Geometric Knots and Oriented Matroids

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Knot theory

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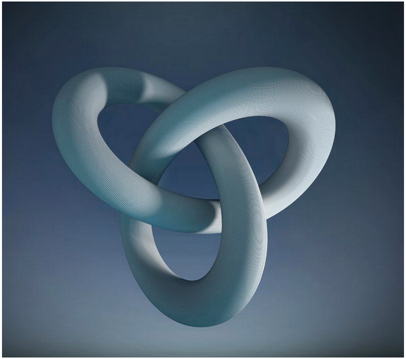


Knot theory : diagrams

Trivial knot
0_1

Trefoil knot
3_1

Figure-eight knot
4_1

Pentafoil knot
5_1

Trivial link
0_1^2

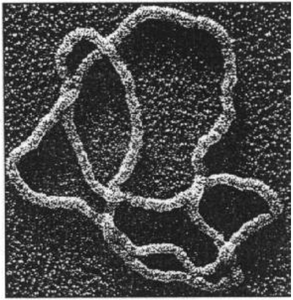
Hopf link
2_1^2

Solomon link
4_1^2

Borromean link
6_2^3

Application : DNA (Deoxyribonucleic Acid)

A fundamental application of this problem can be found in the study of DNA



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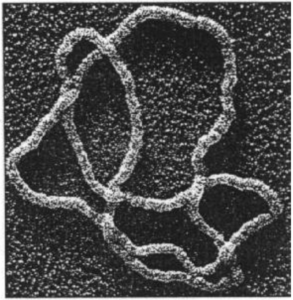

Question (hard) : What is the type of knot arising from a DNA diagram ?

Geometric knots

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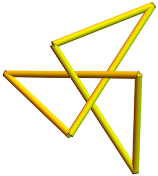

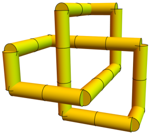

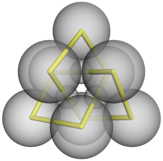

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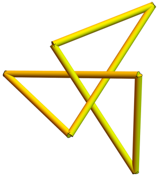
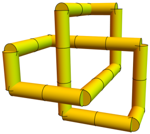
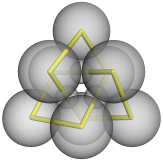

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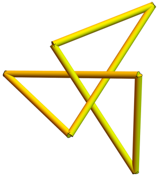

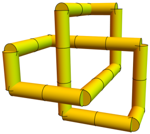

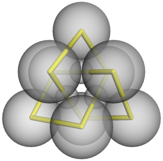

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Geometric knots serve as a model of ring polymers like bacterial DNA.

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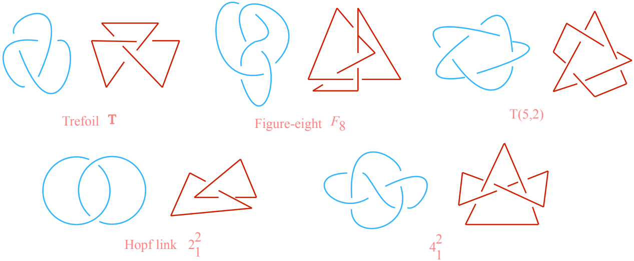

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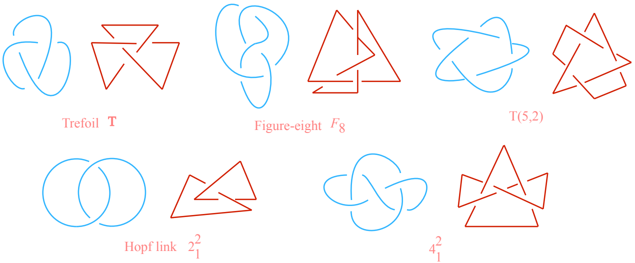

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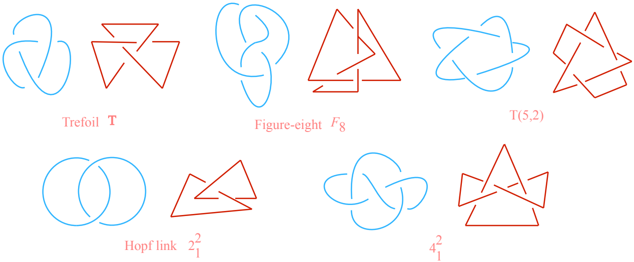

This invariant is only poorly understood. The exact stick number is only known for 30 of the 249 nontrivial knots up to 10 crossings.

Equilateral stick number

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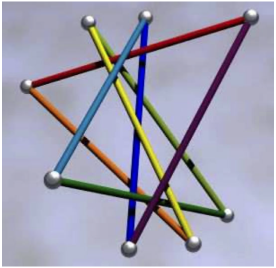

An eight-edge equilateral 8_{19} knot

Matroids : independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

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$$|I_2| \leq dim(W) \leq |I_1| < |I_2| \quad \text{contradiction!!!}$$

Representable Matroids

Let A be the following matrix with coefficients in $\mathbb{R}$.

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\{\emptyset, \{1\}, \{2\}, \{4\}, \{4\}, \{5\}, \{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}\} \subseteq \mathcal{I}(M)$

A matroid obtained form a matrix A with coefficients in $\mathbb{F}$ is denoted by $M(A)$ and is called representable over $\mathbb{F}$ or $\mathbb{F}$-representable.

Circuits

A subset $X \subseteq E$ is said to be minimal dependent if any proper subset of X is independent. A minimal dependent set of matroid M is called circuit of M.

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$\mathcal{C}$ is the set of circuits of a matriod on E if and only if $\mathcal{C}$ verifies the following properties :

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Remark : A matroid can also be defined by its : bases, flats, rank, greedy algorithm, duality, polytope, etc.

Tutte polynomial

The Tutte polynomial of a matroid M is the generating function defined as follows

$$t(M;x,y)=\sum_{X\subseteq E}(x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}.$$

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The Tutte polynomial can be expressed recursively

$$t(M;x,y) = \begin{cases} t(M \setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M \setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

where $M \setminus e$ is the deletion of e and M/e is the contraction of e.

Bracket polynomial

For any link diagram D define a Laurent polynomial $\langle D \rangle$ in one variable A which obeys the following three rules where U denotes the unknot :

i) $\langle U \rangle = 1$

ii) $\langle U + D \rangle = -(A^2 + A^{-2}) \langle D \rangle$

iii) 

Writhe

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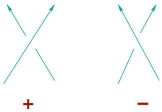

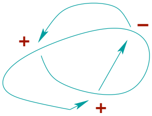

$\omega(D)=1$

Jones' polynomial

Theorem For any link diagram D define the Laurent polynomial

$$f_D(A) = (-A^3)^{\omega(D)} \langle D \rangle$$

Then, $f_D(A)$ is an invariant of ambient isotopy.

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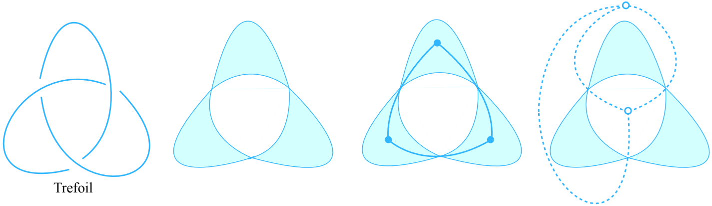
It is known that the so-called Jones' polynomial of an oriented link L is given by

$$V_L(z) = f_D(z^{-1/4})$$

where D is any diagram representing L.

Tait graphs



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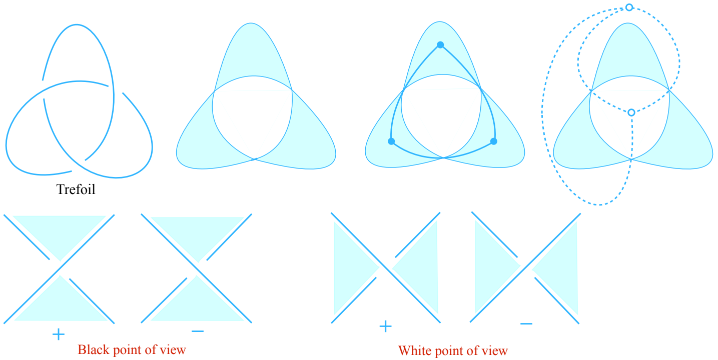

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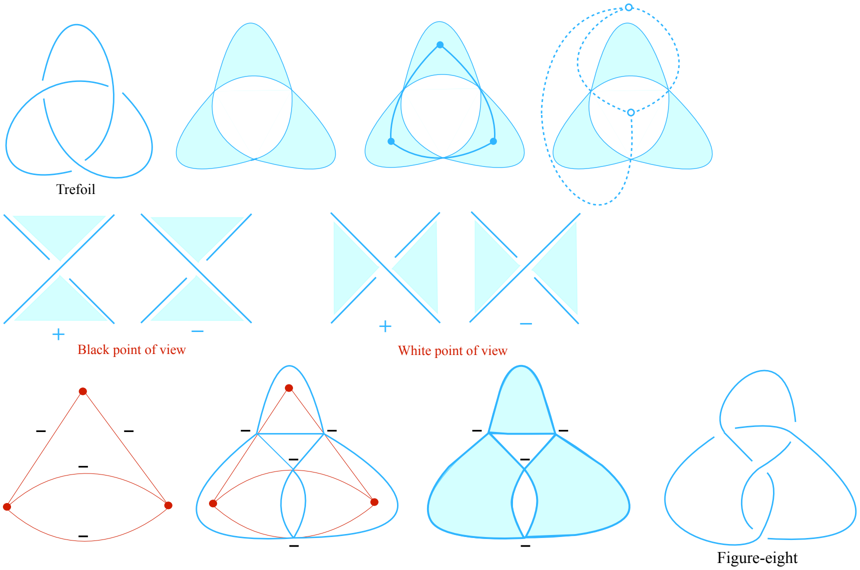
Trefoil
+
−
Black point of view
+
−
White point of view
−
−
−
−
−
−
−
−
−
−
−
−
Figure-eight

Jones' polynomial through Tutte polynomial

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Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram then

$$V_D(z) = (z^{-1/4})^{3\omega(D)-2} t(M(G); -z, -z^{-1})$$

where G is the Tait graph associated to D.

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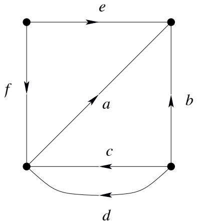
- If signs are not taken into account then these axioms reduced to the circuits axioms $(C0), (C2), (C3)$ of a nonoriented matroid.
- Not all matroids are orientables (for instance, F_7 is not orientable.

Oriented graph

Let G be an oriented graph. We obtain the signed circuits from the cycles of G.



Then,

$$\begin{aligned}\mathcal{C} &= \{(a\bar{b}c),(a\bar{b}d),(a\bar{e}f),(c\bar{d}),(b\overline{ce}f),(b\overline{de}f),\\ &\quad (\bar{a}b\bar{c}),(\bar{a}b\bar{d}),(\bar{a}e\bar{f}),(\bar{c}d),(\bar{b}ce\bar{f}),(\bar{b}de\bar{f})\}.\end{aligned}$$

Vector configuration

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We obtain an oriented matroid on E by considering the signed sets $X = (X^+, X^-)$ where

$$X^+ = \{i : \lambda_i > 0\} \text{ et } X^- = \{i : \lambda_i < 0\}$$

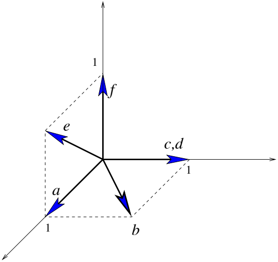
for all minimal dependencies among the v_i.

Example

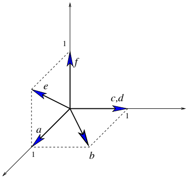
Let

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

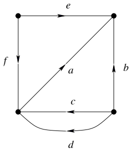
The columns of A correspond to the following vectors



We can check that the circuits of



are the same as those arising from



For exemple, $(a\bar{b}c)$ correspond to the linear combination $a - b + c = 0$ or the circuit $(b\overline{de}f)$ correspond to the linear combination $b - d - e + f = 0$.

Configurations of points

Any configuration of points induce an oriented matroid in the affine space where the signed set of circuits are are the coefficients of minimal affine dependencies of the form

$$\sum_i \lambda_i \mathsf{v}_i = 0 \quad \text{with} \quad \sum_i \lambda_i = 0,\ \lambda_i \in \mathbb{R}$$

$$B = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{pmatrix} \end{array}$$

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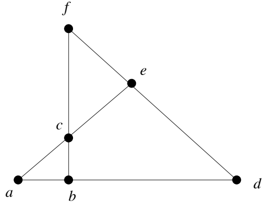

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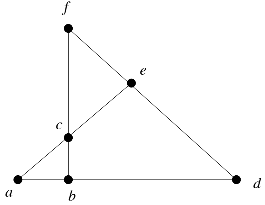

Signed circuits corresponding to the affine oriented matroid associated to B.

$$
\begin{aligned}
\mathcal{C} \quad = \{ & (a\bar{b}d), (b\bar{c}f), (d\bar{e}f), (a\bar{c}e), (\bar{a}b\bar{e}f), (\bar{b}cd\bar{e}), (a\bar{c}df), \\
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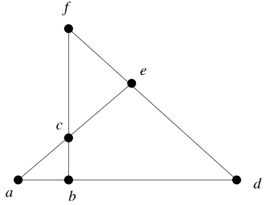

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For instance, circuit $(a\bar{b}d)$ correspond to the affine dependency $3(-1,0)^t - 4(0,0)^t + 1(3,0)^t = (0,0)^t$ with $3 - 4 + 1 = 0$.

Chirotope

$\mathcal{B}$ is the set of bases of an oriented matroid if and only if there is an application, called chirotope, $\chi : E^r \to \{+, -, 0\}$ such that

(B1) $\mathcal{B} \neq \emptyset$;

(B2) for any B and B' in $\mathcal{B}$ and $e \in B \setminus B'$ there existes $f \in B' \setminus B$ such that $B \setminus e \cup f \in \mathcal{B}$;

(B3) $\{b_1, \ldots, b_r\} \in \mathcal{B}$ if and only if $\chi(b_1, \ldots, b_r) \neq 0$

(B4) χ is alternating, i.e.
$\chi(b_{\sigma(1)}, \ldots, b_{\sigma(r)}) = sign(\sigma)\chi(b_1, \ldots, b_r)$ for any $b_1, \ldots, b_r \in E$ for any permutation σ

(B5) χ verifies the Grassmann-Plücker relation

Radon Partitions

Theorem (Radon) Let X be a set of $n \geq d+2$ points in $\mathbb{R}^d$. Then, there always exists a partition $X = A \cup B$ such that $conv(A) \cap conv(B) \neq \emptyset$.

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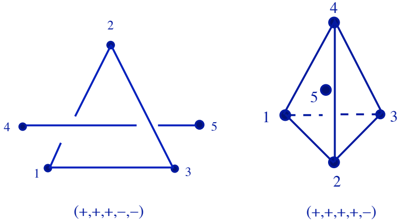
Example : $d = 3$.



These are called minimal Radon partitions

Spatial graphs

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A spatial representation is linear if the curves are line segments

Linear representations

Let $m(L)$ be the smallest integer such that any spatial linear representation of K_n with $n \geq m(L)$ contains cycles isotopic to L.

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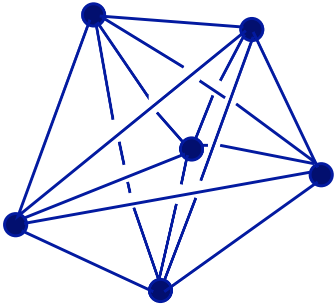

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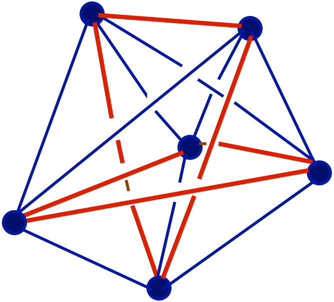

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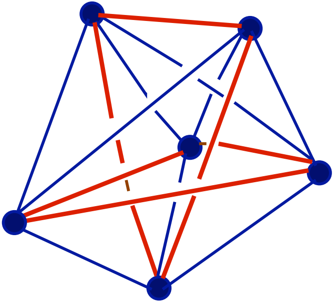

Theorem (R.A. 1998, 2000, 2009)

$m(T \text{or} T^*) = 7,\ m(4_1^2) > 7,\ m(F_8) > 8,\ m(T(5,2)) > 8.$

Isotopy Conjecture

Isotopy Conjecture for Oriented Matroid (Ringel 1956) Is the realization space over the real number field of an oriented matroid path-connected ? In other words, can one given realization of M be continuously deformed, through realizations, to another given one ?

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Theorem (Jaggi, Mani-Levitska, Sturmfels, White 1989) Provide a uniform counterexample of rank 3 on 17 points.

Universality Theorem

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Consequence The isotopy problem has a (very) negative solution even for uniform matroids of rank 3.

Las Vergnas' question

Let $X = (x_0, \ldots, x_{n-1})$ be a n-uple of points in $\mathbb{R}^3$ in general position. Let K_X be the polygonal knot defined by the segments $[x_i, x_{i+1}]$ (addition $\pmod n$)

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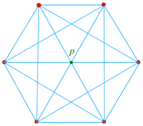
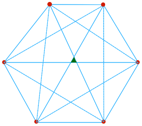
In other words

Question Let X and Y be two sets of n points. Is it true that if there is a bijection $\varphi : X \to Y$ preserving Radon partitions then K_X is isotopic to K_Y ?

Strong geometry

Two configurations of points having the same oriented matroid





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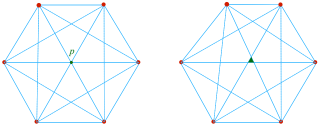
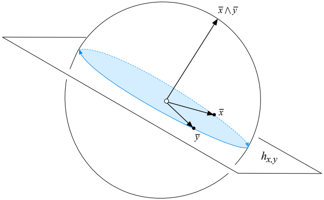


We introduce a new oriented matroid $M_\wedge(X)$ arising from the set of hyperplanes spanned by X.



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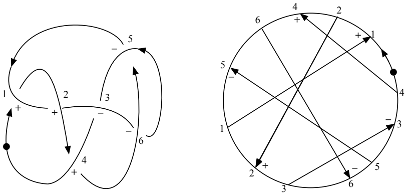
Strong geometries encode nicely the combinatorics of the cells of the arrangement of the spanned lines.

Theorem (Gros, R.A. 2025) The polygonal knot K_X can be completely determined by $SGeom(X)$.

Gauss diagram



Knot 6_3 together with a Gauss code

$U_{+}1\ O_{+}2\ U_{+}4\ O_{-}6\ U_{-}5\ O_{+}1\ U_{+}2\ O_{-}3\ U_{-}6\ O_{-}5\ U_{-}3\ O_{+}4$

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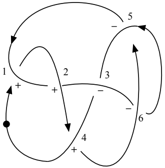

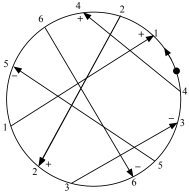

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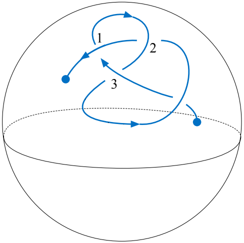
Lemma Any knot diagram on the sphere can be recovered uniquely (up to isotopy) from its Gauss diagram.

Knotoid diagram



The knotoid diagram called line eight-figure

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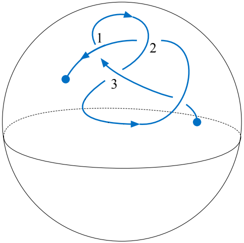

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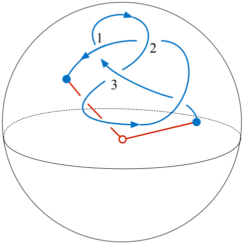
Lemma A knotoid diagram is determined by its Gauss diagram (up to a planar isotopy).



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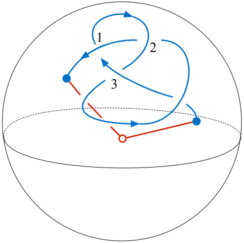

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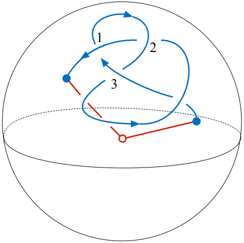

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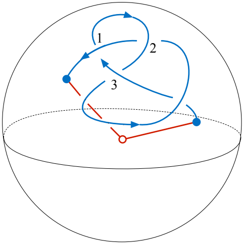

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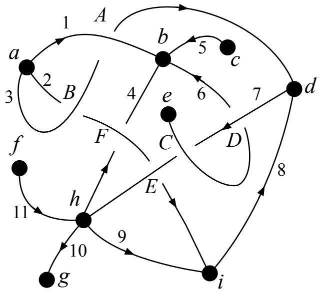

Gauss code

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$a2BU_-FO_+EU_+i$

$a3BO_-AU_+d$

$b4FU_+h$

$b5c$

$b6DU_-CO_-e$

$d7DO_-CU_-EO_+h$

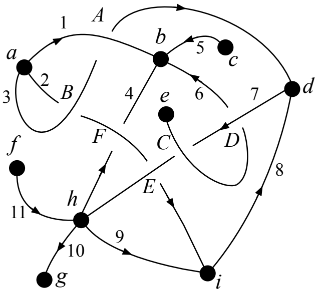
$d8i$

$i9h$

$h10g$

$h11f$

Graphoid diagram



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Spatial Graphs

Let $R(G)$ be a linear spatial representations of G. Let X_G be the set of points in $\mathbb{R}^3$ associated to the vertices of such representation.

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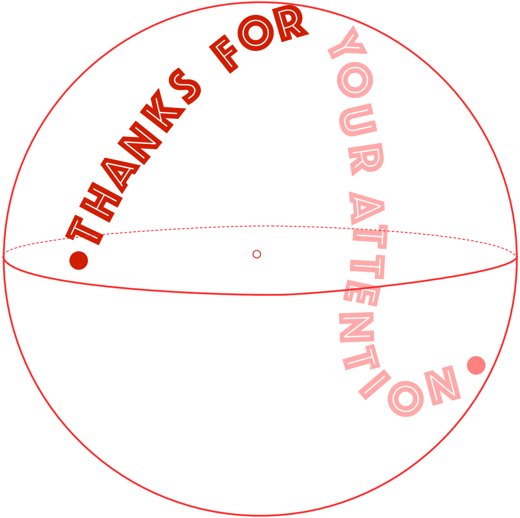
Question (spatial graph version) Is it true that $R(G)$ can be determined by using $M(X_G)$ only ?


THANKS FOR
YOUR ATTENTION