Oriented Matroids

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Introduction

A signed set X is a set $\underline{X}$ divided in two parts (X^+, X^-), where X^+ is the set of the positive elements of X and X^- is the set of the negative elements. The set $\underline{X} = X^+ \cup X^-$ is called the support of X.

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Given a signed set X and a set A we denote by $-_A X$ the signed set defined by $(-_A X)^+ = (X^+ \setminus A) \cup (X^- \cap A)$ and $(-_A X)^- = (X^- \setminus A) \cup (X^+ \cap A)$.

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We say that the signed set $-_A X$ is obtained by a reorientation of A.

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A collection $\mathcal{C}$ of signed set of a finite set E is the set of circuits of an oriented matroid on E if and only if the following axioms are verified :

$(C0)$ $\emptyset \notin \mathcal{C}$,

$(C1)$ (*symmetry*) $\mathcal{C} = -\mathcal{C}$,

$(C2)$ (*incomparability*) for any $X, Y \in \mathcal{C}$, if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$,

$(C3)$ (*weak elimination*) for any $X, Y \in \mathcal{C}, X \neq -Y$, and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

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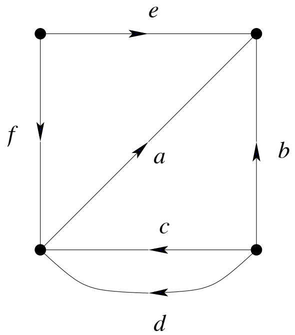
Notation For short, we write $X = a\overline{bc}de$ the signed circuit X defined by $X^+ = \{a, d, e\}$ and $X^- = \{b, c\}$.

Graphs

Let D be the following oriented graph.



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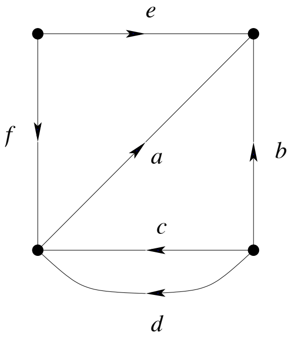

$$\mathcal{C}(D) = \{(a\overline{b}c), (a\overline{b}d), (a\overline{e}f), (c\overline{d}), (b\overline{c}\overline{e}f), (b\overline{d}\overline{e}f),$$
$$(\overline{a}b\overline{c}), (\overline{a}b\overline{d}), (\overline{a}e\overline{f}), (\overline{c}d), (\overline{b}ce\overline{f}), (\overline{b}de\overline{f})\}.$$

Configurations of vectors in the space

Let $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors generating a r-dimensional vector space over a ordered field, says $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^r$.

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We consider the minimal linear dependencies

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = 0$$

with $\lambda_i \in \mathbb{R}$. We obtain an oriented matroid from E by considering the signed sets $X = (X^+, X^-)$ where

$$X^+ = \{i : \lambda_i > 0\} \text{ et } X^- = \{i : \lambda_i < 0\}$$

for all minimal dependencies among $\mathbf{v}_i$.

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for all minimal dependencies among $\mathbf{v}_i$.

This oriented matroid is called vectorial (or linear).

Configurations of points in the space

Any configuration of points in the affine space induces an oriented matroid having as circuits the signed set from the coefficient of minimal dependencies, that is, linear combinations of the form

$$\sum_i \lambda_i \mathbf{v}_i$$

with $\sum_i \lambda_i = 0$, $\lambda_i \in \mathbb{R}$.

Configurations of points in the space

Let us consider the points in $\mathbb{R}^2$ given by the columns of matrix :

$$\begin{array}{c} \quad\; a \quad\; b \quad\; c \quad\; d \quad\; e \quad\; f \\ \overline{A} = \begin{pmatrix} -1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{pmatrix} \end{array}$$

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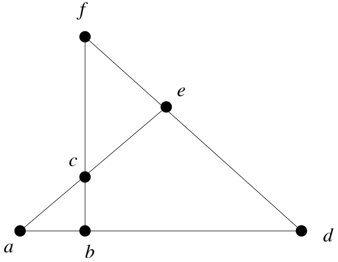
Matrix $\overline{A}$ correspond to points



The set of circuits of the corresponding affine oriented matroid is

$$\begin{aligned}\mathcal{C}(\overline{A}) \quad &= \{(a\bar{b}d), (b\bar{c}f), (d\bar{e}f), (a\bar{c}e), (\bar{a}b\bar{e}f), (\bar{b}cd\bar{e}), (a\bar{c}df), \\ &\quad (\bar{a}b\bar{d}), (\bar{b}c\bar{f}), (\bar{d}e\bar{f}), (\bar{a}c\bar{e}), (a\bar{b}e\bar{f}), (b\bar{c}\bar{d}e), (\bar{a}c\bar{d}\bar{f})\}.\end{aligned}$$

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For instance, $(a\bar{b}d)$ correspond to the affine dependecy $3(-1,0)^t - 4(0,0)^t + 1(3,0)^t = (0,0)^t$ with $3-4+1=0$.

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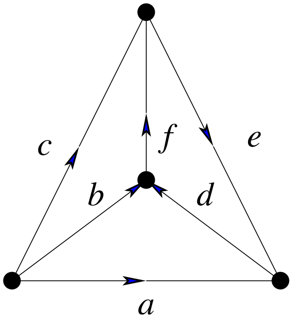
Exemple From the circuit $(a\bar{b}d)$ we see that the point b lies in the segment $[a, b]$ and from circuit $(\bar{a}b\bar{e}f)$ the segment $[a, e]$ intersect the segment $[b, f]$ (in the affine real espace).

We can check that the oriented matroid obtained form K_4 with the orientation illustrated below has the same set of circuits that $M(\overline{A})$



They are isomorphic.

Let us consider the oriented matroid $-_d M(\overline{A})$ obtained by reorienting element d of $M(\overline{A})$. The set of circuits of $-_d M(\overline{A})$ is :

$$\begin{aligned}\mathcal{C} \quad &= \{(a\overline{bd}), (b\overline{c}f), (\overline{de}f), (a\overline{c}e), (\overline{a}b\overline{e}f), (\overline{b}c\overline{de}), (a\overline{cd}f),\\ &\quad (\overline{a}bd), (\overline{b}c\overline{f}), (de\overline{f}), (\overline{a}c\overline{e}), (a\overline{b}e\overline{f}), (b\overline{c}de), (\overline{a}cd\overline{f})\}.\end{aligned}$$

- $-_d M(\overline{A})$ is a graphic oriented matroid since it can be obtained by changing the orientation of the edge d.

• $-_d M(\overline{A})$ is a graphic oriented matroid since it can be obtained by changing the orientation of the edge d.

• Moreover $-_d M(\overline{A})$ correspond to the affine oriented matroid illustrated as before under the permutation $\sigma(a) = b, \sigma(b) = a, \sigma(c) = c, \sigma(d) = d, \sigma(e) = f, \sigma(f) = e$.

Minors

(**Deletion**) Let $M = (E, \mathcal{C})$ be an oriented matroid and let $F \subset E$. Then,

$$\mathcal{C}' = \{X \in \mathcal{C} : \underline{X} \subseteq F\}$$

the set of circuits in M contained in F, is the set of circuits of an oriented matroid in F.

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This oriented matroid is called a sub-matroid induced by F, and denoted by $M|_F$

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(**Contraction**) Let $M = (E, \mathcal{C})$ be an oriented matroid and let $F \subset E$. Then,

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the set of non-empty intersections, minimal by inclusion of the circuits of M with F, is the set of circuits of an oriented matroid in F.

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This oriented matroid is called a contraction of M over F, and it is denoted by M/F

Duality

Two signed sets X et Y are said orthogonal, denoted by $X \perp Y$, if either $\underline{X} \cap \underline{Y} = \emptyset$ or if $X|_{X\cap Y}$ and $Y|_{X\cap Y}$ are neither opposite nor equal, that is, there exists $e, f \in \underline{X} \cap \underline{Y}$ such that $X(e)Y(e) = -X(f)Y(f)$.

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Let $M = (E, \mathcal{C})$ be an oriented matroid, then

(i) there exists a unique signature of $\mathcal{C}^*$ the cocircuits of $\underline{M}$ such that

$$(\perp) \qquad X \perp Y \text{ pour tout } X \in \mathcal{C} \text{ et } Y \in \mathcal{C}^*.$$

(ii) The collection $\mathcal{C}^*$ is the set of circuits of an oriented matroid over E, denoted by M^* and called dual (or orthogonal) of M.

$(iiii)$ We have $M^{**} = M$.

Geometric interpretation of cocircuits

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Let H be a hyperplane of $\underline{M}$, i.e., a closed set of E generating a hyperplane in $\mathbb{R}^d$. We recall that $D = E \setminus H$ is a cocircuit of $\underline{M}$.

Let h be the linear function in $\mathbb{R}^d$ such that $kernel(h)$ is H (unique up to scaling).

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The signature of D in M^* is given by

$$D^+ = \{e \in D : h(e) > 0\} \text{ and } D^- = \{e \in D : h(e) < 0\}.$$

Let $V = \{a, b, c, e, f\}$ be the vectors given in the following matrix

$$\begin{array}{cc} & \begin{array}{ccccc} a & c & f & b & e \end{array} \\ A' = & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

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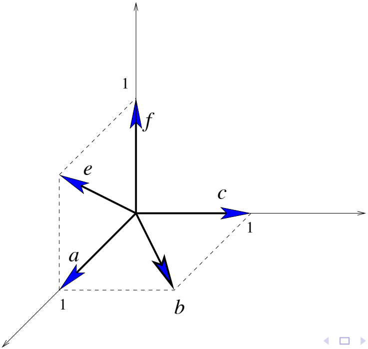
$$

corresponding to vectors



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The vector configuration of the dual space V is given by the columns of

$$A'^{\perp} = \begin{array}{c} \\ \\ \end{array} \begin{array}{ccccc} a^{\perp} & c^{\perp} & f^{\perp} & b^{\perp} & e^{\perp} \\ \end{array}$$

$$A'^{\perp} = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

We thus have that minimal dependencies among the columns of $A'^{\perp}$ are :

$$\begin{aligned} \mathcal{C}(A'^{\perp}) = \mathcal{C}^*(A') \quad &= \{a^{\perp}e^{\perp}b^{\perp}, a^{\perp}e^{\perp}\overline{c^{\perp}}, a^{\perp}\overline{f^{\perp}}b^{\perp}, a^{\perp}\overline{f^{\perp}c^{\perp}}, b^{\perp}c^{\perp}, e^{\perp}f^{\perp}, \\ &\quad \overline{a^{\perp}e^{\perp}b^{\perp}}, \overline{a^{\perp}e^{\perp}}c^{\perp}, \overline{a^{\perp}}f^{\perp}\overline{b^{\perp}}, \overline{a^{\perp}}^{\perp}c^{\perp}, \overline{b^{\perp}c^{\perp}}, \overline{e^{\perp}f^{\perp}}\}. \end{aligned}$$

We notice that the complement of each cocircuit correspond to an hyperplane of $M(A)$.

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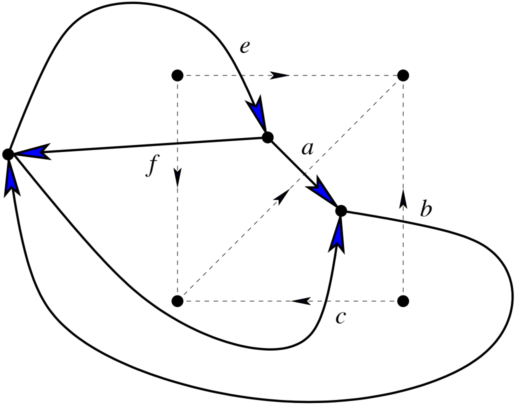
Indeed, $M(A^{\perp})$ is isomorphic to $M(D')$ where D' is the oriented graph dual to the planar signed graph $D \setminus \{d\}$, D' as follows



We can check that the circuits $M(D')$ are the cocircuits of $M(D \setminus \{d\}) = M(A \setminus \{d\})$ corresponding to hyperplanes of $M(D \setminus \{d\})$.

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Example The set $\{e, f\}$ of D' is a minimal cut (and thus a cocircuit) of $D \setminus \{d\}$ corresponding to the hyperplane $E \setminus \{e, f\} = \{a, b, c\}$ of $D \setminus \{d\}$. The set $\{abc\}$ is a hyperplane since $r(\{abc\}) = 2$ and $cl(\{a, b, c\}) = \{a, b, c\}$.

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Geometrically, the vectors $\{a, b, c\}$ generate a hyperplane but they do not form a base.

Geometric interpretation of cocircuits : affine case

Let E be a configuration of points in the $(d-1)$-affine space. Let D be a cocircuit of the oriented matroid of affine linear dependecies of E. The signature of D in M^* is

$$D^+ = D \cap H^+ \text{ et } D^- = D \cap H^-$$

where H^+ and H^- are the two open spaces in $\mathbb{R}^{d-1}$ determined by a hyperplan affine H containing $E \setminus D$.

Bases orientations

A basis orientation of an oriented matroid M is an application from the set of ordered bases of M to $\{-1, +1\}$ verifying

$(B1)$ χ est alternating

(P) (pivotage property) if $(e, x_2, \ldots, x_r)$ and $(f, x_2, \ldots, x_r)$ are two ordered bases of M with $e \neq f$ then,

$$\chi(f, x_2, \ldots x_r) = -C(e)C(f)\chi(e, x_2, \ldots, x_r)$$

where C is one of the two circuits of M in $(e, f, x_2, \ldots, x_r)$.

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Indeed, we can find the signs of the elements $C \in \mathcal{C}(\underline{M})$ from χ as follows : Choose $x_1, \ldots x_r, x_{r+1} \in M$ such that $C \subset \{x_1, \ldots, x_{r+1}\}$ and $\{x_1, \ldots, x_r\}$ is a base of $\underline{M}$. Then,

$$C(x_i) = (-1)^i \chi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+1}) \text{ for any } x_i \in C.$$

We can extend χ to an application defined on E^r, $r = r(M)$ to $\{-1, 0, +1\}$ by setting $\chi(x_1, \ldots, x_r) = 0$ if $\{x_1, \ldots, x_r\} \notin \mathcal{B}(\underline{M})$.

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We also have the dual version for the pivotage property (P) :

(P^*) (pivotage dual property) if $(e, x_2, \ldots, x_r)$ and $(f, x_2, \ldots, x_r)$ are two ordered bases of M with $e \neq f$ then,

$$\chi(f, x_2, \ldots, x_r) = -D(e)D(f)\chi(e, x_2, \ldots, x_r)$$

where D is one of the two cocircuits of M complement to the hyperplane generated by $(x_2, \ldots, x_r)$ in M.

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A chirotope of rank r over E is an application $\chi : E^r \longrightarrow \{-1, 0, +1\}$ verifying

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$(CH0)$ $\chi \neq 0$,

$(CH1)$ χ is alternating, i.e.,
$\chi(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) = sign(\sigma)\chi(x_1, \ldots, x_r)$ for any $x_1, \ldots, x_r \in E^r$ and any permutation σ.

$(CH2)$ for any $x_1, \ldots, x_r, y_1, \ldots, y_r \in E^r$ such that

$\chi(y_i, x_2, \ldots, x_r) \cdot \chi(y_1, \ldots, y_{i_1}, x_1, y_{i+1}, \ldots, y_r) \geq 0$ for any $i = 1, \ldots, r$

then

$$\chi(x_1, \ldots, x_r) \cdot \chi(y_1, \ldots, y_r) \geq 0.$$

If M is an oriented matroid of rank r of the linear dependencies of a set of vectors $E \subset \mathbb{R}^r$, then the corresponding chirotope χ is given by

$$\chi(x_1, \ldots, x_r) = sign(det(x_1, \ldots, x_r))$$

for any $x_1, \ldots, x_r \in E$.

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In this case the axiom $(CH2)$ is an abtraction of the Grassmann-Plücker relation for the determinant claiming that if $x_1, \ldots, x_r, y_1, \ldots, y_r \in \mathbb{R}^r$ then

$$det(x_1, \ldots, x_r) \cdot det(y_1, \ldots, y_r) = \sum_{i=1}^{r} det(y_i, x_2, \ldots, x_r) \cdot det(y_1, \ldots, y_{i_1}, x$$

Theorem Let $r \geq 1$ be an integer and let E be a finite set. An application

$$\chi : E^r \longrightarrow \{-1, 0, +1\}$$

is a basis orientation of an oriented matroid of rank r over E if and only if χ is a chirotope.

Contraction Let $A \subset E$. Recall that $\mathcal{C}/A = Min\{C \setminus A : C \in \mathcal{C}\}$. Let $a_1, \ldots, a_{r-s}$ be a base of A in M. Then,

$$\begin{array}{rccl} \chi/A : & (E \setminus A)^s & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \ldots, x_s) & \longmapsto & \chi(x_1, \ldots, x_s, a_1, \ldots, a_{r-s}) \end{array}$$

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Deletion Let $A \subset E$ and suppose that $M \setminus A$ is of rank $s < r$. Recall that $\mathcal{C} \setminus A = \{C \in \mathcal{C} : C \cap A = \emptyset\}$. Let $a_1, \ldots, a_{r-s} \in A$ such that $E \setminus A \cup \{a_1, \ldots, a_{r-s}\}$ generate M. Then,

$$\begin{array}{rccc} \chi \setminus A : & (E \setminus A)^s & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \ldots, x_s) & \longmapsto & \chi(x_1, \ldots, x_s, a_1, \ldots, a_{r-s}) \end{array}$$

Reorientation Let $A \subset E$ then the set of circuits of $-_A M$ is given by $-_A\mathcal{C} = \{-_A C : C \in \mathcal{C}\}$ where the signature of $-_A C$ is defined by $(-_A C)(x) = (-1)^{|A \cap \{x\}|} \cdot C(x)$. Then

$$\begin{array}{rlcl} -_A\chi : & E^r & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \ldots, x_r) & \longmapsto & \chi(x_1, \ldots, x_r)(-1)^{|A \cap \{x_1, \ldots, x_r\}|} \end{array}$$

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Duality Let $E = \{1, \ldots, n\}$. Given a $(n-r)$-set $(x_1, \ldots, x_{n-r})$, we write $(x'_1, \ldots, x'_r)$ for one permutation of $E \setminus \{x_1, \ldots, x_{n-r}\}$. In particular, $\{x_1, \ldots, x_{n-r}, x'_1, \ldots, x'_r\}$ is a permutation of $\{1, \ldots, n\}$ where its sign, denoted by $sign\{x_1, \ldots, x_{n-r}, x'_1, \ldots, x'_r\}$, is given by the parity of the number of inversions of this set. Then,

$$\begin{array}{rccl} \chi^* : & E^{n-r} & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \ldots, x_{n-r}) & \longmapsto & \chi(x'_1, \ldots, x'_r) sign\{x_1, \ldots, x_{n-r}, x'_1, \ldots, x'_r\} \end{array}$$

Topological Representation

A sphere S of S^{d-1} is a pseudo-sphere if S is homeomorphic to S^{d-2} in a homeomorphisme of S^{d-1}. There are then two connected components in $S^{d-1} \setminus S$, each homeomorphic to a ball of dimension d_1 (called sides of S).

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A finite collection $\{S_1, \ldots, S_n\}$ of pseudo-spheres in S^{d-1} is an arrangement of pseudo-spheres if

$(PS1)$ For all $A \subseteq E = \{1, \ldots, n\}$ the set $S_A = \cap_{e \in A} S_e$ is a topological sphere

$(PS2)$ If $S_A \not\subseteq S_e$ for $A \subseteq E, e \in E$ and S_e^+, S_e^- denote the two sides of S_e then $S_A \cap S_e$ is a pseudo-sphere of S_A having as sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

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• We say that the arrangement is signed if for each pseudo-sphere S_e, $e \in E$ it is choosen a positive and a negative side.

• Every essential arrangement of signed pseudo-sphere $\mathcal{S}$ partition the topological $(d-1)$-sphere in a complexe cellular $\Gamma(\mathcal{S})$. Each cell of $\Gamma(\mathcal{S})$ is uniquely determined by a sign vector in $\{-, 0, +\}^E$ which is the codification of its relative position relative according to each pseudo-sphere S_i. Conversely $\Gamma(\mathcal{S})$ characterize $\mathcal{S}$.

Two arrangements (resp. signed arrangement) are equivalent if they are the same up to a homomorphisme de S^{d_1} (resp. also the homeorphisme preserve the signs). $\mathcal{S}$ is called realizable if there exists arrangement of sphere $\mathcal{S}'$ such that $\Gamma(\mathcal{S})$ is isomorphic to $\Gamma(\mathcal{S}')$.

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Theorem (Topological Representation)) A loop-free oriented matroids of rank $d+1$ (up to isomorphism) are in one-to-one correspondence with arrangements of pseudospheres in S^d (up to topological equivalence) or equivalently to affine arrangements of pseudohyperplanes in $\mathbb{R}^{d-1}$ (up to topological equivalence).

An arrangement of pseudo-lines is simple if three or more lines are not concurrent.

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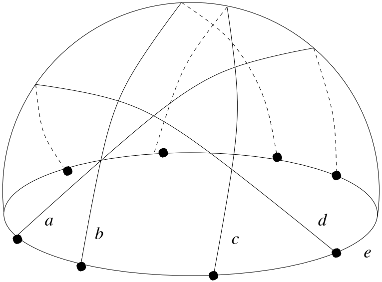

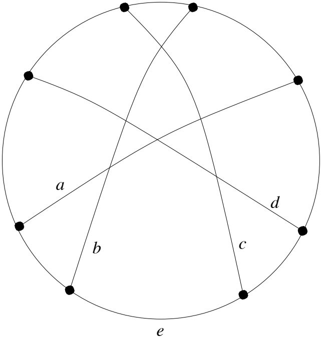

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- The number of subsets A of E such that $-_A M$ are acyclic is equals to $t(M; 2, 0)$.

- The number of subsets A of E such that $-_A M$ are totally cyclic is equals to $t(M; 0, 2)$.