Edge separators for quasi-binary trees

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Abstract

One wishes to remove $k-1$ edges of a vertex-weighted tree T such that the weights of the k induced connected components are approximately the same. How well can one do it? In this paper, we investigate such k-separators for *quasi-binary* trees. We show that, under certain conditions on the total weight of the tree, a particular k-separator can be constructed such that the smallest (respectively the largest) weighted component is lower (respectively upper) bounded. Examples showing optimality for the lower bound are also given.

Keywords: Binary tree, separators

1 Introduction

The seminal paper by Lipton and Tarjan [2] has inspired a number of separator-type problems and applications (we refer the reader to [3] for a recent survey on separators and to [4] for edge separators of graphs with bounded genus).

Let us consider the following question.

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One wishes to split a given embedding of a planar connected graph G into blocks formed by weighted faces (weights might be thought of as the areas of the faces) such that the dual of the planar graph induced by each block is connected and the blocks' weights are approximately the same. How well can this be done?

One way to answer the latter is by considering k-separators on a spanning tree T_G of the vertex-weighted dual graph of G. Indeed, one may want to remove $k-1$ edges of T_G such that the weights of the k induced connected components of T_G are approximately the same.

More formally, let $T = (V, E)$ be a graph, and let $\omega : V(T) \longrightarrow \mathbb{R}^+$ be a *weight* function. Let $\omega(T) = \sum\limits_{v \in V(T)} \omega(v)$, and let $2 \leq k \leq |V| - 1$ be an integer. A *k-separator* of T is a set $F \subset E(T)$ with $|F| = k-1$ whose deletion induces k connected components, say $C_1(F), \ldots, C_k(F)$. If we let $\omega(C_i(F)) = \sum\limits_{v \in V(C_i(F))} \omega(v)$ then $\omega(T) = \sum\limits_{i=1}^{k} \omega(C_i(F))$. Let

$$\beta_k(T) := \max_{F \subseteq E, |F| = k-1} \left\{ \min_{1 \leq i \leq k} \omega(C_i(F)) \right\}$$

and

$$\alpha_k(T) := \min_{F \subseteq E, |F| = k-1} \left\{ \max_{1 \leq i \leq k} \omega(C_i(F)) \right\}.$$

An optimal k-separator is achieved when $\beta_k(T) = \alpha_k(T) = \frac{1}{k}\omega(T)$.

In this paper, we investigate the existence of k-separators with large (resp. small) values for β_k (resp. for α_k) for the class of *quasi-binary* trees. A tree is called *binary* if the degree of any vertex equals three except for *pendant* vertices (vertices of degree one) and a *root* vertex (a vertex of degree two). A tree is say to be *quasi-binary* if it is a connected subgraph of a binary tree. Notice that good k-separators in quasi-binary trees will lead to good k-block separators for triangulated planar graphs in the above question.

Since, for any quasi-binary tree T, the degree $d(v)$ of any $v \in V(T)$ is 1,2 or 3, we may define, for each $i = 1, 2, 3$,

$$V_i := \{v \in V(T) | d(v) = i\}$$

and

$$\omega_i := \max\{\omega(v) | v \in V_j \text{ for each } \; i \leq j \leq 3\}.$$

We will suppose that $V_2, V_3 \neq \emptyset$ and therefore $\omega_2, \omega_3 > 0$. Notice that

$$V(T) = V_1 \cup V_2 \cup V_3, \quad \omega_1 \geq \omega_2 \geq \omega_3 \quad \text{and} \quad \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \geq \omega(T). \tag{1}$$

where $n_i = |V_i|$ for each $i = 1, 2, 3$.

Our main results are the following.

Theorem 1 *Let T be a quasi-binary tree. Let $k \geq 2$ be an integer and $\gamma \in \mathbb{R}$ with $\gamma \geq \omega_3 > 0$. Let $M_\gamma = \max\{\omega_1 + \gamma, 2\omega_2\}$. If*

$$\omega(T) + k\gamma \geq \frac{(k+1)(k-2)M_\gamma}{(k-1)}$$

then

$$\alpha_k(T) \leq \frac{2\omega(T) + (k-1)\gamma}{k+1}.$$

Theorem 2 *Let T be a quasi-binary tree. Let $k \geq 2$ be an integer and $\gamma \in \mathbb{R}$ with $\gamma \geq \omega_3 > 0$. Let $M_\gamma = \max\{\omega_1 + \gamma, 2\omega_2\}$. If*

$$\omega(T) + k\gamma \geq \frac{(2k+1)M_\gamma}{(2)}$$

then

$$\beta_k(T) \geq \frac{\omega(T) - (k-1)\gamma}{2k-1}.$$

We note that the bounds for $\alpha_k(T)$ and $\beta_k(T)$ are not necessarily reached by using the same k-separator. The second author has studied k-separators in a more general setting (for planar graphs with weights on vertices, edges, and faces), where a lower bound for β_k is obtained [5] . We noticed that the conditions given in [5] are different from those presented in Theorem 2, whose proof is distinct (on the same line as that of Theorem 1). The value α_k is not treated in [5] at all. Two-separators for binary trees were also studied in [1, Corollary 2.2] where it is proved that a binary tree T with at least $\lambda + 1$ vertices admits an edge separating a forest F in T satisfying $\lambda \leq |V(F)| < 2\lambda$ for any real number $\lambda > \frac{1}{2}$. In the same spirit, Theorem 1 (resp. Theorem 2) implies, by taking $\omega(v) = 1$ for all vertices v and $\gamma = \lambda - 2$ (resp. $\gamma = \lambda$), the existence of a 2-separator of a binary tree with $\lambda \geq 1$ vertices such that one of the two connected components has at most λ (resp. at least $\frac{2}{3}\lambda$) vertices.

In the following section, we present some preliminary results needed for the rest of the paper. Main results are proved in Section 3. Finally, a family of quasi-binary trees showing optimality of Theorem 2 is constructed in the last section.

2 Preliminary results

Let T be a quasi-binary tree. We may suppose $\omega(T) > 0$ and that $n = |V(T)| > 1$ henceforth. For $i \in \{1, 2, 3\}$, let $n_i = |V_i|$. We observe that

$$n_1 + 2n_2 + 3n_3 = 2|E(T)| = 2(n-1) = 2n_1 + 2n_2 + 2n_3 - 2 \text{ and thus } n_1 = n_3 + 2. \quad (2)$$

Our main theorems will be proved by induction. We need the following result.

Lemma 1 *Let T be a quasi-binary tree with $n = |V(T)| > 1$. Let $\gamma, \eta \in \mathbb{R}$ be such that $\gamma \geq \omega_3$ and $\max\left\{\frac{\omega_1-\gamma}{2}, \omega_2 - \gamma\right\} \leq \eta \leq \frac{\omega(T)}{2}$. Then, there exists an edge $e \in E(T)$ such that*

$$\eta \leq \omega(C_e^i) \leq 2\eta + \gamma$$

for either C_e^1 or C_e^2 where C_e^1, C_e^2 denote the two connected components of $T \setminus \{e\}$.

Proof. The inequality $\eta \leq \omega(C_e^i)$ holds for $i = 1$ or 2 and for any $e \in E(T)$, otherwise $\omega(T) = \frac{\omega(T)}{2} + \frac{\omega(T)}{2} \geq 2\eta > \omega(C_e^1) + \omega(C_e^2) = \omega(T)$, which is a contradiction.

We now prove the right-hand side inequality. Without lost of generality, we suppose that $\omega(C_e^1) \geq \eta$ for each $e \in E(T)$. If we also have that $\omega(C_e^2) \geq \eta$ then we choose indices such that $|V(C_e^1)| \leq |V(C_e^2)|$.

We proceed by contradiction. Suppose that $\omega(C_e^1) > 2\eta + \gamma$ for all $e \in E(T)$. Let $e = \{v_1, v_2\}$ with $v_i \in V(C_e^i)$ be the edge that minimizes $|V(C_e^1)|$. We have three cases.

Case 1) If $d(v_1) = 1$, then $\omega(C_e^1) = \omega(v_1) \leq \omega_1$. Since $\eta \geq \frac{\omega_1-\gamma}{2}$ then $2\eta + \gamma \geq \omega_1 \geq \omega(C_e^2)$, which is a contradiction.

Case 2) If $d(v_1) = 2$, then we let $f = \{v_1, u\} \in E(T), f \neq e$ be the other edge incident to v_1.

Let C_f^i, $i = 1, 2$ be the two connected components of $T \setminus \{f\}$. Since $|V(C_f^1)| \geq |V(C_e^1)|$ then $V(C_e^1)) = V(C_f^2) \cup \{v_1\}$ so $\omega(C_f^2) = \omega(C_e^1) - \omega(v_1) > 2\eta + \gamma - \omega_2 \geq \eta$, and thus $|V(C_f^2)| \geq |V(C_f^1)| \geq |V(C_e^1)|$, which is a contradiction.

Case 3) If $d(v_1) = 3$, then we let $f_1 = \{v_1, u\}, f_2 = \{v_1, v\} \in E(T), f_1, f_2 \neq e$ be the other two edges incident to v_1 with $V(C_{f_1}^2) \cup V(C_{f_2}^2) = V(C_e^1) \setminus \{v_1\}$. So, $\omega(C_{f_1}^2) + \omega(C_{f_2}^2) = \omega(C_e^1) - \omega(v_1) > 2\eta + \gamma - \omega_3 \geq 2\eta$. Without loss of generality, we suppose that $\omega(C_{f_1}^2) \geq \omega(C_{f_2}^2)$, and thus $\omega(C_{f_1}^2) > \eta$ and $|V(C_{f_1}^2)| \geq |V(C_{f_1}^1)| \geq |V(C_e^1)|$ which is a contradiction. □

3 Proofs of main results

We may now prove our main results.

Proof of Theorem 1. We first claim that $\omega(T) > -k\gamma$. Indeed, since $\frac{(k+1)(k-2)}{k-1} \geq \frac{k-1}{2}$ and $\omega_1, \omega_2 > 0$, we have

$$\omega(T) + k\gamma \geq \frac{(k+1)(k-2)}{k-1} M_\gamma \geq \frac{k-1}{2} M_\gamma > 0.$$

We now shall construct the desired k-separator as follows. Let $T_k = T$. We first find an edge $e_k \in E(T_k)$ (by using Lemma 1) such that one of the connected components of $T_k \setminus \{e_k\}$, say T_{k-1}, has a *suitable* weight (the other connected component of $T_k \setminus \{e_k\}$, say R_{k-1}, remains fixed for the rest of the construction). By a suitable weight we mean a weight such that Lemma 1 can be applied to T_{k-1} in order to find an edge $e_{k-1} \in E(T_{k-1})$ such that one of the connected components of $T_{k-1} \setminus \{e_{k-1}\}$, say T_{k-2}, has again a suitable weight (and again the other connected component of $T_{k-1} \setminus \{e_{k-1}\}$, say R_{k-2}, remains fixed for the rest of the construction), and so on. We claim that the weight of component T_j is suitable if

$$\frac{(j-1)(k-1)}{(k+1)(k-2)}\omega(T) + \left(\frac{2(j-1)}{(k+1)(k-2)} - 1\right)\gamma \leq \omega(T_j) \leq \frac{(j+1)}{(k+1)}\omega(T) + \frac{(k-j)}{(k+1)}\gamma. \quad (3)$$

Now, to apply Lemma 1, we need to define an appropriate parameter η_j (that ensures suitable weights throughout the construction). For each $j = k, k-1 \ldots, 2$, we set

$$\eta_j = \frac{(k-3)}{2(2k-j-3)}\omega(T_j) - \frac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) - \frac{(k+3)(k-2) - j(k-1)}{2(2k-j-3)(k+1)}\gamma. \quad (4)$$

We first claim that

$$\max\left\{\frac{\omega_1 - \gamma}{2}, \omega_2 - \gamma\right\} \leq \eta_j \leq \frac{\omega(T_j)}{2}.$$

For the lower bound we have

$$\begin{aligned}
\eta_j &= \tfrac{(k-3)}{2(2k-j-3)}\omega(T_j) - \tfrac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) - \tfrac{(k-3)(k-2)-j(k-1)}{2(2k-j-3)(k+1)}\gamma \\
&\geq \tfrac{(k-3)}{2(2k-j-3)}\left(\tfrac{(j-1)(k-1)}{(k+1)(k-2))}\right)\omega(T) + \tfrac{(k-3)}{2(2k-j-3)}\left(\tfrac{2(j-3)}{(k+1)(k-2)} - 1\right)\gamma \\
&- \tfrac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) - \tfrac{(k-3)(k-2)-j(k-1)}{2(2k-j-3)(k+1)}\gamma \\
&= \tfrac{(k-1)}{2(k+1)(k-2)}\omega(T) - \tfrac{k^2-k-4}{2(k+1)(k-2)}\gamma \\
&\geq \max\left\{\tfrac{\omega_1}{2} - \tfrac{1}{(k+1)(k-2)}\gamma, \omega_2 - \tfrac{k(k-1)}{2(k+1)(k-2)}\gamma\right\} - \tfrac{(k^2-k-4)}{2(k+1)(k-2)}\gamma \\
&= \max\left\{\tfrac{\omega_1}{2} - \tfrac{\gamma}{2}, \omega_2 - \gamma\right\}.
\end{aligned}$$

For the upper bound, we have

$$
\begin{aligned}
\eta_j &= \tfrac{(k-3)}{2(2k-j-3)}\omega(T_j) - \tfrac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) - \tfrac{(k-3)(k-2)-j(k-1)}{2(2k-j-3)(k+1)}\gamma \\
&= \tfrac{\omega(T_j)}{2} - \tfrac{(k-j)}{2(2k-j-3)}\omega(T_j) - \tfrac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) - \tfrac{(k+3)(k-2)-j(k-1)}{2(2k-j-3)(k+1)}\gamma \\
&\le \tfrac{\omega(T_j)}{2} - \tfrac{(k-j)}{2(2k-j-3)}\left(\tfrac{(j-1)(k-1)}{(k+1)(k-2)}\omega(T) + \left(\tfrac{2(j-1)}{(k+1)(k-2)} - 1\right)\gamma\right) \\
&\quad - \tfrac{(j-3)(k-1)}{2(k-j-3)(k+1)}\omega(T) - \tfrac{(k+3)(k-2)-j(k-1)}{2(2k-j-3)(k+1)}\gamma \\
&\le \tfrac{\omega(T_j)}{2} - \tfrac{(j-2)}{(k+1)(k-2)}\left(\tfrac{(k-1)}{2}\omega(T) + \gamma\right) \\
&= \tfrac{\omega(T_j)}{2} - \tfrac{(j-2)}{2k}\omega(T) - \tfrac{(j-2)}{k(k+1)(k-2)}\left(\omega(T) + k\gamma\right) \le \tfrac{\omega(T_j)}{2}.
\end{aligned}
$$

Therefore, by Lemma 1, one of the connected components of $T_j \setminus \{e_j\}$, say R_{j-1}, satisfies

$$\eta_j \le \omega(R_{j-1}) \le 2\eta_j + \gamma, \tag{5}$$

and thus, the weight of the other connected component of $T_j \setminus \{e_j\}$, say T_{j-1}, satisfies

$$\omega(T_j) - 2\eta_j - \gamma \le \omega(T_{j-1}) \le \omega(T_j) - \eta_j.$$

So, the set of edges $e_1, \ldots, e_{k-1}$ chosen as above gives a k-separator T where the connected component with the largest weight is given by $\max_{1 \le i \le k} \{\omega(T_j)\}$. To obtain an upper bound for the latter, we shall show that the components T_j, $j = k, k-1, \ldots, 1$ have suitable weights satisfying both inequalities of (3).

We proceed by induction on j. For $j = k$, the upper bound is obtained immediately. For the lower bound we have,

$$
\begin{aligned}
\omega(T_k) = \omega(T) &= \tfrac{(k-1)(k-1)}{(k+1)(k-2)}\omega(T) + \left(\tfrac{2(k-1)}{(k+1)(k-2)} - 1\right)\gamma + \tfrac{(k-3)}{(k+1)(k-2)}(\omega(T) + k\gamma) \\
&\ge \tfrac{(k-1)(k-1)}{(k+1)(k-2)}\omega(T) + \left(\tfrac{2(k-1)}{(k+1)(k-2)} - 1\right)\gamma. \qquad (6)
\end{aligned}
$$

The latter inequality uses the fact that $\omega(T) > -k\gamma$. Suppose that inequalities hold for some $j \le k$. By using 3,4 and 6, we have

$$
\begin{aligned}
\omega(R_{j-1}) &\ge \omega(R_j) - 2\eta_j - \gamma \\
&= \omega(R_j) - \tfrac{(k-3)}{(2k-j-3)}\omega(R_j) + \tfrac{(j-3)(k-1)}{(2k-j-3)(k+1)}\omega(T) + \tfrac{(k-3)(k-2)-j(k-1)}{(2k-j-3)(k+1)}\gamma - \gamma \\
&\ge \tfrac{(k-j)}{(2k-j-3)}\left(\tfrac{(j-1)(k-1)}{(k+1)(k-2)}\right)\omega(T) + \tfrac{(k-j)}{(2k-j-3)}\left(\tfrac{2(j-1)}{(k+1)(k-2)} - 1\right)\gamma \\
&\quad + \tfrac{(j-3)(k-1)}{(2k-j-3)(k+1)}\omega(T) - \tfrac{(k+3)(k-2)-j(k-1)}{(2k-j-3)(k+1)}\gamma - \gamma \\
&= \tfrac{(j-3)(k-1)}{(k+1)(k-2)}\omega(T) + \left(\tfrac{2(j-2)}{(k+1)(k-2)} - 1\right)\gamma.
\end{aligned}
$$

And

$$\begin{aligned}\omega(R_{j-1}) &\leq \omega(R_j) - \eta_j \\ &= \omega(R_j) - \tfrac{(k-3)}{2(2k-j-3)}\omega(R_j) + \tfrac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) + \tfrac{(k-3)(k-2)-j(k-1)}{(2k-j-3)(k+1)}\gamma \\ &\leq \tfrac{(3k-2j-3)(j+1)}{2(2k-j-3)(k+1)}\omega(T) + \tfrac{(3k-2j-3)(k-j)}{2(2k-j-3)(k+1)}\gamma \\ &\quad + \tfrac{(j-3)(k-1)}{2(2k-j-3)(k+1)}\omega(T) + \tfrac{(k+3)(k-2)-j(k-1)}{2(2k-j-3)(k+1)}\gamma \\ &= \tfrac{j}{(k+1)}\omega(T) + \tfrac{(k-j+1)}{(k+1)}\gamma.\end{aligned}$$

Therefore, (3) holds for all $j = k, \ldots, 2$ when T is decomposed into the k components $T_1, R_k, \ldots, R_2$. So,

$$\begin{aligned}\alpha_k &= \max\left\{\max_{2\leq j\leq k}\{\omega(R_j)\}, \omega(T_1)\right\} \overset{(5)}{\leq} \max\left\{\max_{2\leq j\leq k}\{2\eta_j+\gamma\}, \omega(T_1)\right\} \\ &\overset{(4)}{=} \max\left\{\max_{2\leq j\leq k}\left\{\tfrac{(k-3)}{(2k-j-3)}\omega(T_j) - \tfrac{(j-3)(k-1)}{(2k-j-3)(k+1)}\omega(T) + \tfrac{k^2-2k+3-2j}{(2k-j-3)(k+1)}\gamma\right\}, \omega(T_1)\right\} \\ &\overset{(3)}{\leq} \max\left\{\max_{2\leq j\leq k}\left\{\tfrac{2}{(k+1)}\omega(T) + \tfrac{(k-1)}{(k+1)}\gamma\right\}, \tfrac{2}{(k+1)}\omega(T) + \tfrac{(k-1)}{(k+1)}\gamma\right\} \\ &= \tfrac{2}{(k+1)}\omega(T) + \tfrac{k}{(k+1)}\gamma,\end{aligned}$$

as desired. □

Proof of Theorem 2. We first claim that $\omega(T) > -k\gamma$. Indeed, since $\frac{2k+1}{2} \geq \frac{k-1}{2}$ and $\omega_1, \omega_2 > 0$, we have

$$\omega(T) + k\gamma \geq \frac{2k+1}{2}M_\gamma \geq \frac{k-1}{2}M_\gamma > 0.$$

We shall construct the desired k-separator in a similar way as done in Theorem 1. Let $T_k = T$. We find an edge $e_k \in E(T_k)$ (by using Lemma 1) such that one of the connected components of $T_k \setminus \{e_k\}$, say R_k, has a prescribed weight, which will be fixed for the rest of the construction. By applying Lemma 1 to the other component of $T_k \setminus \{e_k\}$, say T_{k-1}, we find an edge $e_{k-1} \in E(T_{k-1})$ such that one of the connected components of $T_{k-1} \setminus \{e_{k-1}\}$, say R_{k-2}, has a prescribed weight and which will be fixed for the rest of the construction, and so on. The only difference with the procedure in the proof of Theorem 1 is that the value η_j is now fixed for any step of the construction

$$\eta_j = \eta = \tfrac{\omega(T)-(k-1)\gamma}{2k-1} \text{ for all } j = k, k-1, \ldots, 2.$$

First, we claim that $\eta \geq \max\left\{\frac{\omega_1-\gamma}{2}, \omega_2 - \gamma\right\}$. Indeed,

$$\begin{aligned}\eta &= \tfrac{\omega(T)-(k-1)\gamma}{2k-1} \\ &\geq \max\left\{\tfrac{(2k-1)\omega_1-\gamma}{2(2k-1)}, \tfrac{(2k-1)\omega_2-k\gamma}{2(2k-1)}\right\} - \tfrac{(k-1)\gamma}{2k-1} \\ &= \max\left\{\tfrac{\omega_1-\gamma}{2}, \omega_2-\gamma\right\}.\end{aligned}$$

Therefore, at each step (by Lemma 1) one of the connected components of $T_j \setminus \{e_j\}$, say R_{j-1} satisfies

$$\eta = \frac{\omega(T) - (k-1)\gamma}{2k-1} \leq \omega(R_{j-1}) \leq \frac{2\omega(T) + \gamma}{2k-1} = 2\eta + \gamma.$$

The weight of the other connected component of $T_j \setminus \{e_j\}$, say T_{j-1} satisfies

$$\omega(T_j) - \frac{2\omega(T) - \omega}{2k-1} \leq \omega(T_{j-1}) \leq \omega(T_j) - \frac{\omega(T) - (k-1)\gamma}{2k-1}.$$

Since $\omega(T_k) = \omega(T)$, we obtain

$$\omega(T_j) \geq \omega(T) - (k-j)\frac{2\omega(T) + \gamma}{2k-1} = \frac{(2j-1)\omega(T) - (k-j)\gamma}{2k-1}, \quad j = k, k-1, \ldots, 1.$$

We claim that $\eta \leq \frac{\omega(T_j)}{2}$ for each $j = k, k-1, \ldots, 2$. Indeed,

$$\begin{aligned} \tfrac{\omega(T_j)}{2} &\geq \tfrac{(2j-1)\omega(T)-(k-j)\gamma}{2(2k-1)} \\ &= \tfrac{\omega(T)-(k-1)\gamma}{2k-1} + \tfrac{(2j-3)\omega(T)+(k+j-2)\gamma}{2(2k-1)} \\ &= \tfrac{\omega(T)-(k-1)\gamma}{2k-1} + \tfrac{(j-2)}{2k}\omega(T) + \tfrac{(k+j-2)}{2k(2k-1)}(\omega(T)+k\gamma) \\ &\geq \tfrac{\omega(T)-(k-1)\gamma}{2k-1} = \eta. \end{aligned}$$

So, the set of edges $\{e_k, e_{k-1}, \ldots, e_2\}$ chosen as above forms a k-separator S_k of T, where the connected component with the smallest weight is given by

$$\beta(S_k) = \min\{\omega(R_{k-1}), \omega(R_{k-2}), \ldots, \omega(R_1), \omega(T_1)\} \geq \frac{\omega(T) - (k-1)\omega}{2k-1}$$

as desired. □

4 Tightness

In this section, we show that the lower bound presented in Theorem 2 is optimal. We consider the quasi-binary tree T_k consisting of a root vertex r joined by $k-1$ different paths to $k-1$ vertices $x_1, \ldots, x_{k-1}$, each of which is adjacent to exactly two vertices of degree one, see Figure 1.

We set $\omega(x_i) = \omega > 0$ for all i, $\omega(r) = \omega(v) = \omega' \geq \omega > 0$ where $d(v) = 1$ and the weight of any other vertex equals zero. So,

$$\omega(T_k) = (k-1)\omega + 2(k-1)\omega' + \omega' = (k-1)\omega + (2k-1)\omega'.$$

Let F be an optimal k-separator of T. We have that either F contains one of the edges $\{x_i, v\}$, $1 \leq i \leq k-1$ with v a pending vertex (so vertex v will be a connected component

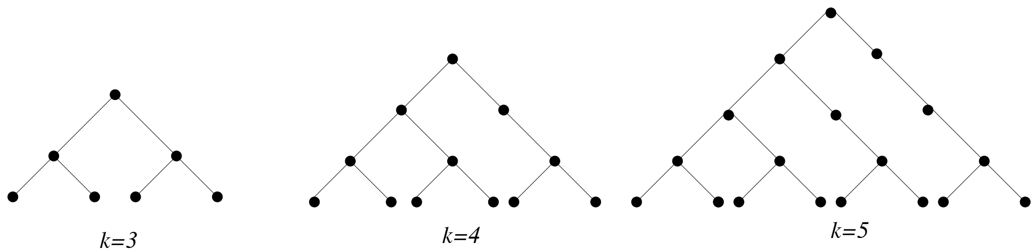

Figure 1: Quasi-binary trees T_2, T_3 and T_3

itself in the separator and thus $\beta_k = \omega'$) or F contains no such edges in which case we find (by an easy analysis of T_k) that the root vertex r will be in a connected component containing just vertices of weight zero in any optimal separator (obtaining again that $\beta_k = \omega'$).

The lower bound of Theorem 2 gives

$$\beta_k \geq \frac{1}{2k-1}\omega(T_k) - \left(\frac{k-1}{2k-1}\right)\omega_3 = \frac{1}{2k-1}\left((k-1)\omega + (2k-1)\omega'\right) - \left(\frac{k-1}{2k-1}\right)\omega = \omega'$$

showing the desired optimality.

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