Ehrhart theory III : Tutte polynomial and zonotopes

J.L. Ramírez Alfonsín

Université de Montpellier

V Mexican School in Discrete Mathematics
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Minkowski's sum

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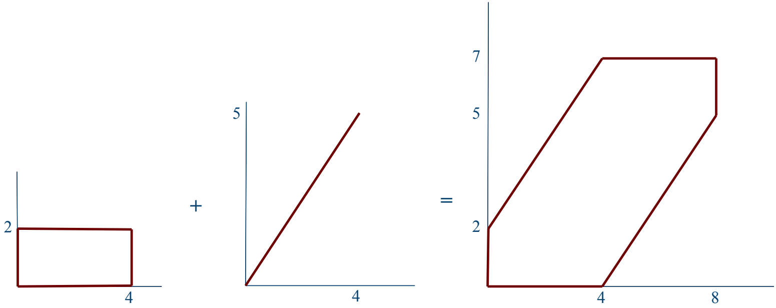
Example $P_1 = [4,0] \times [0,2] \subset \mathbb{R}^2$ and $P_2 = [(0,0),(4,5)] \subset \mathbb{R}^2$.
The vertices of $P_1 + P_2$ are $(0,0),(4,0),(0,2),(4,7),(8,5),(8,7)$.



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Example Take $u_1 = (4,1), u_2 = (3,2), u_3 = (1,3)$.
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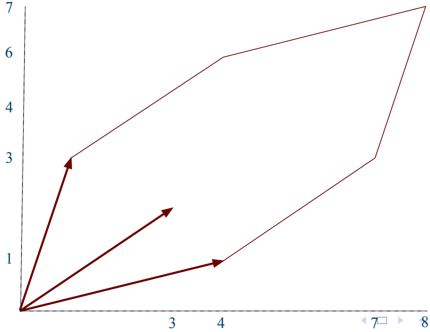
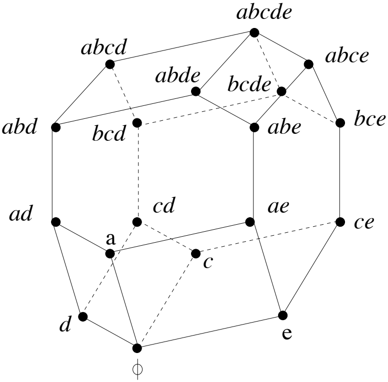


Permutahedron



Zonotopes

We can rewrite as

$$\begin{aligned} Z(u_1,\dots,u_n) &= \{\lambda_1 u_1 + \cdots + \lambda_n u_n : 0 \le \lambda_j \le 1\} \\ &= \left\{(u_1,\dots,u_n)\begin{pmatrix}\lambda_1 \\ \vdots \\ \lambda_n\end{pmatrix} : 0 \le \lambda_j \le 1\right\} \\ &= A[0,1]^n \end{aligned}$$

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where $A = (u_1,\dots,u_n)$.
A zonotope can be defined as a translation of $A[0,1]^n$,

$$A[0,1]^n + b$$

b a vector of $\mathbb{R}^d$.

Zonotopes

We thus have two definitions of a zonotope : via Minkowski's sum and by projecting the unit cube $[0, 1]^n$.

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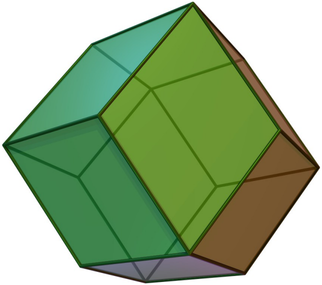

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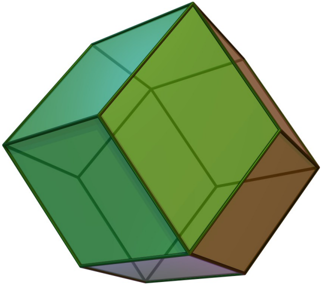

Can we compute the Ehrhart polynomial of zonotopes ?

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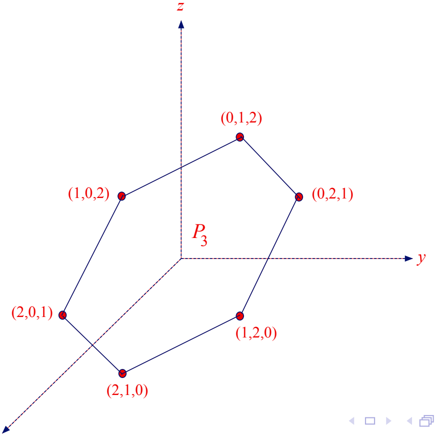

Theorem P_d is a translation of $[e_1, e_2] + [e_1, e_3] + \cdots + [e_{d-1}, e_d]$, that is, P_d is a translation of the Minkowski's sum of the segments formed by each pair of unit vectors in $\mathbb{R}^d$.

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Lemma Let $S \subseteq \{e_1 + e_2, e_1 + e_3, \ldots, e_{d-1} + e_d\}$. We associate a graph G_S with vertices $\{1, \ldots, d\}$ and where two vertices i are j are adjacent if $e_i + e_j \in S$. Then, S is linearly independent if and only if G_S is a forest.

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Theorem The coefficient c_k of $L_{P_d}(t) = c_{d-1}t^{d-1} + \cdots + c_0$ is equals to the number of labeled forests on d vertices with k edges.

Proof (idea). For each linear independent subset S of $\{e_1+e_2, e_1+e_3, \ldots, e_{d-1}+e_d\}$, we associate a half-opened cube

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- The relative volume of each such cube is equals to 1.
- These cubes decompose P_d as follows

$$P_d = 0 + \sum_{S \in \mathcal{I}} \left(\sum_{e_j + e_i \in S} (0, e_j + e_i] \right)$$

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• By Lemma, these subsets correspond to forests in G_S with k edges.

• Therefore, c_k counts of labeled forests on d vertices with k edges.

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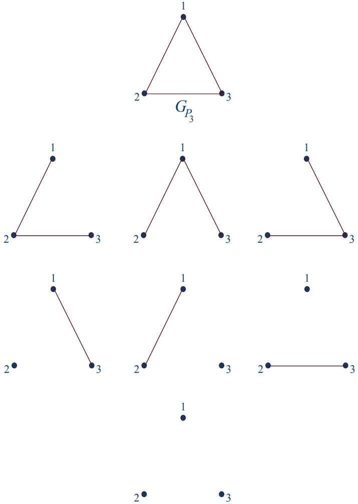
It is known (Cayley theorem) that the number of different labeled trees of K_d is equals to d^{d-2}.

We compute each $c_i, 0 \leq i \leq 2$ in $L_{P_3}(t)$ by counting forests.



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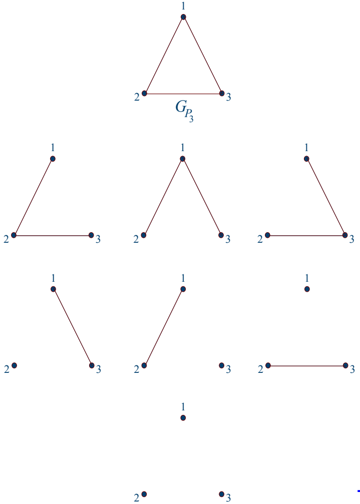

Therefore, $L_{P_3}(t) = 3t^2 + 3t + 1$.

Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

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The rank of a set $X \subseteq E$ is defined by

$$r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

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On one hand, $dim(W) \geq |I_2|$, on the other hand W is contained in the space generated by I_1.

$$|I_2| \leq dim(W) \leq |I_1| < |I_2| \quad !!!$$

Let A be the following matrix with coefficients in $\mathbb{R}$.

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\{\emptyset, \{1\}, \{2\}, \{4\}, \{4\}, \{5\}, \{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}\} \subseteq \mathcal{I}(M)$$

A matroid obtained form a matrix A with coefficients in $\mathbb{F}$ is denoted by $M(A)$ and is called representable over $\mathbb{F}$ or $\mathbb{F}$-representable .

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Theorem Regular matroids are equivalent to totally unimodular matrices.

Non Representable Matroids

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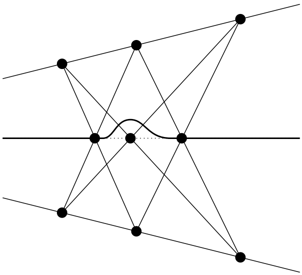
Example (classic) : the rank 3 matroid on 9 elements obtained from the Non-Pappus configuration



Duality

Let M be a matroid on the ground set E and let $\mathcal{B}$ the set of bases of M. Then,

$$\mathcal{B}^* = \{E \backslash B \mid B \in \mathcal{B}\}$$

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The matroid on E having $\mathcal{B}^*$ as set of bases, denoted by M^*, is called the dual of M.

A base of M^* is also called cobase of M.

- $r(M^*) = |E| - r_M$ and $M^{**} = M$.

Operation : deletion

Let M be a matroid on the set E and let $A \subset E$. Then,

$$\{X \subset E \backslash A \mid X \text{ is independent in } M\}$$

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The matroid $(M^* \setminus A)^*$ is called the contraction of the elements of A and it is denoted by M/A.

Tutte Polynomial

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$$t(M;x,y) = \sum_{X\subseteq E}(x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}.$$

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$$\begin{aligned} t(U_{2,3};x,y) &= \sum_{X\subseteq E,\ |X|=0}(x-1)^{2-0}(y-1)^{0-0}+\sum_{X\subseteq E,\ |X|=1}(x-1)^{2-1}(y-1)^{1-1}\\ &+\sum_{X\subseteq E,\ |X|=2}(x-1)^{2-2}(y-1)^{2-2}+\sum_{X\subseteq E,\ |X|=3}(x-1)^{2-2}(y-1)^{3-2}\\ &=(x-1)^2+3(x-1)+3(1)+y-1\\ &=x^2-2x+1+3x-3+3+y-1=x^2+x+y. \end{aligned}$$

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The Tutte polynomial can be expressed recursively as follows

$$t(M;x,y) = \begin{cases} t(M \setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M \setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

Some properties

$t(M^*; x, y) = t(M; y, x)$,
$t(M; 1, 1)$ counts the number of bases of M,
$t(M; 2, 1)$ counts the number of independents of M,
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Some properties

$t(M^*; x, y) = t(M; y, x)$,
$t(M; 1, 1)$ counts the number of bases of M,
$t(M; 2, 1)$ counts the number of independents of M,
$t(M; 1, 2)$ counts the number of generators of M.

The independent polynomial of M is given by

$$I(M, z) = \sum_{i=0}^{|E|} f_i z^i$$

where f_i design the number of independents of size i in M.

Proposition $I(M,z) = z^{r(M)} t\left(M; \frac{1}{z}+1, 1\right).$

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$$t\left(M; \frac{1}{z}+1, 1\right) = \sum_{X\subseteq E} \left(\frac{1}{z}+1-1\right)^{r(M)-r(X)} (1-1)^{|X|-r(X)} = \sum_{X\subseteq E} \left(\frac{1}{z}\right)^{r(M)-r(X)} 0^{|X|-r(X)}$$

But $(\frac{1}{z})^{r(M)-r(X)} 0^{|X|-r(X)}$ is not zero if and only if $|X| = r(X)$, that is, X is independent.

$$t\left(M; \frac{1}{z}+1, 1\right) = \sum_{X\subseteq E, X\in\mathcal{I}(M)} \left(\frac{1}{z}\right)^{r(M)-r(X)} = \left(\frac{1}{z}\right)^{r(M)} \sum_{X\subseteq E, X\in\mathcal{I}(M)} z^{r(X)}.$$

Obtaining,

$$z^{r(M)} t\left(M; \frac{1}{z}+1, 1\right) = z^{r(M)} \left(\frac{1}{z}\right)^{r(M)} \sum_{X\subseteq E, X\in\mathcal{I}(M)} z^{r(X)} \underbrace{=}_{\text{since } r(X)=|X|} \sum_{i=0}^{|E|} f_i z^i = I(M,z).$$

Acyclic Orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

We say that the orientation is acyclic if the oriented graph do not contain an oriented cycle (i.e., a cycle where all its edges are oriented clockwise or anti-clockwise).

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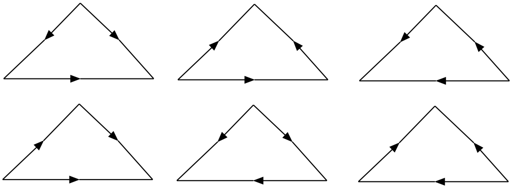
Theorem The number of acyclic orientations of G is equals to

$$t(M(G); 2, 0).$$

Example : There are 6 acyclic orientations of C_3



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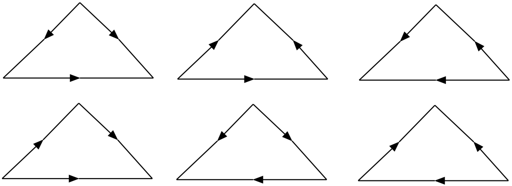

Notice that $M(C_3)$ is isomorphic to $U_{2,3}$.

Since $t(U_{2,3}; x, y) = x^2 + x + y$ then the number of acyclic orientations of C_3 is $t(U_{2,3}; 2, 0) = 2^2 + 2 + 0 = 6$.

Chromatic Polynomial

Let $G = (V, E)$ be a graph and let λ be a positive integer.

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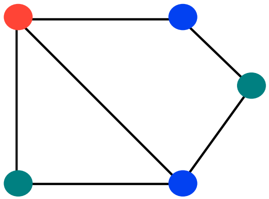

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Theorem $\chi(G,\lambda)$ is a polynomial on λ. Moreover

$$\chi(G,\lambda) = \sum_{X\subseteq E} (-1)^{|X|}\lambda^{\omega(G[X])}$$

where $\omega(G[X])$ denote the number of connected components of the subgraph generated by X.

Proof (idea). By using the inclusion-exclusion formula.

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Theorem If G is a graph with $\omega(G)$ connected components. Then,

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Example : $\chi(K_3,3)=3^1(-1)^{3-1}t(K_3;1-3,0)$
$=3\cdot 1\cdot t(U_{2,3};-2,0)=3((-2)^2-2+0)=6.$

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Let $w_1, \ldots, w_m \in \mathbb{R}^d$ be linear independent and let $\sigma_1, \ldots, \sigma_m \in \{\pm 1\}$. We define

$$\prod_{w_1,\ldots,w_m}^{\sigma_1,\ldots,\sigma_m} := \left\{ \lambda_1 w_1 + \cdots + \lambda_m w_m : \begin{array}{l} 0 \leq \lambda_j < 1 \text{ si } \sigma_j = -1 \\ 0 < \lambda_j \leq 1 \text{ si } \sigma_j = 1 \end{array} \right\}$$

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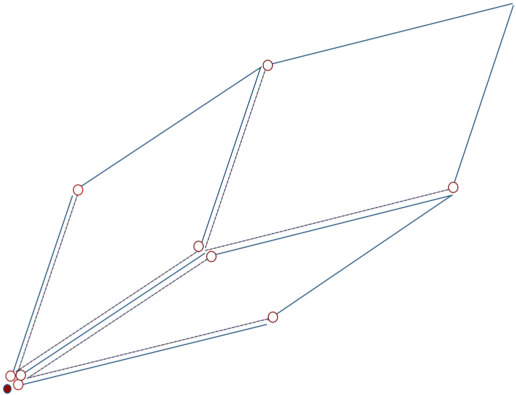
Lemme 1 $Z(u_1, \ldots, u_n)$ can be decomposed in disjoint translations of $\prod_{w_1,\ldots,w_m}^{\sigma_1,\ldots,\sigma_m}$ where $\{w_1, \ldots, w_m\}$ vary over all subset linear independent de $\{u_1, \ldots, u_n\}$ with an appropriate choice of signs $\sigma_1, \ldots, \sigma_m$.

Decomposition of $Z((4,1),(3,2),(1,3))$ in half-opened cubes.



Lemme 2 Let $\{w_1, \ldots, w_d\} \in \mathbb{Z}^d$ a set of vector linear independent and let $\prod := \{\lambda_1 w_1 + \cdots + \lambda_d w_d : 0 \leq \lambda_1, \cdots, \lambda_d < 1\}$. Then,

$$Card\left(\prod \cap \mathbb{Z}^d\right) = vol\left(\prod\right) = |det(w_1, \ldots, w_d)|$$

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By combining Lemmas 1 and 2, we obtain

Corollary Let $Z \subset \mathbb{R}^d$ be a zonotope decomposed in half-opened cubes. Then, the coefficients c_k in $L_Z(t) = c_d t^d + \cdots + c_0$ is equals to the sum of (relative) volumes of the k-dimensional cubes in the decomposition of Z.

Remark If the matrix formed by the vectors $\{w_1, \ldots, w_d\}$ is unimodular then all minors have determinant 0 or ± 1. In this case, the coefficient of t^k is equals to the number of all subset linearly independent of size k.

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So, $(-1)^{r(M)} t(M, 0, 1)$ counts the number of integer points in the interior of $Z(A)$.

Finitude of polytopes

The following transformations of $\mathbb{R}^n$ preserve the lattice $\mathbb{Z}^n$ (and therefore the integer points and integer polytopes) :

• translation by an integer vector,

• linear transformations $x \mapsto Ax$ where A is a matrix in $GL_n(\mathbb{Z})$, that is, $n \times n$ matrices with integer coefficients and determinant ± 1 (such transformation also preserve the euclidean volume.

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Theorem (Lagarias & Ziegler) Let k and n be two positive integer. Then, there is a finite number of equivalent classes of integer polytopes of dimension n with exactly k interior integer points.

For $n = 2$ there are exactly 16 integer polygones with exactly one interior point (modulo $GL_2(\mathbb{Z})$ action)

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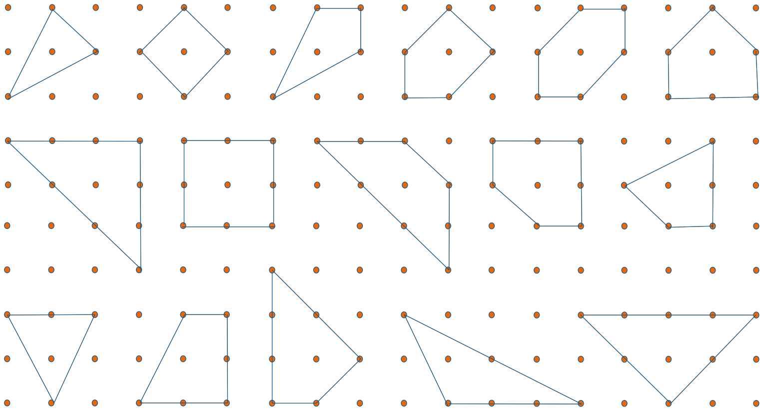

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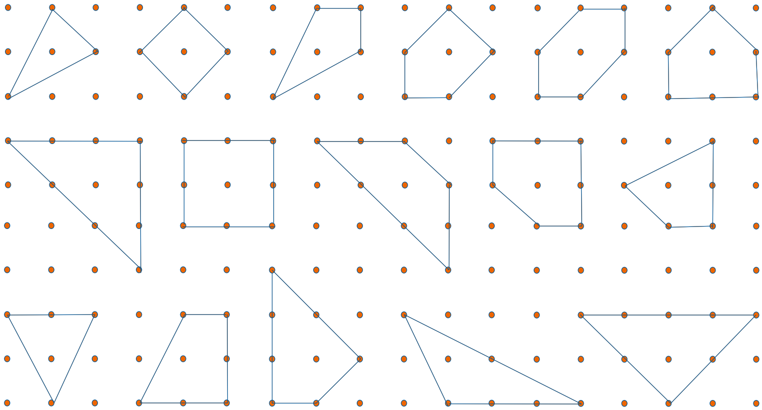

Such a list do not exist in dimension $n \geq 3$ (suspected to be very long and hard to access).

Fano polytopes

A Fano polytope is an integer polytope of dimension n admitting as a unique interior integer point the origin and all its facets have exactement n vertices forming a base for the lattice $\mathbb{Z}^n$.

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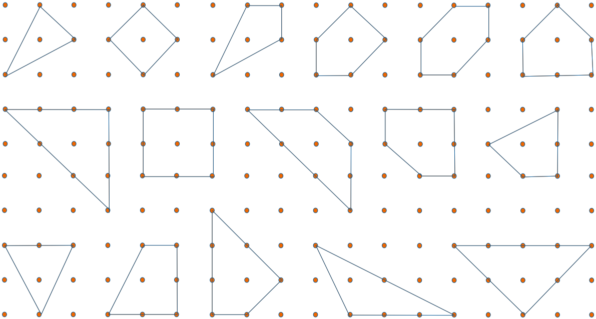

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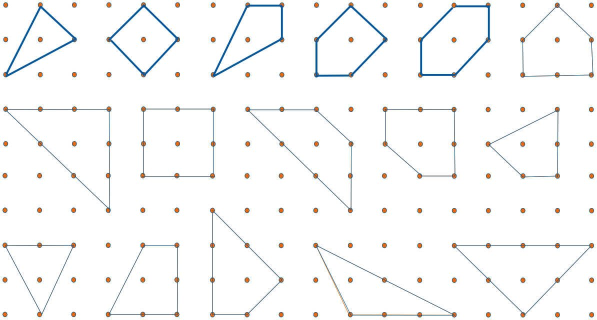

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Moreover, there is equality if and only if n is even and P is of the same type to the cartesian product of $n/2$ copies of the following Fano's polytope.