r. Therefore, for any f in F, one has

$$r = \left|\textstyle\sum_{f'\in F} \mu(f')\varphi(f'f)\right|,$$

thus for any f', f'' in F with $\mu(f') > 0$ and $\mu(f'') > 0$, one has

$$\varphi(f'f) = \varphi(f''f), \ \text{ hence}$$

$$\varphi(f'f) = \zeta\varphi(f). \tag{10.7}$$

Let F' be the set of f in F such that the function $\varphi(f.)$ is a multiple of φ. Then, F' is a subgroup of F and there exists a unique character

χ of F' such that, for any f in F', $\varphi(f.) = \chi(f)\varphi$. As, by (10.7), the group F' contains the support of μ, one has $F' = F$, the function φ is a multiple of χ and, for any f in the support of μ, one has $\chi(f) = \zeta$, hence χ belongs to Ξ, and ζ is a p_μ^{th}-root of unity and the corresponding eigenspace is spanned by the character χ_ζ

Conversely every character χ_ζ is an eigenvector of P with eigenvalue ζ. Since moreover $\|P_\mu\|_\infty = 1$, this eigenvalue is simple.

c) Let us prove that the only eigenvalue of modulus 1 of P^{p_μ} in $\mathbb{C}^{F_\mu}$ is 1 and that the associated eigenspace is the space of constant functions, which implies the result.

Indeed, let φ be a function on F_μ such that $P^{p_\mu}\varphi = \zeta\varphi$, for some ζ with modulus 1. Extend φ to a function on F by setting $\varphi(f) = 0$ for $f \notin F_\mu$. We still have

$$P^{p_\mu}\varphi = \zeta\varphi.$$

Let E be the cyclic space for P spanned by φ. Since the polynomial $t^{p_\mu} - \zeta$ has simple roots, P is diagonalizable in E and its eigenvalues are p_μ^{th}-roots of ζ. Since the eigenvalues of P in $\mathbb{C}^F$ are the p_μ-roots of 1 and the associated eigenfunctions are constant on F_μ, our claim follows. $\square$

The following corollary explains the probabilistic meaning of the spectral properties of the transfer operator: the equidistribution of the walk with exponential speed.

COROLLARY 10.7. *Let μ be an aperiodic probability measure on a finite group F. Then there exists $a < 1$ such that, for all $n \geq 1$ and f in F, one has*

$$\left|\mu^{*n}(\{f\}) - \tfrac{1}{|F|}\right| \leq a^n.$$

10.3. The transfer operator.

In this section we prove that, when the action is μ-contracting, 1 is an isolated eigenvalue of the averaging operator $P = P_\mu$ in a suitable space of Hölder continuous functions. This gives also another way to prove the uniqueness of the μ-stationary measure on X.

Let G be a second countable locally compact semigroup and $s : G \to F$ be a continuous morphism onto a finite group F. Let X be a compact metric G-space which is fibered over F.

We let $\mathcal{C}^0(X)$ be the space of continuous functions on X, equipped with its natural Banach space norm $\|.\|_\infty$, that is, for any φ in $\mathcal{C}^0(X)$,

$$\|\varphi\|_\infty = \max_{x\in X} |\varphi(x)|.$$

Let γ be in $(0,1]$ and Y be a closed subset of X (for example $Y = X$). For $\varphi : Y \to \mathbb{C}$, we set

$$c_\gamma(\varphi) = \sup_{f_x = f_{x'}} \frac{|\varphi(x) - \varphi(x')|}{d(x,x')^\gamma} \text{ and } |\varphi|_\gamma = \|\varphi\|_\infty + c_\gamma(\varphi),$$

where the supremum is taken over the pairs x, x' in Y with $f_x = f_{x'}$ and $x \neq x'$. We let $\mathcal{H}^\gamma(Y)$ be the space of γ-Hölder continuous functions on Y, that is, the space of functions φ on Y such that $c_\gamma(\varphi) < \infty$. The norm $|.|_\gamma$ induces a Banach space structure on $\mathcal{H}^\gamma(Y)$. The following technical lemma will be useful in the proof of Lemma 10.18.*d*.

LEMMA 10.8. *Let γ be in $(0,1]$ and Y be a closed subset of X. Then the restriction map $\mathcal{H}^\gamma(X) \to \mathcal{H}^\gamma(Y)$ is an open surjection.*

The fact that this map is open follows from the open mapping theorem, but will also be a corollary of the proof.

PROOF. Let φ be in $\mathcal{H}^\gamma(Y)$ and let us build ψ in $\mathcal{H}^\gamma(X)$ with $\psi_{|Y} = \varphi$. We can assume φ has real values. For x in X, we set

$$\psi(x) = \inf_{\substack{y \in Y \\ f_y = f_x}} \varphi(y) + c_\gamma(\varphi) d(y,x)^\gamma$$

if there exists y in Y with $f_y = f_x$ and $\psi(x) = 0$ otherwise. By construction, one has $\psi_{|Y} = \varphi$. Now, let x, x' be in X with $f_x = f_{x'}$. If, for all y in Y, $f_y \neq f_x$, we have $\psi(x) = \psi(x') = 0$. Else, for any y in Y with $f_y = f_x$, we have

$$\psi(x) \leq \varphi(y) + c_\gamma(\varphi) d(y,x)^\gamma \leq \varphi(y) + c_\gamma(\varphi) d(y,x')^\gamma + c_\gamma(\varphi) d(x',x)^\gamma,$$

hence,

$$\psi(x) \leq \psi(x') + c_\gamma(\varphi) d(x',x)^\gamma,$$

so that ψ belongs to $\mathcal{H}^\gamma(X)$ as required. □

Fix a Borel probability measure μ on G. As usual, we introduce the following Markov-Feller operator $P = P_\mu$ which is called the *transfer operator* or the *averaging operator*. It is given by, for any φ in $\mathcal{C}^0(X)$ and x in X,

$$P\varphi(x) = \int_G \varphi(gx)\, \mathrm{d}\mu(g). \tag{10.8}$$

The operator P is bounded on $\mathcal{C}^0(X)$, with norm 1. We will now study the eigenvalues of P in $\mathcal{C}^0(X)$ which have modulus 1.

In the sequel, we shall write F_μ and f_μ for $F_{s_*\mu}$ and $f_{s_*\mu}$ and, since X is fibered over F, we will consider $\mathcal{C}^0(F/F_\mu)$ and $\mathcal{C}^0(F)$ as subspaces of $\mathcal{H}^\gamma(X)$. Note that the transfer operators (10.6) and (10.8) coincide on these subspaces.

The following lemma tells us that the averaging operator P preserves $\mathcal{H}^\gamma(X)$ and contracts the seminorm c_γ.

LEMMA 10.9. *Let G be a second countable locally compact semigroup and $s : G \to F$ be a continuous morphism onto a finite group F. Let μ be a Borel probability measure on G such that μ spans F. Let $0 < \gamma \leq \gamma_0$ and let X be a compact metric G-space which is fibered over F and which is (μ, γ_0)-contracting over F.*
a) There exist $0 < \delta < 1$ and $C \geq 0$ such that, for any $\varphi \in \mathcal{H}^\gamma(X)$, $n \in \mathbb{N}$, one has,

$$c_\gamma(P^n\varphi) \leq C\delta^n c_\gamma(\varphi). \tag{10.9}$$

b) One has $P(\mathcal{H}^\gamma(X)) \subset \mathcal{H}^\gamma(X)$ and P is a bounded operator in $\mathcal{H}^\gamma(X)$ with spectral radius 1.

PROOF. *a*) As the action of G on X is (μ, γ)-contracting over F, one can suppose $\gamma = \gamma_0$. Fix $0 < \delta < 1$ and $C \geq 0$ such that (10.3) holds.

Then, for φ in $\mathcal{C}^0(X)$, x, x' in X with $f_x = f_{x'}$ and n in $\mathbb{N}$, one has

$$\begin{aligned} |P^n\varphi(x) - P^n\varphi(x')| &\leq \int_G |\varphi(gx) - \varphi(gx')| \, \mathrm{d}\mu^{*n}(g) \\ &\leq c_\gamma(\varphi) \int_G d(gx, gx')^\gamma \, \mathrm{d}\mu^{*n}(g) \\ &\leq C\delta^n d(x, x')^\gamma c_\gamma(\varphi). \end{aligned} \tag{10.10}$$

Hence $P\varphi$ belongs to $\mathcal{H}^\gamma(X)$ and Inequality (10.9) holds.

b) In particular, for any n in $\mathbb{N}$, one has

$$|P^n\varphi|_\gamma \leq \|\varphi\|_\infty + C\delta^n |\varphi|_\gamma \leq \max(1 + C) |\varphi|_\gamma \,. \tag{10.11}$$

This implies that the spectral radius of P in $\mathcal{H}^\gamma(X)$ is ≤ 1, hence exactly equals 1, since $P\mathbf{1} = \mathbf{1}$. □

The following Proposition tells us that under the contraction hypothesis (10.3), all the p_μ^{th} root of 1 are simple eigenvalues of the averaging operator P in $\mathbb{C}^F$ and that, on an invariant complementary subspace, the operator P has spectral radius < 1.

PROPOSITION 10.10. *Let G be a second countable locally compact semigroup and $s : G \to F$ be a continuous morphism onto a finite group F. Let μ be a Borel probability measure on G such that μ spans F and $p_\mu = |F/F_\mu|$. Let $0 < \gamma \leq \gamma_0$ and let X be a compact metric G-space which is fibered over F and which is (μ, γ_0)-contracting over F.*
a) The eigenvalues ζ of modulus 1 of the operator P in $\mathcal{C}^0(X)$ are the p_μ^{th}-roots of 1. These eigenvalues are simple and the associated eigenline L_ζ is spanned by the character χ_ζ of F/F_μ for which $\chi_\zeta(f_\mu F_\mu) = \zeta$. The direct sum of these eigenlines L_ζ is equal to $\mathcal{C}^0(F/F_\mu)$.

b) There is a unique μ-stationary Borel probability measure ν on X.
c) The operator $N : \mathcal{C}^0(X) \to \mathcal{C}^0(F/F_\mu)$ given by, for any φ in $\mathcal{C}^0(X)$ and f in F,

$$N\varphi(fF_\mu) = p_\mu \int_{\{f_x \in fF_\mu\}} \varphi(x)\,\mathrm{d}\nu(x) \tag{10.12}$$

is the unique P-equivariant projection onto $\mathcal{C}^0(F/F_\mu)$.
d) The restriction of P to $\mathcal{H}^\gamma(X) \cap \operatorname{Ker} N$ has spectral radius < 1.

Note that this spectral radius is computed for the norm $|.|_\gamma$.

Corollary 10.11. *Same notations as in Lemma 10.9. The essential spectral radius of P in $\mathcal{H}^\gamma(X)$ is < 1.*

We recall that the *essential spectral radius* is the infimum of the spectral radii of the restriction of P to a P-invariant finite codimensional subspace. In other words it is the supremum of the $|\lambda|$, where λ is a complex number such that $P - \lambda 1$ is not a Fredholm operator (see Appendix 11.4).

Proof of Proposition 10.10. a) Let φ be in $\mathcal{C}^0(X)$ and ζ be a complex number of modulus 1 with $P\varphi = \zeta\varphi$. According to Formula (10.9), for any n in $\mathbb{N}$, one has $c_\gamma(\varphi) = c_\gamma(P^n\varphi) \xrightarrow[n\to\infty]{} 0$. Thus $c_\gamma(\varphi) = 0$ and the function φ is constant on the fibers of the map $x \mapsto f_x$. By lemma 10.6, ζ is a p_μ^{th}-root of unity and there exists a character χ_ζ of F/F_μ such that φ is proportional to the function $x \mapsto \chi_\zeta(f_xF_\mu)$. Since moreover $\|P_\mu\|_\infty \le 1$, this eigenvalue is simple.

b) We choose a μ-stationary Borel probability measure ν on X. As μ spans F, the image of ν in F is the normalized counting measure. We postpone the proof of the uniqueness of ν until after the proof of *d*).

c) By construction, the operator N is a projection onto $\mathcal{C}^0(F/F_\mu)$. We have to prove that it commutes with P. We compute for φ in $\mathcal{C}^0(F)$ and f in F,

$$\begin{aligned} NP\varphi(f) &= p_\mu \textstyle\int_{G\times X} \varphi(gx)\mathbf{1}_{\{f_x\in fF_\mu\}}\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) \\ &= p_\mu \textstyle\int_{G\times X} \varphi(gx)\mathbf{1}_{\{f_{gx}\in f_\mu fF_\mu\}}\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) \\ &= p_\mu \textstyle\int_X \varphi(x)\mathbf{1}_{\{f_x\in f_\mu fF_\mu\}}\,\mathrm{d}\nu(x) = N\varphi(f_\mu fF_\mu). \end{aligned}$$

where we used the equality $s(g) = f_\mu \bmod F_\mu$, for μ-almost all g in G, to get the second line and the μ-stationarity of ν to get the third one. This proves that $NP = PN$ as required. We postpone the proof of uniqueness of N after the proof of *d*).

d) By Lemma 10.9, the Banach space $E := \mathcal{H}^\gamma(X) \cap \operatorname{Ker} N$ is stable by the action of P, and the spectral radius of P in E for the norm $|.|_\gamma$

is at most 1. We want to prove that this spectral radius of P in E is < 1. Let E' be the finite dimensional subspace of E,

$$E' := \mathcal{C}^0(F) \cap \operatorname{Ker} N.$$

According to Lemma 10.6, the spectral radius of P in E' is < 1. Hence, by Lemma 10.12 below, it is enough to show that *the spectral radius of P in E/E' is* < 1. This quotient Banach space is equal to the space $\mathcal{H}^\gamma(X)/\mathcal{C}^0(F)$. Since

$$\mathcal{C}^0(F) = \{\varphi \in \mathcal{H}^\gamma(X) \mid c_\gamma(\varphi) = 0\},$$

the seminorm c_γ defines a norm on this quotient Banach space. This norm is equivalent to the norm induced by $|.|_\gamma$. Indeed, choosing a point x_f in each fiber of the map $x \mapsto f_x$, the closed subspace

$$E'' := \{\varphi \in E \mid \varphi(x_f) = 0 \text{ for all } f \text{ in } F\}$$

satisfies $\mathcal{H}^\gamma(X) = \mathcal{C}^0(F) \oplus E''$ and there exists $C' > 0$, such that, one has

$$\|\varphi\|_\infty \leq C'\, c_\gamma(\varphi) \ \ , \ \text{ for all } \varphi \text{ in } E''.$$

Hence, according to Equation (10.9), the spectral radius of P in E/E' is < 1, as required.

e) We prove now the uniqueness of both N and ν. By *d*), for any φ in $\mathcal{H}^\gamma(X) \cap \operatorname{Ker} N$, we have

$$P^n\varphi \xrightarrow[n\to\infty]{} 0 \text{ uniformly on } X. \tag{10.13}$$

Since the subspace $\mathcal{H}^\gamma(X)$ is dense in $\mathcal{C}^0(X)$ and since the operator N is a projection onto a subspace of $\mathcal{H}^\gamma(X)$, the intersection $\mathcal{H}^\gamma(X) \cap \operatorname{Ker} N$ is dense in $\operatorname{Ker} N$ for the uniform topology. Since $\|P\|_\infty = 1$, the convergence (10.13) holds for any continuous φ in $\operatorname{Ker} N$. This gives uniqueness of N.

Now, from (10.12), one gets, for every $\varphi \in \mathcal{C}^0(X)$,

$$\nu(\varphi) = \tfrac{1}{p_\mu} \textstyle\sum_{F/F_\mu} N\varphi(fF_\mu)$$

and uniqueness of ν follows from the uniqueness of N. $\square$

In this proof, we used the following lemma.

LEMMA 10.12. *Let E be a Banach space, E' be a closed subspace and T be a bounded operator of E preserving E'. Then, the spectrum of T is included in the union of the spectra of the two operators $T_{E'}$ and $T_{E/E'}$ induced by T in E' and in E/E'.*

Proof. By the open mapping theorem, the spectrum of T is the set of complex numbers λ for which $T-\lambda$ is not bijective. Hence our statement follows from the following elementary fact: *if $T_{E'}$ and $T_{E/E'}$ are bijective, then T is also bijective.* □

Remark 10.13. *The spectral radius of P in $\operatorname{Ker} N$ with respect to the norm $\|.\|_\infty$ may be equal to* 1. Let, as in Example 10.4, X be the compact space $X=\{0,1\}^{\mathbb{N}}$, G be the semigroup spanned by the two prefix maps $s_i : x \mapsto ix$, $i=0,1$, and $\mu := \frac{1}{2}(\delta_{s_0}+\delta_{s_1})$. In this case, the action of G on X is μ-contracting and the only μ-stationary probability measure ν on X is the Bernoulli probability measure $\nu = (\frac{1}{2}(\delta_0+\delta_1))^{\otimes\mathbb{N}}$, so that

$$\operatorname{Ker} N = \{\varphi \in \mathcal{C}^0(X) \mid \textstyle\int \varphi \,\mathrm{d}\mu = 0\}.$$

The averaging operator P_μ is given by, for φ in $\mathcal{C}^0(X)$ and x in X,

$$P_\mu\varphi(x) = \tfrac{1}{2}(\varphi(s_0x)+\varphi(s_1x)).$$

By Proposition 10.10, this operator P_μ has spectral radius smaller than 1 in $\mathcal{H}^\gamma(X)\cap \operatorname{Ker} N$. Nevertheless, it has spectral radius 1 in $\operatorname{Ker} N$. Indeed, let $S : X\to X$ be the shift map, $\varphi : X \to \mathbb{C}$ be the function given by $\varphi(x) := (-1)^{x_1}$. The continuous functions $\varphi_k := \varphi\circ S^k$ have zero average and satisfy $P_\mu^k\varphi_k = \varphi$ and $\|\varphi_k\|_\infty = \|\varphi\|_\infty = 1$, hence P_μ^k has norm 1, for all $k\geq 0$. Similar examples can be constructed with $G := \mathrm{SL}(2,\mathbb{K})$ and $X := \mathbb{P}^1(\mathbb{K})$, for any local field $\mathbb{K}$. See Example 12.21 when $\mathbb{K}=\mathbb{Q}_p$.

10.4. Cocycles over μ-contracting actions.

In this section, we introduce a suitable moment conditions for cocycles over μ-contracting actions. We prove that under these conditions the random trajectories of this cocycle do not depend on the starting point except for a bounded error term.

We also claim that these cocycles are special. The proof will be given in Sections 10.5 and 10.6.

Let E be a real finite dimensional Euclidean vector space, and $\sigma : G\times X \to E$ be a continuous cocycle. We set E^* to be the dual vector space of E, $E_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}} E$ and $E^*_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}} E^*$.

Recall that we defined the *sup-norm* $\sigma_{\sup}$ of σ as

$$\sigma_{\sup}(g) = \sup\nolimits_{x\in X} \|\sigma(g,x)\|.$$

We now define the *fibered Lipschitz constant of the cocycle* σ_{Lip} on G by, for g in G,

$$\sigma_{\mathrm{Lip}}(g) = \sup_{f_x = f_{x'}} \frac{\|\sigma(g,x) - \sigma(g,x')\|}{d(x,x')}$$

where the supremum is taken over the pairs x, x' in X with $f_x = f_{x'}$ and $x \neq x'$.

DEFINITION. We shall say that the *sup-norm of the cocycle σ has a finite exponential moment* if there exists $\alpha > 0$ such that

$$\int_G e^{\alpha \sigma_{\sup}(g)} \, \mathrm{d}\mu(g) < \infty. \tag{10.14}$$

We shall say that the *Lipschitz constant of the cocycle σ has a finite moment* if there exists $\alpha > 0$ such that

$$\int_G \sigma_{\mathrm{Lip}}(g)^{\alpha} \, \mathrm{d}\mu(g) < \infty. \tag{10.15}$$

We describe now how the behavior of these cocycles depends on the starting point x.

LEMMA 10.14 (Bounded dependance on the starting point).
Let G be a second countable locally compact semigroup, $s : G \to F$ be a continuous morphism onto a finite group F, and E be a finite dimensional real vector space. Let μ be a Borel probability measure on G such that μ spans F. Let X be a compact metric G-space which is fibered over F and which is μ-contracting over F, and $\sigma : G \times X \to E$ be a continuous cocycle whose Lipschitz constant has a finite moment.
a) There exist $\gamma > 0$ and $I_\gamma > 0$ such that, for any x, x' in X with $f_x = f_{x'}$, for any $n \geq 1$, one has

$$\int_G \|\sigma(g,x) - \sigma(g,x')\|^{\gamma} \, \mathrm{d}\mu^{*n}(g) \leq I_\gamma. \tag{10.16}$$

b) For any x, x' in X with $f_x = f_{x'}$, for β-almost any b in B, one has

$$\sup_{n \geq 1} \|\sigma(b_n \cdots b_1, x) - \sigma(b_n \cdots b_1, x')\| < \infty. \tag{10.17}$$

c) For any x, x' in X with $f_x = f_{x'}$, one has

$$\lim_{C \to \infty} \inf_{n \geq 1} \mu^{*n}(\{g \in G \mid \|\sigma(g,x) - \sigma(g,x')\| \leq C\}) = 1. \tag{10.18}$$

PROOF. *a*) Using the cocycle relation (2.6), one gets, for any $g_1, \ldots, g_n$ in G,

$$\begin{aligned} &\|\sigma(g_n \cdots g_1, x) - \sigma(g_n \cdots g_1, x')\| \\ &\qquad \leq \textstyle\sum_{k=1}^{n} \sigma_{\mathrm{Lip}}(g_k) d(g_{k-1} \cdots g_1 x, g_{k-1} \cdots g_1 x'). \end{aligned}$$

This gives the following domination of the left hand-side L of (10.16)

$$\begin{array}{rcl} L & = & \int_{G^n} \|\sigma(g_n\cdots g_1, x) - \sigma(g_n\cdots g_1, x')\|^\gamma \,\mathrm{d}\mu(g_1)\cdots \mathrm{d}\mu(g_n) \\ & \leq & \sum_{k=1}^n \int_G d(gx, gx')^\gamma \,\mathrm{d}\mu^{*(k-1)}(g) \int_G \sigma_{\mathrm{Lip}}^\gamma \,\mathrm{d}\mu. \end{array}$$

Using now the μ-contraction condition (10.3) and the moment condition (10.15), if γ is small enough, one can find $C_0 > 0$ and $\delta < 1$ such that

$$L \leq \sum_{k=1}^\infty C_0 \delta^{k-1} d(x, x')^\gamma \int_G \sigma_{\mathrm{Lip}}^\gamma \,\mathrm{d}\mu < \infty.$$

b) Fix $\alpha > 0$ such that, by the moment condition (10.15), the function $\sigma_{\mathrm{Lip}}^\alpha$ is μ-integrable. As a corollary of Birkhoff ergodic theorem for the Bernoulli dynamical system (B, β, T), for β-almost every b in B, the sequence $\sigma_{Lip}(b_k)^\alpha/k$ converges to 0. In particular, for k large, one has $\sigma_{\mathrm{Lip}}(b_k) \leq k^{1/\alpha}$. Hence using the cocycle property as in *a*) and the bound (10.5), one can find a constant $M(b) > 0$ such that the left hand-side L' of (10.17) is bounded by :

$$\begin{array}{rcl} L' & \leq & \sum_{k=1}^\infty \sigma_{\mathrm{Lip}}(b_k) d(b_{k-1}\cdots b_1 x, b_{k-1}\cdots b_1 x') \\ & \leq & M(b) + \sum_{k\geq 1} k^{1/\alpha} e^{-\gamma(k-1)} < \infty. \end{array}$$

The constant $M(b) > 0$ in this computation takes into account the finitely many terms we cannot control.

c) Our statement follows either from the bound

$$\mu^{*n}(\{g \in G \mid \|\sigma(g, x) - \sigma(g, x')\| \geq C\}) \leq C^{-\gamma} I_\gamma$$

based on *a*) or from the bound

$$\lim_{C\to\infty} \beta(\{b \in G \mid \sup_{n\geq 1} \|\sigma(b_n..b_1, x) - \sigma(b_n..b_1, x')\| \leq C\}) = 1$$

that can be deduced from *b*). □

The following proposition gives a sufficient condition for a cocycle to be special (as in Section 2.4). This proposition will be applied to the Iwasawa cocycle.

PROPOSITION 10.15. *Let G be a second countable locally compact semigroup, $s : G \to F$ be a continuous morphism onto a finite group F, and E be a finite dimensional real vector space. Let μ be a Borel probability measure on G such that μ spans F. Let X be a compact metric G-space which is fibered over F and which is μ-contracting over F.*

Let $\sigma : G \times X \to E$ be a continuous cocycle whose sup-norm has a finite exponential moment (10.14) *and whose Lipschitz constant has a finite moment* (10.15). *Then the cocycle σ is special.*

The proof of Proposition 10.15 will last up to the end of Section 10.6. It relies on the study of the leading eigenvalue λ_θ of a family of linear operators P_θ called the complex transfer operators. The tools that we will develop to prove Proposition 10.15 will be useful to prove the Central Limit Theorem 11.1.

10.5. The complex transfer operator.

In this section, we introduce the complex transfer operator P_θ. We prove that it depends analytically on the parameter θ and deduce that, for θ small enough, it has a leading eigenvalue λ_θ which also depends analytically on θ.

We keep the notations of section 10.3 and we assume that the action of G on X is μ-contracting.. Let $\sigma : G\times X \to E$ be a continuous cocycle as in section 10.4.

According to the finite moment conditions (10.14) and (10.15), one can choose $\alpha \in (0,1)$ such that the function κ_0 on G

$$g \mapsto \kappa_0(g) := \max(\sigma_{\sup}(g), \log \sigma_{\mathrm{Lip}}(g)) \tag{10.19}$$

has a finite exponential moment:

$$\textstyle\int_G e^{\alpha\kappa_0(g)}\,\mathrm{d}\mu(g) < \infty. \tag{10.20}$$

If one assumes α to be smaller than γ_0 from Definition in 10.1, using the cocycle property, one easily checks that κ_0 also has a finite exponential moment for all the measures μ^{*n} with $n \geq 1$:

$$\textstyle\int_G e^{\alpha\kappa_0(g)}\,\mathrm{d}\mu^{*n}(g) < \infty. \tag{10.21}$$

For θ in $E^*_{\mathbb{C}}$ with $\|\Re\theta\| < \alpha$, for φ in $\mathcal{C}^0(X)$ and x in X, we set

$$P_\theta\varphi(x) = \textstyle\int_G e^{\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu(g). \tag{10.22}$$

Then, P_θ is a bounded operator of $\mathcal{C}^0(X)$ called the *complex transfer operator*. Since σ is a cocycle, for any $n \geq 1$, we have

$$P_\theta^n\varphi(x) = \textstyle\int_G e^{\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu^{*n}(g). \tag{10.23}$$

We shall now fix γ with $0 < \gamma < \min(\gamma_0, \alpha)/2$.

Lemma 10.16. *Same assumptions as in Proposition 10.15. For any θ in $E^*_{\mathbb{C}}$ with $\|\Re\theta\| < \min(\alpha/2, \alpha-\gamma)$, one has $P_\theta\mathcal{H}^\gamma(X) \subset \mathcal{H}^\gamma(X)$ and P_θ is a continuous operator of $\mathcal{H}^\gamma(X)$, which depends analytically on θ.*

Proof. We fix θ in $E^*_{\mathbb{C}}$ with $\|\Re\theta\| < \min(\alpha/2, \alpha - \gamma)$. We choose an orthogonal basis $e_1, \ldots, e_r$ of E and decompose any element $\varepsilon \in E^*_{\mathbb{C}}$

along the dual basis: $\varepsilon = \varepsilon_1 + \ldots + \varepsilon_r$ with $\varepsilon_i \in E^*_{\mathbb{C}}$ and $\varepsilon_i(e_j) = \delta_{i,j}\varepsilon(e_j)$ for all i, j. We will consider elements $\varepsilon \in E^*_{\mathbb{C}}$ with

$$r\|\varepsilon\| < \alpha/2 - \gamma - \|\Re\theta\|. \tag{10.24}$$

We will use the standard multiindices notation: for $m = (m_1, \ldots, m_r) \in \mathbb{N}^r$, we set

$$|m| = m_1 + \ldots + m_r\ ,\quad m! = m_1! \cdots m_r!\ ,\quad \varepsilon^m = \varepsilon_1^{m_1} \cdots \varepsilon_r^{m_r} \in S^{|m|}E^*_{\mathbb{C}}$$

and we introduce the operator $P_{\theta,\varepsilon,m}$ on $\mathcal{C}^0(X)$ given by, for $\varphi \in \mathcal{C}^0(X)$ and $x \in X$,

$$P_{\theta,\varepsilon,m}\varphi(x) = \int_G \varepsilon^m(\sigma(g,x))\, e^{\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu(g).$$

Note that for $m = 0$ this operator is equal to P_θ. Now, since, for any v in E,

$$e^{(\theta+\varepsilon)(v)} = \sum_{m\in\mathbb{N}^r} \tfrac{1}{m!}\varepsilon^m(v)e^{\theta(v)},$$

to get analycity of P in the neighborhood of θ, it suffices to check, for $\varphi \in \mathcal{H}^\gamma(X)$, the absolute convergence:

$$\sum_{m\in\mathbb{N}^r} \tfrac{1}{m!}\, |P_{\theta,\varepsilon,m}\varphi|_\gamma \le M|\varphi|_\gamma\ , \tag{10.25}$$

for some finite constant M independent of φ and ε. We first bound the sup norm: one has

$$\|P_{\theta,\varepsilon,m}\varphi\|_\infty \le \int_G \|\varepsilon\|^{|m|}\,\kappa_0(g)^{|m|}\,e^{\|\Re(\theta)\|\kappa_0(g)}\,\|\varphi\|_\infty\,\mathrm{d}\mu(g)$$

and hence, using (10.20) and (10.24),

$$\sum_{m\in\mathbb{N}^r} \tfrac{1}{m!}\,\|P_{\theta,\varepsilon,m}\varphi\|_\infty \le \int_G e^{(r\|\varepsilon\|+\|\Re(\theta)\|)\kappa_0(g)}\,\|\varphi\|_\infty\,\mathrm{d}\mu(g) \le M_\alpha\|\varphi\|_\infty.$$

Now it remains to bound, for $x \neq x'$ in X with $f_x = f_{x'}$:

$$\tfrac{P_{\theta,\varepsilon,m}\varphi(x)-P_{\theta,\varepsilon,m}\varphi(x')}{d(x,x')^\gamma} = A_m + B_m + C_m \quad \text{where}$$

$$\begin{aligned}
A_m &= \int_G \tfrac{\varepsilon^m(\sigma(g,x))-\varepsilon^m(\sigma(g,x'))}{d(x,x')^\gamma}\, e^{\theta(\sigma(g,x))}\,\varphi(gx)\,\mathrm{d}\mu(g)\\
B_m &= \int_G \varepsilon^m(\sigma(g,x'))\,\tfrac{e^{\theta(\sigma(g,x))}-e^{\theta(\sigma(g,x'))}}{d(x,x')^\gamma}\,\varphi(gx)\,\mathrm{d}\mu(g)\\
C_m &= \int_G \varepsilon^m(\sigma(g,x'))\,e^{\theta(\sigma(g,x'))}\,\tfrac{\varphi(gx)-\varphi(gx')}{d(x,x')^\gamma}\,\mathrm{d}\mu(g).
\end{aligned}$$

Since

$$\|a^m - b^m\| \le 2^{1-\gamma}|m|\max(\|a\|,\|b\|)^{|m|-\gamma}\|a-b\|^\gamma$$

for all $a, b \in \mathbb{C}^r$, one gets

$$|A_m| \le 2\int_G |m|\|\varepsilon\|^{|m|-\gamma}\,\kappa_0(g)^{|m|-\gamma}\,e^{\gamma\kappa_0(g)}\,e^{\|\Re(\theta)\|\kappa_0(g)}\,\|\varphi\|_\infty\,\mathrm{d}\mu(g),$$

and, using the equality

$$\sum_{m\in\mathbb{N}^r} \tfrac{|m|}{m!}x^{|m|-1} = re^{rx} \ \text{ for } x > 0,$$

one gets

$$\sum_{m\in\mathbb{N}^r} \tfrac{1}{m!}|A_m| \leq 2r\|\varepsilon\|^{1-\gamma}\,\|\varphi\|_\infty \int_G \kappa_0(g)^{1-\gamma} e^{(r\|\varepsilon\|+\|\Re(\theta)\|+\gamma)\kappa_0(g)}\,\mathrm{d}\mu(g).$$

This quantity is bounded by a uniform multiple of $\|\varphi\|_\infty$.

Since

$$|e^a - e^b| \leq 2^{1-\gamma}\max(|a|,|b|)^{1-\gamma}\max(e^{\Re a}, e^{\Re b})|a-b|^\gamma$$

for all a, b in $\mathbb{C}$, one gets

$$|B_m| \leq 2\int_G \|\varepsilon\|^{|m|}\,\kappa_0(g)^{|m|+1-\gamma}\,e^{\|\Re(\theta)\|\kappa_0(g)} e^{\gamma\kappa_0(g)}\;\|\varphi\|_\infty\,\mathrm{d}\mu(g),$$

hence,

$$\sum_{m\in\mathbb{N}^r} \tfrac{1}{m!}|B_m| \leq 2\,\|\varphi\|_\infty \int_G \kappa_0(g)^{1-\gamma} e^{(r\|\varepsilon\|+\|\Re(\theta)\|+\gamma)\kappa_0(g)}\,\mathrm{d}\mu(g).$$

Again, this quantity is bounded by a uniform multiple of $\|\varphi\|_\infty$.

Finally one also has

$$|C_m| \leq \int_G \|\varepsilon\|^{|m|}\,\kappa_0(g)^{|m|}\,e^{\|\Re(\theta)\|\kappa_0(g)}\,c_\gamma(\varphi)\,\tfrac{d(gx,gx')^\gamma}{d(x,x')^\gamma}\,\mathrm{d}\mu(g),$$

hence,

$$\begin{aligned}\sum_{m\in\mathbb{N}^r} \tfrac{1}{m!}|C_m| &\leq c_\gamma(\varphi)\int_G e^{(r\|\varepsilon\|+\|\Re(\theta)\|)\kappa_0(g)}\tfrac{d(gx,gx')^\gamma}{d(x,x')^\gamma}\,\mathrm{d}\mu(g)\\ &\leq c_\gamma(\varphi)\left(\int_G e^{\alpha\kappa_0(g)}\,\mathrm{d}\mu(g)\right)^{1/2}\left(\int_G \mathrm{Lip}(g)^{\gamma_0}\,\mathrm{d}\mu(g)\right)^{1/2},\end{aligned}$$

where we used Cauchy-Schwartz inequality, and we are done. □

As $P_0 = P$, using elementary perturbation theory and the preceding analysis of P, we can prove the following structure result for P_θ with small θ. For a p_μ^{th}-root of unity ζ in U_{p_μ}, we still denote by χ_ζ the character of F which is constant with value ζ on $f_\mu F_\mu$.

Lemma 10.17. *Same assumptions as in Proposition 10.15.*
a) There exist $\varepsilon > 0$, a convex bounded open neighborhood U of 0 in $E_{\mathbb{C}}^*$ and analytic maps on U

$$\theta \mapsto \lambda_\theta \in \mathbb{C}\ \ ,\ \ \theta \mapsto \varphi_\theta \in \mathcal{H}^\gamma(X)\ \text{ and }\ \theta \mapsto N_\theta \in \mathcal{L}(\mathcal{H}^\gamma(X)).$$

such that, for any θ in U,
(i) $\lambda_0 = 1$, $\varphi_0 = \mathbf{1}$ and $N_0 = N$ and $|\lambda_\theta - 1| \leq \varepsilon$,
(ii) $P_\theta\varphi_\theta = \lambda_\theta\varphi_\theta$ and $\nu(\varphi_\theta) = 1$,
(iii) $P_\theta N_\theta = N_\theta P_\theta$, the map N_θ is a projection onto the p_μ-dimensional subspace $\oplus\mathbb{C}\,\chi_\zeta\varphi_\theta \subset \mathcal{H}^\gamma(X)$, where the direct sum is over the p_μ^{th}-roots of unity, and the restriction of P_θ to $\operatorname{Ker} N_\theta$ has spectral radius $\leq 1-\varepsilon$.
b) The fuctions $\chi_\zeta\varphi_\theta$ are eigenvectors of P_θ with eigenvalues $\zeta\lambda_\theta$.

PROOF. By construction, the function χ_ζ satisfies the following equivariance property: for every g in the support of μ and $x \in X$

$$\chi_\zeta(gx) = \zeta\chi_\zeta(x).$$

Hence for every θ in a small neighborhood $U \subset E^*_{\mathbb{C}}$ of 0 and any $\varphi \in \mathcal{H}^\gamma(X)$, one has

$$P_\theta(\chi_\zeta\varphi) = \zeta\chi_\zeta P_\theta(\varphi). \tag{10.26}$$

Now we use the functional calculus of operators. Thanks to Proposition 10.10, the projection N has finite rank p_μ and commutes with the transfer operator P, the restriction of P to $\operatorname{Im} N$ has simple eigenvalues equal to the p_μ^{th}-roots of unity, and we can choose ε small enough so that the specral radius of the restriction of the transfer operator P to $\operatorname{Ker} N \cap \mathcal{H}^\gamma(X)$ is $\leq 1 - 2\varepsilon$. For a in $\mathbb{C}$, $r \geq 0$, we denote by $C(a, r)$ the positively oriented circle with center a and radius r. When U is small enough, the following expressions, with ζ a p_μ^{th}-root of unity,

$$Q_\theta = \tfrac{1}{2i\pi} \oint_{C(0,1-\varepsilon)} (z - P_\theta)^{-1}\, \mathrm{d}z \quad\text{and}\quad N_{\zeta,\theta} = \tfrac{1}{2i\pi} \oint_{C(\zeta,\varepsilon)} (z - P_\theta)^{-1}\, \mathrm{d}z$$

define disjoint projections of $\mathcal{H}^\gamma(X)$, which commute with P_θ, whose sum is the identity operator and which depend analytically on θ.

We claim that, if θ is small enough, each of the $N_{\zeta,\theta}$ has rank 1. Indeed, if θ is small enough, the operator $Q_0 Q_\theta$ is an automorphism of $\operatorname{Ker} N$. In particular, the image of Q_θ has codimension at most p_μ, whereas, if θ is small enough, each of the $N_{\zeta,\theta}$ is nonzero, and hence has rank ≥ 1. Therefore, they all have rank 1.

We set λ_θ to be the eigenvalue of P_θ in $\operatorname{Im} N_{1,\theta}$. If θ is small enough we can define a generator φ_θ of this line by requiring that $\nu(\varphi_\theta) = 1$. Because of the equivariance property (10.26), for each p_μ^{th}-root of unity ζ, the function $\chi_\zeta\varphi_\theta$ spans the eigenline $\operatorname{Im} N_{\zeta,\theta}$ and the associated eigenvalue is $\zeta\lambda_\theta$. We let N_θ be the projection

$$N_\theta = \textstyle\sum_\zeta N_{\zeta,\theta}$$

and we are done. □

Note that, since, for any φ in $\mathcal{C}^0(X)$, one has $P_{\bar\theta}\bar\varphi = \overline{P_\theta\varphi}$, where $\bar{\cdot}$ denotes complex conjugation, for θ in E, one has $\lambda_\theta \in \mathbb{R}$.

10.6. Second derivative of the leading eigenvalue.

The proof of Proposition 10.15 now essentially relies on the local study near $\theta = 0$ of the leading eigenvalue λ_θ and the leading eigenfunction φ_θ of the complex transfer operator P_θ in $\mathcal{H}^\gamma(X)$.

We denote by $\dot{\lambda}_\theta \in E_{\mathbb{C}}$ the derivative of the function $\theta \mapsto \lambda_\theta$ and by $\ddot{\lambda}_\theta \in \mathrm{S}^2(E_{\mathbb{C}})$ its second derivative. One has $\dot{\lambda}_0 \in E$ and $\ddot{\lambda}_\theta \in \mathrm{S}^2(E)$. We denote also by $\dot{\varphi}_\theta$ and $\ddot{\varphi}_\theta$ the first and second derivatives of the map $\theta \mapsto \varphi_\theta$. These are respectively Hölder continuous functions on X with values in $E_{\mathbb{C}}$ and $\mathrm{S}^2 E_{\mathbb{C}}$. Similarly we will use the notations $\dot{P}_\theta$ and $\ddot{P}_\theta$.

In the following lemma, we prove that the cocycle σ is special and we relate the objects that have been introduced in sections 2.3 and 2.4 with the derivatives at $\theta = 0$ of the functions $\theta \mapsto \lambda_\theta$ and $\theta \mapsto \varphi_\theta$. We recall that ν is the unique μ-stationary probability measure on X (see for instance Lemma 10.5).

Lemma 10.18. *Same assumptions as in Proposition 10.15.*
a) The derivative of λ_θ at $\theta = 0$ is the average of σ: $\dot{\lambda}_0 = \sigma_\mu$. The cocycle σ is special. More precisely, the cocycle $\sigma_0 : G \times X \to E$ defined, for any (g, x) in $G \times X$, by

$$\sigma(g,x) = \sigma_0(g,x) + \dot{\varphi}_0(x) - \dot{\varphi}_0(gx) \tag{10.27}$$

has constant drift.
b) The recentered second derivative $\ddot{\lambda}_0 - \dot{\lambda}_0^2 \in \mathrm{S}^2 E$ is a non-negative 2-tensor that is equal to the covariance 2-tensor

$$\Phi_\mu = \int_{G \times X} (\sigma_0(g,x) - \sigma_\mu)^2 \, \mathrm{d}\mu(g) \, \mathrm{d}\nu(x). \tag{10.28}$$

c) Let $E_\mu \subset E$ be the linear span of Φ_μ (see Section 2.4). Then, for all g in $\operatorname{Supp} \mu$ and x in the support S_ν of ν, one has

$$\sigma_0(g,x) = \sigma_\mu \mod E_\mu. \tag{10.29}$$

d) For any $\theta \in U$ and $\theta' \in E_\mu^\perp$ with $\theta + \theta' \in U$, one has

$$\lambda_{\theta+\theta'} = e^{\theta'(\sigma_\mu)} \lambda_\theta \, .$$

Conclusion c) roughly means that the 2-tensor Φ_μ is non-degenerate except if in some direction the cocycle is the sum of a constant and a coboundary. Conclusion d) means that the function $\theta \mapsto e^{-\theta(\sigma_\mu)} \lambda_\theta$ is invariant by translations in the direction of the orthogonal $E_\mu^\perp$ of E_μ in the dual space E^*. Recall that this space $E_\mu^\perp$ is also the kernel of Φ_μ seen as a quadratic form on E^*.

Proof. Using the trick (2.9), we may assume that $\sigma_\mu = 0$. This will simplify a little the computations.

a) Differentiating the equation

$$\lambda_\theta \varphi_\theta = P_\theta \varphi_\theta \ \text{ and } \ \nu(\varphi_\theta) = 1 \quad (\theta \in U),$$

one gets

$$\dot\lambda_\theta\varphi_\theta + \lambda_\theta\dot\varphi_\theta = \dot P_\theta\varphi_\theta + P_\theta\dot\varphi_\theta \text{ and } \nu(\dot\varphi_\theta) = 0\,. \tag{10.30}$$

Substituting $\theta = 0$, one gets

$$\dot\lambda_0 + \dot\varphi_0 = \dot P_0\mathbf{1} + P_0\dot\varphi_0. \tag{10.31}$$

Setting $\sigma_0(g,x) = \sigma(g,x) - \dot\varphi_0(x) + \dot\varphi_0(gx)$, Equation (10.31) can be rewritten as, for any $x \in X$,

$$\dot\lambda_0 = \textstyle\int_G \sigma_0(g,x)\,\mathrm{d}\mu(g). \tag{10.32}$$

Hence the cocycle σ_0 has constant drift and the cocycle σ is special. Applying ν to (10.31), one gets, since ν is μ-stationary, the equality in E

$$\dot\lambda_0 = \textstyle\int_X\int_G \sigma(g,x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) = \sigma_\mu = 0\,.$$

b) Differentiating Equation (10.30), one gets

$$\ddot\lambda_\theta\varphi_\theta + 2\dot\lambda_\theta\dot\varphi_\theta + \lambda_\theta\ddot\varphi_\theta = \ddot P_\theta\varphi_\theta + 2\dot P_\theta\dot\varphi_\theta + P_\theta\ddot\varphi_\theta \text{ and } \nu(\ddot\varphi_\theta) = 0\,.$$

Substituting $\theta = 0$ and applying ν, one gets the equalities in S^2E

$$\begin{aligned}\ddot\lambda_0 &= \nu(\ddot P_0\mathbf{1}) + 2\,\nu(\dot P_0\dot\varphi_0)\\ &= \textstyle\int_X\int_G(\sigma(g,x)^2 + 2\,\sigma(g,x)\,\dot\varphi_0(gx))\;\mathrm{d}\mu(g)\,\mathrm{d}\nu(x)\\ &= \textstyle\int_X\int_G(\sigma(g,x) + \dot\varphi_0(gx))^2\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) - \int_X \dot\varphi_0(x)^2\,\mathrm{d}\nu(x),\end{aligned}$$

where the first equality follows from the μ-sationarity of ν applied to the function $\ddot\varphi_0$, and where the last equality follows from the μ-stationarity of ν applied to the function $\dot\varphi_0^2$. Now using Equation (10.27), one gets the equalities in S^2E

$$\ddot\lambda_0 = \textstyle\int_X\int_G \sigma_0(g,x)^2\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) = \Phi_\mu.$$

Hence this quadratic form on E^* is non-negative.

c) By the above formula, since E_μ is the linear span of Φ_μ, for $\mu\otimes\nu$-almost every (g,x) in $G\times X$, $\sigma_0(g,x)$ belongs to E_μ.

d) By *c*), one has

$$\theta'(\sigma(g,x)) = \theta'(\sigma_\mu) + \theta'(\dot\varphi_0(x)) - \theta'(\dot\varphi_0(gx)) \tag{10.33}$$

for any g in the support of μ and x in S_ν.

First, assume $S_\nu = X$. One has

$$P_{\theta+\theta'} = e^{\theta'(\sigma_\mu)} M_{e^{\theta'(\dot\varphi_0)}} P_\theta M_{e^{-\theta'(\dot\varphi_0)}},$$

where M_ψ denotes the operator of multiplication by a function ψ. In other words, the operator $P_{\theta+\theta'}$ is conjugated to a multiple of P_θ. By uniqueness of the eigenvalue of P_θ that is close to one, one gets $\lambda_{\theta+\theta'} = e^{\theta'(\sigma_\mu)}\lambda_\theta$ if θ and θ' are small enough.

In general, let us prove that the operator $P_{\theta+\theta'}$ is conjugated to a multiple of P_θ in the Banach space $\mathcal{H}^\gamma(S_\nu)$. Indeed, let $\mathcal{F}$ be the closed subspace of those ψ in $\mathcal{H}^\gamma(X)$ whose restriction to S_ν is 0. By Lemma 10.8, the restriction map induces a topological isomorphism between the Banach spaces $\mathcal{H}^\gamma(X)/\mathcal{F}$ and $\mathcal{H}^\gamma(S_\nu)$. Since $P_\theta\mathcal{F} \subset \mathcal{F}$ (or since $\Gamma_\mu S_\nu \subset S_\nu$), one may consider P_θ as a continuous operator on $\mathcal{H}^\gamma(S_\nu)$. Besides, one has $\varphi_\theta \notin \mathcal{F}$ since $\nu(\varphi_\theta) = 1$, and hence λ_θ is also an eigenvalue of the operator P_θ acting on $\mathcal{H}^\gamma(S_\nu) = \mathcal{H}^\gamma(X)/\mathcal{F}$. Now, still by (10.33), the operator $P_{\theta+\theta'}$ is conjugated to a multiple of P_θ in $\mathcal{H}^\gamma(S_\nu)$. By the uniqueness of the eigenvalue of P_θ that is close to one, one still gets $\lambda_{\theta+\theta'} = e^{\theta'(\sigma_\mu)}\lambda_\theta$ if θ and θ' are small enough. $\square$

The following corollary tells us that the asymptotic behavior of the cocycle σ is controlled by its average and by its component on the vector space E_μ.

Corollary 10.19. *Same assumptions as in Proposition 10.15. There exists $C \geq 0$ such that, for any n in $\mathbb{N}$, for any g in the support of μ^{*n} and for any x in the support S_ν of ν, one has*

$$d(\sigma(g,x) - n\sigma_\mu, E_\mu) \leq C. \tag{10.34}$$

Proof. This follows from (10.27) and (10.29). $\square$

Remark 10.20. The upper bound (10.34) cannot be extended beyond the support of ν i.e. to any x in X. For example, there exists a cocycle $\sigma : G \times X \to \mathbb{R}$ which satisfies the assumptions of Proposition 10.15 and such that $\sigma = 0$ on $\Gamma_\mu \times S_\nu$ but σ is unbounded on $\Gamma_\mu \times X$. Such an example is obtained by applying the recentering trick 2.9 to the Iwasawa cocycle for the compactly supported probability μ on $G = \mathrm{SL}(2, \mathbb{Q}_p)$ described in Example 12.21 (see Remark 12.22).

11. Limit Laws for cocycles

In this chapter we prove three limit laws (CLT, LIL and LDP) for cocycles over contracting actions that have suitable moments. The starting point of the proof is a formula relating the Fourier transform of the law of the cocycle at time n with the n^{th}-power of the complex transfer operator P_θ (Formula (0.24) or (11.4)). The proof relies then on the spectral properties of P_θ proven in Chapter 10.

We will apply these limit laws to the Iwasawa cocycle in Chapter 12.

11.1. Statement of the limit laws.

We now state the three limit laws that we will prove in this chapter.

We keep the notations of the preceding chapter. We set N_μ for the *Gaussian law* on E whose covariance 2-tensor is Φ_μ. This law is supported by E_μ. It can also be described by the formula

$$N_\mu := (2\pi)^{-\frac{e_\mu}{2}} e^{-\frac{1}{2}\Phi_\mu^*(v)}\, \mathrm{d}v \tag{11.1}$$

where $e_\mu = \dim E_\mu$, Φ_μ^* is the positive quadratic form on E_μ that is dual to Φ_μ and $\mathrm{d}v$ is the Lebesgue measure on E_μ that gives mass 1 to the *unit cubes* of Φ_μ^*, i.e. the parallelepipeds of E_μ whose sides form an orthonormal basis of Φ_μ.

For every sequence $(v_n)_{n\geq 1}$ in E, we denote by $C(v_n)$ its set of cluster points, that is, $C(v_n) := \{v \in E \mid \exists n_k \to \infty \quad \lim_{k\to\infty} v_{n_k} = v\}$.

THEOREM 11.1. *Let G be a second countable locally compact semigroup, $s : G \to F$ be a continuous morphism onto a finite group F, and E be a finite dimensional real vector space. Let μ be a Borel probability measure on G which is aperiodic in F. Let X be a compact metric G-space which is fibered over F and μ-contracting over F and ν the unique μ-stationary Borel probability measure on X.*

Let $\sigma : G \times X \to E$ be a continuous cocycle whose sup-norm has a finite exponential moment (10.14) *and whose Lipschitz constant has a finite moment* (10.15). *Let $\sigma_\mu \in E$ be the average $\sigma_\mu := \int_{G\times X} \sigma\, \mathrm{d}\mu \otimes \nu$, Φ_μ be the covariance 2-tensor $\Phi_\mu := \lim_{n\to\infty} \frac{1}{n} \int_{G\times X} (\sigma - \sigma_\mu)^2\, \mathrm{d}\mu^{*n} \otimes \mathrm{d}\nu$, E_μ the linear span of Φ_μ and N_μ the Gaussian law on E whose covariance 2-tensor is Φ_μ.*

(*i*) **Central limit theorem for σ with target**. *For any bounded continuous function ψ on $X \times E$, uniformly for x in X,*

$$\int_G \psi\left(gx, \tfrac{\sigma(g,x) - n\sigma_\mu}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{X\times E} \psi(y, v)\, \mathrm{d}\nu(y)\, \mathrm{d}N_\mu(v), \tag{11.2}$$

(*ii*) **Law of the iterated logarithm**. *Let $K_\mu := \{v \in E_\mu \mid v^2 \leq \Phi_\mu\}$ be the unit ball of Φ_μ (see* (2.15)*). For any x in X, for β-almost any b in B, the following set of cluster point is equal to K_μ*

$$C\left(\frac{\sigma(b_n \cdots b_1, x) - n\sigma_\mu}{\sqrt{2n \log\log n}}\right) = K_\mu. \tag{11.3}$$

(*iii*) **Large deviations**. *For any x in X and $t_0 > 0$, one has*

$$\limsup_{n\to\infty} \sup_{x\in X} \mu^{*n}(\{g \in G \mid \|\sigma(g, x) - n\sigma_\mu\| \geq nt_0\})^{\frac{1}{n}} < 1.$$

REMARK 11.2. The existence of the limit covariance 2-tensor Φ_μ follows from Theorem 2.13 and Proposition 10.15. This limit Φ_μ can be computed with Formula (2.16) where σ_0 is the unique cocycle (10.27) with constant drift which is equivalent to σ.

Remark 11.3. We only made the assumption that μ is aperiodic in F to get a simpler formulation of the Central Limit Theorem. The whole Theorem 11.1 can easily be extended to probability measures μ that are not assumed to be aperiodic in F. Indeed, one can replace F by the subgroup spanned by the image of μ and use the fact that the random walk moves in a deterministic and cyclic way in the quotient cyclic group F/F_μ. Note that the statement of the law of the iterated logarithm and the large deviations principle would remain unchanged for μ non-aperiodic.

In Chapter 12 we will apply this abstract theorem to Iwasawa cocycles of reductive groups. We will then need the following

Corollary 11.4. *Same assumptions as in Theorem 11.1. We assume moreover that E is equipped with a linear action of the finite group F and that X is equipped with a continuous right action of F which commutes with the action of G* [1] *and that, for all f in F, the cocycles $(g,x) \mapsto \sigma(g,xf)$ and $(g,x) \mapsto f^{-1}\sigma(g,x)$ are cohomologous. Then*
a) The average $\sigma_\mu \in E$ is F-invariant.
b) The covariance 2-tensor Φ_μ on E is F-invariant.
c) The vector subspace $E_\mu \subset E$ is stable by F.

Proof of Corollary 11.4. This follows from Lemmas 2.10 and 2.17. $\square$

11.2. The Central Limit Theorem.

We prove in this section the Central Limit Theorem. As in the case of the sum of independent real random variables, the proof relies on the convergence of the corresponding characteristic functions thanks to the continuity method.

Let ν be a finite Borel measure on E. For θ in E^*, we set

$$\widehat{\nu}(\theta) = \int_E e^{i\theta(x)}\, \mathrm{d}\nu(x)$$

and we call $\widehat{\nu}$ the *characteristic function* of ν. In particular, for the Gaussian law N_μ, we have

$$\widehat{N_\mu}(\theta) = \exp(-\frac{1}{2}\Phi_\mu(\theta)).$$

[1] This amounts to saying that the G-action on X is isomorphic to the diagonal action on a product $F \times X_c$ of $F = G/H$ with some other G-space X_c.

The following classical lemma tells us that the weak convergence can be detected thanks to the pointwise convergence of the characteristic functions.

LEMMA 11.5 (Lévy continuity method). *Let $E = \mathbb{R}^r$. Let ν_n and ν_∞ be finite Borel measures on E such that the characteristic functions satisfy $\widehat{\nu}_n(\theta) \xrightarrow[n\to\infty]{} \widehat{\nu}_\infty(\theta)$ for all $\theta \in E^*$. Then one has $\nu_n(\psi) \xrightarrow[n\to\infty]{} \nu_\infty(\psi)$ for any bounded continuous function ψ on E.*

PROOF. Equip once for all E and E^* with coherent Lebesgue measures (that is, if the unit cube of a basis of E has volume 1, so has the unit cube of the dual basis). If ψ is a Schwartz function on E and θ is in E^*, set

$$\widehat{\psi}(\theta) = \int_E \psi(x)e^{-i\theta(x)}\,\mathrm{d}x,$$

so that, by the Fourier inversion formula, we have, for any x in E,

$$\psi(x) = (2\pi)^{-r}\int_{E^*} \widehat{\psi}(\theta)e^{i\theta(x)}\,\mathrm{d}\theta.$$

¿From this formula we get, for any n,

$$\int_E \psi\,\mathrm{d}\nu_n = (2\pi)^{-r}\int_{E^*} \widehat{\psi}(\theta)\widehat{\nu_n}(\theta)\,\mathrm{d}\theta.$$

Since $\sup|\widehat{\nu_n}| = \nu_n(0) \xrightarrow[n\to\infty]{} \nu_\infty(0)$, we can apply Lebesgue dominated convergence theorem and we get

$$\int_E \psi\,\mathrm{d}\nu_n \xrightarrow[n\to\infty]{} \int_E \psi\,\mathrm{d}\nu_\infty.$$

The result follows by classical approximation arguments. □

PROOF OF THE CENTRAL LIMIT ESTIMATE IN THEOREM 11.1. By the recentering trick (2.9), we may assume that $\sigma_\mu = 0$.

We want to understand the limit of the law of the random variables

$$(g_n\cdots g_1 x, \tfrac{\sigma(g_n\cdots g_1,x)}{\sqrt{n}}) \in X\times E$$

By standard approximation arguments, it suffices to prove the convergence of (11.2) for functions ψ of the form $(y,v)\mapsto \varphi(y)\rho(v)$, where φ and ρ are bounded continuous functions on X and E. We may also assume that φ is γ-Hölder continuous and non-negative. For any n in $\mathbb{N}$ and x in X, we want to understand the limit for $n\to\infty$ of the measures $\mu^\varphi_{n,x}$ given, for any bounded continuous function ρ on E, by

$$\int_E \rho\,\mathrm{d}\mu^\varphi_{n,x} = \int_G \varphi(gx)\rho\left(\tfrac{\sigma(g,x)}{\sqrt{n}}\right)\,\mathrm{d}\mu^{*n}(g).$$

Note that, when $\varphi = \mathbf{1}$, the measure $\mu^{\varphi}_{n,x}$ is nothing but the law of the random variable $\frac{\sigma(g_n\cdots g_1,x)}{\sqrt{n}}$.

We will determine the limit of these measures $\mu^{\varphi}_{n,x}$ by computing their charateristic functions. By (10.23), for any θ in E^*, one has the following expression for the charateristic function $\widehat{\mu}^{\varphi}_{n,x}$ of $\mu^{\varphi}_{n,x}$:

$$\widehat{\mu}^{\varphi}_{n,x}(\theta) = \int_G \varphi(gx) e^{i\theta(\sigma(g,x)/\sqrt{n})}\, \mathrm{d}\mu^{*n}(g).$$

This formula can be rewritten as

$$\widehat{\mu}^{\varphi}_{n,x}(\theta) = P^n_{\frac{i\theta}{\sqrt{n}}}\varphi(x). \tag{11.4}$$

By Lévy's continuity Theorem (Lemma 11.5), we have to check that, for any $\theta \in E^*$, the sequence of characteristic functions evaluated at θ converges uniformly in x:

$$\widehat{\mu}^{\varphi}_{n,x}(\theta) \xrightarrow[n\to\infty]{} e^{-\frac{\Phi_\mu(\theta)}{2}} \textstyle\int_X \varphi\, \mathrm{d}\nu. \tag{11.5}$$

Let U be a small neighborhood of 0 in $E^*_{\mathbb{C}}$ as in Lemma 10.17. For every $\theta \in E^*$, for large n, the element $\frac{i\theta}{\sqrt{n}}$ belongs to U. Then, by this lemma, we can decompose the function $\varphi \in \mathcal{H}^\gamma(X)$ as

$$\varphi = N_{\frac{i\theta}{\sqrt{n}}}\varphi + Q_{\frac{i\theta}{\sqrt{n}}}\varphi \tag{11.6}$$

(where, as in the proof of Lemma 10.17, $Q_\theta = P_\theta - N_\theta$).

On the one hand, since μ is aperiodic in F, by Lemma 10.17, for $\theta \in U$, the operator N_θ has rank one and $\lambda_\theta^{-1}P_\theta$ acts trivially on the line $\mathrm{Im}\,(N_\theta)$. Since the function $N_0\varphi = (\int_X \varphi\, \mathrm{d}\nu)\mathbf{1}$ is P_0-invariant, one gets

$$\lambda_\theta^{-n} P_\theta^n N_\theta \varphi \xrightarrow[\theta\to 0]{} (\textstyle\int_X \varphi\, \mathrm{d}\nu)\mathbf{1} \ \text{ in } \mathcal{H}^\gamma(X), \ \text{ uniformly for } n \geq 1.$$

Hence, for every $\theta \in E^*$, one has

$$\lambda^{-n}_{\frac{i\theta}{\sqrt{n}}} P^n_{\frac{i\theta}{\sqrt{n}}} N_{\frac{i\theta}{\sqrt{n}}}\varphi \xrightarrow[n\to\infty]{} (\textstyle\int_X \varphi\, \mathrm{d}\nu)\mathbf{1} \ \text{ in } \mathcal{H}^\gamma(X). \tag{11.7}$$

We notice also that, according to the computation of the first two derivatives of the analytic function $\theta \to \lambda_\theta$ in Lemma 10.18, by the Taylor-Young formula, one has, since $\sigma_\mu = 0$,

$$n \log \lambda_{\frac{i\theta}{\sqrt{n}}} + \tfrac{1}{2}\Phi_\mu(\theta) \xrightarrow[n\to\infty]{} 0,$$

that is,

$$\lambda^n_{\frac{i\theta}{\sqrt{n}}} \xrightarrow[n\to\infty]{} e^{-\frac{\Phi_\mu(\theta)}{2}}. \tag{11.8}$$

On the other hand, by Lemma 10.17,

$$P_\theta^n Q_\theta \varphi \xrightarrow[n\to\infty]{} 0 \text{ in } \mathcal{H}^\gamma(X), \text{ uniformly for } \theta \in U.$$

Hence for every $\theta \in E^*$,

$$P^n_{\frac{i\theta}{\sqrt{n}}} Q_{\frac{i\theta}{\sqrt{n}}} \varphi \xrightarrow[n\to\infty]{} 0 \text{ in } \mathcal{H}^\gamma(X). \tag{11.9}$$

Putting together Equations (11.6), (11.7), (11.8) and (11.9), one gets (11.5) as required. □

11.3. The upper law of the iterated logarithm.

In this section, we prove the upper bound in the law of the iterated logarithm, i.e. the fact that the cluster set is included in K_μ.

We begin by two reductions

(11.10) *We can assume that σ has constant zero drift.*

(11.11) *We can assume that $E = \mathbb{R}$ and $\int_{G\times X} \sigma^2 \,\mathrm{d}\mu \,\mathrm{d}\nu = 1$.*

Proof of (11.10). By the recentering trick (2.9), we can assume that $\sigma_\mu = 0$. We know by Lemma 10.18 that σ is special: we can write σ as a sum $\sigma(g,x) = \sigma_0(g,x) + \dot\varphi_0(x) - \dot\varphi_0(gx)$ where σ_0 has constant zero drift and $\dot\varphi_0$ is a α-Hölder continuous function on X for some $\alpha \in]0,1]$. In order to apply Theorem 11.1.*ii* to σ_0, it remains to check that the sup norm of $\sigma_0 - \sigma$ has a finite exponential moment and that its Lipschitz constant has a finite moment. The control of the sup norm follows from the boundedness of the function $\dot\varphi_0$. To control the Lipschitz constant, we replace the distance d by the distance d^α. Now, we get the required bound from the fact that $\dot\varphi_0$ is α-Hölder continuous and from (10.1). □

Proof of (11.11). First assume the covariance 2-tensor Φ_μ is zero. Since σ has constant zero drift, by Formula (2.16), one has $\sigma = 0$ on $\Gamma_\mu \times S_\nu$, so that Theorem 11.1.*ii* holds for x in S_ν. Now, by Lemma 10.14.b), it holds for any x.

Hence we can assume that the covariance 2-tensor Φ_μ is nonzero. Then we can find a countable set D of elements θ in E^* with $\Phi_\mu(\theta) = 1$ such that the unit ball K_μ of Φ_μ is equal to

$$K_\mu = \{v \in E \mid \theta(v) \le 1 \text{ for all } \theta \in D\}.$$

Still by (2.16), the real-valued cocycles $\sigma_\theta := \theta \circ \sigma$ satisfy

$$\int_{G\times X} \sigma_\theta^2 \,\mathrm{d}\mu \,\mathrm{d}\nu = 1.$$

Thus, if Theorem 11.1.*ii* holds for the cocycles σ_θ, for β-almost all b, for any θ in D, one has

$$C\left(\frac{\sigma_\theta(b_n\cdots b_1,x)}{\sqrt{2n\log\log n}}\right)\subset[-1,1].$$

Hence one has $C(v_n)\subset K_\mu$. $\square$

We write S_n for the random variable $(b,x)\mapsto\sigma(b_n\cdots b_1,x)$, omitting the dependance on (b,x) and we use the notation $\mathbb{P}_x$ and $\mathbb{E}_x$ as in section 2.2. This will allow us to lighten our notations, for instance for $x\in X$ and $t>0$, we will have

$$\begin{aligned}\mathbb{P}_x(|S_n|<t)&=\beta(\{b\in B\mid|\sigma(b_n\cdots b_1,x)|<t\})\\&=\mu^{*n}(\{g\in G\mid|\sigma(g,x)|<t\}).\end{aligned}$$

Let $a_n>0$ be a non-decreasing sequence such that

$$\lim_{n\to\infty}\tfrac{a_n^2}{n}=\lim_{n\to\infty}\tfrac{n}{a_n}=\infty. \tag{11.12}$$

For instance, $a_n=\sqrt{2n\log\log n}$ for $n\geq 3$. We set $S_n^*=\sup\limits_{1\leq k\leq n}S_k$.

We will prove successively the four following lemmas in which we assume both (11.10) and (11.11) to hold.

Lemma 11.6. *For all $\varepsilon>0$, there exists n_0, such that, for $n\geq n_0$ and x in X,*

$$\min_{k\leq n}\mathbb{P}_x(|S_k|\leq\varepsilon\,a_n)\geq\tfrac12.$$

Lemma 11.7. *For all $\varepsilon>0$, $c>0$, there exists n_0, such that, for $n\geq n_0$ and x in X,*

$$\mathbb{P}_x(S_n^*\geq(c+\varepsilon)\,a_n)\leq 2\,\mathbb{P}_x(S_n\geq c\,a_n).$$

Lemma 11.8. *For all $c>0$ and $c'>1$, one has*

$$\sup_{x\in X}\mathbb{P}_x(S_n\geq ca_n)=O(e^{-c^2a_n^2/(2c'n)}).$$

Lemma 11.9. *For all x in X, one has* $\mathbb{P}_x(\limsup\limits_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}\leq 1)=1$.

We will often use the cocycle relation for these random variables S_n on the forward dynamical system under the form

$$S_{m+n}=S_m\circ(T^X)^n+S_n.$$

Proof of Lemma 11.6. According to the Central Limit Theorem 11.1.i, since $\frac{a_k}{\sqrt{k}} \xrightarrow[k\to\infty]{} \infty$, there exists $n_1 \geq 1$ such that, for every $n_1 \leq k \leq n$, for all x in X, one has

$$\mathbb{P}_x(|S_k| \leq \varepsilon\, a_n) \geq \mathbb{P}_x(\tfrac{|S_k|}{\sqrt{k}} \leq \varepsilon\, \tfrac{a_k}{\sqrt{k}}) \geq \tfrac{1}{2}.$$

Now, we choose a compact subset K of G such that, for any $0 \leq k < n_1$, one has $\mu^{*k}(K) \geq \frac{1}{2}$. Since $a_n \xrightarrow[n\to\infty]{} \infty$ and $\sup_{K\times X} |\sigma| < \infty$, one can find $n_0 \geq 1$ such that, for all $n \geq n_0$, for all x in X, one has

$$\mathbb{P}_x(|S_k| \leq \varepsilon\, a_n) \geq \tfrac{1}{2} \quad \text{when } k < n_1.$$

This proves our claim. □

Proof of Lemma 11.7. We want to bound $\mathbb{P}_x(A_n)$ where $A_n \subset B \times X$ is the union $A_n = \cup_{1\leq k\leq n} A_{n,k}$ with

$$A_{n,k} := \{S_k \geq (c+\varepsilon)\, a_n \;\text{ and }\; S_j < (c+\varepsilon)\, a_n \;\text{ for }\; 1 \leq j < k\}.$$

We introduce also the sets

$$B_{n,k} = \{|S_n - S_k| \leq \varepsilon\, a_n\} \;\text{ and }\; C_n = \{S_n \geq c\, a_n\}.$$

These sets C_n contain the disjoint union

$$C_n \supset \cup_{k=1}^{n} A_{n,k} \cap B_{n,k}.$$

According to the Markov property and to the cocycle property, one has

$$\mathbb{P}_x(B_{n,k} \mid A_{n,k}) \geq \inf_{y\in X} \mathbb{P}_y(|S_{n-k}| \leq \varepsilon\, a_n).$$

Hence, by Lemma 11.6, one can find $n_0 \geq 1$ such that, for all $n \geq n_0$, $k \leq n$ and x in X, one has

$$\mathbb{P}_x(B_{n,k} \mid A_{n,k}) \geq \tfrac{1}{2}.$$

Thus one has

$$\mathbb{P}_x(A_n) \leq \textstyle\sum_{k=1}^{n} \mathbb{P}_x(A_{n,k}) \leq 2 \sum_{k=1}^{n} \mathbb{P}_x(A_{n,k} \cap B_{n,k}) \leq 2\, \mathbb{P}(C_n),$$

as required. □

Proof of Lemma 11.8. By Theorem 2.13 and (11.11), one can find $n_1 \geq 1$ such that

$$\mathbb{E}_y(S_{n_1}^2) = \textstyle\int_G \sigma(g,y)^2\, \mathrm{d}\mu^{*n_1}(g) \leq n_1 \tfrac{1}{2}(c'+1) \quad \text{for all } y \text{ in } X.$$

Now, by Lebesgue convergence theorem, since σ depends continuously on x and since X is compact, one can find $\alpha_0 > 0$ such that,

$$\textstyle\int_G \sigma(g,y)^2 e^{\alpha_0 \sigma_{\sup}(g)}\, \mathrm{d}\mu^{*n_1}(g) \leq n_1 c' \quad \text{for all } y \text{ in } X.$$

Using the upperbound $e^t \leq 1 + t + \frac{t^2}{2}\, e^{|t|}$, for all t in $\mathbb{R}$, and using the zero drift condition (11.10), one computes, for $0 < t < \alpha_0$ and y in X,

$$\begin{aligned} \mathbb{E}_y(e^{tS_{n_1}}) &\leq 1 + t\,\mathbb{E}_y(S_{n_1}) + \tfrac{t^2}{2}\,\mathbb{E}_y(S_{n_1}^2\, e^{t|S_{n_1}|}) \\ &\leq 1 + \tfrac{n_1 c' t^2}{2} \leq e^{n_1 c' t^2/2}. \end{aligned}$$

We will denote by I_{α_0} the integral $I_{\alpha_0} = \int_G e^{\alpha_0 \sigma_{\sup}(g)}\, \mathrm{d}\mu(g)$. Writing $n = q_1 n_1 + r_1$ with $r_1 < n_1$, using Chebyshev inequality, the Markov property and the cocycle property, one gets for $t_n < \alpha_0$,

$$\begin{aligned} \mathbb{P}_x(S_n > c\, a_n) &\leq e^{-t_n c a_n}\, \mathbb{E}_x(e^{t_n S_n}) \\ &\leq e^{-t_n c a_n} \sup_{y \in X} \mathbb{E}_y(e^{t_n S_{n_1}})^{q_1}\, I_{\alpha_0}^{r_1} \\ &\leq e^{-t_n c a_n + n c' t_n^2/2}\, I_{\alpha_0}^{r_1}. \end{aligned}$$

Since $\frac{a_n}{n} \to 0$, for n large one has $t_n := \frac{c\, a_n}{c' n} < \alpha_0$, so that

$$\mathbb{P}_x(S_n > c\, a_n) \leq e^{-c^2 a_n^2/(2c'n)}\, I_{\alpha_0}^{n_1},$$

as required. □

Proof of Lemma 11.9. We now set $a_n = \sqrt{2n \log\log n}$. We fix $1 < \alpha < c$ and set n_k to be the integral part $n_k := [\alpha^{2k}]$. One has the inclusion of subsets of $B \times X$, in which *i.o.* stands for "infinitely often",

$$\begin{aligned} \{S_n \geq c^3\, a_n \;\; i.o.\} &\subset \{S_{n_k}^* \geq c^3\, a_{n_{k-1}} \;\; i.o.\} \\ &\subset \{S_{n_k}^* \geq c^2\, a_{n_k} \;\; i.o.\}. \end{aligned}$$

We want to prove that this set has $\mathbb{P}_x$-measure zero. By Borel-Cantelli Lemma, it is enough to check that the series $\sum p_k$ is convergent, where

$$p_k := \mathbb{P}_x(S_{n_k}^* \geq c^2\, a_{n_k}).$$

By Lemmas 11.7 and 11.8 with $c' = c$, for k large enough, one has the upperbound

$$p_k \leq 2\,\mathbb{P}_x(S_{n_k} \geq c\, a_{n_k}) = O(k^{-c}).$$

Hence this series $\sum p_k$ is convergent. □

11.4. The lower law of the iterated logarithm.

In this section, we prove the lower bound in the law of the iterated logarithm, i.e. the fact that the cluster set contains K_μ.

We keep the notations of the previous paragraph. Because of the upperbound, we can replace the cocycle σ by any projection of it on E_μ. Hence,

(11.13) *we can assume that Φ_μ is non-degenerate.*

We still denote by Φ_μ^* the quadratic form on E^* that is dual to Φ_μ. We will prove successively the following two lemmas for a sequence a_n which satisfies (11.12).

LEMMA 11.10. *For all v in E and $R > 0$, one has*

$$\liminf_{n\to\infty} \tfrac{2n}{a_n^2} \inf_{x\in X} \log \mathbb{P}_x(|S_n/a_n - v| \le R) \;\ge\; -\Phi_\mu^*(v).$$

LEMMA 11.11. *For all v in E with $\Phi_\mu^*(v) < 1$, for all $R > 0$ and x in X, one has*

$$\mathbb{P}_x(|\tfrac{S_n}{\sqrt{2n\log\log n}} - v| \le R \;\; i.o.) = 1.$$

Lemma 11.10 is a kind of converse to Lemma 11.8.

PROOF OF LEMMA 11.10. We set $r = R/2$, $V_r = B(v,r)$ and $B_r = B(0,r)$. Fix $t > 0$ and set

$$p_n = \left[\tfrac{n^2t^2}{a_n^2}\right] \;\text{ and }\; q_n = \left[\tfrac{a_n^2}{nt^2}\right],$$

so that p_n goes to ∞ and

$$p_nq_n \le n \;\text{ and }\; n - p_nq_n = O(\tfrac{n^2}{a_n^2} + \tfrac{a_n^2}{n}).$$

Decomposing the interval $[1,n]$ into q_n intervals of length p_n plus a remaining interval of length at most p_n, using the Markov property and the cocycle property, one gets the lower bound

$$\inf_{x\in X} \mathbb{P}_x(S_n \in a_n\, V_R) \ge \lambda_n^{q_n}\lambda_n' \quad \text{where}$$

$$\lambda_n = \inf_{x\in X} \mathbb{P}_x(S_{p_n} \in \tfrac{a_n}{q_n}\, V_r) \;\text{ and }\; \lambda_n' = \inf_{x\in X} \mathbb{P}_x(S_{n-p_nq_n} \in a_n\, B_r)\,.$$

According to Theorem 2.13, the following constant M_0 is finite :

$$M_0 = \sup_{n\ge 1}\sup_{x\in X} \tfrac{1}{n}\mathbb{E}_x(S_n^2) < \infty\,.$$

Hence, since, by Chebyshev inequality, one has

$$\mathbb{P}_x(S_{n-p_nq_n} \notin a_n\, B_r) \le a_n^{-2}r^{-2}\mathbb{E}_x(S_{n-p_nq_n}^2),$$

one gets

$$1 - \lambda_n' \le a_n^{-2}r^{-2}(n - p_nq_n)M_0 = O(\tfrac{n^2}{a_n^4} + \tfrac{1}{n}) \xrightarrow[n\to\infty]{} 0.$$

We want a lower bound for the left hand side

$$L := \liminf_{n\to\infty} \tfrac{2n}{a_n^2} \inf_{x\in X} \log \mathbb{P}_x(S_n/a_n \in V_R).$$

We have already proved that

$$L \ge \liminf_{n\to\infty} \tfrac{2nq_n}{a_n^2} \inf_{x\in X} \log \mathbb{P}_x(S_{p_n} \in \tfrac{a_n}{q_n}V_r).$$

Using the Central Limit Theorem 11.1.i, the fact (11.12) that p_n goes to ∞ and the equivalence $\sqrt{p_n} \sim \frac{a_n}{q_n t}$, one gets

$$L \geq \tfrac{2}{t^2} \log N_\mu(t\, V_r),$$

where N_μ is the limit normal law. According to Jensen inequality, one has

$$N_\mu(t\, V_r) \geq e^{-\frac{t^2}{2}\Phi^*_\mu(v)} N_\mu(t\, B_r).$$

Hence one has, for all $t > 0$,

$$L \geq -\Phi^*_\mu(v) + \tfrac{2}{t^2} \log N_\mu(t\, B_r).$$

Since $\lim\limits_{t\to\infty} N_\mu(t\, B_r) = 1$, one gets $L \geq -\Phi^*_\mu(v)$. $\square$

Proof of Lemma 11.11. We set $a_n = \sqrt{2n \log\log n}$. We will prove that the event $S_n \in a_n\, V_R$ occurs infinitely often along the sequence $n = n_k = k^k$. Because of the upperbound and the choice of this sequence, one has

$$\limsup_{k\to\infty} S_{n_{k-1}}/a_{n_k} \;\leq\; \limsup_{k\to\infty} a_{n_{k-1}}/a_{n_k} \;\limsup_{n\to\infty} S_n/a_n \;=\; 0.$$

Hence we only have to check that, $\mathbb{P}_x$-almost surely, the event

$$A_k := \{S_{n_k} - S_{n_{k-1}} \in a_{n_k}\, V_R\}$$

occurs infinitely often. According to Borel-Cantelli Lemma it is enough to check that, for all $k_0 \geq 1$, the following series diverges:

$$\textstyle\sum_{k\geq k_0} \mathbb{P}_x(A_k \mid \cap_{j=k_0}^{k-1} A_j^c) = \infty.$$

By the Markov property and the cocycle property, one has the lower bound

$$\mathbb{P}_x(A_k \mid \cap_{j=k_0}^{k-1} A_j^c) \geq p_k \quad \text{where}$$

$$p_k = \inf_{y\in X} \mathbb{P}_y(S_{n_k - n_{k-1}} \in a_{n_k}\, V_R).$$

We choose α with $\Phi_\mu(v) < \alpha < 1$. By Lemma 11.10, for k large, one has

$$p_k \geq e^{-\alpha \log\log(n_k - n_{k-1})} = \log(n_k - n_{k-1})^{-\alpha} \sim (k \log k)^{-\alpha},$$

and the series $\sum p_k$ diverges as required. $\square$

This proof of the law of the iterated logarithm gives also the following

PROPOSITION 11.12. *Same assumptions as in Theorem 11.1. Let a_n be a non-decreasing sequence such that $\lim_{n\to\infty} \frac{n}{a_n} = \lim_{n\to\infty} \frac{a_n^2}{n} = \infty$. For every open convex subset $C \subset E$ with $C \cap E_\mu \neq \emptyset$, one has the convergence*

$$\frac{2n}{a_n^2} \log \mathbb{P}_x\left(\frac{S_n}{a_n} \in C\right) \xrightarrow[n\to\infty]{} -\inf_{v\in C\cap E_\mu} \Phi_\mu^*(v), \tag{11.14}$$

uniformly for x in X. For instance, one has the convergence

$$\frac{1}{\log\log n} \log \mathbb{P}_x\left(\frac{S_n}{\sqrt{2n\log\log n}} \in C\right) \xrightarrow[n\to\infty]{} -\inf_{v\in C\cap E_\mu} \Phi_\mu^*(v), \tag{11.15}$$

uniformly for x in X.

PROOF. This follows from Lemmas (11.8) and (11.10). □

11.5. Large deviations estimates.

This last section is devoted to the proof of the large deviations principle for cocycles over a contracting action.

PROOF OF THEOREM 11.1.(*iii*). As for random walks on $\mathbb{R}$, the proof relies on the Laplace-Fourier transform of the law and on the Chebyshev inequality. The new ingredient is again Formula (0.24) expressing this Laplace-Fourier transform thanks to the transfer operator.

We may assume $\sigma_\mu = 0$. Fix $t_0 > 0$ and introduce the following sets for $x \in X$, $n \in \mathbb{N}$, $t_0 > 0$ and $\theta \in E^*$,

$$H_{x,n}^{t_0} := \{g \in G \mid \|\sigma(g,x)\| \geq nt_0\},$$

$$K_{x,n}^{\theta} := \{g \in G \mid \theta(\sigma(g,x)) \geq n\}.$$

We want to prove

$$\limsup_{n\to\infty} \sup_{x\in X} \tfrac{1}{n} \log \mu^{*n}(H_{x,n}^{t_0}) < 0.$$

Notice that there exists a finite set $\Theta_{t_0} \subset E^*$ such that the set $H_{x,n}^{t_0}$ is included in the union of the sets $K_{x,n}^{\theta}$ for θ in Θ_{t_0}. Hence it is enough to check, for every θ in E^*,

$$\limsup_{n\to\infty} \sup_{x\in X} \tfrac{1}{n} \log \mu^{*n}(K_{x,n}^{\theta}) < 0. \tag{11.16}$$

Fix θ in E^* and choose $t > 0$ small enough. Using Chebyshev inequality, one has the bound

$$\mu^{*n}(K_{x,n}^{\theta}) \ \leq\ e^{-tn} \int_G e^{t\theta(\sigma(g,x))}\, \mathrm{d}\mu^{*n}(g).$$

This inequality can be rewritten as

$$\mu^{*n}(K_{x,n}^{\theta}) \ \leq\ e^{-tn} P_{t\theta}^{n}\mathbf{1}(x).$$

When t is small enough, the element $t\theta$ belongs to U and, by Lemma 10.17, one has

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \|P_{t\theta}^n \mathbf{1}\|_\infty \leq \log \lambda_{t\theta}.$$

Hence one has

$$\limsup_{n\to\infty} \sup_{x\in X} \tfrac{1}{n} \log \mu^{*n}(K_{x,n}^{\theta}) \leq \log \lambda_{t\theta} - t.$$

Since $\sigma_\mu = 0$, according to Lemma 10.18 the derivative of the map $t \mapsto \log \lambda_{t\theta}$ at $t = 0$ is zero. Hence, when t is small enough, the right-hand side is negative. This proves the bound (11.16) and ends the proof. □

12. Limit laws for products of random matrices

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. In this chapter, we continue the study of random walks on G using freely the notations of Chapter 9. We will apply Theorem 11.1 in order to prove limit laws both for the Iwasawa cocycle σ and for the Cartan projection κ.

12.1. Lipschitz constant of the cocycle.

We first check that the partial Iwasawa cocycle σ_{Θ_μ} on the partial flag variety $\mathcal{P}_{\Theta_\mu}$ satisfies the finite moment conditions needed in Theorem 11.1 .

To this aim, we need to introduce a distance on the partial flag varieties $\mathcal{P}_\Theta$, $\Theta \subset \Pi$. Let us first deal with distances on projective spaces. Let $\mathbb{K}$ be a local field and V be a finite dimensional $\mathbb{K}$-vector space.

If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, fix a Euclidean norm $\|.\|$ on V. Then, there exists a unique Euclidean norm on $\wedge^2 V$ such that, for any orthogonal decomposition $V = V_1 \oplus V_2$, the decomposition $\wedge^2 V = \wedge^2 V_1 \oplus V_1 \wedge V_2 \oplus \wedge^2 V_2$ is orthogonal and that, for any v_1 in V_1 and v_2 in V_2, one has $\|v_1 \wedge v_2\| = \|v_1\|\,\|v_2\|$.

If $\mathbb{K}$ is non-archimedean, fix a ultrametric norm $\|.\|$ on V and say a decomposition $V = \bigoplus_{1\leq i\leq k} V_i$ is *good* if, for any $v = \sum_{1\leq i\leq k} v_i$ in V, one has $\|v\| = \max_{1\leq i\leq k} \|v_i\|$. Then, there exists a unique ultrametric norm on $\wedge^2 V$ such that, for any good decomposition $V = V_1 \oplus V_2$, the decomposition $\wedge^2 V = \wedge^2 V_1 \oplus V_1 \wedge V_2 \oplus \wedge^2 V_2$ is good and that, for any v_1 in V_1 and v_2 in V_2, one has $\|v_1 \wedge v_2\| = \|v_1\|\,\|v_2\|$.

In all cases, set, for any $x = \mathbb{K}v$, $x' = \mathbb{K}v'$ in $\mathbb{P}(V)$,

$$d(x, x') = \tfrac{\|v\wedge v'\|}{\|v\|\|v'\|}. \tag{12.1}$$

The function d is a distance which induces the usual compact topology on $\mathbb{P}(V)$.

For any g in $\mathrm{GL}(V)$ and x, x' in $\mathbb{P}(V)$, one has

$$d(gx, gx') \leq \left\|\wedge^2 g\right\| \left\|g^{-1}\right\|^2 d(x,x') \leq \|g\|^2 \left\|g^{-1}\right\|^2 d(x,x'). \tag{12.2}$$

Let $\Theta \subset \Pi$ be an F-invariant subset. We recall, from Sections 7.4 and 7.6, that the G-equivariant map

$$\mathcal{P}_\Theta \to \prod_{\alpha\in\Pi} \bigcup_{f\in F} \mathbb{P}\left(V_{\alpha,f}\right), \eta \mapsto V_{\alpha,\eta}$$

is a closed immersion. For any η, η' in $\mathcal{P}_\Theta$, set

$$d(\eta,\eta') = \begin{cases} \max_{\alpha\in\Theta} d(V_{\alpha,\eta}, V_{\alpha,\eta'}) & \text{if } f_\eta = f_{\eta'}. \\ 1 & \text{if } f_\eta \neq f_{\eta'}. \end{cases} \tag{12.3}$$

Note that, by Lemma 7.18, Corollary 7.20 and (12.2), there exist constants $C_1, C_2 > 0$ such that, for any g in G and η, η' in $\mathcal{P}_\Theta$, one has

$$d(g\eta, g\eta') \leq C_1 e^{C_2 \|\kappa(g)\|} d(\eta,\eta'). \tag{12.4}$$

This inequality will be useful in Section 12.2 for checking the condition (10.1).

The following lemma gives an estimation for the Lipschitz constant of the Iwasawa cocycle.

LEMMA 12.1. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. Let Θ be an F-invariant subset of Π. There exist $p, q > 0$ such that, for any g in G, η, η' in $\mathcal{P}_\Theta$ with $f_\eta = f_{\eta'}$, one has*

$$\|\sigma_\Theta(g,\eta) - \sigma_\Theta(g,\eta')\| \leq p\, e^{q\,\|\kappa(g)\|} d(\eta,\eta'). \tag{12.5}$$

To prove this lemma, we will proceed to an analysis of the norm cocycle associated to a given representation.

LEMMA 12.2. *Let $\mathbb{K}$ be a local field and V be a normed finite dimensional $\mathbb{K}$-vector space. There exists a constant $C > 0$ such that, for any g in $\mathrm{GL}(V)$ and v, v' in $V \smallsetminus \{0\}$, one has*

$$\left|\log \tfrac{\|gv\|}{\|v\|} - \log \tfrac{\|gv'\|}{\|v'\|}\right| \leq C\, \|g\|\, \|g^{-1}\|\, d(\mathbb{K}v, \mathbb{K}v'). \tag{12.6}$$

In this Lemma 12.2, we do not assume the norm to be Euclidean or ultrametric.

REMARK 12.3. Note that one cannot bound the left-hand side of (12.6), uniformly in v and v', by a linear expression in $\log(N(g))\, d(\mathbb{K}v, \mathbb{K}v')$. For instance for $V = \mathbb{R}^2$, $v = (1, \varepsilon)$, $v' = (1,0)$ and $g = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ with $\varepsilon, s, t > 0$, the left-hand side of (12.6) is $|\frac{1}{2} \log \frac{1+(s^{-1}t\varepsilon)^2}{1+\varepsilon^2}|$ which is not bounded uniformly in $\varepsilon \in [0,1]$ by a multiple of $(|\log s| + |\log t|)\,\varepsilon$.

Proof. We first note that there exists a constant $c \geq 1$ such that, for any x, x' in $\mathbb{P}(V)$,

$$c^{-1}d(x,x') \leq \min_{v,v'} \|v'-v\| \leq c\, d(x,x'), \tag{12.7}$$

where the minimum is taken over all the nonzero vectors v in x and v' in x' with $\|v\| \geq 1$ and $\|v'\| \geq 1$. Hence we can assume that the vectors v and v' in (12.6) satisfy

$$\|v\| \geq 1 \ , \ \|v'\| \geq 1 \ \text{ and } \ \|v'-v\| \leq c\, d(\mathbb{K}v, \mathbb{K}v'). \tag{12.8}$$

Since Inequality (12.6) is symmetric in v and v', we only have to prove the upper bound

$$\log \tfrac{\|gv'\|}{\|gv\|} + \log \tfrac{\|v\|}{\|v'\|} \leq C\, \|g\|\, \|g^{-1}\|\, \|v'-v\|\, . \tag{12.9}$$

We set L for the left-hand side of (12.9), $w := v'-v$ and compute

$$L \leq \log(1+\tfrac{\|gw\|}{\|gv\|}) + \log(1+\tfrac{\|w\|}{\|v'\|}) \leq \tfrac{\|gw\|}{\|gv\|} + \tfrac{\|w\|}{\|v'\|} \leq 2\, \|g\|\, \|g^{-1}\|\, \|w\|.$$

This proves the wanted inequality (12.9). □

Proof of Lemma 12.1. This follows from Lemmas 7.15, 7.18 and 12.2. □

This implies that the moment assumptions of Theorem 11.1 are satisfied. Recall that if μ is a Zariski dense probability measure on G, we defined Θ_μ as the set of α in Π such that the set $\alpha^\omega(\kappa(\Gamma_\mu)) \subset \mathbb{R}_+$ is unbounded.

Corollary 12.4. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F = G/G_c$ and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Then, the corresponding partial Iwasawa cocycle $\sigma_{\Theta_\mu} : G \times \mathcal{P}_{\Theta_\mu} \to \mathfrak{a}_{\Theta_\mu}$ satisfies the finite moment conditions* (10.14) *and* (10.15).

Proof. Condition (10.14) follows from the bound (7.16) and from the finite exponential moment assumption (9.3). Condition (10.15) follows from the bound (12.5) with $\Theta = \Theta_\mu$ and from the same finite exponential moment assumption (9.3). □

12.2. Contraction speed on the flag variety.

In this section, we check the μ-contraction property on the partial flag variety $\mathcal{P}_{\Theta_\mu}$ also needed in Theorem 11.1.

Lemma 12.5. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F = G/G_c$ and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Then, there exists $\gamma_0 > 0$ such that the action of G on $\mathcal{P}_{\Theta_\mu}$ is (μ, γ_0)-contracting over F.*

The proof uses the following elementary

LEMMA 12.6. *Let $(X, \mathcal{X}, \chi)$ be a probability space, Φ be a set of real measurable functions on $(X, \mathcal{X})$, and $t_0 > 0$ such that*

$$\int_X \sup_{\varphi\in\Phi} e^{t_0|\varphi|}\,\mathrm{d}\chi < \infty \quad \textit{and} \quad \sup_{\varphi\in\Phi} \int_X \varphi\,\mathrm{d}\chi < 0.$$

Then there exists $0 < t \le t_0$ with

$$\sup_{\varphi\in\Phi} \int_X e^{t\varphi}\,\mathrm{d}\chi < 1.$$

PROOF. The key ingredient in this proof is the Law of Large Numbers and the regularity of the Lyapunov vector (Theorem 9.9). We set $\psi = \sup_{\varphi\in\Phi}|\varphi|$ and $\varepsilon = -\sup_{\varphi\in\Phi}\int_X \varphi\,\mathrm{d}\chi > 0$. For any $a \in \mathbb{R}$, one has $e^a \le 1 + a + a^2 e^{|a|}$, thus, for any $t > 0$, one has

$$\int_X e^{t\varphi}\,\mathrm{d}\chi \le 1 + t\int_X \varphi\,\mathrm{d}\chi + t^2\int_X \psi^2 e^{t\psi}\,\mathrm{d}\chi.$$

The result follows, by taking $t > 0$ such that $t\int_X \psi^2 e^{t\psi}\,\mathrm{d}\chi < \varepsilon$. $\square$

PROOF OF LEMMA 12.5. First note that the moment assumption and Inequality (12.4) imply that (10.1) holds for small enough γ_0. Let us check that (10.2) is verified for some $n \ge 1$. Recall, for any $\eta \ne \eta'$ in $\mathcal{P}_{\Theta_\mu}$ with $f_\eta = f_{\eta'}$, the distance $d(\eta, \eta')$ is given by (12.3).

For g in G and $\alpha \in \Theta_\mu$, by Lemma 7.18 and Formula (12.1), we have

$$d(gV_{\alpha,\eta}, gV_{\alpha,\eta'}) \le e^{a_{\alpha,\eta,\eta'}(g)}\, d(V_{\alpha,\eta}, V_{\alpha,\eta'}) \ \text{ where}$$
$$a_{\alpha,\eta,\eta'}(g) := (2\chi_\alpha^\omega - \alpha^\omega)(\kappa(g\,\tau_{f_\eta})) - \chi_\alpha^\omega(\sigma(g,\eta) + \sigma(g,\eta')).$$

Thus,

$$\log \tfrac{d(g\eta,g\eta')}{d(\eta,\eta')} \le a_{\eta,\eta'}(g) \ \text{ where } \ a_{\eta,\eta'}(g) := \max_{\alpha\in\Theta_\mu} a_{\alpha,\eta,\eta'}(g).$$

We need to prove that there exist $\gamma_0 > 0$ and $n \ge 1$ such that one has

$$\sup_{f_\eta = f_{\eta'}} \int_G \tfrac{d(g\eta,g\eta')^{\gamma_0}}{d(\eta,\eta')^{\gamma_0}}\,\mathrm{d}\mu^{*n}(g) < 1,$$

where the supremum is taken is over the pairs η, η' in $\mathcal{P}_{\Theta_\mu}$ with $f_\eta = f_{\eta'}$ and $\eta \ne \eta'$. According to Lemma 12.6, it suffices to check that

$$\sup_{f_\eta = f_{\eta'}} \int_G \log \tfrac{d(g\eta,g\eta')}{d(\eta,\eta')}\,\mathrm{d}\mu^{*n}(g) < 0 \ \text{ for some integer } n. \tag{12.10}$$

We will use once again the one-sided Bernoulli space (B, β) with alphabet (G, μ), and denote by $b = (b_1, \dots, b_n, \dots)$ its elements. According to Theorem 9.9, one has

$$\tfrac{1}{n}\,\kappa(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \sigma_\mu \ \text{ in } L^1(B, \beta, \mathfrak{a}),$$

and the limit σ_μ belongs to $\mathfrak{a}_{\Theta_\mu}^{++}$. By Corollary 7.20.c), one also gets

$$\tfrac{1}{n}\,\kappa(b_n\cdots b_1\tau_f) \xrightarrow[n\to\infty]{} \sigma_\mu \;\text{ in } L^1(B,\beta,\mathfrak{a}),$$

for any f in F. The same theorem 9.9 tells us that, uniformly for η in $\mathcal{P}$, one has the convergence

$$\tfrac{1}{n}\,\sigma(b_n\cdots b_1,\eta) \xrightarrow[n\to\infty]{} \sigma_\mu \;\text{ in } L^1(B,\beta,\mathfrak{a}).$$

As a consequence, for every α in Θ_μ, uniformly for $\eta \neq \eta' \in \mathcal{P}_{\Theta_\mu}$ with $f_\eta = f_{\eta'}$, one has,

$$\tfrac{1}{n}\,a_{\alpha,\eta,\eta'}(b_n\cdots b_1) \xrightarrow[n\to\infty]{} -\alpha^\omega(\sigma_\mu) \;\text{ in } L^1(B,\beta,\mathfrak{a}),$$

and hence, one also has

$$\tfrac{1}{n}\,a_{\eta,\eta'}(b_n\cdots b_1) \xrightarrow[n\to\infty]{} -\min_{\alpha\in\Theta_\mu}\alpha^\omega(\sigma_\mu) \;\text{ in } L^1(B,\beta,\mathfrak{a}).$$

and, using the regularity of the Lyapunov vector (Theorem 9.9.e),

$$\tfrac{1}{n}\int_G a_{\eta,\eta'}(g)\,\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} -\min_{\alpha\in\Theta_\mu}\alpha^\omega(\sigma_\mu) < 0.$$

Thus, for n large enough, one has

$$\sup_{f_\eta = f_{\eta'}} \int_G a_{\eta,\eta'}(g)\,\mathrm{d}\mu^{*n}(g) < 0$$

This proves (12.10) and ends the proof. □

Corollary 12.7. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Then, the corresponding partial Iwasawa cocycle $\sigma_{\Theta_\mu} : G\times\mathcal{P}_{\Theta_\mu} \to \mathfrak{a}_{\Theta_\mu}$ is special.*

Proof. This follows from Proposition 10.15. Indeed, the contraction assumption has been checked in Lemma 12.5, and the moment assumptions (10.14) and (10.15) have been checked in Corollary 12.4. □

12.3. Comparing the Iwasawa cocycle with its projection.

In this section, we compare the behavior of the Iwasawa cocycle σ with the behavior of its projection on $\mathfrak{a}_{\Theta_\mu}$.

The reader who is interested only in real Lie groups G can skip this section because, by (8.1), when $\mathcal{S} = \{\mathbb{R}\}$, for any Zariski dense subsemigroup Γ of G, one has $\Theta_\Gamma = \Pi$ and $\mathfrak{a}_{\Theta_\Gamma} = \mathfrak{a}$.

The first lemma is similar to Corollary 10.19.

Recall that the *limit set* Λ_Γ in $\mathcal{P}_{\Theta_\Gamma}$ of a Zariski dense subsemigroup Γ of G is the smallest non-empty Γ-invariant closed subset of $\mathcal{P}_{\Theta_\Gamma}$ (see Section 12.7).

LEMMA 12.8. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, Γ be a Zariski dense subsemigroup of G and $S_\Gamma \subset \mathcal{P}$ be the pullback of the limit set $\Lambda_\Gamma \subset \mathcal{P}_{\Theta_\Gamma}$. There exists $C \geq 0$ such that, for any n in $\mathbb{N}$, for any g in Γ and for any η in S_Γ, one has*

$$d(\sigma(g,\eta), \mathfrak{a}_{\Theta_\Gamma}) \leq C. \tag{12.11}$$

Eventhough this lemma is similar to it Corollary 10.19, it cannot be seen as a consequence of Corollary 10.19 applied to a probability measure μ on G such that $\Gamma_\mu = \Gamma$ because the action of G on the full flag variety $\mathcal{P}$ might not be μ-contracting over F when G is not a real Lie group.

PROOF OF LEMMA 12.8. First note that, since σ is a continuous cocycle, for any g_0 in G, one has

$$\sup_{g\in G, \eta\in\mathcal{P}} \|\sigma(g,\eta) - \sigma(g_0 g, \eta)\| < \infty.$$

Hence, we can assume that $G = G_c$ is connected. Now, fix α in Θ_Γ^c and let us prove that

$$\sup_{g\in\Gamma_\mu, \eta\in S_\Gamma} |\alpha^\omega(\sigma(g,\eta))| < \infty.$$

We will apply Lemma 3.2 to the representation (ρ_α, V_α) of G from Lemma 7.15. By definition, the proximal dimension r of $\rho_\alpha(\Gamma)$ is the dimension of the space V_α^Γ that is the sum of weight spaces of V_α that are associated to weights of the form $\chi_\alpha - \rho$, where ρ is a positive combination of elements of Θ_Γ^c. The map $g \mapsto gV_\alpha^\Gamma$ factors as a map

$$\mathcal{P}_{\Theta_\Gamma} \to \mathbb{G}_r(V_\alpha); \eta \mapsto V_{\alpha,\eta}^\Gamma.$$

Now, by definition, the image of S_Γ in $\mathcal{P}_{\Theta_\Gamma}$ is the limit set Λ_Γ which is included in the limit set $\Lambda^r_{\rho_\alpha(\Gamma)}$ from Lemma 3.2. Thus, from this Lemma, we get the existence of $C \geq 1$ such that, for any g in Γ, η in S_Γ and $v, v' \neq 0$ in $V_{\alpha,\eta}^\Gamma$, one has

$$\tfrac{1}{C}\tfrac{\|\rho_\alpha(g)v'\|}{\|v'\|} \leq \tfrac{\|\rho_\alpha(g)v\|}{\|v\|} \leq C\tfrac{\|\rho_\alpha(g)v'\|}{\|v'\|}. \tag{12.12}$$

To conclude, we will make the same computation as in the proof of Lemma 7.17.

Let k be in K_c such that $\eta = k\xi_\Pi$ and k' be in K_c, z in Z and u in U with $gk = k'zu$. We have $\omega(z) = \sigma(g,\eta)$. Let v and v' be nonzero vectors in V_{α,χ_α} and $V_{\alpha,\chi_\alpha-\alpha}$ and set $v'' = \rho_\alpha(u)^{-1}v'$. By construction, we have $\rho_\alpha(k)v, \rho_\alpha(k)v', \rho_\alpha(k)v'' \in V_{\alpha,\eta}^\mu$ and

$$\|\rho_\alpha(gk)v\| = \|\rho_\alpha(z)v\| = e^{\chi_\alpha^\omega(\omega(z))}\|v\|.$$

Besides, on the one hand,

$$\|\rho_\alpha(gk)v'\| = \|\rho_\alpha(zu)v'\| \geq e^{(\chi_\alpha^\omega - \alpha^\omega)(\omega(z))}\|v'\|,$$

where the latter inequality follows from the fact that

$$\rho_\alpha(zu)v' \in \rho_\alpha(z)v' + V_{\alpha,\chi_\alpha}.$$

By (12.12), this gives

$$\alpha^\omega(\sigma(g,\eta)) \geq -\log C.$$

On the other hand, since $v'' \in v' + V_{\alpha,\chi_\alpha}$, we have $\|v''\| \geq \|v'\|$ and

$$\begin{array}{rcl} \|\rho_\alpha(gk)v''\| & = & \|\rho_\alpha(zu)v''\| \; = \; \|\rho_\alpha(z)v'\| \\ & = & e^{(\chi_\alpha^\omega - \alpha^\omega)(\omega(z))}\|v'\| \; \leq \; e^{(\chi_\alpha^\omega - \alpha^\omega)(\omega(z))}\|v''\|, \end{array}$$

which, again by (12.12), gives

$$\alpha^\omega(\sigma(g,\eta)) \leq \log C.$$

Together, we get $|\alpha^\omega(\sigma(g,\eta))| \leq \log C$ as required. □

The upper bound (12.11) cannot be extended beyond the set S_Γ i.e. to any η in $\mathcal{P}$. Here is an Example.

EXAMPLE 12.9. There exists a finitely generated and Zariski dense subsemigroup Γ of a simple algebraic p-adic Lie group G such that

$$\sup_{g\in\Gamma}\sup_{\eta\in\mathcal{P}} d(\sigma(g,\eta),\mathfrak{a}_{\Theta_\Gamma}) = \infty.$$

PROOF. Here is an example with $G = \mathrm{SL}(3,\mathbb{Q}_p)$: choose Γ to be spanned by finitely many elements in a small compact open neighborhood of the matrix

$$g_0 = \begin{pmatrix} p^{-1} & 0 & 0 \\ 0 & p^{-1} & 0 \\ 0 & 0 & p^2 \end{pmatrix}$$

so that the simple root $\alpha := e_1^* - e_2^*$ is not in Θ_Γ. Choose η_0 to be the flag $\langle e_2\rangle \subset \langle e_2, e_3\rangle$ in $\mathbb{Q}_p^3$. One computes, for $n \geq 1$,

$$\alpha(\sigma(g_0^n,\eta_0)) = 2\,\log\|g_0^n|_{\langle e_2\rangle}\| - \log\|g_0^n|_{\langle e_2\wedge e_3\rangle}\| = n\log p$$

which is not bounded. □

Despite this remark, one has the following lemma which is similar to Lemma 10.14. In this lemma, we do not assume the starting point η to belong to the set $\mathcal{S}_\Gamma$.

LEMMA 12.10. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Let $\alpha \in \Pi \smallsetminus \Theta_\mu$ and $\eta \in \mathcal{P}$.*
a) For β-almost every b in B, the sequence $n \mapsto \alpha(\sigma(b_n \cdots b_1, \eta))$ is bounded.
b) One has $\lim\limits_{C\to\infty} \inf\limits_{n\geq 1} \mu^{*n}(\{g \in G \mid |\alpha(\sigma(g,\eta))| \leq C\}) = 1$.

PROOF. *a*) By Lemma 7.6, we may assume that η belongs to $\mathcal{P}_c$. By Theorem 9.9, for β-almost any b in B, the sequence

$$n \mapsto \|\sigma(b_n \cdots b_1, \eta) - \kappa(b_n \cdots b_1)\|$$

is bounded and, by the definition (8.1) of Θ_μ, the set $\alpha(\kappa(\Gamma_\mu))$ is also bounded.

b) This follows from from the bound

$$\lim_{C\to\infty} \beta(\{b \in B \mid \sup_{n\geq 1} |\alpha(\sigma(b_n..b_1, \eta))| \leq C\}) = 1$$

based on *a*). □

12.4. Limit laws for the Iwasawa cocycle.

We can now state and prove the limit laws (CLT, LIL, LDP) for the Iwasawa cocycle on the full flag variety $\mathcal{P}$. Remember that, when $\mathbb{K} = \mathbb{R}$, the action of G on $\mathcal{P}$ is μ-contracting.

¿From Lemma 12.1 and 12.5, we deduce that, if μ is a Zariski dense Borel probability measure on G with a finite exponential moment, then the Iwasawa cocycle

$$\sigma_{\Theta_\mu} : G \times \mathcal{P}_{\Theta_\mu} \to \mathfrak{a}_{\Theta_\mu}$$

satisfies the assumptions of Theorem 11.1 (note that, in this case, the uniqueness of the μ-stationary Borel probability measure on $\mathcal{P}_{\Theta_\mu}$ is already warranted by Lemma 1.24 and Proposition 3.18).

We let $\sigma_\mu \in \mathfrak{a}^+_{\Theta_\mu}$ be the average of σ_{Θ_μ}, $\Phi_\mu \in \mathrm{S}^2(\mathfrak{a}_{\Theta_\mu})$ be the covariance 2-tensor (2.16) of the cocycle with constant drift which is cohomologous to σ_{Θ_μ}, $\mathfrak{a}_\mu \subset \mathfrak{a}_{\Theta_\mu}$ be the linear span of this 2-tensor and N_μ be the Gaussian law on $\mathfrak{a}$ with covariance 2-tensor Φ_μ. By definition, the support of the Gaussian law N_μ is the vector subspace $\mathfrak{a}_\mu$.

We now reformulate Theorem 11.1 for the Iwasawa cocycle σ on the full flag variety $\mathcal{P}$.

THEOREM 12.11. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F := G/G_c$ and μ be a Zariski dense Borel probability measure on G with a finite exponential moment which is aperiodic in F. Let ν be the μ-stationary measure on the partial flag variety $\mathcal{P}_{\Theta_\mu}$.*

Then the average σ_μ, the covariance 2-tensor Φ_μ, the linear span $\mathfrak{a}_\mu$ and the Gaussian law N_μ are F-invariant and one has the following asymptotic estimates for the Iwasawa cocycle σ on the full flag variety $\mathcal{P}$.

(i) **Central limit theorem for σ with target**. *For any bounded continuous function ψ on $\mathcal{P}_{\Theta_\mu} \times \mathfrak{a}$, uniformly for η in $\mathcal{P}$,*

$$\int_G \psi\left(g\eta, \tfrac{\sigma(g,\eta)-n\sigma_\mu}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathcal{P}_{\Theta_\mu}\times\mathfrak{a}} \psi \,\mathrm{d}\nu\, \mathrm{d}N_\mu. \tag{12.13}$$

(ii) **Law of the iterated logarithm**. *Let $K_\mu \subset \mathfrak{a}_\mu$ be the unit ball of Φ_μ. For any η in $\mathcal{P}$, for β-almost any b in B, the following set of cluster points is equal to K_μ*

$$C\left(\frac{\sigma(b_n\cdots b_1,\eta)-n\sigma_\mu}{\sqrt{2n\log\log n}}\right) = K_\mu. \tag{12.14}$$

(iii) **Large deviations**. *For any $t_0 > 0$, one has*

$$\limsup_{n\to\infty} \sup_{\eta\in\mathcal{P}} \mu^{*n}(\{g\in G \mid \|\sigma(g,\eta)-n\sigma_\mu\| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{12.15}$$

In the left-hand side of Equality (12.13), the function ψ is viewed as a function on $\mathcal{P}\times\mathfrak{a}$ via the natural projection $\mathcal{P}\to\mathcal{P}_{\Theta_\mu}$.

REMARK 12.12. When moreover G is a real Lie group, we have already seen that the flag variety is the full flag variety $\mathcal{P}_{\Theta_\mu} = \mathcal{P}$, the Lyapunov vector σ_μ belongs to the open Weyl chamber $\mathfrak{a}^{++}$ and we will see soon that the support $\mathfrak{a}_\mu$ of the limit Gaussian law N_μ is equal to $\mathfrak{a}$.

PROOF OF THEOREM 12.11. (i) and (ii) The limit laws follow from Theorem 11.1 applied to the cocycle σ_{Θ_μ} on the partial flag variety $\mathcal{P}_{\Theta_\mu}$. We know that the contraction and the moment assumptions in Theorem 11.1 are satisfied because of Corollary 12.4 and Lemma 12.5. To deduce the conclusions of Theorem 11.1.i and 11.1.ii for the Iwasawa cocycle σ on the full flag variety $\mathcal{P}$, from the same results for σ_{Θ_μ}, we use the comparison Lemma 12.10. The F-invariance follows from Lemma 7.22 and Corollary 11.4.

(iii) Theorem 11.1.iii gives a similar conclusion with σ_{Θ_μ} in place of σ : for any $t_0 > 0$, one has

$$\limsup_{n\to\infty} \sup_{\eta\in\mathcal{P}} \mu^{*n}(\left\{g\in G \mid \left\|\sigma_{\Theta_\mu}(g,\eta)-n\sigma_\mu\right\| \geq nt_0\right\})^{\frac{1}{n}} < 1. \tag{12.16}$$

When G is a real Lie group this finishes the proof since $\Theta_\mu = \Pi$. In general, our conclusion follows from Proposition 12.13 below whose proof uses both the large deviations inequality (12.16) for σ_{Θ_μ} and the large deviations inequality (12.29) for κ that we will prove in the next section. $\square$

PROPOSITION 12.13 (Large deviations away from $\mathfrak{a}_{\Theta_\mu}$).
Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Let σ be the Iwasawa cocycle on the full flag variety $\mathcal{P}$. Then, for any $\alpha_0 \in \Pi \smallsetminus \Theta_\mu$, and $t_0 > 0$, one has

$$\limsup_{n\to\infty} \sup_{\eta\in\mathcal{P}} \mu^{*n}(\{g \in G \mid \ |\alpha_0(\sigma(g,\eta))| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{12.17}$$

In the proof of Proposition 12.13, we will also need the following Lemma 12.14 which gives a property valid for any root system. In order to lighten notations, we forget in this lemma the superscript ω, identifying Σ with the root system $\Sigma^\omega \subset \mathfrak{a}^*$. For a subset $\Theta \subset \Pi$ of the set Π of simple roots, we set $\Theta^c = \Pi \smallsetminus \Theta$, Σ_Θ to be the root subsystem generated by Θ, Σ_Θ^+ the corresponding set of positive roots and $\delta_\Theta = \sum_{\alpha\in\Sigma_\Theta^+} \alpha$ the sum of these positive roots. For α in Π, we set $\varpi_\alpha \in \mathfrak{a}^*$ for the corresponding fundamental weight (by definition, χ_α is an integer multiple of ϖ_α).

LEMMA 12.14. *Let $\mathfrak{a}$ be a Euclidean real vector space, $\Sigma \subset \mathfrak{a}^*$ a root system, Π a set of simple roots, and Θ a subset of Π.*
a) Then there exist integers $n_{\Theta,\alpha} \geq 0$, $\alpha \in \Theta$, such that

$$\delta_{\Theta^c} = 2\textstyle\sum_{\alpha\in\Theta^c}(2m_\alpha - 1)\pi_\alpha \ - \ \sum_{\alpha\in\Theta} n_{\Theta,\alpha}\,\pi_\alpha \tag{12.18}$$

(where $m_\alpha = \sharp(\Sigma^+ \cap \mathbb{R}\alpha) \in \{1,2\}$).
b) There exists $c > 0$ such that, for any $\alpha_0 \in \Theta^c$, any point $p \in \mathfrak{a}^+$ in the Weyl chamber and any point $q \in \mathrm{Conv}(Wp)$ in the convex hull of the Weyl orbit of p, one has the upper bound

$$|\alpha_0(q)| \leq c\textstyle\sum_{\alpha\in\Theta^c} \alpha(p) + c\sum_{\alpha\in\Theta} \pi_\alpha(p-q). \tag{12.19}$$

PROOF OF LEMMA 12.14. *a)* If α is in Π, β is in Σ^+ and s_α is the orthogonal symmetry associated to α, one has $s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$. Since $s_\alpha(\beta)$ belongs $\Sigma^+ \cup -\Sigma^+$, either $\beta \in \{\alpha, 2\alpha\}$ or $s_\alpha(\beta) \in \Sigma^+$. If moreover β is simple and $\neq \alpha$, one gets $(\alpha,\beta) \leq 0$.

Therefore, if α belongs to Θ^c, s_α preserves the set $\Sigma_{\Theta^c}^+ \smallsetminus \{\alpha, 2\alpha\}$ and sends α to $-\alpha$. This proves that $s_\alpha(\delta_{\Theta^c}) = \delta_{\Theta^c} - 2(2m_\alpha - 1)\alpha$. Hence one has $2\frac{(\alpha,\delta_{\Theta^c})}{(\alpha,\alpha)} = 2(2m_\alpha - 1)$.

If α belongs to Θ, one has $(\alpha,\beta) \leq 0$ for any β in $\Sigma_{\Theta^c}^+$, hence $(\alpha,\delta_{\Theta^c}) \leq 0$.

Since $(\pi_\alpha)_{\alpha\in\Pi}$ is the dual basis of $(\frac{2\alpha}{(\alpha,\alpha)})_{\alpha\in\Pi}$ with respect to the scalar product, this proves (12.18).

b) According to (7.22) one has the bound $\pi_\alpha(q) \leq \pi_\alpha(p)$ for all α in Π. Applying Equality (12.18) to the point $p - q$, one gets then

$$\delta_{\Theta^c}(q) \leq \delta_{\Theta^c}(p) + c \textstyle\sum_{\alpha\in\Theta} \pi_\alpha(p-q), \tag{12.20}$$

as soon as $c \geq \max_{\alpha\in\Theta} n_{\Theta,\alpha}$.

Applying this bound (12.20) to the point $q' = w^{-1}q$ with w in the Weyl group W_{Θ^c} of Σ_{Θ^c} such that

$$\alpha(q') \geq 0\ , \quad \text{for all } \alpha \text{ in } \Theta^c,$$

one gets,

$$|\alpha_0(q)| \leq \delta_{\Theta^c}(q') \leq \delta_{\Theta^c}(p) + c \textstyle\sum_{\alpha\in\Theta} \pi_\alpha(p-q). \tag{12.21}$$

Inequality (12.19) follows. □

Proof of Proposition 12.13. ¿From Lemma 7.22, we may assume that the η's which occur in Formula (12.17) belong to $\mathcal{P}_c$. By Corollary 7.20, for such an η, the point $q := \sigma(g, \eta)$ is in the convex hull of the W-orbit of the point $p := \kappa(g)$. Then (12.19) tells us that, for any α_0 in Θ_μ^c, one has

$$|\alpha_0^\omega(\sigma(g,\eta)| \leq \sum_{\alpha\in\Theta_\mu^c} \alpha^\omega(\kappa(g)) + c \sum_{\alpha\in\Theta_\mu} \pi_\alpha^\omega(\kappa(g) - \sigma(g,\eta)),$$

for some constant $c > 0$ depending only on G. Now, Equation (12.17) follows from the following three bounds,

$$\sup_{g\in\Gamma_\mu} \alpha^\omega(\kappa(g)) < \infty \ \text{ for all } \alpha \text{ in } \Theta_\mu^c, \tag{12.22}$$

and, for all α in Θ_μ and $t_0 > 0$,

$$\limsup_{n\to\infty} \mu^{*n}(\{g \in G \mid \ |\pi_\alpha^\omega(\kappa(g) - n\sigma_\mu)| \geq nt_0\})^{\frac{1}{n}} < 1 \quad \text{and} \tag{12.23}$$

$$\limsup_{n\to\infty} \sup_{\eta\in\mathcal{P}} \mu^{*n}(\{g \in G \mid \ |\pi_\alpha^\omega(\sigma(g,\eta) - n\sigma_\mu)| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{12.24}$$

The bound (12.22) follows from the Definition (8.1) of Θ_μ.

The bound (12.23) follows from the large deviations estimate (12.29) for κ from Theorem 12.17 below (note that the proof of (12.29) only relies on the large deviations estimates for σ_{Θ_μ}).

The bound (12.24) follows from the large deviations estimate (12.16) for σ_{Θ_μ} (note that, for $\alpha \in \Theta_\mu$, since π_α is s_β-invariant for any $\beta \neq \alpha$ in Π, one has $\pi_\alpha \circ \sigma = \pi_\alpha \circ \sigma_{\Theta_\mu}$). □

When the point $\eta \in \mathcal{P}$ belongs to the support of a μ-stationary measure one has a much stronger control than the one given in Proposition 12.13:

LEMMA 12.15. *Same assumptions as in Theorem 12.11. Let ν be a μ-stationary measure on $\mathcal{P}$. There exists $C \geq 0$ such that, for any n in $\mathbb{N}$, for any g in the support of μ^{*n} and for any η in the support S_ν of ν, one has*

$$d(\sigma(g,\eta)-n\sigma_\mu, \mathfrak{a}_\mu) \leq C. \tag{12.25}$$

PROOF. This follows from Corollary 10.19 applied to the cocycle σ_{Θ_μ} and from Lemma 12.8. □

As we have already noted in Remark 10.20 and Example 12.9, one cannot extend the bound (12.25) to any η in $\mathcal{P}$.

12.5. Iwasawa cocycle and Cartan projection.

Now, for g in G, we will define a subset $\mathcal{Q}_{\Theta,g}$ of $\mathcal{P}_{\Theta,c}$ outside of which we will be able to control the difference between the Cartan projection and the Iwasawa cocycle.

We need more notations. Recall, from Section 7.7, that, for any p in $\mathcal{S}$, we fixed a good maximal compact subgroup $K_c = \prod_{p\in\mathcal{S}} K_{p,c}$ of G_c and a Cartan decomposition $G_c = K_c Z^+ K_c$. We also defined a section $\tau : F \to G$ of the quotient map $s : G \to F$. which takes values in P. For any g in G_c, we fix once for all elements k_g and l_g of K_c and $z_g \in Z^+$ such that $g = k_g z_g l_g$. We can also suppose $k_{g^{-1}} = l_g^{-1}$. For g in G, we set $k_g = k_{\tau_{s(g)}^{-1} g}$ and $l_g = l_{\tau_{s(g)}^{-1} g}$

Fix $\Theta \subset \Pi$ and set $\Theta^\vee = \iota(\Theta)$ to be the image of Θ by the opposition involution. We let ξ_Θ be the fixed point of $P_{\Theta,c}$ in $\mathcal{P}_{\Theta,c}$ and $\mathcal{Q}_\Theta$ be the set of those η in $\mathcal{P}_{\Theta,c}$ such that, for some α in Θ, in the representation space V_α given in section 7.4.5, the line $V_{\alpha,\eta}$ is contained in the A-invariant hyperplane $\bigoplus_{\chi\neq\chi_\alpha} V_\alpha^\chi$ that is complementary to $V_\alpha^{\chi_\alpha}$. For g in G, we set

$$\xi^M_{\Theta,g} = k_g \xi_\Theta \text{ and } \mathcal{Q}^m_{\Theta,g} = l_g^{-1}\mathcal{Q}_\Theta. \tag{12.26}$$

Note that, when $\min_{\alpha\in\Theta} \alpha^\omega(\kappa(g)) > 0$, the point $\xi^M_{\Theta,g}$ and the subset $\mathcal{Q}^m_{\Theta,g}$ do not depend on the choice of k_g and l_g.

We let $P^\vee_{\Theta,c}$ be the parabolic subgroup of type $\Theta^\vee$ of G_c which is opposite to $P_{\Theta,c}$ with respect to A. One checks that $\mathcal{Q}_\Theta$ is the complement of the open $P^\vee_{\Theta,c}$-orbit in $\mathcal{P}_{\Theta,c}$ and hence that the map from G into the subsets of $\mathcal{P}_\Theta$, $g \mapsto g\mathcal{Q}_\Theta$ factors as a map from $\mathcal{P}_{\Theta^\vee} \simeq G/P^\vee_{\Theta,c}$ into the subsets of $\mathcal{P}_\Theta$,

$$\zeta \mapsto \mathcal{Q}_{\Theta,\zeta}. \tag{12.27}$$

These subsets $\mathcal{Q}_{\Theta,\zeta}$ are called the maximal Schubert cells of $\mathcal{P}_\Theta$. By construction, for any g in G, $\mathcal{Q}^m_{\Theta,g}$ is equal to a maximal Schubert cell of $\mathcal{P}_{\Theta,c}$. For instance, if g belongs to G_c, one has

$$\mathcal{Q}^m_{\Theta,g} = \mathcal{Q}_{\Theta,\xi^M_{\Theta^\vee,g^{-1}}}.$$

LEMMA 12.16. *For any $\varepsilon > 0$, there exists $M \geq 0$ such that, for any g in G and η in $\mathcal{P}_{\Theta,c}$ with $d\left(\eta, \mathcal{Q}^m_{\Theta,g}\right) \geq \varepsilon$, one has*

$$\|\sigma_\Theta(g,\eta) - p_\Theta(\kappa(g))\| \leq M.$$

The distance on $\mathcal{P}_\Theta$ is defined in (12.3) by using the map (7.26) constructed with the family of representations V_α with α in Θ, where the V_α were defined in section 7.4.5.

PROOF. The proof relies on the interpretation, in Section 7.5, of the Iwasawa cocycle and the Cartan projection via representations of G.

By construction, one can assume that g belongs to G_c and it suffices to prove the result for the elements g of Z^+. Let α be in Θ and (ρ_α, V_α) be the representation introduced in 7.4.5. Equip V_α with a (ρ_α, A, K_c)-good norm. Let $V' := V_\alpha^{\chi_\alpha}$ be the dominant eigenline and let V'' be its A-stable complementary subspace. For any $v \neq 0$ in V_α, writing $v = v' + v''$ with $v' \in V'$ and v'' in V'', we have $d(\mathbb{K}v, \mathbb{P}(V'')) = \frac{\|v'\|}{\|v\|}$. For g in Z^+ and η in $\mathcal{P}_{\Theta,c}$, picking a vector v in $V_{\alpha,\eta}$, using Lemma 7.17, one gets

$$\begin{aligned} e^{\chi^\omega_\alpha(\kappa(g))} = \|\rho_\alpha(g)\| \geq e^{\chi^\omega_\alpha(\sigma(g,\eta))} = \tfrac{\|\rho_\alpha(g)v\|}{\|v\|} \\ \geq \frac{\|\rho_\alpha(g)v'\|}{\|v\|} = e^{\chi^\omega_\alpha(\kappa(g))}\frac{\|v'\|}{\|v\|} = e^{\chi^\omega_\alpha(\kappa(g))} d(V_{\alpha,\eta}, W'). \end{aligned}$$

Hence one has

$$\chi^\omega_\alpha(\kappa(g)) + \log d(\eta, \mathcal{Q}_{\Theta,g}) \leq \chi^\omega_\alpha(\sigma(g,\eta)) \leq \chi^\omega_\alpha(\kappa(g)).$$

Our lemma follows. $\square$

12.6. Limit laws for the Cartan projection.

We can now extend the three limit laws to the Cartan projection under the same assumptions as in Theorem 12.11.

THEOREM 12.17. (Limit laws for $\kappa(g)$) *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F := G/G_c$ and μ be a Zariski dense Borel probability measure on G with a finite exponential moment which is aperiodic in F. One has the following asymptotic estimates for the Cartan projection $\kappa : G \to \mathfrak{a}$.*

(i) **Central limit theorem**. *For any bounded continuous function ψ on $\mathfrak{a}$,*

$$\int_G \psi\left(\tfrac{\kappa(g)-n\sigma_\mu}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathfrak{a}} \psi \,\mathrm{d}N_\mu,$$

where N_μ is the Gaussian law on $\mathfrak{a}_\mu$ whose covariance 2-tensor is Φ_μ.
(ii) **Law of the iterated logarithm**. *Let K_μ be the unit ball of Φ_μ. For β-almost any b in B, the following set of cluster points is equal to K_μ*

$$C\left(\frac{\kappa(b_n\cdots b_1)-n\sigma_\mu}{\sqrt{2n\log\log n}}\right) = K_\mu. \tag{12.28}$$

(iii) **Large deviations**. *For any $t_0>0$, one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{g\in G \mid \|\kappa(g)-n\sigma_\mu\| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{12.29}$$

The same argument below gives also a Central Limit Theorem for κ with target similar to (12.13). We leave the details to the reader.

Proof. (i) Central limit estimate. By usual aproximation arguments, it suffices to prove the result for compactly supported functions on $\mathfrak{a}$. Let ψ be such a function and η be in $\mathcal{P}_c$. According to Theorem 12.11, it is enough to prove that the following integral

$$I_n := \int_G \left|\psi(\tfrac{\sigma(g,\eta)-n\sigma_\mu}{\sqrt{n}}) - \psi(\tfrac{\kappa(g)-n\sigma_\mu}{\sqrt{n}})\right| \mathrm{d}\mu^{*n}(g) \tag{12.30}$$

converges to 0. Fix $\varepsilon>0$. By uniform continuity of ψ, there exists $\delta>0$ such that, for any v,w in $\mathfrak{a}$ with $\|v-w\|\leq\delta$, one has $|\psi(v)-\psi(w)|\leq\varepsilon$. Since η belongs to $\mathcal{P}_c$, by Theorem 9.9, for β-almost any b in B, the sequence

$$\|\sigma(b_n\cdots b_1,\eta)-\kappa(b_n\cdots b_1)\| \text{ is bounded.}$$

Hence, there exist $M>0$ and $n_0\geq 1$ such that, for all $n\geq n_0$,

$$\mu^{*n}(\{g\in G \mid \|\sigma(g,\eta)-\kappa(g)\|\geq M\})\leq\varepsilon.$$

Choosing $n\geq\max(n_0,\frac{M^2}{\delta^2})$ and cutting the integral I_n as the sum of the integrals over this set and its complement, one gets

$$I_n \leq 2\varepsilon\|\psi\|_\infty+\varepsilon.$$

This proves that $I_n \xrightarrow[n\to\infty]{} 0$ as required.

(ii) The law of the iterated logarithm is proved in the same way.

(iii) In what concerns the large deviations estimate, it is important to notice that the following proof relies only on (12.16) and not on (12.15) whose proof used (12.29).

By compactness, there exist $\varepsilon>0$ and $\eta_1,\ldots,\eta_r$ in $\mathcal{P}_{\Theta_\mu,c}$ such that, for any ζ in $\mathcal{P}_{\Theta_\mu^\vee,c}$, there exists $1\leq i\leq r$ with $d(\eta_i,\mathcal{Q}_{\Theta,\zeta})>\varepsilon$. Thus, by Lemma 12.16 and as $\sup_{g\in\Gamma_\mu} d(\kappa(g), \mathfrak{a}_{\Theta_\mu}) < \infty$, there exists $M \geq 0$ such that, for any g in Γ_μ, there exists $1 \leq i \leq r$ with

$$\left\|\sigma_{\Theta_\mu}(g,\eta_i) - \kappa(g)\right\| \leq M. \tag{12.31}$$

Now, by Equation (12.16), for any $t_0 > 0$, there exist $\alpha > 0$ and n_0 in $\mathbb{N}$ such that, for any $1 \leq i \leq r$, for any $n \geq n_0$, one has

$$\mu^{*n}(\{g \in G \mid \left\|\sigma_{\Theta_\mu}(g,\eta_i) - n\sigma_\mu\right\| \geq nt_0\}) \leq e^{-\alpha n}.$$

Thus, for any $n \geq \max(n_0, \frac{M}{t_0})$, we get

$$\mu^{*n}(\{g \in G \mid \|\kappa(g) - n\sigma_\mu\| \geq 2nt_0\}) \leq re^{-\alpha n}.$$

The result follows. $\square$

One also has the following control analogous to Lemma 12.15.

LEMMA 12.18. *Same assumptions as in Theorem 12.11. There exists $C \geq 0$ such that, for any n in $\mathbb{N}$, for any g in the support of μ^{*n}, one has*

$$d(\kappa(g) - n\sigma_\mu, \mathfrak{a}_\mu) \leq C. \tag{12.32}$$

PROOF. Let ν be the μ-stationary probability measure on the partial flag variety $\mathcal{P}_{\Theta_\mu}$. According to Proposition 9.1, one can find points $\eta_1, \ldots, \eta_r$ in the support S_ν of ν such that Equation (12.31) is satisfied. Our statement then follows from Corollary 10.19 applied to the Iwasawa cocycle σ_{Θ_μ} and the points η_i. $\square$

12.7. The support of the covariance 2-tensor.

In order to complete this chapter, we give some precisions on the linear span $\mathfrak{a}_\mu$ of the covariance 2-tensor Φ_μ.

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. As in Section 8.4, for any s in $\mathcal{S}$, we set $\mathfrak{b}_s$ to be the orthogonal in $\mathfrak{a}_s$ of the subspace of $\mathfrak{a}_s^*$ spanned by the algebraic characters of the center of G_s. We set $\mathfrak{b}_\mathbb{R}$ to be this subspace $\mathfrak{b}_s$ when the local field is $\mathbb{K}_s = \mathbb{R}$.

PROPOSITION 12.19. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Then the vector space $\mathfrak{a}_\mu$ contains $\mathfrak{b}_\mathbb{R}$.*

In particular, when G is an algebraic semisimple real Lie group, one has $\mathfrak{a}_\mu = \mathfrak{a}$, that is the Gaussian law N_μ is non-degenerate.

This result is proved in Goldsheid Guivarc'h [**55**] when G is $\mathrm{SL}(n,\mathbb{R})$ and in [**60**] when G is real semisimple.

PROOF OF PROPOSITION 12.19. Recall, from Proposition 9.2, that there exists a unique μ-stationary Borel probability measure ν on $\mathcal{P}_{\Theta_\mu}$ and, from Lemma 9.3, that the support of ν is Λ_{Γ_μ}.

By Lemmas 12.1 and 12.5, we know that the assumptions of Lemma 10.18 are satisfied. Therefore, by this lemma, there exists a Hölder continuous function $\dot{\varphi}_0 : \mathcal{P}_{\Theta_\mu} \to \mathfrak{a}$ such that, for $\mu \otimes \nu$-almost any (g, η) in $G \times \mathcal{P}_{\Theta_\mu}$, one has

$$\sigma_{\Theta_\mu}(g, \eta) - \dot{\varphi}_0(\eta) + \dot{\varphi}_0(g\eta) \in \sigma_\mu + \mathfrak{a}_\mu.$$

Since the function $\dot{\varphi}_0$ is continuous, by Lemma 9.3, we get, for any $n \geq 1$, any g in $\mathrm{Supp}\mu^{*n} \cap G_c$ and η in Λ_{Γ_μ},

$$\sigma_{\Theta_\mu}(g, \eta) - \dot{\varphi}_0(\eta) + \dot{\varphi}_0(g\eta) \in n\sigma_\mu + \mathfrak{a}_\mu.$$

In particular, when g is Θ_μ-proximal and $\eta = \xi^+_{\Theta_\Gamma, g}$, this gives

$$\lambda(g) = \sigma_{\Theta_\mu}(g, \xi^+_{\Theta_\mu, g}) \in n\sigma_\mu + \mathfrak{a}_\mu. \tag{12.33}$$

Now, by Proposition 8.8, the closed subgroup of $\mathfrak{a}$ spanned by the elements $\lambda(gh) - \lambda(g) - \lambda(h)$, when g, h and gh are Θ_Γ-proximal elements of Γ contains $\mathfrak{b}_\mathbb{R}$. Combining this Proposition 8.8 with (12.33), one gets the inclusion $\mathfrak{a}_\mu \supset \mathfrak{b}_\mathbb{R}$, which completes the proof. □

REMARK 12.20. From (12.33), one always has

$$\lambda(\Gamma) \subset \mathbb{N}\sigma_\mu + \mathfrak{a}_\mu.$$

By using the Central Limit Theorem 12.17 and elementary properties of Zariski dense subsemigroups, one can prove that the subspace of $\mathfrak{a}$ spanned by $\lambda(\Gamma)$ is

$$\langle \lambda(\Gamma) \rangle = \mathbb{R}\sigma_\mu + \mathfrak{a}_\mu.$$

12.8. A p-adic example.

The aim of this section is to construct an example where the Gaussian law in the Central Limit Theorem does not have full support.

EXAMPLE 12.21. Let $G = \mathrm{SL}(2, \mathbb{Q}_p)$ with $p < \infty$. There exists a Zariski dense probability measure μ on G with finite support such that $\mathfrak{a}_\mu = 0$ and Γ_μ is not bounded.

In other words, in this example, the Gaussian measure which appears in the Central Limit Theorem is a Dirac mass, whereas the set $\lambda(\Gamma_\mu)$ is not bounded.

PROOF. In this example we choose $\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2})$ with

$$g_1 = \begin{pmatrix} p & 0 \\ 1 & p^{-1} \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} p & 1 \\ 0 & p^{-1} \end{pmatrix}. \tag{12.34}$$

The semigroup Γ of $G = \mathrm{SL}(2, \mathbb{Q}_p)$ generated by g_1 and g_2 is Zariski dense and unbounded. Now, the flag manifold of $\mathrm{SL}(2, \mathbb{Q}_p)$ is the projective line $\mathbb{P}^1(\mathbb{Q}_p)$. As usual, we identify $\mathbb{P}^1(\mathbb{Q}_p)$ with $\mathbb{Q}_p \cup \{\infty\}$ by sending any $x \neq \infty$ to the line $\mathbb{Q}_p(x, 1)$ and ∞ to the line $\mathbb{Q}_p(1, 0)$. Then, the action of g_1 and g_2 read as the homographies

$$x \mapsto \frac{1}{px+1} p^2 x \quad \text{and} \quad x \mapsto p^2 x + p,$$

so that one has

$$g_1 \mathbb{Z}_p \subset p^2 \mathbb{Z}_p \;\text{ and }\; g_2 \mathbb{Z}_p \subset p + p^2 \mathbb{Z}_p.$$

In particular, Γ is the free semigroup with generators g_1 and g_2. For g in Γ, we denote by $|g|$ its length as a word in g_1 and g_2.

The limit set of Γ, which, by Lemma 9.3 is the support of the μ-stationary probability measure, is contained in the closed Γ-invariant set $\mathbb{Z}_p$.

Let K_c be the maximal compact subgroup $\mathrm{SL}(2, \mathbb{Z}_p)$ and A be the group of diagonal matrices. Then the usual norm on $\mathbb{Q}_p^2$ is good for the standard representation. Identify $\mathfrak{a}$ with $\mathbb{R}$ by setting

$$\omega \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix} = \log p.$$

Then, by Lemma 7.17, for any g in $\mathrm{SL}(2, \mathbb{Q}_p)$ and $v \neq 0$ in $\mathbb{Q}_p^2$, one has

$$\sigma(g, \mathbb{Q}_p v) = \log \tfrac{\|gv\|}{\|v\|}.$$

If g is g_1 or g_2 and $v = (x, 1)$ with x in $\mathbb{Z}_p$, this gives

$$\sigma(g_1, x) = \sigma(g_2, x) = \log p$$

and hence by the cocycle property, for g in Γ,

$$\sigma(g, x) = |g| \log p.$$

Therefore, for β-almost every b in B, for all x in S_ν and $n \geq 1$, one has

$$\sigma(b_n \cdots b_1, x) = n \log p$$

Hence this random sequence is deterministic with speed $\sigma_\mu = \log p$ and one has $\mathfrak{a}_\mu = 0$. $\square$

Remark 12.22. Note that, in this example, one has $g_2 \infty = \infty$ and $\sigma(g_2, \infty) = -\log p$, so that, for any n, $\sigma(g_2^n, \infty) = -n \log p = -n\sigma_\mu$. This validates Remark 10.20. One could also easily give explicit formulae for the functions $\sigma(g_1, .)$ and $\sigma(g_2, .)$ on $\mathbb{Q}_p \cup \{\infty\}$.

12.9. A non-connected example.

The aim of this section is to construct an enlightening example of a probability measure μ which illustrates the asymptotic behavior of the random product when the reductive group $\mathbf{G}$ is not connected, and, more precisely, when one deals with irreducible representations that are not strongly irreducible.

Over the field $\mathbb{R}$, this example will be similar to the one in Remark 3.10 but with a semisimple group G. Over the field $\mathbb{Q}_p$, it will give the example for Remark 3.19.

12.9.1. *Construction of the example.* Let $G_c = \mathrm{SL}(3, \mathbb{K})$ and G be the group generated by G_c and an element s of order two such that, for every g in G_c, $sgs = {}^t g^{-1}$. Let (ρ, V) be the 6-dimensional representation of G given by

$$\rho(g) = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} \text{ and } \rho(gs) = \begin{pmatrix} 0 & g \\ {}^t g^{-1} & 0 \end{pmatrix}. \tag{12.35}$$

We decompose V as a direct sum $V = V_1 \oplus V_2$ of irreducible representations of G_c.

Let μ be a Zariski dense probability measure on G with a finite exponential moment and (B, β) be the Bernoulli shift with alphabet (G, μ).

12.9.2. *Comparing various norms in Example* (12.35). We claimed in remarks 3.10 and 3.22 that, when $\mathbb{K} = \mathbb{R}$, for β-almost every b in B,

(12.36) *the set of cluster points in* $\mathbb{P}(\mathrm{End}(V))$ *of the sequence* $\mathbb{R}\rho(b_n \cdots b_1)$ *contains both rank* 1 *and rank* 2 *matrices,*

$$\sup_{n \geq 1} \tfrac{\|\rho(b_n \cdots b_1)|_{V_1}\|}{\|\rho(b_n \cdots b_1)|_{V_2}\|} = \infty \quad \text{and} \quad \inf_{n \geq 1} \tfrac{\|\rho(b_n \cdots b_1)|_{V_1}\|}{\|\rho(b_n \cdots b_1)|_{V_2}\|} = 0. \tag{12.37}$$

Proof of Claims (12.36) and (12.37). This statement follows from the results that we proved in the preceding chapters. We introduced the induced probability measure μ_c on G_c and proved that it has an exponential moment (Corollary 4.6). We can only consider subsequences associated to μ_c, i.e. setting (B_c, β_c) for the Bernoulli space with alphabet (G_c, μ_c), we only have to prove that (12.36) and (12.37) are true for β_c-almost every b in B_c. According to Proposition 3.7, all nonzero limit point of sequences $\lambda_n b_n \cdots b_1$ and $\lambda'_n\, {}^t b_n^{-1} \cdots {}^t b_1^{-1}$ with λ_n, λ'_n in $\mathbb{R}$, have rank one.

We introduce the sequences

$$S_n := \log \|b_n \cdots b_1\| \text{ and } S'_n := \log \|{}^t b_n^{-1} \cdots {}^t b_1^{-1}\|.$$

We have to prove that, for β_c-almost every b in B_c, the sequence $S_n - S'_n$ does not go to ∞, is not bounded above and is not bounded below.

On the one hand, according to Theorem 9.9 the limits $\lim_{n\to\infty} \frac{1}{n} S_n$ and $\lim_{n\to\infty} \frac{1}{n} S'_n$ exist and are equal. Let ν be the unique μ-stationary probability measure on the flag variety $\mathcal{P}_c$ of G_c. According to Lemma 2.18, for $\beta \otimes \nu$-almost every (b, η) in $B_c \times \mathcal{P}_c$, denoting by $\mathbb{R}v$ and $\mathbb{R}f$ the corresponding lines in $\mathbb{R}^3$ and its dual space, the sequence

$$\log \|b_n \cdots b_1 v\| - \log \|{}^t b_n^{-1} \cdots {}^t b_1^{-1} f\| \text{ does not go to } \infty.$$

By Theorem 3.28, this sequence remains at bounded distance of the sequence $S_n - S'_n$, hence $S_n - S'_n$ cannot go to ∞.

On the other hand, according to the Law of the Iterated Logarithm (Theorem 12.17) the upper limit $\limsup \frac{S_n - S'_n}{\sqrt{2n \log\log n}}$ is finite and positive. This proves that $S_n - S'_n$ is not bounded above. Similarly the sequence $S_n - S'_n$ is not bounded below. $\square$

12.9.3. *Stationary measures fo Example* (12.35). We note also that in this example,

(12.38) *when $\mathbb{K} = \mathbb{R}$ there exists only one μ-stationary probability on $\mathbb{P}(V)$.*

(12.39) *when $\mathbb{K} = \mathbb{Q}_p$, for suitable μ, there exist infinitely many μ-stationary probability on $\mathbb{P}(V)$.*

These claims (12.38) and (12.39) are special cases of more general results in [**13**]. The second claim (12.39) was annonced in Remark 3.19.

Sketch of proof of (12.38) and (12.39). See [**13**] for more details.

Assume $\mathbb{K} = \mathbb{R}$. The only μ-stationary probability on $\mathbb{P}(V)$ is the one supported by $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$. Indeed, there are no other μ-stationary probability since, by the Central Limit Theorem, for every x in $\mathbb{P}(V) \smallsetminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$ for every compact $K \subset \mathbb{P}(V) \smallsetminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$ one has $\lim_{n\to\infty} \mu^{*n} * \delta_x(K) = 0$.

Assume $\mathbb{K} = \mathbb{Q}_p$. Let $e_1 = (1, 0, 0) \in V_1$ and $e_2 = (0, 0, 1) \in V_2$. One can construct a probability measure μ on G such that, for every integer $\ell \geq 1$ the compact sets

$$K_\ell := \begin{Bmatrix} x = \mathbb{K}(v_1, v_2) \in \mathbb{P}(V) \ \mid \ \|v_2\| = p^\ell \|v_1\|, \\ d(\mathbb{K}v_1, \mathbb{K}e_1) \leq p^{-10}, \ \ d(\mathbb{K}v_2, \mathbb{K}e_2) \leq p^{-10} \end{Bmatrix}$$

are invariant by the semigroup Γ_μ. Hence each of these compact sets supports at least one μ-stationary probability. $\square$

12.9.4. *The Central Limit Theorem for Example* (12.35). The assumption of "strong irreducibility" in the Central Limit Theorem 0.7 cannot be weakened to an "irreducibility" assumption. Indeed, let σ_μ be the first Lyapunov exponent of μ. One can check that, the laws of the above sequence $\frac{\log\|\rho(b_n\cdots b_1)\|-n\sigma_\mu}{\sqrt{n}}$ converge to a law which is not Gaussian but which is the maximum of two independent Gaussian laws (see [**18**, Ex. 4.15] for details).

13. Regularity of the stationary measure

In this chapter, we prove a Hölder regularity property for stationary measures due to Guivarc'h [**58**]. We use a different method inspired by [**26**]. We will use this method all over this chapter.

We will first prove the Law of Large Numbers for the coefficients and for the spectral radius in Sections 13.4 and 13.5.

We will then give a new formula for the variance of the limit Gaussian Law in Section 13.6.

We will also prove the CLT, LIL and GDP for the norm of matrices, the norm of vectors, the coefficients and the spectral radius in sections 13.7, 13.8 and 13.9.

13.1. Regularity on the projective space.

We first prove a Hölder regularity property for stationary measures on projective spaces.

We recall quickly the notations from section 3.1. Let $\mathbb{K}$ be a local field and V be a finite dimensional $\mathbb{K}$-vector space endowed with a *good norm*. This means that we fix a basis $e_1, \ldots, e_d$ of V and the following norm on V. For $v = \sum v_i e_i \in V$ one has $\|v\|^2 = \sum |v_i|^2$ when $\mathbb{K}$ is archimedean and $\|v\| = \max(|v_i|)$ when $\mathbb{K}$ is non-archimedean. We denote by $e_1^*, \ldots, e_d^*$ the dual basis of V^* and we use the same symbol $\|.\|$ for the norms induced on V^*, $\mathrm{End}(V)$, $\wedge^2 V$, etc. We equip $\mathbb{P}(V)$ with the distance d given, for $x = \mathbb{K}v$, $x' = \mathbb{K}v'$ in $\mathbb{P}(V)$, by

$$d(x, x') = \tfrac{\|v\wedge v'\|}{\|v\|\|v'\|}.$$

For $x = \mathbb{K}v$ in $\mathbb{P}(V)$ and $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$, we set $y^\perp = \mathbb{P}(\mathrm{Ker}\, f) \subset \mathbb{P}(V)$ and

$$\delta(x, y) = \tfrac{|f(v)|}{\|f\|\|v\|}. \tag{13.1}$$

This quantity is also equal to the distance

$$\delta(x, y) = d(x, y^\perp) := \min_{x' \in y^\perp} d(x, x')$$

in $\mathbb{P}(V)$ and to the distance $d(y, x^\perp)$ in $\mathbb{P}(V^*)$.

Theorem 13.1. *Let μ be a Borel probability measure on $G = \mathrm{GL}(V)$ with a finite exponential moment and such that Γ_μ is proximal and strongly irreducible. Let ν be the unique μ-stationary Borel probability measure on $X = \mathbb{P}(V)$. Then, there exists $t > 0$ such that*

$$\sup_{y\in\mathbb{P}(V^*)} \int_X \delta(x,y)^{-t}\,\mathrm{d}\nu(x) < \infty. \tag{13.2}$$

In particular, there exists $C > 0$ and $t > 0$ such that, for any x in $\mathbb{P}(V)$ and $r > 0$, one has

$$\nu(B(x,r)) \leq C\,r^t. \tag{13.3}$$

A positive measure ν satisfying this condition (13.3) is sometimes called a *Frostman* measure.

As usual, we introduce the group K of isometries of $(V, \|.\|)$, and the semigroup

$$A^+ := \{\mathrm{diag}(a_1,\dots,a_d) \mid |a_1| \geq \cdots \geq |a_d|\}$$

(where, by a diagonal endomorphism, we mean an endomorphism that is diagonal in the basis $e_1,\dots,e_d$). For every element g in GL(V), we choose a decomposition

$$g = k_g a_g \ell_g \tag{13.4}$$

with k_g, ℓ_g in K and a_g in A^+. We denote by $x_g^M \in \mathbb{P}(V)$ the *density point* of g, that is

$$x_g^M := \mathbb{K}k_g e_1\,,$$

and by $y_g^m \in \mathbb{P}(V^*)$ the *density point* of ${}^t g$, that is

$$y_g^m := \mathbb{K}\,{}^t\ell_g e_1^*\,.$$

We denote by $\gamma_{1,2}(g)$ the *gap* of g, that is

$$\gamma_{1,2}(g) := \frac{\|\wedge^2 g\|}{\|g\|^2}.$$

The proof of Theorem 13.1 relies on the following Lemma 13.2 and Proposition 13.3. This Proposition 13.3 will be even more useful in the applications than Theorem 13.1.

Lemma 13.2. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. For every g in $\mathrm{GL}(V)$, $x = \mathbb{K}v$ in $\mathbb{P}(V)$ and $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$, one has*
(i) $\delta(x,y_g^m) \leq \frac{\|gv\|}{\|g\|\|v\|} \leq \delta(x,y_g^m) + \gamma_{1,2}(g)$
(ii) $\delta(x_g^M,y) \leq \frac{\|{}^t g f\|}{\|g\|\|f\|} \leq \delta(x_g^M,y) + \gamma_{1,2}(g)$
(iii) $d(gx,x_g^M)\,\delta(x,y_g^m) \leq \gamma_{1,2}(g)$.

PROOF. For all these inequalities, we can assume that g belongs to A^+, i.e. $g = \mathrm{diag}(a_1, \dots, a_d)$ with $|a_1| \geq \dots \geq |a_d|$. We write $v = v_1 + v_2$ with v_1 in $\mathbb{K}e_1$ and v_2 in $\mathrm{Ker} e_1^*$. One has then

$$\|g\| = |a_1| \ , \ \ \gamma_{1,2}(g) = \tfrac{|a_2|}{|a_1|}, \ \text{ and } \ \delta(x, y_g^m) = \tfrac{\|v_1\|}{\|v\|}.$$

(i) follows from $\|g\| \, \|v_1\| \leq \|gv\| \leq \|g\| \, \|v_1\| + |a_2| \, \|v_2\|$.
(ii) follows from (i) by replacing V by V^* and g by ${}^t g$.
(iii) follows from $d(gx, x_g^M) \, \delta(x, y_g^m) = \frac{\|gv_2\|}{\|gv\|} \frac{\|v_1\|}{\|v\|} \leq \frac{|a_2|}{|a_1|}$. □

Let $\sigma_\mu = (\lambda_1, \cdots, \lambda_d) \in \mathbb{R}^d$ be the Lyapunov vector of μ given by the Law of Large Numbers for reductive groups (Theorem 9.9). Since Γ_μ is proximal, according to Corollary 9.15, one has

$$\lambda_1 > \lambda_2.$$

PROPOSITION 13.3. *Let μ be a Borel probability measure on $G = \mathrm{GL}(V)$ with a finite exponential moment and such that Γ_μ is proximal and strongly irreducible. For any $\varepsilon > 0$, there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, x in $\mathbb{P}(V)$ and y in $\mathbb{P}(V^*)$, one has*

$$\mu^{*n}(\{g \in G \mid \delta(x, y_g^m) \geq e^{-\varepsilon n}\}) \geq 1 - e^{-cn}, \tag{13.5}$$

$$\mu^{*n}(\{g \in G \mid d(gx, x_g^M) \leq e^{-(\lambda_1 - \lambda_2 - \varepsilon)n}\}) \geq 1 - e^{-cn}, \tag{13.6}$$

$$\mu^{*n}(\{g \in G \mid \delta(x_g^M, y) \geq e^{-\varepsilon n}\}) \geq 1 - e^{-cn}, \tag{13.7}$$

$$\mu^{*n}(\{g \in G \mid \delta(gx, y) \geq e^{-\varepsilon n}\}) \geq 1 - e^{-cn}. \tag{13.8}$$

PROOF. We can assume $\varepsilon < \frac{1}{2}(\lambda_1 - \lambda_2)$. According to the large deviations principles for the Iwasawa cocycle (Theorem 12.11) and for the Cartan projection (Theorem 12.17), there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $x = \mathbb{K}v$ in $\mathbb{P}(V)$ and $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$ with $\|v\| = \|f\| = 1$, there exists a subset $G_{n,x,y} \subset G$ with

$$\mu^{*n}(G_{n,x,y}) \geq 1 - e^{-cn},$$

such that for g in $G_{n,x,y}$, the four quantities

$$\left|\lambda_1 - \tfrac{\log \|g\|}{n}\right|, \left|\lambda_1 - \tfrac{\log \|gv\|}{n}\right|, \left|\lambda_1 - \tfrac{\log \|{}^t g f\|}{n}\right|, \left|\lambda_1 - \lambda_2 - \tfrac{\log \gamma_{1,2}(g)}{n}\right|$$

are bounded by $\frac{\varepsilon}{8}$. We will check that, provided n_0 is large enough, for any g in $G_{n,x,y}$, one has

$$\delta(x, y_g^m) \geq e^{-\varepsilon n} \ , \ \ d(gx, x_g^M) \leq e^{-(\lambda_1 - \lambda_2 - \varepsilon)n} \ ,$$

$$\delta(x_g^M, y) \geq e^{-\varepsilon n} \ \text{ and } \ \delta(gx, y) \geq e^{-\varepsilon n}.$$

We first notice that, according to Lemma 13.2.i, one has

$$\delta(x, y_g^m) \geq e^{-\frac{\varepsilon}{4}n} - e^{-(\lambda_1-\lambda_2-\frac{\varepsilon}{8})n},$$

hence, if n_0 is large enough,

$$\delta(x, y_g^m) \geq e^{-\frac{\varepsilon}{2}n}. \tag{13.9}$$

This proves (13.5).

Now, using Lemma 13.2.iii one gets, for n_0 large enough,

$$d(gx, x_g^M) \leq e^{-(\lambda_1-\lambda_2-\frac{\varepsilon}{8})n} e^{\frac{\varepsilon}{3}n} \leq e^{-(\lambda_1-\lambda_2-\varepsilon)n}. \tag{13.10}$$

This proves (13.6).

Applying the same argument as above to ${}^tg$ acting on $\mathbb{P}(V^*)$, Inequality (13.9) becomes

$$\delta(x_g^M, y) \geq e^{-\frac{\varepsilon}{2}n}. \tag{13.11}$$

This proves (13.7).

Hence, combining (13.11) with (13.10), one gets, for n_0 large enough,

$$\begin{aligned}\delta(gx, y) &\geq \delta(x_g^M, y) - d(gx, x_g^M) \\ &\geq e^{-\frac{\varepsilon}{2}n} - e^{-(\lambda_1-\lambda_2-\varepsilon)n} \geq e^{-\varepsilon n}.\end{aligned}$$

This proves (13.8). □

Proof of Theorem 13.1. We choose ε, c, n_0 as in Proposition 13.3. We first check that, for $n \geq n_0$ and y in $\mathbb{P}(V^*)$, one has

$$\nu(\{x \in X \mid \delta(x, y) \geq e^{-\varepsilon n}\}) \geq 1 - e^{-cn}. \tag{13.12}$$

Indeed, since $\nu = \mu^{*n} * \nu$, by using (13.8) one computes

$$\begin{aligned}\nu(\{x \in X | \delta(x, y) \geq e^{-\varepsilon n}\}) &= \textstyle\int_X \mu^{*n}(\{g \in G | \delta(gx, y) \geq e^{-\varepsilon n}\}) \,\mathrm{d}\nu(x) \\ &\geq \textstyle\int_X (1 - e^{-cn}) \,\mathrm{d}\nu(x) = 1 - e^{-cn}.\end{aligned}$$

Then, choosing $t < \frac{c}{\varepsilon}$ and cutting the integral (13.2) along the subsets

$$A_{n,y} := \{x \in X \mid e^{-\varepsilon(n+1)} \leq \delta(x, y) < e^{-\varepsilon n}\},$$

one gets the upperbound

$$\begin{aligned}\textstyle\int_X \delta(x, y)^{-t} \,\mathrm{d}\nu(x) &\leq e^{tc\varepsilon n_0} + \textstyle\sum_{n \geq n_0} e^{t\varepsilon(n+1)} \nu(A_{n,y}) \\ &\leq e^{t\varepsilon n_0} + \textstyle\sum_{n \geq n_0} e^{t\varepsilon} e^{-(c-t\varepsilon)n} < \infty.\end{aligned}$$

This proves (13.2). □

13.2. Regularity on the flag variety.

In this section, we deduce from Theorem 13.1 a Hölder regularity property for the stationary measure on the flag variety.

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group. Let Θ be a subset of the set of simple restricted roots Π. Recall that we defined a G_c-equivariant embedding (7.26) using the family of representations V_α defined in section 7.4.5

$$\mathcal{P}_{\Theta,c} \to \prod_{\alpha\in\Theta} \mathbb{P}\left(V_\alpha\right), \eta \mapsto (V_{\alpha,\eta})_{\alpha\in\Theta}.$$

In the same way, one has a G_c-equivariant embedding

$$\mathcal{P}_{\Theta^\vee,c} \to \prod_{\alpha\in\Theta} \mathbb{P}\left(V_\alpha^*\right), \eta \mapsto (V_{\alpha,\eta}^*)_{\alpha\in\Theta}.$$

For any η in $\mathcal{P}_{\Theta,c}$ and ζ in $\mathcal{P}_{\Theta^\vee,c}$, we set

$$\delta(\eta,\zeta) = \min_{\alpha\in\Theta} \delta(V_{\alpha,\eta}, V_{\alpha,\zeta}^*). \tag{13.13}$$

One has then the equivalence, using Notation (12.27),

$$\delta(\eta,\zeta) = 0 \Longleftrightarrow \eta \in \mathcal{Q}_{\Theta,\zeta}.$$

Let μ be a Zariski dense Borel probability measure on G. From Proposition 9.1, we know that there exists a unique μ-stationary Borel probability measure ν on $\mathcal{P}_{\Theta_\mu,c}$ and that, for any ζ in $\mathcal{P}_{\Theta_\mu^\vee,c}$, one has $\nu(\mathcal{Q}_{\Theta,\zeta}) = 0$. We deduce from Theorem 13.1 the following

THEOREM 13.4. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with a finite exponential moment.*

Let ν be the unique μ-stationary Borel probability measure on $\mathcal{P}_{\Theta_\mu,c}$. There exists $t > 0$ such that

$$\sup_{\zeta\in\mathcal{P}_{\Theta_\mu^\vee,c}} \int_{\mathcal{P}_{\Theta_\mu,c}} \delta(\eta,\zeta)^{-t}\, \mathrm{d}\nu(\eta) < \infty.$$

PROOF. Let μ_c be the measure induced by μ on the finite index subgroup G_c of G defined in section 7.23. From Lemma 4.7, we know that ν is μ_c-stationary and, from Lemma 9.8, that μ_c has a finite exponential moment. Hence, the proof of Theorem 13.4 is reduced to the case where $G = G_c$.

Then, we just notice that, for $t > 0$, $\eta \in \mathcal{P}_\Theta$ and $\zeta \in \mathcal{P}_{\Theta^\vee}$, one has

$$\delta(\eta,\zeta)^{-t} \le \sum_{\alpha\in\Theta} \delta(V_{\alpha,\eta}, V_{\alpha,\zeta}^*)^{-t}.$$

Since V_α is a strongly irreducible proximal representation of Γ_μ, our claim follows from Theorem 13.1. $\square$

13.3. Regularity on the Grassmann variety.

In this section, we deduce from Theorem 13.1 a Hölder regularity property for the stationary measure on the limit set Λ_Γ^r in the Grassmann variety $\mathbb{G}_r(V)$ where r is the proximal dimension of Γ.

We will use the notations of Lemma 3.38.

Theorem 13.5. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that μ has a finite exponential moment and $\Gamma := \Gamma_\mu$ is strongly irreducible. Let $r \geq 1$ be the proximal dimension of Γ in V and ν_r be the unique μ-stationary probability measure on the limit set Λ_Γ^r in the Grassmann variety $\mathbb{G}_r(V)$. Then, there exists $t > 0$ such that*

$$\sup_{y\in\mathbb{P}(V^*)} \int_X \mathbf{d}(z,y)^{-t}\,\mathrm{d}\nu_r(z) < \infty. \tag{13.14}$$

Here, the "distance" $\mathbf{d}(z,y)$ is defined as the maximum

$$\mathbf{d}(z,y) := \max_{x\in z} \delta(x,y) \tag{13.15}$$

where $\delta(x,y)$ is as in (13.1).

The bound (13.14) does not depend on the choice of the norm on V. Hence we may assume that the norm on V is good i.e. it is an Euclidean norm when $\mathbb{K}$ is archimedean and a sup-norm when $\mathbb{K}$ is non-archimedean. We assume also that V^* and $\wedge^r V$ are endowed with compatible good norms. Now there are two others equivalent definitions for the quantity (13.15).

First, let $z^\perp$ be the subspace $z^\perp := \{y' = \mathbb{R}f' \text{ such that } f'|_z = 0\}$ orthogonal to z in $\mathbb{P}(V^*)$. One has the equality

$$\mathbf{d}(z,y) = d(y,z^\perp) := \min_{y'\in z^\perp} d(y,y') \tag{13.16}$$

Second, let $i_r : \mathbb{G}_r(V) \to \mathbb{P}(\wedge^r V)$ be the natural embedding. For any hyperplane $y \in \mathbb{P}(V^*)$ we denote by y_r the subspace $y_r := \mathbb{P}(\wedge^r y)$ of $\mathbb{P}(\wedge^r V)$. One has the equality

$$\mathbf{d}(z,y) = d(i_r(z),y_r) := \min_{z'\in y_r} d(i_r(z),z') \tag{13.17}$$

The proofs of (13.16) and (13.17) are left to the reader. Note that if the norms are not assumed to be good, the equalities (13.16) and (13.17) are true only up to a uniformly bounded multiplicative factor.

Proof of Theorem 13.5. According to Lemma 3.36, there exists a strongly irreducible and proximal representation $\rho' : \Gamma \to \mathrm{GL}(V'_r)$, in a $\mathbb{K}$-vector space V'_r and a Γ-equivariant embedding $i'_r : \Lambda_\Gamma^r \to \mathbb{P}(V'_r)$.

This representation is constructed as a quotient $V'_r = V_r/U_r$ where V_r and U_r are Γ-invariant subspaces of $\wedge^r V$ and the embedding i'_r is induced by the natural Γ-equivariant embedding $i_r : \Lambda^r_\Gamma \to \mathbb{P}(\wedge^r V)$ whose image is included in $\mathbb{P}(V_r)$ and does not meet $\mathbb{P}(U_r)$ (see Lemma 3.36).

Since Γ acts irreducibly on V, the subspace y_r never contains $\mathbb{P}(V_r)$ and is never included in $\mathbb{P}(U_r)$. Hence it defines a non-trivial proper subspace y'_r of $\mathbb{P}(V'_r)$. Using (13.17), for any z in Λ^r_Γ, one gets the bound

$$d(i'_r(z), y'_r) \leq d(i_r(z), y_r) = \mathbf{d}(z, y). \tag{13.18}$$

The image of ν_r by i'_r is the unique μ-stationary probability measure on $\mathbb{P}(V'_r)$. The bound (13.14) follows from (13.18) and from the bound (13.2) applied to this representation V'_r. □

Using the same method we can also prove the following Proposition 13.6.

PROPOSITION 13.6. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *with a finite exponential moment and such that Γ_μ is strongly irreducible. For any $\varepsilon > 0$, there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and v in $V \smallsetminus \{0\}$, one has*

$$\mu^{*n}(\{g \in G \mid \tfrac{\|gv\|}{\|g\|\|v\|} \geq e^{-n\varepsilon}\}) \geq 1 - e^{-cn}. \tag{13.19}$$

REMARK 13.7. When Γ_μ is proximal, we obtained a formula similar to (13.19) in the the proof of Proposition 13.3 as a consequence of the Large Deviation Principle for the Iwasawa cocycle. When Γ_μ is not assumed to be proximal, we will first prove Formula (13.19) and we will use it in the proof of the Large Deviation Principle for the norm cocycle in Theorem 13.19.

Before to start the proof of Proposition 13.6, we need a few notations. Fix $1 \leq r \leq d$. Let $e_1, \ldots, e_d$ be the standard basis of $V = \mathbb{K}^d$, For every element g in $\mathrm{GL}(V)$, we fix a Cartan decomposition $g = k_g a_g \ell_g$ as in (13.4). We set $z^M_g \in \mathbb{G}_r(V)$ to be the *density r-dimensional subspace of g*

$$z^M_g = k_g(\mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_r) \tag{13.20}$$

i.e. z^M_g is the r-dimensional subspace given by the density point of $\wedge^r g$. Similarly, we set $z^m_g \in \mathbb{G}_{d-r}(V)$ to be the *density $(d-r)$-dimensional subspace of ${}^t g$*

$$z^m_g = \ell_g^{-1}(\mathbb{K}e_{r+1} \oplus \cdots \oplus \mathbb{K}e_d) \tag{13.21}$$

i.e. z^m_g is the $(d–r)$-dimensional subspace of V orthogonal to the density r-dimensional subspace $z^M_{{}^t g}$ of ${}^t g$ in V^*. Once r is fixed, these density subspaces z_g^M and z_g^m are uniquely defined when the r^{th}-singular value $\kappa_r(g)$ is larger than $\kappa_{r+1}(g)$. In general they depend on the choice of the decomposition (13.4).

Proof of Proposition 13.6. This follows from Lemma 13.8.b) below where r is the proximal dimension of Γ_μ and from Proposition 13.9.b). Note that, by Lemma 9.16, the ratios of singular values $\frac{\kappa_1(g)}{\kappa_r(g)}$ for g in Γ_μ, are uniformly bounded. $\square$

We used the following lemma which is a variation of Lemma 13.2.

Lemma 13.8. *Let $\mathbb{K}$ be a local field, $V = \mathbb{K}^d$ and $x = \mathbb{K}v$ be a point in $\mathbb{P}(V)$. Fix $1 \leq r \leq d$ and let $c_0 > 0$ and g be an element of* $\mathrm{GL}(V)$.
a) Assume that the r first singular values are equal $\kappa_1(g) = \ldots = \kappa_r(g)$. Then one has the inequality

$$\frac{\|gv\|}{\|g\|\|v\|} \geq d(x, z_g^m). \tag{13.22}$$

b) More generally, assuming that $\kappa_r(g) \geq c_0\, \kappa_1(g)$, one has

$$\frac{\|gv\|}{\|g\|\|v\|} \geq c_0\, d(x, z_g^m). \tag{13.23}$$

Proof of Lemma 13.8. Same proof as for Lemma 13.2. $\square$

We also used the following proposition 13.9 which is a variation of Proposition 13.3.

Proposition 13.9. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on* $\mathrm{GL}(V)$ *with a finite exponential moment such that Γ_μ is strongly irreducible. Let r be the proximal dimension of Γ_μ. For any $\varepsilon > 0$, there exist $c > 0$ and $n_0 \in \mathbb{N}$ satisfying the following.*
a) For all $n \geq n_0$ and y in $\mathbb{P}(V^*)$, one has

$$\mu^{*n}(\{g \in G \mid \ \mathbf{d}(z_g^M, y) \geq e^{-n\varepsilon}\}) \geq 1 - e^{-cn}. \tag{13.24}$$

b) For all $n \geq n_0$ and x in $\mathbb{P}(V)$, one has

$$\mu^{*n}(\{g \in G \mid \ d(x, z_g^m) \geq e^{-n\varepsilon}\}) \geq 1 - e^{-cn}. \tag{13.25}$$

Proof of Proposition 13.9. a) We recall that the distance $\mathbf{d}(z, y)$ has been defined in (13.15). According to (13.18) and its notations, one has the inequality $\mathbf{d}(z_g^M, y) \geq d(i'_r(z_g^M), y_r)$. Since the point $i'_r(z_g^M)$ is the density point of $\rho'(g)$ in the proximal representation V'_r, our assertion follows from (13.7).

b) This follows from a) applied to the dual representation and from (13.16). $\square$

13.4. Law of Large Numbers for the coefficients.

We use the regularity properties of the Furstenberg measure from Section 13.1 to prove the Law of Large Numbers for the coefficients.

Let $\mathbb{K}$ be a local field, $V = \mathbb{K}^d$ and μ be a Borel probability measure on $\mathrm{GL}(V)$. We recall that Γ_μ is the closed subsemigroup of $\mathrm{GL}(V)$ spanned by the support of μ and that $B := \{b = (b_1, \ldots, b_n, \ldots)\} = \Gamma_\mu^{\mathbb{N}^*}$ is the Bernoulli space endowed with the Bernoulli probability measure $\beta := \mu^{\otimes\mathbb{N}^*}$. We fix a norm $\|.\|$ on V. We recall that the limit

$$\lambda_1 = \lambda_{1,\mu} := \lim_{n\to\infty} \frac{1}{n} \int_G \log \|g\| \, \mathrm{d}\mu^{*n}(g) \tag{13.26}$$

exists and is called the first Lyapunov exponent of μ.

Theorem 13.10. *Let $\mathbb{K}$ be a local field, $V = \mathbb{K}^d$, and μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that μ has a finite exponential moment and that Γ_μ is proximal and strongly irreducible. For v in $V \smallsetminus \{0\}$, f in $V^* \smallsetminus \{0\}$, for β-almost all b in B, one has*

$$\lim_{n\to\infty} \tfrac{1}{n} \log |f(b_n \cdots b_1 v)| = \lambda_{1,\mu}, \tag{13.27}$$

$$\lim_{n\to\infty} \tfrac{1}{n} \log |f(b_1 \cdots b_n v)| = \lambda_{1,\mu}, \tag{13.28}$$

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|b_1 \cdots b_n v\| = \lambda_{1,\mu}. \tag{13.29}$$

Moreover these sequences converge in $L^1(B, \beta)$.

It is plausible that the assumption that Γ_μ is proximal in Theorem 13.10 can be weakened into the assumption that, Γ_μ is *absolutely strongly irreducible* i.e. that, for any field extension $\mathbb{L} \supset \mathbb{K}$, the action of Γ_μ in $\mathbb{L}^d$ is still strongly irreducible. It is also plausible that the finite exponential moment assumption can be weakened into a finite first moment assumption.

The main new difficulty when one compares statement (13.27) with the Law of Large Numbers for the norm (0.14) is that one has to control the relative position of the vector $b_n \cdots b_1 v$ and of the hyperplane $\mathrm{Ker} f$. This is done in the following Lemma which will also be useful in Section 13.8. We recall the notation $\delta(x, y) = \frac{|f(v)|}{\|f\|\|v\|}$ as in (13.1), when $x = \mathbb{K}v \in \mathbb{P}(V)$ and $y = \mathbb{K}f \in \mathbb{P}(V^*)$.

Lemma 13.11. *Let $\mathbb{K}$ be a local field, $V = \mathbb{K}^d$, and μ be a Borel probability measure on* $\mathrm{GL}(V)$ *such that μ has a finite exponential moment and that Γ_μ is proximal and strongly irreducible.*

For all $\varepsilon > 0$, there exists $c > 0$, $\ell_0 > 0$ such that for all $n \geq \ell \geq \ell_0$, one has, for all x in $\mathbb{P}(V)$, y in $\mathbb{P}(V^*)$,

$$\mu^{*n}(\{g \in G \mid \delta(gx, y) \geq e^{-\varepsilon\ell}\}) \;\geq\; 1 - e^{-c\ell}. \tag{13.30}$$

Proof of Lemma 13.11. When $n = \ell$, this is (13.8) in Proposition 13.3. Since

$$\mu^{*n}(\{g \mid \delta(gx, y) \geq e^{-\varepsilon\ell}\}) = \int_G \mu^{*\ell}(\{g \mid \delta(ghx, y) \geq e^{-\varepsilon\ell}\})\, \mathrm{d}\mu^{*(n-\ell)}(h),$$

the case $n \geq \ell$ follows. □

Proof of Theorem 13.10. Write $x = \mathbb{K}v$ and $y = \mathbb{K}f$. According to the Law of Large Numbers in Theorem 3.28.b, for β-almost all b in B, one has

$$\lim_{n\to\infty} \tfrac{1}{n} \log \tfrac{\|b_n \cdots b_1 v\|}{\|v\|} = \lambda_{1,\mu}. \tag{13.31}$$

According to Lemma 13.11 with $n = \ell$, there exists $c > 0$ and $\ell_0 \in \mathbb{N}$ such that, for $n \geq \ell_0$, one has

$$\beta(\{b \in B \mid \delta(b_n \cdots b_1 x, y) \leq e^{-\varepsilon n}\}) \leq e^{-cn}.$$

Hence, by Borel-Cantelli lemma, for β-almost all b in B, one has

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \delta(b_n \cdots b_1 x, y) \;\;\geq\;\; -\varepsilon, \;\text{ i.e.}$$

$$\lim_{n\to\infty} \tfrac{1}{n} \log \frac{|f(b_n \cdots b_1 v)|}{\|f\|\|b_n \cdots b_1 v\|} = 0.$$

Combined with (13.31), this proves (13.27).

One deduces (13.28) from (13.27) by exchanging the roles of V and V^*.

Finally, according to Lemma 3.27, for β-almost all b in B, one also has

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|b_1 \cdots b_n\| = \lambda_{1,\mu}.$$

One deduces (13.29) from (13.28), and from the above limit since one has the lower and upper bounds :

$$|f(b_1 \cdots b_n v)| \leq \|f\|\, \|b_1 \cdots b_n v\| \leq \|f\|\, \|b_1 \cdots b_n\|\, \|v\|.$$

The convergence in $L^1(B, \beta)$ follows from the almost sure convergence and from Lemma 1.2, since the three sequences in (13.27), (13.28) and (13.29) are uniformly integrable. Indeed they are bounded above by the sequence $\frac{1}{n}\sum_{1\leq i\leq n} \log \|b_i\|$ which converges in $L^1(B, \beta)$ according to the classical Law of Large Numbers in Theorem 1.5. □

13.5. Law of Large Numbers for the spectral radius.

We now prove the Law of Large Numbers for the spectral radius. As in Section 13.4, this relies on the regularity properties of the Furstenberg measure from Section 13.1.

We recall that $\mathbb{K}$ is a local field, that $V = \mathbb{K}^d$, that $\lambda_1(g)$ denotes the spectral radius of an element g in $\mathrm{GL}(V)$ and that $\lambda_{1,\mu}$ denotes the first Lyapunov exponent of a probability measure μ on $\mathrm{GL}(V)$.

THEOREM 13.12. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that μ has a finite exponential moment and that Γ_μ is strongly irreducible. For β-almost all b in B, one has*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \lambda_1(b_n \cdots b_1) \;=\; \lambda_{1,\mu}. \tag{13.32}$$

Moreover this sequence converges in $L^1(B,\beta)$.

When Γ_μ is proximal, the main new difficulty when one compares statement (13.32) with the Law of Large Numbers for the coefficients (13.27) is that one has to ensure that $b_n \cdots b_1$ is proximal and to control the relative position of the attractive fixed point $x^+_{b_n\cdots b_1}$ and of the repulsing hyperplane $y^<_{b_n\cdots b_1}$. This is done in the proof of the following Lemma.

LEMMA 13.13. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that μ has a finite exponential moment and that Γ_μ is strongly irreducible. Then for all $\varepsilon > 0$, there exist $c > 0$ and $\ell_0 \geq 1$ such that for all $n \geq \ell \geq \ell_0$, one has*

$$\mu^{*n}\big(\{g \in G \mid \frac{\lambda_1(g)}{\|g\|} \geq e^{-\varepsilon\ell}\,\}\big) \;\geq\; 1 - e^{-c\ell}, \tag{13.33}$$

and, when Γ_μ is proximal,

$$\mu^{*n}(\{g \in G \mid g \text{ is proximal }\}) \;\geq\; 1 - e^{-cn}. \tag{13.34}$$

In this section we will only need Lemma 13.13 with $n = \ell$. This more general formulation with $n \geq \ell$ will be needed in Section 13.9.

We will say that *a property $P_n(\ell, b)$ is true except on an exponentially small set* if there exist $c > 0$ and $\ell_0 \geq 1$ such that, for all $n \geq \ell \geq \ell_0$, one has

$$\beta(\{b \in B \mid P_n(\ell, b) \text{ is true}\}) \;\geq\; 1 - e^{-c\ell}. \tag{13.35}$$

PROOF OF LEMMA 13.13. Let r be the proximal dimension of Γ_μ. According to Lemma 3.36, there exists a proximal and strongly irreducible representation ρ' of Γ_μ in a vector space V'_r such that, for all

g in Γ_μ, one has $\lambda_1(\rho(g)) = \lambda_1(g)^r$ and $\|\rho(g)\| \leq \|g\|^r$. Hence with no loss of generality, one can assume Γ_μ to be proximal.

We want to prove that, for all $\varepsilon > 0$, the property

$$b_n \cdots b_1 \text{ is proximal and } \frac{\lambda_1(b_n \cdots b_1)}{\|b_n \cdots b_1\|} \geq e^{-\varepsilon \ell} \tag{13.36}$$

is true except on an exponentially small set.

We keep the notations $d(x,x')$, $\delta(x,y)$, x_g^M, y_g^m, $\gamma_{12}(g)$ from Section 13.1. We fix x_0 in $\mathbb{P}(V)$, y_0 in $\mathbb{P}(V^*)$ and a very small $\epsilon > 0$ to be determined later.

We first notice that, by the Large Deviation Principle in Theorem 12.17, the following property (13.37) is true except on an exponentially small set :

$$\gamma_{1,2}(b_n \cdots b_1) \;\; \leq \;\; e^{-(\lambda_{1,\mu} - \lambda_{2,\mu} - \varepsilon)\ell}, \tag{13.37}$$

where $\lambda_{1,\mu}$ and $\lambda_{2,\mu}$ are the two first Lyapunov exponents of μ. Since Γ_μ is proximal, according to Corollary 9.15, one has $\lambda_1 > \lambda_2$.

We claim now that the following property (13.38) is true except on an exponentially small set :

$$\delta(x^M_{b_n \cdots b_1}, y^m_{b_n \cdots b_1}) \;\; \geq \;\; e^{-\varepsilon \ell}, \tag{13.38}$$

Here is a rough sketch of the proof of (13.38): we decompose the product $g = b_n \cdots b_1$ as $g = g_2 g_1$ with $g_2 = b_n \cdots b_{[n/2]+1}$ and $g_1 = b_{[n/2]} \cdots b_1$, where $[n/2]$ denotes the floor integer of $n/2$. We want to check that the density point $x^M_{g_2 g_1}$ is not too close to the density hyperplane $y^m_{g_2 g_1}$. We will check successively that the density points $x^M_{g_2 g_1}$ and $x^M_{g_2}$ are very lose (this will be Equations (13.39) and (13.40)), that the density hyperplanes $y^m_{g_2 g_1}$ and $y^m_{g_1}$ are very close (this will be Equations (13.41) and (13.42)), and that the density point $x^M_{g_2}$ is not too close to the density hyperplane $y^m_{g_1}$ (this will be Equations (13.43)). This last assertion is easier to check than the claim (13.38) since $x^M_{g_2}$ and $y^m_{g_1}$ are independant variables.

Now, here is the precise proof of (13.38). Applying twice Equation (13.6), the following properties (13.39) and (13.40) are true except on an exponentially small set :

$$d(x^M_{b_n \cdots b_1}, b_n \cdots b_1 x_0) \;\; \leq \;\; e^{-(\lambda_{1,\mu} - \lambda_{2,\mu} - \varepsilon)\ell}, \tag{13.39}$$

$$d(x^M_{b_n \cdots b_{[n/2]+1}}, b_n \cdots b_1 x_0) \;\; \leq \;\; e^{-(\lambda_{1,\mu} - \lambda_{2,\mu} - \varepsilon)\ell/2}. \tag{13.40}$$

By the same arguments in the dual space V^*, the following properties (13.41) and (13.42) are true except on an exponentially small set :

$$d(y^m_{b_n\cdots b_1}, {}^t(b_n\cdots b_1)y_0) \;\leq\; e^{-(\lambda_{1,\mu}-\lambda_{2,\mu}-\varepsilon)\ell}, \tag{13.41}$$

$$d(y^m_{b_{[n/2]}\cdots b_1}, {}^t(b_n\cdots b_1)y_0) \;\leq\; e^{-(\lambda_{1,\mu}-\lambda_{2,\mu}-\varepsilon)\ell/2}. \tag{13.42}$$

According to Equation (13.7), the following property (13.43) is also true except on an exponentially small set :

$$\delta(x^M_{b_n\cdots b_{[n/2]+1}}, y^m_{b_{[n/2]}\cdots b_1}) \;\geq\; e^{-\varepsilon\ell}, \tag{13.43}$$

These five equations imply our claim (13.38).

Finally, when ε is small enough, the two assertions (13.37) and (13.38) imply (13.34) and (13.33) because of Lemma 13.14 below. □

When g is a proximal element in $\mathrm{GL}(V)$, we will denote as in Section 3.1, by x_g^+ the attractive fixed point of g in $\mathbb{P}(V)$ and by $y_g^<$ the attractive fixed point of ${}^tg$ in $\mathbb{P}(V^*)$.

LEMMA 13.14. *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let $g \in \mathrm{GL}(V)$. Set $\gamma_0 = \gamma_{1,2}(g)$ and $\delta_0 := \delta(x_g^M, y_g^m)/2$. Assume that $\gamma_0 < \delta_0^2$. Then g is proximal and one has*

$$d(x_g^+, x_g^M) \leq \tfrac{\gamma_0}{\delta_0}\ ,\ d(y_g^<, y_g^m) \leq \tfrac{\gamma_0}{\delta_0}\ \text{ and} \tag{13.44}$$

$$\tfrac{\lambda_1(g)}{\|g\|} \geq \delta_0. \tag{13.45}$$

PROOF OF LEMMA 13.14. For $r > 0$, let

$$b_g^M(r) := \{x \in \mathbb{P}(V) \mid d(x, x_g^M) \leq r\,\},$$
$$B_g^m(r) := \{x \in \mathbb{P}(V) \mid \delta(x, y_g^m) \geq r\,\}.$$

By definition, one has $b_g^M(\delta_0) \subset B_g^m(\delta_0)$. Moreover, using the decomposition (13.4), one checks that, for any $x = \mathbb{K}v$ and $x' = \mathbb{K}v'$ in $B_g^m(\delta_0)$, the images gx and gx' belong to $b_g^M(\frac{\gamma_0}{\delta_0})$, one has

$$\tfrac{\|gv\|}{\|g\|\|v\|} \geq \delta_0\ \text{ and} \tag{13.46}$$

$$d(gx, gx') \leq \gamma_0\,\delta_0^{-2}\,d(x, x') \tag{13.47}$$

(the distance estimate (13.47) relies on the norm estimate (13.46) and the definition of the distance (12.1)).

The contraction property (13.47) implies that g has an attractive fixed point x_g^+ in the ball $b_g^M(\frac{\gamma_0}{\delta_0})$. Arguing in the same way with the action on $\mathbb{P}(V^*)$, this proves (13.44). The norm estimate (13.46) then implies the lower bound (13.45) for the spectral radius. □

Proof of Theorem 13.12. According to the Law of Large Numbers in Theorem 3.28.a, for β-almost all b in B, one has

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|b_n \cdots b_1\| = \lambda_{1,\mu}.$$

Using Lemma 13.13 with $n = \ell$ and using Borel-Cantelli Lemma, one also has, for β-almost all b in B,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \frac{\lambda_1(b_n \cdots b_1)}{\|b_n \cdots b_1\|} = 0.$$

The limit (13.32) is a direct consequence of these two equalities.

The convergence of the sequence (13.32) in $L^1(B, \beta)$ follows from Lemma 1.2. Indeed this sequence is uniformly integrable since it is dominated by the sequence $\frac{1}{n} \sum_{1\leq i\leq n} \log \|b_i\|$ which converges in $L^1(B, \beta)$. □

We give now a reformulation of Theorem 13.12 in the language of reductive groups. We use the notations of Sections 9.4 and 12.4.

Theorem 13.15. (Law of Large Numbers for the Jordan projection) *Let G be a connected algebraic reductive $\mathcal{S}$-adic Lie group, $\lambda : G \to \mathfrak{a}^+$ be the Jordan projection, and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Let σ_μ be the Lyapunov vector of μ. For β-almost all b in B, one has*

$$\lim_{n\to\infty} \tfrac{1}{n}\lambda(b_n \cdots b_1) \;=\; \sigma_\mu. \tag{13.48}$$

Moreover this sequence converges in $L^1(B, \beta, \mathfrak{a})$.

Proof of Theorem 13.15. Let (V, ρ) be an irreducible representation of G and χ be its highest weight. According to Lemma 7.17, one has the equality, for all g in G, $\log \lambda_1(\rho(g)) = \chi^\omega(\lambda(g))$. Hence, by Theorem 13.12 and Corollary 9.12, for β-almost all b in B, one has

$$\lim_{n\to\infty} \tfrac{1}{n}\chi^\omega(\lambda(b_n \cdots b_1)) \;=\; \chi^\omega(\sigma_\mu). \tag{13.49}$$

By Lemma 7.15, the dual space $\mathfrak{a}^*$ is spanned by the highest weights χ^ω of the irreducible representations of G. This proves (13.48). □

13.6. A formula for the variance.

In this section, we give a formula for the variance of the limit Gaussian law in the Central Limit Theorem.

We give first the formula for the variance in the language of matrices as it will occur in the Central Limit Theorem 13.18.

PROPOSITION 13.16. *Let* $\mathbb{K}$ *be a local field and* $V = \mathbb{K}^d$. *Let* μ *be a Borel probability measure on* $\mathrm{GL}(V)$ *such that* μ *has a finite exponential moment and that* Γ_μ *is strongly irreducible. Let* $\lambda_{1,\mu}$ *be its first Lyapunov exponent. Then the following limit exists*

$$\Phi_{1,\mu} := \lim_{n\to\infty} \frac{1}{n} \int_G (\log \|g\| - n\,\lambda_{1,\mu})^2 \,\mathrm{d}\mu^{*n}(g)\,. \tag{13.50}$$

Moreover, when Γ_μ *is proximal, the norm cocycle* $(g, \mathbb{K}v) \mapsto \log \frac{\|gv\|}{\|v\|}$ *on* $G \times \mathbb{P}(V)$ *is special and its covariance* 2*-tensor* (2.17) *is equal to* $\Phi_{1,\mu}$.

The main difference between Formula (13.50) and Formula (2.17) applied to the norm cocycle is that the quantity $\log \frac{\|gv\|}{\|v\|}$ has been replaced by $\log \|g\|$. The key point in the proof of Proposition 13.16 is to dominate the L^2-norm of the difference of these two quantities.

PROOF OF PROPOSITION 13.16. Using Lemma 3.36, one can assume Γ_μ to be proximal. The fact that the norm cocycle (3.10) on $G \times \mathbb{P}(V)$ is special follows from Proposition 10.15 applied with $F = \{1\}$. Indeed, the contraction assumption can be checked as in Lemma 12.5, and the moment assumptions (10.14) and (10.15) can be checked as in Corollary 12.4.

Let $\mathrm{d}x$ be a Borel probability measure on $\mathbb{P}(V)$ that is invariant under a maximal compact subgroup of $\mathrm{GL}(V)$. We introduce the following integrals

$$I_n := \int_G (\log \|g\| - n\,\lambda_{1,\mu})^2 \,\mathrm{d}\mu^{*n}(g)\,,$$

$$J_n := \int_{G\times\mathbb{P}(V)} (\log \tfrac{\|gv\|}{\|v\|} - n\,\lambda_{1,\mu})^2 \,\mathrm{d}\mu^{*n}(g)\,\mathrm{d}x$$

where $x = \mathbb{K}v$ sits in $\mathbb{P}(V)$. Since Γ_μ is proximal, Proposition 3.7 and Theorem 2.13 imply that, the limit

$$\Phi_{1,\mu} := \lim_{n\to\infty} \tfrac{1}{n} J_n$$

exists. On the other hand, using Lemma 13.2.*i* and Minkowski inequality, one has the bound

$$\begin{aligned}
(\sqrt{I_n} - \sqrt{J_n})^2 \;&\le\; \int_{G\times\mathbb{P}(V)} (\log \tfrac{\|gv\|}{\|g\|\|v\|})^2 \,\mathrm{d}\mu^{*n}(g)\,\mathrm{d}x \\
&\le\; \int_{G\times\mathbb{P}(V)} (\log \delta(x, y_g^m))^2 \,\mathrm{d}\mu^{*n}(g)\,\mathrm{d}x \\
&\le\; C := \sup_{y\in\mathbb{P}(V^*)} \int_{\mathbb{P}(V)} (\log \delta(x,y))^2 \,\mathrm{d}x\,.
\end{aligned}$$

Since the function $t \mapsto (\log |t|)^2$ is locally integrable on $\mathbb{K}$, this constant C which does not depend on μ is finite. In particular, one has

$$|I_n - J_n| \;\; \leq \;\; (\sqrt{C} + 2\sqrt{J_n})\sqrt{C} = O(\sqrt{n})$$

and $\lim_{n\to\infty} \frac{1}{n} I_n = \Phi_{1,\mu}$. □

We give now the formula for the variance in the language of reductive groups. We use the notations of Sections 9.4 and 12.4.

PROPOSITION 13.17. *Let G be a connected algebraic reductive $\mathcal{S}$-adic Lie group, $\kappa : G \to \mathfrak{a}^+$ be the Cartan projection, and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Let σ_μ be the Lyapunov vector of μ. Then the variance $\Phi_\mu \in S^2(\mathfrak{a})$ of the Gaussian law in the Central Limit Theorem 12.11 is given by*

$$\Phi_\mu := \lim_{n\to\infty} \frac{1}{n} \int_G (\kappa(g) - n\,\sigma_\mu)^2 \,\mathrm{d}\mu^{*n}(g)\,. \tag{13.51}$$

PROOF OF PROPOSITION 13.17. Let (V, ρ) be an irreducible representation of G and χ be its highest weight. According to Lemma 7.17, one has the equality, for all g in G, $\log \|\rho(g)\| = \chi^\omega(\kappa(g))$. Hence, by Corollary 9.12 and Proposition 13.16, the limit

$$\lim_{n\to\infty} \frac{1}{n} \int_G (\chi^\omega(\kappa(g)) - n\,\chi^\omega(\sigma_\mu))^2 \,\mathrm{d}\mu^{*n}(g)$$

exists and is the variance of the gaussian law for the central limit theorem for the variables $\log \|\rho(b_n \cdots b_1)\|$. Hence this limit is equal to $\Phi_\mu(\chi^\omega)$ where the covariance tensor Φ_μ is seen as a quadratic form on $\mathfrak{a}^*$. According to Lemma 7.15, the space $S^2\mathfrak{a}^*$ is spanned by the square $(\chi^\omega)^2$ of the highest weights of the irreducible representations of G. This proves (13.51). □

13.7. Limit laws for the norms.

We give now corollaries of the limit laws stated in Theorems 12.11 and 12.17. These corollaries are concrete formulations of the limit laws as in Introduction 0.5. We quote them here over any local field, allowing as always positive characteristic.

For $\Phi \geq 0$, we denote by N_Φ the centered Gaussian probability measure on $\mathbb{R}$ with variance Φ. i.e.

$$\begin{aligned} N_\Phi \;&:=\; \tfrac{1}{\sqrt{2\pi\Phi}} e^{-\frac{t^2}{2\Phi}}\,\mathrm{d}t \;\; \text{when } \Phi > 0\,, \\ N_\Phi \;&:=\; \delta_0 \qquad\qquad\quad\; \text{when } \Phi = 0\,. \end{aligned} \tag{13.52}$$

Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $\mathrm{GL}(V)$. We fix a norm $\|.\|$ on V. We recall that Γ_μ is the closed subsemigroup of G spanned by the support of μ and that $B := \Gamma_\mu^{\mathbb{N}^*}$ is the Bernoulli space endowed with the Bernoulli probability measure $\beta := \mu^{\otimes \mathbb{N}^*}$.

We recall that the limit $\lambda_{1,\mu} := \lim\limits_{n\to\infty} \frac{1}{n} \int_G \log \|g\| \, \mathrm{d}\mu^{*n}(g)$ exists and is called the first Lyapunov exponent of μ. We recall also from (13.50) that the limit $\Phi_{1,\mu} := \lim\limits_{n\to\infty} \frac{1}{n} \int_G (\log \|g\| - n\,\lambda_{1,\mu})^2 \, \mathrm{d}\mu^{*n}(g)$ exists when Γ_μ is strongly proximal.

Theorem 13.18. (Limit laws for $\log \|g\|$) *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $\mathrm{GL}(V)$ with a finite exponential moment such that Γ_μ is strongly irreducible.*
(i) **Central limit theorem**. *For any bounded continuous function ψ on $\mathbb{R}$, one has*

$$\int_G \psi\left(\tfrac{\log \|g\| - n\lambda_{1,\mu}}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \psi \, \mathrm{d}N_{\Phi_{1,\mu}}.$$

(ii) **Law of the iterated logarithm**. *For β-almost all b in B, the set of cluster points of the sequence*

$$\frac{\log \|b_n \cdots b_1\| - n\lambda_{1,\mu}}{\sqrt{2\,n \log\log n}}$$

is equal to the interval $[-\sqrt{\Phi_{1,\mu}}, \sqrt{\Phi_{1,\mu}}]$.
(iii) **Large deviations**. *For any $t_0 > 0$, one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{g \in G \mid \ |\log \|g\| - n\lambda_{1,\mu}| \geq nt_0\})^{\frac{1}{n}} < 1.$$

Moreover, when Γ_μ is an unbounded subsemigroup of $\mathrm{SL}(V)$ and when $\mathbb{K} = \mathbb{R}$, one has $\lambda_{1,\mu} > 0$ and $\Phi_{1,\mu} > 0$.

The assumption that Γ_μ is strongly irreducible is crucial in Theorem 13.18 as we explained in Example 12.9.4.

Proof. These statements do not depend on the choice of the norm on V. Hence we can assume that this norm is good and we can use Lemma 7.17. The statements follow then from Theorem 12.17, and, for the last statement, from Corollary 3.32 and Proposition 12.19. $\square$

Theorem 13.19. (Limit laws for $\log \|gv\|$) *Let $\mathbb{K}$ be a local field and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $\mathrm{GL}(\mathbb{K}^d)$ with a finite exponential moment such that Γ_μ is strongly irreducible. Let v in $V \smallsetminus \{0\}$.*

(i) **Central limit theorem**. *For any bounded continuous function* ψ *on* $\mathbb{R}$, *one has*

$$\int_G \psi\left(\tfrac{\log\|gv\|-n\lambda_{1,\mu}}{\sqrt{n}}\right)\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \psi \,\mathrm{d}N_{\Phi_{1,\mu}}.$$

(ii) **Law of the iterated logarithm**. *For* β*-almost all* b *in* B, *the set of cluster points of the sequence*

$$\frac{\log\|b_n\cdots b_1 v\| - n\lambda_{1,\mu}}{\sqrt{2\,n\log\log n}}$$

is equal to the interval $[-\sqrt{\Phi_{1,\mu}}, \sqrt{\Phi_{1,\mu}}]$.
(iii) **Large deviations**. *For any* $t_0 > 0$, *one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{g\in G \mid \ |\log\|gv\| - n\lambda_{1,\mu}| \geq nt_0\})^{\frac{1}{n}} < 1.$$

When Γ_μ is proximal this Theorem 13.19 may be seen as a direct consequence of the general Limit Laws in Theorem 11.1 for a cocycle over a μ-contracting action. The main issue in the proof is to explain how to get rid of the proximality assumption.

Proof of Theorem 13.19. These statements can be deduced from those in Theorem 13.18.
For (i) and (ii), this follows from Proposition 3.21.
For (iii), this follows from Proposition 13.6. □

13.8. Limit laws for the coefficients.

We explain how to deduce the Central Limit Theorem, Law of Iterated Logarithms and Large Deviation Principle for the coefficients from the analog results for the norms.

We keep the notations $\lambda_{1,\mu}$, $\Phi_{1,\mu}$, $N_{\Phi_{1,\mu}}$ from Section 13.7.

Theorem 13.20. (Limit laws for $\log|f(gv)|$) *Let* $\mathbb{K}$ *be a local field,* $V = \mathbb{K}^d$, *and* μ *be a Borel probability measure on* $\mathrm{GL}(V)$ *such that* Γ_μ *is proximal and strongly irreducible and* μ *has a finite exponential moment. Let* v *in* $V \smallsetminus \{0\}$ *and* f *in* $V^* \smallsetminus \{0\}$.
(i) **Central limit theorem**. *For any bounded continuous function* ψ *on* $\mathbb{R}$, *one has*

$$\int_G \psi\left(\tfrac{\log|f(gv)|-n\lambda_{1,\mu}}{\sqrt{n}}\right)\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \psi \,\mathrm{d}N_{\Phi_{1,\mu}}.$$

(ii) **Law of the iterated logarithm**. *For* β*-almost all* b *in* B, *the set of cluster points of the sequence*

$$\frac{\log|f(b_n\cdots b_1 v)| - n\lambda_{1,\mu}}{\sqrt{2\,n\log\log n}}$$

is equal to the interval $[-\sqrt{\Phi_{1,\mu}}, \sqrt{\Phi_{1,\mu}}]$.
(iii) **Large deviations**. *For any* $t_0 > 0$*, one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{g \in G \mid \,||f(gv)| - n\lambda_{1,\mu}| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{13.53}$$

It is plausible that the assumption that Γ_μ is proximal in Theorem 13.20 can be weakened into the assumption that Γ_μ is absolutely strongly irreducible.

Proof of Theorem 13.20. We deduce these statements from Theorem 13.19 and Lemma 13.11.

For (i) we apply Lemma 13.11 with $\ell = [\sqrt{n}]$, and we obtain

$$\mu^{*n}(\{g \in G \mid \log \tfrac{|f(gv)|}{\|f\|\|gv\|} \leq -\varepsilon\sqrt{n}\}) \xrightarrow[n\to\infty]{} 0.$$

Hence the random variables $\frac{\log \|b_n \cdots b_1 v\| - n\lambda_{1,\mu}}{\sqrt{n}}$ and $\frac{\log |f(b_n \cdots b_1 v)| - n\lambda_{1,\mu}}{\sqrt{n}}$ have the same limit in law.

For (ii), we apply Lemma 13.11 with $\ell = [\sqrt{n \log\log n}]$, and we obtain

$$\sum_{n\geq 1} \mu^{*n}(\{g \in G \mid \log \tfrac{|f(gv)|}{\|f\|\|gv\|} \leq -\varepsilon\sqrt{n\log\log n}\}) < \infty,$$

and we apply Borel Cantelli Lemma.

For (iii) we apply Lemma 13.11 with $\ell = n$, and we obtain

$$\mu^{*n}(\{g \in G \mid \log \tfrac{|f(gv)|}{\|f\|\|gv\|} \leq -\varepsilon n\}) \leq e^{-cn}.$$

This proves (13.53). □

13.9. Limit laws for the spectral radius.

We explain how to deduce the Central Limit Theorem, Law of Iterated Logarithms and Large Deviation Principle for the spectral radius from the analog results for the norms.

We keep the notations $\lambda_{1,\mu}$, $\Phi_{1,\mu}$, $N_{\Phi_{1,\mu}}$ from Section 13.7.

Theorem 13.21. (Limit laws for $\log \lambda_1(g)$) *Let* $\mathbb{K}$ *be a local field,* $V = \mathbb{K}^d$*, and* μ *be a Borel probability measure on* $\mathrm{GL}(V)$ *such that* Γ_μ *is strongly irreducible and* μ *has a finite exponential moment.*
(i) **Central limit theorem**. *For any bounded continuous function* ψ *on* $\mathbb{R}$*, one has*

$$\int_G \psi\left(\tfrac{\log \lambda_1(g) - n\lambda_{1,\mu}}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \psi \, \mathrm{d}N_{\Phi_{1,\mu}}.$$

(*ii*) **Law of the iterated logarithm**. *For β-almost all b in B, the set of cluster points of the sequence*

$$\frac{\log \lambda_1(b_n\cdots b_1) - n\lambda_{1,\mu}}{\sqrt{2\,n\log\log n}}$$

is equal to the interval $[-\sqrt{\Phi_{1,\mu}}, \sqrt{\Phi_{1,\mu}}]$.
(*iii*) **Large deviations**. *For any $t_0 > 0$, one has*

$$\limsup_{n\to\infty} \mu^{*n}(\{g \in G \mid \ |\lambda_1(g) - n\lambda_{1,\mu}| \geq nt_0\})^{\frac{1}{n}} < 1. \tag{13.54}$$

Proof of Theorem 13.21. Using Lemma 3.36, one can assume Γ_μ to be proximal. We deduce these statements from Theorem 13.18 and Lemma 13.13.

For (*i*) we apply Lemma 13.13 with $\ell = [\sqrt{n}]$, and we obtain

$$\mu^{*n}(\{g \in G \mid \log \tfrac{\lambda_1(g)}{\|g\|} \leq -\varepsilon\sqrt{n}\}) \xrightarrow[n\to\infty]{} 0.$$

Hence the random variables $\frac{\log \|b_n\cdots b_1\| - n\lambda_{1,\mu}}{\sqrt{n}}$ and $\frac{\log \lambda_1(b_n\cdots b_1) - n\lambda_{1,\mu}}{\sqrt{n}}$ have the same limit in law.

For (*ii*), we apply Lemma 13.13 with $\ell = [\sqrt{n\log\log n}]$, and we obtain

$$\sum_{n\geq 1} \mu^{*n}(\{g \in G \mid \log \tfrac{\lambda_1(g)}{\|g\|} \leq -\varepsilon\sqrt{n\log\log n}\}) < \infty,$$

and we apply Borel Cantelli Lemma.

For (*iii*) we apply Lemma 13.13 with $\ell = n$, and we obtain

$$\mu^{*n}(\{g \in G \mid \log \tfrac{\lambda_1(g)}{\|g\|} \leq -\varepsilon n\}) \leq e^{-cn}.$$

This proves (13.54). □

When we reformulate Theorem 13.21 in the language of reductive groups we obtain the following limit laws for the Jordan projection. We keep the notations σ_μ, Φ_μ, N_μ, K_μ of Sections 12.6.

Theorem 13.22. (Limit laws for $\lambda(g)$) *Let G be a connected algebraic reductive $\mathcal{S}$-adic Lie group, $\lambda : G \to \mathfrak{a}^+$ be the Jordan projection, and μ be a Zariski dense Borel probability measure on G with a finite exponential moment.*
(*i*) **Central limit theorem**. *For any bounded continuous function ψ on $\mathfrak{a}$,*

$$\int_G \psi\left(\tfrac{\lambda(g) - n\sigma_\mu}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathfrak{a}} \psi \,\mathrm{d}N_\mu.$$

(ii) **Law of the iterated logarithm**. *Let K_μ be the unit ball of Φ_μ. For β-almost any b in B, the following set of cluster points is equal to K_μ*

$$C\left(\frac{\lambda(b_n\cdots b_1)-n\sigma_\mu}{\sqrt{2n\log\log n}}\right)=K_\mu. \tag{13.55}$$

(iii) **Large deviations**. *For any $t_0>0$, one has*

$$\limsup_{n\to\infty}\mu^{*n}(\{g\in G \mid \|\lambda(g)-n\sigma_\mu\|\geq nt_0\})^{\frac{1}{n}}<1. \tag{13.56}$$

Proof of Theorem 13.22. This follows from the limit laws for the Cartan projection in Theorem 12.17 and the following comparison Lemma 13.23, in the same way as we deduced Theorem 13.21. from the limit laws for the norm in Theorem 13.18 and the comparison Lemma 13.13. □

Lemma 13.23. *Let G be a connected algebraic reductive $\mathcal{S}$-adic Lie group, κ and λ be the Cartan and Jordan projection, and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Then for all $\varepsilon>0$, there exist $c>0$ and $\ell_0\geq 1$ such that for all $n\geq\ell\geq\ell_0$, one has*

$$\mu^{*n}(\{g\in G \mid \|\kappa(g)-\lambda(g)\|\geq\varepsilon\ell\})\quad\leq\quad e^{-c\ell}. \tag{13.57}$$

Proof of Lemma 13.23. This follows from Lemma 13.13 using sufficiently many irreducible representations of G as in the proof of Theorem 13.15. □

Part 4

Local Limit Theorem

14. Spectrum of the complex transfer operator

We come back in this chapter and the next one to the abstract framework of Chapters 10 and 11, studying the cocycles over a μ-contrating action. The proofs of the three limit theorems discussed in Chapter 11, were based on spectral properties of the complex transfer operator P_θ for small values of the parameter θ discussed in Chapter 10.

We study in this chapter the spectral properties of P_θ for all pure imaginary values of the parameter θ. We will use these properties in Chapter 15 to prove a local limit theorem for cocycles.

14.1. The essential spectral radius of $P_{i\theta}$.

We first show that the spectral radius of the transfer operator $P_{i\theta}$ is strictly less than 1 except if $P_{i\theta}$ has eigenvalues of modulus 1.

The following lemma is an extension of Corollary 10.11. In this lemma, the assumptions are the same as in Proposition 10.15.

LEMMA 14.1. *Let G be a second countable locally compact semigroup and $s : G \to F$ be a continuous morphism onto a finite group F. Let μ be a Borel probability measure on G such that μ spans F. Let $0 < \gamma \le \gamma_0$ and let X be a compact metric G-space which is fibered over F and (μ, γ_0)-contracting over F.*

Let $\sigma : G \times X \to E$ be a continuous cocycle whose sup-norm has a finite exponential moment (10.14) *and whose Lipschitz constant has a finite moment* (10.15).

Then, there exists γ_0 in $(0, 1]$ such that, for $0 < \gamma \le \gamma_0$, there exists $\delta \in (0, 1)$ such that, for any θ in E^*, the operator $P_{i\theta}$ has spectral radius ≤ 1 and essential spectral radius $\le \delta$ in $\mathcal{H}^\gamma(X)$.

PROOF. We fix $0 < \gamma \le \gamma_0$ where γ_0 is as in Definition 10.1. According to Ionescu-Tulcea-Marinescu Theorem 2.26 and to Lemma 2.13 in Appendix 2, it is enough to check that there exists $\delta \in (0, 1)$, $C > 0$ such that for any $n \ge 1$, there exists $C_n > 0$ with, for every $\varphi \in \mathcal{H}^\gamma(X)$,

$$\|P_{i\theta}^n \varphi\|_\gamma \le C\delta^n \|\varphi\|_\gamma + C_n \|\varphi\|_\infty. \tag{14.1}$$

We recall that the complex transfer operator $P_{i\theta}$ is defined by

$$P_{i\theta}\varphi(x) = \textstyle\int_G e^{i\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu(g) \tag{14.2}$$

and that its powers are given by

$$P_{i\theta}^n\varphi(x) = \int_G e^{i\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu^{*n}(g).$$

In particular, one has

$$\|P_{i\theta}^n \varphi\|_\infty \leq \|\varphi\|_\infty.$$

It remains to bound, for $x \neq x'$ in X with $f_x = f_{x'}$:

$$\tfrac{P_{i\theta}^n\varphi(x)-P_{i\theta}^n\varphi(x')}{d(x,x')^\gamma} = A_n + B_n \text{ where}$$

$$A_n = \int_H \tfrac{e^{i\theta(\sigma(g,x))}-e^{i\theta(\sigma(g,x'))}}{d(x,x')^\gamma}\, \varphi(gx)\, \mathrm{d}\mu^{*n}(g)$$
$$B_n = \int_H e^{i\theta(\sigma(g,x'))}\, \tfrac{\varphi(gx)-\varphi(gx')}{d(x,x')^\gamma}\, \mathrm{d}\mu^{*n}(g).$$

In order to bound A_n, we compute, using (10.19), for g in G and $x \neq x'$ in X with $f_x = f_{x'}$,

$$\begin{aligned}\left|e^{i\theta(\sigma(g,x))} - e^{i\theta(\sigma(g,x'))}\right| &\leq 2^{1-\gamma}\left|e^{i\theta(\sigma(g,x))} - e^{i\theta(\sigma(g,x'))}\right|^\gamma \\ &\leq 2^{1-\gamma}\|\theta\|^\gamma\|\sigma(g,x)-\sigma(g,x')\|^\gamma \\ &\leq 2^{1-\gamma}\|\theta\|^\gamma e^{\gamma\kappa_0(g)} d(x,x')^\gamma.\end{aligned}$$

Hence one gets, using (10.21),

$$|A_n| \leq C_n'\|\varphi\|_\infty \quad \text{with} \quad C_n' = 2^{1-\gamma}\|\theta\|^\gamma \int_G e^{\gamma\kappa_0(g)}\, \mathrm{d}\mu^{*n}(g) < \infty.$$

In order to bound B_n, we use the contraction property under the form (10.3), and we get, for some $\delta \in (0,1)$ and $C > 0$,

$$|B_n| \leq c_\gamma(\varphi) \int_G \tfrac{d(gx,gx')^\gamma}{d(x,x')^\gamma}\, \mathrm{d}\mu^{*n}(g) \leq C\delta^n c_\gamma(\varphi).$$

This proves (14.1) with $C_n = C_n' + 1$ □

As a direct corollary of Lemma 14.1, one gets.

Corollary 14.2. *Same assumptions as in Lemma 14.1 For any θ in E^*, the complex transfer operator $P_{i\theta}$ has spectral radius* 1 *in $\mathcal{H}^\gamma(X)$ if and only if it has an eigenvalue of modulus* 1*.*

14.2. Eigenvalues of modulus 1 of $P_{i\theta}$.

We study now the eigenspaces in $\mathcal{H}^\gamma(X)$ of the transfer operator $P_{i\theta}$ associated to the eigenvalues of modulus 1.

The following lemma tells us that these eigenspaces are obtained by solving a cohomological equation on S_ν and that the measurable and integrable solutions of this cohomological equation are automatically Hölder regular.

Let $S_\nu \subset X$ denote the support of the unique μ-stationary Borel probability measure ν on X (see Proposition 10.10). Let $p_\mu = |F/F_\mu|$.

Lemma 14.3. *Same assumptions as in Lemma 14.1. Let $\theta \in E^*$ and $u \in \mathbb{C}$ with $|u| = 1$.*
a) Let $\varphi \in \mathcal{H}^\gamma(X)$ be an eigenfunction of $P_{i\theta}$ with eigenvalue u, i.e. a function satisfying $P_{i\theta}\varphi = u\varphi$. Then the function $|\varphi|$ is constant on S_ν with value $\|\varphi\|_\infty$ and, for any (g, x) in $\mathrm{supp}(\mu) \times S_\nu$, one has

$$\varphi(gx) = ue^{-i\theta(\sigma(g,x))}\varphi(x). \tag{14.3}$$

Moreover, for any $\mathrm{p}_\mu^{\mathrm{th}}$-root of unity ζ, the function $\chi_\zeta\varphi$ is an eigenfunction of $P_{i\theta}$ with eigenvalue ζu.
b) Conversely, if there exists a nonzero function φ in $\mathrm{L}^1(X, \nu)$ satisfying (14.3) for $\mu\otimes\nu$-almost any (g, x) in $G \times X$, then u is an eigenvalue of $P_{i\theta}$ in $\mathcal{H}^\gamma(X)$ and φ is ν-almost surely equal to an eigenfunction of $P_{i\theta}$ in $\mathcal{H}^\gamma(X)$.
c) In this case, the eigenvalues of $P_{i\theta}$ of modulus 1 are exactly the ζu, where ζ is a $\mathrm{p}_\mu^{\mathrm{th}}$ root of 1. For any such ζ, the corresponding eigenspace has dimension 1 and is generated by $\chi_\zeta\varphi$.
d) In particular, if μ is aperiodic in F, $P_{i\theta}$ has at most one eigenvalue of modulus 1.

Remark 14.4. When G is an algebraic semisimple real Lie group, μ a Zariski dense probability measure on G, X the flag variety and σ the Iwasawa cocycle, we will see in Proposition 16.1 that, for every nonzero $\theta \in \mathfrak{a}^*$, the operator $P_{i\theta}$ has no eigenvalue of modulus 1.

When G is an algebraic semisimple p-adic Lie group, X the flag variety and σ the Iwasawa cocycle, there always exists a Zariski dense probability measure μ on G with finite support such that, for every $\theta \in \mathfrak{a}^*$, the operator $P_{i\theta}$ has an eigenvalue $\lambda_{i\theta}$ of modulus 1. For instance, when $G = \mathrm{SL}(2, \mathbb{Q}_p)$ and $\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2})$ is the probability given in Example 12.21.

Note that in this example 12.21, when $\theta(\sigma_\mu) \notin 2\pi\mathbb{Z}$, the eigenfunction associated to the eigenvalue of modulus 1 of $P_{i\theta}$ in $\mathcal{H}^\gamma(X)$ does not have constant modulus and does not satisfy (14.3) on the whole variety $X = \mathbb{P}^1(\mathbb{Q}_p)$. The reason is that the functions $x \mapsto e^{i\theta(\sigma(g_1,x))}$ and $x \mapsto e^{i\theta(\sigma(g_2,x))}$ are equal on the support S_ν but not on the whole variety X.

Proof of Lemma 14.3 . a) By assumption, for any $x \in X$, one has

$$u\varphi(x) = \textstyle\int_G e^{i\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu(g) \tag{14.4}$$

Taking moduli in this equation, we get

$$|\varphi| \leq P\,|\varphi|\,, \tag{14.5}$$

thus, for any n in $\mathbb{N}$, one has $|\varphi| \leq P^n |\varphi|$. By Proposition 10.10, we have the convergence in $\mathcal{H}^\gamma(X)$, $P^{np_\mu} |\varphi| \xrightarrow[n\to\infty]{} N|\varphi|$, and therefore

$$|\varphi| \leq N |\varphi| ,$$

i.e. for any x in X,

$$|\varphi(x)| \leq p_\mu \int_{\{f_{x'} \in f_x F_\mu\}} |\varphi(x')| \, \mathrm{d}\nu(x').$$

Hence, for any f in F, the function $|\varphi|$ is constant on the set $\{x \in S_\nu \mid f_x \in fF_\mu\}$. Denoting by C_{fF_μ} the value of this constant, Equation (14.5) becomes

$$C_{fF_\mu} \leq C_{f_\mu f F_\mu} \ , \ \text{ for any } f \text{ in } F.$$

Therefore this inequality is an equality and the function $|\varphi|$ is equal to a constant C on S_ν. As, everywhere on X, one has $|\varphi| \leq N|\varphi| = C$, this constant value is

$$C = \|\varphi\|_\infty .$$

Moreover, if x belongs to S_ν, the left-hand side of (14.4) has modulus $\|\varphi\|_\infty$, so that, for μ-almost any g in G,

$$u\varphi(x) = e^{i\theta(\sigma(g,x))}\varphi(gx),$$

which proves (14.3).

Finally, since one has

$$\chi_\zeta(gx) = \zeta\, \chi_\zeta(x) \ , \ \text{ for } \mu\text{-almost all } g \text{ in } G \text{ and all } x \text{ in } X,$$

one gets $P_{i\theta}(\chi_\zeta\varphi) = \zeta\chi_\zeta\varphi$ as required.

b) We first remark that, since ν is μ-stationary, Formula (14.2) defines a continuous operator $P_{i\theta}$ of $\mathrm{L}^1(X,\nu)$ with norm at most 1. By Equation (14.3), the function φ is an eigenvector in $\mathrm{L}^1(X,\nu)$ for this operator $P_{i\theta}$.

We claim that, then, the operator $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(X)$. Indeed, if this is not the case, for any ψ in $\mathcal{H}^\gamma(X)$, one has

$$P_{i\theta}^n\psi \xrightarrow[n\to\infty]{} 0 \ \text{ in } \ \mathcal{H}^\gamma(X),$$

therefore, by density, for any ψ in $\mathrm{L}^1(X,\nu)$, one has

$$P_{i\theta}^n\psi \xrightarrow[n\to\infty]{} 0 \ \text{ in } \ \mathrm{L}^1(X,\nu),$$

which contradicts the existence of the eigenvector φ. Thus, $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(X)$ and hence, by Lemma 14.1, it admits an eigenvector φ' associated to an eigenvalue u' with modulus 1.

We claim that the ratio $\zeta := u/u'$ is a p_μ^{th} root of unity. Indeed, since φ' has constant modulus on S_ν, the function $\varphi'' = \varphi/\varphi'$ is in $\mathrm{L}^1(X,\nu)$ and, by 14.3, for $(\mu\otimes\nu)$-almost any (g,x) in $G\times X$, one has

$$\varphi''(gx) = \zeta\,\varphi''(x).$$

This means that φ'' is an eigenvector of P in $\mathrm{L}^1(X,\nu)$ with eigenvalue ζ. Now, Formula (10.12) defines a continuous operator N of the space $\mathrm{L}^1(X,\nu)$. By Proposition 10.10, for any ψ in $\mathcal{H}^\gamma(X)$, one has

$$P^{np_\mu}\psi \xrightarrow[n\to\infty]{} N\psi \ \text{ in } \ \mathcal{H}^\gamma(X),$$

therefore, by density, for any ψ in $\mathrm{L}^1(X,\nu)$, one has

$$P^{np_\mu}\psi \xrightarrow[n\to\infty]{} N\psi \ \text{ in } \ \mathrm{L}^1(X,\nu).$$

Since $P^n\varphi'' = \zeta^n\varphi''$, we get $\zeta^{p_\mu} = 1$, $\varphi'' = N\varphi''$ and φ'' is ν-almost surely equal to a multiple of χ_ζ, which was to be shown.

c) and d) follow from the previous discussion. □

Remark 14.5. The operator $P_{i\theta}$ is also a bounded operator in the space $\mathrm{L}^\infty(X,\nu)$ with norm at most 1. As a consequence of this proof $P_{i\theta}$ has the same eigenvalues of modulus 1 in each of the Banach spaces $\mathcal{H}^\gamma(X)$, $\mathcal{C}^0(X)$, $\mathcal{H}^\gamma(S_\nu)$, $\mathcal{C}^0(S_\nu)$, $\mathrm{L}^\infty(X,\nu)$ and $\mathrm{L}^1(X,\nu)$.

The following corollary tells us that, when θ is in $E_\mu^\perp$, the associated eigenfunctions can easily been described.

Corollary 14.6. *Same assumptions as in Lemma 14.1. Let $\sigma_\mu\in E$, $E_\mu\subset E$ and $\dot\varphi_0\in\mathcal{H}^\gamma(X)$ be as in Lemma 10.18. For any θ in $E_\mu^\perp$, the operator $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(X)$. Its eigenvalues of modulus 1 are the $\zeta e^{i\theta(\sigma_\mu)}$, where ζ is a p_μ^{th} root of 1, and the restriction of the associated eigenfunctions to S_ν are multiples of the function $x\mapsto \chi_\zeta(x)e^{i\theta(\dot\varphi_0(x))}$.*

Proof. According to Formula (10.29), for $(\mu\otimes\nu)$-almost any (g,x) in $G\times X$, one has

$$\sigma(g,x) = \sigma_\mu - \dot\varphi_0(gx) + \dot\varphi_0(x) \mod E_\mu.$$

Hence, when $\theta\in E^*$ is orthogonal to E_μ, the function $\varphi : x\mapsto e^{i\theta(\dot\varphi_0(x))}$ satisfies, for $(\mu\otimes\nu)$-almost any (g,x) in $G\times X$,

$$\varphi(gx) = e^{i\theta(\sigma_\mu)}e^{-i\theta(\sigma(g,x))}\varphi(x).$$

which is Equation (14.3) with $u = e^{i\theta(\sigma_\mu)}$. Our claim follows from Lemma 14.3. □

For technical reasons, when studying the Iwasawa cocycle of reductive $\mathcal{S}$-adic Lie groups that have both real and non-archimedean components, in the proof of Proposition 16.4, we will need the following.

COROLLARY 14.7. *Same assumptions as in Lemma 14.1. Assume moreover that* Y *is another compact metric* G*-space, which is fibered over* F *and* μ*-contracting over* F*, and that* $\pi : Y \to X$ *is a* G*-equivariant continuous map such that* $f_{\pi(y)} = f_y$ *for any* y *in* Y*. We also denote by* σ *the lifted cocycle on* $G \times Y$*. Then, for any* θ *in* E^**, the operator* $P_{i\theta}$ *has spectral radius* 1 *in* $\mathcal{H}^\gamma(Y)$ *if and only if it has spectral radius* 1 *in* $\mathcal{H}^\gamma(X)$*.*

PROOF. Assume $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(X)$. By Lemma 14.1, it has an eigenfunction $\varphi \in \mathcal{H}^\gamma(X)$ associated to an eigenvalue of modulus 1. Then the function $\psi = \varphi \circ \pi \in \mathcal{C}^0(Y)$ is an eigenfunction of $P_{i\theta}$ for the same eigenvalue. Hence by Lemma 14.3, $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(Y)$.

Conversely, assume $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(Y)$. For any ψ in $H^\gamma(Y)$, set

$$p(\psi) = \sup_{\pi(y)=\pi(y')} |\psi(y) - \psi(y')| \,,$$

where the supremum is taken over the pairs y, y' in Y with $\pi(y) = \pi(y')$. Since σ is constant on the fibers of π, using the contraction property as in (10.10), for any n, one has

$$p(P_{i\theta}^n \psi) \le \delta^n c_\gamma(\psi) C, \tag{14.6}$$

for some fixed $C > 0$.

According to Lemma 14.1, $P_{i\theta}$ has an eigenfunction $\psi \in \mathcal{H}^\gamma(Y)$ associated to an eigenvalue of modulus 1. Hence, by (14.6), one has

$$p(\psi) = \lim_{n\to\infty} p(P_{i\theta}^n \psi) = 0.$$

This means that there exists a function φ in $\mathcal{C}^0(X)$ such that $\psi = \varphi \circ \pi$. This function φ is an eigenfunction of $P_{i\theta}$ for the same eigenvalue. Hence by Lemma 14.3, $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(X)$. □

14.3. The residual image Δ_μ of the cocycle.

We introduce in this section a subgroup Δ_μ of E called the μ-residual image of the cocycle σ. This group is important since it preserves the limit measure that will occur in the Local Limit Theorem 15.1.

We will give two definitions of Δ_μ. The first one in Proposition 14.8 describes Δ_μ as the orthogonal of the set of parameters θ for which the complex transfer operator $P_{i\theta}$ has spectral radius 1 in $\mathcal{H}^\gamma(X)$. The second one in Corollary 14.10 describes Δ_μ as the smallest subgroup for which one can find a cocycle cohomologous to σ with values in a translate of Δ_μ.

We keep the notations that have been introduced along Chapter 10 and Section 14.1. We keep also the assumptions of Lemma 14.1. As the cocycle σ may be cohomologous to a cocycle taking values in a coset of a proper subgroup of E_μ, before stating the main result of this chapter, we must proceed to some reductions of σ.

When Δ is a closed subgroup of E, we let $\Delta^\perp$ be the subgroup of E^* consisting of those θ in E^* with $\theta(v) \in 2\pi\mathbb{Z}$, for any v in Δ. Here are a few basic properties of $\Delta^\perp$.
(i) One has $\Delta^{\perp\perp} = \Delta$.
(ii) Δ is connected $\Longleftrightarrow$ $\Delta^\perp$ is connected. In this case both Δ and $\Delta^\perp$ are vector spaces and $\Delta^\perp$ is the usual orthogonal subspace of Δ in E^*.
(iii) Δ is discrete $\Longleftrightarrow$ $\Delta^\perp$ is compact.
(iv) The map that sends some θ in E^* to the character $v \mapsto e^{i\theta(v)}$ of Δ identifies $E^*/\Delta^\perp$ with the dual group of Δ.

According to Lemma 14.1, for θ in E^*, the operator $P_{i\theta}$ has spectral radius ≤ 1 in $\mathcal{H}^\gamma(X)$. The next lemma describes the set of θ such that it has spectral radius exactly 1.

PROPOSITION 14.8. *Same assumptions as in Lemma 14.1.*
a) The set

$$\Lambda_\mu := \{\theta \in E^* \mid P_{i\theta} \text{ has spectral radius 1 in } \mathcal{H}^\gamma(X)\}$$

is a closed subgroup of E^* *whose connected component is* $E_\mu^\perp$.
b) Its dual group $\Delta_\mu := \Lambda_\mu^\perp$ *is a closed cocompact subgroup of* E_μ.
c) If moreover μ *is aperiodic in* F *i.e.* $p_\mu = 1$, *then there exists an element* v_μ *of* E_μ *and a Hölder continuous function* $\overline{\varphi}_0 : S_\nu \to E/\Delta_\mu$ *such that, for any* (g,x) *in* Supp$\mu \times S_\nu$, *one has*

$$\sigma(g,x) = \sigma_\mu + v_\mu - \overline{\varphi}_0(gx) + \overline{\varphi}_0(x) \mod \Delta_\mu. \tag{14.7}$$

The group Δ_μ is called the *μ-residual image of the cocycle σ*. This notion is different from the essential image of a cocycle in [**111**]. The cocycle σ is said to be *non-degenerate* if $E_\mu = E$. It is said to be *aperiodic* if

$$\Delta_\mu = E. \tag{14.8}$$

REMARK 14.9. Equation (14.7) gives a reduction of the cocycle σ to a smaller subgroup than Equation (10.29).

PROOF OF PROPOSITION 14.8. *a*) According to Lemma 14.3, an element $\theta \in E^*$ belongs to Λ_μ if and only if there exist a function $\varphi_{i\theta} \in \mathcal{H}^\gamma(S_\nu)$ of modulus 1 and $\lambda_{i\theta} \in \mathbb{C}$ with $|\lambda_{i\theta}| = 1$ such that for any (g, x) in $\mathrm{supp}(\mu) \times S_\nu$, one has

$$\varphi_{i\theta}(gx) = \lambda_{i\theta} e^{-i\theta(\sigma(g,x))} \varphi_{i\theta}(x).$$

Now, take θ, θ' in Λ_μ and set $\theta'' = \theta - \theta'$. The ratio $\lambda_{i\theta''} := \lambda_{i\theta}/\lambda_{i\theta'}$ of the eigenvalues and the ratio $\varphi_{i\theta''} := \varphi_{i\theta}/\varphi_{i\theta'}$ of the corresponding eigenfunctions satisfy

$$\varphi_{i\theta''}(gx) = \lambda_{i\theta''} e^{-i\theta''(\sigma(g,x))} \varphi_{i\theta''}(x),$$

for any (g, x) in $\mathrm{supp}(\mu) \times S_\nu$. Hence $\theta - \theta'$ belongs also to Λ_μ and Λ_μ is a group. According to Corollary 14.6 and Lemma 10.18, the group Λ_μ contains the vector space $E_\mu^\perp$ as an open subgroup. In particular the quotient group $\Lambda_\mu/E_\mu^\perp$ is discrete in $E^*/E_\mu^\perp$. This proves that the group Λ_μ is closed in E^* and that its connected component is $E_\mu^\perp$.

b) By duality, since $\Delta_\mu^\perp$ contains $E_\mu^\perp$, the group Δ_μ is included in E_μ. Moreover since $\Delta_\mu^\perp/E_\mu^\perp$ is discrete, the quotient E_μ/Δ_μ is compact.

c) We assume now that μ is aperiodic in F i.e. $p_\mu = 1$. By Lemma 14.3, for any θ in Λ_μ, the eigenvalue $\lambda_{i\theta}$ of modulus 1 of $P_{i\theta}$ is uniquely determined by θ. By the above construction, for any θ, θ' in Λ_μ, one has

$$\lambda_{i\theta+i\theta'} = \lambda_{i\theta}\lambda_{i\theta'}$$

and $\theta \mapsto \lambda_{i\theta}$ is a character of the group Λ_μ whose restriction to $E_\mu^\perp$ is, according to Corollary 14.6, given by $\theta \mapsto e^{i\theta(\sigma_\mu)}$. Hence there exists an element v_μ of E_μ such that

$$\lambda_{i\theta} = e^{i\theta(\sigma_\mu + v_\mu)} \text{ for any } \theta \text{ in } \Lambda_\mu.$$

Fix x_0 in S_ν. By Lemma 14.3, for any θ in Λ_μ, there exists a unique eigenfunction $\varphi_{i\theta} \in \mathcal{H}^\gamma(X)$ of $P_{i\theta}$ such that $\varphi_{i\theta}(x_0) = 1$. For any (g, x) in $\mathrm{supp}(\mu) \times S_\nu$, one has

$$\varphi_{i\theta}(gx) = e^{i\theta(\sigma_\mu+v_\mu)} e^{-i\theta(\sigma(g,x))} \varphi_{i\theta}(x) \quad \text{and} \quad |\varphi_{i\theta}(x)| = 1. \tag{14.9}$$

By the above construction, for any θ, θ' in Λ_μ and x in S_ν, one has

$$\varphi_{i\theta+i\theta'}(x) = \varphi_{i\theta}(x)\, \varphi_{i\theta'}(x).$$

Hence, for any x in S_ν, there exists a unique element $\overline{\varphi}_0(x)$ in E/Δ_μ such that

$$\varphi_{i\theta}(x) = e^{i\theta(\overline{\varphi}_0(x))}$$

Using (14.9), one gets, for any (g,x) in $\mathrm{supp}(\mu)\times S_\nu$,

$$\overline{\varphi}_0(gx)=\sigma_\mu+v_\mu-\sigma(g,x)+\overline{\varphi}_0(x)\ \text{ in } E/\Delta_\mu$$

as required. □

The following corollary explains why this group Δ_μ is called the μ-residual image of σ: it tells us that Δ_μ, is the smallest closed subgroup Δ of E for which there exists a cocycle cohomologous to σ taking almost surely its values in a translate of Δ. It tells us also that the decomposition (14.7) is unique.

Corollary 14.10. *Same assumptions as in Lemma 14.1. Suppose μ is aperiodic in F. Let Δ be a closed subgroup of E, v be an element of E/Δ and $\varphi : S_\nu \to E/\Delta$ be a continuous function such that, for $\mu\otimes\nu$ every (g,x) in $G\times X$, one has*

$$\sigma(g,x)=\sigma_\mu+v-\varphi(gx)+\varphi(x)\mod \Delta.$$

Then, one has $\Delta\supset\Delta_\mu$, $v\in v_\mu+\Delta$ and the function φ is equal to $\overline{\varphi}_0+\Delta$ up to a constant.

Proof. Let θ be in $\Delta^\perp$. By construction, for $\mu\otimes\nu$ every (g,x) in $G\times X$, one has

$$e^{i\theta(\varphi(gx))}=e^{i\theta(\sigma_\mu+v)}e^{-i\theta(\sigma(g,x))}e^{i\theta(\varphi(x))},$$

so that, by Lemma 14.3, θ belongs to Λ_μ. We get $\Lambda_\mu\supset\Delta^\perp$, which amounts to $\Delta_\mu\subset\Delta$.

We combine our assumption with (14.7). To simplify notations, we still denote by v, v_μ and $\overline{\varphi}_0$ the images of these quantities in E/Δ. For every x in S_ν, for any $n\geq 1$, for μ^{*n}-every g in G, we get, in E/Δ,

$$(\overline{\varphi}_0-\varphi)(gx)=n(v_\mu-v)+(\overline{\varphi}_0-\varphi)(x), \tag{14.10}$$

hence, if y is another point of S_ν,

$$(\overline{\varphi}_0-\varphi)(gx)-(\overline{\varphi}_0-\varphi)(gy)=(\overline{\varphi}_0-\varphi)(x)-(\overline{\varphi}_0-\varphi)(y). \tag{14.11}$$

Now, by Lemma 10.5, if $f_x=f_y$, for β-almost any b in B, one has $d(b_n\cdots b_1x,b_n\cdots b_1y)\xrightarrow[n\to\infty]{}0$ and hence, in E/Δ, by (14.11),

$$\overline{\varphi}_0(x)-\varphi(x)=\overline{\varphi}_0(y)-\varphi(y),$$

that is, there exists $\psi:F\to E/\Delta$ such that, for x in S_ν,

$$\overline{\varphi}_0(x)-\varphi(x)=\psi(f_x).$$

Now, (14.10) gives, for μ-almost any g in G, for all f in F,

$$\psi(s(g)f)=v_\mu-v+\psi(f).$$

Thus, if θ belongs to $\Delta^\perp$, the function $f \mapsto e^{i\theta(\psi(f))}$ is an eigenvector of P in $\mathbb{C}^F$ associated to the eigenvalue $e^{i\theta(v_\mu - v)}$ of modulus 1. Since we assumed μ to be aperiodic, by Lemma 10.6, $\theta \circ \psi$ is constant and $\theta(v - v_\mu) \in 2\pi\mathbb{Z}$. As this is true for any θ, we get that $\varphi - \overline{\varphi}_0$ is constant mod Δ and $v = v_\mu$ mod Δ as required. □

REMARK 14.11. By Corollary 14.6, when θ belongs to $E_\mu^\perp$, the eigenfunction $\varphi_{i\theta}$ of $P_{i\theta}$ is given by, for any x in S_ν,

$$\varphi_{i\theta}(x) = e^{i\theta(\dot\varphi_0(x) - \dot\varphi_0(x_0))}.$$

Hence, by Corollary 14.10, one has

$$\overline{\varphi}_0(x) = \dot\varphi_0(x) - \dot\varphi_0(x_0) \mod E_\mu.$$

In the application in Chapter 16 where X is the flag variety of a reductive group, the following consequence of Corollary 14.10, which is similar to Corollary 11.4, will be useful.

COROLLARY 14.12. (F-invariance) *Same assumptions as in Proposition 14.8. We assume moreover that E is equipped with a linear action of the finite group F and that X is equipped with a continuous right action of F which commutes with the action of G and that, for all f in F, the cocycles $(g, x) \mapsto \sigma(g, xf)$ and $(g, x) \mapsto f^{-1}\sigma(g, x)$ are cohomologous. Then*
a) The subgroups Λ_μ and Δ_μ are stable by F.
b) The image of v_μ in E_μ/Δ_μ is F-invariant.

REMARK 14.13. The element $v_\mu \in E_\mu$ cannot always be chosen to be F-invariant.

For example, let F be a finite group which acts on a finite-dimensional real vector space E. We set $G = F \ltimes E$ and $X = G/E = F$. We define a function $\sigma : G \times F \to E$ by setting, for $g = fv$ in G and x in F, $\sigma(g, x) = x^{-1}v$ where x is viewed as an element of F which acts on E. One easily checks that σ is a F-equivariant cocycle. Now assume, for example, $E = \mathbb{R}$ and $F = \mathbb{Z}/2\mathbb{Z} = \{1, \varepsilon\}$ acts on $\mathbb{R}$ by multiplication by -1. We let μ be the probability measure on G given by $\mu = \frac{1}{2}(\delta_{\frac{1}{2}} + \delta_{\varepsilon\frac{1}{2}})$. Then one checks that $\sigma_\mu = 0$, $\Delta_\mu = \mathbb{Z}$ and $v_\mu = \frac{1}{2} + \mathbb{Z}$ whereas $\mathbb{R}$ does not admit any nonzero F-invariant element.

15. Local limit theorem for cocycles

Using the spectral properties of the complex transfer operator proven in Chapter 14, we prove now a local limit theorem with moderate deviations for cocycles over a μ-contracting action. This theorem is an extension of the local limit theorem of Breuillard in [**30**, Théorème 4.2] for classical random walks on the line.

15.1. Local limit theorem.

In this section we state the local limit theorem (Theorem 15.1) for the cocycle σ. It will be deduced from a local limit theorem with target (Proposition 15.6) for a cocycle $\widetilde{\sigma}$ taking values in a translate of the μ-residual image Δ_μ of σ.

We keep the assumptions and notations of Proposition 14.8. Let ν be the unique μ-stationary Borel probability measure on X (see Proposition 10.10). Let σ_μ be the average of σ given by Formula (2.14). Since by Proposition 10.15 the cocycle σ is special, we can introduce the covariance 2-tensor Φ_μ which is given by Formulas (2.16) and (2.17). Let $E_\mu \subset E$ be the linear span of Φ_μ.

For $n \geq 1$ and $x \in S_\nu$, we want to understand the behavior of the measure $\mu_{n,x}$ on E given by, for every $\psi \in \mathcal{C}_c(E)$,

$$\mu_{n,x}(\psi) = \int_G \psi(\sigma(g,x) - n\sigma_\mu)\, \mathrm{d}\mu^{*n}(g) \tag{15.1}$$

i.e. we want to compute the rate of decay of the probability that the recentered variable $\sigma(g_n \cdots g_1, x) - n\sigma_\mu$ belongs to a fixed convex set C. To emphasize its role, this convex set C is often called a *window*.

We first define precisely the renormalization factor G_n and the limit measure Π_μ that occur in the statement of the Local Limit Theorem 15.1.

As in (11.1) we introduce the Lebesgue measure $\mathrm{d}v$ on E_μ that gives mass one to the unit cubes of Φ_μ^*. For $n \geq 1$, we denote by G_n the density of the Gaussian law N_μ^{*n} on E_μ with respect to $\mathrm{d}v$,

$$G_n(v) = (2\pi n)^{-\frac{e_\mu}{2}} e^{-\frac{1}{2n}\Phi_\mu^*(v)}, \text{ for all } v \text{ in } E_\mu, \tag{15.2}$$

where $e_\mu := \dim E_\mu$ and Φ_μ^* is the positive definite quadratic form on E_μ that is dual to Φ_μ.

Let Λ_μ be the group of elements θ in E^* such that $P_{i\theta}$ has spectral radius 1 and $\Delta_\mu = \Lambda_\mu^\perp$ (see Proposition 14.8). According to Proposition 14.8, there exist v_μ in E_μ and a Hölder continuous function $\overline{\varphi}_0 : S_\nu \to E/\Delta_\mu$ such that Equation (14.7) holds.

We now assume that the cocycle σ has the *lifting property :* this means that the function $\overline{\varphi}_0$ admits a continuous lift $\widetilde{\varphi}_0 : S_\nu \to E$. Equivalently, we assume that there exist an element v_μ of E_μ and a Hölder continuous function $\widetilde{\varphi}_0 : S_\nu \to E$ such that, for any (g,x) in $\mathrm{Supp}\mu \times S_\nu$, one has

$$\sigma(g,x) = \sigma_\mu + v_\mu - \widetilde{\varphi}_0(gx) + \widetilde{\varphi}_0(x) \mod \Delta_\mu. \tag{15.3}$$

The group Δ_μ is cocompact in E_μ. We let π_μ be the Haar measure of Δ_μ that gives mass one to the intersection of the unit cubes of Φ_μ^* with the connected component Δ_μ° of Δ_μ. We let Π_μ be the average measure on E such that, for any Borel subset C of E, one has

$$\Pi_\mu(C) = \int_X \pi_\mu(C+\widetilde{\varphi}_0(x'))\,\mathrm{d}\nu(x'). \tag{15.4}$$

Here is our first version of the local limit theorem for σ.

Theorem 15.1. (Local limit theorem for σ) *Let G be a second countable locally compact semigroup and $s : G \to F$ be a continuous morphism onto a finite group F. Let μ be a Borel probability measure on G which is aperiodic in F. Let X be a compact metric G-space which is fibered over F and μ-contracting over F.*

Let $\sigma : G \times X \to E$ be a continuous cocycle whose sup-norm has a finite exponential moment (10.14) *and whose Lipschitz constant has a finite moment* (10.15). *We also assume the existence* (15.3) *of a lift $\widetilde{\varphi}_0$. We fix a bounded convex subset $C \subset E$ and $R > 0$. Then one has the limit*

$$\lim_{n\to\infty} \tfrac{1}{G_n(v_n)}\,\mu_{n,x}(C+v_n) - \Pi_\mu(C+v_n-nv_\mu-\widetilde{\varphi}_0(x)) \;=\; 0. \tag{15.5}$$

This limit is uniform for $x \in S_\nu$ and $v_n \in E_\mu$ with $\|v_n\| \le \sqrt{Rn\log n}$.

Remark 15.2. In this theorem we allow moderate deviations i.e. we allow the window $C + v_n$ to jiggle moderately, since our result is uniform for

$$\|v_n\| \le R\sqrt{n\log n}. \tag{15.6}$$

These moderate deviations are crucial for the concrete applications in Sections 16.4 and 16.5. They are also used in [**15**].

Remark 15.3. When the deviation satisfies the condition (15.6), we get the following lower bound for the denominator (15.2) of the left hand side of (15.5)

$$G_n(v_n) \ge A_0\, n^{-R-\frac{e_\mu}{2}} \tag{15.7}$$

where the constant A_0 depends only on μ and R. This lower bound will allow us to neglect in the calculation of $\mu_{n,x}(C+v_n)$ any term that decays faster than this power of n.

Theorem 15.1 is a special case of the local limit theorem with target 15.15 that we will state and prove in section 15.4.

REMARK 15.4. We could give a general version of this theorem without the assumption that μ is aperiodic in F, but this would make the statement heavy, since we would have to restrict our attention to integers n in arithmetic sequences $k + \mathbb{Z}p_\mu$.

Theorem 15.1 may be true without the assumption (15.3) that a lift $\widetilde{\varphi}_0$ exists. This condition is satisfied in our main application in Chapter 16, but this is not always the case, as shown by the following example.

EXAMPLE 15.5. There exists a cocycle $\sigma : G \times X \to E$ which satisfies the assumptions of Proposition 10.15 but for which there does not exist any function $\widetilde{\varphi}_0 : S_\nu \to E$ wich fulfills (15.3).

PROOF. We choose G to be a free group on two generators g_1 and g_2, $\mu = \frac{1}{4}(\delta_{g_1} + \delta_{g_2} + \delta_{g_1^{-1}} + \delta_{g_2^{-1}})$ and $X = \mathbb{P}(\mathbb{R}^2)$. We let G acts faithfully on X via a dense subgroup of $\mathrm{SL}(2,\mathbb{R})$, so that $S_\nu = X$. We identify the universal cover of X with $\mathbb{R}$ by setting, for any $t \in \mathbb{R}$, $x_t := \mathbb{R}(\cos t, \sin t) \in X$. For $i = 1, 2$, we choose a continuous lift $\widetilde{g}_i : \mathbb{R} \to \mathbb{R}$ of g_i: it satisfies $x_{\widetilde{g}_i t} = g_i(x_t)$. For any $g \in G$, we set $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ for the corresponding word in $\widetilde{g}_1$, $\widetilde{g}_2$.

We let $\sigma : G \times X \to E = \mathbb{R}$ to be the cocycle given by, for $g \in G$,

$$\sigma(g, x_t) = \widetilde{g}\,t - t \ \text{ for all } \ t \in \mathbb{R}. \tag{15.8}$$

For θ in $2\mathbb{Z}$, the function φ_θ on X such that $\varphi(x_t) = e^{i\theta t}$, $t \in \mathbb{R}$, satisfies, for any g in G and x in X,

$$e^{i\theta\sigma(g,x)} = \varphi(gx)\varphi(x)^{-1},$$

so that, by Corollary 14.10, one has $\pi\mathbb{Z} \supset \Delta_\mu$. However, we claim that one cannot write σ under the form (15.3) with a continuous $\widetilde{\varphi}_0 : X \to \mathbb{R}$. Indeed, if this was the case, since the space X is connected, for any g in G, the function

$$x \mapsto \sigma(g, x) - \widetilde{\varphi}_0(x) + \widetilde{\varphi}_0(gx)$$

would be constant with a value $c(g)$. By the coycle property, the map c would be a morphism $G \to \mathbb{R}$. In particular, c would be trivial on the derived group $[G, G]$ of G. Now, since $\mathrm{SL}(2,\mathbb{R})$ is equal to its derived group, $[G, G]$ has dense image in $\mathrm{SL}(2,\mathbb{R})$ and one can find g in $[G, G]$ that acts on $\mathbb{P}(\mathbb{R}^2)$ as a non-trivial rotation, so that $|\sigma(g^n, x)| \xrightarrow[n\to\infty]{} \infty$ uniformly in X. This contradicts the fact that, since $c(g) = 0$, one has

$$\sigma(g, x) = \widetilde{\varphi}_0(x) - \widetilde{\varphi}_0(gx) \ \text{ for all } \ x \in X. \qquad \square$$

We now begin the proof of Theorem 15.1 and of its extension : Theorem 15.15. We introduce the cocycle

$$
\begin{aligned}
(15.9)\qquad \widetilde{\sigma} : G \times S_\nu &\to E; \\
(g,x) &\mapsto \widetilde{\sigma}(g,x) := \sigma(g,x) + \widetilde{\varphi}_0(gx) - \widetilde{\varphi}_0(x).
\end{aligned}
$$

It satisfies

$$
(15.10)\qquad \widetilde{\sigma}(g,x) \in \sigma_\mu + v_\mu + \Delta_\mu \quad \text{for all } (g,x) \text{ in } \mathrm{Supp}\mu \times S_\nu.
$$

We first need a notation similar to (15.1) for the cocycle $\widetilde{\sigma}$. For $\varphi \in \mathcal{H}^\gamma(X)$, $n \geq 1$ and $x \in S_\nu$, we introduce the measure $\widetilde{\mu}^{\varphi}_{n,x}$ on E_μ given by, for every $\psi \in \mathcal{C}_c(E_\mu)$,

$$
(15.11)\qquad \widetilde{\mu}^{\varphi}_{n,x}(\psi) = \int_G \psi(\widetilde{\sigma}(g,x) - n\sigma_\mu)\varphi(gx)\, \mathrm{d}\mu^{*n}(g).
$$

The main advantage in first considering this measure $\widetilde{\mu}^{\varphi}_{n,x}$ is that it is concentrated on $nv_\mu + \Delta_\mu \subset E_\mu$.

We will first prove an analogous local limit theorem for the cocycle $\widetilde{\sigma}$. For any v in E_μ, we denote by π^v_μ the image of π_μ under the translation by v.

Proposition 15.6 (Local limit theorem for $\widetilde{\sigma}$ with target). *Same assumptions as in Theorem 15.1. We fix $\varphi \in \mathcal{H}^\gamma(X)$, a bounded convex subset $C \subset E$, and $R > 0$. Then one has the limit,*

$$
\lim_{n\to\infty} \tfrac{1}{G_n(v_n)}\, \widetilde{\mu}^{\varphi}_{n,x}(C+v_n) \;-\; \nu(\varphi)\, \pi^{nv_\mu}_\mu(C+v_n) \;=\; 0.
$$

This limit is uniform for $x \in S_\nu$ and $v_n \in E_\mu$ with $\|v_n\| \leq \sqrt{Rn\log n}$.

The proof of Proposition 15.6 will occupy the main part of this chapter. Note that, in the course of the proof, the assumption that x belongs to S_ν is only used in relation to the construction of $\overline{\varphi}_0$, so that we can drop it when the cocycle σ is aperiodic i.e. satisfies (14.8):

Corollary 15.7 (Local limit theorem for aperiodic cocycles). *Let G be a second countable locally compact semigroup, μ be a Borel probability measure on G. Let X be a compact metric G-space which is μ-contracting. Let $\sigma : G \times X \to E$ be a continuous cocycle whose sup-norm has a finite exponential moment* (10.14) *and whose Lipschitz constant has a finite moment* (10.15). *We assume that σ is aperiodic. Let π_μ be the Lebesgue measure of E which gives mass one to the unit cubes of Φ^*_μ.*

We fix a bounded convex subset $C \subset E$ and $R > 0$. Then, the sequence

$$
\tfrac{1}{G_n(v_n)}\, \mu^{*n}(\{g \in G \mid \sigma(g,x) - n\sigma_\mu \in C + v_n\})
$$

converges uniformly to $\pi_\mu(C)$ when n goes to ∞, as soon as $x \in X$ and $v_n \in E$ with $\|v_n\| \leq \sqrt{Rn \log n}$.

15.2. Local limit theorem for smooth functions.

We will first prove a smoothened variation (Lemma 15.11) of the local limit theorem with target (Proposition 15.6) for $\widetilde{\sigma}$ where we replace the convex set C by an adequate smooth function ψ on E_μ.

Let ψ be a Borel function on E_μ, such that

$$\sup_{v \in E_\mu} \int_E |\psi| \, \mathrm{d}\pi_\mu^v < \infty. \tag{15.12}$$

For any v in E_μ, we introduce the partial Fourier transform $\widehat{\psi}_v$ given by, for θ in E^*,

$$\widehat{\psi}_v(\theta) = \int_{E_\mu} \psi(w) e^{-i\theta(w)} \, \mathrm{d}\pi_\mu^v(w).$$

Note that, for θ in E^* and θ' in Λ_μ, we have

$$\widehat{\psi}_v(\theta + \theta') = e^{-i\theta'(v)} \widehat{\psi}_v(\theta)$$

and hence $\widehat{\psi}_v$ may be seen as a function on $E_\mu^* \simeq E^*/E_\mu^\perp$ and $\left|\widehat{\psi}_v\right|$ may be seen as a function on E^*/Λ_μ.

Definition 15.8. A Borel function ψ on E_μ is called Δ_μ-admissible if

- For any k in $\mathbb{N}$, one has $\sup\limits_{v \in E_\mu} (1 + \|v\|)^k |\psi(v)| < \infty$.

- There exist compact subsets K of E_μ and K^* of E^* such that ψ has support in $K + \Delta_\mu^\circ$ and, for any v in E_μ, $\widehat{\psi}_v$ has support in $K^* + (\Delta_\mu^\circ)^\perp$.

See the beginning of Section 15.3 for examples of such functions.

Remark 15.9. When $\Delta_\mu = E$ (i.e. when the cocycle is aperiodic, which is the case for the Iwasawa cocycle of an algebraic semisimple real Lie group), an admissible function on E is a Schwartz function whose Fourier transform has compact support.

When Δ_μ is a discrete subgroup of E, an admissible function is a compactly supported bounded Borel function on E_μ.

The general case is a mixture of those two cases since one has the following dual sequences of injections

$$0 \longrightarrow \Delta_\mu^\circ \xrightarrow{\text{codiscrete}} \Delta_\mu \xrightarrow{\text{cocompact}} E_\mu \longrightarrow E\,,$$

$$0 \longrightarrow \Lambda_\mu^\circ = E_\mu^\perp \xrightarrow{\text{codiscrete}} \Lambda_\mu = \Delta_\mu^\perp \xrightarrow{\text{cocompact}} (\Delta_\mu^\circ)^\perp \longrightarrow E^*\,.$$

REMARK 15.10. When ψ is an admissible function and ρ is a finite Borel measure on E_μ supported by $v+\Delta_\mu$ for some v in E_μ, to compute $\rho(\psi) = \int_{v+\Delta_\mu} \psi \, \mathrm{d}\rho$, we will use the following Fourier inversion formula

$$\int_{v+\Delta_\mu} \psi \, \mathrm{d}\rho = (2\pi)^{-e_\mu} \int_{E^*/\Lambda_\mu} \widehat{\psi}_v(\theta) \widehat{\rho}(\theta) \, \mathrm{d}\theta. \tag{15.13}$$

Note that the right-hand side of (15.13) is well defined. Indeed, the characteristic function $\widehat{\rho} : \theta \mapsto \rho(e^{i\theta})$ satisfies, for θ in E^* and θ' in Λ_μ,

$$\widehat{\rho}(\theta + \theta') = e^{i\theta'(v)} \widehat{\rho}(\theta),$$

hence $\widehat{\psi}_v \, \widehat{\rho}$ may be seen as a function on E^*/Λ_μ.

We will apply Formula (15.13) to the measure $\rho = \widetilde{\mu}^{\varphi}_{n,x}$ from (15.11). This is allowed since this measure is concentrated on $nv_\mu + \Delta_\mu$.

Here is the smoothened variation of the Local Limit Theorem for $\widetilde{\sigma}$ where the convex set C has been replaced by a smooth function.

LEMMA 15.11. *Same assumptions as in Theorem 15.1. Let φ be in $\mathcal{H}^\gamma(X)$ and $r \geq 2$. There exists a sequence $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ such that, for any non-negative Δ_μ-admissible function ψ on E_μ, $n \geq 1$ and x in S_ν, one has*

$$\left| \widetilde{\mu}^{\varphi}_{n,x}(\psi) - \nu(\varphi) \, \pi_\mu^{nv_\mu}(\psi \, G_n) \right| \leq \varepsilon_n \, \pi_\mu^{nv_\mu}(\psi \, G_n) + O_\psi\left(\tfrac{1}{n^{r/2}}\right),$$

where the O_ψ is uniform in x and over the translates of the function ψ by elements of E_μ.

We recall that G_n is the Gaussian function given by (15.2).

The proof of this lemma relies on the following asymptotic expansion of the quantities appearing in Lemma 10.17 (compare with [**30**, p. 48]).

LEMMA 15.12. *Same assumptions as in Theorem 15.1. Fix $r \geq 2$. There exist polynomial functions A_k on E^*, $0 \leq k \leq r-1$, with degree at most $3k$ and no constant term for $k > 0$, with values in the space $\mathcal{L}(\mathcal{H}^\gamma(X))$ of bounded endomorphisms of $\mathcal{H}^\gamma(X)$ and such that, for any $M > 0$, uniformly for θ in E^* with $\|\theta\| \leq \sqrt{M \log n}$ and φ in $\mathcal{H}^\gamma(X)$, one has, in $\mathcal{H}^\gamma(X)$, $A_0(\theta)\varphi = N\varphi$ and*

$$e^{\frac{\Phi_\mu(\theta)}{2}} e^{-i\sqrt{n}\theta(\sigma_\mu)} \lambda^n_{\frac{i\theta}{\sqrt{n}}} N_{\frac{i\theta}{\sqrt{n}}} \varphi = \sum_{k=0}^{r-1} \tfrac{A_k(\theta)\varphi}{n^{k/2}} + O\left(\tfrac{(\log n)^{3r/2} |\varphi|_\gamma}{n^{r/2}}\right).$$

PROOF. Using the trick (2.9), we may assume $\sigma_\mu = 0$.

Now, on one hand, by Lemmas 10.17, 10.18 and Taylor-Young Formula, there exists a polynomial function P on E^*, with degree $\leq r+1$ and whose homogeneous components of degree 0, 1 and 2 are equal to

0, and there exists an analytic function ρ_1, defined in a neighborhood of zero in $E^*_{\mathbb{C}}$ with

$$\rho_1(\theta) = O(\|\theta\|^{r+2}),$$

such that, for any θ close enough to zero, one has

$$\log \lambda_\theta - \tfrac{1}{2}\Phi_\mu(\theta) = P(\theta) + \rho_1(\theta).$$

Thus, when n is large enough and $\theta \in E^*$ with $\|\theta\| \leq \sqrt{M \log n}$, we get

$$\begin{aligned} e^{\frac{1}{2}\Phi_\mu(\theta)} \lambda^n_{\frac{i\theta}{\sqrt{n}}} &= e^{nP\left(\frac{i\theta}{\sqrt{n}}\right) + n\rho_1\left(\frac{i\theta}{\sqrt{n}}\right)} \\ &= 1 + \textstyle\sum_{k=1}^{r-1} \frac{n^k}{k!} P\left(\frac{i\theta}{\sqrt{n}}\right)^k + O\left(\frac{(\log n)^{3r/2}}{n^{r/2}}\right). \end{aligned}$$

On the other hand, by lemma 10.17 and Taylor-Young Formula, there exist a polynomial function Q on E^*, with degree $\leq r-1$ and no constant term, with values in $\mathcal{L}(\mathcal{H}^\gamma(X))$ and an analytic function ρ_2, defined in a neighborhood U of zero in $E^*_{\mathbb{C}}$, with values in $\mathcal{L}(\mathcal{H}^\gamma(X))$, such that, uniformly for $\varphi \in \mathcal{H}^\gamma(X)$, for θ in U, one has

$$\begin{aligned} \rho_2(\theta)\varphi = O(\|\theta\|^r)|\varphi|_\gamma \quad \text{and} \\ N_{i\theta}\varphi = N\varphi + Q(\theta)\varphi + \rho_2(\theta)\varphi. \end{aligned}$$

The proof follows by writing, for $1 \leq k \leq r-1$,

$$n^k P\left(\frac{i\theta}{\sqrt{n}}\right)^k Q\left(\frac{i\theta}{\sqrt{n}}\right) \text{ and } n^k P\left(\frac{i\theta}{\sqrt{n}}\right)^k N$$

as the sum of homogeneous terms of degree at least $3k$ in θ and only keeping the ones that have degree $\leq \frac{r-1}{2}$ in n^{-1}. $\square$

Proof of lemma 15.11. We may again assume $\sigma_\mu = 0$. We may also assume that E_μ has dimension $e_\mu \geq 1$. We fix φ in $\mathcal{H}^\gamma(X)$ and x in X. For any θ in E^*, the characteristic function of $\mu^\varphi_{n,x}$ is given by

$$\widehat{\widetilde{\mu}^\varphi_{n,x}}(\theta) = \textstyle\int_G e^{i\theta(\sigma(g,x))}\varphi(gx)\,\mathrm{d}\mu^{*n}(g) = P^n_{i\theta}\varphi(x). \tag{15.14}$$

Let $s \leq e_\mu$ be the rank of the free abelian group $\Lambda_\mu/E^\perp_\mu$. Choose a basis $\theta_1, \ldots, \theta_{e_\mu}$ of a complementary subspace to $E^\perp_\mu$ in E^* such that $\theta_1, \ldots, \theta_s$ span Λ_μ mod $E^\perp_\mu$. The quadratic form Φ_μ induces a norm on this complementary subspace which we denote by $\|.\|$. Define

$$L := \{\theta = \textstyle\sum_{\ell=1}^{e_\mu} t_\ell\,\theta_\ell \in E^* \text{ such that } |t_\ell| \leq \frac{1}{2} \text{ when } 1 \leq \ell \leq s\},$$

so that L is a fundamental domain for the projection $E^* \to E^*/\Lambda_\mu$. If ψ is a Δ_μ-admissible function on E, we compute, from Formulae (15.13) and (15.14), the integral

$$I_n := (2\pi)^{e_\mu}\widetilde{\mu}^\varphi_{n,x}(\psi) = \textstyle\int_L \widehat{\psi}_{nv_\mu}(\theta)\, P^n_{i\theta}\varphi(x)\,\mathrm{d}\theta.$$

We decompose this integral as the sum of four terms

$$I_n = I_n^1 + I_n^2 + I_n^3 + I_n^4.$$

We now bound individually these four terms. Each time we will use implicitely the fact that the function $\theta \to \widehat{\psi}_{nv_\mu}(\theta)$ is uniformly bounded by (15.12).

First, we keep the notations from Lemma 10.17 and we choose some large enough $T > 0$. On the one hand, since ψ is admissible and since Λ_μ is cocompact in $(\Delta_\mu^\circ)^\perp$, there exists a compact subset K^* of E^* such that, for any v in E_μ, $\widehat{\psi}_v$ has support in $K^* + \Lambda_\mu$. On the other hand, by definition of L and Λ_μ, for any neighborhood V of 0 in L, there exists $0 \leq \omega < 1$ such that for any θ in $((K^* + \Lambda_\mu) \cap L) \smallsetminus V$, $P_{i\theta}$ has spectral radius $< \omega$. Hence, for n large enough, for any θ in $((K^* + \Lambda_\mu) \cap L) \smallsetminus V$, $P_{i\theta}^n$ has norm $\leq \omega^n$ and

$$I_n^1 := \textstyle\int_{L \smallsetminus V} \widehat{\psi}_{nv_\mu}(\theta)\, P_{i\theta}^n \varphi(x)\, \mathrm{d}\theta = O_\psi(\omega^n)$$

(note that this O_ψ is uniform over the translates of ψ by elements of E_μ).

Second, by Lemma 10.18, one can choose V small enough so that, for n large enough, for any θ in V, $P_{i\theta}$ has spectral radius $< e^{-\frac{1}{4}\Phi_\mu(\theta)}$. Hence, for n large enough, for any θ in V, $P_{i\theta}^n$ has norm $\leq e^{-\frac{n}{4}\Phi_\mu(\theta)}$ and one has,

$$I_n^2 := \int_{\substack{\theta \in V \\ \|\theta\|^2 \geq T \frac{\log n}{n}}} \widehat{\psi}_{nv_\mu}(\theta)\, P_{i\theta}^n \varphi(x)\, \mathrm{d}\theta = O_\psi(n^{-\frac{T}{4}}).$$

Third, by Lemma 10.17, there exists $0 < \delta < 1$ such that, for any θ in V, $P_{i\theta} - \lambda_{i\theta} N_{i\theta}$ has spectral radius $< \delta$. Hence, for n large enough, one has,

$$I_n^3 := \int_{\substack{\theta \in V \\ \|\theta\|^2 \leq T \frac{\log n}{n}}} \widehat{\psi}_{nv_\mu}(\theta)(P_{i\theta}^n - \lambda_{i\theta}^n N_{i\theta})\varphi(x)\, \mathrm{d}\theta = O_\psi(\delta^n).$$

It remains to control the fourth term:

$$I_n^4 := \int_{\substack{\theta \in V \\ \|\theta\|^2 \leq T \frac{\log n}{n}}} \widehat{\psi}_{nv_\mu}(\theta)\, \lambda_{i\theta}^n\, N_{i\theta} \varphi(x)\, \mathrm{d}\theta.$$

By Lemma 15.12, since $\sigma_\mu = 0$, one has

$$I_n^4 = \int_{\substack{\theta \in V \\ \|\theta\|^2 \leq T \frac{\log n}{n}}} \widehat{\psi}_{nv_\mu}(\theta) \textstyle\sum_{k=0}^{r-1} \frac{\widehat{G}_n(\theta) A_k(\sqrt{n}\theta)\varphi(x)}{n^{k/2}}\, \mathrm{d}\theta + O_\psi\left((\frac{\log^3 n}{n})^{\frac{r+e_\mu}{2}} \right),$$

where the Fourier transform $\widehat{G}_n$ of the Gaussian function G_n is given, for $\theta \in E_\mu^*$, by

$$\widehat{G}_n(\theta) = e^{-\frac{n}{2}\Phi_\mu(\theta)}.$$

Since, for any $0 \leq k \leq r-1$, A_k has degree at most $3k$, we get

$$\int_{\substack{\theta \in E_\mu^* \\ \|\theta\|^2 \geq T\frac{\log n}{n}}} \widehat{\psi}_{nv_\mu}(\theta) \tfrac{\widehat{G}_n(\theta) A_k(\sqrt{n}\theta)\varphi(x)}{n^{k/2}} \,\mathrm{d}\theta = O_\psi\left(\tfrac{\log n^{(3k+e_\mu)/2}}{n^{(T+k+e_\mu)/2}}\right).$$

Thus, since $e_\mu \geq 1$, choosing T large enough, we have established that

$$I_n = \int_{E_\mu^*} \widehat{\psi}_{nv_\mu}(\theta) \textstyle\sum_{k=0}^{r-1} \tfrac{\widehat{G}_n(\theta) A_k(\sqrt{n}\theta)\varphi(x)}{n^{k/2}} \,\mathrm{d}\theta + O_\psi\left(\tfrac{1}{n^{r/2}}\right). \tag{15.15}$$

Now, for $0 \leq k \leq r-1$, there exists a polynomial function B_k on E_μ, with values in $\mathcal{H}^\gamma(X)$, such that B_k has degree at most $3k$ and, for any x in S_ν, the function on E_μ^* given by $\theta \mapsto e^{-\frac{1}{2}\Phi_\mu(\theta)} A_k(\theta)\varphi(x)$ is the Fourier transform of the function $v \mapsto G_1(v)\, B_k(v)(x)$. Therefore, we get, from (15.15) and the Fourier inversion formula (15.13),

$$I_n = (2\pi)^{e_\mu}\int_{E_\mu} \psi(v) G_n(v) \textstyle\sum_{k=0}^{r-1} \tfrac{B_k(\frac{v}{\sqrt{n}})(x)}{n^{k/2}} \,\mathrm{d}\pi_\mu^{nv_\mu}(v) + O_\psi\big(\tfrac{1}{n^{r/2}}\big). \tag{15.16}$$

For any $0 \leq k \leq r-1$, on the one hand one has

$$\int_{\substack{v \in E_\mu \\ \|v\|^2 \geq T\, n \log n}} \psi(v) G_n(v) \tfrac{B_k(\frac{v}{\sqrt{n}})(x)}{n^{k/2}} \,\mathrm{d}\pi_\mu^{nv_\mu}(v) = O\left(\tfrac{\log n^{(3k+e_\mu)/2}}{n^{(T-k)/2}}\right) \|\psi\|_\infty,$$

and on the other hand, since ψ is nonnegative, one has

$$\int_{\substack{v \in E_\mu \\ \|v\|^2 \leq T\, n \log n}} \psi(v) G_n(v) \tfrac{B_k(\frac{v}{\sqrt{n}})(x)}{n^{k/2}} \,\mathrm{d}\pi_\mu^{nv_\mu}(v) = O\big(\tfrac{\log n^{3k/2}}{n^{k/2}}\big)\, \pi_\mu^{nv_\mu}(\psi\, G_n).$$

In particular, choosing T large enough, the leading term in (15.16) is the one with $k = 0$. Since one has $A_0(\theta) = N$ and $N\varphi = \nu(\varphi)$, one gets $B_0(v)(x) = \nu(\varphi)$ and, if T is large enough,

$$I_n = (2\pi)^{e_\mu}\nu(\varphi)\, \pi_\mu^{nv_\mu}(\psi\, G_n) + o(\pi_\mu^{nv_\mu}(\psi\, G_n)) + O_\psi\left(\tfrac{1}{n^{r/2}}\right).$$

Our claim follows. □

15.3. Approximation of convex sets.

We explain in this section how to deduce the local limit theorem with target (Proposition 15.6) for $\widetilde{\sigma}$ from its smoothened version (Lemma 15.11). The key point is a regularization procedure for a convex set C of E.

We fix a nonnegative Schwartz function α on Δ_μ° with $\int_{\Delta_\mu^\circ} \alpha \, \mathrm{d}\pi_\mu = 1$ and whose Fourier transform has compact support and, for any $\varepsilon > 0$ and v in Λ_μ°, we set $\alpha_\varepsilon(v) = \frac{1}{\varepsilon^r}\alpha(\frac{v}{\varepsilon})$, where r is the dimension of Δ_μ°. If C is a bounded Borel subset of E_μ, the convolution product

$$\psi_{\varepsilon,C} := (\alpha_\varepsilon \pi_\mu) * \mathbf{1}_C$$

is given by the formula, for all v in E_μ,

$$\psi_{\varepsilon,C}(v) = \int_{\Delta_\mu^\circ} \alpha_\varepsilon(w) \mathbf{1}_C(v-w) \, \mathrm{d}\pi_\mu(w).$$

This function $\psi_{\varepsilon,C}$ is a Δ_μ-admissible function on E_μ.

The following lemma tells us that the functions $\psi_{\varepsilon,C}$ are good approximations of the function $\mathbf{1}_C$.

Lemma 15.13. *Same assumptions as in Theorem 15.1. Let C be a bounded Borel subset of E_μ and let $R \geq 0$ be a real number. One has*

$$\frac{1}{G_n(v)} \pi_\mu^u(\psi_{\varepsilon,C+v} G_n) - \pi_\mu^u(C+v) \xrightarrow[n\to\infty]{} 0 \tag{15.17}$$

uniformly for $u \in E_\mu$, $v \in E_\mu$, $\|v\| \leq \sqrt{Rn \log n}$ and $\varepsilon \in (0,1]$.

Proof. Let us compute, for $n \geq 1$, u, v in E_μ with $\|v\| \leq \sqrt{Rn\log n}$ and $\varepsilon \in (0,1]$, the left-hand side of Formula (15.17)

$$J_n := \frac{1}{G_n(v)} \pi_\mu^u(\psi_{\varepsilon,C+v} G_n) - \pi_\mu^u(C+v).$$

As the measure π_μ^u is invariant under the translations by the elements of Δ_μ° and as $\int_{\Delta_\mu^\circ} \alpha_\varepsilon \, \mathrm{d}\pi_\mu = 1$, one has

$$J_n = \int_{\Delta_\mu^\circ \times E_\mu} \alpha_\varepsilon(w) \mathbf{1}_{C+v}(w'-w) \left(\tfrac{G_n(w')}{G_n(v)} - 1\right) \mathrm{d}(\pi_\mu \otimes \pi_\mu^u)(w,w').$$

We decompose this integral as a sum $J_n = J_n^1 + J_n^2$ with

$$J_n^1 = \int_{\|w\| \leq n^{1/4}} \alpha_\varepsilon(w) \mathbf{1}_{C+v}(w'-w) \left(\tfrac{G_n(w')}{G_n(v)} - 1\right) \mathrm{d}(\pi_\mu \otimes \pi_\mu^u)(w,w'),$$

$$J_n^2 = \int_{\|w\| \geq n^{1/4}} \alpha_\varepsilon(w) \mathbf{1}_{C+v}(w'-w) \left(\tfrac{G_n(w')}{G_n(v)} - 1\right) \mathrm{d}(\pi_\mu \otimes \pi_\mu^u)(w,w').$$

In order to control J_n^1, we use the fact that

$$\frac{G_n(w')}{G_n(v)} = e^{\frac{1}{2n}\langle v+w', v-w'\rangle} \xrightarrow[n\to\infty]{} 1$$

uniformly for $v - w' \in C+w$, $\|w\| \leq n^{\frac14}$ and $\|v\| \leq \sqrt{Rn\log n}$. We get

$$J_n^1 \xrightarrow[n\to\infty]{} 0 \ \text{ uniformly.}$$

In order to control J_n^2, we use the bound

$$\frac{G_n(w')}{G_n(v)} \leq e^{\frac{\Phi_\mu^*(v)}{2n}} \leq n^{R/2}$$

for $\|v\| \leq \sqrt{Rn\log n}$. Setting $z = \varepsilon^{-1}w$, we get, uniformly for $\varepsilon \in (0,1]$,

$$J_n^2 \leq n^{R/2}\,\pi_\mu^u(C+v)\,\int_{\|w\|\geq n^{1/4}} \alpha_\varepsilon(w)\,\mathrm{d}\pi_\mu(w) \xrightarrow[n\to\infty]{} 0$$

since α is a Schwartz function. $\square$

To approximate open convex sets in measure, we shall also need the following

LEMMA 15.14. *Let E be a Euclidean space and π be a Lebesgue measure on E. Then, for any $\rho > 0$, the map $C \mapsto \pi(C)$ is uniformly continuous on the set of open convex subsets C of E with diameter $\leq \rho$, equipped with the Hausdorff distance.*

PROOF. Let d be the dimension of E. By Steiner's formula (see [**110**, III.13.3]), for any bounded convex subset $C \subset E$ and any integer $i \in [0,d]$, there exists $w_i(C) > 0$ such that, for $\varepsilon > 0$, the volume of the ε-neighborhood C^ε of C is given by

$$\pi(C^\varepsilon) = \textstyle\sum_{i=0}^d w_i(C)\varepsilon^i,$$

and the w_i's are non-decreasing functions of C. The result follows. $\square$

We can now conclude the

PROOF OF PROPOSITION 15.6. Roughly speaking, the main idea is to use the equality

$$\widetilde{\mu}_{n,x}^{\varphi}(\psi_{\varepsilon,C}) = \int_{\Delta_\mu^\circ} \alpha_\varepsilon(w)\,\widetilde{\mu}_{n,x}^{\varphi}(C+w)\,\mathrm{d}\pi_\mu(w) \tag{15.18}$$

where C is a bounded open convex subset of E_μ and $\varepsilon > 0$ is small. Using (15.18), we will get upper and lower bounds for the quantity $\widetilde{\mu}_{n,x}^{\varphi}(C)$ by means of $\widetilde{\mu}_{n,x}^{\varphi}(\psi_{\varepsilon,C'})$, where C' is a convex set that is very close to C and then we will apply the estimates of Lemmas 15.11, 15.13 and 15.14. The main technical issue which wheighs the proof is the fact that the test function α does not have compact support, since its Fourier transform has compact support. Let us proceed precisely.

We set $B(\varepsilon)$ for the open ball with radius ε and center 0 in Δ_μ° and

$$C^\varepsilon = C+B(\varepsilon) \;\text{ and }\; C_\varepsilon = \textstyle\bigcap_{w\in B(\varepsilon)} C-w. \tag{15.19}$$

For $\rho > 0$ and $\varepsilon > 0$, we set

$$V_\rho = \sup\{\pi_\mu(C) \mid C \subset E_\mu \text{ convex},\ \operatorname{diam} C \leq 2\rho\}.$$

$$\theta_\rho(\varepsilon) = \sup\{\pi_\mu(C^\varepsilon) - \pi_\mu(C_\varepsilon) \mid C \subset E_\mu \text{ convex},\ \operatorname{diam} C \leq 2\rho\},$$

By Lemma 15.14, for every $\rho > 0$, one has

$$\theta_\rho(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0. \tag{15.20}$$

Finally, we assume that φ is real and non-negative and $\|\varphi\|_\infty \leq 1$.

First step: We will first prove the upperbound: *for every* $R, \rho, \varepsilon_0 > 0$ *there exists* n_0 *such that for* $n \geq n_0$, $x \in S_\nu$, $v \in E_\mu$ *with* $\|v\| \leq \sqrt{Rn \log n}$ *and* C *a convex subset included in the ball* $B(\rho)$, *one has*

$$\tfrac{1}{G_n(v)}\, \widetilde{\mu}^{\varphi}_{n,x}(C{+}v) \leq \nu(\varphi)\, \pi^{nv_\mu}_{\mu}(C{+}v) + \varepsilon_0. \tag{15.21}$$

We can choose $\varepsilon \in (0,1]$ small enough so that $\int_{\|w\|\geq \frac{1}{\sqrt{\varepsilon}}} \alpha(w)\, \mathrm{d}\pi_\mu(w) \leq \varepsilon$. We note that, for w in Δ°_μ with $\|w\| \leq \sqrt{\varepsilon}$, we have $C \subset C^{\sqrt{\varepsilon}} + w$ and we deduce from (15.18) the inequality

$$(1{-}\varepsilon)\, \widetilde{\mu}^{\varphi}_{n,x}(C{+}v) \leq \widetilde{\mu}^{\varphi}_{n,x}(\psi_{\varepsilon, C^{\sqrt{\varepsilon}}+v}). \tag{15.22}$$

We also keep in mind the bound

$$G_n(v)^{-1} \leq (2\pi)^{\frac{e_\mu}{2}}\; n^{\frac{1}{2}(e_\mu+R)}.$$

Using successively (15.22), Lemma 15.11, Lemma 15.13 and (15.20), choosing first ε small enough and then n large enough, we get

$$\begin{aligned}
G_n(v)^{-1}\, \widetilde{\mu}^{\varphi}_{n,x}(C{+}v) &\leq \tfrac{1}{1-\varepsilon} G_n(v)^{-1}\, \widetilde{\mu}^{\varphi}_{n,x}(\psi_{\varepsilon, C^{\sqrt{\varepsilon}}+v}) \\
&\leq \tfrac{\nu(\varphi)+\varepsilon_0}{1-\varepsilon}\, G_n(v)^{-1}\, \pi^{nv_\mu}_{\mu}(\psi_{\varepsilon, C^{\sqrt{\varepsilon}}+v} G_n) + \varepsilon_0 \\
&\leq \tfrac{\nu(\varphi)+\varepsilon_0}{1-\varepsilon}\, \pi^{nv_\mu}_{\mu}(C^{\sqrt{\varepsilon}}{+}v) + 2\varepsilon_0 \\
&\leq \nu(\varphi)\, \pi^{nv_\mu}_{\mu}(C{+}v) + 3\varepsilon_0 + 2V_\rho\, \varepsilon_0.
\end{aligned}$$

Letting ε_0 go to 0, this proves the upper bound (15.21).

Second step: We will now prove the lower bound: *for every positive* R, ρ, ε_0 *there exists* n_0 *such that for* $n \geq n_0$, $x \in S_\nu$, $v \in E_\mu$ *with* $\|v\| \leq \sqrt{Rn \log n}$ *and* C *a convex subset included in the ball* $B(\rho)$, *one has*

$$\tfrac{1}{G_n(v)}\, \widetilde{\mu}^{\varphi}_{n,x}(C{+}v) \geq\; \nu(\varphi)\, \pi^{nv_\mu}_{\mu}(C{+}v) - \varepsilon_0. \tag{15.23}$$

Again, we will first choose $\varepsilon \in (0,1]$ very small and then n very large. As above, we can assume that $\int_{\|w\|\geq \frac{1}{\sqrt{\varepsilon}}} \alpha(w)\, \mathrm{d}\pi_\mu(w) \leq \varepsilon$. We notice that, for w in E_μ with $\|w\| \leq \sqrt{\varepsilon}$, we have $C_{\sqrt{\varepsilon}}{+}w \subset C$ and we deduce from (15.18)

$$\begin{aligned}
\widetilde{\mu}^{\varphi}_{n,x}(C{+}v) &\geq \int_{\|w\|\leq\sqrt{\varepsilon}} \alpha_\varepsilon(w)\, \widetilde{\mu}^{\varphi}_{n,x}(C_{\sqrt{\varepsilon}}{+}v{+}w)\, \mathrm{d}\pi_\mu(w) \\
&\geq \widetilde{\mu}^{\varphi}_{n,x}(\psi_{\varepsilon, C_{\sqrt{\varepsilon}}+v}) - K^1_n - K^2_n,
\end{aligned} \tag{15.24}$$

where

$$K_n^1 = \int_{\sqrt{\varepsilon}\le\|w\|\le n^{1/4}} \alpha_\varepsilon(w)\, \widetilde{\mu}_{n,x}^{\varphi}(C_{\sqrt{\varepsilon}}+v+w)\, \mathrm{d}\pi_\mu(w),$$
$$K_n^2 = \int_{\|w\|\ge n^{1/4}} \alpha_\varepsilon(w)\, \widetilde{\mu}_{n,x}^{\varphi}(C_{\sqrt{\varepsilon}}+v+w)\, \mathrm{d}\pi_\mu(w).$$

First, using the upperbound (15.21), we have, reasoning as in the proof of lemma 15.13, for n large,

$$\begin{aligned}\tfrac{K_n^1}{G_n(v)} &\le \textstyle\int_{\sqrt{\varepsilon}\le\|w\|\le n^{\frac14}} \alpha_\varepsilon(w)\tfrac{G_n(v+w)}{G_n(v)}(\pi_\mu^{nv_\mu}(C+v+w)+\varepsilon_0)\,\mathrm{d}\pi_\mu(w)\\ &\le \varepsilon(1+\varepsilon_0)(V_{2\rho}+\varepsilon_0)\le\varepsilon_0.\end{aligned}$$

Second, using the bound $\|v\| \le \sqrt{Rn\log n}$ and the fact that α is a Schwartz function, one gets, for n large,

$$\tfrac{K_n^2}{G_n(v)} \le n^{\frac{R}{2}} \textstyle\int_{\|w\|\ge n^{\frac14}} \alpha_\varepsilon(w)\,\mathrm{d}\pi_\mu(w) \le \varepsilon_0.$$

Now, using successively inequality (15.24), Lemma 15.11, Lemma 15.13 and the limit (15.20), we get,

$$\begin{aligned} G_n(v)^{-1}\,\widetilde{\mu}_{n,x}^{\varphi}(C+v) &\ge G_n(v)^{-1}\,(\widetilde{\mu}_{n,x}^{\varphi}(\psi_{\varepsilon,C_{\sqrt{\varepsilon}}+v}) - K_n^1 - K_n^2)\\ &\ge (\nu(\varphi)-\varepsilon_0)G_n(v)^{-1}\,\pi_\mu^{nv_\mu}(\psi_{\varepsilon,C_{\sqrt{\varepsilon}}+v}) - 3\varepsilon_0\\ &\ge (\nu(\varphi)-\varepsilon_0)\;\pi_\mu^{nv_\mu}(C_{\sqrt{\varepsilon}}+v) - 4\varepsilon_0\\ &\ge \nu(\varphi)\,\pi_\mu^{nv_\mu}(C+v) - 5\varepsilon_0 - V_\rho\,\varepsilon_0.\end{aligned}$$

Letting ε_0 go to 0, this proves the lower bound (15.23) and ends the proof of Proposition 15.6. □

15.4. Local limit theorem for σ with target.

We will now state and prove a Local Limit Theorem with target for the cocycle σ (Theorem 15.15) which generalizes the Local Limit Theorem for σ (Theorem 15.1).

For φ in $\mathcal{H}^\gamma(X)$, $n \ge 1$ and $x \in S_\nu$, we want to describe the behavior of the measure $\mu_{n,x}^{\varphi}$ on E analogous to (15.1), given by, for every $\psi \in \mathcal{C}_c(E)$,

$$\mu_{n,x}^{\varphi}(\psi) = \int_G \psi(\sigma(g,x)-n\sigma_\mu)\varphi(gx)\,\mathrm{d}\mu^{*n}(g). \tag{15.25}$$

We let Π_μ^φ be the average measure on E analogous to (15.4), given, for $C\subset E$, by

$$\Pi_\mu^\varphi(C) = \int_X \pi_\mu(C+\widetilde{\varphi}_0(x'))\varphi(x')\,\mathrm{d}\nu(x'), \tag{15.26}$$

where $\widetilde{\varphi}_0$ is as in (15.3).

Here is our final version of the local limit theorem with moderate deviations.

Theorem 15.15. (Local limit theorem for σ with target) *Same assumptions as in Theorem 15.1. We fix $\varphi \in \mathcal{H}^\gamma(X)$, a bounded convex subset $C \subset E$ and $R > 0$. Then one has the limit*

$$\lim_{n\to\infty} \tfrac{1}{G_n(v_n)}\, \mu^{\varphi}_{n,x}(C+v_n) - \Pi^{\varphi}_{\mu}(C+v_n-nv_\mu-\widetilde{\varphi}_0(x)) \;=\; 0.$$

This limit is uniform for $x \in S_\nu$ and $v_n \in E_\mu$ with $\|v_n\| \le \sqrt{Rn\log n}$.

Proof. Roughly speaking, this follows from (15.9) and from Proposition 15.6. Here are more details.

We can assume φ to be real-valued. We fix $\varepsilon_0 > 0$ and, using (15.20), choose $\varepsilon > 0$ such that $\theta_{2\rho}(2\varepsilon) < \varepsilon_0$. We write $\varphi = \sum_{i=1}^{\ell} \varphi_i$ where $\varphi_i \in \mathcal{H}^\gamma(X)$ has support contained in a ball $B_i \subset X$ with center x_i such that $\sup_{y,z\in B_i} \|\widetilde{\varphi}_0(y) - \widetilde{\varphi}_0(z)\| \le \varepsilon$.

Now, we get, for n large enough, using Proposition 15.6,

$$\begin{aligned}
\tfrac{1}{G_n(v_n)}\, \mu^{\varphi}_{n,x}(C+v_n) &\le \textstyle\sum_{i=1}^{\ell} \tfrac{1}{G_n(v_n)}\, \widetilde{\mu}^{\varphi_i}_{n,x}(C^\varepsilon+v_n-\widetilde{\varphi}_0(x)+\widetilde{\varphi}_0(x_i))\\
&\le \textstyle\sum_{i=1}^{\ell} \nu(\varphi_i)\pi_\mu^{nv_\mu}(C^\varepsilon+v_n-\widetilde{\varphi}_0(x)+\widetilde{\varphi}_0(x_i)) + \varepsilon_0\\
&\le \textstyle\int_X \pi_\mu^{nv_\mu}(C^{2\varepsilon}+v_n-\widetilde{\varphi}_0(x)+\widetilde{\varphi}_0(y))\varphi(y)\,\mathrm{d}\nu(y)+\varepsilon_0\\
&\le \Pi^{\varphi}_{\mu}(C+v_n-nv_\mu-\widetilde{\varphi}_0(x))+2\varepsilon_0.
\end{aligned}$$

One concludes by replacing φ with $-\varphi$. □

16. Local limit theorem for products of random matrices

We come back to the notations of Chapter 12. The first two sections deal with $\mathcal{S}$-adic Lie groups. Starting from the third section, we will deal only with real Lie groups.

The aim of this chapter is to prove, using the results of Chapter 15, the Local Limit Theorem 16.6 with target and with moderate deviations for products of random matrices, and to give various applications of this theorem. These applications are the Local Limit Theorems for the random variables given by the Cartan projection in Section 16.4, by the norms of matrices and the norms of vectors in Section 16.5.

The moderate deviations in Theorem 16.6, will be crucial in these applications.

16.1. Lifting the coboundary.

In this section, we give more information on the μ-residual image Δ_μ, and we prove the lifting property (15.3) for the Iwasawa cocycle.

Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F := G/G_c$ and μ be a Zariski dense Borel probability measure on G with a finite exponential moment and which is aperiodic in F. In order to apply Theorem 15.1 to the Iwasawa cocycle $\sigma_{\Theta_\mu} : G \times \mathcal{P}_{\Theta_\mu} \to \mathfrak{a}_{\Theta_\mu}$, we will need the following Proposition 16.1 which refines Proposition 12.19 and which tells us that, when $\mathcal{S} = \{\mathbb{R}\}$, the complex transfer operator $P_{i\theta}$, with $\theta \neq 0$, does not have eigenvalues of modulus 1. Equivalently, the cohomological equation (14.3) has no solutions. We will use the vector subspace $\mathfrak{b}_{\mathbb{R}}$ of $\mathfrak{a}$ introduced in Sections 8.4 and 12.7.

Proposition 16.1. *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group and μ be a Zariski dense Borel probability measure on G with a finite exponential moment. Let $\Delta_\mu \subset \mathfrak{a}_\mu$ be the μ-residual image of the Iwasawa cocycle σ_{Θ_μ}. Then this subgroup Δ_μ contains $\mathfrak{b}_{\mathbb{R}}$.*

In particular, when $\mathcal{S} = \{\mathbb{R}\}$ and G is semisimple, the Iwasawa cocycle σ on the full flag variety $\mathcal{P}$ is aperiodic i.e. $\Delta_\mu = \mathfrak{a}_\mu = \mathfrak{a}$.

Proof of Proposition 16.1. We first assume that the finite set $\mathcal{S}$ does not contain the local field $\mathbb{C}$. Keep the notations of sections 15.1. Recall that, by definition, Δ_μ is the orthogonal in $\mathfrak{a}$ of the group

$$\Lambda_\mu := \{\theta \in \mathfrak{a}^* \mid P_{i\theta} \text{ has spectral radius } 1\}.$$

We also keep the notations of the proof of Proposition 12.19. We know from Equation (14.7) that, for any $n \in \mathbb{N}$, $g \in \mathrm{Supp}(\mu^{*n}) \cap G_c$,

$$\lambda(g) = \sigma_{\Theta_\mu}(g, \xi^+_{\Theta_\mu, g}) \in n(v_\mu + \sigma_\mu) + \Delta_\mu. \tag{16.1}$$

For any s in $\mathcal{S}$, the image of Γ_μ in G_s is a Zariski dense subsemigroup of G_s. We write

$$\lambda(g) = (\lambda_s(g))_{s\in\mathcal{S}} \in \mathfrak{a} = \textstyle\prod_{s\in\mathcal{S}} \mathfrak{a}_s\,.$$

Now, by Proposition 8.8, the closed subgroup of $\mathfrak{a}$ spanned by the elements $\lambda(gh) - \lambda(g) - \lambda(h)$, when g, h and gh are Θ_Γ-proximal elements of Γ contains $\mathfrak{b}_{\mathbb{R}}$. Combining this Proposition 8.8 with (16.1), one gets the inclusion $\Delta_\mu \supset \mathfrak{b}_{\mathbb{R}}$, as required.

The general case reduces to the case where the finite set $\mathcal{S}$ does not contain the local field $\mathbb{C}$, because every complex algebraic Lie group G can be seen as a real algebraic Lie group. Indeed one just has to use Lemmas 16.2 and 16.3, which tell us that the "real Zariski closure" H of a "complex Zariski dense" subgroup of G is still a real algebraic reductive group, and that the flag variety of H can be seen as a closed H-orbit in the flag variety $\mathcal{P}_{\Theta_\Gamma} = \mathcal{P}_{\Theta_H}$ of G. □

The following lemma compares the closure of a subgroup for the real and for the complex Zariski topology.

LEMMA 16.2. *Let G be an algebraic simple complex Lie group, let Γ be a subgroup of G which is dense for the complex Zariski topology, and let H be the closure of Γ for the real Zariski topology. Then H is an algebraic simple real Lie group. More precisely, either one has $H = G$, or there exists a simple algebraic group $\mathbf{H}$ defined over $\mathbb{R}$ such that $H = \mathbf{H}(\mathbb{R})$ and $G = \mathbf{H}(\mathbb{C})$.*

PROOF OF LEMMA 16.2. By assumption G is the group of complex points $G = \mathbf{G}(\mathbb{C})$ of an algebraic group $\mathbf{G}$ defined over $\mathbb{C}$. The Lie algebra $\mathfrak{h}$ of H is a Γ-invariant real Lie subalgebra of the complex Lie algebra $\mathfrak{g}$ of G. Since Γ is dense in G for the complex Zariski toplogy, the complex Lie subalgebras $\mathfrak{h} + i\,\mathfrak{h}$ and $\mathfrak{h} \cap i\,\mathfrak{h}$ are ideals of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, one has $\mathfrak{h} + i\,\mathfrak{h} = \mathfrak{g}$ and one has $\mathfrak{h} \cap i\,\mathfrak{h} = \mathfrak{g}$ or $\{0\}$. In the first case, one has $H = G$. In the second case, $\mathfrak{h}$ is a real form of $\mathfrak{g}$, and H is the group of real point of an algebraic group $\mathbf{H}$ defined over $\mathbb{R}$ which is isomorphic to $\mathbf{G}$ over $\mathbb{C}$. $\square$

The following lemma embeds the full flag variety $\mathcal{P}$ of an algebraic simple real Lie group H as a closed orbit in the partial flag variety $\mathcal{P}_{\Theta_H}$ of the complexification G of H.

LEMMA 16.3. *Let $\mathbf{H}$ be a simble algebraic group defined over $\mathbb{R}$, let $H = \mathbf{H}(\mathbb{R})$ and $G = \mathbf{H}(\mathbb{C})$, let $\mathfrak{h} \subset \mathfrak{g}$ be their Lie algebras, $\mathfrak{a}_{\mathfrak{h}} \subset \mathfrak{a}$ be Cartan subspaces of $\mathfrak{h}$ and $\mathfrak{g}$. Choose a system of simple roots $\Pi_{\mathfrak{h}}$ of $\mathfrak{a}_{\mathfrak{h}}$ in $\mathfrak{h}$ and a compatible system of simple roots Π of $\mathfrak{a}$ in $\mathfrak{g}$, i.e. such that the restriction to $\mathfrak{a}_{\mathfrak{h}}$ of the simple roots $\alpha \in \Pi$ belong to $\Pi_{\mathfrak{h}} \cup \{0\}$.*
a) Using the notation (8.1), one has $\Theta_H = \{\alpha \in \Pi \mid \alpha^{\omega}(\mathfrak{a}_{\mathfrak{h}}) \neq 0\,\}$.
b) Let P_{Θ_H} be the parabolic subgroup of G as in Section 7.6. Then the intersection $P_H := H \cap P_{\Theta_H}$ is a minimal parabolic subgroup of H.
c) One has a H-equivariant embedding $H/P_H \hookrightarrow G/P_{\Theta_H}$.

PROOF OF LEMMA 16.3. *a*) One can choose a Cartan involution of G that preserves H. The corresponding Cartan projection κ of G satisfies $\kappa(H) = \kappa(\exp(\mathfrak{a}_{\mathfrak{h}}))$ and hence $\alpha^{\omega}(\kappa(H))$ is bounded if and only if $\alpha^{\omega}(\mathfrak{a}_{\mathfrak{h}}) = 0$.

b) Let $\mathfrak{p}_{\Theta_H}$ be the parabolic Lie subalgebra of $\mathfrak{h}$ associated to the subset Θ_H of Π. According to *a*), the Lie algebra $\mathfrak{p}_{\Theta_H}$ is defined over $\mathbb{R}$ and the intersection $\mathfrak{p}_H = \mathfrak{h} \cap \mathfrak{p}_{\Theta_H}$ is the minimal parabolic Lie subalgebra of $\mathfrak{h}$ associated to $\Pi_{\mathfrak{h}}$. Hence its normalizer $P_H = H \cap P_{\Theta_H}$ is the minimal parabolic subgroup of H associated to Π.

c) This follows from point *b*). $\square$

Now, we still let $S_\nu \subset \mathcal{P}_{\Theta_\mu}$ denote the support of the μ-stationary measure ν, $\sigma_\mu \in \mathfrak{a}$ the average of σ, Φ_μ the covariance 2-tensor of σ_{Θ_μ}, $\mathfrak{a}_\mu$ its linear span. Let Δ_μ, $\overline{\varphi}_0 : S_\nu \to \mathfrak{a}_\mu/\Delta_\mu$, $v_\mu \in \mathfrak{a}_\mu$ be as in Proposition 14.8.

PROPOSITION 16.4. *Same assumptions as in Proposition 16.1.*
a) The subgroup Δ_μ is F-stable and the image of the vector v_μ in $\mathfrak{a}_\mu/\Delta_\mu$ is F-invariant.
b) The lifting property (15.3) *holds. More precisely, there exists a Hölder continuous function $\widetilde{\varphi}_0 : S_\nu \to \mathfrak{a}$ such that, for all (g,η) in* $\mathrm{Supp}\mu \times S_\nu$,

$$\sigma_{\Theta_\mu}(g,\eta) \in \sigma_\mu + v_\mu - \widetilde{\varphi}_0(g\eta) + \widetilde{\varphi}_0(\eta) + \Delta_\mu.$$

PROOF. *a*) The F-invariance follows from Corollaries 11.4 and 14.12.

b) As in the proof of Proposition 16.1, we can assume, using Lemmas 16.2 and 16.3, that the finite set $\mathcal{S}$ does not contain the local field $\mathbb{C}$. Let, for any s in $\mathcal{S}$, $\mathfrak{c}_s$ be the subspace of $\mathfrak{a}_s$ spanned by the image under ω of the center of $G_{s,c}$, so that one has

$$\mathfrak{a}_s = \mathfrak{b}_s \oplus \mathfrak{c}_s.$$

Set $\mathfrak{c} = \bigoplus_{s\in\mathcal{S}} \mathfrak{c}_s$ and $\mathfrak{b}_f = \bigoplus \mathfrak{b}_s$ where the sum is over the non-archimedean local fields $\mathbb{K}_s$. Since the set $\mathcal{S}$ does not contain $\mathbb{C}$, one has

$$\mathfrak{a} = \mathfrak{b}_\mathbb{R} \oplus \mathfrak{b}_f \oplus \mathfrak{c}.$$

By Proposition 14.8, we already know that there exist an element v_μ of E_μ and a Hölder continuous function $\overline{\varphi}_0 : S_\nu \to E/\Delta_\mu$ such that, for any (g,η) in $\mathrm{Supp}\mu \times S_\nu$, one has

$$\sigma(g,\eta) = \sigma_\mu + v_\mu - \overline{\varphi}_0(g\eta) + \overline{\varphi}_0(\eta) \mod \Delta_\mu. \tag{16.2}$$

Let σ' be the projection of σ_{Θ_μ} on $\mathfrak{b}_f \oplus \mathfrak{c}$ in this direct sum. By construction, the cocycle σ' is invariant under $G_{\infty,c}$, that is, $\sigma'(g,h\eta) = \sigma'(g,\eta)$ for any g in G, h in $G_{\infty,c}$ and η in $\mathcal{P}_{\Theta_\mu}$. Let X' be the compact metric G-space

$$X' := G_{\mathbb{R},c}\backslash\mathcal{P}_{\Theta_\mu} \text{ and } \pi : \mathcal{P}_{\Theta_\mu} \to X'$$

be the natural map. Note that X' is totally discontinuous. We can consider σ' as a cocycle $G \times X' \to \mathfrak{b}_f \oplus \mathfrak{c}$. By Proposition 16.1, the group Δ_μ contains $\mathfrak{b}_\mathbb{R}$. By Corollary 14.7, the μ-residual image Δ'_μ of the cocycle σ' on X' is equal to $\Delta_\mu/\mathfrak{b}_\mathbb{R}$. Now Equation (14.7) reads as

$$\sigma'(g,\pi(\eta)) = \sigma_\mu + v_\mu - \overline{\varphi}_0(g\eta) + \overline{\varphi}_0(\eta) \mod \Delta_\mu,$$

for g in G and η in S_ν. By Corollary 14.10, for any η, η' in S_ν with $\pi(\eta) = \pi(\eta')$, one has $\overline{\varphi}_0(\eta) = \overline{\varphi}_0(\eta')$. In particular, $\overline{\varphi}_0$ factors as a

Hölder continuous function from a totally discontinuous space to $\mathfrak{a}/\Delta_\mu$. Hence, it can be lifted as a Hölder continuous function $\widetilde{\varphi}_0 : S_\nu \to \mathfrak{a}$. This ends the proof when $\mathcal{S}$ does not contain the local field $\mathbb{C}$. □

16.2. Local limit theorem for $\mathcal{S}$-adic Lie groups.

We can now state and prove the local limit theorem for products of random matrices in $\mathcal{S}$-adic Lie groups.

For $n \geq 1$ and η in the support S_ν of ν, we will describe the behavior of the measure $\mu_{n,\eta}$ on $\mathfrak{a}$ given by, for every $\psi \in \mathcal{C}_c(\mathfrak{a})$,

$$\mu_{n,\eta}(\psi) = \int_G \psi(\sigma_{\Theta_\mu}(g,\eta) - n\sigma_\mu)\, \mathrm{d}\mu^{*n}(g) \tag{16.3}$$

If $\mathcal{S} = \{\mathbb{R}\}$ and G is semisimple, we set $\Delta_\mu = \mathfrak{b}_\mu = \mathfrak{a}$, $v_\mu = 0$ and we denote by π_μ the Lebesgue measure on $\mathfrak{a}$ defined above.

In general, because of the non-archimedean factors of G, and the eventual periodicity phenomena in the center of G_c, the group Δ_μ is only cocompact in $\mathfrak{a}_\mu$. We let π_μ be the Haar measure of Δ_μ which gives mass one to the unit cubes of Φ_μ^* in the connected component of Δ_μ.

Let Π_μ be the average measure, given, for $C \subset \mathfrak{a}$, by

$$\Pi_\mu(C) = \int_{\mathcal{P}_{\Theta_\mu}} \pi_\mu(\widetilde{\varphi}_0(\eta')+C)\, \mathrm{d}\nu(\eta').$$

THEOREM 16.5 (Local limit theorem for $\sigma_{\Theta_\mu}(g)$). *Let G be an algebraic reductive $\mathcal{S}$-adic Lie group, $F := G/G_c$ and μ be a Zariski dense Borel probability measure on G with a finite exponential moment and which is aperiodic in F.*

We fix a bounded convex subset $C \subset \mathfrak{a}$ and $R > 0$. Then one has the limit

$$\lim_{n\to\infty} \tfrac{1}{G_n(v_n)}\mu_{n,\eta}(C+v_n) \;-\; \Pi_\mu(v_n - nv_\mu + \widetilde{\varphi}_0(\eta)+C) \;=\; 0.$$

This limit is uniform for $\eta \in S_\nu$ and $v_n \in \mathfrak{a}_\mu$ with $\|v_n\| \leq \sqrt{Rn\log n}$.

In an analogous way, we leave to the reader the task to translate the local limit theorem with target 15.15 in this case.

PROOF. Theorem 16.5 follows from Theorem 15.1 applied to the cocycle σ_{Θ_μ}. The contraction condition and the moment condition were checked in Lemmas 12.1 and 12.5. The lifting condition of this cocycle over the limit set S_ν was checked in Proposition 16.4. □

16.3. Local Limit Theorem for the Iwasawa cocycle.

¿From now on in this chapter, the base field is $\mathbb{K} = \mathbb{R}$, and we will state various versions of the Local Limit Theorem. In this section we will state the Local Limit Theorem for the Iwasawa cocycle. We will allow a target and a moderate deviation.

In this section and the next one, we keep the following notations from Sections 5.7 and 7.2. The group G is an algebraic semisimple real Lie group, $G = K \exp \mathfrak{a} N$ is the Iwasawa decomposition, $G = K \exp \mathfrak{a}^+ K_c$ is the Cartan decomposition, $\mathcal{P} = G/P$ is the flag variety, $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is the Iwasawa cocycle, and $\kappa : G \to \mathfrak{a}^+$ is the Cartan projection.

We also keep the following notations from Sections 9.4 and 12.4. We let μ be a Borel probability measure on G which is Zariski dense in G and has a finite exponential moment. We set ν for the μ-stationary probability measure on $\mathcal{P}$, $\sigma_\mu \in \mathfrak{a}^{++}$ for its Lyapunov vector, N_μ for the Gaussian probability measure with full support on $\mathfrak{a}$ which occurs in the Central Limit Theorem 12.11, $\Phi_\mu \in S^2(\mathfrak{a})$ for its covariance 2-tensor.

In the following version of the Local Limit Theorem for the Iwasawa cocycle, we allow a target φ and a moderate deviation v_n.

Theorem 16.6. (Local Limit Theorem for $\sigma(g)$) *Let G be an algebraic semisimple real Lie group, μ be a Zariski dense Borel probability measure on G with a finite exponential moment and ν be the μ-stationary probability measure on $\mathcal{P}$. We fix, a continuous function $\varphi \in \mathcal{C}^0(\mathcal{P})$, an open bounded convex subset $C \subset \mathfrak{a}$ and $R > 0$. Then, one has*

$$\lim_{n\to\infty} \tfrac{1}{N_\mu^{*n}(C+v_n)} \int_{\{\sigma(g,\eta)-n\sigma_\mu \in C+v_n\}} \varphi(g\eta)\, \mathrm{d}\mu^{*n}(g) = \nu(\varphi). \tag{16.4}$$

This limit is uniform for all η in $\mathcal{P}$ and all $v_n \in \mathfrak{a}$ with $\|v_n\| \leq \sqrt{Rn \log n}$.

In particular when $v_n = o(\sqrt{n})$, one has,

$$\lim_{n\to\infty} \sqrt{(2\pi n)^r \det \Phi_\mu}\; \mu^{*n}(\{g \mid \sigma(g,\eta) - n\sigma_\mu \in C + v_n\}) = |C|.$$

Here, $|C|$ denotes the volume of C for a Lebesgue measure on $\mathfrak{a}$, and the *determinant* $\det \Phi_\mu$ is computed with respect to the same Lebesgue measure.

It will be crucial for the applications in the next three sections to have allowed a target φ and a moderate deviation v_n.

The main reason to deal only with the field $\mathbb{K} = \mathbb{R}$ is that in this case the statements are much simpler.

Proof of Theorem 16.6. We begin by assuming that the measure μ is aperiodic in $F := G/G_c$. In this case these claims follow from Theorem 16.5 and the following two remarks.

First, the limit measure π_μ is a Lebesgue measure on the whole Cartan subspace $\mathfrak{a}$ because of the aperiodicity of the Iwasawa cocycle (Proposition 16.1).

Second, the fact that the convergence is uniform for η in the whole flag variety $\mathcal{P}$ and not just the the limit set S_ν follows from Corollary 15.7. Indeed, since the Iwasawa cocycle is aperiodic, the function $\overline{\varphi}_0$ can be defined on the whole flag variety as the zero function $\overline{\varphi}_0 := 0$.

We now deal with a measure μ which is not aperiodic. We will deduce our claims from the first case. We recall that F_μ is the normal subgroup of the finite group $F = G/G_c$ introduced in Lemma 10.6 and that p_μ is the cardinality of the cyclic group F/F_μ. Let G' be the algebraic subgroup of G containing G_c whose image in F is F_μ. The probability measure $\mu' := \mu^{*p_\mu}$ is Zariski dense in G' and, by Lemma 10.6, the measure μ' is aperiodic in F_μ. We decompose $n = n'p_\mu + r$ with $0 \leq r < p_\mu$, we rewrite the integral I_n in the left hand side of (16.4) as

$$\int_{\{\|\kappa(g_1)\|\leq(\log n)^2\}} \left(\int_{\left\{ \substack{\sigma(g_2,g_1\eta)-n'p_\mu\sigma_\mu\in \\ C+v_n-\sigma(g_1,\eta)-r\sigma_\mu} \right\}} \varphi(g_2g_1\eta)\, \mathrm{d}\mu'^{*n'}(g_2) \right) \mathrm{d}\mu^{*r}(g_1) + R_n.$$

We claim that, uniformly in η and v_n, the error terms R_n satisfies, $R_n = o(n^{-A})$ for all $A > 0$.

Indeed, we choose a small $t_0 > 0$ and we compute, using Chebyshev inequality,

$$\begin{array}{rcl} |R_n| & \leq & \mu^{*r}(\{g_1 \in G \mid \|\kappa(g_1)\| \geq (\log n)^2\})\, \|\varphi\|_\infty \\ & \leq & e^{-t_0(\log n)^2}\, \|\varphi\|_\infty \int_G e^{t_0\|\kappa(g_1)\|}\, \mathrm{d}\mu^{*r}(g_1). \end{array}$$

Since, thanks to the bound (7.17), the measure μ^{*r} also has a finite exponential moment (9.3), we deduce that $|R_n| = o(n^{-A})$, for all $A > 0$.

In view of Remark 15.3, we can neglect the error term R_n and apply the first case to the measure μ' in order to estimate the integral in between the parenthesis. □

16.4. Local Limit Theorem for the Cartan projection.

We explain in this section how one can deduce the Local Limit Theorem for the Cartan projection from the Local Limit Theorem for the Iwasawa cocycle.

We keep the notations of Section 16.3.

THEOREM 16.7. (Local Limit Theorem for $\kappa(g)$) *Let G be an algebraic semisimple real Lie group, μ be a Zariski dense Borel probability measure on G with a finite exponential moment. We fix an open bounded convex subset $C \subset \mathfrak{a}$ and $R > 0$. Then, one has*

$$\lim_{n\to\infty} \frac{\mu^{*n}(\{g \in G \mid \kappa(g) - n\sigma_\mu \in C + v_n\})}{N_\mu^{*n}(C + v_n)} = 1. \tag{16.5}$$

This limit is uniform for all $v_n \in \mathfrak{a}$ with $\|v_n\| \leq \sqrt{Rn\log n}$.

In particular when $v_n = o(\sqrt{n})$, one has,

$$\lim_{n\to\infty} \sqrt{(2\pi n)^r \det \Phi_\mu}\, \mu^{*n}(\{g \mid \kappa(g) - n\sigma_\mu \in C + v_n\}) = |C|.$$

The main idea in the proof of Theorem 16.7 is to write the variable $\kappa(b_n \cdots b_1)$ as the sum of three variables $\sigma(b_n \cdots b_{\ell+1}, x_\ell) + \kappa(b_\ell \cdots b_1) + r_n$ where $x_\ell = b_\ell \cdots b_1 x$ and $\ell = [(\log n)^2]$ and where the error term r_n decays to zero outside a set whose probability decays faster than any power of n. We will deal with the first term thanks to the Local Limit Theorem for the Iwasawa cocycle. The second term will be seen as a moderate deviation.

Again, a key ingredient in the proof of Theorem 16.7 will be the following lower bound for the denominator of the left hand side of (16.5) (see Remark 15.3)

$$N_\mu^{*n}(C + v_n) \geq A_0\, n^{-R-\frac{r}{2}} \tag{16.6}$$

where the constant A_0 depends only on μ, R and C. This lower bound will allow us to neglect subsets S_n of G whose measure $\mu^{*n}(S_n)$ decays faster than any power of n.

The proof will also rely on the following lemma which gives a very precise estimate of the Cartan projection in terms of the Iwasawa cocycle.

LEMMA 16.8. *Let G be an algebraic semisimple real Lie group, μ be a Zariski dense Borel probability measure on G with a finite exponential moment. For all $\varepsilon > 0$, there exists $c > 0$ and $\ell_0 > 0$ such that, for all $n \geq \ell \geq \ell_0$, for all η in $\mathcal{P}$, there exists a subset $S_{n,\ell,\eta} \subset G \times G$ with*

$$\mu^{*(n-\ell)} \otimes \mu^{*\ell}(S_{n,\ell,\eta}) \geq 1 - e^{-c\ell}$$

and for all (g_2, g_1) in $S_{n,\ell,\eta}$, one has

$$\|\kappa(g_2 g_1) - \sigma(g_2, g_1\eta) - \kappa(g_1)\| \leq e^{-\varepsilon\ell}. \tag{16.7}$$

Using the phrasing of (13.35), Lemma 16.8 tells us that, uniformly for η in $\mathcal{P}$, the following property is true except on an exponentially small set

$$\|\kappa(b_n\cdots b_1)-\sigma(b_n\cdots b_{\ell+1},b_\ell\cdots b_1\eta)-\kappa(b_\ell\cdots b_1)\|\leq e^{-\varepsilon\ell}. \tag{16.8}$$

Proof of Lemma 16.8. In this proof we will assume G to be connected. The general case is left to the reader. Using the interpretation of the Iwasawa cocycle and the Cartan projection in terms of norms in various representations of G given in Lemmas 5.32 and 5.33, we only have to check the following claim.

Let $V=\mathbb{R}^d$ and μ be a probability measure on $\mathrm{GL}(V)$ with a finite exponential moment such that Γ_μ is proximal and strongly irreducible. Then, uniformly for nonzero v in V, the following property is true except on an exponentially small set

$$|\log\|b_n\cdots b_1\|-\log\tfrac{\|b_n\cdots b_1v\|}{\|b_\ell\cdots b_1v\|}-\log\|b_\ell\cdots b_1\|\,|\leq e^{-\varepsilon\ell}. \tag{16.9}$$

Indeed we will prove successively that, uniformly for $x=\mathbb{R}v$ in $\mathbb{P}(V)$, the following properties (16.10) to (16.15) are true except on an exponentially small set.

First, according to the simplicity of the first Lyapunov exponent (Corollary 9.15) and to the Large Deviation Principle (Theorem 12.17), the property

$$\gamma_{1,2}(b_\ell\cdots b_1)\leq e^{-\varepsilon\ell} \tag{16.10}$$

is true except on an exponentially small set. Hence, using Lemma 13.2 and its notations, the properties

$$|\log\|b_n\cdots b_1\|-\log\tfrac{\|b_n\cdots b_1v\|}{\|v\|}-\log\delta(x,y^m_{b_n\cdots b_1})\,|\leq e^{-\varepsilon\ell}\quad\text{and} \tag{16.11}$$

$$|\log\|b_\ell\cdots b_1\|-\log\tfrac{\|b_\ell\cdots b_1v\|}{\|v\|}-\log\delta(x,y^m_{b_\ell\cdots b_1})\,|\leq e^{-\varepsilon\ell} \tag{16.12}$$

are true except on an exponentially small set.

Second, let $\lambda_{1,\mu}>\lambda_{2,\mu}$ be the two first Lyapunov exponents of μ. According to (13.5) and (13.6), the properties

$$\delta(x,y^m_{b_n\cdots b_1})\geq e^{-\varepsilon\ell}\quad\text{and} \tag{16.13}$$

$$d(y^m_{b_\ell\cdots b_1},y^m_{b_n\cdots b_1})\leq e^{-(\lambda_{1,\mu}-\lambda_{2,\mu}-\varepsilon)\ell} \tag{16.14}$$

(with $x=\mathbb{R}v$) are true except on an exponentially small set. These two bounds (16.13) and (16.14) imply that the property

$$|\log\delta(x,y^m_{b_n\cdots b_1})-\log\delta(x,y^m_{b_\ell\cdots b_1})|\leq e^{-(\lambda_{1,\mu}-\lambda_{2,\mu}-2\varepsilon)\ell} \tag{16.15}$$

is true except on an exponentially small set.

Now, the bounds (16.11), (16.12), (16.15) imply the claim (16.9). $\square$

Proof of Theorem 16.7. Our claims follow from the Local Limit Theorem 16.6 for the Iwasawa cocycle and from Lemma 16.8.

We write $n = m + \ell$ with $\ell = [(\log n)^2]$, and $g = g_2 g_1$ with $g_2 = b_n \cdots b_{\ell+1}$ and $g_1 = b_\ell \cdots b_1$. We first prove the upper bound in (16.5). We fix $\varepsilon > 0$ and introduce the ε-neighborhood C^ε of C.

Let $M = 2\,\|\sigma_\mu\|$. According to the Large Deviation Principle (Theorem 12.17), the following property is true except on an exponentially small set

$$\|\kappa(b_\ell \cdots b_1)\| \leq M\ell. \tag{16.16}$$

Combining (16.8) with (16.16), one gets the following upper bound for the numerator N_n of the left hand side of (16.5)

$$\int_{\{\|\kappa(g_1)\|\leq M\ell\}} \mu^{*(n-\ell)}(\{g_2 \mid \sigma(g_2, g_1 x) + \kappa(g_1) - n\sigma_\mu \in C^\varepsilon + v_n\})\, \mathrm{d}\mu^{*\ell}(g_1) + R_n$$

where, uniformly in v_n, the error term R_n decays exponentially in ℓ and hence decays faster than any power of n.

Hence the left-hand side of (16.5) is bounded, uniformly in v_n, by

$$\limsup_{n\to\infty} \sup_{\substack{\|w\|\leq M(\log n)^2 \\ \|v\|^2 \leq R\, n \log n}} \frac{N_\mu^{*(n-\ell)}(C^\varepsilon + v + w)}{N_\mu^{*n}(C + v)} = \frac{|C^\varepsilon|}{|C|} \xrightarrow[\varepsilon\to 0]{} 1.$$

This proves the upper bound in (16.5). The lower bound is proved in the same way using the convex sets C_ε introduced in (15.19). □

16.5. Local Limit Theorem for the norm.

We explain in this section how one can prove the Local Limit Theorem both for the norm of the matrices and for the norm of vectors using the Local Limit Theorem for the Iwasawa cocycle.

In this section and the next one we come back to the assumptions and keep the notation $\lambda_{1,\mu}$, $\Phi_{1,\mu}$ and $N_{\Phi_{1,\mu}}$ from Section 13.7. We assume moreover that $\mathbb{K} = \mathbb{R}$, that the Borel probability measure μ is supported by $\mathrm{SL}(V)$ and that Γ_μ is unbounded. These conditions ensure that the Zariski closure G of Γ_μ is a non-compact reductive group with compact center, that $\lambda_{1,\mu} > 0$ and that $\Phi_{1,\mu} > 0$. We assume also that the Euclidean norm $\|.\|$ in V is good for G as defined in Lemma 5.33. Note that the construction given in this Lemma 5.33 proves the existence of such a good norm for any strongly irreducible representation of a reductive algebraic real Lie group.

Theorem 16.9. (Local Limit Theorem for $\log\|g\|$) *Let $V = \mathbb{R}^d$ and μ be a Borel probability measure on $\mathrm{SL}(V)$ with a finite exponential* *moment such that* Γ_μ *is unbounded and strongly irreducible. Let* $a_1 < a_2$ *and* $R > 0$. *Then, one has*

$$\lim_{n\to\infty} \frac{\mu^{*n}(\{g \in G \mid \log\|g\| - n\lambda_{1,\mu} \in [a_1, a_2] + t_n\})}{N^{*n}_{\Phi_{1,\mu}}([a_1, a_2] + t_n)} = 1.$$

This limit is uniform for all $t_n \in \mathbb{R}$ *with* $|t_n| \le \sqrt{Rn\log n}$.

In particular when $t_n = o(\sqrt{n})$, *one has*

$$\lim_{n\to\infty} \sqrt{2\pi\Phi_{1,\mu}\, n}\; \mu^{*n}(\{g \in G \mid \log\|g\| - n\lambda_{1,\mu} \in [a_1, a_2] + t_n\}) = a_2 - a_1.$$

Proof of Theorem 16.9. This is a straightforward application of the Local Limit Theorem for the Cartan projection (Theorem 16.7) combined with the interpretation of the Cartan projection in terms of representations (Lemmas 5.32, 5.33 and Section 7.2). □

Theorem 16.10. (Local Limit Theorem for $\log\|gv\|$) *Let* $V = \mathbb{R}^d$ *and* μ *be a Borel probability measure on* $\mathrm{SL}(V)$ *with a finite exponential moment such that* Γ_μ *is unbounded and strongly irreducible. Let* $a_1 < a_2$ *and* $R > 0$. *Then, one has*

$$\lim_{n\to\infty} \frac{\mu^{*n}(\{g \in G \mid \log\|gv\| - n\lambda_{1,\mu} \in [a_1, a_2] + t_n\})}{N^{*n}_{\Phi_{1,\mu}}([a_1, a_2] + t_n)} = 1. \tag{16.17}$$

This limit is uniform for all v *in* V *with* $\|v\| = 1$ *and all* $t_n \in \mathbb{R}$ *with* $|t_n| \le \sqrt{Rn\log n}$. *In particular when* $t_n = o(\sqrt{n})$, *one has*

$$\lim_{n\to\infty} \sqrt{2\pi\Phi_{1,\mu}\, n}\; \mu^{*n}(\{g \in G \mid \log\|gv\| - n\lambda_{1,\mu} \in [a_1, a_2] + t_n\}) = a_2 - a_1.$$

When Γ_μ is proximal this Theorem 16.10 may be seen as a direct consequence of the general Local Limit Theorem 15.1 for a cocycle over a μ-contracting action applied to the norm cocycle

$$\sigma_1(g, x) = \log \tfrac{\|gv\|}{\|v\|} \quad \text{where } x = \mathbb{R}v.$$

The main issue in the proof of Theorem 16.10 is to control the norm cocycle σ_1 without this proximality assumption. Roughly speaking, the idea is to write the variable $\sigma_1(b_n \cdots b_1, x)$ as the sum of two variables $\sigma_1(b_n \cdots b_{\ell+1}, x_\ell) + \sigma_1(b_\ell \cdots b_1, x)$ with $x_\ell = b_\ell \cdots b_1 x$ and $\ell = [(\log n)^2]$. The point x_ℓ will be very quickly approximated by another point x'_ℓ living on a r-dimensional subspace z'_ℓ which belongs to the limit set Λ^r_Γ where r is the proximal dimension of Γ_μ. For this point, the norm cocycle can be computed thanks to the Iwasawa cocycle. The second term will be seen as a moderate deviation.

We will need the following Lemma 16.11 in which we keep the notations $z^m_g \in \mathbb{G}_{d-r}(V)$ for the density $(d-r)$-dimensional subspace of ${}^t g$ introduced in Lemma 13.8.

LEMMA 16.11. *Let $V = \mathbb{R}^d$, $x = \mathbb{R}v$, $x' = \mathbb{R}v'$ in $\mathbb{P}(V)$ and g be an element of* $\mathrm{GL}(V)$ *whose r first singular values are equal. Then one has the bound*

$$\left|\log \frac{\|gv\|}{\|v\|} - \log \frac{\|gv'\|}{\|v'\|}\right| \leq \frac{\sqrt{2}\, d(x,x')}{\min(d(x,z_g^m), d(x',z_g^m))}. \tag{16.18}$$

PROOF OF LEMMA 16.11. With no loss of generality, we can choose the vectors such that $\|v\| = \|v'\| = 1$, such that $\|v - v'\| \leq \sqrt{2}\, d(x,x')$ and such that $\|gv'\| \geq \|gv\|$. Using the bound $\log(1+t) \leq t$ for all $t \geq 0$, and using Lemma 13.8, one computes

$$\log \frac{\|gv'\|}{\|gv\|} \leq \frac{\|g\|\,\|v-v'\|}{\|gv\|} \leq \frac{\sqrt{2}\, d(x,x')}{d(x,z_g^m)}$$

This proves (16.18). □

We will also need a few facts and notations from the previous chapters. Since $\mathbb{K} = \mathbb{R}$, by Lemma 5.23, the proximal dimension r of Γ_μ is also the proximal dimension of G. Since V is strongly irreducible, V has a highest weight χ. The corresponding weight space $V^\chi \subset V$ has dimension r. For any $\eta = gP_c$ in the flag variety $\mathcal{P} = G/P_c$, we denote by V_η the space $V_\eta := gV^\chi$ as in (5.10). The map $\eta \mapsto V_\eta$ is a G-equivariant map from $\mathcal{P}$ to $\mathbb{G}_r(V)$. By construction, the image of this map is the limit set Λ_G^r defined in Lemma 3.2. We introduce the closed subset of $\mathbb{P}(V)$,

$$Z_G := \{x \in \mathbb{P}(V) \mid \exists \eta \in \mathcal{P}\ ,\ x \in \mathbb{P}(V_\eta)\} = \textstyle\bigcup_{z \in \Lambda_G^r} z.$$

Since the norm on V is good, according to Lemma 5.33, for g in G, η in $\mathcal{P}$ and v nonzero in V_η, one has,

$$\log \tfrac{\|gv\|}{\|v\|} = \chi(\sigma(g,\eta)) \tag{16.19}$$

where σ is the Iwasawa cocycle.

Let $\lambda_{1,\mu} \geq \ldots \geq \lambda_{d,\mu}$ be the Lyapunov exponents of μ. We recall that, according to Corollary 9.15, one has $\lambda_{1,\mu} = \cdots = \lambda_{r,\mu} > \lambda_{r+1,\mu}$ where r is the proximal dimension of Γ_μ. The following Lemma 16.12 tells us that, uniformly in $x \in \mathbb{P}(V)$, the property

$$d(b_\ell \cdots b_1 x, Z_G) \leq e^{-(\lambda_{1,\mu} - \lambda_{r+1,\mu} + \varepsilon)\ell}$$

is true except on an exponentially small set.

LEMMA 16.12. *Let $V = \mathbb{R}^d$ and μ be a Borel probability measure on* $\mathrm{SL}(V)$ *such that Γ_μ is unbounded, strongly irreducible and μ has a finite exponential moment. For all $\varepsilon > 0$, there exists $c > 0$ and*

$\ell_0 > 0$ *such that, for all* $\ell \geq \ell_0$*, for all* x *in* $\mathbb{P}(V)$*, there exists a subset* $S_{\ell,x} \subset G$ *with*

$$\mu^{*\ell}(S_{\ell,x}) \geq 1 - e^{-c\ell}$$

and for all g_1 *in* $S_{\ell,x}$*, there exists a point* x'_{g_1x} *in* Z_G *such that*

$$d(g_1x, x'_{g_1x}) \leq e^{-(\lambda_{1,\mu} - \lambda_{r+1,\mu} + \varepsilon)\ell}. \tag{16.20}$$

Proof of Theorem 16.12. The proof is similar to the one of (13.6). The point x'_{g_1x} is (a measurable choice of) a point on Z_G whose distance to g_1x is minimal. □

Proof of Theorem 16.10. We write $n = m+\ell$ with $\ell = [(\log n)^2]$, and $g = g_2g_1$ with $g_2 = b_n \cdots b_{\ell+1}$ and $g_1 = b_\ell \cdots b_1$. We first prove the upper bound in (16.17). We fix $\varepsilon > 0$ and introduce the ε-neighborhood I^ε of the interval $I := [a_1, a_2]$.

Let $M = 2\,\lambda_{1,\mu}$. According to the Large Deviation Principle (Theorem 12.17), the following property is true except on an exponentially small set

$$\|b_\ell \cdots b_1\| \leq M\ell. \tag{16.21}$$

According to (13.25), uniformly for x' in $\mathbb{P}(V)$, the following property is true except on an exponentially small set

$$d(x', z^m_{b_n\cdots b_{\ell+1}}) \geq e^{-\varepsilon\ell}. \tag{16.22}$$

Combining (16.18), (16.20), (16.21) and (16.22), one gets the following upper bound for the numerator N_n of the left hand side of (16.17)

$$\int_{\left\{\substack{g_1 \in S_{\ell,x} \\ \log\|g_1\| \leq M\ell}\right\}} \mu^{*m}\left(\left\{g_2 \mid \substack{\sigma_1(g_2, x'_{g_1x}) + \sigma_1(g_1,x) - n\lambda_{1,\mu} \in I^\varepsilon + t_n \\ \delta(g_1x, y^m_{g_2}) \geq e^{-\varepsilon\ell}}\right\}\right) \mathrm{d}\mu^{*\ell}(g_1) + R_n$$

where, uniformly in t_n, the error term R_n decays exponentially in ℓ and hence decays faster than any power of n.

Hence, using (16.19) and the Local Limit Theorem 16.6 for the Iwasawa cocycle, one can bound, uniformly in t_n, the left-hand side of (16.17) by

$$\limsup_{n\to\infty} \sup_{\substack{s \leq M(\log n)^2 \\ t^2 \leq R\,n\log n}} \frac{N^{*(n-\ell)}_{\Phi_{1,\mu}}(I^\varepsilon + t + s)}{N^{*n}_{\Phi_{1,\mu}}(I + t)} = \frac{|I^\varepsilon|}{|I|} \xrightarrow[\varepsilon\to 0]{} 1.$$

This proves the upper bound in (16.17). The lower bound is proved in the same way using smaller intervals I_ε. □

It is plausible that the assumption that the Euclidean norm is good in Theorem 16.9 and 16.10 can be removed when Γ_μ is absolutely strongly irreducible.

Part 5

Appendix

1. Convergence of sequences of random variables

In this appendix, we establish more or less classical, purely probabilistic results about convergence of sequences of random variables.

1.1. Uniform integrability.

> The concept of uniform integrability is a tool which is useful for proving convergence of integrals when one cannot apply directly Lebesgue Convergence Theorem.

We first recall a usual lemma that we used in Section 3.5.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. A subset $A \in \mathcal{B}$ is sometimes called an *event*. A measurable function $\psi : \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{B})$ is sometimes called a *random variable*. The *law* of ψ is the probability measure on $\mathbb{R}$ which is the image of $\mathbb{P}$ by ψ. We will write $\mathbb{E}|\psi| := \int_\Omega |\psi| \, \mathrm{d}\mathbb{P}$ for the L^1-norm of ψ and, when this norm is finite, we will write $\mathbb{E}(\psi) := \int_\Omega \psi \, \mathrm{d}\mathbb{P}$ for the expectation or space average of this random variable ψ.

A subset $\mathcal{I}$ of $L^1(\Omega, \mathcal{B}, \mathbb{P})$ is said to be *uniformly integrable* if it is bounded and if, for any sequence A_n in $\mathcal{B}$ with $\mathbb{P}(A_n) \xrightarrow[n\to\infty]{} 0$, one has $\sup\limits_{\psi \in \mathcal{I}} \mathbb{E}(|\psi| \mathbf{1}_{A_n}) \xrightarrow[n\to\infty]{} 0$.

Example 1.1. Let $p > 1$. A sequence ψ_n of functions which is bounded in $\mathrm{L}^p(\Omega, \mathcal{B}, \mathbb{P})$ i.e. such that $\sup_{n\geq 1} \mathbb{E}(|\psi_n|^p) < \infty$ is uniformly integrable. Indeed this follows from Hölder inequality

$$\mathbb{E}(|\psi_n| \mathbf{1}_{A_n}) \leq \mathbb{E}(|\psi_n|^p)^{\frac{1}{p}} \, \mathbb{P}(A_n)^{1-\frac{1}{p}} \xrightarrow[n\to\infty]{} 0.$$

By Dunford-Pettis Theorem, a subset of $L^1(\Omega, \mathcal{B}, \mathbb{P})$ is uniformly integrable if and only if it is relatively compact for the weak topology. See [**38**, Chap. II, Thm T25]. We will only use the following Lemma 1.2 which is an easy consequence of Dunford-Pettis Theorem.

Lemma 1.2. (Uniform integrability) *Let ψ_n be a sequence of integrable functions on Ω which converges $\mathbb{P}$-almost surely. Then this sequence converges in $\mathrm{L}^1(\Omega, \mathcal{B}, \mathbb{P})$ if and only if it is uniformly integrable.*

Proof of Lemma 1.2. We just sketch the proof of this classical result. See [**91**, Chap. II-5].

$\Longrightarrow$ Set ψ for the limit. Since by assumption $\lim\limits_{n\to\infty} \mathbb{E}(|\psi_n - \psi|) = 0$, we may assume $\psi_n = \psi$ for all $n \geq 1$. Since, by Lebesgue's convergence theorem, one has $\lim\limits_{N\to\infty} \mathbb{E}(|\psi| \mathbf{1}_{|\psi|\geq N}) = 0$, our assertion follows from the bound $\mathbb{E}(|\psi| \mathbf{1}_{A_n}) \leq N\mathbb{P}(A_n) + \mathbb{E}(|\psi| \mathbf{1}_{|\psi|\geq N})$.

$\Longleftarrow$ By assumption one has $\sup_{n\geq 1} \mathbb{E}(|\psi_n|) < \infty$. By Fatou Lemma the limit ψ is integrable. Hence using the first implication, we can assume $\psi = 0$. Since ψ_n converges almost surely to 0, the sets $A_n := \{|\psi_n| \geq 1\}$ satisfy $\lim\limits_{n\to\infty} \mathbb{P}(A_n) = 0$. Hence by assumption one has $\lim\limits_{n\to\infty} \mathbb{E}(|\psi_n|\mathbf{1}_{A_n}) = 0$, and by Lebesgue convergence theorem one has $\lim\limits_{n\to\infty} \mathbb{E}(|\psi_n|\mathbf{1}_{A_n^c}) = 0$. Adding these equations proves that ψ_n converges to 0 in L^1. □

1.2. Martingale convergence Theorem.

We begin by recalling Doob martingale convergence Theorem that we use both in Sections 1.5 and 1.3.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. When $\mathcal{B}'$ is a sub-σ-algebra of $\mathcal{B}$, we write $\mathbb{E}(\psi \mid \mathcal{B}')$ for the *conditional expectation* of a random variable ψ with respect to $\mathcal{B}'$ (when it is defined) and $\mathbb{P}(A \mid \mathcal{B}') := \mathbb{E}(\mathbf{1}_A \mid \mathcal{B}')$ for the *conditional probability* of an event A.

Let $(\mathcal{B}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{B}$. We recall that a *martingale* with respect to $\mathcal{B}_n$ is a sequence ψ_n of $\mathbb{P}$-integrable functions on Ω such that, for all $n \geq 1$, ψ_n is the conditional expection of ψ_{n+1} with respect to $\mathcal{B}_n$, that is,

$$\psi_n = \mathbb{E}(\psi_{n+1} \mid \mathcal{B}_n).$$

THEOREM 1.3. (Doob martingale convergence theorem) *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, $\mathcal{B}_n$ an increasing sequence of sub-σ-algebras of $\mathcal{B}$ and ψ_n a martingale with respect to $\mathcal{B}_n$.*
a) If $\sup_{n\geq 1}\mathbb{E}|\psi_n| < \infty$, then there exists a $\mathbb{P}$-integrable function ψ_∞ on Ω such that $\psi_n \xrightarrow[n\to\infty]{} \psi_\infty$ $\mathbb{P}$-almost surely.
b) If the ψ_n are uniformly integrable, then one has $\mathbb{E}|\psi_n - \psi_\infty| \xrightarrow[n\to\infty]{} 0$.

The proof of 1.3 will use the following maximal inequality

LEMMA 1.4. *Let ψ_n be a martingale and $\varepsilon > 0$. Then*

$$\mathbb{P}(\sup_{1\leq k\leq n}|\psi_k| \geq \varepsilon) \leq \varepsilon^{-1}\, \mathbb{E}(|\psi_n|).$$

PROOF. We want to bound $\mathbb{P}(A)$ for $A = \cup_{1\leq k\leq n} A_k$ where

$$A_k = \{|\psi_1| < \varepsilon, \ldots, |\psi_{k-1}| < \varepsilon, |\psi_k| \geq \varepsilon\} \in \mathcal{B}_k.$$

We compute, using Chebyshev inequality and the martingale property,

$$\begin{aligned}\mathbb{P}(A) = \textstyle\sum_{k=1}^n \mathbb{P}(A_k) \leq \varepsilon^{-1} \sum_{k=1}^n \mathbb{E}(|\psi_k|\mathbf{1}_{A_k}) \leq \varepsilon^{-1} \sum_{k=1}^n \mathbb{E}(|\psi_n|\mathbf{1}_{A_k})\\ \leq \varepsilon^{-1}\mathbb{E}(|\psi_n|),\end{aligned}$$

which is the required inequality. □

Proof of Theorem 1.3 for L^2-bounded martingales. Since we will only use this theorem in this case we will give the proof under the assumption: $\sup_{n\geq 1}\mathbb{E}(\psi_n^2) < \infty$.

Using the martingale property, one has for $m \leq n$,

$$\mathbb{E}((\psi_n - \psi_m)^2) = \mathbb{E}((\psi_n)^2) - \mathbb{E}((\psi_m)^2).$$

Hence the sequence $\mathbb{E}((\psi_n)^2)$ is non decreasing, hence it is convergent, hence the sequence ψ_n is a Cauchy sequence in L^2, and hence ψ_n converges in L^2-norm to some function $\psi_\infty \in L^2$. Note that ψ_n converges also to ψ_∞ in L^1-norm.

According to Lemma 1.4, for $\varepsilon > 0$ and $m \geq 1$, one has

$$\mathbb{P}(\sup_{n\geq m}|\psi_n - \psi_m| \geq \varepsilon) \leq \varepsilon^{-1}\,\mathbb{E}(|\psi_\infty - \psi_m|) \xrightarrow[m\to\infty]{} 0.$$

This proves that the sequence ψ_n converges also $\mathbb{P}$-almost surely towards ψ_∞. $\square$

For a general proof, see for example [**62**].

1.3. Kolmogorov's Law of Large Numbers.

We now recall briefly Kolmogorov's law of large numbers and we explain how it can be deduced from Doob's martingale convergence theorem.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. Two sub-σ-algebras $\mathcal{B}'$ and $\mathcal{B}''$ of $\mathcal{B}$ are said to be *independent* if for every $B' \in \mathcal{B}'$ and $B'' \in \mathcal{B}''$ one has $\mathbb{P}(B' \cap B'') = \mathbb{P}(B')\mathbb{P}(B'')$. A sequence of functions φ_n on B is said to be *independent* if, for every $n \geq 1$, the sub-σ-algebra generated by φ_{n+1} is independent from the sub-σ-algebra $\mathcal{B}_n$ generated by $\varphi_1, \ldots, \varphi_n$.

We have the classical

Theorem 1.5. (Kolmogorov's Law of Large Numbers) *Let $(\varphi_n)_{n\geq 1}$ be a sequence of integrable random variables which are independent and have the same law. Then one has $\mathbb{P}$-almost surely*

$$\tfrac{1}{n}(\varphi_1 + \cdots + \varphi_n) \xrightarrow[n\to\infty]{} \mathbb{E}(\varphi_1).$$

This sequence converges also in L^1 i.e.

$$\mathbb{E}|\tfrac{1}{n}(\varphi_1 + \cdots + \varphi_n) - \mathbb{E}(\varphi_1)| \xrightarrow[n\to\infty]{} 0.$$

We will need a stronger version of Theorem 1.5 where the same conclusion is obtained under much weaker assumptions: the assumption that the variables have the *same law* is replaced by a domination by an integrable law and the *independence* assumption is replaced by a conditional recentering.

THEOREM 1.6. (Kolmogorov's Law of Large Numbers bis) *Let* $(\varphi_n)_{n\geq 1}$ *be a sequence of random variables and* $\mathcal{B}_n$ *be an increasing sequence of sub-*σ*-algebras such that* φ_n *is* $\mathcal{B}_n$*-measurable. Assume that there exists an integrable random variable* φ *such that, for every* $t \geq 0$, $n \geq 1$, *one has almost surely*

$$\mathbb{P}(\{|\varphi_n| > t\} \mid \mathcal{B}_{n-1}) \leq \mathbb{P}(\varphi > t). \tag{1.1}$$

Then one has almost surely

$$\tfrac{1}{n}\textstyle\sum_{k=1}^n (\varphi_k - \mathbb{E}(\varphi_k \mid \mathcal{B}_{k-1})) \xrightarrow[n\to\infty]{} 0.$$

This sequence converges also in L^1 *with a speed depending only on* φ, *i.e. there exist a sequence* $c_n = c_n(\varphi) \xrightarrow[n\to\infty]{} 0$ *such that*

$$\mathbb{E}|\tfrac{1}{n}\textstyle\sum_{k=1}^n (\varphi_k - \mathbb{E}(\varphi_k \mid \mathcal{B}_{k-1}))| \leq c_n(\varphi). \tag{1.2}$$

We note that Condition (1.1) implies that for every $t \geq 0$, $n \geq 1$, one has

$$\mathbb{P}(|\varphi_n| > t) \leq \mathbb{P}(\varphi > t). \tag{1.3}$$

We will need the following elementary trick:

LEMMA 1.7. (Kronecker) *Let* $(v_n)_{n\geq 1}$ *be a sequence in a normed vector space such that the series* $\sum_{k=1}^\infty \frac{1}{k} v_k$ *converges. Then the sequence* $\frac{1}{n}\sum_{k=1}^n v_k$ *converges to* 0.

PROOF. By assumption, the sequence $\psi_n := \sum_{k=1}^n \frac{1}{k} v_k$ converges. Hence, its Cesaro average converges to the same limit. Now, we have

$$\tfrac{1}{n}\textstyle\sum_{k=1}^n \psi_k = \tfrac{1}{n}\sum_{k=1}^n \sum_{\ell=1}^k \tfrac{1}{\ell} v_\ell = \tfrac{1}{n}\sum_{\ell=1}^n \tfrac{n-\ell+1}{\ell} v_\ell = \tfrac{n+1}{n}\psi_n - \tfrac{1}{n}\sum_{\ell=1}^n v_\ell.$$

The result follows. □

PROOF OF THEOREM 1.6. **First step**: We introduce the truncated random variables

$$\overline{\varphi}_n := \varphi_n \min(1, \tfrac{n}{|\varphi_n|}).$$

These functions $\overline{\varphi}_n$ are equal to φ_n when $|\varphi_n| \leq n$, to n when $\varphi_n \geq n$ and to $-n$ when $\varphi_n \leq -n$. We check that *almost surely* $\varphi_n - \overline{\varphi}_n$ *is equal to* 0 *except for finitely many* n. We also check that $\varphi_n - \overline{\varphi}_n$ *converges to* 0 *in* L^1.

The first statement follows from Borel-Cantelli Lemma since one computes using (1.3)

$$\textstyle\sum_{n\geq 1} \mathbb{P}(\varphi_n \neq \overline{\varphi}_n) = \sum_{n\geq 1} \mathbb{P}(|\varphi_n| > n) \leq \sum_{n\geq 1} \mathbb{P}(\varphi > n) \leq \mathbb{E}(\varphi)$$

which is finite since φ is integrable. The second statement follows from a similar computation using (1.3)

$$\begin{aligned}\mathbb{E}(|\varphi_n - \overline{\varphi}_n|) &= \textstyle\int_n^\infty \mathbb{P}(|\varphi_n| > t)\,\mathrm{d}t \\ &\le \textstyle\int_n^\infty \mathbb{P}(\varphi > t)\,\mathrm{d}t \le \mathbb{E}(\varphi \mathbf{1}_{\{\varphi > n\}})\end{aligned}$$

which goes to 0 for $n \to \infty$ by Lebesgue convergence theorem.
Second step: We introduce the random variables

$$\Phi_n := \mathbb{E}(\varphi_n \mid \mathcal{B}_{n-1}) \;\text{ and }\; \overline{\Phi}_n := \mathbb{E}(\overline{\varphi}_n \mid \mathcal{B}_{n-1})$$

and we check that *outside a null subset, the sequence* $\Phi_n - \overline{\Phi}_n$ *converges uniformly to* 0. Indeed this follows from a similar computation outside a null subset using (1.1)

$$\begin{aligned}|\Phi_n - \overline{\Phi}_n| &= \textstyle\int_n^\infty \mathbb{P}(\{|\varphi_n| > t\} \mid \mathcal{B}_{n-1})\,\mathrm{d}t \\ &\le \textstyle\int_n^\infty \mathbb{P}(\varphi > t)\,\mathrm{d}t \le \mathbb{E}(\varphi \mathbf{1}_{\{\varphi > n\}})\end{aligned}$$

which goes to 0 for $n \to \infty$.
Third step: We introduce the random variables

$$\psi_n = \textstyle\sum_{k=1}^n \frac{1}{k}(\overline{\varphi}_k - \overline{\Phi}_k)$$

and we check that this sequence ψ_n *converges almost surely and in* L^1 *towards a function* ψ_∞. This follows from Doob's martingale convergence theorem 1.3: by construction ψ_n is a martingale with respect to $\mathcal{B}_n$. We only have to check that *the sequence* ψ_n *is bounded in* L^2 *and hence uniformly integrable.* Hence we compute using orthogonality properties of the conditional expectation

$$\mathbb{E}(\psi_n^2) = \textstyle\sum_{k=1}^n \frac{1}{k^2}\mathbb{E}((\overline{\varphi}_k - \overline{\Phi}_k)^2) \le R := \sum_{k=1}^\infty \frac{1}{k^2}\mathbb{E}(\overline{\varphi}_k^2).$$

It remains to check that this right-hand side R is finite. For $t > 0$, we set $F_k(t) = \mathbb{P}(|\varphi_k| > t)$ and $F(t) := \mathbb{P}(\varphi > t)$. As in the first steps, but in a more tricky way, using integration by parts and (1.1), we get

$$\begin{aligned}R &= \textstyle\sum_{k=1}^\infty \frac{1}{k^2}\int_0^k 2t\, F_k(t)\,\mathrm{d}t \le \sum_{k=1}^\infty \frac{1}{k^2}\int_0^k 2t\, F(t)\,\mathrm{d}t \\ &\le \textstyle\sum_{m=1}^\infty \big(\sum_{k=m}^\infty \frac{1}{k^2}\big)\int_{m-1}^m 2t\, F(t)\,\mathrm{d}t \le \sum_{m=1}^\infty \frac{4}{m}\int_{m-1}^m t\, F(t)\,\mathrm{d}t \\ &\le 8\textstyle\int_0^\infty F(t)\,\mathrm{d}t = 8\,\mathbb{E}(\varphi) < \infty.\end{aligned}$$

Fourth step: We just combine the three first steps:

Set $c_{1,n} := \mathbb{E}(\varphi \mathbf{1}_{\{\varphi > n\}})$. By taking a Cesaro average in the first step, the sequence $\frac{1}{n}\sum_{k=1}^n(\varphi_k - \overline{\varphi}_k)$ converges to 0 almost surely and one has the L^1-bound

$$\mathbb{E}|\tfrac{1}{n}\textstyle\sum_{k=1}^n(\varphi_k - \overline{\varphi}_k)| \le \frac{1}{n}\sum_{k=1}^n c_{1,k}.$$

Using the second step in the same way, the sequence $\frac{1}{n}\sum_{k=1}^{n}(\Phi_k - \overline{\Phi}_k)$ converges to 0 almost surely and one also has the L^1-bound

$$\mathbb{E}|\tfrac{1}{n}\textstyle\sum_{k=1}^{n}(\Phi_k - \overline{\Phi}_k)| \leq \tfrac{1}{n}\textstyle\sum_{k=1}^{n} c_{1,k}.$$

Using Lemma 1.7, we deduce from the third step that the sequence $\frac{1}{n}\sum_{k=1}^{n}(\overline{\varphi}_k - \overline{\Phi}_k)$ converges to 0 almost surely. Using the same computation as in the proof of Lemma 1.7, one gets the equality

$$\begin{aligned} \tfrac{1}{n}\textstyle\sum_{k=1}^{n}(\overline{\varphi}_k - \overline{\Phi}_k) &= \psi_n - \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\psi_k \\ &= \tfrac{1}{n}\psi_\infty + (\psi_n - \psi_\infty) - \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}(\psi_k - \psi_\infty)\ , \end{aligned}$$

and the L^1-bound

$$\mathbb{E}|\tfrac{1}{n}\textstyle\sum_{k=1}^{n}(\overline{\varphi}_k - \overline{\Phi}_k)| \leq \tfrac{1}{n}\mathbb{E}|\psi_\infty| + \mathbb{E}|\psi_\infty - \psi_n| + \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1}\mathbb{E}|\psi_\infty - \psi_k|.$$

Now, reasoning as in the third step, one gets

$$\begin{aligned} \mathbb{E}((\psi_\infty - \psi_n)^2) &\leq \textstyle\sum_{k=n+1}^{\infty}\tfrac{1}{k^2}\mathbb{E}(\overline{\varphi}_k^2) \\ &\leq d_n := \textstyle\sum_{k=n+1}^{\infty}\tfrac{2}{k^2}\int_0^k t\,F(t)\,\mathrm{d}t. \end{aligned}$$

This sequence $d_n = d_n(\varphi)$ converges to 0 for $n \to \infty$, since the following series is convergent:

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty}\tfrac{2}{k^2}\int_0^k t\,F(t)\,\mathrm{d}t &\leq \textstyle\sum_{m=1}^{\infty}\tfrac{8}{m}\int_{m-1}^{m} t\,F(t)\,\mathrm{d}t \\ &\leq 8\textstyle\int_0^\infty F(t)\,\mathrm{d}t \leq 8\,\mathbb{E}(\varphi) < \infty. \end{aligned}$$

Besides, still by the third step and Cauchy-Schwarz inequality, one has also

$$\mathbb{E}|\psi_\infty| \leq 2\,\mathbb{E}(\varphi)^{1/2}.$$

Now, (1.2) follows with

$$c_n = \tfrac{2}{n}\textstyle\sum_{k=1}^{n} c_{1,k} + \tfrac{2}{n}\mathbb{E}(\varphi)^{1/2} + d_n^{1/2} + \tfrac{1}{n}\textstyle\sum_{k=1}^{n} d_k^{1/2}.$$

□

The following statement is not a direct consequence of Theorem 1.6 but its proof is similar and much simpler since no truncation step is needed.

Corollary 1.8. *Let $(\varphi_n)_{n\geq 1}$ be a sequence of random variables which are bounded in* L^2 *and such that,*

$$\mathbb{E}(\varphi_n \mid \varphi_1, \ldots, \varphi_{n-1}) = 0 \ \text{ for all } n \geq 1.$$

Then the sequence $\frac{1}{n}\sum_{k=1}^{n}\varphi_k$ converges to 0 *almost surely and in* L^2.

Proof. By assumption, the sequence of random variables

$$\psi_n = \sum_{k=1}^{n} \frac{1}{k}\varphi_k$$

is a martingale with respect to $\mathcal{B}_n$. This martingale is bounded in L^2 since

$$\mathbb{E}(\psi_n^2) = \sum_{k=1}^{n} \frac{1}{k^2}\mathbb{E}(\varphi_k^2) \leq (\sum_{k=1}^{\infty} \frac{1}{k^2}) \sup_{k\geq 1} \mathbb{E}(\varphi_k^2) < \infty.$$

Hence by Doob's martingale convergence theorem, ψ_n converges almost surely and in L^2. We conclude thanks to Lemma 1.7 that $\frac{1}{n}\sum_{k=1}^{n} \varphi_k$ converges to 0 almost surely and in L^2 when $n \to \infty$. □

2. Essential spectrum of bounded operators

Let E be a (complex) Banach space and T be a bounded endomorphism of E. In this chapter, we will introduce a nonempty closed subset $\sigma_e(T)$ of the spectrum $\sigma(T)$ of T, called the essential spectrum of T. The essential spectral radius $\rho_e(T)$ of T will be defined as the largest modulus of an element of the essential spectrum. If λ is a spectral value of T whose modulus is larger than $\rho_e(T)$, then λ is an eigenvalue of T. Now, the essential spectral radius may be computed by using a formula, due to Nussbaum. We will then apply this formula for dominating the essential spectral radius under certain assumptions which are natural in a dynamical setting. This result was used in Chapters 14 and 15 for proving the Local Limit Theorem.

In this appendix, we will freely use the basic results of Functional Analysis as in Rudin books [**107**] and [**108**] .

2.1. Compact operators.

In this section, we recall the definition of compact operators and some elementary properties.

Let E be a complex Banach space. For any x in E and $r > 0$, we let $B_E(x, r)$ (or $B(x, r)$ when there is no ambiguity) denote the closed ball with center x and radius r in E.

Let E, F be Banach spaces. We let $\mathcal{B}(E, F)$ denote the space of *bounded linear operators* from E to F, equipped with its natural Banach space structure. When $E = F$, we write $\mathcal{B}(E)$ for $\mathcal{B}(E, E)$. It carries a natural srtructure of Banach algebra.

A bounded operator $T : E \to F$ is said to be *compact* if the set $TB(0, 1)$ is relatively compact in F (for the norm topology). This amounts to say that the image under T of any bounded subset of E is relatively compact in F. We let $\mathcal{K}(E, F)$ (or $\mathcal{K}(E)$ when $E = F$) denote the set of compact operators from E to F.

LEMMA 2.1. *Let E, F, G be Banach spaces. The set $\mathcal{K}(E, F)$ of compact operators from E to F is a closed subspace of $\mathcal{B}(E, F)$. One has*

$$\mathcal{B}(F, G)\mathcal{K}(E, F) \subset \mathcal{K}(E, G) \text{ and } \mathcal{K}(F, G)\mathcal{B}(E, F) \subset \mathcal{K}(E, G).$$

In particular, the space $\mathcal{K}(E)$ is an ideal in the Banach algebra $\mathcal{B}(E)$.

The proof of closedness of the space of compact operators (such as several other proofs below) uses the following classical characterization of relatively compact subsets of complete metric spaces: a subset Y of a complete metric space (X, d) is relatively compact if and only if, for every $\varepsilon > 0$, Y is contained in a finite union of balls of X with radius ε.

PROOF OF LEMMA 2.1. Any scalar multiple of a compact operator is clearly compact. If S and T are compact operators from E to F, $S + T$ is compact since the sum map $F \times F \to F$ is continuous.

Assume T is a compact operator from E to F and S is any operator in $\mathcal{B}(F, G)$. Then, since S is continuous and $TB(0, 1)$ is relatively compact in F, $STB(0, 1)$ is relatively compact, hence ST is compact. Now, assume T is in $\mathcal{K}(F, G)$ and S is in $\mathcal{B}(E, F)$. Since $SB(0, 1)$ is bounded and T is compact, $TSB(0, 1)$ is compact. Hence TS is compact.

It remains to check that $\mathcal{K}(E, F)$ is closed in $\mathcal{B}(E, F)$. Let (T_n) be a sequence in $\mathcal{K}(E, F)$ that converges in the norm topology towards an operator T and let us prove that T is compact. We will use the characterization above of relatively compact subsets of F. Fix $\varepsilon > 0$. Chose n such that $\|T - T_n\| \leq \varepsilon$. Then, since $T_n B(0, 1)$ is relatively compact in F, there exist $y_1, \dots, y_p$ in F with

$$T_n B(0, 1) \subset B(y_1, \varepsilon) \cup \cdots \cup B(y_p, \varepsilon).$$

As $\|T - T_n\| \leq \varepsilon$, we get

$$TB(0, 1) \subset B(y_1, 2\varepsilon) \cup \cdots \cup B(y_p, 2\varepsilon).$$

Since this holds for any ε, $TB(0, 1)$ is compact, which completes the proof. □

Let E and F be Banach spaces and E^* and F^* be their topological dual spaces. For any T in $\mathcal{B}(E, F)$, we let T^* denote its *adjoint operator*: this is the bounded operator

$$\begin{aligned} F^* &\to E^* \\ f &\mapsto f \circ T. \end{aligned}$$

We will sometimes use duality arguments which rely on the

Lemma 2.2. *A bounded operator $T : E \to F$ is compact if and only if T^* is compact.*

Proof. Assume T is compact. Fix $\varepsilon > 0$ and $y_1, \dots, y_p$ in F with

$$TB_E(0,1) \subset B_F(y_1, \varepsilon) \cup \cdots \cup B_F(y_p, \varepsilon).$$

Consider the finite-dimensional subspace G of F spanned by $y_1, \dots, y_p$. Since the dual space of G is also finite dimensional, its unit ball is compact and there exist linear functionals $f_1, \dots, f_q$ in $B_{G^*}(0,1)$ such that

$$B_{G^*}(0,1) \subset B_{G^*}(f_1, \varepsilon/M) \cup \cdots \cup B_{G^*}(f_q, \varepsilon/M),$$

where $M = \max_{1\leq i\leq p} \|y_i\|$. By Hahn-Banach theorem, $f_1, \dots, f_q$ may be extended as linear functionals on F which have norm ≤ 1 (which we still denote by $f_1, \dots, f_q$).

Now, pick f in $B_{F^*}(0,1)$. By construction, there exists $1 \leq j \leq q$ with

$$|\langle f - f_j, y\rangle| \leq \varepsilon \|y\|$$

for any y in G. We claim that we have $\|T^*f - T^*f_j\| \leq 3\varepsilon$ in E^*. Indeed, for any x in $B_E(0,1)$, there exists $1 \leq i \leq p$ with $\|Tx - y_i\| \leq \varepsilon$. We then have

$$\langle T^*f - T^*f_j, x\rangle = \langle f, Tx - y_i\rangle + \langle f - f_j, y_i\rangle - \langle f_j, Tx - y_i\rangle,$$

hence $|\langle T^*f - T^*f_j, x\rangle| \leq 3\varepsilon$. Thus, we have

$$TB_{F^*}(0,1) \subset B_{E^*}(T^*f_1, 3\varepsilon) \cup \cdots \cup B_{E^*}(T^*f_q, 3\varepsilon),$$

and T^* is compact since this holds for any $\varepsilon > 0$.

Conversely, assume T^* is compact. By the result above, the bounded operator T^{**} between the bidual spaces E^{**} and F^{**} is compact. If E and F are reflexive, we are done. In general, E and F embed isometrically as closed subspaces in E^{**} and F^{**} and $TB_E(0,1)$ is contained in the intersection of $T^{**}B_{E^{**}}(0,1)$ with the image of F in F^{**}. As $T^{**}B_{E^{**}}(0,1)$ is relatively compact in F^{**}, so is $TB_E(0,1)$ in F, which completes the proof. □

2.2. Bounded operators and their adjoints.

We recall classical properties of the adjoint operators of bounded operators.

Let E be a Banach space and E^* be the topological dual space of E. If F is a closed subspace of E, we let $F^\perp$ denote the *orthogonal subspace* of F in E^*, that is, the space of linear functionals f on E such that f is 0 on F. We recall that the *weak-$*$ topology* on E^* is the topology of locally convex vector space defined by the family of seminorms on E^* given by $f \mapsto |f(x)|$ where x varies in E.

To be able to describe the spectral structure of compact operators, we shall need elementary properties of adjoint operators, which are summarized in the following lemma.

LEMMA 2.3. *Let E, F be Banach spaces and $T : E \to F$ be a bounded linear operator.*
a) We have $(\operatorname{Im} T)^{\perp} = \operatorname{Ker} T^*$ and $\operatorname{Im} T^*$ is weak-$*$ dense in $(\operatorname{Ker} T)^{\perp}$.
b) In particular, the operator T has closed image if and only if T^* has closed image. In this case, one has $(\operatorname{Ker} T)^{\perp} = \operatorname{Im} T^*$.

The proof of this Lemma uses quotients of Banach spaces. In all the sequel, if E is a Banach space and F is a closed subspace of E, we equip the *quotient space* E/F with the norm defined by, for any x in E,

$$\|x + F\| = \inf_{y \in F} \|x + y\| . \tag{2.1}$$

This induces a Banach space structure on E/F. Since Formula (2.1) defines a norm, there exists a vector $x \in E$ with $\|x\| = 2$ and $\|x - y\| \geq 1$ for any y in F. Such a vector x will be useful in the next sections. Indeed it will play the role of an almost-normal direction to F eventhough E is not assumed to be a Hilbert space. Note that the natural maps $F^{\perp} \to (E/F)^*$ and $E^*/F^{\perp} \to F^*$ are isometries (in the second case, this follows by the Hahn-Banach Theorem).

PROOF OF LEMMA 2.3. *a)* For any f in E^*, we have

$$T^* f = 0 \Leftrightarrow \forall x \in E \quad \langle f, Tx \rangle = 0 \Leftrightarrow f_{|\operatorname{Im} T} = 0,$$

hence one has the equality $(\operatorname{Im} T)^{\perp} = \operatorname{Ker} T^*$.

Now, observe that one has $\operatorname{Im} T^* \subset (\operatorname{Ker} T)^{\perp}$: indeed, if f is in F^* and x is in $\operatorname{Ker} T$, one has

$$\langle T^* f, x \rangle = \langle f, Tx \rangle = 0.$$

Hence by Hahn-Banach theorem applied in F, one has

$$\operatorname{Ker} T = \{x \in E | \forall f \in F^* \quad \langle T^* f, x \rangle = 0\}.$$

Then by Hahn-Banach theorem applied to the weak-$*$ topology on E^*, the space $(\operatorname{Ker} T)^{\perp}$ is the weak-$*$ closure of $\operatorname{Im} T^*$.

b) Assume now T has closed image. Then T factors as a composition

$$E \to E/\operatorname{Ker} T \to F,$$

where, by the open mapping Theorem, the second map is an isomorphism with its image. We thus have a factorisation of T^* as

$$F^* \to (E/\operatorname{Ker} T)^* \to E^*.$$

where the first map is an isomorphism. Therefore, the space $\operatorname{Im} T^*$ is closed in E^* and equal to $(\operatorname{Ker} T)^\perp$.

It remains to prove that if $\operatorname{Im} T^*$ is closed in E^*, $\operatorname{Im} T$ is closed in F.

Assume first that T has dense image, so that, since $\operatorname{Ker} T = (\operatorname{Im} T)^\perp$, T^* is injective. Then, since we assumed that T^* has closed image, by the open mapping Theorem, there exists $\varepsilon > 0$ such that, for every f in F^*, one has $\|T^* f\| \geq \varepsilon \|f\|$. We claim that one has

$$\overline{TB_E(0,1)} \supset B_F(0,\varepsilon). \tag{2.2}$$

We will argue as in the proof of the open mapping Theorem. Indeed, as $TB_E(0,1)$ is convex, by the Hahn-Banach Theorem, for every y in $F \smallsetminus \overline{TB_E(0,1)}$, there exists f in F^* with

$$|\langle f, y\rangle| > \sup_{x\in B_E(0,1)} |\langle f, Tx\rangle| = \|T^* f\| \geq \varepsilon \|f\| .$$

We get $\|y\| > \varepsilon$, hence the claim (2.2). This implies that one has

$$TB_E(0,2) \supset B_F(0,\varepsilon) \tag{2.3}$$

For any $y = y_0$ in $B_F(0,\varepsilon)$, one can find x_0 in $B_E(0,1)$ such that

$$y_1 = y_0 - Tx_0$$

has norm $\leq \varepsilon/2$. Iterating this process, one construct a sequence (x_n) such that, for any n, $\|x_n\| \leq 2^{-n}$ and

$$y_{n+1} = y_0 - T(x_0 + \cdots + x_n)$$

has norm $\leq \varepsilon 2^{-n-1}$. As $\sum_{n\geq 0} \|x_n\| \leq 2$, $x = \sum_{n\geq 0} \|x_n\|$ belongs to $B_E(0,2)$ and by construction, $Tx = y$. This proves (2.3). In particular, T is surjective and we are done with the result under the assumption that T has dense image.

In general, we set $G = \overline{\operatorname{Im} T}$, so that T may be written as a composition of maps

$$E \xrightarrow{T} G \hookrightarrow F,$$

where the first one has dense image. The corresponding decomposition for T^* is of the form

$$F^* \to G^* \xrightarrow{T^*} E^*.$$

In this decomposition, the first map is surjective and the second one has closed image. In other words, the adjoint of the operator $E \to G, x \mapsto Tx$ has closed image. Hence, by the first part of the proof, this operator is surjective, which completes the proof. $\square$

2.3. Spectrum of compact operators.

In this section, we describe the structure of the spectrum of compact operators.

We will now assume $E = F$. If T is a bounded operator of E, we let $\sigma_E(T)$ (or $\sigma(T)$ when there is no ambiguity) denote its *spectrum*, that is, the set of λ in $\mathbb{C}$ such that $T - \lambda$ is not invertible, and we let $\rho(T)$ denote the *spectral radius* of T that is, the radius of the smallest disc centered at 0 in $\mathbb{C}$ which contains $\sigma(T)$. We assume that E has infinite dimension (else, every operator is compact and the spectral result below is trivial).

PROPOSITION 2.4. *Let T be a compact bounded operator of E. Then $\sigma(T)$ is the union of 0 and an at most countable subset of $\mathbb{C}$ with 0 as its unique cluster point. For every $\lambda \neq 0$ in $\sigma(T)$, the space E splits uniquely as a direct sum $E = E_\lambda \oplus F_\lambda$ where E_λ and F_λ are T-stable closed subspaces of E, E_λ has finite dimension, $\sigma_{E_\lambda}(T) = \{\lambda\}$ and $\sigma_{F_\lambda}(T) = \sigma(T) \smallsetminus \{\lambda\}$.*

The proof relies on a succession of lemmas where we will prove that the spaces $E_\lambda = \bigcup_r \operatorname{Ker}(T-\lambda)^r$ and $F_\lambda = \bigcap_r \operatorname{Im}(T-\lambda)^r$ have the required properties.

First, we study eigenspaces of T.

LEMMA 2.5. *Let T be a compact bounded operator of E and λ be a nonzero complex number. For any $r \geq 1$, the space* $\operatorname{Ker}(T-\lambda)^r$ *is finite-dimensional.*

PROOF. First assume we have $r = 1$. Set $F = \operatorname{Ker}(T-\lambda)$. We have $TB_F(0,1) = B_F(0,|\lambda|)$. Therefore, $B_F(0,|\lambda|)$ is relatively compact in F. As $|\lambda| \neq 0$, Riesz's Theorem implies that F has finite dimension.

Now, in general, set $S = (T-\lambda)^r - (-\lambda)^r$, so that $(T-\lambda)^r = S + (-\lambda)^r$. By Lemma 2.1, S is compact, hence $\operatorname{Ker}(T-\lambda)^r$ is finite-dimensional. □

Now duality allows to recover informations on $\operatorname{Im}(T-\lambda)$.

LEMMA 2.6. *Let T be a compact bounded operator of E and λ be a nonzero complex number. For any $r \geq 1$, the space* $\operatorname{Im}(T-\lambda)^r$ *is closed with finite codimension.*

PROOF. Again, as in the proof of Lemma 2.5, it suffices to deal with the case $r = 1$.

First, let us prove that $\operatorname{Im}(T-\lambda)$ is closed. Set $F = E/\operatorname{Ker}(T-\lambda)$ and let $S : F \to E$ be the bounded injective operator induced by $(T-\lambda)$. We claim that there exists $\varepsilon > 0$ with $\|Sy\| \geq \varepsilon \|y\|$ for any y in F (which implies the result). Indeed, if this is not the case, there exists a sequence (y_n) of unit vectors in F with $\|Sy_n\| \xrightarrow[n\to\infty]{} 0$. Let $\pi : E \to F$ be the quotient map. For any n, pick x_n in E with $\pi(x_n) = y_n$ and $1 \leq \|x_n\| \leq 2$. By the definition of S, we have

$$Tx_n - \lambda x_n \xrightarrow[n\to\infty]{} 0.$$

As (x_n) is bounded in E and T is compact, after having extracted a subsequence, we can assume that there exists z in E with

$$Tx_n \xrightarrow[n\to\infty]{} z.$$

We also get

$$\lambda x_n \xrightarrow[n\to\infty]{} z.$$

Hence, if we set $t = \frac{1}{\lambda}z$, we have $x_n \xrightarrow[n\to\infty]{} t$ and $Tt = \lambda t$, that is $t \in \mathrm{Ker}(T-\lambda)$. Applying π gives

$$y_n = \pi(x_n) \xrightarrow[n\to\infty]{} \pi(t) = 0,$$

a contradiction. Theferore S has closed image and $\mathrm{Im}(T-\lambda)$ is closed.

Set $G = \mathrm{Im}(T-\lambda)$. By Lemma 2.3, we have $G^\perp = \mathrm{Ker}(T^* - \lambda)$. Since, by Lemma 2.2, the operator T^* is compact, Lemma 2.5 implies that $G^\perp$ is finite-dimensional. As $G^\perp$ may be seen as the topological dual space of E/G, the codimension of $\mathrm{Im}(T-\lambda)$ is finite. □

Now, we prove that the non-increasing sequence of subspaces from Lemma 2.7 eventually becomes stationary.

Lemma 2.7. *Let T be a compact bounded operator of E and λ be a nonzero complex number. There exits $r \geq 0$ with* $\mathrm{Im}\,(T-\lambda)^r = \mathrm{Im}\,(T-\lambda)^{r+1}$.

Proof. Assume this is not the case and set, for any r, $G_r = \mathrm{Im}\,(T-\lambda)^r$, which is a closed subspace of E by Lemma 2.6. By assumption, we have $G_{r+1} \subsetneq G_r$. Since Formula (2.1) defines a norm, there exists $x_r \in G_r$ with $\|x_r\| = 2$ and $\|x_r - y\| \geq 1$ for any y in G_{r+1}.

For $r < s$, we have

$$Tx_r - Tx_s = \lambda x_r + (Tx_r - \lambda x_r - Tx_s),$$

hence, as $Tx_r - \lambda x_r - Tx_s$ belongs to G_{r+1}, $\|Tx_r - Tx_s\| \geq |\lambda|$. In particular, the sequence (Tx_r) has no converging subsequence, which contradicts the compactness of T. □

Finally, we prove the dual statement to the one of Lemma 2.7:

LEMMA 2.8. *Let T be a compact bounded operator of E and λ be a nonzero complex number. There exits $r \geq 0$ with* $\mathrm{Ker}\,(T-\lambda)^r = \mathrm{Ker}\,(T-\lambda)^{r+1}$.

PROOF. We prove this statement by duality. Indeed, let $r \geq 0$. By Lemma 2.6 the operator $(T-\lambda)^r$ has closed image. Hence, by Lemma 2.3, the orthogonal subspace of $\mathrm{Ker}(T-\lambda)^r$ in E^* is $\mathrm{Im}(T^*-\lambda)^r$. Now, by Lemma 2.2, T^* is compact, so that, by Lemma 2.7, there exists $r \geq 0$ with $\mathrm{Im}\,(T^*-\lambda)^r = \mathrm{Im}\,(T^*-\lambda)^{r+1}$ and we are done. □

We now have all the tools in hand to establish the

PROOF OF PROPOSITION 2.4. Let λ be a nonzero complex number. By Lemmas 2.7 and 2.8, we can fix $r \geq 0$ so that, for all $s \geq r$,

$$\mathrm{Ker}\,(T-\lambda)^r = \mathrm{Ker}\,(T-\lambda)^s \quad \text{and} \quad \mathrm{Im}\,(T-\lambda)^r = \mathrm{Im}\,(T-\lambda)^s.$$

We set

$$E_\lambda = \mathrm{Ker}\,(T-\lambda)^r \quad \text{and} \quad F_\lambda = \mathrm{Im}\,(T-\lambda)^r.$$

By Lemma 2.5, E_λ has finite dimension and, by Lemma 2.6, F_λ is closed with finite codimension.

We claim that $E_\lambda \cap F_\lambda = \{0\}$. Indeed, if x belongs to this intersection, we may write $x = (T-\lambda)^r y$ for some y. As $(T-\lambda)^r x = 0$, we get $(T-\lambda)^{2r} y = 0$, hence, by the choice of r, $(T-\lambda)^r y = 0$, that is $x = 0$, which was to be proved.

We claim that $E_\lambda \oplus F_\lambda = E$. Indeed, let x be in E and let us prove that x may be written as a sum of an element of E_λ and of one of F_λ. By definition $(T-\lambda)^r x$ belongs to F_λ. Since $(T-\lambda)F_\lambda = F_\lambda$, there exists y in F_λ with $(T-\lambda)^r x = (T-\lambda)^r y$. We get $x - y \in E_\lambda$ and we are done.

By definition the only spectral value of T on E_λ is λ. We claim that λ is not a spectral value of T on F_λ. Indeed by definition this operator $T-\lambda$ is surjective on F_λ and we have just seen that this operator $T-\lambda$ is injective on F_λ. Hence $T-\lambda$ is an automorphism of F_λ as required.

Now, assume λ is a nonzero spectral value of T. To complete the proof of Proposition 2.4, it only remains to prove that λ is an isolated point of the spectrum. Indeed if $\mu \neq \lambda$ is a complex number that is close enough to λ, since $T-\lambda$ is invertible on F_λ, $T-\mu$ is invertible on F_λ. As $\mu \neq \lambda$, $T-\mu$ is also invertible on E_λ and the result follows. □

2.4. Fredholm operators and the essential spectrum.

We now introduce Fredholm operators: these are the operators which are invertible modulo the ideal of compact operators. In the same spirit, we define the essential spectral radius of an operator: this is the spectral radius of the image of the operator in the Calkin algebra.

Definition 2.9. Let E be a Banach space. The quotient of the Banach algebra of bounded operators on E by the ideal of compact operators

$$\mathcal{C}(E) := \mathcal{B}(E)/\mathcal{K}(E)$$

is a Banach algebra called the *Calkin algebra.*

Let T be a bounded linear operator in E. We say that T is *Fredholm* if there exists a bounded operator S such that $TS - 1$ and $ST - 1$ are compact operators. In other words, T is Fredholm if and only if its image in the Calkin algebra $\mathcal{C}(E)$ is invertible.

Lemma 2.10. *The product T_1T_2 of two Fredholm operators T_1 and T_2 of E is also Fredholm.*

Proof. As in any ring, the product x_1x_2 of two invertible elements x_1 and x_2 of the Calkin algebra is also invertible. □

Proposition 2.11. *Let T be a bounded linear operator in E. Then T is Fredholm if and only if* $\operatorname{Ker} T$ *has finite dimension and* $\operatorname{Im} T$ *is closed with finite codimension.*

Proof. Assume $\operatorname{Ker} T$ is finite-dimensional and $\operatorname{Im} T$ is closed with finite codimension. Chose closed subspaces F and G of E such that

$$E = F \oplus \operatorname{Ker} T = G \oplus \operatorname{Im} T.$$

The action of T induces an isomorphism from F onto $\operatorname{Im} T$. We define R as the inverse operator $\operatorname{Im} T \to F$. For any x in E, if $x = y + z$ with y in $\operatorname{Im} T$ and z in G, we set $Sx = Ry$. Let us check that $ST - 1$ and $TS - 1$ are compact; we will even prove that they have finite rank. Indeed, for any x in F, we have $STx = x$. Therefore $\operatorname{Ker}(ST - 1) \supset F$ and $ST - 1$ has finite rank since F has finite codimension. In the same way, for any x in $\operatorname{Im} T$, $TSx = x$ and $TS - 1$ has finite rank. Thus T is Fredholm.

Conversely, assume that T is Fredholm and let S be such that $K = ST - 1$ and $L = TS - 1$ are compact operators. Then we have $\operatorname{Ker} T \subset \operatorname{Ker}(K + 1)$, hence, by Lemma 2.5, $\operatorname{Ker} T$ has finite dimension. In the same way, we have $\operatorname{Im} T \supset \operatorname{Im}(L + 1)$, hence, by Lemma 2.6, $\operatorname{Im} T$ is closed with finite codimension. □

Corollary 2.12. *A bounded linear operator T of E is Fredholm if and only if T^* is.*

Proof. Assume T is Fredholm and let S be an inverse of T modulo compact operators. By Lemma 2.2, the operators $S^*T^* - 1 = (TS - 1)^*$ and $T^*S^* - 1 = (ST - 1)^*$ are compact. Thus, T^* is Fredholm.

Conversely, assume T^* is Fredholm. By Proposition 2.11, $\operatorname{Im} T^*$ is closed, so that by Lemma 2.3, $\operatorname{Im} T$ is closed, $(\operatorname{Im} T)^\perp = \operatorname{Ker} T^*$ and $(\operatorname{Ker} T)^\perp = \operatorname{Im} T^*$. As, again by Proposition 2.11, $\operatorname{Ker} T^*$ has finite dimension and $\operatorname{Im} T^*$ has finite codimension, $\operatorname{Im} T$ has finite codimension and $\operatorname{Ker} T$ has finite dimension. Now Proposition 2.11 tells us that the operator T is Fredholm. □

Let T be a bounded operator of E. We define the *essential spectrum* $\sigma_e(T)$ of T as the set of complex numbers λ such that $T - \lambda$ is not Fredholm. In other words, $\sigma_e(T)$ is the spectrum of the image of T in the Calkin algebra $\mathcal{C}(E)$. In particular $\sigma_e(T)$ is a non-empty closed subset of $\sigma(T)$. By Corollary 2.12, we have $\sigma_e(T^*) = \sigma_e(T)$.

We also define the *essential spectral radius* $\rho_e(T)$ of T as the radius of the smallest disc centered at 0 in $\mathbb{C}$ which contains $\sigma_e(T)$: in other words $\rho_e(T)$ is the spectral radius of the image of T in the Calkin algebra $\mathcal{C}(E)$.

Lemma 2.13. *Let T be a bounded operator in E. For all $n \geq 1$, the essential spectral radius of T^n is given by $\rho_e(T^n) = \rho_e(T)^n$.*

Proof. As in any Banach algebra, the spectral radius $\rho(x)$ of an element x of the Calkin algebra $\mathcal{C}(E)$ is given by $\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$ and hence satisfies $\rho(x^n) = \rho(x)^n$, for all positive integer n. □

If T is a compact operator, its essential spectrum is $\{0\}$. Thus, Proposition 2.4 may be seen as a description of the spectral values of T whose modulus is $> \rho_e(T)$. This description may be extended in general:

Proposition 2.14. *Let T be a bounded operator of E. Then the set of spectral values of T with modulus $> \rho_e(T)$ is at most countable and all its cluster points have modulus $\rho_e(T)$. For every λ in $\sigma(T)$ with $|\lambda| > \rho_e(T)$, the space E splits uniquely as a direct sum $E = E_\lambda \oplus F_\lambda$ where E_λ and F_λ are T-stable closed subspaces of E, E_λ has finite dimension, $\sigma_{E_\lambda}(T) = \{\lambda\}$ and $\sigma_{F_\lambda}(T) = \sigma(T) \smallsetminus \{\lambda\}$.*

The following example is important to keep in mind while reading the proof of Proposition 2.14. The reader is strongly encouraged to check the details of this example.

Example 2.15. Let $E = \ell^2(\mathbb{N})$ be the Hilbert space of square-integrable complex sequences and $T : E \to E$ be the shift operator: for any $x = (x_k)_{k\in\mathbb{N}}$ in E, $Tx = (x_{k+1})_{k\in\mathbb{N}}$. The spectrum $\sigma(T)$ of T is the unit disc in $\mathbb{C}$. Its essential spectrum is the unit circle in $\mathbb{C}$.

The proof of Proposition 2.14 is completely analogue to the one of Proposition 2.4. We easily extend Lemmas 2.5 and 2.6.

LEMMA 2.16. *Let T be a compact bounded operator of E and λ be a nonzero complex number. For any $r \geq 1$, the space* $\mathrm{Ker}\,(T-\lambda)^r$ *is finite-dimensional and the space* $\mathrm{Im}\,(T-\lambda)^r$ *is closed with finite codimension.*

PROOF. This follows from Proposition 2.11 since, by Lemma 2.10, the operator $(T-\lambda)^r$ is Fredholm. □

The only difficulty is to extend Lemma 2.7. This is done by

LEMMA 2.17. *Let T be a bounded operator of E and λ be a complex number with $|\lambda| > \rho_e(T)$. There exists $r \geq 0$ with* $\mathrm{Im}(T-\lambda)^r = \mathrm{Im}(T-\lambda)^{r+1}$.

PROOF. This proof is a refinement of the one of Lemma 2.7, which uses the spectral radius formula in the Calkin algebra $\mathcal{C}(E)$.

We again assume that the conclusion is false, and we set for any $r \geq 0$, $G_r = \mathrm{Im}(T-\lambda)^r$. Since λ is not an essential spectral value of T, by Lemma 2.10 and Proposition 2.11, for any $r \geq 0$, the space $G_r = \mathrm{Im}(T-\lambda)^r$ is closed in E. For any r, we fix a vector x_r in G_r with $\|x_r\| = 2$ and $\|x_r - y\| \geq 1$ for any y in G_{r+1}.

We pick θ with $\rho_e(T) < \theta < |\lambda|$. By the spectral radius formula in the Calkin algebra $\mathcal{C}(E)$, for any large enough n, there exists a compact operator S_n of E such that

$$\|T^n - S_n\| \leq \theta^n.$$

Let us prove that, if n is sufficiently large, the sequence $(S_n x_r)_{r\geq 0}$ has no converging subsequence: the result follows from this contradiction. Indeed, for any $r < s$, we have

$$\begin{aligned} S_n x_r - S_n x_s &= T^n x_r - T^n x_s + (S_n - T^n)(x_r - x_s) \\ &= \lambda^n x_r + (T^n x_r - \lambda^n x_r - T^n x_s) + (S_n - T^n)(x_r - x_s). \end{aligned}$$

As $T^n - \lambda^n = (T-\lambda)(T^{n-1} + \cdots + \lambda^{n-1})$, the element

$$y := T^n x_r - \lambda^n x_r - T^n x_s$$

belongs to G_{r+1}. Hence, one has $\|\lambda^n x_r + y\| \geq |\lambda|^n$ and

$$\|S_n x_r - S_n x_s\| \geq |\lambda|^n - \|S_n - T^n\|\,\|x_r - x_s\| \geq |\lambda|^n - 4\theta^n.$$

Since $\theta < |\lambda|$, for large n, we have $|\lambda|^n - 4\theta^n > 0$ and we are done. □

As above, the dual result is

LEMMA 2.18. *Let T be a bounded operator of E and λ be a complex number with $|\lambda| > \rho_e(T)$. There exists $r \geq 0$ with* $\mathrm{Ker}(T-\lambda)^r = \mathrm{Ker}(T-\lambda)^{r+1}$.

PROOF. Again, as, by Proposition 2.11, $(T-\lambda)^r$ has closed image, we have $(\mathrm{Ker}(T-\lambda)^r)^\perp = \mathrm{Im}(T^*-\lambda)^r$ by Lemma 2.3. The result follows since $T^*-\lambda$ is Fredholm by Corollary 2.12. □

PROOF OF PROPOSITION 2.14. This follows from Lemmas 2.16, 2.17, 2.18 as Proposition 2.4 followed from Lemmas 2.5, 2.6, 2.7 and 2.8. □

The following Corollary extends the conclusion of Proposition 2.14 to a larger set of complex numbers λ.

COROLLARY 2.19. *Let T be a bounded operator of E and denote by Ω the unbounded connected component of $\mathbb{C} \smallsetminus \sigma_e(T)$. Then the set of spectral values of T belonging to Ω is at most countable and is discrete in Ω. For every λ in $\sigma(T) \cap \Omega$, the space E splits uniquely as a direct sum $E = E_\lambda \oplus F_\lambda$ where E_λ and F_λ are T-stable closed subspaces of E, E_λ has finite dimension, $\sigma_{E_\lambda}(T) = \{\lambda\}$ and $\sigma_{F_\lambda}(T) = \sigma(T) \smallsetminus \{\lambda\}$.*

Since we will not use this Corollary we just sketch its proof.

PROOF. Let K be the compact set $K := \mathbb{C} \smallsetminus \Omega$. Fix a complex value λ in $\sigma(T) \cap \Omega$. According to Mergelyan Theorem (see [**107**]), there exists a polynomial function P with complex coefficients such that
$$|P(\lambda)| > \sup_{z \in K} |P(z)|.$$
The corollary now follows by applying Proposition 2.14 to the operator $P(T)$ and its spectral value $P(\lambda)$. □

2.5. The measure of non-compactness.

We introduce the seminorm γ on operators which measures how far they are from being compact. This seminorm allows to give an analogue of the spectral radius formula for the essential spectral radius: this is Nussbaum's formula.

Let T be a bounded operator of the Banach space E. We let $\gamma(T)$ be the infimum of the set of $r \geq 0$ such that $TB(0,1)$ is contained in a finite union of balls with radius r. This infimum $\gamma(T)$ is called the *measure of non-compactness* of T. By definition, one has $\gamma(T) \leq \|T\|$.

Lemma 2.20. *The function γ is a seminorm on $\mathcal{B}(E)$ which cancels exactly on $\mathcal{K}(E)$. For any S, T in $\mathcal{B}(E)$, we have $\gamma(ST) \leq \gamma(S)\gamma(T)$.*

Remark 2.21. The seminorm γ factors as a norm on the Calkin algebra $\mathcal{C}(E)$, but it is not clear wether this norm is complete, hence it is not clear wether this norm is equivalent to the quotient norm on $\mathcal{C}(E)$.

Proof. By definition, if T is a bounded operator, $\gamma(T) = 0$ if and only if T is compact. Besides, γ is clearly homogeneous.

Let S, T be in $\mathcal{B}(E)$ and let $s > \gamma(S)$ and $t > \gamma(T)$. We want to prove that

$$\gamma(S + T) < s + t \text{ and } \gamma(S\,T) < s\,t\,.$$

We can find $x_1, \dots, x_m$ and $y_1, \dots, y_n$ in E with

$$SB(0,1) \subset \bigcup_{i=1}^{m} B(x_i, s) \text{ and } TB(0,1) \subset \bigcup_{j=1}^{n} B(y_j, t).$$

On one hand, we have

$$(S + T)B(0,1) \subset \bigcup_{i,j} (B(x_i, s) + B(y_j, t)) = \bigcup_{i,j} B(x_i + y_j, s + t).$$

On the other hand, we have

$$STB(0,1) \subset \bigcup_{j} (Sy_j + tSB(0,1)) \subset \bigcup_{i,j} B(tx_i + Sy_j, st).$$

The result follows. $\square$

Eventhough the seminorm γ does not factor as the usual norm on the Calkin algebra $\mathcal{C}(E)$, it may be used to compute the essential spectral radius:

Theorem 2.22 (Nussbaum). *Let T be a bounded operator of E. We have*

$$\rho_e(T) = \lim_{n \to \infty} \gamma(T^n)^{\frac{1}{n}}.$$

Note that the limit exists from Lemma 2.20 and a classical subadditivity argument.

The remainder of the section will be devoted to the proof of Theorem 2.22. We temporarily set

$$\eta(T) = \lim_{n \to \infty} \gamma(T^n)^{\frac{1}{n}}.$$

Since $\gamma(T) \leq \|T\|$, we clearly have $\eta(T) \leq \rho(T)$. The more precise inequality $\eta(T) \leq \rho_e(T)$ will essentially follow from Proposition 2.14. We will first focus on the reverse inequality.

We need to prove that, if λ is a complex number with $|\lambda| > \eta(T)$, then $T - \lambda$ is Fredholm. The main step in this proof is

LEMMA 2.23. *Let T be a bounded operator of E and λ be a complex number with $|\lambda| > \eta(T)$. The operator $T - \lambda$ is proper on bounded subsets of E. More precisely, for any compact subset K of E, the set of x in $B(0,1)$ with $(T - \lambda)x \in K$ is compact.*

PROOF. By replacing T with $\lambda^{-1}T$, we can assume $\lambda = 1$.

We set $L = B(0,1) \cap (T-1)^{-1}K$. For x in L we set $y = Tx - x$ so that $y \in K$. For any $n \geq 1$, we have

$$T^n x - x = y + \cdots + T^{n-1}y,$$

that is,

$$x = -y - \cdots - T^{n-1}y + T^n x.$$

We get

$$L \subset -K - \cdots - T^{n-1}K + T^n B(0,1).$$

Fix $\varepsilon > 0$. As $\eta(T) < 1$, we have $\gamma(T^n) \xrightarrow[n\to\infty]{} 0$ and we can find $n \geq 1$ with $\gamma(T^n) < \varepsilon$. As $-K - \cdots - T^{n-1}K$ is a compact subset of E, it can be covered by a finite number of balls with radius ε. Therefore, L can be covered by a finite number of balls with radius 2ε. As this is true for any ε and as L is clearly closed, L is compact. □

Now, operators which are proper on bounded subsets may be easily described:

LEMMA 2.24. *Let T be a bounded operator of E. Then T is proper on bounded subsets if and only if* $\operatorname{Ker} T$ *has finite dimension and* $\operatorname{Im} T$ *is closed.*

PROOF. Assume $\operatorname{Ker} T$ has finite dimension and $\operatorname{Im} T$ is closed. Then the projection map $E \to E/\operatorname{Ker} T$ is proper on bounded subsets and, as T factors as a composition of this map with an isomorphism from $E/\operatorname{Ker} T$ onto a closed subspace of E, T is proper on bounded subsets.

Conversely, assume that T is proper on bounded subsets. As we have $B_{\operatorname{Ker} T}(0,1) = B_E(0,1) \cap T^{-1}\{0\}$, $B_{\operatorname{Ker} T}(0,1)$ is compact and, by Riesz Theorem, $\operatorname{Ker} T$ has finite dimension. Let F be a closed subspace of E such that $E = F \oplus \operatorname{Ker} T$. We have $\operatorname{Im} T = TF$, hence it suffices to prove that TF is closed in E. We claim that there exists $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon \|x\|$ for any x in F: this implies that TF is closed. Indeed, if this is not the case, there exists a sequence (x_n) of unit vectors in F with

$$\|Tx_n\| \xrightarrow[n\to\infty]{} 0.$$

Then, the set $K = \{0\} \bigcup \{x_n | n \geq 0\}$ is compact in E. As (x_n) is bounded and T is proper on bounded subsets, (x_n) admits a subsequence which converges to some y in F. Since the (x_n) are unit vectors, we have $\|y\| = 1$. Since $\|Tx_n\| \xrightarrow[n\to\infty]{} 0$, we have $Ty = 0$, which contradicts the fact that $F \cap \operatorname{Ker} T = \{0\}$. □

To conclude from Lemmas 2.23 and 2.24, we again need to apply a duality argument. This relies on

Lemma 2.25. *Let T be a bounded operator of E. Then we have $\gamma(T^*) \leq 2\gamma(T)$.*

Proof. This is obtained by taking care of constants in the proof of Lemma 2.2. Let us write it.

Fix $r > \gamma(T)$ and $y_1, \dots, y_p$ in E with

$$TB_E(0,1) \subset B_E(y_1, r) \cup \cdots \cup B_E(y_p, r).$$

Consider the finite-dimensional subspace F of E spanned by $y_1, \dots, y_p$. Pick $\varepsilon > 0$. Since the dual space of F is also finite dimensional, its unit ball is compact and there exist linear functionals $f_1, \dots, f_q$ in $B_{F^*}(0,1)$ such that

$$B_{F^*}(0,1) \subset B_{F^*}(f_1, \varepsilon/M) \cup \cdots \cup B_{F^*}(f_q, \varepsilon/M),$$

where $M = \max_{1\leq i\leq p} \|y_i\|$. By Hahn-Banach theorem, $f_1, \dots, f_q$ may be extended as linear functionals on E which have norm ≤ 1 (which we still denote by $f_1, \dots, f_q$).

Now, pick f in $B_{E^*}(0,1)$. By construction, there exists $1 \leq j \leq q$ with

$$|\langle f - f_j, y\rangle| \leq \varepsilon \|y\|$$

for any y in F. We claim that we have $\|T^*f - T^*f_j\| \leq 2r + \varepsilon$ in E^*. Indeed, for any x in $B_E(0,1)$, there exists $1 \leq i \leq p$ with $\|Tx - y_i\| \leq r$. We then have

$$\langle T^*f - T^*f_j, x\rangle = \langle f, Tx - y_i\rangle + \langle f - f_j, y_i\rangle - \langle f_j, Tx - y_i\rangle,$$

hence $|\langle T^*f - T^*f_j, x\rangle| \leq 2r + \varepsilon$. Thus, we have

$$B_{E^*}(0,1) \subset B_{E^*}(T^*f_1, 2r + \varepsilon) \cup \cdots \cup B_{E^*}(T^*f_q, 2r + \varepsilon).$$

Since this holds for any $\varepsilon > 0$ and $r > \gamma(T)$, the result follows. □

We now can conclude the

Proof of Theorem 2.22. We first prove that we have $\eta(T) \leq \rho_e(T)$. Pick $\theta > \rho_e(T)$. By Proposition 2.14, we may find a splitting of E as a direct sum $F \oplus G$, where F and G are closed, T-stable subspaces, F has finite dimension and all the spectral values of T in G

have modulus $\leq \theta$. We clearly have $\eta_E(T) = \max(\eta_F(T), \eta_G(T))$. As F is finite-dimensional, we have $\eta_F(T) = 0$. As $\eta_G(T) \leq \rho_G(T)$, we get $\eta(T) \leq \theta$. As this is true for any $\theta > \rho_e(T)$, we get $\eta(T) \leq \rho_e(T)$.

Conversely, let us prove that $\eta(T) \geq \rho_e(T)$. We fix λ in $\mathbb{C}$ with $|\lambda| > \eta(T)$ and we will prove that $T - \lambda$ is Fredholm. By Lemma 2.23, $T - \lambda$ is proper on bounded subsets. By Lemma 2.24, $T - \lambda$ has finite-dimensional kernel and closed image. Now, by Lemma 2.25, we have $\eta(T^*) \leq \eta(T)$, hence $|\lambda| > \eta(T^*)$. Therefore, again by Lemmas 2.23 and 2.24, $T^* - \lambda$ has finite-dimensional kernel. As $\mathrm{Ker}(T^* - \lambda) = \mathrm{Im}(T - \lambda)^\perp$, $\mathrm{Im}(T - \lambda)$ has finite codimension. By Proposition 2.11, $T - \lambda$ is Fredholm and the Theorem follows. □

2.6. The result by Ionescu-Tulcea and Marinescu.

We will now use Nussbaum's formula to give a proof of a result due to Ionescu-Tulcea and Marinescu, which we used in our proof of the local limit theorem. This proof is due to Hennion in [**66**] (see also [**68**]).

Let E and F be Banach spaces. A compact embedding from E to F is an injective bounded operator $E \to F$ which is compact. Given such an embedding, we identify E with its image in F.

THEOREM 2.26 (Ionescu-Tulcea and Marinescu). *Let $E \hookrightarrow F$ be a compact embedding of Banach spaces. Let T be a bounded operator in F. We assume that $TE \subset E$ and that there exist $\theta > 0$ and $M > 0$ such that, for any x in E, one has*

$$\|Tx\|_E \leq \theta \|x\|_E + M \|x\|_F .$$

Then T has essential spectral radius $\leq \theta$ in E. In particular, if T has spectral radius $\rho > \theta$, it admits an eigenvalue with modulus ρ.

PROOF. We will apply Nussbaum's Formula to the operator T in E. To this aim, we need to control the action of the powers of T. For any $n \geq 1$, set

$$M_n = M \sum_{k=0}^{n-1} \theta^k \|T\|_F^{n-1-k} .$$

An easy induction argument gives, for any x in E,

$$\|T^n x\|_E \leq \theta^n \|x\|_E + M_n \|x\|_F .$$

As the embedding of E in F is compact, there exists $x_1, \ldots, x_r$ in $B_E(0,1)$ such that, for any x in $B_E(0,1)$, one can find $1 \leq i \leq r$ with $\|x - x_i\|_F \leq \theta^n / M_n$. One then gets

$$\|T^n x - T^n x_i\|_E \leq 3\,\theta^n ,$$

hence $\gamma(T^n) \leq 3\,\theta^n$. By Nussbaum's Formula 2.22, we get $\rho_e(T) \leq \theta$ in E.

The last statement follows from Proposition 2.14. $\square$

3. Bibliographical Comments

We want to cite here our sources. This is not an easy task since we have mixed in this text ideas coming from various old fashioned books, inaccessible articles, lost preprints, drowsy seminars, endless discussions and silly reflections. An excellent general reference is the monograph [**25**] by Bougerol and Lacroix.

Chapter 1. Markov chains is a very classical topic in Probability theory (see the book of Dynkin [**42**], Neveu [**91**], Meyn and Tweedie [**89**] or the survey of Kaimanovich and Vershik [**75**]). They have been introduced by Markov for countable state spaces X, and have been generalized since then to any standard state spaces. The relation between P-invariant functions and P-invariant subsets in Lemma 1.3 is proved in Foguel's book [**46**]. The construction of the dynamical systems of forward trajectories is classical (see for instance Neveu's book [**91**]). The various characterizations of P-ergodicity in Proposition 1.8 and their interpretation in terms of ergodicity of the forward dynamical system in Proposition 1.9, are well-known by specialists. The Markov-Kakutani argument in the proof of existence of stationary measures in Lemma 1.10 finds its roots in the theorem of Bogoliubov and Krylov in [**21**]. The construction and the properties of the limit measures ν_b in Lemmas 1.17, 1.19, 1.21 are due to Furstenberg in [**49**]. Corollary 1.22 is the famous Choquet-Deny Theorem in [**33**] or [**40**]. For another proof using Hewitt-Savage zero-one law, see [**17**]. The backward dynamical system is a crucial tool in [**14**].

Chapter 2. The Law of Large Numbers for functions over a Markov chain (Corollaries 2.4, 2.6, 2.7) is due to Breiman in [**29**]. The Law of Large Numbers for cocycles over a semigroup action (Theorem 2.9) is due to Furstenberg in [**49**, Lemme 7.3]. The convergence of the covariance 2-tensor (Theorem 2.13) is due to Raugi in [**103**]. The divergence of Birkhoff sums (Lemma 2.18) goes back to Kesten in [**77**] and Atkinson in [**3**] and can also be found in [**111**].

Chapter 3. The existence of proximal elements (Lemma 3.1) can be found in [**2**] and the technical but useful Lemma 3.2 is proved in [**13**]. The Law of Large Numbers for the norm (Theorem 3.28) and the positivity of the first Lyapunov exponent (Theorem 3.31) are due to Furstenberg in [**49**]. The uniqueness of the stationary measure on the projective space for proximal groups (Proposition 3.7) is also due to Furstenberg in [**50**]. When the representation is not irreducible, related results are proved by Furstenberg and Kifer in [**53**]. See also Ledrappier's course [**82**].

Chapter 4. The main input of this Chapter is a comparison of averages in Lemmas 4.1 and 4.4 due to Kac in [**73**]. The first hitting times and the induced Markov chains are well known and useful tools to study Markov chains (see for instance [**89**]).

Chapter 5. The existence of loxodromic elements in Proposition 5.11 and Theorem 5.36 is due to Benoist and Labourie in [**12**]. The original proof relied on the previous works of Goldsheid, Margulis in [**56**] and Guivarc'h, Raugi in [**61**]. Later a much simpler proof was given by Prasad in [**97**]. The proof given here is slightly different since it relies on the simultaneous proximality Lemma 5.25 which is due to Abels, Margulis and Soifer in [**2**, Lemma 5.15]. The short proof of Lemma 5.25 given here is in [**10**, Lemma 3.1]

The structure theory of semisimple Lie groups over $\mathbb{R}$, is due to E. Cartan (see for instance [**64**]). The Iwasawa decomposition was developed later by Iwasawa in [**71**]. The classification of the finite dimensional representations of a real or complex semisimple Lie group is due to E. Cartan.

Chapter 6. The convexity and non-degeneracy of the limit cone L_Γ (Theorem 6.2) are due to Benoist in [**10**]. The density of the group spanned by the Jordan projection (Theorem 6.4) is due to Benoist in [**11**]. Both original proofs relied on Hardy fields. We give here simpler proofs due to Quint in [**102**]. These proofs replace the use of Hardy fields by suitable asymptotic expansions of the Jordan projection of well-chosen words.

Chapter 7. The theory of algebraic reductive groups over a general field was developped by Borel and Tits in [**23**]. The Cartan and Iwasawa decomposition for connected algebraic reductive groups over a non-archimedean local field is due to Bruhat-Tits in [**32**]. The classification of the algebraic representations of G over an arbitrary base field is due to Tits in [**121**]. The use of these representations in order to control the Cartan projection, the Iwasawa cocycle, and also the Jordan decomposition, as in Lemma 7.17, was introduced in [**10**].

Chapter 9. For a product of random matrices with irreducible Γ_μ, the "maximal simplicity" of the Lyapunov exponents (as in Corollary 9.15) is due to Guivarc'h in [**57**] and Guivarc'h-Raugi in [**61**] under the assumption that there exists a "contracting sequence" in Γ_μ. Goldsheid and Margulis found out in [**56**] that this condition depends only on the Zariski closure of the group Γ_μ.

Chapter 10. The content of this Chapter can be seen as a general strategy for proving limit theorems (CLT, LIL and LDP) for Hölder continuous observables over Markov-Feller chains with strong contraction properties. The relevance of the Hölder continuity condition and of the spectral theory of the transfer operator in similar contexts was already noticed by Fortet for the doubling map on the circle in [**47**], and by Sinai for geodesic flows in [**116**]. The method presented here follows the lines of the one introduced for hyperbolic dynamical systems by Ruelle in [**105**] (see also Parry and Pollicott's book [**93**]). The adaptation of this method in the context of products of random matrices is due to Le Page in [**84**], Guivarc'h, Goldsheid in [**55**] and Guivarc'h in [**60**]. The perturbation theory of quasicompact operators (Lemma 10.17) is a classical result from functional analysis (see [**74**]).

Chapter 11. Thanks to the tools of Chapter 10, the proof of the limit theorems for cocycles now follows the lines of the classical proof for sums of random variables. The classical Central Limit Theorem has a very long and well documented history (see [**45**]). The proof of the Central Limit Theorem in Section 11.2 follows this classical approach using Fourier analysis and Lévy continuity method. The classical Law of the Iterated Logarithm goes back to Khinchin in [**78**] and Kolmogorov in [**79**]. It was developed later by Hartman and Wintner in [**63**] and many other mathematicians. The proof of the Law of the Iterated Logarithm given in Sections 11.3 and 11.4 does not follow the approach via Fourier analysis and Berry-Esseen inequality as in [**84**]. It follows instead the strategy of Kolmogorov in [**79**] (see also Wittman in [**124**] or de Acosta in [**37**]). The classical Large Deviations Principle is due to Cramér in [**35**] (see [**39**] for a modern account of the LDP). The very short proof of the upper bound given in Section 11.5 follows this classical approach.

Chapter 12. The search for Central Limit Theorems for products of random matrices (Theorems 12.11 and 12.17) started in the early fifties. The existence of a "non-commutative CLT" was guessed by Bellman in [**8**]. Such a CLT was first proved by Furstenberg and Kesten in [**52**] for the norm of products of random positive matrices. This CLT was then extended by Le Page in [**84**] to more general semigroups. The general central limit theorem for the Iwasawa cocycle was proved by Goldsheid and Guivarc'h in [**55**]. The nondegeneracy of the Gaussian limit law N_μ is proved in [**55**] for $G = \mathrm{SL}(n, \mathbb{R})$ and in [**60**] when G is a real semisimple linear group. One key ingredient is the fact from [**10**] that the so-called limit cone of a Zariski dense subsemigroup of a semisimple real Lie group is convex with non-empty interior. The new feature in the Central Limit Theorems 12.11 and 12.17 is that they are valid over any local field even in positive characteristic and for any Zariski dense probability measure μ. In these Central Limit Theorems 12.11 and 12.17 there remains an unnecessary assumption, namely, that μ has a finite exponential moment (9.3). Recently the authors have replaced this assumption [**18**] by the optimal assumption that μ has a finite second moment. The irreducible example 12.9 where the limit law is not Gaussian is borrowed from [**18**].

Chapter 13. The Hölder regularity of the stationary measure on projective spaces (Theorem 13.1) is due to Guivarc'h in [**58**]. The new proof given here borrows ideas from [**26**].

Chapter 14. We go on here the general strategy we began in Chapter 10 in view of the last limit theorem (LLT), and the comments of Chapter 10 are also valid for this one. Inequality (14.1) already appears in the context of Markov chains in Doeblin-Fortet [**41**].

Chapter 15. The classical Local Limit Theorem is due to Gnedenko in the lattice case (see [**54**] or [**94**]) and is due to Stone in the aperiodic case in [**120**]. Recently Breuillard in [**30**] extended this theorem by allowing moderate deviations. The first version of the Local Limit Theorem for the norm cocycle over products of random matrices is due to Le Page in [**84**] under an aperiodicity assumption similar to (14.8). The new features in our local limit theorems 15.1, 15.15 and Corollary 15.7 for cocycles, are that we deal with multidimensional cocycles, we allow moderate deviations and the choice of a target in the base space. All these improvements are crucial for the applications. The proof is a mixture of the arguments of Le Page based on spectral gap properties for the complex transfer operator $P_{i\theta}$ and the arguments of Breuillard based on the Edgeworth asymptotic expansion of the Fourier transform in Lemma 15.12.

Chapter 16. In order to apply the local limit theorem for the Iwasawa cocycle, it only remains to describe the essential image of the cocycle. In particular, for real semisimple groups, one has to check that this cocycle is aperiodic. This was the aim of Chapter 8.

Appendix 1 The ubiquitous Martingale Theorem 1.3 is due to Doob. The very general version, Theorem 1.5 of the law of large numbers presented here is due to Kolmogorov.

Appendix 2 Fredholm operators first occured in the context of integral functional equations as a nice class of bounded linear operators which generalizes both compact operators and contracting operators. A good reference for the spectral Theory of Fredholm operators is [**109**]. The main result of this appendix is Theorem 2.26 which is due to Ionescu-Tulcea and Marinescu. The proof of Theorem 2.22 is due to Nussbaum in [**92**]. The application of Nussbaum's formula to the Ionescu-Tulcea and Marinescu Theorem is due to Hennion in [**66**] (see also [**68**]).

Index

Bibliography

[1] J. Aaronson, An introduction to infinite ergodic theory, *Math. Surv. Mon.* **50** (1996).

[2] H. Abels, G. Margulis, G. Soifer, Semigroups containing proximal linear maps *Isr. Jour. Math.* **91** (1995) 1-30.

[3] G. Atkinson, Recurrence of cocycles and random walks *Jour. London. Math. Soc.* **13** (1976) 486-488.

[4] K. Azuma, Weighted sums of certain dependent random variables, *Tohoku Math. J.* **19** (1967) 357-367.

[5] M. Babillot, F. Ledrappier, Lalley's theorem on periodic orbits of hyperbolic flows, *Ergodic theory and dynamical systems* **18** (1998), 17-39.

[6] W. Ballman, Lectures on spaces of non positive curvature, *DMV Seminar* **25** Birkhäuser (1995).

[7] L. Baum, M. Katz, Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** (1965) 108-123.

[8] R. Bellman, Limit theorems for non-commutative operations I, *Duke Math. Jour.* **21** (1954) 491-500.

[9] Y. Benoist, Actions propres sur les espaces homogènes réductifs, *Annals of Math.* **144** (1996) 315-347.

[10] Y. Benoist, Propriétés asymptotiques des groupes linéaires, *Geom. Funct. Anal.* **7** (1997) 1-47.

[11] Y. Benoist, Propriétés asymptotiques des groupes linéaires II. *Adv. Stud. Pure Math.* **26** (2000) 33-48.

[12] Y. Benoist, F. Labourie, Sur les difféomorphismes d'Anosov à feuilletages différentiables, *Inv. Math.* **111** (1993) 285-308.

[13] Y. Benoist, J.-F. Quint, Random walk on projective spaces, preprint (2012).

[14] Y. Benoist, J.-F. Quint, Mesures stationnaires et fermés invariants des espaces homogènes (I), *Ann. of Math.* **174** (2011) 1111-1162.

[15] Y. Benoist, J.-F. Quint, Stationary measures and invariant subsets of homogeneous spaces (II), *Journ. Am. Math. Soc.* **26** (2013) 659-734.

[16] Y. Benoist, J.-F. Quint, Stationary measures and invariant subsets of homogeneous spaces (III), *Ann. of Math.* **178** (2013) 1017-1059.

[17] Y. Benoist, J.-F. Quint, Introduction to random walks on homogeneous spaces, *Japan. J. Math.* **7** (2012) 135-166.

[18] Y. Benoist, J.-F. Quint, Central limit theorem for linear groups, Annals of Proba. (2015).

[19] Y. Benoist, J.-F. Quint, Central limit theorem for hyperbolic groups, preprint (2014).

[20] M. Bjorklund, Central limit theorems for Gromov hyperbolic groups, *J. Theoret. Probab.* **23** (2010), 871-887.

[21] N. Bogoliubov, N. Krylov, La théorie générale de la mesure dans son application à l'étude de systèmes dynamiques de la mécanique non-linéaire, *Ann. Math.* **38** (1937) 65-113.
[22] A. Borel, Linear algebraic groups, *GTM* **126** (1991).
[23] A. Borel, J. Tits, Groupes réductifs, Publ. Math. IHES **27** (1965) 55-150.
[24] P. Bougerol, Théorème central limite local sur certains groupes de Lie *Ann. Sc. ENS.* **14** (1981) 403-432.
[25] P. Bougerol, J. Lacroix, Products of Random Matrices with Applications to Schrödinger Operators, *Birkhauser* (1985).
[26] J. Bourgain, A. Furman, E. Lindenstrauss, S. Mozes, Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus *J. Amer. Math. Soc.* **24** (2011), 231-280.
[27] J. Bourgain, A. Gamburd, A spectral gap theorem in SU(d), *J. Eur. Math. Soc.* **14** (2012) 1455-1511.
[28] l. Breiman, Probability (1968).
[29] L. Breiman, The strong Law of Large Numbers for a class of Markov chains, *Ann. Math. Statist.* **31** (1960), 801-803.
[30] E. Breuillard, Distributions diophantiennes et théorème limite local sur $\mathbb{R}^d$, *Probab. Theory Related Fields* **132** (2005) 39-73.
[31] B. Brown, Martingale central limit theorem, *Ann. Math. Stat.* **42** (1971) 59-66.
[32] F. Bruhat, J. Tits, Groupes réductifs sur un corps local *Publ. Math. IHES* **41** (1972) 5-251 and **60** (1984) 5-184.
[33] G. Choquet and J. Deny, Sur l'équation de convolution $\mu = \mu * \sigma$, C.R. Acad. Sci. **250** (1960) 799-801.
[34] J.-P. Conze, Y. Guivarc'h, Limit sets of groups of linear transformations, *Sankhyā Series A* **62** (2000), 367-385.
[35] H. Cramér, Sur un nouveau théorème limite de la théorie des probabilités, Act. Sc. (1938) 5-23
[36] H. Cramér, Random variables and probability distributions (1937)
[37] A. De Acosta, A new proof of the Hartman-Wintner law of the iterated logarithm *Ann. Prob.* **11** (1983) 270-276.
[38] C. Dellacherie, P.-A. Meyer, Probabilities and Potential, *North-Holland* (1978).
[39] A. Dembo, O. Zeitouni, Large Deviations Techniques and Applications, Springer (1998).
[40] J. Deny, Sur l'équation de convolution $\mu = \mu * \sigma$,(1959) 11 p. Available on Numdam, http://www.numdam.org/item?id=SBCD_1959-1960 __4__A5_0.
[41] W. Doeblin, R. Fortet, Sur des chaines a liaisons completes, Bulletin SMF **65** (1937) 132-148.
[42] E. Dynkin, Markov Processes, Springer (1965).
[43] P. Erdös, On a theorem of Hsu and Robbins. *Ann. Math. Statist.* **20** (1949) 286-291.
[44] W. Feller, An introduction to probability theory and its applications. *John Wiley & Sons, Inc.* (1971).
[45] H. Fisher, A History of the central limit theorem, Springer (2011).
[46] S. Foguel, The ergodic theory of Markov processes, *Van Nostrand* (1969).
[47] R. Fortet, Sur une suite également répartie, Studia Mathematica **9** (1940) 54-69.

[48] A. Furman, Random walks on groups and random transformations *Handbook of dynamical systems* **1** (2002) 931-1014.

[49] H. Furstenberg, Noncommuting random products, *Trans. Amer. Math. Soc* **108** (1963) 377-428.

[50] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces. Am. Math. Soc. (1973) 193-229.

[51] H. Furstenberg, Stiffness of group actions, *Lie groups and ergodic theory*, Tata Inst. Fund. Res. Stud. Math. **14** (1998), 105-117.

[52] H. Furstenberg, H. Kesten, Products of Random Matrices *Ann. Math. Statist.* **31** (1960) 457-469.

[53] H. Furstenberg, Y. Kifer, Random matrix products and measures on projective spaces *Isr. Jour. Math.* **46** (1983) 12-32.

[54] B. Gnedenko, A. Kolmogorov, Limit distributions for sums of independent random variables, Addison-Wesley (1954)

[55] I. Goldsheid, Y. Guivarc'h, Zariski closure and the dimension of the Gaussian law..., *Prob. Th. Rel. Fields* **105** (1996) 109-142.

[56] I. Goldsheid, G. Margulis, Lyapunov Indices of a Product of Random Matrices, *Russian Math. Surveys* **44** (1989) 11-81.

[57] Y. Guivarc'h, Sur les exposants de Liapounoff des marches aléatoires à pas markovien. *C. R. Acad. Sci.* **292** (1981) 327-329.

[58] Y. Guivarc'h, Produits de matrices aléatoires et applications, *Ergodic Th. Dyn. Syst.*, **10** (1990) 483-512.

[59] Y. Guivarc'h, On Contraction Properties for Products of Markov Driven Random Matrices, *Journal Math. Phys. Anal. Geom.*, **4** (2008) 457-489.

[60] Y. Guivarc'h, On the Spectrum of a Large Subgroup of a Semisimple Group, *J. Modern Dynamics* **2** (2008) 15-42.

[61] Y. Guivarc'h, A. Raugi, Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence, *Z. Wahrsch. Verw. Gebiete* **69** (1985) 187-242.

[62] P. Hall, C. Heyde, Martingale Limit Theory and its Applications, Acad. Press (1980).

[63] P. Hartman, A. Wintner, On the law of the iterated logarithm, *Amer. J. Math.* **63** (1941) 169-176.

[64] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Ac. Pr. **34** (1964).

[65] H. Hennion, Loi des grands nombres et perturbations pour des produits réductibles de matrices aléatoires indépendantes, *Z. Wahrsch. verw. Gebiete* **67** (1984) 265-278

[66] H. Hennion, Sur un théorème spectral et son application aux noyaux lipchitziens, *Proc. Amer. Math. Soc.* **118** (1993) 627-634.

[67] H. Hennion, Limit theorems for products of positive random matrices, *Ann. Probab.* **25** (1997) 1545-1587.

[68] H. Hennion, L. Hervé, Limit theorems for Markov chains and stochastic properties of dynamical systems by quasicompactness, *Springer Lect. Notes* **1766** (2001).

[69] P. Hsu, H. Robbins, Complete convergence and the law of large numbers. *Proc. Natl. Acad. Sci.* **33** (1947) 25-31.

[70] C. Ionescu-Tulcea, G. Marinescu, Théorie ergodique pour des classes d'opérations non complètement continues, *Annals of Math.* **52** (1950) 140-147.

[71] K. Iwasawa, On some types of topological groups, *Annals of Math.* **50** (1949) 507-558.

[72] C. Jan, Vitesse de convergence dans le TCL pour des chaines de Markov ..., *C. R. Acad. Sci.* **331** (2000) 395-398.

[73] M. Kac, On the notion of recurrence in discrete stochastic processes, Bull. Amer. Math. Soc. **53** (1947) 1002-1010.

[74] T. Kato, Perturbation theory of linear operators, Springer (1966).

[75] V. Kaimanovich, A. Vershik, Random walks on discrete groups, boundary and entropy, *Annals of Prob.* **11** (1983) 457-490.

[76] H. Kesten, Symmetric random walks on groups, *Trans. AMS* **92** (1959) 336-354.

[77] H. Kesten, Sequence of stationary sequences cannot grow slower than linearly, *Proc. AMS* **49** (1975) 205-211.

[78] A. Khinchin, Über das Gesetz der großen Zahlen, Math. Ann. **96** (1927) 152-168.

[79] A. Kolmogorov, Uber das Gesetz des iterierten logarithmus, Math. Ann. **101** (1929) 126-135.

[80] A. Kolmogorov, Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer (1933).

[81] B. Kostant, On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sc. ENS* **6** (1973) 413-455.

[82] F. Ledrappier, Quelques propriétés des exposants caractéristiques, *Springer LN* **1097** (1982) 305-396.

[83] F. Ledrappier, Some asymptotic properties of random walks on free groups, in Topics in probability and Lie groups: boundary theory, *CRM Proc. Lecture Notes* **28** (2001) 117-152 .

[84] E. Le Page, Théorèmes limites pour les produits de matrices aléatoires, *Springer LN* **928** (1982) 258-303.

[85] P. Lévy, Théorie de l'adition des variables aléatoires, Gautier-Villars (1937).

[86] M. Loeve, *Probability theory*, PUP (1963).

[87] G. Margulis, Discrete subgroups of semisimple Lie groups, *Springer* (1991).

[88] I. Melbourne, M. Nicol, A vector-valued almost sure invariance principle for hyperbolic dynamical systems, *Ann. Proba.* **37** (2009) 478-505.

[89] S. Meyn, R. Tweedie, Markov chain and stochastic stability, Springer (1993).

[90] L. Nachbin, The Haar integral, van Nostrand (1965).

[91] J. Neveu, Calcul des probabilités, *Masson* (1970).

[92] R. Nussbaum, The radius of the essential spectrum, *Duke Math. J.* **37** (1970) 473-478.

[93] W. Parry, M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Asterisque* **187-188** (1990).

[94] V. Petrov, Sums of independant random variables, Springer (1975).

[95] M. Pollicott, R. Sharp, Linear actions of free groups, *Annales de l'institut Fourier* **51** (2001), 131-150.

[96] M. Pollicott, R. Sharp, Asymptotic expansions for closed orbits in homology classes, *GeometriæDedicata* **87** (2001), 123-160.

[97] G. Prasad, $\mathbb{R}$-regular elements in Zariski-dense subgroups, *Quart. J. Math. Oxford Ser.* (2) **45** (1994) 541-545.

[98] J.-F. Quint, Divergence exponentielle des sous-groupes discrets en rang supérieur, *Comm. Math. Helv.* **77** (2002) 563-608.

[99] J.-F. Quint, Cônes limites des sous-groupes discrets des groupes réductifs sur un corps local, *Transform. Groups* **7** (2002) 247-266.

[100] J.-F. Quint, Mesures de Patterson-Sullivan en rang supérieur, *Geom. Funct. Anal.* **12** (2002) 776-809.

[101] J.-F. Quint, L'indicateur de croissance des groupes de Schottky, *Ergodic theory and dynamical systems* **23** (2003), 249-272.

[102] J.-F. Quint, Groupes de Schottky et comptage, *Ann. Inst. Fourier (Grenoble)* **55** (2005) 373-429.

[103] A. Raugi, Théorème ergodique multiplicatif. Produits de matrices aléatoires indépendantes. *Publ. Math. Rennes* (1996) 1-43.

[104] Th. Roblin, Ergodicité et équidistribution en courbure négative, *Mémoires de la Société Mathématique de France* **95** (2003).

[105] D. Ruelle, Thermodynamic formalism, Addison Wesley (1978).

[106] Raghunathan, Discrete subgroups of Lie groups, *Springer* (1972)

[107] W. Rudin, Real and Complex Analysis, *Mc Graw-Hill* (1970).

[108] W. Rudin, Functional Analysis, *Mc Graw-Hill* (1991).

[109] A. Ruston, Fredholm theory in Banach spaces, *Camb. Univ. Press* (1986),

[110] L. A. Santaló, Integral geometry and geometric probability, *Cambridge University Press* (2004).

[111] K. Schmidt, On recurrence, *Zeit. Wahr. verw. Geb.* **68** (1984) 75-95.

[112] S. Sawyer, T. Steger, The rate of escape for anisotropic random walks in a tree, *Probab. Theory Relat. Fields* **76** (1987) 207-230.

[113] C. Series, The infinite word problem and limit sets in Fuchsian groups, *Ergodic theory and dynamical systems* **1** (1981), 337-360.

[114] I. Shafarevich, Basic Algebraic Geometry, GMW **213** Springer (1977).

[115] Y. Shalom, Random ergodic theorems, invariant means and unitary representations *TIFR Stud. Math* **14** (1998), 273-314.

[116] J. Sinai, The central limit theorem for geodesic flows on manifolds of constant negative curvature, *Soviet Math. Dokl.* **1** (1960) 983-987.

[117] F. Spitzer, A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.* **82** (1956), 323-339.

[118] F. Spitzer, *Principles of random walks* GTM **34** Springer (1964).

[119] M. Steele, Kingman's subadditive ergodic theorem *Ann. Inst. H. Poincaré Probab. Statist.* **25** (1989) 93-98.

[120] C. Stone, A Local Limit Theorem for Nonlattice Multi-Dimensional Distribution Functions, Ann. Math. Statist. **36** (1965) 546-551.

[121] J. Tits, Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque, *J. Reine Angew. Math.* **247** (1971) 196-220.

[122] Varadarajan, Lie groups, Lie algebras and their representations, GTM **102** Springer (1974).

[123] A. Virtzer On the simplicity of the spectrum of the Lyapunov characteristic indices of a product of random matrices *Th. Probab. Appl.* **28** (1983), 122-135.

[124] R. Wittmann, A General Law of Iterated Logarithm, *Zeit. Wahr. verw. Geb.* **68** (1985) 521-543.

[125] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser (1984).