Theory of matroids and applications : II

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Transversal Matroid

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A set $X \subseteq S$ is called partial transversal of $\mathcal{A}$ if there exists $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}$ such that X is a transversal of $\{A_{i_1}, \ldots, A_{i_l}\}$.

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Let $G = (S, \mathcal{A}; E)$ be a bipartite graph constructed from $S = \{e_1, \ldots, e_n\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}$ and two vertices $e_i \in S$, $A_j \in \mathcal{A}$ are adjacent if and only if $e_i \in A_j$.

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A matching in a graph is a set of edges without common vertices.

Partial transversal $\mathcal{A}$ correspond to matchings in $G = (S, \mathcal{A}; E)$.

Transversal Matroid

$E = \{e_1, \ldots, e_6\}$ et $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ with $A_1 = \{e_1, e_2, e_6\}$, $A_2 = \{e_3, e_4, e_5, e_6\}$, $A_3 = \{e_2, e_3\}$ and $A_4 = \{e_2, e_4, e_6\}$.

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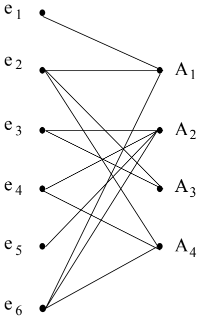

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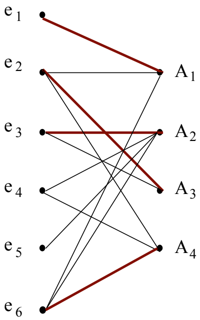

$\{e_1, e_3, e_2, e_6\}$ is a transversal of $\mathcal{A}$.

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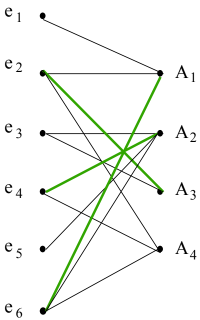

$X = \{e_6, e_4, e_2\}$ is a partial transversal of $\mathcal{A}$ since X is a transversal of $\{A_1, A_2, A_3\}$.

Transversal Matroid

Theorem Let $S = \{e_1, \ldots, e_n\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}, A_i \subseteq S$. Then, the set of partial transversals of $\mathcal{A}$ is the set of independents of a matroid.

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Proof Exercise.

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Such matroid is called transversal matroid.

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Problem : Assign the tasks to the agents in an optimal way (maximizing the priorities).

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- Transversal Matroid $M = (\mathcal{I}, \{t_1, t_2, t_3, t_4\})$ where $\mathcal{I}$ is given by the set of matchings of the bipartite graph $G = (U, V; E)$ with $U = \{t_1, t_2, t_3, t_4\}$, $V = \{e_1, e_2, e_3\}$.

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- By applying the greedy algorithm to M we have $X_0 = \emptyset, X_1 = \{t_1\}, X_2 = \{t_1, t_4\}$ and $X_3 = \{t_1, t_4, t_2\}$.

Duality

Let M be a matroid on the ground set E and let $\mathcal{B}$ the set of bases of M. Then,

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The matroid on E having $\mathcal{B}^*$ as set of bases, denoted by M^*, is called the dual of M.

A base of M^* is also called cobase of M.

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- The rank function of M^* is given by

$$r_{M^*}(X) = |X| + r_M(E \backslash X) - r_M,$$

for $X \subset E$.

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The matroid obtained on this way is called the matroid of cocycle of G or bond matroid, denoted by $B(G)$.

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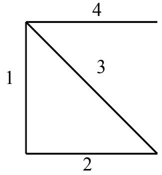

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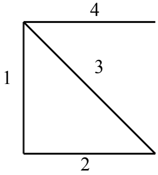

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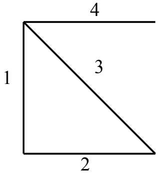

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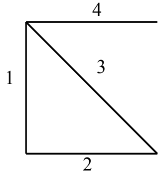

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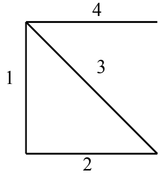

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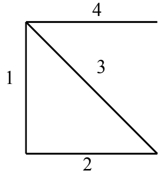

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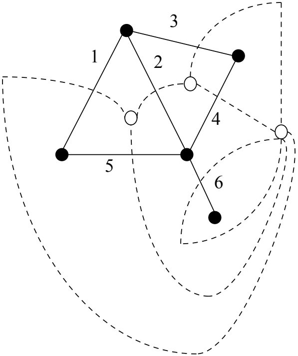

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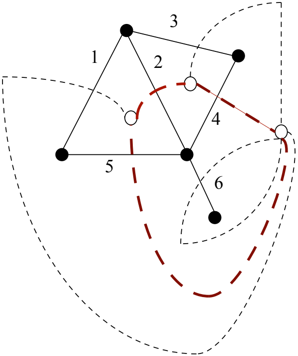

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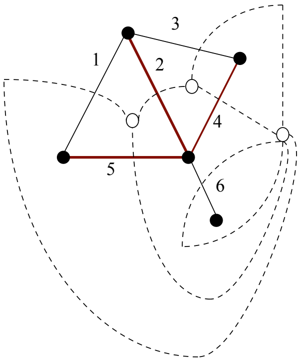

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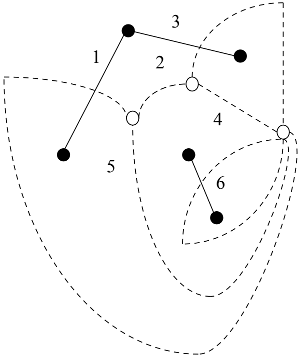

Remark The dual of a graphic matroid is not necessarly graphic.

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(Exercise) M^* can be obtained from the set of columns of the matrix

$$(-{}^tA \mid I_{n-r})$$

where I_{n-r} is the identity $(n-r) \times (n-r)$ and ${}^tA$ is the transpose of A.

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The orthogonal space of the space generated by the columns of $(I \mid A)$ is given by the space generated by the columns of $(-{}^tA \mid I_{n-r})$.