Ehrhart theory I : introduction

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Lattice notions

Let $B = \{b_1, \ldots, b_n\}$ be a family of linear independent vectors in $\mathbb{R}^d$. The lattice generated by B is the set

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We say that B is a base of Λ of dimension n and that Λ is of full rank if $n = d$.

Lattice notions

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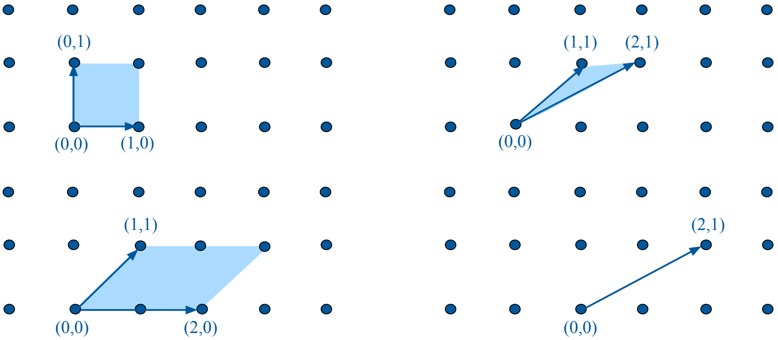

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belong to Λ. But, x' also belong to $P(B)$ and thus $x' = 0$, implying that $y_i - \lfloor y_i \rfloor = 0$ and thus all y_i are integers. Therefore, x is an integer combination of $b_1, \ldots, b_n$.

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- $\det(\Lambda)$ is well defined in the sense that it is independent of the choice of the base.
- $\det(\Lambda)$ is proportionally inverse to its density : smaller is the determinant denser will be the lattice.

Blichfeldt's theorem

Theorem Let Λ be a lattice generated by a base B and let $S \subseteq span(B)$. If $vol(S) > \det(\Lambda)$ then there are $z_1, z_2 \in S$ such that $z_1 - z_2 \in \Lambda$.

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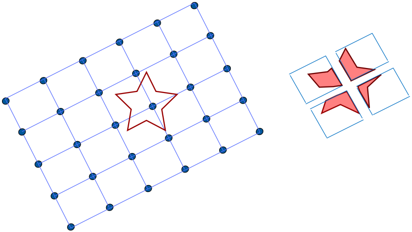
Proof Consider $S_x = S \cap (x + P(B))$ with $x \in \Lambda$. They are pairwise disjoint and $S = \bigcup_{x \in \Lambda} S_x$.



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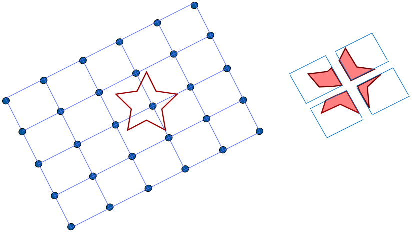

In particular we have $vol(S) = \sum\limits_{x \in \Lambda} vol(S_x)$.

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Let $z \in (S_x - x) \cap (S_y - y)$ and define

$$z_1 = z + x \in S_x \subseteq S \text{ and } z_2 = z + y \in S_y \subseteq S.$$

These vectors satisfy $z_1 - z_2 = x - y \in \Lambda$.

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$$z_1 - z_2 = \frac{2z_1 - 2z_2}{2} \in S$$

is a non-zero lattice point vector contained in S.

Integer points and volume

Theorem Let K be a convex in $\mathbb{R}^n$ containing 0 and such that $K \cap \mathbb{Z}^n$ is not included in a hyperplan. Then, $Card(K \cap \mathbb{Z}^n) \leq n + n! vol(K)$.

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Among the rest $k-1$ integer points in K, we pick the closest to S_0, say x_1.

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We carry on this way to construct k simplex S_j, obtaining

$$vol(K) \geq \sum_{i=0}^{k-1} vol(S_i) = k/n!$$

Therefore

$$Card(K \cap \mathbb{Z}^n) \leq n + n!vol(K).$$

Pick's theorem

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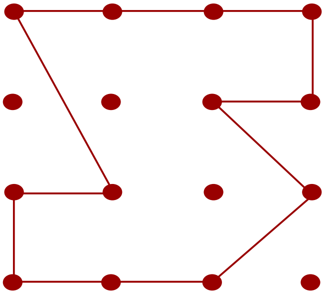

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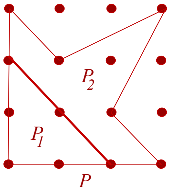
Proof (idea) Decompose P and use induction.



Let P be an integer polygon. By using Pick's theorem we obtain

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Theorem (Scott) Let $k \neq 0$ be the number of interior points of P then

$$Card(P \cap \mathbb{Z}^2) \leq 3k + 6$$

except if P is equivalent to a triangle with vertices $(0,0),(3,0),(0,3)$.

Theorem (Hensley) There exists a constant $B(n,k)$ depending only on k and n such that for any n-dimensional polytope P having exactly k integer points in its interior we have

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It is known that

$$B(n, k) \approx k(7(k+1))n2^n + 1.$$

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Example Let T_h be the tetrahedra with vertices $a = (0, 0, 0), b = (1, 0, 0), c = (0, 1, 0)$ et $d = (1, 1, h)$, $h \in \mathbb{Z}_{>0}$.

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Moreover, the only integer points in T_h are its 4 vertices.
Therefore, T_h only admits 4 integer points but the volume grows arbitrairily big as $h \to \infty$.

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For $k \in \mathbb{N}$, let

$$L_P(k) := \#(kP \cap \mathbb{Z}^d)$$

Ehrhart theory

The first investigations of $L_P(k)$ date back to the 1960's work of Ehrhart. As a lycée teacher he did many of his investigations as an amateur mathematician.

Eugène Ehrhart (1906-2000)

Ehrhart theory

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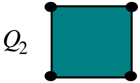

k	1
$L_{Q_2}(k)$	4

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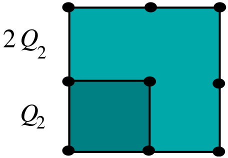

k	1	2
$L_{Q_2}(k)$	4	9

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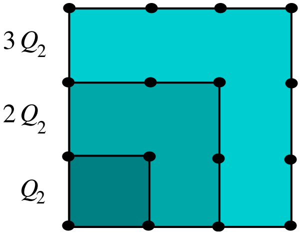

k	1	2	3
$L_{Q_2}(k)$	4	9	16

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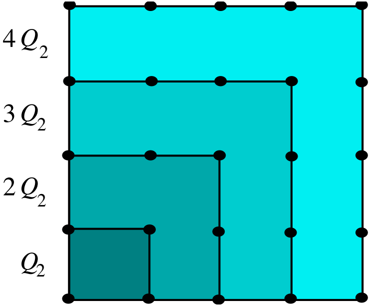

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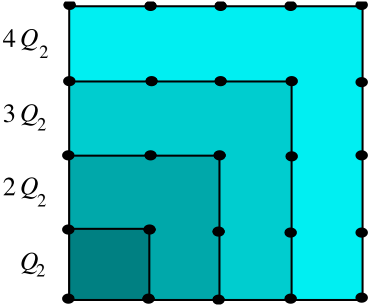

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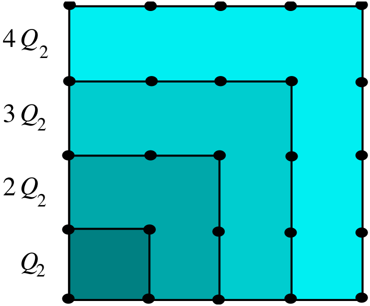

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***d*-dimensional cube :** $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Ehrhart polynomial

Theorem (Ehrhart 1962) Let P be an integer polytope. Then, $L_P(k)$ is a polynomial on k of degree $dim(P)$ with rational coefficients and constante term equals 1.

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Proof Let S be a simplex with vertices $0, s_1, \ldots, s_r$.

The set mS is the disjoint union of

$$S_{m,j} = \left\{ \lambda_1 s_1 + \cdots + \lambda_r s_r \mid \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i \leq m, \sum_{i=1}^{r} \lfloor \lambda_i \rfloor = m - j \right\}$$

for each $j \in \{0, \ldots, m\}$.

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$$a_j = \left\{ \beta_1 s_1 + \cdots + \beta_r s_r \mid 0 \leq \beta_i < 1, \ \sum_{i=1}^{r} \beta_i \leq j \right\}.$$

Ehrhart polynomial

Since $a_j = a_r$ for all $j \geq r$, we have

$$
\begin{aligned}
Card(mS \cap \mathbb{Z}^n) &= \sum_{j=0}^{m} a_j \binom{m-j+r-1}{r-1} \\
&= \sum_{j=0}^{r-1} a_j \binom{m-j+r-1}{r-1} + a_r \sum_{j=r}^{m} \binom{m-j+r-1}{r-1} \\
&= \sum_{j=0}^{r-1} (a_j - a_r) \binom{m-j+r-1}{r-1} + a_r \sum_{j=0}^{m} \binom{m-j+r-1}{r-1} \\
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&= \sum_{j=0}^{r-1} (a_j - a_r) \binom{m-j+r-1}{r-1} + a_r \binom{m+r}{r}.
\end{aligned}$$

Therefore, $Card(mP \cap \mathbb{Z}^n)$ is a polynomial function on m of degree r and its value is equals 1 when $m = 0$. Moreover, its leading term is equals $\frac{1}{r!} a_r$.

Example

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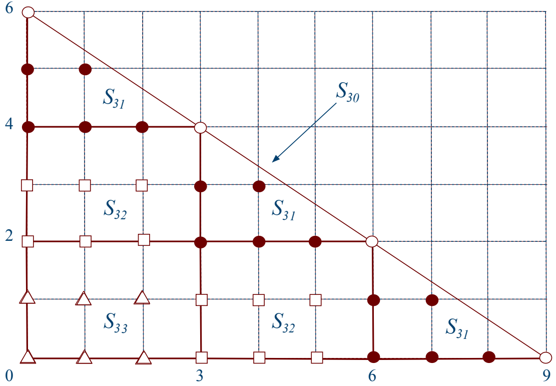

Decomposition of S in $S_{m,j}$. Points marked with $\circ, \bullet, \square, \triangle$ correspond to solutions of a_0, a_1, a_2 and a_3 respectively.

Example

$$
\begin{aligned}
L_S(m) &= \sum_{j=0}^{1}(a_j - a_2)\binom{m-j+2-1}{2-1} + a_2\binom{m+2}{2} \\
&= (a_0 - a_2)\binom{m+1}{1} + (a_1 - a_2)\binom{m}{1} + a_2\binom{m+2}{2} \\
&= (1-6)(m+1) + (5-6)m + 6\frac{(m+2)(m+1)}{2} \\
&= -5m - 5 - m + 3m^2 + 3m + 6m + 6 \\
&= 3m^2 + 3m + 1.
\end{aligned}
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By induction on the number of vertices, we obtain

$$\begin{aligned} L_{P^\circ}(m) &= L_{Q^\circ}(m) + L_{R^\circ}(m) + L_{F^\circ}(m) \\ &= (-1)^r L_Q(-m) + (-1)^r L_R(-m) + (-1)^{r-1} L_F(-m) \\ &= (-1)^r (L_Q(-m) + L_R(-m) - L_F(-m)) \\ &= (-1)^r L_P(-m). \end{aligned}$$

Some coefficients

Proposition Let P be an integer polytope of dimension d in $\mathbb{R}^d$ and let $c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$ be its Ehrhart polynomial. Then,

$$c_d = vol_d(P) \text{ and } c_{d-1} = \frac{1}{2} \sum_{F \subset P} vol_{d-1}(F).$$

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- The volume can be approximated by counting theses cubes or, equivalently, the integer points in $(\frac{1}{t}\mathbb{Z})^d$.

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Obtaining

$$vol(P) = \lim_{t\to\infty} \frac{c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0}{t^d} = c_d.$$

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[second equality] By the reciprocity law, we have

$$L_{P^\circ}(t) = a_d t^d - a_{d-1} t^{d-1} + \cdots .$$

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and the equality follows by same arguments as above.

Relative affine volume

Let $S \subset \mathbb{R}^d$ of dimension $m < d$. As before, we can compute the volume with respect to the induced sublattice of $span(S) \cap \mathbb{Z}^d$.

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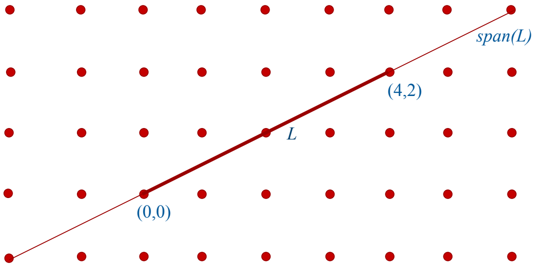

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$n = 3$ None of the coefficients a_i can be expressed in terms of $\sum_{F \subset P} vol_k(F)$. a_k is a linear combination of the $vol_k(F)$.

Voting theory

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Will PV yield another candidate as winner as PRV for a given voting situation ?And if so, what is the probability of this happening ?

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Consider an election on three candidates $\{a, b, c\}$. Assume that voters have complete linear preference ranking on these candidates. There are 6 possible preferences orders :

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

In order to compute the probability that some event takes place, we assume that all voting situation is equally likely to occur (called Impartial Anonymous Culture condition).
Suppose that a voting situation occurs where PV will denote a as winner while PRV will claim that b has won.

$n_{abc} + n_{acb} > n_{bac} + n_{bca}$	a beats b
$n_{bac} + n_{bca} > n_{cab} + n_{cba}$	b beats c
$n_{abc} + n_{acb} + n_{cab} < N/2$	a loses the second round
$n_{abc} + n_{acb} + n_{bac} + n_{bca} + n_{cab} + n_{cba} = N$	all votes add up N
$n_i \geq 0$	for all $i \in S_{abc}$

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- The last makes sure that the number people with certain preference is not negative.

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• The above describe the situation where a is the PV winner while b is the PRV winner but there are 6 possible pairs of PV winner and PRV winner Therefore,

$$Prob(PV\, and\, PRV\, disagree) = 6\frac{\#(P_d \cap \mathbb{Z}^d)}{\#(P_t \cap \mathbb{Z}^d)}.$$