Transversals to the convex hull of all k set of discrete subsets of $\mathbb{R}^n$

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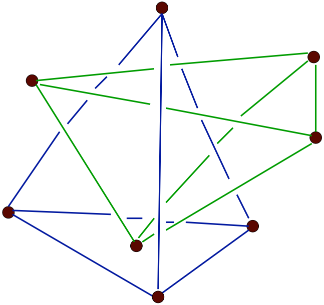
(joint work with J. Arocha, J. Bracho and L. Montejano)

Let us consider 8 points in $\mathbb{R}^3$ general position.



Question : Is there a transversal line to all tetrahedra ?

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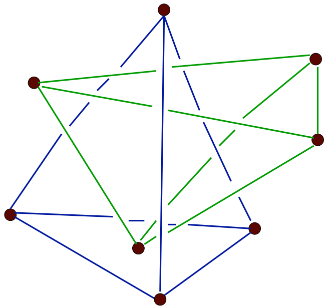
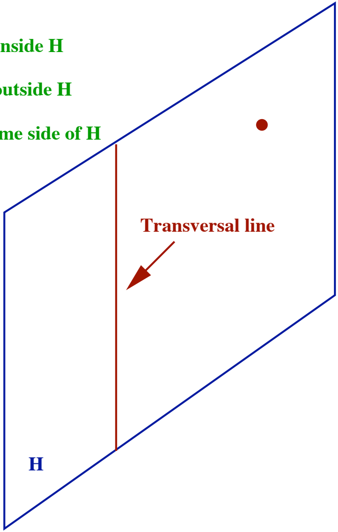


Question : Is there a transversal line to all tetrahedra ?

NEVER

- **There are at most 3 points inside H**
- **There are at least 5 points outside H**
- **There are 3 points in the same side of H**



Question : Let A be a set of 6 points in $\mathbb{R}^3$ in general position. Is there a transversal line to all tetrahedra of A?

ALWAYS

Let $x \in A$ and let T_1 be the set of tetrahedras containing x and let T_2 be the set of tetrahedras not containing x.

T_2 has the 4-Helly property, and therefore, there exists a point y in the intersection of all tetrahedras in T_2.

So, the line passing through x and y gives the desired transversal.

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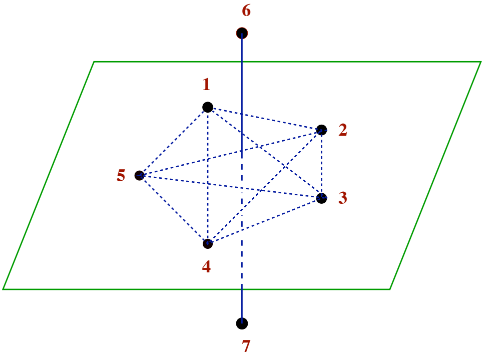

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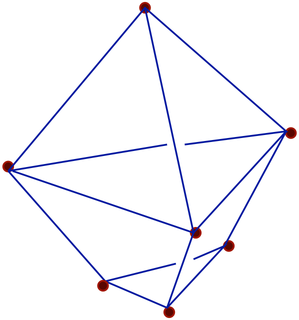

Definitions

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$M(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hull of all k-set have a transversal $(d - \lambda)$-plane.

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ the minimum positive integer n such that for every set of n points in general position in $\mathbb{R}^d$ the convex hull of the k-sets does not have a transversal $(d - \lambda)$-plane.

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- $M(k,d,\lambda) < m(k,d,\lambda)$.
- $M(4,3,2) = 6$ and $m(4,3,2) = 8$.

Theorem (Arocha, Bracho, Montejano, R.A.)

$$m(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

Proof (idea) :

1) $m(k,d,\lambda) \leq \{ \cdots$ by using similar arguments as before.

2) $m(k,d,\lambda) \geq \{ \cdots$ by using a classical result of Gale.

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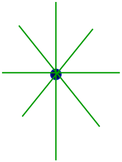
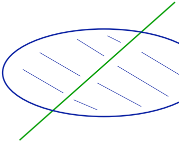
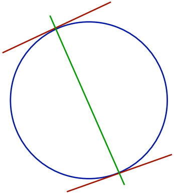


Lines through a point



lines dividing the area in half



diametral lines

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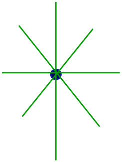

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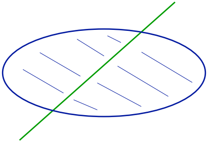

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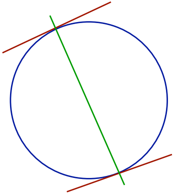

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Fact : Two systems of lines in $\mathbb{R}^2$ coincide in some direction.

System of planes in $\mathbb{R}^d$ is a continuous selection of one plane in every direction. Fact : Three systems of planes in $\mathbb{R}^d$ coincide in some direction.

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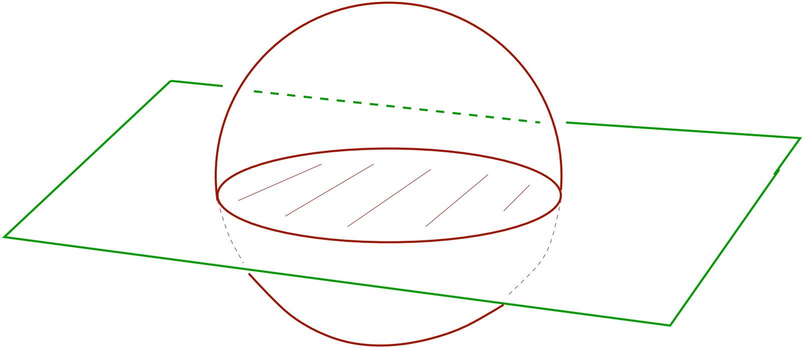

Planes dividing volume (or surface) in half

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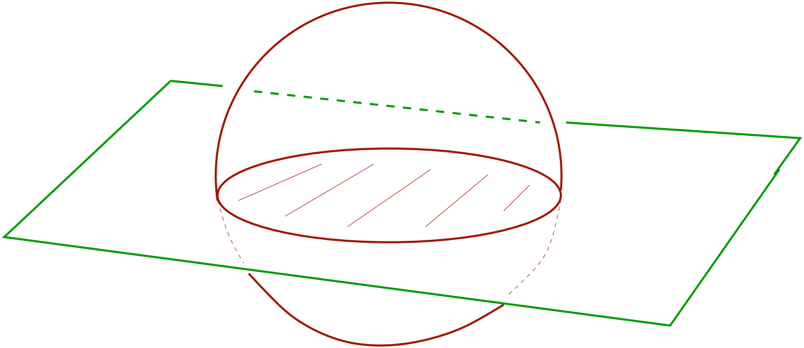

Planes dividing volume (or surface) in half

Fact : Three systems of planes in $\mathbb{R}^d$ coincide in some direction.

System of λ-planes for every λ-plane H through the origin in $\mathbb{R}^d$, we choose continuously a λ-plane $\Phi(H)$ parallel to H

$$\{\Phi(H)\}_{H\in G(\lambda,d)}$$

Theorem (Dol'nikov, Bracho-Montejano) Let $\{\Phi_0(H)\},\ldots,\{\Phi_\lambda(H)\}$ be $\lambda+1$ systems of λ-plane in $\mathbb{R}^d$. Then, they coincide in some direction, that is, there is a λ-plane H' through the origin such that

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A family of convex sets $\{A_i\}_1^n$ in $\mathbb{R}^d$ has λ-Helly property if every subfamily of $\{A_i\}_1^n$ of size $\lambda + 1$ is intersecting.

Remark Suppose that family $F = \{A_i\}_1^n$ in $\mathbb{R}^d$ has the λ-Helly property with $\lambda \leq d$. Then, there is a system of $(d - \lambda)$-planes in $\mathbb{R}^d$ transversal to F.

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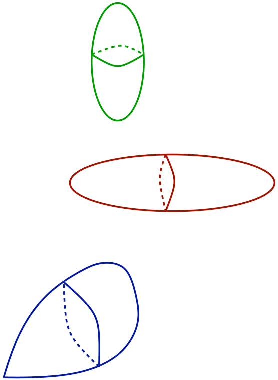

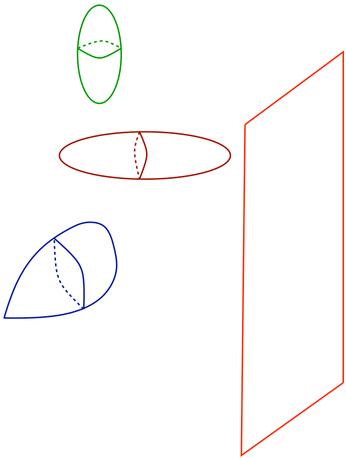

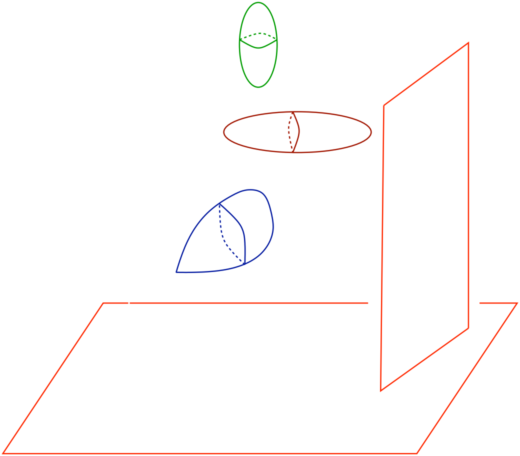

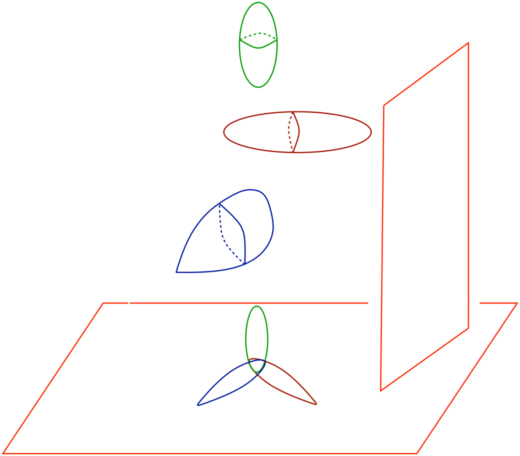

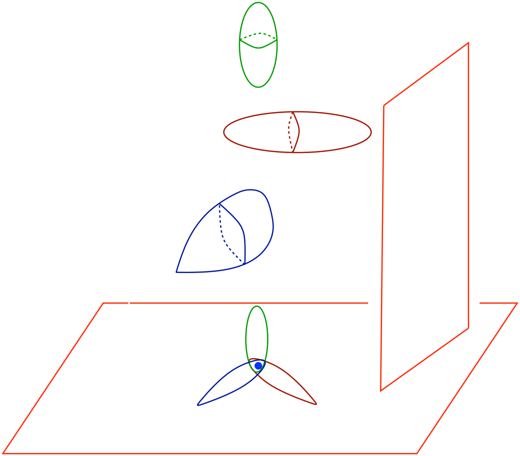

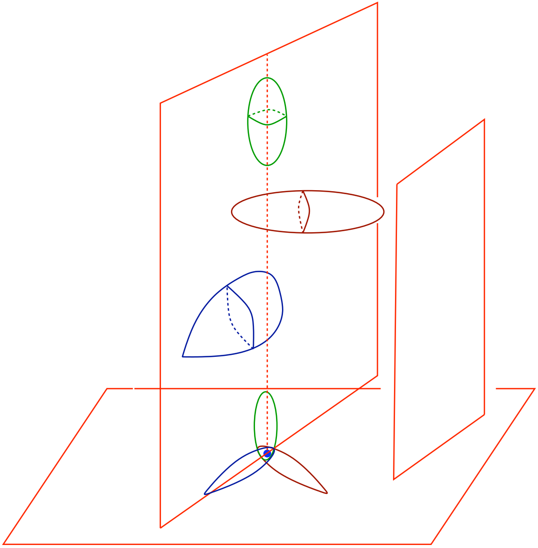

A coloration of F is λ-admissable if every subfamily of the convex sets with the same color has the λ-Helly property.

Proposition Let F be a family of convex sets in $\mathbb{R}^d$ and suppose that F has λ-admissable coloration with $d - \lambda + 1$ colors, $\lambda \leq d$. Then, F admits a transversal $(d - \lambda)$-plane.

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Kneser hypergraphs

A hypergraph H is a pair $(V, \mathcal{H})$ where V (*vertices*) is a finite set and $\mathcal{H}$ (*hyperedges*) is a collection of subsets of V.

The Kneser hypergraph $K^{\lambda+1}(n, k)$ is the hypergraph $(V, \mathcal{H})$ where V is the collection of all k-elements subsets of a n-set and $\mathcal{H} = \{(S_1, \ldots, S_\rho) |\ 2 \leq \rho \leq \lambda + 1,\ S_1 \cap \cdots \cap S_\rho = \emptyset\}$.

Remark Kneser graphs are obtained when $\lambda = 1$.

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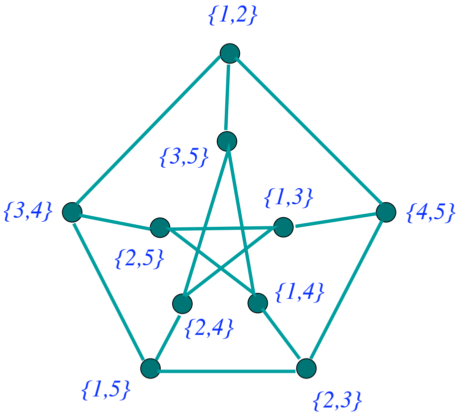
Kneser hypergraph when $n = 5$, $k = 2$ and $\lambda = 1$ (Petersen graph)



A coloring of a hypergraph H is a function that assigns colors to the vertices such that each hyperedge of H is *heterochromatic*.

A collection of vertices $\{S_1, \ldots, S_\rho\}$ of $K^{\lambda+1}(n,k)$ are in the same color class if and only if either

a) $\rho \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\rho \neq \emptyset$ or

b) $\rho > \lambda + 1$ and any $(\lambda+1)$-subfamily $\{S_{i_1}, \ldots, S_{i_{\lambda+1}}\}$ of $\{S_1, \ldots, S_\rho\}$ is such that $S_{i_1} \cap \cdots \cap S_{i_{\lambda+1}} \neq \emptyset$ (that is, they satisfy the λ-Helly property).

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Proposition If $\chi(K^{\lambda+1}(n,k)) \leq d-\lambda+1$ then $n \leq M(k,d,\lambda)$.

Theorem $\chi(K^{\lambda+1}(n,k)) \leq n-k-\lceil\frac{k}{\lambda}\rceil+2$.

Corollary $d-\lambda+k+\lceil\frac{k}{\lambda}\rceil-1 \leq M(k,d,\lambda)$.

Corollary

$$\chi(K^{\lambda+1}(n,k)) > \begin{cases} n-2k+\lambda & \text{if } k \geq \lambda, \\ n-2k & \text{if } k \leq \lambda. \end{cases}$$

Theorem (Lovász) $\chi(K^2(n,k)) = n-2k+2$.

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Conjecture $M(k,d,\lambda) = d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1.$

Theorem (Arocha, Bracho, Montejano, R.A.)
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a) $d = \lambda$ or
b) $\lambda = 1$ or
c) $k \leq \lambda$ or
d) $\lambda = k - 1$ or
e) $k = 2, 3.$

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c) $k \leq \lambda$ or

d) $\lambda = k - 1$ or

e) $k = 2, 3.$