Theory of matroids and applications : III

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Tutte Polynomial

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$$t(M; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)}.$$

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$$\begin{aligned} t(U_{2,3};x,y) &= \sum_{X\subseteq E,\ |X|=0}(x-1)^{2-0}(y-1)^{0-0}+\sum_{X\subseteq E,\ |X|=1}(x-1)^{2-1}(y-1)^{1-1} \\ &+\sum_{X\subseteq E,\ |X|=2}(x-1)^{2-2}(y-1)^{2-2}+\sum_{X\subseteq E,\ |X|=3}(x-1)^{2-2}(y-1)^{3-2} \\ &=(x-1)^2+3(x-1)+3(1)+y-1 \\ &=x^2-2x+1+3x-3+3+y-1=x^2+x+y. \end{aligned}$$

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The Tutte polynomial can be expressed recursively as follows

$$t(M;x,y) = \begin{cases} t(M \setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M \setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

Acyclic Orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

We say that the orientation is acyclic if the oriented graph do not contain an oriented cycle (i.e., a cycle where all its edges are oriented clockwise or anti-clockwise).

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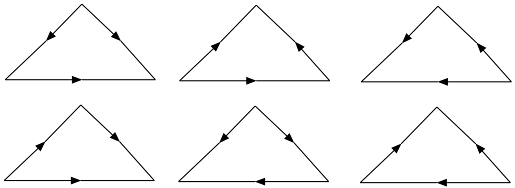
Theorem The number of acyclic orientations of G is equals to

$$t(M(G); 2, 0).$$

Acyclic Orientations

Example : There are 6 acyclic orientations of C_3



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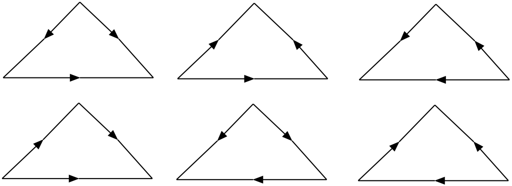

Notice that $M(C_3)$ is isomorphic to $U_{2,3}$.

Since $t(U_{2,3}; x, y) = x^2 + x + y$ then the number of acyclic orientations of C_3 is $t(U_{2,3}; 2, 0) = 2^2 + 2 + 0 = 6$.

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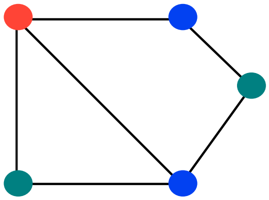

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Theorem $\chi(G, \lambda)$ is a polynomial on λ. Moreover

$$\chi(G, \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\omega(G[X])}$$

where $\omega(G[X])$ denote the number of connected components of the subgraph generated by X.

Proof (idea) By using the inclusion-exclusion formula.

Chromatic Polynomial

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Theorem If G is a graph with $\omega(G)$ connected components. Then,

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Exemple : $\chi(K_3,3)=3^1(-1)^{3-1}t(K_3;1-3,0)$
$= 3\cdot 1\cdot t(U_{2,3};-2,0)=3((-2)^2-2+0)=6.$

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A polytope is called integer if all its vertices have integer coordinates.
Ehrhart studied the function i_P that counts the number of integer points in the polytope P *dilated* by a factor of t

$$\begin{aligned} i_P : \quad & \mathbb{N} \longrightarrow \mathbb{N}^* \\ & t \mapsto \left|tP \cap \mathbb{Z}^d\right| \end{aligned}$$

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

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All others coefficients remain a mystery ! !

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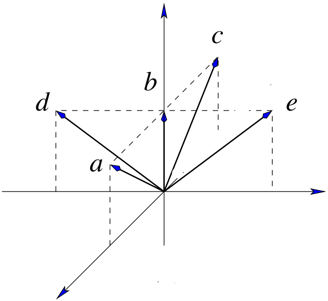
Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.
A zonotope generated by A, denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

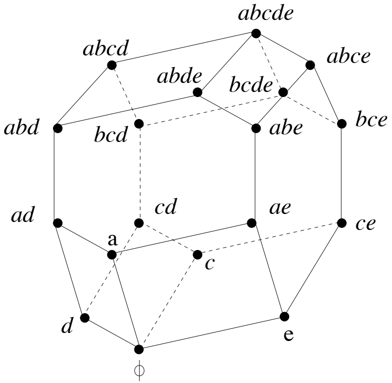
$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

Ehrhart Polynomial



Ehrhart Polynomial

Permutahedron



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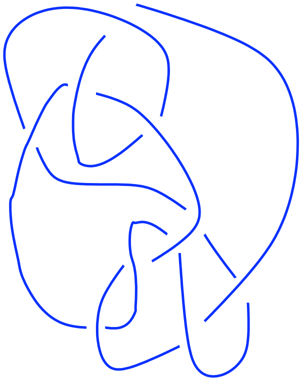
Theorem Let M be a regular matroid and let A be one of its representation matrix. Then, the Ehrhart polynomial associated to the zonotope $Z(A)$ is given by

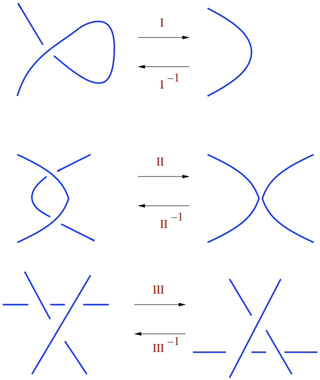
$$i_{Z(A)}(q) = q^{r(M)} t\left(M; 1 + \frac{1}{q}, 1\right).$$

Knots



Knots

Reidemeister moves



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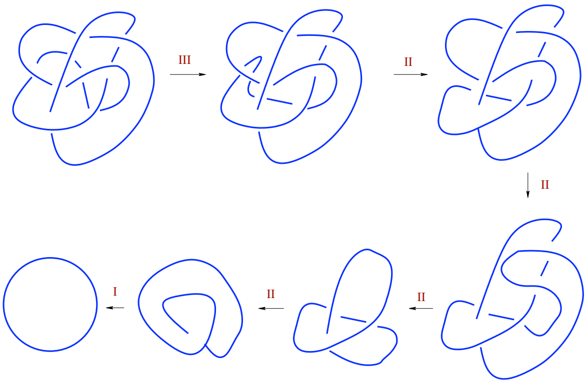

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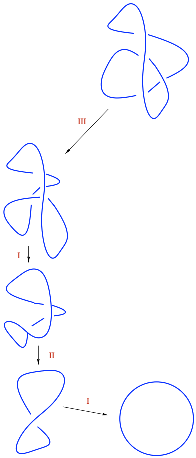

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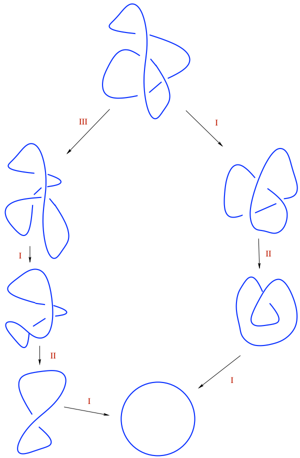

Bracket polynomial

For any link diagram D define a Laurent polynomial $< D >$ in one variable A which obeys the following three rules where U denotes the unknot :

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i) $\left\langle U \right\rangle = 1$

ii) $\left\langle U + D \right\rangle = -(A^2 + A^{-2}) \left\langle D \right\rangle$

iii) 

Knots

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i) - (iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves II and III

but it is not invariant under Reidemeister move I!!

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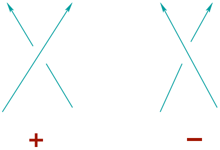
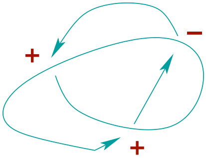
The writhe of an oriented link diagram D is the sum of the signs at the crossings of D (denoted by $\omega(D)$).

Knots





$\omega(D)=1$

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Theorem For any link L define the Laurent polynomial

$$f_D(A) = (-A^3)^{\omega(D)} < L >$$

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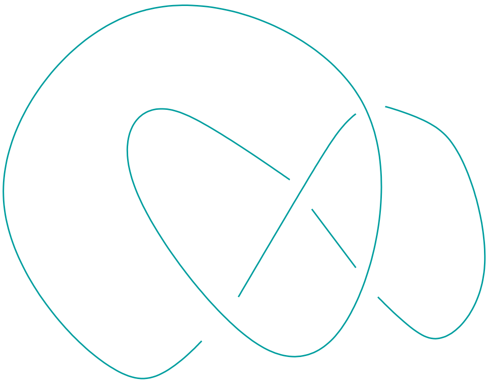
Now, define for any link L

$$V_L(z) = f_D(z^{-1/4})$$

where D is any diagram representing L. Then $V_L(z)$ is the Jones polynomial of the oriented link L.

Knots



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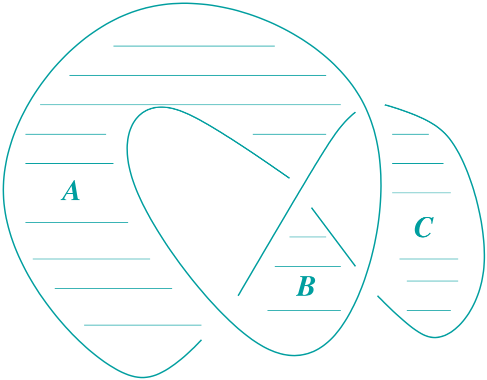

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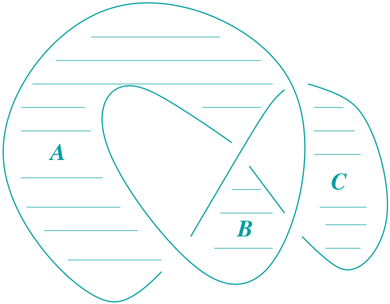

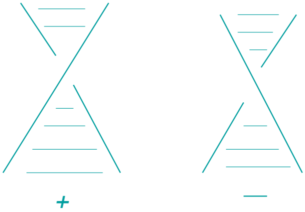

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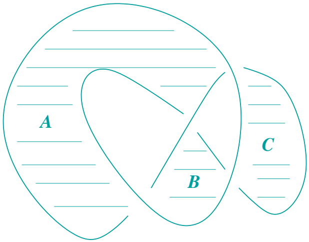

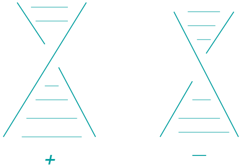

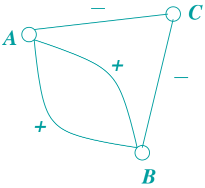

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Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram then

$$V_L(z) = (z^{-1/4})^{3\omega(D)-2} t(M(G); -z, -z^{-1})$$

where G is the graph associated to the knot diagram.

More applications

- Code theory
- Flow polynomial
- Bicycle space of a graph
- Statistical mechanics
- Arrangements of hyperplanes

$\vdots$