Matroid base polytope decomposition

J.L. Ramírez Alfonsín

IMAG, Université de Montpellier

San Luis, Argentina
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Introduction

Let $M = (E, \mathcal{B})$ be a matroid on $E = \{1, \ldots, n\}$ where $\mathcal{B} = \mathcal{B}(M)$ denote the collection of bases.

The set $\mathcal{B}$ verifies the base exchange axiom :

if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there exists $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.

Let $P(M)$ be the matroid base polytope of M defined as the convex hull of the incidence vector of bases of M, that is,

$$P(M) := conv\left\{\sum_{i \in B} e_i : B \in \mathcal{B}\right\}$$

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Remarks :
(a) $P(M)$ is a polytope of dimension at most $n-1$.
(b) $P(M)$ is a facet of the independent polytope of M obtained as the convex hull of the incidence vectors of the independent sets of M.

A decomposition of $P(M)$ is a decomposition of the form

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where each $P(M_i)$ is a matroid base polytope for some matroid M_i, and for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is also a matroid base polytope for some matroid (a facet of both $P(M_i)$ and $P(M_j)$).

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A decomposition is called hyperplane split if $t = 2$.

Motivations

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(Ardila, Fink and Rincon) There exist functions that behave like *valuation* on the associated base polytope decomposition.

Known results

(Kapranov 1993)

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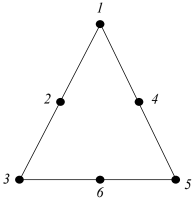
• Presented five rank 3 matroids on 6 elements such that each of the corresponding base polytope is indecomposable.

• Provided a decomposition into three indecomposable pieces of $P(W)$ that cannot be obtained via hyperplane splits.

W :



Combinatorial decomposition

A base decomposition of a matroid M is a decomposition of the form

$$\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$$

where $\mathcal{B}(M_k)$, $1 \leq k \leq t$ and $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$, $1 \leq i \neq j \leq t$ are collections of bases of matroids.

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M is said to be combinatorial decomposable if it has a base decomposition.
We say that the decomposition is *nontrivial* if $\mathcal{B}(M_i) \neq \mathcal{B}(M)$ for all i.

• If $P(M)$ is decomposable then clearly M is combinatorial decomposable.

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• A combinatorial decomposition do not necessarily induce a base polytope decomposition.

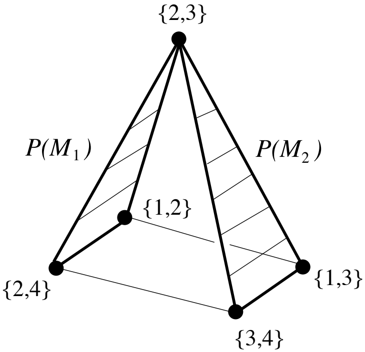
Example :

$\mathcal{B}(M) = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$ admit the combinatorial decomposition

$\mathcal{B}(M_1) = \{\{1,2\},\{2,3\},\{2,4\}\}$ and
$\mathcal{B}(M_2) = \{\{1,3\},\{2,3\},\{3,4\}\}$

We can verify that $\mathcal{B}(M_1), \mathcal{B}(M_2)$ and $\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{2,3\}$ are collection of bases of matroids.

However, $P(M_1)$ and $P(M_2)$ do not decompose $P(M)$.


{2,3}
P(M1)
P(M2)
{1,2}
{2,4}
{1,3}
{3,4}

Some geometry

Proposition Let P be a d-polytope with set of vertices X. Let H be a hyperplane such that $H \cap P \neq \emptyset$ with H not supporting de P. Then, H divides P into two polytopes P_1 and P_2, that is, $H \cap P = P_1 \cap P_2 = F \neq \emptyset$. Also, H partition X into two sets X_1 et X_2 with $X_1 \cap X_2 = W$. Then, for each edge $[u, v]$ of P we have $\{u, v\} \subset X_i$ for $i = 1$ or 2 if and only if $F = conv(W)$.

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Corollary $F = conv(W)$ if and only if $P_i = conv(X_i)$, $i = 1, 2$ (and thus $P = P_1 \cup P_2$ with P_1 and P_2 polytopes of the same dimension as P and sharing one facet).

Let (E_1, E_2) be a partition of E, that is, $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Let $r_i > 1$, $i = 1, 2$ be the rank of $M|_{E_i}$.

(E_1, E_2) is a good partition if there exist integers $0 < a_1 < r_1$ and $0 < a_2 < r_2$ such that :

(P1) $r_1 + r_2 = r + a_1 + a_2$ and

(P2) for any $X \in \mathcal{I}(M|_{E_1})$ with $|X| \leq r_1 - a_1$ and
for any $Y \in \mathcal{I}(M|_{E_2})$ with $|Y| \leq r_2 - a_2$
we have $X \cup Y \in \mathcal{I}(M)$.

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Lemma Let (E_1, E_2) be a good partition of E. Let

$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq r_1 - a_1\}$

$\mathcal{B}(M_2) = \{B \in \mathcal{B}(M) : |B \cap E_2| \leq r_2 - a_2\}$

with r_i the rank of $M|_{E_i}$, $i = 1, 2$ and a_1, a_2 verifying (P1) et (P2).

Then, $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ are the collections of bases of two matroids, say M_1 and M_2.

Theorem (Chatelain and R.A. 2011) Let $M = (E, \mathcal{B})$ be a matroid and let (E_1, E_2) be a good partition of E. Then, $P(M) = P(M_1) \cup P(M_2)$ is a nontrivial hyperplane split where M_1 and M_2 are the matroids defined in the previous lemma.

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Proof (idea) (i) $\mathcal{B}(M) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2)$,

(ii) $\mathcal{B}(M_1), \mathcal{B}(M_2) \subset \mathcal{B}(M)$,

(iii) $\mathcal{B}(M_1), \mathcal{B}(M_2) \not\subseteq \mathcal{B}(M_1) \cap \mathcal{B}(M_2)$,

(iv) $\mathcal{B}(M_1), \mathcal{B}(M_2), \mathcal{B}(M_1) \cap \mathcal{B}(M_2)$ are collections of bases,

(v) there exists a hyperplane containing the vertices corresponding to $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$ and not supporting $P(M)$,

(vi) each edge of $P(M)$ is an edge of either $P(M_1)$ or $P(M_2)$.

We say that two hyperplane splits $P(M_1) \cup P(M_2)$ and $P(M_1') \cup P(M_2')$ of $P(M)$ are equivalente if $P(M_i)$ is combinatorially equivalent to $P(M_i')$, $i = 1, 2$. They are different otherwise.

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Corollary (Chatelain and R.A. 2011) Let $n \geq r + 2 \geq 4$ be integers and let $h(U_{r,n})$ be the number of different hyperplane splits of $P(U_{r,n})$. Then,

$$h(U_{r,n}) \geq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Example. We consider $U_{2,4}$. Then, $E_1 = \{1,2\}$ and $E_2 = \{3,4\}$ is a good partition (and thus $r_1 = r_2 = 2$) with $a_1 = a_2 = 1$.
We have $\mathcal{B}(M_1) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$,
$\mathcal{B}(M_2) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ and
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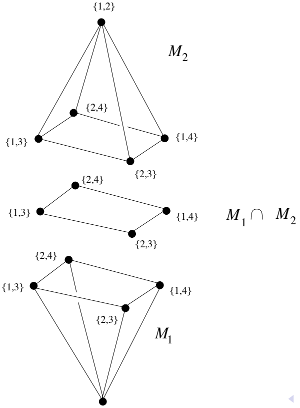

Lattice path matroid

Let $m = 3$ and $r = 4$ and let $M[Q, P]$ be the transversal matroid on $\{1, \ldots, 7\}$ with presentation $(N_i : i \in \{1, \ldots, 4\})$ where $N_1 = [1, 2, 3, 4]$, $N_2 = [3, 4, 5]$, $N_3 = [5, 6]$ and $N_4 = [7]$.

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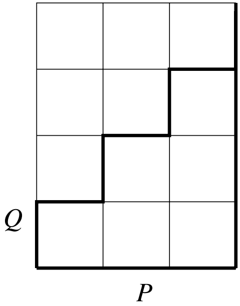
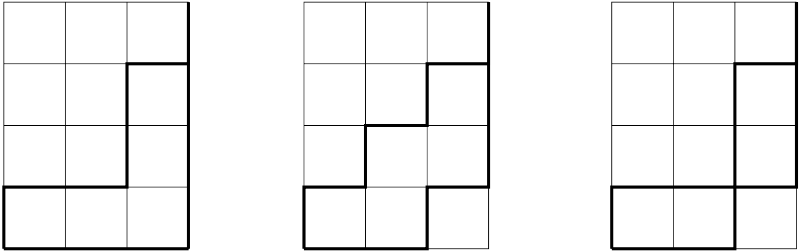


Example. Transversal matroids (a) M_1, (b) M_2 and (c) $M_1 \cap M_2$.



(a) (b) (c)

Theorem (Chatelain and R.A. 2011) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be two matroids of ranks r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a nontrivial hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a nontrivial hyperplane split.

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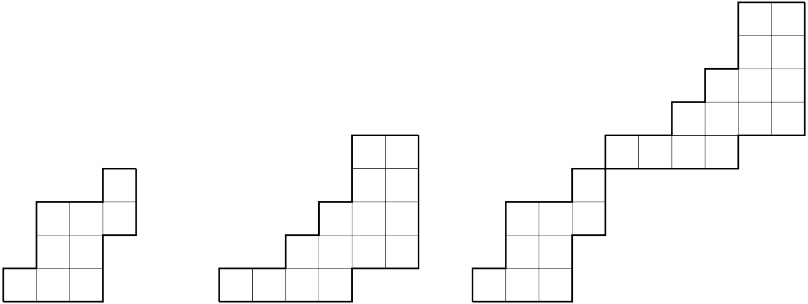

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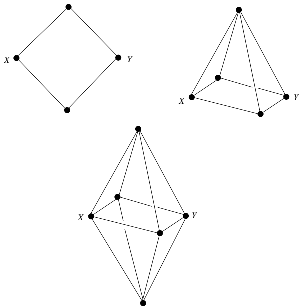

Lemma Let $M=(E,\mathcal{B})$ be a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid. If $X \in \mathcal{B}_1$ and all the neighbors of X (that is, the set of vertices of $G(M)$ adjacent to X) are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

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Corollary Let $P(M)$ be the polytope base polytope of the matroid M having as 1-skeleton the d-hypercube. Then, $P(M)$ is indecomposable.

Multi-decompositions

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Example :
$\mathcal{B}(M_1) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$
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but
$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\{1,3\},\{2,3\},\{2,4\}\}$ is not a matroid.

Let $t \geq 2$ be an integer with $r \geq t$. Let $E = \bigcup_{i=1}^{t} E_i$ be a t-partition of $E = \{1, \ldots, n\}$ and let $r_i = r(M|_{E_i}) > 1$, $i = 1, \ldots, t$.

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We say that $\bigcup_{i=1}^{t} E_i$ is a good t-partition if there exist integers $0 < a_i < r_i$ with the following properties :

$(P1)$ $r = \sum_{i=1}^{t} a_i$,

$(P2)$

(a) For any j with $1 \leq j \leq t-1$

if $X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq a_1$ and
$Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})$ with $|Y| \leq a_2$,
then $X \cup Y \in \mathcal{I}(M)$.

$(P2)$

(b) For any pair j, k with $1 \leq j < k \leq t-1$

$$\text{if } X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j}) \quad \text{with } |X| \leq \sum_{i=1}^{j} a_i,$$

$$Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_k}) \quad \text{with } |Y| \leq \sum_{i=j+1}^{k} a_i,$$

$$Z \in \mathcal{I}(M|_{E_{k+1} \cup \cdots \cup E_t}) \quad \text{with } |Z| \leq \sum_{i=k+1}^{t} a_i,$$

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then $X \cup Y \cup Z \in \mathcal{I}(M)$.

Notice that the good 2-partitions provided by $(P2)$ case (a) with $t = 2$ are the *good partitions*

Lemma Let $t \geq 2$ be an integer and let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, i=1,. . .,t. Let

$$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1\}$$

and, for each $j = 2, \ldots, t$, let

$$\mathcal{B}(M_j) = \Big\{B \in \mathcal{B}(M) \quad : |B \cap E_1| \geq a_1, \ldots, |B \cap \bigcup_{i=1}^{j-1} E_i| \geq \sum_{i=1}^{j-1} a_i, \; |B \cap \bigcup_{i=1}^{j} E_i| \leq \sum_{i=1}^{j} a_i \Big\}.$$

Then, $\mathcal{B}(M_j)$ is the collection of bases of a matroid for each $j = 1, \ldots, t$.

Theorem (Chatelain and R.A. 2014) Let $t \geq 2$ be an integer and let $M = (E, \mathcal{B})$ be a matroid of rank r. Let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, \ldots, t$. Then, $P(M)$ has a sequence of hyperplane splits yielding the decomposition

$$P(M) = \bigcup_{i=1}^{t} P(M_i),$$

where M_i, $1 \leq i \leq t$, are the matroids defined in previous lemma

Uniform matroid

Corollary (Chatelain and R.A. 2014) Let $n, r, t \geq 2$ be integers with $n \geq r + t$ and $r \geq t$. Let $p_t(n)$ be the number of different decompositions of the integer n of the form $n = \sum_{i=1}^{t} p_i$ with $p_i \geq 2$ and let $h_t(U_{n,r})$ be the number of *different* decompositions of $P(U_{r,n})$ into t pieces. Then,

$$h_t(U_{r,n}) \geq p_t(n).$$

Rank 3 matroids

Corollary (Chatelain and R.A. 2014) Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2$ be a partition of the points of the geometric representation of M such that

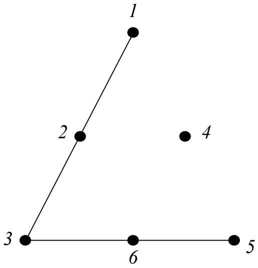
1) $r(M|_{E_1}) \geq 2$ and $r(M|_{E_2}) = 3$;

2) for each line l of M, if $|l \cap E_1| \neq \emptyset$, then $|l \cap E_2| \leq 1$.

Then, $E = E_1 \cup E_2$ is a 2-good partition.

Example

Let M be the rank-3 matroid arising from the configuration of points given below.



It can be easily checked that $E_1 = \{1, 2\}$ and $E_2 = \{3, 4, 5, 6\}$ verify the conditions of the previous Corollary. Thus, $E_1 \cup E_2$ is a 2-good partition.

Rank 3 matroids

Corollary (Chatelain and R.A. 2014) Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2 \cup E_3$ be a partition of the points of the geometric representation of M such that

1) $r(M|_{E_i}) \geq 2$ for each $i = 1, 2, 3$,

2) for each line l with at least 3 points of M,

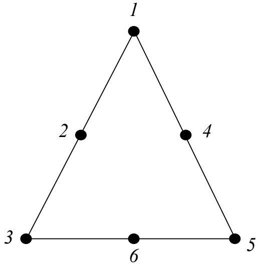
a) if $|l \cap E_1| \neq \emptyset$ then $|l \cap (E_2 \cup E_3)| \leq 1$,

b) if $|l \cap E_3| \neq \emptyset$ then $|l \cap (E_1 \cup E_2)| \leq 1$.

Then, $E = E_1 \cup E_2 \cup E_3$ is a 3-good partition.

Example

Let W be the matroid shown below

W : 

It can be checked that $E_1 = \{1,6\}$, $E_2 = \{2,5\}$, and $E_3 = \{3,4\}$ verify the conditions of the previous Corollary. Thus, $E_1 \cup E_2 \cup E_3$ is a good 3-partition.

Direct sum

Theorem (Chatelain and R.A. 2014) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ admits a sequence of t hyperplane splits if either $P(M_1)$ or $P(M_2)$ admits a sequence of t hyperplane splits.