On a scissors congruence phenomenon for some polytopes

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joint work with
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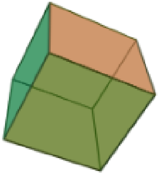
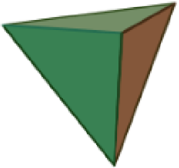


$\not\sim_{sc}$



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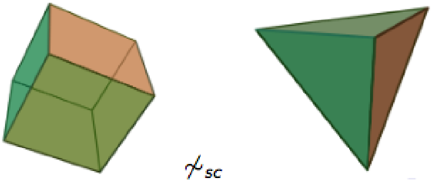

Done by using the Dehn invariant of a polyhedron depending on edge lengths and edge dihedral angles.

Hilbert's third problem for the unimodular group

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Question (Haase and McAllister, 2008) Is there a decomposition of $\mathcal{P}_1$ in a finite number of polytopes Q_i and a set of affine unimodular transformations U_i such that the union of all $U_i(Q_i)$ is equal to $\mathcal{P}_2$?

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Motivation : The Ehrahrt polynomial of an integer polytope is invariant under affine unimodular transformation.

Ehrhart polynomial

Let $\mathcal{P}$ be an integer polytope.

Ehrhart defined a function $L_{\mathcal{P}}(t)$ on the integer parameter t which is the number of integer points inside the dilation $t\mathcal{P}$.

Theorem (Ehrhart).
For every integer polytope $\mathcal{P}$ of dimension d,
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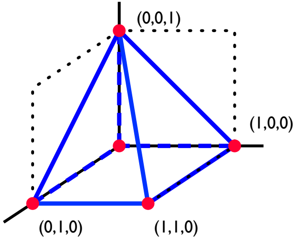

$$\begin{aligned} L_{\mathcal{P}}(t) &= 1+4+\cdots+(t+1)^2 \\ &= \sum_{i=1}^{t+1} i^2 \\ &= \frac{1}{3}t^3+\frac{3}{2}t^2+\frac{13}{6}t+1 \end{aligned}$$

Polytope of Liu and Osserman

$\{1,3\}$-graphs : degree of every vertex is 1 or 3

For each degree-3 vertex v of a $\{1,3\}$-graph G,
let a, b, and c be the edges incident to v.

$S(v)$ is the system defined on the variables w_a, w_b, and w_c :

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\begin{aligned}
w_a + w_b + w_c &\leq 1 \\
w_a &\leq w_b + w_c \\
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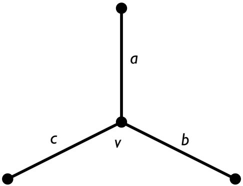

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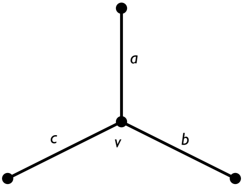

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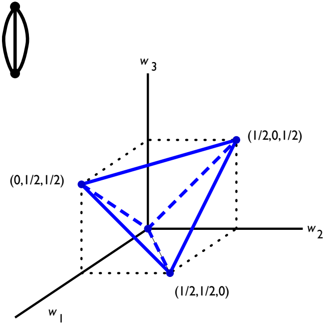
Properties of this polytope are related to a work in algebraic geometry by Mochizuki, 1999.

Examples : polytopes of cubic graphs on two vertices



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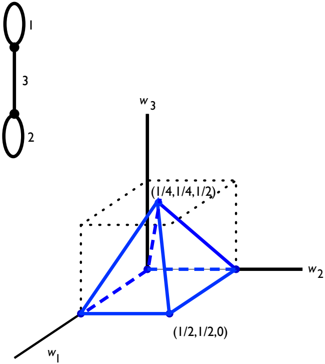

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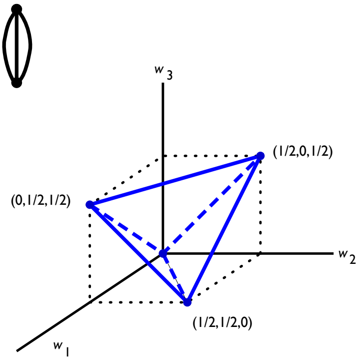

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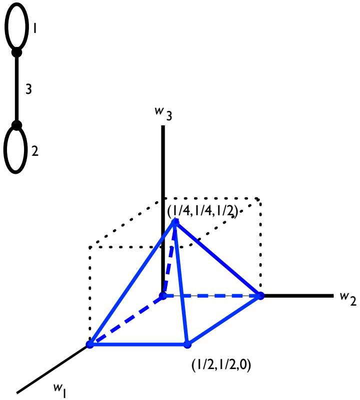

$$2w_1 + w_3 \leq 1$$
$$w_3 \leq 2w_1$$
$$w_3 \geq 0$$
$$2w_2 + w_3 \leq 1$$
$$w_3 \leq 2w_2$$

Main result

Theorem (Fernandes, De Pina, Robins, R.A., 2018) Let G_1 and G_2 be two same-size connected $\{1,3\}$-graphs. Then, $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ are unimodular equidecomposable.

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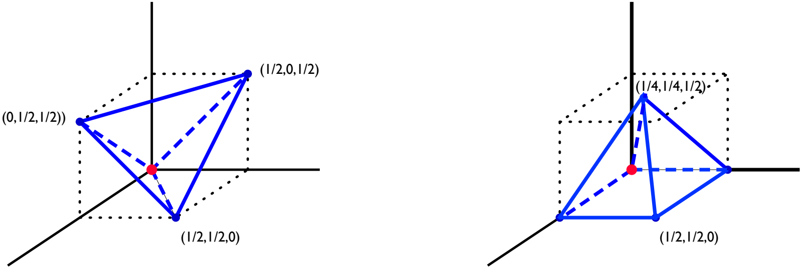

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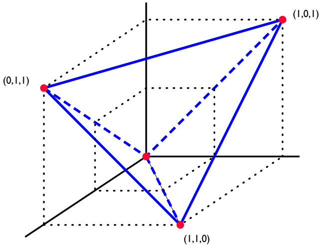
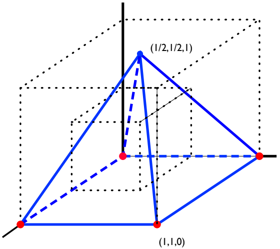
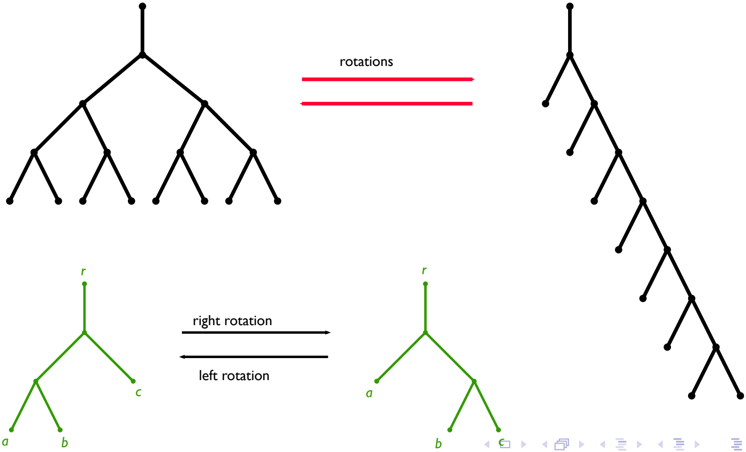




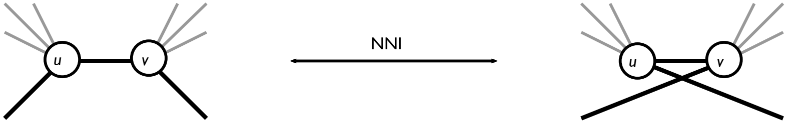
Binary trees and rotations

Theorem (Culik and Wood, 1982) Any two binary trees with the same number of vertices can be transformed into one another through a finite series of rotations.



Nearest neighbor interchange

NNI : nearest neighbor interchange



u and v adjacent vertices

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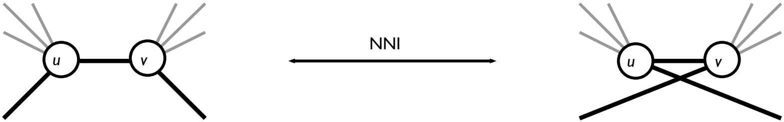

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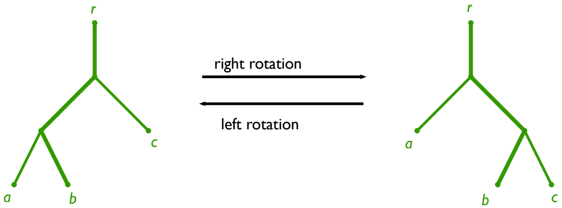

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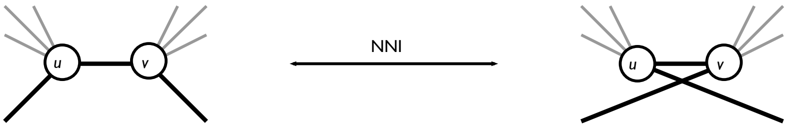

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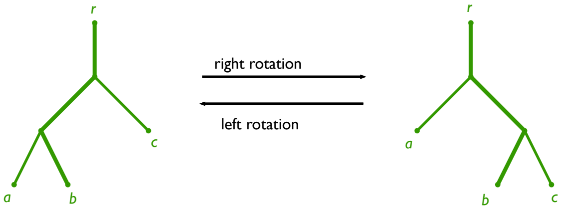

Loops and parallel edges allowed

First result

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An edge is external if it is incident to a leaf, otherwise it is internal.

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(a) G can be transformed into G' through a series of NNI moves.

(b) One can choose a spanning tree in G and a spanning tree in G' and require that all the pivots of the NNI moves are internal edges of both of these spanning trees.

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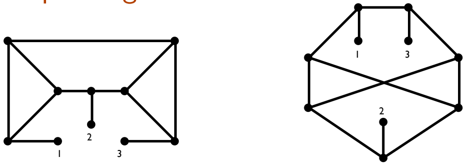

Extension of a theorem for cubic graphs by Tsukui (1996).

Cutting an edge

For the proof of the previous theorem...

G	:	a connected graph that is not a tree.
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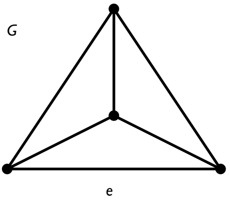
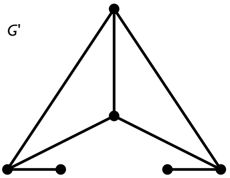
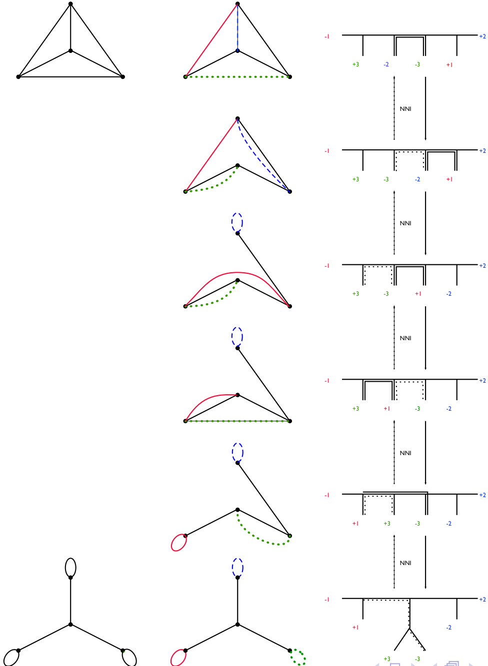


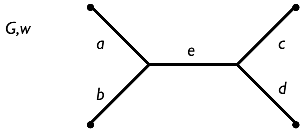


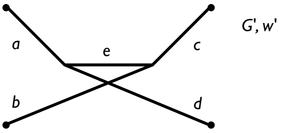
the graph G' resulting from the splitting of e into two edges, each connecting one of the ends of e to one of two new leaves.


NNI
-1
+2
+3
-2
-3
+1

Weighted NNIs

G : $\{1,3\}$-graph
e : edge between two degree-3 vertices of G





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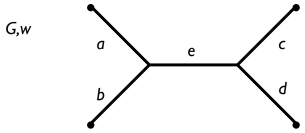

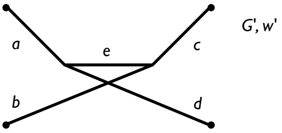

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Note that if w has integer values, so does w'.

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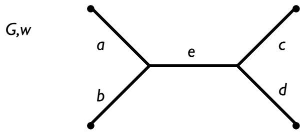

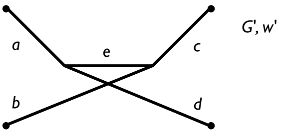

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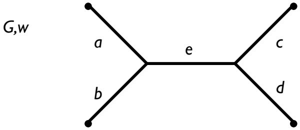

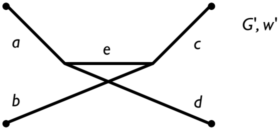

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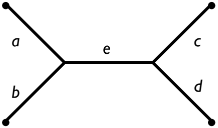
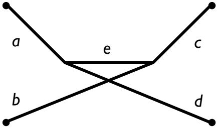
Thus $w'_e = w_e + w_b - w_d$.

Scketch of the proof of the lemma

G,w





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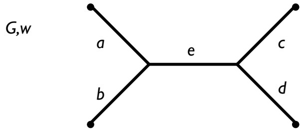

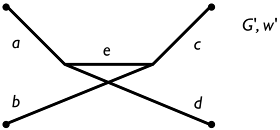

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For example, if $w_e + w_a + w_b \leq t$, then

$$\begin{aligned} w'_e + w_b + w_c &\leq w'_e + w_a + w_d \\ &= (w_e + w_b - w_d) + w_a + w_d \\ &= w_e + w_a + w_b \leq t. \end{aligned}$$

Scissors congruence

We think a weighted NNI as a function $\psi(G, w) : \mathbb{R}^m \to \mathbb{R}^m$

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Clearly $\psi(G,w)=(G',w')$ is piecewise linear, namely, for $w\in\mathbb{R}^m, w'_f = w_f$ for every $f\neq e$ and

$w'_e = w_e + w_b - w_d$ if $w_a + w_b \geq w_c + w_d$ and $w_a + w_d \geq w_b + w_c$,
$w'_e = w_e + w_a - w_c$ if $w_a + w_b \geq w_c + w_d$ and $w_a + w_d < w_b + w_c$,
$w'_e = w_e + w_c - w_a$ if $w_a + w_b < w_c + w_d$ and $w_a + w_d \geq w_b + w_c$,
$w'_e = w_e + w_d - w_b$ if $w_a + w_b < w_c + w_d$ and $w_a + w_d < w_b + w_c$,

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$$w'_e = w_e + w_b - w_d \text{ if } w_a + w_b \geq w_c + w_d \text{ and } w_a + w_d \geq w_b + w_c,$$
$$w'_e = w_e + w_a - w_c \text{ if } w_a + w_b \geq w_c + w_d \text{ and } w_a + w_d < w_b + w_c,$$
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$$w'_e = w_e + w_d - w_b \text{ if } w_a + w_b < w_c + w_d \text{ and } w_a + w_d < w_b + w_c,$$

We associate to ψ the hyperplanes $w_a + w_b - w_c - w_d = 0$ and $w_a - w_b - w_c + w_d = 0$, which are either the same hyperplane (if $a = b$ or $c = d$) or two orthogonal hyperplanes.

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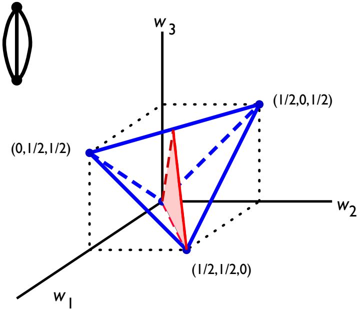
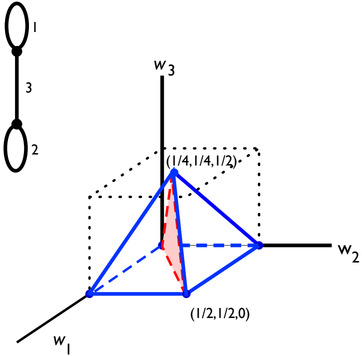
Clearly $\psi(G, w) = (G', w')$ is piecewise linear, namely, for $w \in \mathbb{R}^m, w'_f = w_f$ for every $f \neq e$ and

$$
\begin{aligned}
w'_e &= w_e + w_b - w_d \text{ if } w_a + w_b \geq w_c + w_d \text{ and } w_a + w_d \geq w_b + w_c,\\
w'_e &= w_e + w_a - w_c \text{ if } w_a + w_b \geq w_c + w_d \text{ and } w_a + w_d < w_b + w_c,\\
w'_e &= w_e + w_c - w_a \text{ if } w_a + w_b < w_c + w_d \text{ and } w_a + w_d \geq w_b + w_c,\\
w'_e &= w_e + w_d - w_b \text{ if } w_a + w_b < w_c + w_d \text{ and } w_a + w_d < w_b + w_c,
\end{aligned}
$$

We associate to ψ the hyperplanes $w_a + w_b - w_c - w_d = 0$ and $w_a - w_b - w_c + w_d = 0$, which are either the same hyperplane (if $a = b$ or $c = d$) or two orthogonal hyperplanes.
Moreover, the matrix that gives the linear transformation in each case is unimodular.

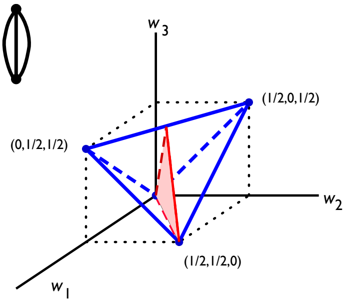
Scissors congruence

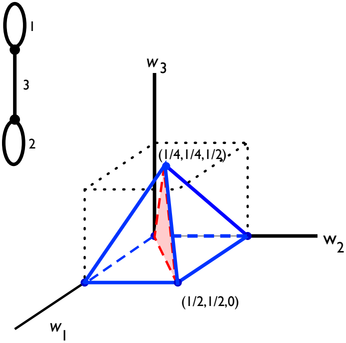




$$
\begin{aligned}
w_3' &= w_3 + \max\{w_1 + w_2, w_1 + w_2\} - \max\{2w_1, 2w_2\} \\
&= w_3 + w_1 + w_2 - 2\max\{w_1, w_2\} \\
&= \begin{cases} w_3 - w_1 + w_2 & \text{if } w_1 \geq w_2 \\ w_3 + w_1 - w_2 & \text{if } w_1 \leq w_2 \end{cases}
\end{aligned}
$$

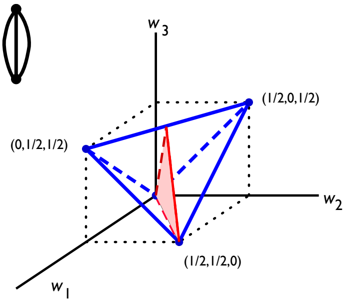
Scissors congruence

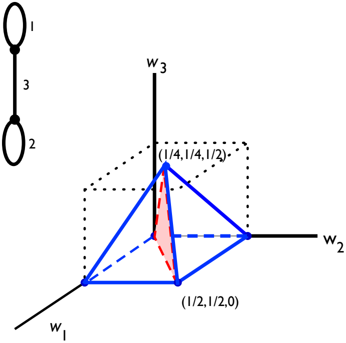




$$w_3' = \begin{cases} w_3 + w_1 - w_2 & \text{if } w_1 \le w_2 \\ w_3 - w_1 + w_2 & \text{if } w_1 \ge w_2 \end{cases}$$

Scissors congruence

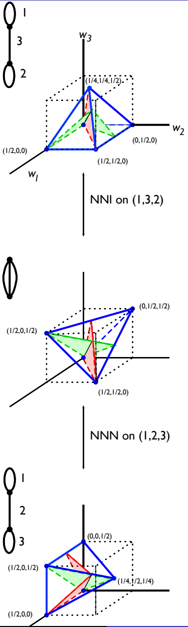




$$w_3' = \begin{cases} w_3 + w_1 - w_2 & \text{if } w_1 \le w_2 \\ w_3 - w_1 + w_2 & \text{if } w_1 \ge w_2 \end{cases}$$

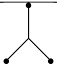
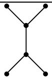
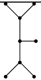
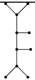
$$U_{w_1 \le w_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad U_{w_1 \ge w_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Scissors congruence



Tiavintío ! !
(Thanks in Mixteca)

Ehrhart polynomials

G	Ehrhart polynomial $\mathcal{P}_G$
	$\frac{1}{24}t^3 + \frac{1}{4}t^2 + \begin{cases} \frac{5}{6}t + 1, & \text{if } t \text{ is even} \\ \frac{11}{24}t + \frac{1}{4}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{1}{240}t^5 + \frac{1}{24}t^4 + \begin{cases} \frac{5}{24}t^3 + \frac{7}{12}t^2 + \frac{11}{10}t + 1, & \text{if } t \text{ is even} \\ \frac{1}{6}t^3 + \frac{1}{3}t^2 + \frac{79}{240}t + \frac{1}{8}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{17}{40320}t^7 + \frac{17}{2880}t^6 + \begin{cases} \frac{59}{1440}t^5 + \frac{25}{144}t^4 + \frac{179}{360}t^3 + \frac{173}{180}t^2 + \frac{93}{70}t + 1, & \text{if } t \text{ is even} \\ \frac{103}{2880}t^5 + \frac{35}{288}t^4 + \frac{1439}{5760}t^3 + \frac{893}{2880}t^2 + \frac{791}{3360}t + \frac{1}{16}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{31}{725760}t^9 + \frac{31}{40320}t^8 + \begin{cases} \frac{829}{120960}t^7 + \frac{37}{960}t^6 + \frac{653}{4320}t^5 + \frac{103}{240}t^4 + \frac{20413}{22680}t^3 + \frac{1723}{1260}t^2 + \frac{193}{126}t + \\ \frac{43}{6912}t^7 + \frac{19}{640}t^6 + \frac{3181}{34560}t^5 + \frac{123}{640}t^4 + \frac{39205}{145152}t^3 + \frac{9923}{40320}t^2 + \frac{379}{2880}t \end{cases}$