Combinatorics for Knots

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1 Basic notions

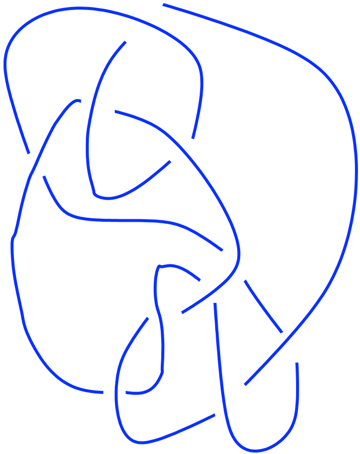
2 Matroid

3 Knot coloring and the unknotting problem

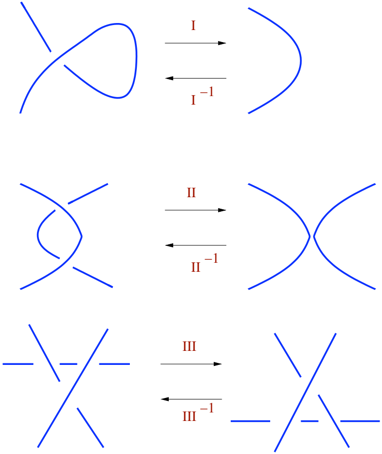
4 Oriented matroids

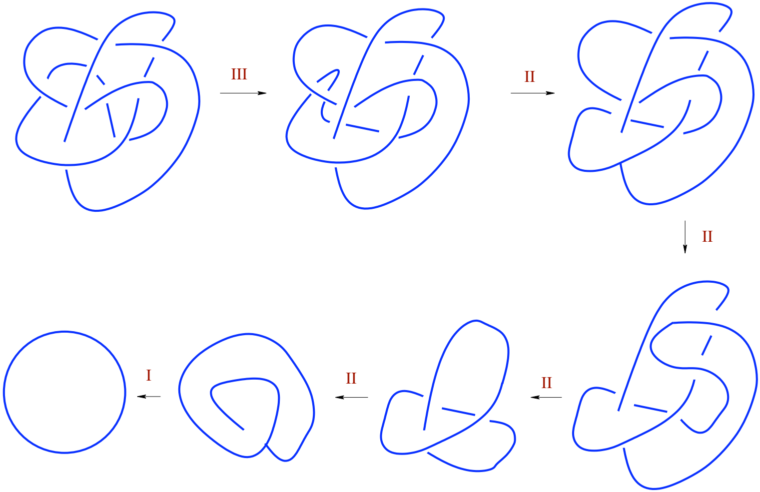
5 Spatial graphs

6 Ropes and thickness



Reidemeister moves

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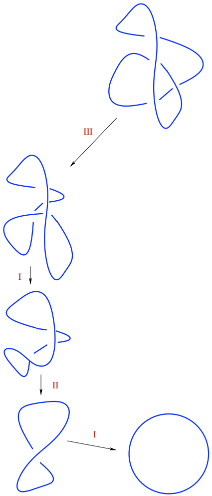
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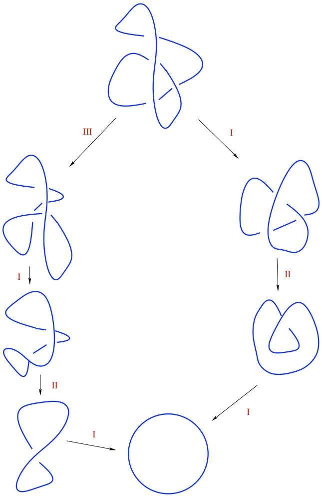
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Bracket polynomial

For any link diagram D define a Laurent polynomial $< D >$ in one variable A which obeys the following three rules where U denotes the unknot :

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i) $$\langle U \rangle = 1$$

ii) $$\langle U + D \rangle = -(A^2 + A^{-2}) \langle D \rangle$$

iii) 

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i)-(iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves II and III

but it is not invariant under Reidemeister move I!!

The writhe of an oriented link diagram D is the sum of the signs at the crossings of D (denoted by $\omega(D)$).

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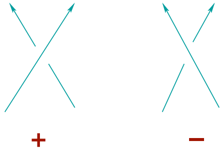

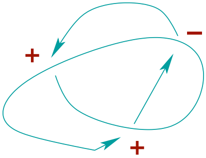

$\omega(D)=1$

Theorem For any link L define the Laurent polynomial

$$f_D(A) = (-A^3)^{\omega(D)} < L >$$

Then, $f_D(A)$ is an invariant of ambient isotopy.

Now, define for any link L

$$V_L(t) = f_D(t^{-1/4})$$

where D is any diagram representing L. Then $V_L(t)$ is the Jones polynomial of the oriented link L.

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Matroids

Let E a finite set. A matroid is a family $\mathcal{B}$ of subsets of E verifying certain axioms (the family $\mathcal{B}$ is called the bases of the matroid)

There is a natural way to obtain a matroid M from a graph G (the set of bases on M is given by the family of all spanning trees of G).

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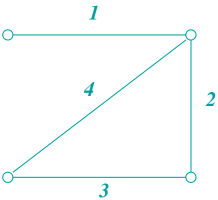
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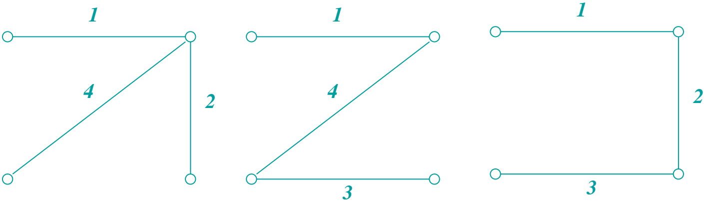
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The Tutte polynomial of a matroid $M(E)$ is defined to be the 2-variable polynomial

$$T(M;x,y) = \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}$$

where r is the rank function of M.

Remark :The evaluation of the Tutte polynomial in certain points may count something.

Example : $T(1,1)$ counts the number of bases in the matroid.

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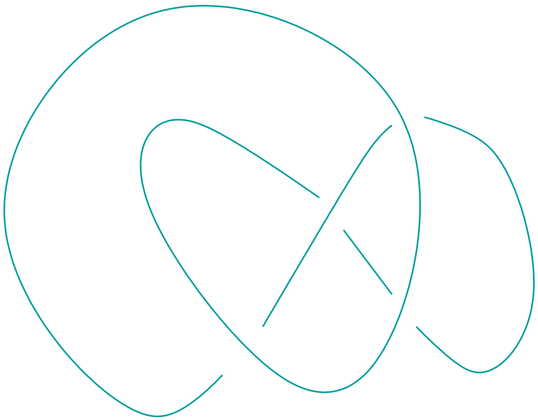

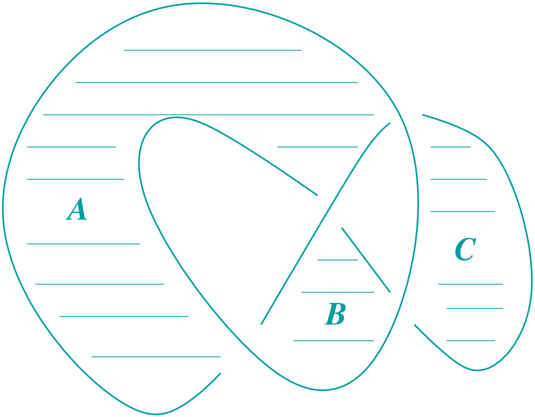
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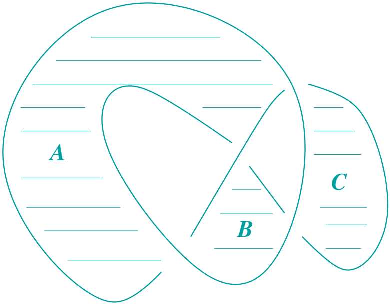
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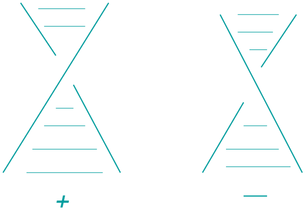
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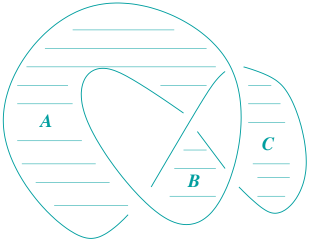
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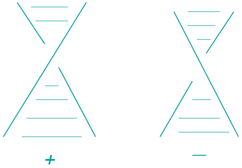
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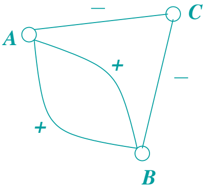
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A link diagram is alternating if the crossings alternate under-over-under-over ... as the link is traversed.

A link is alternating if there is an alternating link diagram representing L.

Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram and G denotes its associated unsigned 'blackface' graph then

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A knot diagram K is called colorable if each arc can be drawn using one of three colors (say, **red**, **blue**, **green**) in such a way that
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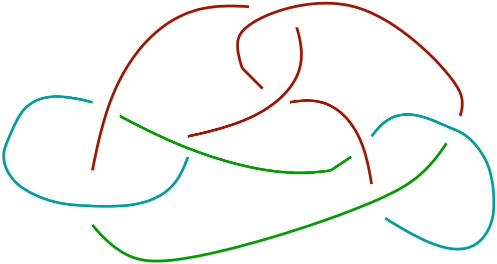

A knot diagram K is called colorable mod n if each arc can be labeled with an integer from 0 to $p-1$ such a way that
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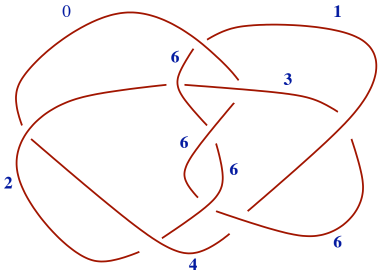

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Tait 1877 Knot classification.
Papakyriakopoulos 1957 A knot is trivial if and only if the fundamental group of its complement is abelian.

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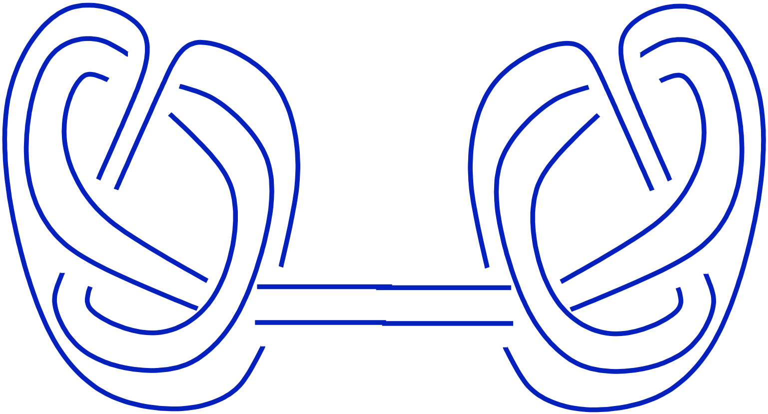

Question (Welsh) Is there a function f such that any diagram of the trivial knot with n crossngs can be transformed to a a diagram without crossing by using at most $f(n)$ Reidemeister moves ?

Theorem (Hass and Lagarias 2001) $f(n) = 2^{cn}$ where $c = 10^{11}$.

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There is a natural way to obtain an oriented matroid from a configuration of points in $\mathbb{R}^d$.

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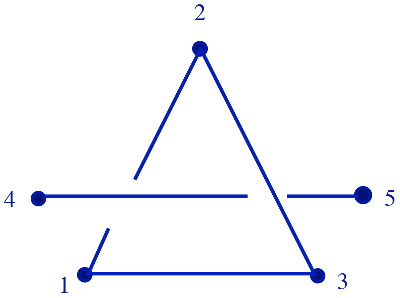
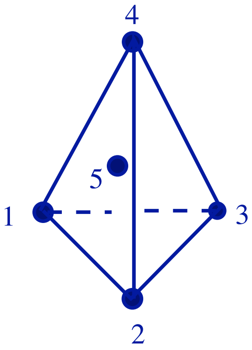
Example : $d = 3$.



(+,+,+,−,−)



(+,+,+,+,−)

Cyclic Polytope

Let $t_1, \ldots, t_n \in \mathbb{R}$. The cyclic polytope of dimension d with n vertices is defined as

$$C_d(t_1, \ldots, t_n) := conv(x(t_1), \ldots, x(t_n))$$

where $x(t_i) = (t_i, t_i^2, \ldots, t_i^d)$ are points of the moment curve

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Upper bound theorem (McMullen 1970) The number of j-faces of a d-dimensional polytope with n vertices is maximal for $C_d(n)$.

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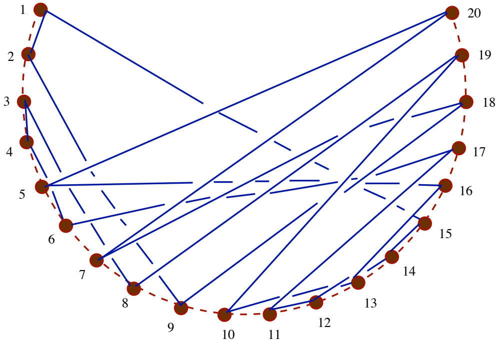

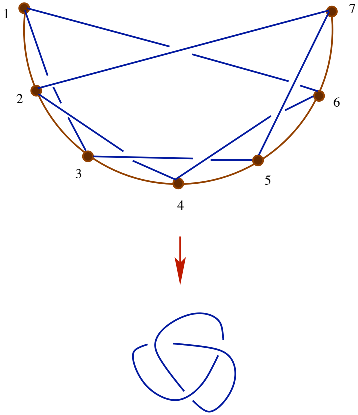
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Geometric algorithm

Given a diagram of a knot K.

Construct a cycle C in $C_3(m)$ isotopic to K.

Detect useless edges in C by using the circuits of the oriented matroid associated to $C_3(m)$.

Theorem (R.A. 2010) This method detect if a diagram with n crossing is trivial and its order is $O(2^{cn})$ where c is a constant.

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Spatial graphs

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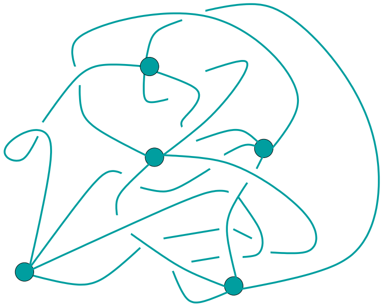
Spatial representation of K_5.



Let $m(L)$ be the smallest integer such that any spatial representation of K_n with $n \geq m(L)$ contains cycles isotopic to L.

Theorem (Conway and Gordon 1983)

- For any spatial representation of K_6, it holds

$$\sum_{(\lambda_1,\lambda_2)} lk(\lambda_1, \lambda_2) \equiv 1 \bmod 2$$

where (λ_1, λ_2) is a 2-component link contained in K_6 and lk denotes the linking number.

- For any spatial representation of K_7, it holds

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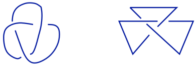
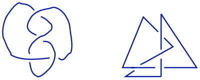
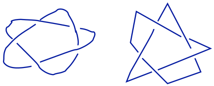
Trefoil
(T)



Figure–eight
F_8

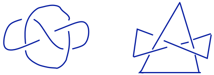


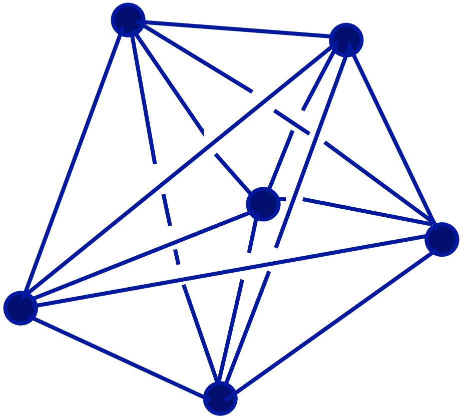
$T(5,2)$

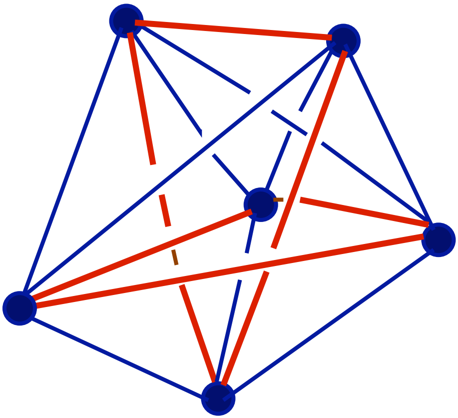


Hopf link
2^2_1

4^2_1







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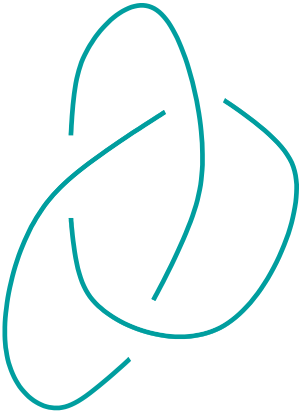

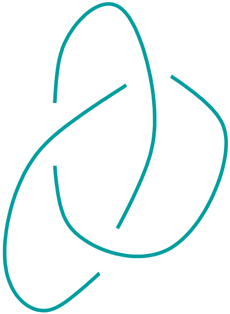

A number $r > 1$ is nice if for any distinct points x and y on K we have $D(x, r) \cap D(y, r) = \emptyset$. The disk thickness of K is defined to be $t(K) = sup\{r | r \text{is nice}\}$.

A thick realization K_0 of K is a knot of unit thickness which is of the same type as K.

The rope length $L(K)$ of K is the infimum of the length of K_0 taken over all thick realizations of K.

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$$L(K) \leq c \cdot (Cr(K))^{3/2}$$

where $Cr(K)$ is the crossing number of K.

The cubic lattice consists of all points in $\mathbb{R}^3$ with integral coordinates and all unit line segments joining these points.

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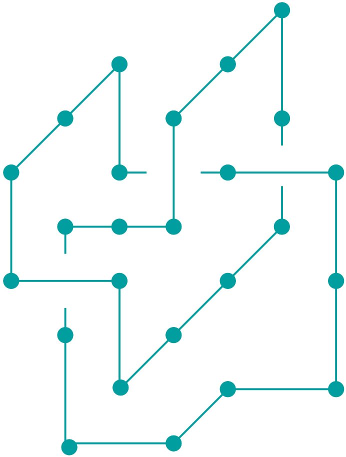
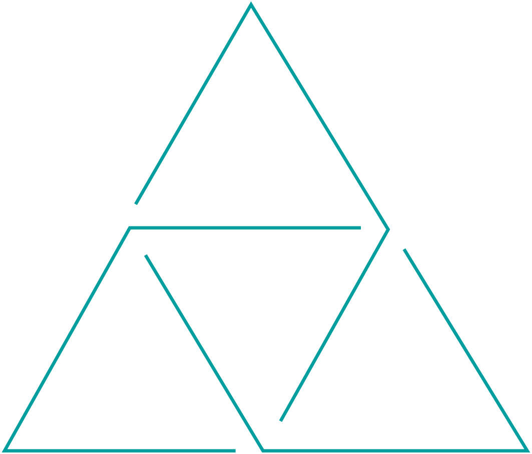
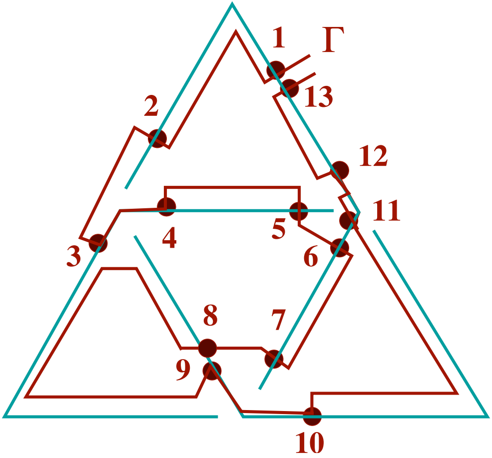
The trefoil represented in the cubic lattice.



Theorem (Diao, Ernst and Yu 2004) Let K be a knot. Then, K can be embedded into the cubic lattice with length at most

$$136\,(Cr(K))^{3/2} + 84\,Cr(K) + 22\sqrt{Cr(K)} + 11.$$

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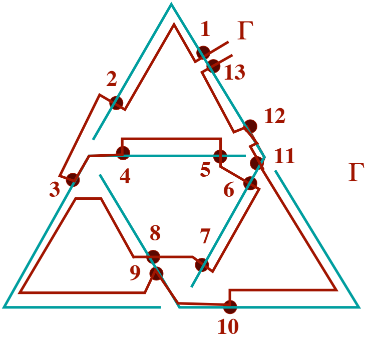
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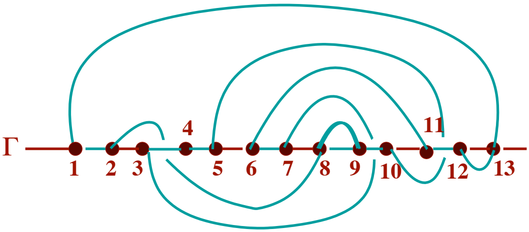
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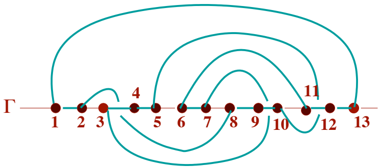
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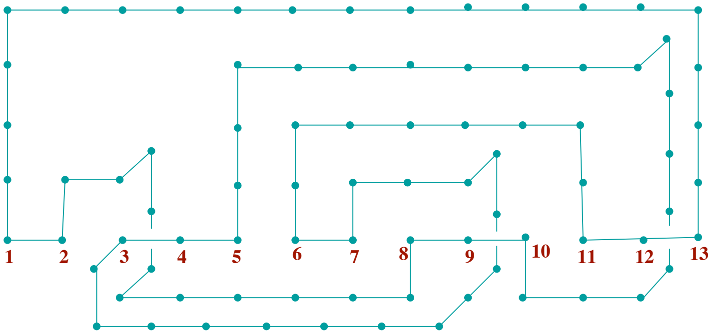
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