Polytopes from $\{1, 3\}$-graphs: Ehrhart quasi-polynomials and scissors congruence phenomenon

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joint work with
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For $k \in \mathbb{N}$, let

$$L_P(k) := \#(kP \cap \mathbb{Z}^d)$$

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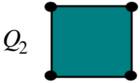

k	1
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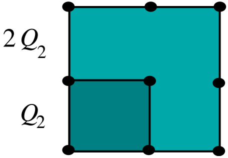

k	1	2
$L_{Q_2}(k)$	4	9

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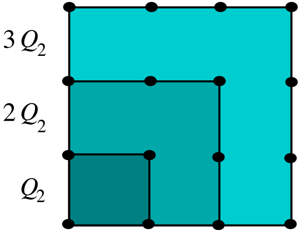

k	1	2	3
$L_{Q_2}(k)$	4	9	16

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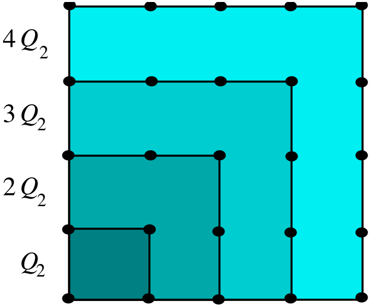

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$L_{Q_2}(k)$	4	9	16	25

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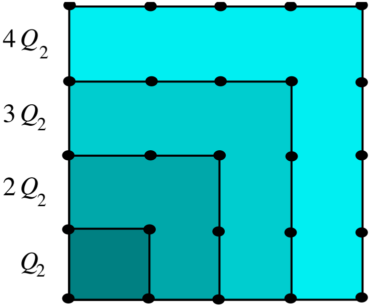

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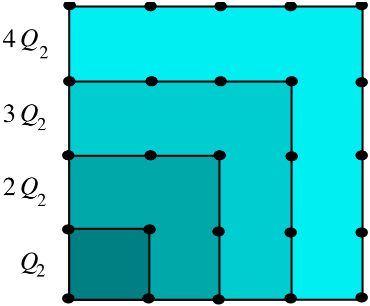

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***d*-dimensional cube :** $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

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As a lycée teacher Ehrhart did many of his investigations as an amateur mathematician.

Eugène Ehrhart (1906-2000)

Quasi-polynomial

A periodic rational number $c(n)$ is a function $c : \mathbb{Z} \to \mathbb{Q}$ with a period q such that $c(n) = c(n')$ when $n \equiv n' \pmod q$.

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A quasi-polinomial f of degree d is a function

$$f(n) = c_d(n)n^d + \cdots + c_1(n)n + c_0$$

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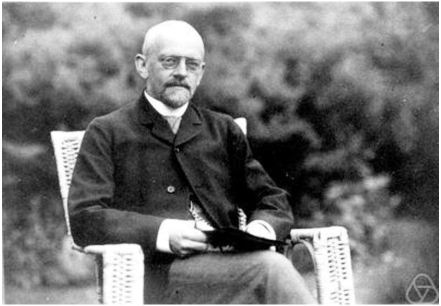
Theorem (Ehrhart) If P is a rational d-polytope then $L_P(t)$ is a quasi-polynomial of degree d. The period of $L_P(t)$ divides the denominator of P.

Hilbert's Third Problem

In a famous lecture delivered at the International Congress of Mathematics at Paris in 1900, Hilbert posed 23 problems.



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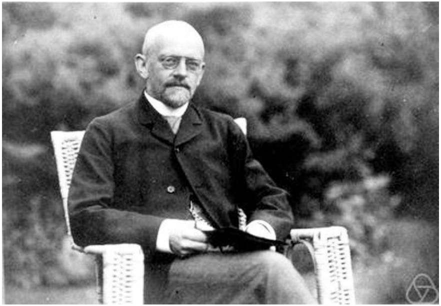

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Hilbert's Third Problem Are polyhedra in $\mathbb{R}^3$ of same volume scissors congruent ?

Analogue for $d = 2$

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Theorem (Wallace-Bolyai-Gerwien, 1833) Two polygons are equidecomposable if and only if they have the same area.
The idea of the proof is nicely explained in the video
https://www.youtube.com/watch?v=ysV6iF3Rmjo

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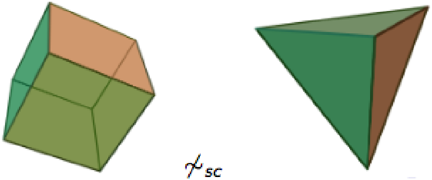

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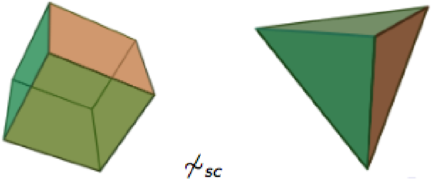

Done by using the Dehn invariant of scissors congruence (depending on edge lengths and edge dihedral angles).

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Motivation : The Ehrahrt polynomial of an integer polytope is invariant under affine unimodular transformation.

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For each degree-3 vertex v of a $\{1,3\}$-graph G,
let a, b, and c be the edges incident to v.

$S(v)$ is the system defined on the variables w_a, w_b, and w_c :

$$w_a + w_b + w_c \leq 1$$

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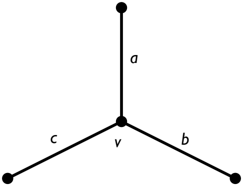

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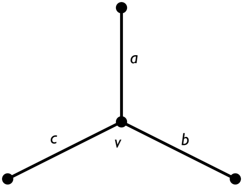

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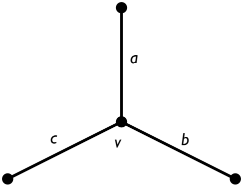

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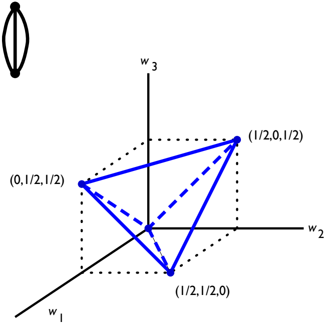
Properties of this polytope are related to a work in algebraic geometry by Mochizuki, 1999.

Examples : polytopes of cubic graphs on two vertices



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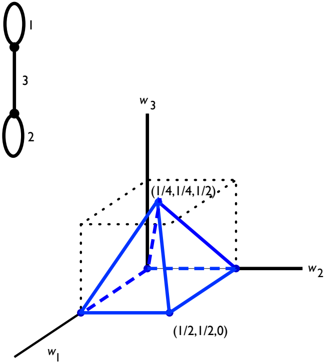

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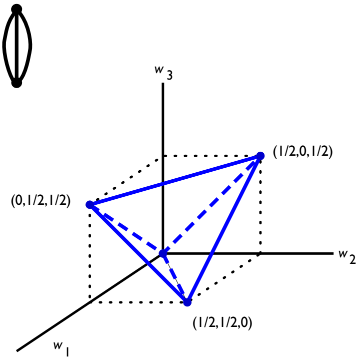

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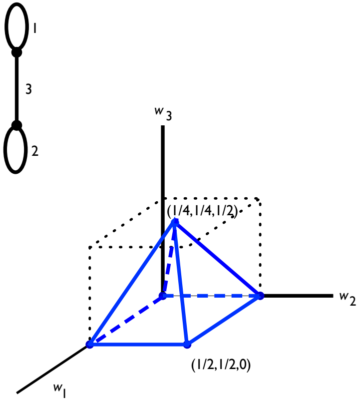

$$2w_1 + w_3 \leq 1$$
$$w_3 \leq 2w_1$$
$$w_3 \geq 0$$
$$2w_2 + w_3 \leq 1$$
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First result

Theorem (Fernandes, De Pina, Robins, R.A., 2021) Let G_1 and G_2 be two same-size connected $\{1,3\}$-graphs. Then, $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ are unimodular equidecomposable.

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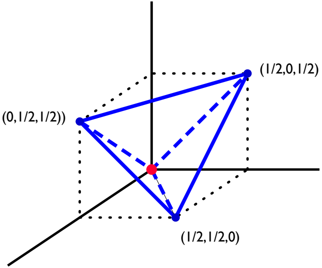

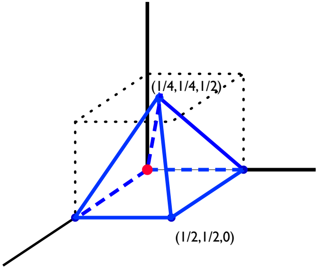

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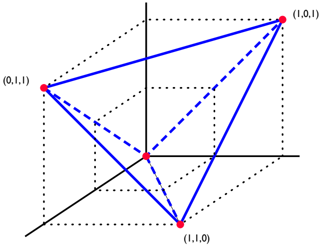
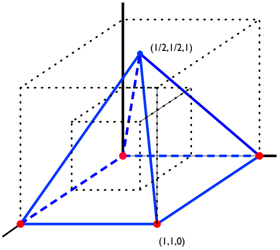
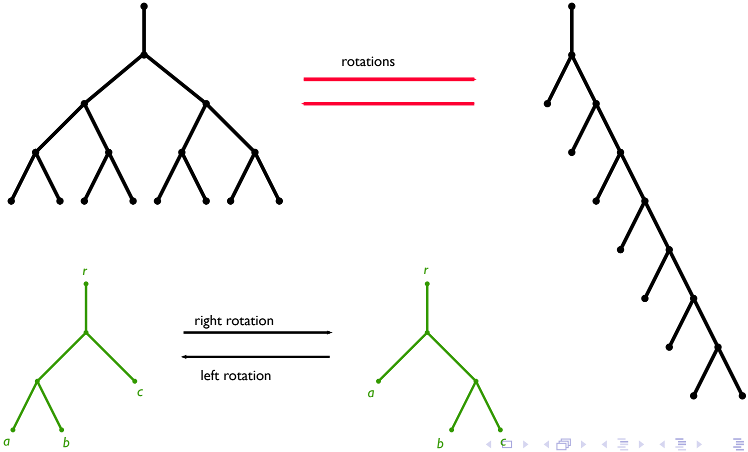




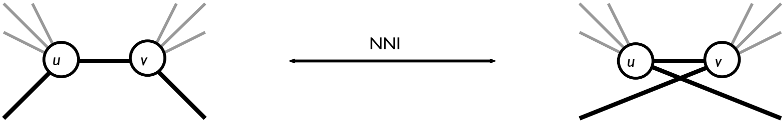
Binary trees and rotations

Theorem (Culik and Wood, 1982) Any two binary trees with the same number of vertices can be transformed into one another through a finite series of rotations.



Nearest neighbor interchange

NNI : nearest neighbor interchange



u and v adjacent vertices

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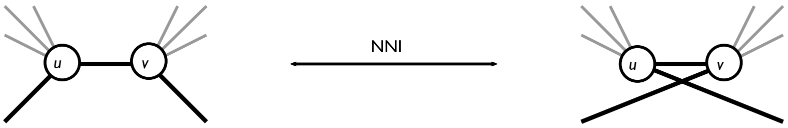

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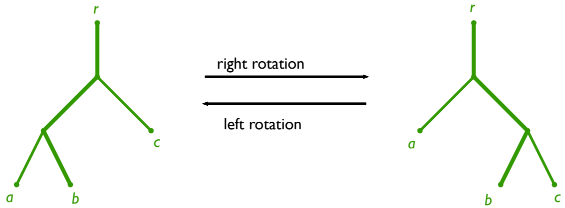

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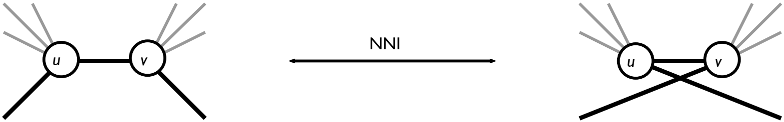

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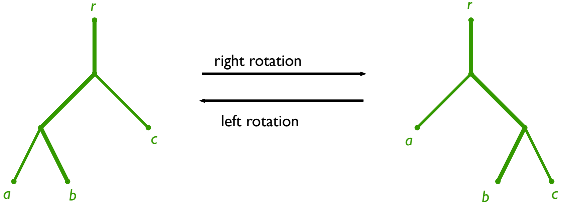

Loops and parallel edges allowed

Graph result

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(b) One can choose a spanning tree in G and a spanning tree in G' and require that all the pivots of the NNI moves are internal edges of both of these spanning trees.

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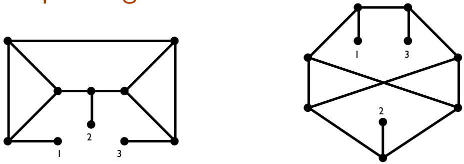

Extension of a theorem for cubic graphs by Tsukui (1996).

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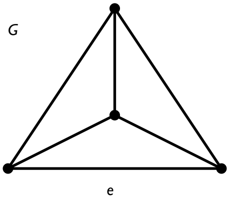

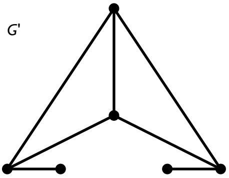

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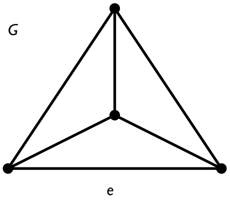
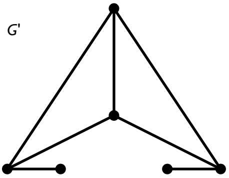
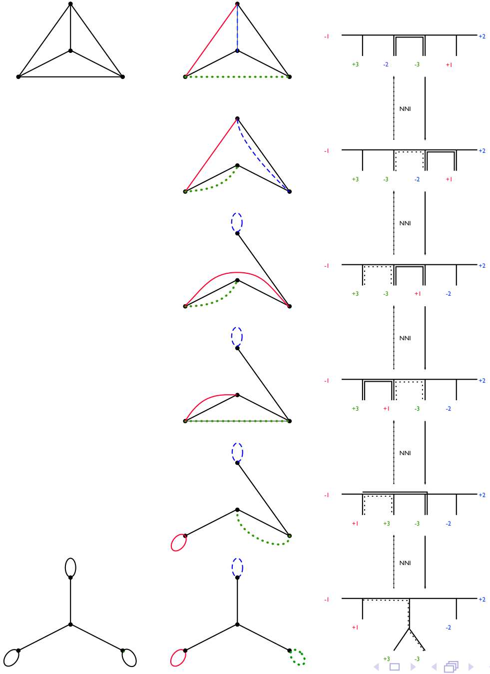




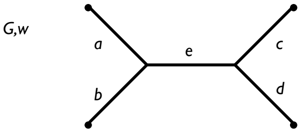
Then $r(G') = r(G) - 1$.

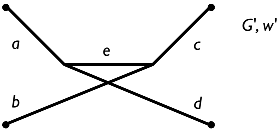


-1
+2
+3
-2
-3
+1
NNI

Weighted NNIs

G : $\{1,3\}$-graph
e : edge between two degree-3 vertices of G





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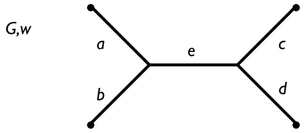

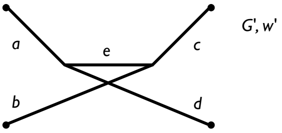

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Note that if w has integer values, so does w'.

Weighted NNIs

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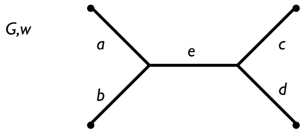

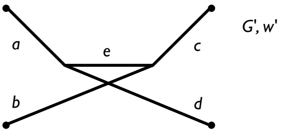

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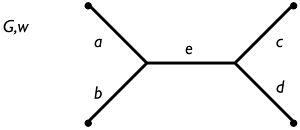

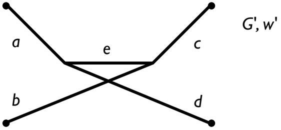

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Assume that $w_a + w_b \geq w_c + w_d$ and $w_a + w_d \geq w_b + w_c$.
Thus $w'_e = w_e + w_b - w_d$. If $w_e + w_a + w_b \leq t$, then

$$\begin{aligned} w'_e + w_b + w_c &\leq w'_e + w_a + w_d \\ &= (w_e + w_b - w_d) + w_a + w_d \\ &= w_e + w_a + w_b \leq t. \end{aligned}$$

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We associate to ψ the hyperplanes $w_a + w_b - w_c - w_d = 0$ and $w_a - w_b - w_c + w_d = 0$, which are either the same hyperplane (if $a=b$ or $c=d$) or two orthogonal hyperplanes.

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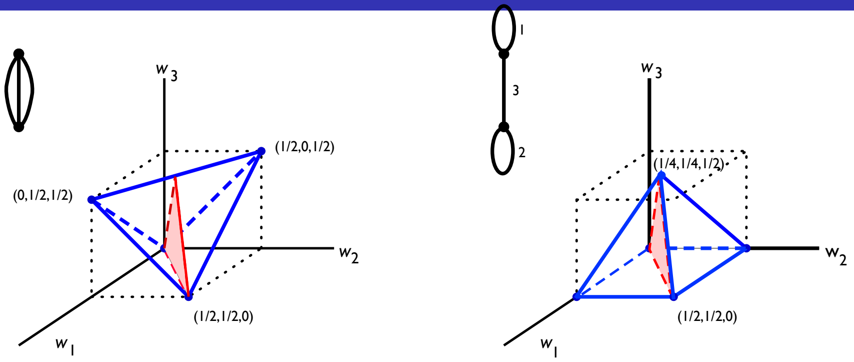
Moreover, the matrix that gives the linear transformation in each case is unimodular : the identity matrix substituting row e by row $\chi^e + \chi^b - \chi^d$ (in the first case) an so on.

Scissors congruence



$$
\begin{aligned}
w_3' &= w_3 + \max\{w_1 + w_2, w_1 + w_2\} - \max\{2w_1, 2w_2\} \\
&= w_3 + w_1 + w_2 - 2\max\{w_1, w_2\} \\
&= \begin{cases} w_3 - w_1 + w_2 & \text{if } w_1 \geq w_2 \\ w_3 + w_1 - w_2 & \text{if } w_1 \leq w_2 \end{cases}
\end{aligned}
$$

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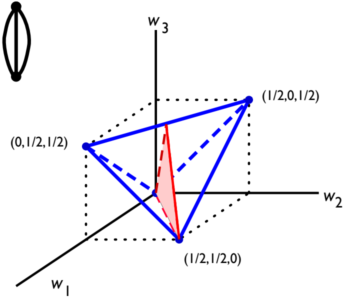

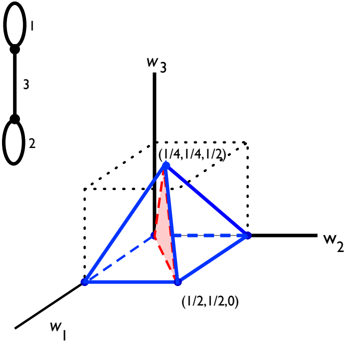

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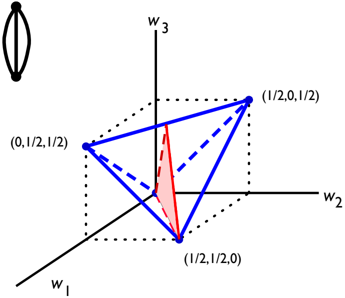

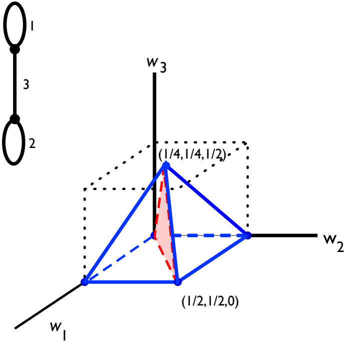

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$$U_{w_1 \leq w_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad U_{w_1 \geq w_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

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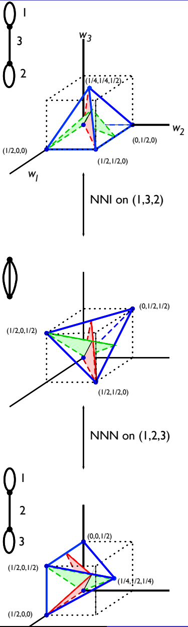
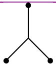
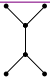
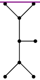
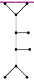


Ehrhart quasi-polynomials

G	Ehrhart quasi-polynomial $\mathcal{P}_G$
	$\frac{1}{24}t^3 + \frac{1}{4}t^2 + \begin{cases} \frac{5}{6}t + 1, & \text{if } t \text{ is even} \\ \frac{11}{24}t + \frac{1}{4}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{1}{240}t^5 + \frac{1}{24}t^4 + \begin{cases} \frac{5}{24}t^3 + \frac{7}{12}t^2 + \frac{11}{10}t + 1, & \text{if } t \text{ is even} \\ \frac{1}{6}t^3 + \frac{1}{3}t^2 + \frac{79}{240}t + \frac{1}{8}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{17}{40320}t^7 + \frac{17}{2880}t^6 + \begin{cases} \frac{59}{1440}t^5 + \frac{25}{144}t^4 + \frac{179}{360}t^3 + \frac{173}{180}t^2 + \frac{93}{70}t + 1, & \text{if } t \text{ is even} \\ \frac{103}{2880}t^5 + \frac{35}{288}t^4 + \frac{1439}{5760}t^3 + \frac{893}{2880}t^2 + \frac{791}{3360}t + \frac{1}{16}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{31}{725760}t^9 + \frac{31}{40320}t^8 + \begin{cases} \frac{829}{120960}t^7 + \frac{37}{960}t^6 + \frac{653}{4320}t^5 + \frac{103}{240}t^4 + \frac{20413}{22680}t^3 + \frac{1723}{1260}t^2 + \frac{193}{126}t + \\ \frac{43}{6912}t^7 + \frac{19}{640}t^6 + \frac{3181}{34560}t^5 + \frac{123}{640}t^4 + \frac{39205}{145152}t^3 + \frac{9923}{40320}t^2 + \frac{379}{2880}t \end{cases}$

Further results

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Theorem (Fernandes, De Pina, Robins, R.A., 2021)
The period of the Ehrhart quasi-polynomial of $\mathcal{P}_G$ is 2 if and only if G is cubic or a tree. Otherwise its period is 4.

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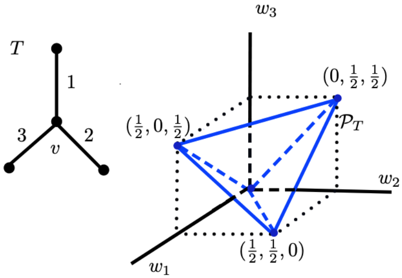
- The degree of each vertex of $\mathcal{P}_T$ is $\binom{\ell}{2}$ where ℓ is the number of leaves of T.
- Let w and w' be two distinct vertices of $\mathcal{P}_T$. Then w and w' are adjacent in the 1-skeleton of $\mathcal{P}_T$ if and only if $H_w \Delta H_{w'}$ is a leaf-path.

Example

Disjoint leaf-paths : $\{\emptyset, \{1,2\}, \{2,3\}, \{1,3\}\}$
Vertex set of $\mathcal{P}_T$: $\{(0,0,0), (\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$



High symmetry

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Theorem Let H be a collection of disjoint leaf-paths in T and let $h_H : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ where

$$h_H(w) = \begin{cases} \frac{1}{2} - w_e & \text{if } e \in E(H) \\ w_e & \text{otherwise.} \end{cases}$$

Then h_H is an isometry of $\mathcal{P}_T$.

Polytope $\mathcal{Q}_G$

For each degree-3 vertex v of a $\{1,3\}$-graph G,
let a, b, and c be the edges incident to v.

$S'(v)$ is the system defined on the variables w_a, w_b, and w_c :

$$w_a + w_b + w_c \leq 1$$

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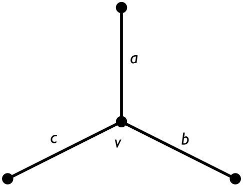

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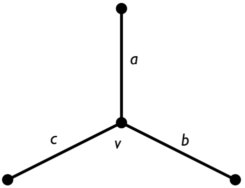

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$\mathcal{Q}_G$: solutions of the union of $S'(v)$

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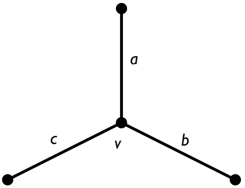
$S'(v)$ is the system defined on the variables w_a, w_b, and w_c :

$$w_a + w_b + w_c \leq 1$$

$$\begin{aligned} w_a &\leq w_b + w_c \\ w_b &\leq w_a + w_c \\ w_c &\leq w_a + w_b \end{aligned}$$



$w_a + w_b + w_c = 2z_v$ (parity contraint)
$z_v \leq t$ (auxiliary variable)

$\mathcal{Q}_G$: solutions of the union of $S'(v)$

Remark $\mathcal{Q}_G \subset \mathbb{R}^E \times \mathbb{R}^I$ with I set of internal nodes of G.

Connection between $\mathcal{P}_G$ and $\mathcal{Q}_G$

If t is a nonnegative odd integer then

$$L_G^{\mathcal{Q}}(t) = N_G L_G^{\mathcal{P}}(t)$$

where N_G is the number of internally Eulerian subgraphs of G, that is, the degree of every internal node is equal to zero or two.

Interpretation : arrangements of pseudocircles

Let T be a triangulation of the 2-sphere and let T^* its dual.

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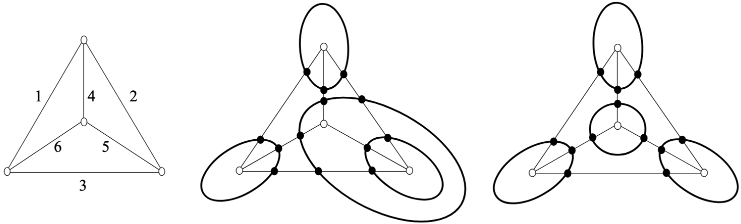
Example Consider the triangulation induced by K_4. It can be checked that$w_1 = w_4 = w_6 = 2, w_2 = w_3 = 3, w_5 = 1, z_{v_1} = 2, z_{v_2} = z_{v_3} = z_{v_4} = 3$ and $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 2$, $z_{v_1} = z_{v_2} = z_{v_3} = z_{v_4} = 3$ are two integer points in $3\mathcal{Q}_{K_4^*}$.



Thanks for your attention ! !