Ideals and simplicial complexes of matroids

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CombinatoireS Summer School

Paris, June 29 - July 3 2015

Toric ideal associated to a matroid

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A binomial in R is a difference of two monomials, an ideal generated by binomials is called a *binomial ideal*.

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We consider the homomorphism of k-algebras $\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

$$y_B \mapsto \prod_{i \in B} x_i.$$

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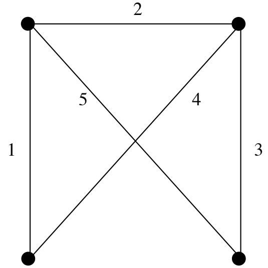
Observation Let b be the number of bases of a matroid M on n elements. Then, I_M is generated by the kernel of the integer $n \times b$ matrix whose columns are the zero-one incidence vectors of the bases of M.

Example

Matroid $M(G)$ associated to graph G. We have $r(M(G)) = 3$.



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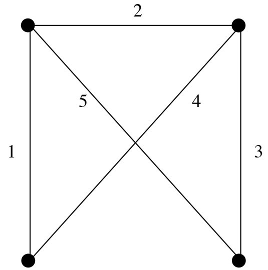

$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

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$$\begin{array}{c}
\begin{array}{cccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{array} \\
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By considering $\varphi : k[y_{B_1}, \ldots, y_{B_8}] \longrightarrow k[x_1, \ldots, x_5]$ we have that

$y_{B_1} \mapsto x_1x_2x_3, \quad y_{B_2} \mapsto x_1x_2x_5, \quad y_{B_3} \mapsto x_1x_3x_4, \quad \ldots$

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An element of the kernel of φ ($i.e.$, $I_{M(G)}$) is : $y_{B_7}y_{B_4} - y_{B_2}y_{B_8}$.

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Observation Since $R/I_M \simeq S_M$, it follows that the height of I_M is $\mathrm{ht}(I_M) = |\mathcal{B}| - \dim(S_M) = |\mathcal{B}| - (n - c + 1)$, where c is the number of connected components of M.

White's conjecture

Let $\mathcal{B}$ denote the set of bases of M. By definition $\mathcal{B}$ is not empty and satisfies the following exchange axiom :

For every $B_1, B_2 \in \mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.

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Brualdi proved that the exchange axiom is equivalent to the symmetric exchange axiom :

> *For every B_1, B_2 in $\mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that both $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$ and $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$.*

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Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

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We say that the quadratic binomial $y_{B_1} y_{B_2} - y_{D_1} y_{D_2}$ correspond to a symmetric exchange.

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Conjecture (White 1980) For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

White's conjecture

Observation for $B_1, \ldots, B_s, D_1, \ldots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets.

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Since I_M is a homogeneous binomial ideal, it follows that

$$I_M = \left(\{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \mid B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \textit{ as multisets}\}\right)$$

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Observation White's conjecture does not depend on the field k.

Example continued

We had $\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$.

We also had that $y_{B_7}y_{B_4} - y_{B_2}y_{B_8} \in I_{M(G)}$.

We can check that $B_7 \cup B_4 = \{2, 4, 5, 1, 3, 5\} = B_2 \cup B_8$.

Results of White's conjecture

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- Lasoń, Michałek (2014) proved for strongly base orderables matroids.

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The n-base graph of M, which is denoted by $G_n(M)$, has as its vertex set the set of all sets of n disjoint bases (a set of n bases $\{B_1, \ldots, B_n\}$ of M is disjoint if and only if

$$|E| = \bigcup_{i=1}^{n} B_i.$$

There is an edge between $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ if and only if $B_i = D_j$ for some i, j.

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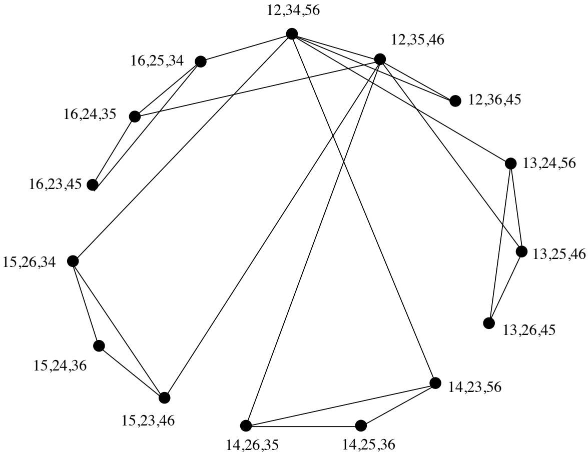

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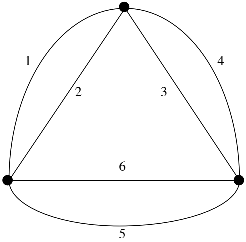

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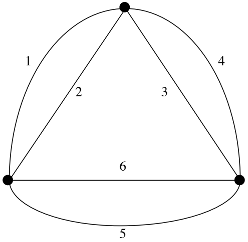

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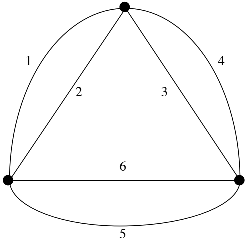

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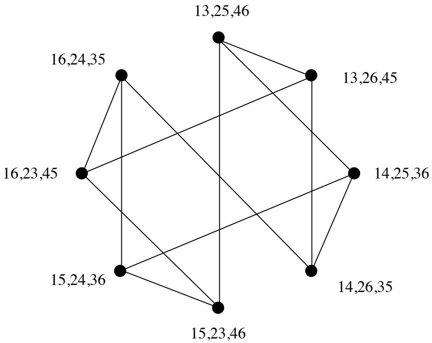
$G_3(M(G))$



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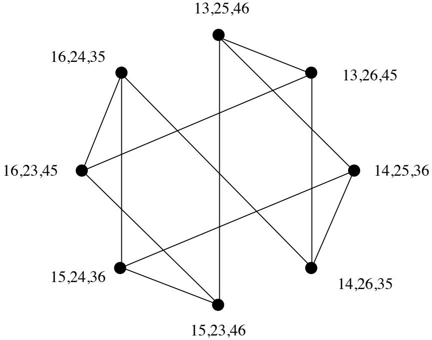

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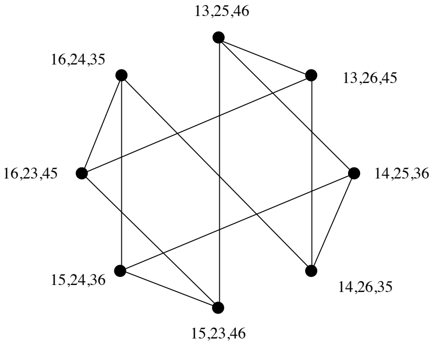

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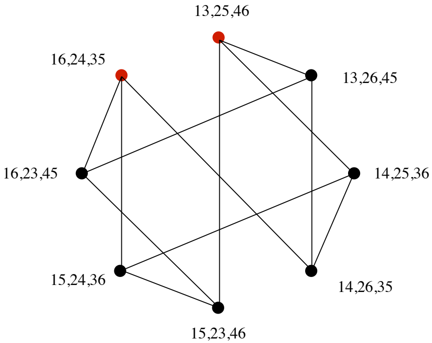

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Blasiak's reduction

Lemma (Blasiak) Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.

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This will prove the result because I_M, as a toric ideal, is generated by binomials.

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The binomial b is necessarily of the form $b = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i}$ for some bases $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ of M such that the B_i and D_i have the same multiset union.

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By induction the degree $n-1$ binomials are in the ideal generated by the quadratics of I_M so this will complete the proof.

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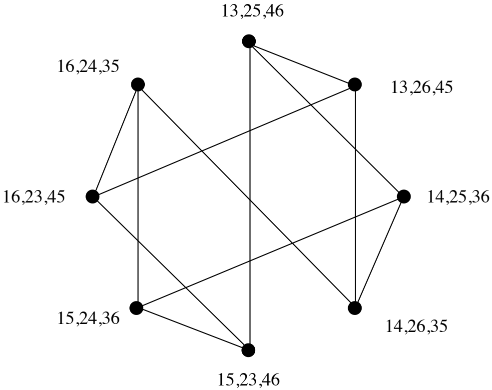

$y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

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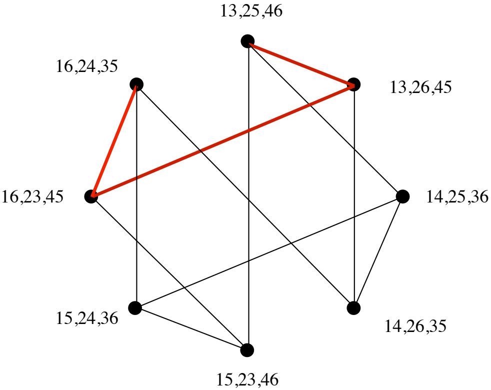

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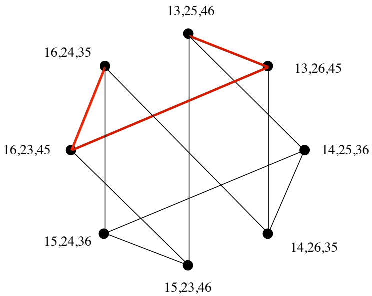

By following the path we construct
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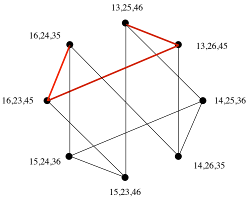

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Or equivalently
$y_{16}(y_{24}y_{35} - y_{23}y_{45}) + y_{45}(y_{16}y_{23} - y_{13}y_{26}) + y_{13}(y_{26}y_{55} - y_{25}y_{46}) = y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Variants of White's conjecture

Conjecture 1 For any matroid M, the toric ideal I_M is generated by quadratics binomials.

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Remark : Conjectures 1 and 2 together imply White's conjecture.

Complete Intersection

The toric ideal I_M is a complete intersection if and only if there exists a set of homogeneous binomials $g_1, \ldots, g_s \in R$ such that $s = \mathrm{ht}(I_M)$ and $I_M = (g_1, \ldots, g_s)$.

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Equivalently, I_M is a complete intersection if

$$\mu(I_M) = \mathrm{ht}(I_M) = |\mathcal{B}| - (n - c + 1)$$

where $\mu(I_M)$ denotes the minimal number of generators of I_M and c the number of connected components of M.

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The number of connected components of a matroid M is given by the number of equivalent classes induced by the relation $\mathcal{R}$ defined as follows : $a\mathcal{R}b$ if and only if there exist a circuit of M containing both $a, b \in M$.

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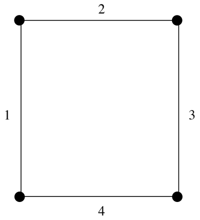

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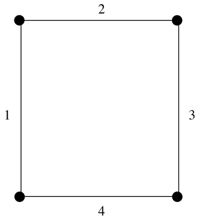

We have $\mathcal{B}(M(G)) = \{123, 124, 134, 234\}$. There is one equivalent classe, and thus $\mathrm{ht}(I_M) = 4 - (4 - 1 + 1) = 0$.

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Recall that

$$I_M = \left(\{y_{B_1}\cdots y_{B_s} - y_{D_1}\cdots y_{D_s} \mid B_1\cup\cdots\cup B_s = D_1\cup\cdots\cup D_s\}\right) \quad (1)$$

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Thus, we only consider the case $r \leq n-2$.

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Proposition Let M' be a minor of M. If I_M is a complete intersection, then $I_{M'}$ also is.

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Example :
$\mathcal{B}(U_{2,4}) = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{1,4\}, B_4 = \{2,3\}, B_5 = \{2,4\}, B_6 = \{3,4\}\}$

$$\begin{array}{c} \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

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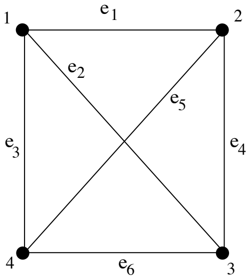

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Theorem (I. Bermejo, I. Garcia-Marco, E. Reyes) Whenever $I_{H(M)}$ is a complete intersection, then H_M does not contain $K_{2,3}$ as subgraph.

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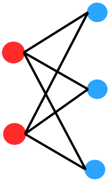

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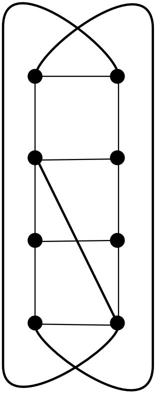

G

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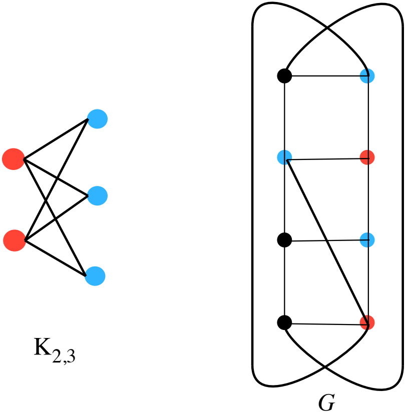

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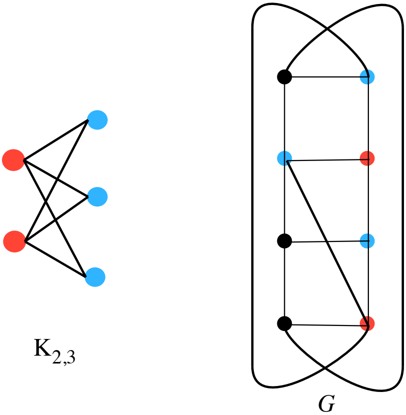

Therefore I_G is not complete intersection.

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($\Leftarrow$) By computer.

Complete Intersection : general case

Theorem Let M be a matroid without loops or coloops and with $n > r + 1$. Then, I_M is a complete intersection if and only if $n = 4$ and M is the matroid whose set of bases is :

1. $\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$,
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3. $\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}$, i.e., $M = U_{2,4}$.

Detecting minors

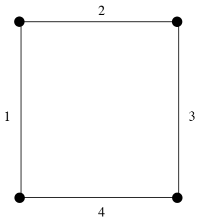
We consider the following binary equivalence relation $\sim$ on the set of pairs of bases :

$$\{B_1, B_2\} \sim \{B_3, B_4\} \iff B_1 \cup B_2 = B_3 \cup B_4 \text{ as multisets,}$$

and we denote by $\Delta_{\{B_1,B_2\}}$ the cardinality of the equivalence class of $\{B_1, B_2\}$.

Detecting minors

We consider the graph



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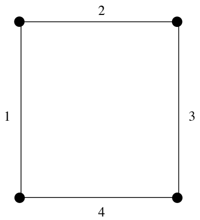

Therefore,
$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{124\}, B_3 = \{134\}, B_4 = \{234\}\}.$

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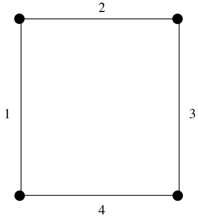

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$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{124\}, B_3 = \{134\}, B_4 = \{234\}\}$.
It can be checked that the equivalent class of $\{B_i, B_j\}$ is $\{B_i, B_j\}$, that is, $\Delta_{\{B_i, B_j\}} = 1$ for any pair $1 \leq i \neq j \leq 4$.

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Lemma (bounds) For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.

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Proof of the lower bound Take $e \in B_1 \setminus B_2$. By the multiple symmetric exchange property, for every A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, there exists $A_2 \subset B_2$ such that both $B_1' := (B_1 \cup A_2) \setminus A_1$ and $B_2' := (B_2 \cup A_1) \setminus A_2$ are bases.

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Since $B_1 \cup B_2 = B_1' \cup B_2'$ as multisets, we derive that $\Delta_{\{B_1,B_2\}}$ is greater or equal to the number of sets A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, which is exactly 2^{d-1}.

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Lemma Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1, B_2\}}$.

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Theorem M has a minor $M' \simeq U_{3,6}$ if and only if $\Delta_{\{B_1,B_2\}} = 10$ for some $B_1, B_2 \in \mathcal{B}$.

System of generators

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1. $\mu(I_M) \geq (b^2 - b - 2s)/2$, where $b := |\mathcal{B}|$, and
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Moreover, in both cases equality holds whenever I_M is generated by quadratics.

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Question Can we characterize those matroids M with $\nu(I_M) = 1$?

The basis graph of a matroid M is the undirected graph G_M with vertex set $\mathcal{B}$ and edges $\{B, B'\}$ such that $|B \setminus B'| = 1$. The diameter of a graph is the maximum distance between two vertices of the graph.

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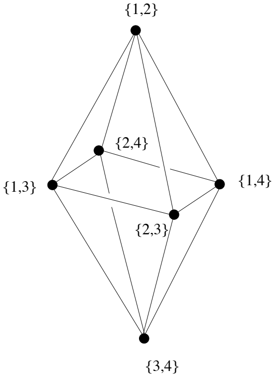
Basis graph $G_{U_{2,4}}$



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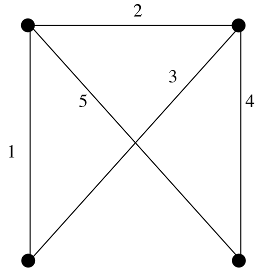
($\Leftarrow$) More complicated.

Example

Matroid $M(G)$ associated to graph G.



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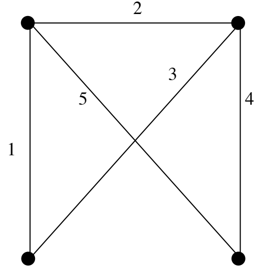
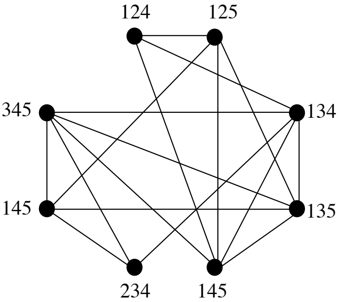


$\mathcal{B}(M(G)) = \{B_1 = \{124\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{235\}, B_8 = \{345\}\}$

Example

The base graph $G_{M(G)}$



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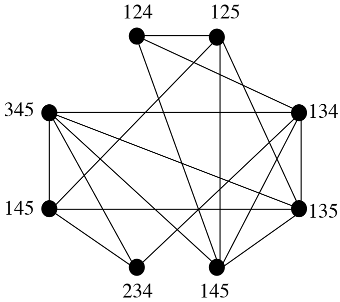

Since diameter of $G_{M(G)}$ is at most two, and $M(G)$ is binary then $\nu(I_M) = 1$.

Simplicial complexes

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If $\{v\} \in \Delta$ then we call v a vertex of Δ.

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta| \dim F = i\}|$ is the number of i-dimensional faces in Δ.

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v is called the apex of $C\Delta$.

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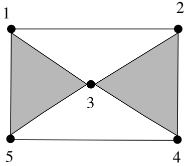
• Typically, we will describe a simplicial complex by listing its facets.

Example

Simplicial complexe Δ of dimension 2



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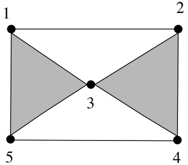

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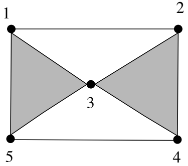

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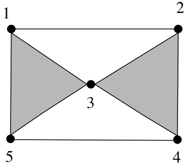

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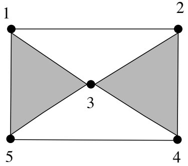

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• The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

Matroid complex

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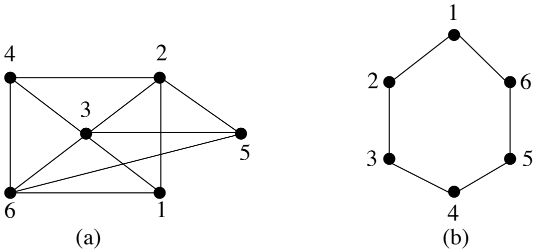
Examples

Two 1-dimensional simplicial complexes.



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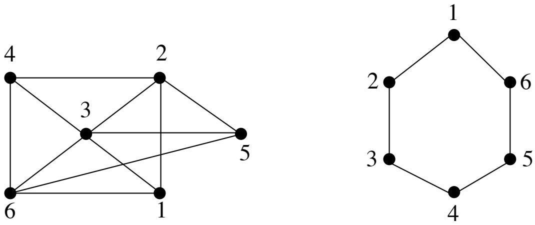

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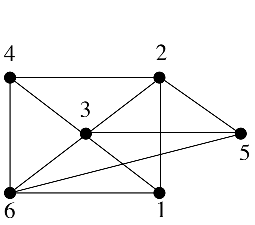

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(b) is not a matroid complex since it admits a restriction that is not pure, for instance, the facets of $\Delta_{1,3,4}$ are $\{1\}$ and $\{3, 4\}$ and so this restriction is not pure.

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A matroid complex Δ_M is a cone if and only if M has a coloop (or isthme), which corresponds to the apex defined above.

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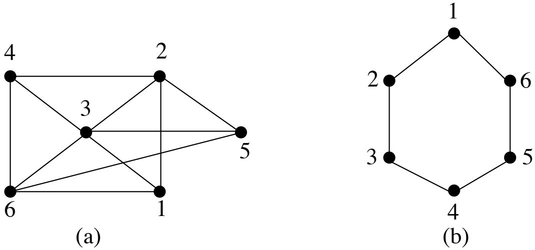

Stanley-Reisner ideal

Let k be a field. We can associate to a simplicial complex Δ, a square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

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The ideal I_Δ is called the Stanley-Reisner ideal of Δ and S/I_Δ the Stanley-Reisner ring of Δ.

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Facts

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In particular, for any $j = 0, \ldots, d$, we have

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-1} h_i$$

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-1} f_{i-1}.$$

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Remark v_j is externally passive in B if it is internally passive in $E \setminus B$ in M^*.

h-vector of simplicial complexes

Bjorner proved that

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)}$$

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Alternatively,

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- h-vector of a matroid complex Δ_M is actually a specialization of the Tutte polynomial of the corresponding matroid ; precisely we have $T(M; x, 1) = h_0 x^d + h_1 x^{d_1} + \cdots + h_d$

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Obtaining that $h(\Delta) = (1, 1, 1)$.

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A vector $\mathbf{h} = (h_0, \ldots, h_d)$ is a pure O-sequence if there is a pure ideal $\mathcal{O}$ such that $\mathbf{h} = F(\mathcal{O})$.

Example

The pure monomial order ideal inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

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Hence the h-vector of X is the pure O-sequence $h = (1, 3, 6, 7, 5, 2)$.

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Conjecture hold for several families of matroid complexes :

(Merino, Noble, Ramirez-Ibañez, Villarroel, 2010) Paving matroids

(Merino, 2001) Cographic matroids

(Oh, 2010) Cotranversal matroids

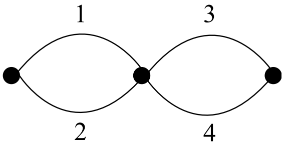
(Schweig, 2010) Lattice path matroids

(Stokes, 2009) Matroids of rank at most three

(De Loera, Kemper, Klee, 2012) for all matroids on at most nine elements all matroids of corank two.

Example

We consider the matroid complexe Δ associated to the rank 2 matroid induced by the graph G



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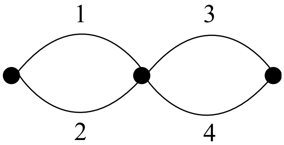

We have that $\dim \Delta = 1$ and $f_{-1} = 1, f_0 = 4$ and $f_1 = 4$.

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Obtaining the h-vector $h(1,2,1)$. Since $\mathcal{O} = (1, x_1, x_2, x_1 x_2)$ is an order ideal then $h(1,2,1)$ is pure O-sequence.