Self-dual maps

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joint work with
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Combinatorics and Related Topics
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Planar graphs and others

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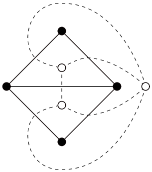
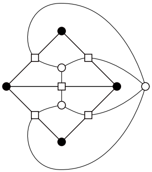


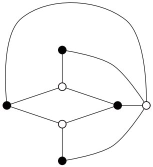
G and G^*

planar and dual graphs



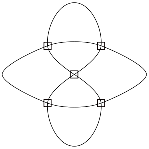
$G^{\square}$

Square graph graph



$I(G)$

Vertex-face incident graph



$med(G)$

medial graph

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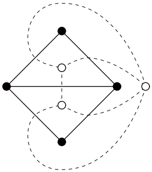
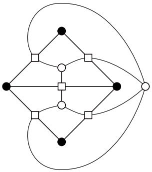


G and G^*

planar and dual graphs



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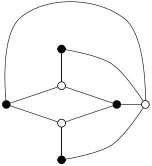

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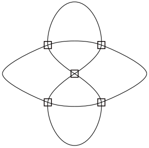

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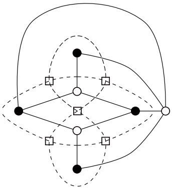
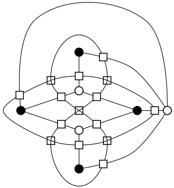


$I(G)$ and $med(G)$



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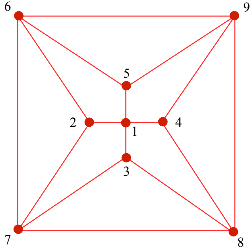

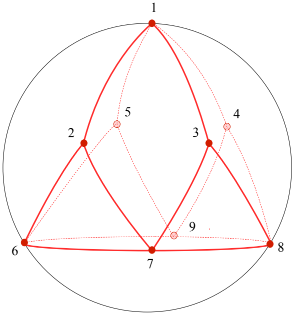

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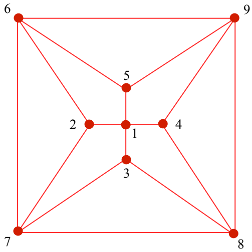

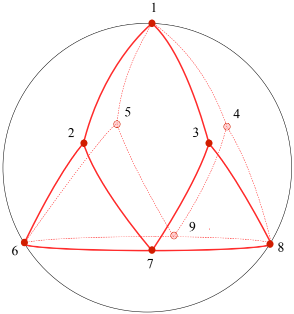

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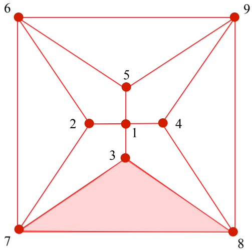

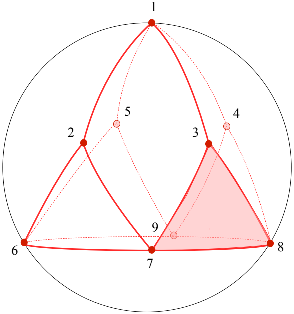

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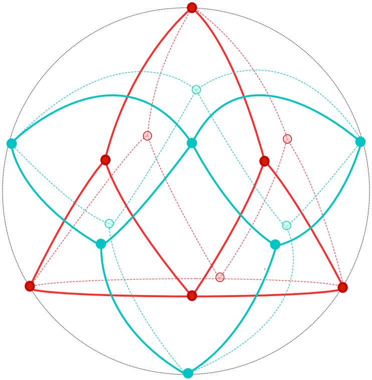
Self-dual maps

An embedding of G and its dual G^* in $\mathbb{S}^2$.



Antipodally self-dual maps

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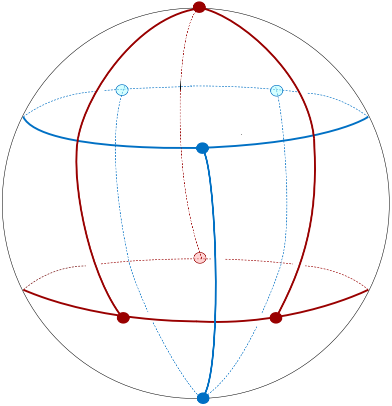

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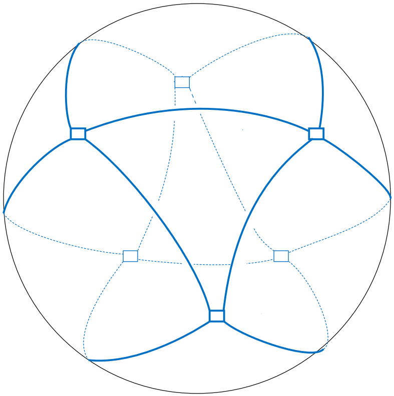

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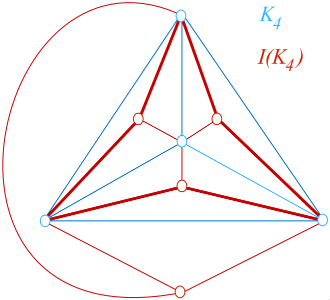

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(iv) $\{\Lambda^{-1}(x_i), \Lambda^{-1}(x_j)\} \in E$ if and only if $\{\Lambda^{-1}(\overline{x}_i), \Lambda^{-1}(\overline{x}_j)\} \in E$ where $\Lambda^{-1}(x_i) := \{v \in V \mid x_i \in \Lambda(v)\}$.

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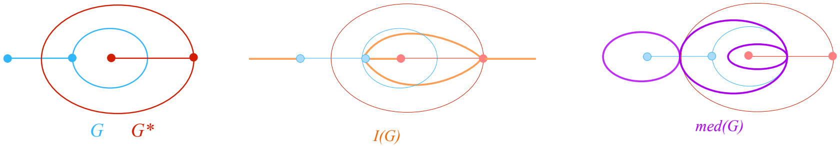

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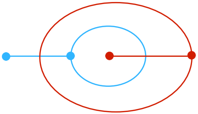
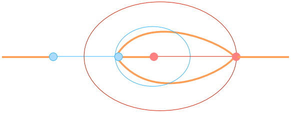
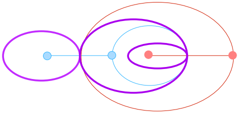


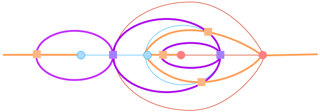
G G^*



$I(G)$



med(G)



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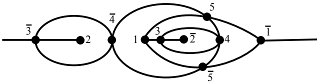
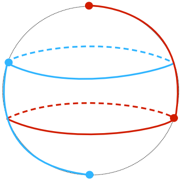
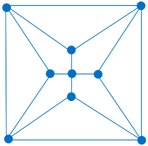


involutive labeling

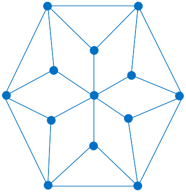


antipodally embedding

Antipodally self-dual : infinite families



4-ear



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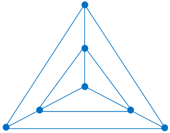
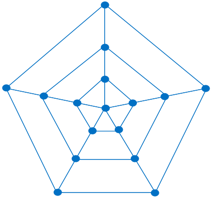


(3,2)-pancake



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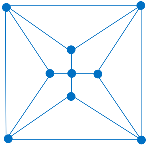
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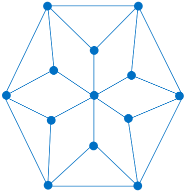
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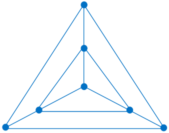
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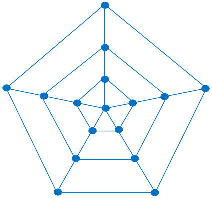
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Proposition (Montejano, R.A., Rasskin, 2022) The n-ear is antipodally self-dual if and only if $n \geq 4$ is even.

Proposition (Montejano, R.A., Rasskin, 2022) The (n, l)-pancake is antipodally self-dual if and only if $n \geq 3$ is odd for all integer $l \geq 1$.

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• Connected with ball polyhedra.

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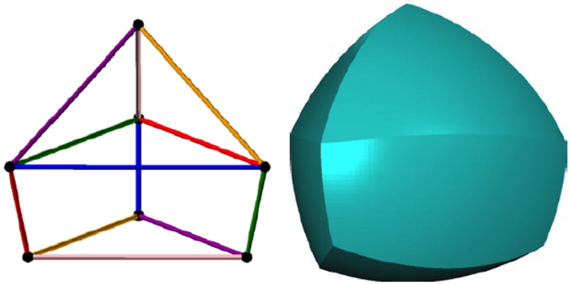

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Proposition (Montejano, R.A., Rasskin, 2022) Let G be a self-dual map. Then, G is strongly involutive if and only if $I(G)$ admits an involutive labeling without edges whose ends are labeled by k and $\bar{k}$.

Knot theory : quick overview

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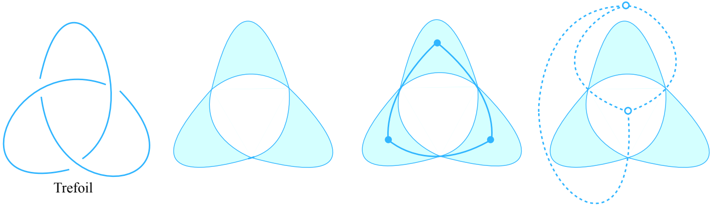

Trefoil

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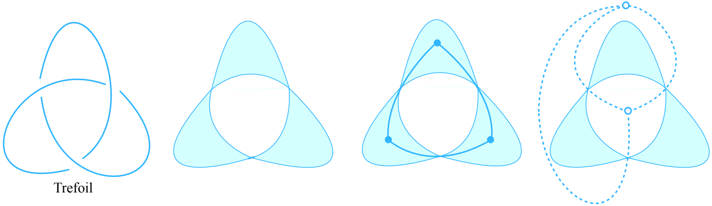

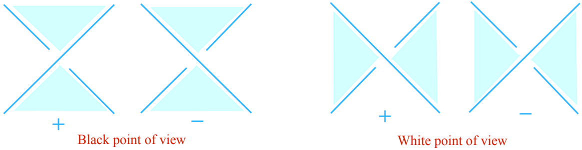

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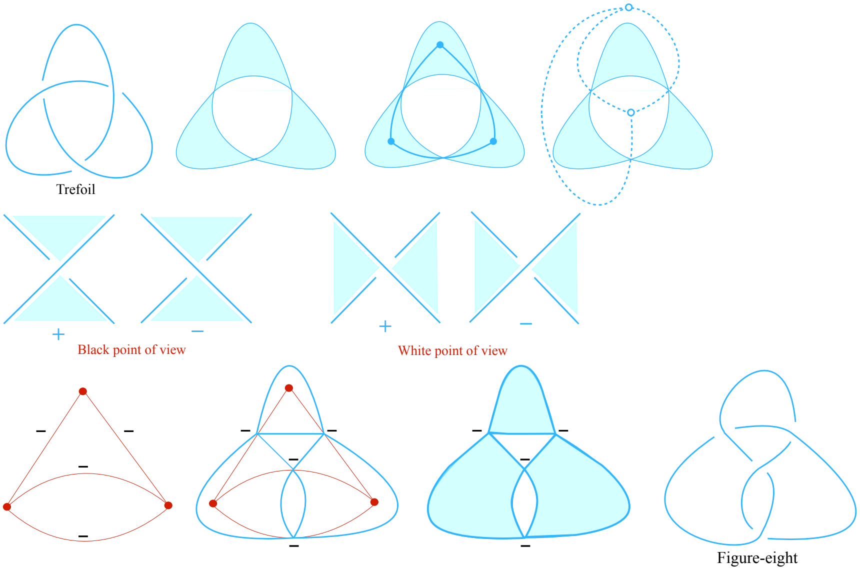

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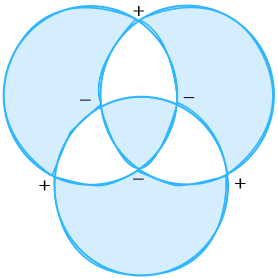
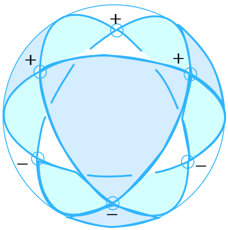
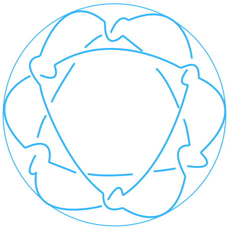
Remark : the Trefoil is not achiral while the Figure-eight is achiral.







Theorem (Montejano, R.A., Rasskin, 2022) Let G be an edge-signed map and suppose that $med(G)$ is antipodally symmetric (realized by a map α). If either

(a) α is color-preserving and sign-reversing ; or

(b) α is color-reversing and sign-preserving,

then the link L obtained from G is achiral.

Achirality

Theorem (Montejano, R.A., Rasskin, 2022) Let G be an edge-signed map and suppose that $med(G)$ is antipodally symmetric (realized by a map α). If either

(a) α is color-preserving and sign-reversing ; or

(b) α is color-reversing and sign-preserving,

then the link L obtained from G is achiral.

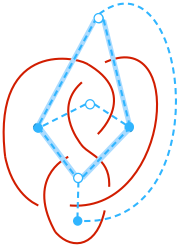
Theorem (Montejano, R.A., Rasskin, 2022) Let G be an edge-signed map and suppose that $I(G)$ admits either

(a) color-preserving and sign-reversing *reflexive curve* ; or

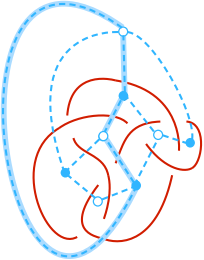
(b) color-reversing and sign-preserving *reflexive curve*,

then the link L obtained from G is achiral.

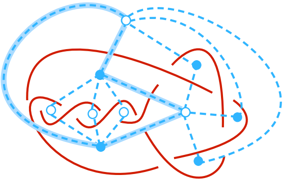
Some achiral knots



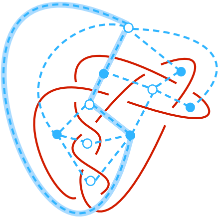
4_1 (figure-eight)



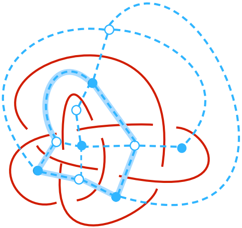
6_3



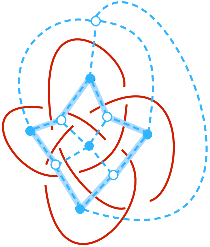
8_3



8_9



8_{12}



8_{18}


THANKS FOR
YOUR ATTENTION