ON THE NUMBER OF VERTICES OF PROJECTIVE POLYTOPES

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Abstract. Let X be a set of n points in $\mathbb{R}^d$ in general position. What is the maximum number of vertices that $\mathsf{conv}(T(X))$ can have among all the possible permissible projective transformations T? In this paper, we investigate this and other related questions. After presenting several upper bounds, obtained by using oriented matroid machinery, we study a closely related problem (via Gale transforms) concerning the maximal number of minimal Radon partitions of a set of points. The latter led us to a result supporting a positive answer to a question of Pach and Szegedy asking whether *balanced* 2-colorings of points in the plane maximize the number of induced multicolored Radon partitions. We also discuss a related problem concerning the size of topes in arrangements of hyperplanes as well as a tolerance-type problem of finite sets.

1 Introduction

A *projective transformation* $T:\mathbb{R}^d\to\mathbb{R}^d$ is a function such that $T(x)=\frac{Ax+b}{\langle c,x\rangle+\delta}$, where A is a linear transformation of $\mathbb{R}^d$, $b,c\in\mathbb{R}^d$ and $\delta\in\mathbb{R}$, is such that at least one of $c\neq 0$ or $\delta\neq 0$. T is said to be *permissible* for a set $X\subset\mathbb{R}^d$ if $\langle c,x\rangle+\delta\neq 0$ for all $x\in X$. T is *nonsingular* if and only if the matrix $\begin{pmatrix} A & b^t \\ c & \delta \end{pmatrix}$ is nonsingular. We refer the reader to [20, Appendix 2.6] for a nice discussion on this notion.

Consider the following question

> Given a set of n points in general position $X\subset\mathbb{R}^d$, what is the maximum number of k-faces that $\mathsf{conv}(T(X))$ can have among all the possible permissible projective transformations T?

More precisely, let $d>k\geq 0$ be integers and let $X\subset\mathbb{R}^d$ be a set of points in general position, we define the number of *projective k–faces* of X as

$$h_k(X,d)=\max_T\{f_k(\mathsf{conv}(T(X)))\}, \tag{1}$$

where the maximum is taken over all possible permissible projective transformations T of X and $f_k(P)$ denotes the number of k-faces of a polytope P.

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We define now the function $H_k(n,d)$ which determines the *maximum* number of *projective k-faces* that *any* X configuration of n points in $\mathbb{R}^d$ must have as,

$$H_k(n,d) = \min_{X\subset\mathbb{R}^d, |X|=n} \{h_k(X,d)\}.$$

In this paper, we focus our attention on the behavior of $H_0(n,d)$ (the number of projective vertices). It turns out that $H_0(n,d)$ is the source of several applications.

1.1 Scope/general interest

The function $H_0(n,d)$ is closely connected with different notions/problems : McMullen's problem, bounds on $H_k(n,d)$, a connection of $H_{d-1}(n,d)$ with minimal Radon partitions and its relation with an open question due to Pach and Szegedy, tolerance-type problems of finite sets and topes in arrangements of hyperplanes.

1.1.1 McMullen's problem

$H_0(n,d)$ is a natural generalization of the following well-known problem of McMullen [10]:

What is the largest integer $\nu(d)$ such that any set of $\nu(d)$ points in general position, $X\subset\mathbb{R}^d$, can de mapped by a permissible projective transformation onto the vertices of a convex polytope?

The best known bounds on McMullen's problem are:

$$2d+1 \leq \nu(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil. \tag{2}$$

The lower bound was given by Larman [10] while the upper bound was provided by Ramírez Alfonsín [14]. In the same spirit, the following function has also been investigated:

> $\nu(d,k)$:= the largest integer n such that any set of n points in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a k–neighborly polytope.

Recall that a d-polytope P is *k-neighborly* if for a fixed natural $k \leq \lfloor \frac{d}{2} \rfloor$, every subset of at most k vertices form a face of P. As a consequence of [7, Lemma 9] and [7, Equation (1)] it can be obtained that

$$\nu(d,k) \geq d + \left\lfloor \frac{d}{k} \right\rfloor + 1 \tag{3}$$

This inequality will be useful later for our purposes.

Let $t \geq 0$ be an integer. We define the following function.

> $n(t,d)$:= the largest integer n such that any set of n points in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation onto the vertices of a convex polytope with at most t points in its interior.

The function $n(t,d)$ will allow us to study $H_0(n,d)$ in a more general setting, that of oriented matroids.

We notice that

$$n(0,d) = \nu(d) \text{ and } H_0(n(t,d),d) = n(t,d) - t. \tag{4}$$

Our first main contribution is the following

Theorem 1.1. *Let $d, t \geq 1$ and $n \geq 2$ be integers. Then,*

$$H_0(n,d) \begin{cases} \leq 7 & \text{if } d = 3,\ n \geq 7, \\ \leq n - 1 - t & \text{if } d \geq 4,\ n \geq 2d + t(d-2) + 2,\ t \geq 1. \end{cases}$$

Also, it will be proved that $H_0(n,2) = 5$ if $n \geq 5$ (Corollary 6.2) as well as $H_0(n,1) = 2$ if $n \geq 2$, $H_0(n,d) = n$ if $n \leq 2d+1$ and $H_0(n,d) < n$ if $n \geq 2d + \lceil \frac{d+1}{2} \rceil$ (Proposition 2.1).

We use $C_d(n)$ to denote the d-dimensional *cyclic polytope* with n vertices. Let $C_d(n)$ be the polytope obtained as the convex hull of n distinct points in the moment curve $x(t) := (t, t^2, \ldots, t^d)$. We use $P_d(n)$ to denote the d-dimensional *stacked polytope* with n vertices. Let $P_d(n)$ be the polytope formed from a simplex by repeatedly gluing another simplex onto one of its facets. We denote by $f_k(P)$ the number of k-faces of polytope P.

As a consequence, we will obtain the following.

Corollary 1.2. *Let $d, t \geq 1$, $n \geq 2$ and $1 \leq k \leq d-1$ be integers. Then,*

$$H_k(n,d) \begin{cases} = 5 & \text{if } d = 2,\ n \geq 5, \\ \leq f_k(C_3(7)) & \text{if } d = 3,\ n \geq 7, \\ \leq f_k(C_d(n)) & \text{if } d \geq 2,\ n \leq 2d+1, \\ \leq f_k(C_d(n-1)) & \text{if } d \geq 4,\ n \geq 2d + \lceil \frac{d+1}{2} \rceil, \\ \leq f_k(C_d(n-1-t)) & \text{if } d \geq 4,\ n \geq 2d + t(d-2) + 2,\ t \geq 1. \end{cases}$$

Moreover, $H_k(n,d) \geq f_k(P_d(n))$ for $n \leq 2d+1, d \geq 2$. In particular, $H_1(7,3) = 15$ and $H_2(7,3) = 10$.

1.1.2 Minimal Radon partitions

Let $X = \{x_1, \ldots, x_n\}, n \geq d+2$ be a set of points in general position in $\mathbb{R}^d$. We recall that $A, B \subset X$ is a *Radon partition* of X if $X = A \cup B$, $A \cap B = \emptyset$ and $\mathsf{conv}(A) \cap \mathsf{conv}(B) \neq \emptyset$.

It happens that $H_{d-1}(n,d)$ is very useful to count minimal Radon partitions. More specifically, let $X = A \cup B$ be any partition of X, we define $r_X(A,B)$ as the number of $(d+2)$-element subsets $S \subset X$ such that $\mathsf{conv}(A \cap S) \cap \mathsf{conv}(B \cap S) \neq \emptyset$, that is, as the number of *minimal (size) Radon partitions* induced by A and B.

We define the functions

$$r(X) := \max_{\{(A,B) | A \cup B = X\}} r_X(A,B) \quad \text{and} \quad r(n,d) := \min_{X \subset \mathbb{R}^d, |X| = n} r(X).$$

Our second main result establishes a connection of minimal Radon partitions with $H_{d-1}(n,d)$.

Theorem 1.3. *Let $d, n \geq 1$ be integers. Then,*

$$r(n,d) = H_{d'-1}(n,d') \text{ where } d' = n-d-2.$$

We shall prove this by using the duality between *Gale transforms* and projective transformations. Theorem 1.3 might be useful to study of a problem due to Pach and Szegedy [13].

1.1.3 Pach and Szegedy's question

In [13], Pach and Szegedy investigated the probability that a triangle induced by 3 randomly and independently selected points in the plane contains the origin in its interior. They remarked [13, last paragraph] that in order to generalize their arguments to 3-space the following problem should be solved.

Question 1.4. *Given n points in general position in the plane, coloured red and blue, maximize the number of multicoloured 4-tuples with the property that the convex hull of its red elements and the convex hull of its blue elements have at least one point in common. In particular, show that when the maximum is attained, the number of red and blue elements are roughly the same.*

This question may be studied in any dimension. However, if the dimension and the number of points are very similar then optimal partitions can be unbalanced. For example, one may consider $d+2$ points in $\mathbb{R}^d$ with one point contained in the simplex spanned by the remaining $d+1$ points. The optimal partition will have 1 red point and $d+1$ blue points and becomes arbitrarily unbalanced as d goes to infinity. Nevertheless, it is not clear whether for a large set of points with respect to the dimension it is also possible that very unbalanced partitions optimize the maximum number of induced Radon partitions.

On this direction, our third main result provides support for a positive answer of Question 1.4.

Theorem 1.5. *Let $X \subset \mathbb{R}^2$ be a set of points in general position with $|X| = n \geq 8$. Then, for any partition A, B of X such that $r_X(A,B) = r(X)$, we have that $|A|, |B| \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$.*

1.1.4 Tolerance

Let us consider the following function

> $\lambda(t,d)$:= the smallest number λ such that for any set X of λ points in $\mathbb{R}^d$ there exists a partition of X into two sets A,B and a subset $P \subseteq X$ of cardinality $\lambda - i$, for some $0 \leq i \leq t$, such that $\mathsf{conv}(A \setminus y) \cap \mathsf{conv}(B \setminus y) \neq \emptyset$ for every $y \in P$ and $\mathsf{conv}(A \setminus y) \cap \mathsf{conv}(B \setminus y) = \emptyset$ for every $y \in X \setminus P$.

Our fourth main result is an interesting relationship between $n(t,d)$ and $\lambda(t,d)$.

Theorem 1.6. *Let $t \geq 0$ and $d \geq 1$ be integers. Then,*

$$n(t,d) = \max_{m\in\mathbb{N}}\{m \mid \lambda(t, m-d-1) \leq m\}$$

and

$$\lambda(t,d) = \min_{m\in\mathbb{N}}\{m \mid m \leq n(t, m-d-1)\}.$$

This theorem can be considered as a generalization of a result due to Larman [10] obtained when $t = 0$.

The parameter $\lambda(t,d)$ can be thought of as a generalization of the *tolerant* Radon theorem stating that there is a minimal positive integer $N = N(t,d)$ so that any set $X \subset \mathbb{R}^d$ with $|X| = N$ allows a partition into two pairwise disjoint subsets $X = A \cup B$ such that after deleting any t points from X the convex hulls of remaining parts intersect, that is,

$$\mathsf{conv}(A \setminus Y) \cap \mathsf{conv}(B \setminus Y) \neq \emptyset \text{ for any } Y \subset X,\ |Y| = t.$$

The information on $\lambda(d,t)$ sheds light on the understanding of the tolerant Radon theorem as well as a more general version known as the *tolerant Tverberg theorem*; see [8, 19].

1.1.5 Arrangements of (pseudo)hyperplanes

A *projective* d-arrangement of n pseudo-hyperplanes $\mathcal{H}(d,n)$ is a finite collection of pseudo-hyperplanes in the projective space $\mathbb{P}^d$ such that no point belongs to every hyperplane of $\mathcal{H}(d,n)$. Any such arrangement, $\mathcal{H}$ decomposes $\mathbb{P}^d$ into a d–dimensional cell complex. A cell of dimension d is usually called a *tope* of the arrangement $\mathcal{H}$. The *size* of a tope is the number of pseudo-hyperplanes bordering it.

A classic research topic is to study the combinatorics of the topes in arrangements of hyperplanes. For instance, it is known [17, 18] that arrangements of n hyperplanes (that is, realizable oriented matroids) always admit n topes of size $d+1$ (a simplex). In [15], Richter proved that the number of simplices in an arrangement of $4k$ pseudo hyperplanes in $\mathbb{P}^3$ is at most $3k+1$ for $k \geq 2$. Finding a sharp lower bound for the number of simplices in the non-realizable case is an open problem for $d \geq 3$. Las Vergnas conjectured that in fact every arrangement of (pseudo) hyperplanes in $\mathbb{P}^d$ admits at least one simplex. In [16], Roudneff proved that the number of *complete topes* (a tope touching all the hyperplanes) of the *cyclic arrangement* on dimension d with n hyperplanes, is at least $\sum\limits_{i=0}^{d-2} \binom{n-1}{i}$ and conjectured [16, Conjecture 2.2] that for every d-arrangement of $n > 2d+1 > 5$ (pseudo)hyperplanes has at most this number of complete topes; see [12] for the proof of this conjecture for an infinite family of arrangements.

It happens that the function $H_0(n,d)$ is very helpful to investigate the size's behavior of topes in arrangements of (pseudo)hyperplanes. Let us consider the following questions :

> Are there simple arrangements of n (pseudo)hyperplanes in $\mathbb{P}^d$ in which every tope is of at most *certain size* ?
>
> Which arrangements of n (pseudo)hyperplanes in $\mathbb{P}^d$ contain a tope of at least *certain size* ?

We will partially answer these questions for small values of d.

1.2 Paper's organization

The structure of the paper is as follows. In the next section we give some straightforward values and bounds for both $H_0(n,d)$ and $H_{d-1}(n,d)$ (Propositions 2.1 and 2.2).

In Section 3, we discuss the treatment of the function $n(t,d)$ in the oriented matroid setting. We also recall several notions and results on oriented matroids and, specifically, on the special class of *Lawrence oriented matroids* (LOM) that are needed for the rest of the paper.

In Section 4, we present several upper bounds based on specific constructions of LOM (Theorems 4.2, 4.4, 4.8). The latter yield the proofs of Theorem 1.1 and Corollary 1.2 also presented in this section.

After recalling the relationship between Gale transforms and projective transformations, we prove Theorem 1.3 in Section 5. We also present values and bounds for $r(n,2)$ (Theorem 5.4) that we use to prove Theorem 1.5 and present our Tolerant result (Theorem 1.6) at the end of this section.

Finally, in Section 6, we present some results concerning the size of *topes* in arrangements of (pseudo) hyperplanes.

2 Some basic results

It is known [9] that the number of k-faces of the d-dimensional cyclic polytope with n vertices $C_d(n)$ for $d \geq 2$ and $0 \leq k \leq d-1$ is given by

$$f_k(C_d(n)) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \binom{j}{d-k-1}\binom{n-d+j-1}{j} + \sum_{j=\lfloor \frac{d}{2} \rfloor+1}^{d} \binom{j}{d-k-1}\binom{n-j-1}{d-j} \tag{5}$$

The well-known Upper Bound Theorem (UBT) [20, Theorem 8.23] [11] states that for any d-polytope P with $n = f_0$ vertices, for every $1 \leq k \leq n$,

$$f_{k-1}(P) \leq f_{k-1}(C_d(n)).$$

Here equality for some k with $\lfloor \frac{d}{2} \rfloor \leq k \leq d$ implies that P is neighborly.

We notice that (UBT) implies that the number of faces of an arbitrary polytope can never be more than the number of faces of a cyclic (or *neighborly*) polytope with the same dimension and number of vertices.

Since $H_0(n,d)$ is the maximal number of projective vertices obtained from any set of n points in $\mathbb{R}^d$ then, by the UBT, the number of k-faces of a projective polytope on $H_0(n,d)$ vertices is bounded by the number of k-faces of $C_d(H_0(n,d))$. We thus have that

$$H_k(n,d) \leq f_k(C_d(H_0(n,d))) \text{ for all } n \geq 1. \tag{6}$$

Analogously to the (UBT), the Lower Bound Theorem [1, 2] states that for all $1 \leq k \leq d-1$,

$$f_k(P_d(n)) \leq f_k(P)$$

among all simplicial (convex) polytopes $P \subset \mathbb{R}^d$ with n vertices (recall that $P_d(n)$ denotes the d-dimensional stacked polytope with n vertices).

For $d \geq 2$ and $0 \leq k \leq d-1$, the number of k-faces of $P_d(n)$ with n vertices is

$$f_k(P_d(n)) = \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{if } 0 \leq k \leq d-2, \\ (d-1)n - (d+1)(d-2) & \text{if } k = d-1. \end{cases} \tag{7}$$

As we will explain in Section 3, we know that if X is a set of points in general position then $T(X)$ is also in general position. Therefore, $\mathsf{conv}(T(X))$ gives a simplicial polytope. We may thus deduce that

$$f_k(P_d(H_0(n,d))) \leq H_k(n,d). \tag{8}$$

Proposition 2.1. *Let $d \geq 2, n \geq 1$ be integers. Then,*

$$H_0(n,d) \begin{cases} = 2 & \text{if } d = 1, n \geq 2, \\ = n & \text{if } n \leq 2d+1, \\ < n & \text{if } n \geq 2d + \lceil \frac{d+1}{2} \rceil. \end{cases}$$

Proof. • If $d = 1, n \geq 2$, then we clearly have that $H_0(n,1) = 2$ since every convex set in dimension 1 has as support only two vertices.

• If $n \leq 2d+1$, then by the lower bound of $\nu(d)$ given in (2), it follows that any set of points of cardinality n can be mapped to the vertices of a convex polytope by a permissible projective transformations, and thus $H_0(n,d) = n$.

• If $n \geq 2d + \lceil \frac{d+1}{2} \rceil$, then by the upper bound of $\nu(d)$ given in (2), there exists a set of n points that cannot be mapped to the vertices of a convex polytope by any permissible projective transformation, and thus $H_0(n,d) \leq n-1$.

□

Combining Proposition 2.1 and (6), we have the following easy consequence.

Proposition 2.2. *Let $d \geq 2, n \geq 1$ and $1 \leq k \leq d-1$ be integers. Then,*

$$H_k(n,d) \begin{cases} \leq f_k(C_d(n)) & \text{if } n \leq 2d+1, \\ \leq f_k(C_d(n-1)) & \text{if } n \geq 2d + \lceil \frac{d+1}{2} \rceil. \end{cases}$$

Moreover, if $n \leq 2d+1, d \geq 2$ then, by Proposition 2.1, $H_0(n,d) = n$ and so $f_k(P_d(H_0(n,d))) = f_k(P_d(n))$. By (8), we obtain

$$H_k(n,d) \geq f_k(P_d(n)). \tag{9}$$

3 Oriented matroid setting

Let us briefly give some basic notions and definitions on oriented matroid theory needed for the rest of the paper. We refer the reader to [3] for background on oriented matroid theory.

3.1 Oriented matroid preliminaries

Let E be a finite set of $\mathbb{R}^d$. We can naturally associate to E two oriented matroids: the *linear* matroid (of rank d), denoted by $\mathrm{Lin}(E)$, arising from the linear dependencies of E in $\mathbb{R}$ and the *affine* matroid (of rank $d+1$), denoted by $\mathrm{Aff}(E)$, arising from the affine dependencies of E in $\mathbb{R}$.

Let M be an oriented matroid on a finite set E. The matroid M is *acyclic* if it does not contain positive circuits (otherwise, M is called *cyclic*). A *reorientation* of M on $A \subseteq E$ is performed by changing the signs of the elements in A in all the circuits of M. It is easy to check that the new set of signed circuits is also the set of circuits of an oriented matroid, usually denoted by $_{-A}M$. A reorientation is *acyclic* if $_{-A}M$ is acyclic. An element $e \in E$ of an acyclic oriented matroid is *interior* if there exists a signed circuit $C = (C^+, C^-)$ with $C^- = \{e\}$.

Cordovil and Da Silva [4] proved that a permissible projective transformation on a set n points in $\mathbb{R}^d$ corresponds to an acyclic reorientation of its oriented matroid of affine dependencies M of rank $r = d+1$ and that the converse also holds. Indeed, in [4] was checked that for any permissible transformation $T(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$ an acyclic reorientation can be obtained by changing signs to all x of a given set such that $\langle c,x\rangle + \delta > 0$. Moreover, it can be checked that T preserves linear dependency, in other words, T gives an isomorphism of the unoriented matroids $\mathrm{Aff}(E)$ and $\mathrm{Aff}(T(E))$. We thus have that T sends points in general position to points in general position.

Recall that an oriented matroid on n elements of rank r is *uniform* if its set of bases consists of all r-element subsets of $[n]$. As a consequence of Cordovil and Da Silva's result it is evident that the natural generalization of $n(t,d)$ in terms of oriented matroids is given by the following function:

> $\bar{n}(t,d) :=$ the largest integer m such that for any uniform oriented matroid M of rank $d+1$ with m elements there is an acyclic reorientation of M with at most t interior elements.

We notice that if in the definition of $\bar{n}(t,d)$ we insist that M is not only uniform but also affine then $\bar{n}(t,d)$ coincide with $n(t,d)$. Indeed, the affinity translate to points in the space (instead of elements of the matroid) and the uniformity implies that the points are in general position.

In this section we shall provide examples of uniform oriented matroids with the property that in any of their acyclic reorientations there are at least $t+1$ interior elements. These examples provide upper bounds for $\bar{n}(t,d)$. With this aim, we will briefly outline some facts about *Lawrence oriented matroids.* For further details and proofs on this special class of matroids we refer the reader to [3, Section 7.6], [14] and also to [6] where arrangements reconstructions are investigated.

3.2 Lawrence oriented matroid

A *Lawrence oriented matroid* (LOM) M of rank r on the totally ordered set $E = \{1,\dots,n\}$, $r \leq n$, is a uniform oriented matroid obtained as the union of r uniform oriented matroids $M_1,\dots,M_r$ of rank 1 on $(E,<)$. LOMs can also be defined via the signature of their bases, that is, via their chirotope χ. Indeed, the chirotope χ corresponds to some LOM, M_A, if and only if there exists a matrix $A = (a_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n$ with entries from $\{+1,-1\}$ (where the i-th row corresponds to the chirotope of the oriented matroid $\mathcal{M}_i$) such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i}$$

where B is an ordered r-tuple, $j_1 < \cdots < j_r$, of elements of E.

Proposition 3.1. *(a) Acyclic LOMs are realizable as configurations of points, that is, they are affine matroids (since they are unions of realizable oriented matroids)* [3, Proposition 8.2.7].

(b) The LOM corresponding to the reorientation of an element $c \in E$, ${}_{\bar{c}}M_A$ is obtained by reversing the sign of all the coefficients of a column c in A.

Some of the following definitions and lemmas, which highlight the properties of A and facilitate the study of this type of matroid, were introduced and proved in [14].

A *Top Travel*, denoted by TT, in A is a subset of the entries of A,

$$\{[a_{1,1},a_{1,2},\dots,a_{1,j_1}],[a_{2,j_1},a_{2,j_1+1},\dots,a_{2,j_2}],\dots,[a_{s,j_{s-1}},a_{s,j_{s-1}+1},\dots,a_{s,j_s}]\},$$

with the following characteristics.

(1) $a_{i,j_{i-1}} \times a_{i,j} = 1$, for all $j_{i-1} \leq j < j_i$, $1 \leq i \leq s$, where we define $j_0 = 1$;

(2) $a_{i,j_i-1} \times a_{i,j_i} = -1$, for all $1 \leq i \leq s-1$ and either

- (a) $j_{s-1} = n$; then $[a_{s,j_{s-1}},a_{s,j_{s-1}+1},\dots,a_{s,j_s}] = [a_{s,j_s}]$ or
- (b) $j_{s-1} < n$ and $1 \leq s < r$; then $j_s = n$ and $a_{s,j_s-1} \times a_{s,j_s} = 1$ or
- (c) $j_{s-1} < n$, $s = r$ and $j_s < n$; then $a_{s,j_s-1} \times a_{s,j_s} = -1$ or
- (d) $j_{s-1} < n$, $s = r$ and $j_s = n$; then $a_{s,j_s-1} \times a_{s,j_s} = 1$ or -1.

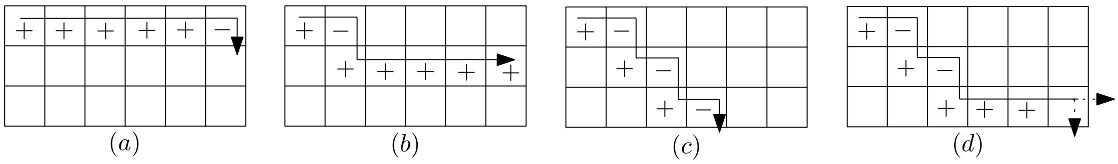


FIGURE 1. The TT of (a), (b), (c) and (d) satisfies the situations 2(a), 2(b), 2(c) and 2(d), respectively, of the definition of the Top Travel.

Figure 1 shows some examples of matrices with their corresponding TT. For instance, the TT of example (c) have $3 = j_{s-1} < n = 6$, $s = r = 3$ and $4 = j_s < n = 6$, then we notice that it happen the situation (2)(c) of the definition of the Top Travel.

A *Bottom Travel*, denoted by BT, in A is a subset of the entries of A,

$\{[a_{r,n}, a_{r,n-1}, \ldots, a_{r,j_r}], [a_{r-1,j_r}, a_{r-1,j_r-1}, \ldots, a_{r-1,j_{r-1}}], \ldots, [a_{s,j_{s+1}}, a_{s,j_{s+1}-1}, \ldots, a_{s,j_s}]\}$,

with the following characteristics.

(1) $a_{i,j_{i+1}} \times a_{i,j} = 1$, for all $j_i < j \leq j_{i+1}$, $s \leq i \leq r$, where we define $j_{r+1} = n$;

(2) $a_{i,j_i+1} \times a_{i,j_i} = -1$, for all $s-1 \leq i \leq r$ and either

- (a) $j_{s+1} = 1$; then $[a_{s,j_{s+1}}, a_{s,j_{s+1}-1}, \ldots, a_{s,j_s}] = [a_{s,j_s}]$ or
- (b) $j_{s+1} > 1$ and $1 < s \leq r$; then $j_s = 1$ and $a_{s,j_s+1} \times a_{s,j_s} = 1$ or
- (c) $j_{s+1} > 1$, $s = 1$ and $1 < j_s$; then $a_{s,j_s+1} \times a_{s,j_s} = -1$ or
- (d) $j_{s+1} > 1$, $s = 1$ and $1 = j_s$; then $a_{s,j_s+1} \times a_{s,j_s} = 1$ or -1.

Travel TT (resp. BT) may be thought of as a travel starting at $a_{1,1}$ (resp. at $a_{r,n}$) making horizontal movements to the right (resp. to the left) and vertical movements to the bottom (resp. to the top) of A according with the above constructions.

Any matrix A admits exactly one TT and one BT. These travels carry surprising information about M_A.

Lemma 3.2. [14, Lemma 2.1] *Let A be a $(r \times n)$-matrix. Then, the following statements are equivalent:*

(a) M_A is cyclic*;*

(b) TT ends at $a_{r,s}$ for some $1 \leq s < n$ or $a_{r,n-1}, a_{r,n} \in TT$ and $a_{r,n-1} \times a_{r,n} = -1$;

(c) BT ends at $a_{1,s'}$ for some $1 < s' \leq n$ or $a_{1,1}, a_{1,2} \in BT$ and $a_{1,1} \times a_{1,2} = -1$.

Remark 3.3. *If M_A is acyclic then TT never goes below BT (by construction of TT and BT and Lemma 3.2).*

Let $a_{i,k-1}, a_{i,k}, a_{i,k+1} \in TT$, we say that TT and BT are *parallel* at column k if either $a_{i,k-1}, a_{i,k}, a_{i,k+1} \in BT$ or $a_{i+1,k-1}, a_{i+1,k}, a_{i+1,k+1} \in BT$, with $2 \leq k \leq n-1$, $1 \leq i \leq r$.

Lemma 3.4. [14, Lemma 2.2] *Let A be a $(r \times n)$-matrix. Then, k is an interior element of M_A if and only if* *(a)* $BT = (a_{r,n}, \ldots, a_{1,2}, a_{1,1})$ *for* $k = 1$,

(b) $TT = (a_{1,1}, \ldots, a_{r,n-1}, a_{r,n})$ *for* $k = n$,

(c) TT *and* BT *are parallel at* k *for* $2 \leq k \leq n-1$.

Lemma 3.4 implies that we can identify acyclic reorientations and interior elements of an oriented matroid M_A by studying the behaviour of the TT and BT in the re-orientations of A.

Example 3.5. Let M_A be the LOM associated to the matrix A described in Figure 2. M_A is acyclic, and 4, 5 and 6 are interior elements.

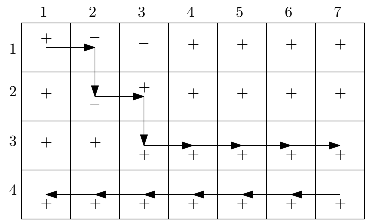


Figure 2. Top and Bottom travels in matrix A.

Furthermore, all possible re-orientations of the matroid can be identified with yet another simple object.

A *Plain Travel* in A, denoted by PT, is a subset of the entries of A of the form

$$PT = \{[a_{1,1}, a_{1,2}, \ldots, a_{1,j_1}], [a_{2,j_1}, a_{2,j_1+1}, \ldots, a_{2,j_2}], \ldots, [a_{s,j_{s-1}}, a_{s,j_{s-1}+1}, \ldots, a_{s,j_s}]\}$$

with $2 \leq j_{i-1} < j_i \leq n$ for all $1 \leq i \leq r$, $1 < s \leq r$ and $j_s = n$.

Lemma 3.6. [14, Lemma 3.1] *There is a bijection between the set of all plain travels of* A *and the set of all acyclic reorientations of* M_A*, it is defined by associating to each* PT *the set of column indices of* A *that should be reoriented in order to transform* A *into a new matrix* $\mathcal{A}$ *whose* TT *is identical to* PT.

The *chessboard* $B[A]$ of A is another useful object that can be constructed from the entries of A. It is defined by a black and white board of size $(r-1) \times (n-1)$, such that the square $s(i,j)$ has its upper left hand corner at the intersection of row i and column j in the matrix A; a square $s(i,j)$, with $1 \leq i \leq r-1$ and $1 \leq j \leq n-1$ will be said to be *black* if the product of the entries in A, $a_{i,j}, a_{i,j+1}, a_{i+1,j}, a_{i+1,j+1}$ is -1, and *white* otherwise. Figure 3 illustrate an example.

A plain travel of A may be considered as a travel going along the corners of the squares in the chessboard $B[A]$ (see Figure 3). The square-coloring of $B[A]$ will be helpful to determine the behavior of the movements of a plain travel in A. The following result was obtained in [14].

Proposition 3.7. *Let* $B[A]$ *be a board of size* $(r-1) \times (n-1)$ *and suppose that* TT *is a plain travel in* A.

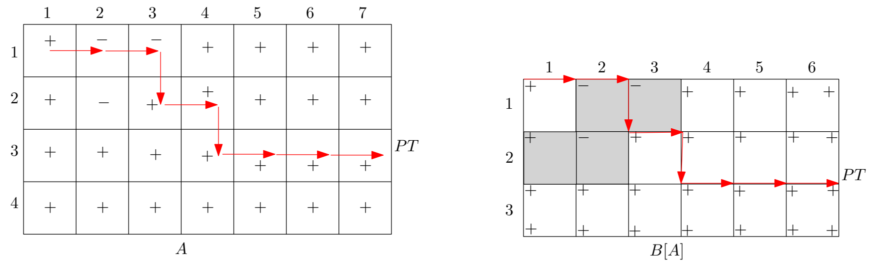


FIGURE 3. A matrix A and its chessboard $B[A]$ with the same plain travel PT.

(a) The re-orientation of any set of elements do not change the chessboard. Therefore, if A_1 and A_2 are matrices such that M_{A_1} and M_{A_2} are in the same orientation class then $B[A_1] = B[A_2]$.

(b) Suppose that $a_{i,j}$, $a_{i,j+1} \in TT$ with $a_{i,j} \times a_{i,j+1} = 1$, $1 \le i \le r-1$ and $1 \le j \le n-1$, and that $a_{k,j}, a_{k,j+1} \in BT$ for some $i < k \le r$.

- *If $s(l,j)$ is white for all $i \le l \le k-1$, then $a_{k,j-1} \in BT$.*
- *If $s(l,j)$ is black for some $i \le l \le k-1$ and $s(m,j)$ is white for all $i \le m \le k-1$, $m \ne l$, then $a_{k-1,j-1}, a_{k-1,j} \in BT$.*

(c) Suppose that $a_{i,j}$, $a_{i,j+1} \in TT$ with $a_{i,j} \times a_{i,j+1} = -1$, $1 \le i \le r-1$ and $1 \le j \le n-1$, and that $a_{k,j}, a_{k,j+1} \in BT$ for some $i < k \le r$.

- *If $s(l,j)$ is white for all $i \le l \le k-1$ then $a_{k-1,j-1}, a_{k-1,j} \in BT$.*
- *If $s(l,j)$ is black for some $i \le l \le k-1$ and $s(m,j)$ is white for all $i \le m \le k-1$, $m \ne l$ then $a_{k,j-1} \in BT$.*

We notice that (a) can be easily checked since a re-orientation of an element translates to swap the signs of all entries of the corresponding column in A, say c. The latter do not change the color of any square (the parity of a square with two corners in column c is invariant under the swapping). Assertions (b) and (c) can be deduced by a simple parity argument; see Figures 4 and 5.

4 Upper bounds for $\bar{n}(t, d)$

Our general strategy to bound $\bar{n}(t, d)$ is as follows. We construct special LOMs with the property that in any of their acyclic reorientations there are at least $t + 1$ interior elements. As LOMs are realizable, such a construction provides us with a configuration of the desired point set in $\mathbb{R}^d$. In its turn, to construct the desired LOMs, we will construct their chessboard first. The existence of LOMs with given chessboards is straightforward and it is omitted here. We will then use Lemma 3.6. Namely, we will consider such a re-orientation of the matrix of LOM that a given Plain Travel is the Top Travel (according to Lemma 3.6, it is equivalent to considering all acyclic re-orientations). Then, by using Proposition 3.7 and the properties of TT and BT, we will estimate the number of interior elements.

4.1 Small dimension d

We first show that $\bar{n}(t,d) \leq 2d+1+t$ for $d = 2,3$ and every $t \geq 0$. The following simple result will be very useful throughout this section.

Claim 4.1. *Any acyclic reorientation of a rank 2 oriented matroid on n elements has $n-2$ interior elements.*

Given a matrix $A = A_{r,n}$ and its Top and Bottom Travels, TT and BT, respectively, we say that column j is *flat* if the following two properties hold:

- TT and BT do not make a vertical movement at the j-th column;
- j is not an interior element of A.

We denote by $A^{+}_{i,j}$ the sub-matrix of A obtained by deleting rows $i+1,\dots,r$ and columns $j+1,\dots,n$. Similarly, we denote by $A^{-}_{i,j}$ the sub-matrix of A obtained by deleting rows $1,\dots,i-1$ and columns $1,\dots,j-1$.

Theorem 4.2. $\bar{n}(t,2) < t+6$ *for every integer* $t \geq 0$.

Proof. Consider a chessboard of size $2 \times (t+5)$ containing exactly one black square in each column, a matrix $A = A_{3,t+6}$ having this chessboard and consider an acyclic reorientation $\mathcal{A}$ of the matrix A. We will show that $\mathcal{A}$ has at least $t+1$ interior elements. Let TT and BT be the Top and Bottom travels of $\mathcal{A}$, respectively.

Since both TT and BT make at most 2 vertical movements, there are at least $t+2$ columns in which they do not make vertical movements. There is nothing to prove if all of these columns are interior in $\mathcal{A}$, thus, we may assume that for some integer j, column j is flat. We claim that we can find j such that $1 < j < t+6$. If $j = 1$ then BT does not arrive at the first row at column one, otherwise, by Lemma 3.4 (a), column one would be an interior element of $\mathcal{A}$. Then, BT makes at most one vertical movement, and thus obtaining that there are at least $t+3$ columns in which TT and BT do not make vertical movements. Hence, we can assume that there are at least 3 columns that are flat (otherwise, we would have at least $t+1$ interior elements and the result holds). So, we may choose one of these columns different from columns 1 and $t+6$. The case $j = t+6$ can be treated in a similar fashion.

We thus may suppose that $1 < j < t+6$. By Remark 3.3, TT never goes below BT, then we have that TT and BT arrive at the j-th column at the first and third rows, respectively, otherwise, by Lemma 3.4 (c), j would be an interior element in $\mathcal{A}$. Then, it can be deduced from Proposition 3.7 (b) and (c) that TT and BT make a vertical movement at columns $j+1$ and $j-1$, respectively (see Figure 4). So, the Top and Bottom Travels of the matrices $\mathcal{A}^{+}_{2,j-1}$ and $\mathcal{A}^{-}_{2,j+1}$ coincides with the corresponding parts of TT and BT of $\mathcal{A}$ and thus, each interior element of $\mathcal{A}^{+}_{2,j-1}$ and $\mathcal{A}^{-}_{2,j+1}$ is an interior element of $\mathcal{A}$. As the matrices $\mathcal{A}^{+}_{2,j-1}$ and $\mathcal{A}^{-}_{2,j+1}$, of sizes $2 \times (j-1)$ and $2 \times (t+6-j)$, have $\max\{j-3,0\}$ and $\max\{t+4-j,0\}$ interior elements, respectively, by Claim 4.1, we conclude that $\mathcal{A}$ has at least $t+1$ interior elements and the result follows. $\square$

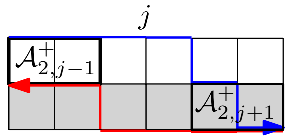


FIGURE 4. A chessboard considered in Theorem 4.2, for $t = 1$ and $j = 4$ and the matrices $\mathcal{A}^+_{2,j-1}$ and $\mathcal{A}^-_{2,j+1}$ with interior elements at columns 2 and 7.

Given a matrix $A = A_{r,n}$, we say that the chess board $B[A]$ has *sequence* $(x_1, x_2, \ldots, x_{r-1})$ if the square $s(i,j)$ is black if and only if $\sum_{k=0}^{i-1} x_k + 1 \le j \le \sum_{k=0}^{i} x_k$ with $1 \le i \le r-1$ and $1 \le j \le n-1$ and taking $x_0 = 0$.

Example 4.3. Figure 6 illustrates a chessboard with sequence $(2, 3, 2, 3)$.

Theorem 4.4. $\bar{n}(t, 3) < t + 8$ *for every integer* $t \ge 0$.

Proof. Consider a chessboard of size $3 \times (t+7)$ with sequence $(2, t+3, 2)$, a matrix $A = A_{4,t+8}$ with this chessboard and consider an acyclic re-orientation $\mathcal{A}$ of the matrix A. We will show that $\mathcal{A}$ has at least $t+1$ interior elements. Let TT and BT be the Top and Bottom travels of $\mathcal{A}$, respectively. Since both TT and BT make at most 3 vertical movements, there are at least $t+2$ columns in which they do not make vertical movements. There is nothing to prove if all of these columns are interior in $\mathcal{A}$, thus, we may assume that for some integer j, column j is flat. We claim that we can find j such that $1 < j < t+8$. If $j = 1$, we observe that BT does not arrive at the first row at column one, otherwise $\mathcal{A}$ would have an interior element at column one by Lemma 3.4 (a). Then, BT makes at most two vertical movement in this case, obtaining that there are at least $t+3$ columns in which TT and BT do not make vertical movements. Hence, we can assume that there are at least 3 columns that are flat, otherwise the theorem holds. So, we may choose some of these columns different from columns 1 and $t+8$. The case $j = t+8$ can be treated in a similar fashion.

Thus, we may consider $1 < j < t+8$. By Remark 3.3, TT never goes below BT. We thus have that either TT arrives at the j-th column at the first row or BT arrives at the j-th column at the fourth row, otherwise j would be an interior element in $\mathcal{A}$ by Lemma 3.4 (c). We may suppose that TT arrives at the j-th column at the first row since the other case can be treated analogously. Therefore, BT arrives at the j-th column at rows 3 or 4.

Case 1. BT arrives at the j-th column at row 4.

As $a_{1,j-2}, a_{1,j-1}, a_{1,j}, a_{1,j+1} \in TT$ and $a_{4,j-1}, a_{4,j}, a_{4,j+1}, a_{4,j+2} \in BT$, it can be deduced from Proposition 3.7 (b) and (c) that TT makes vertical movements at columns $j+1$ and $j+2$ and that BT makes vertical movements at columns $j-1$ and $j-2$, concluding that $a_{3,j+2} \in TT$ if $j = 2, \ldots, t+6$ and $a_{2,j-2} \in BT$ if $j = 3, \ldots, t+7$ (see Figure 5 (a)). Moreover, we notice that the Top and Bottom Travels of the matrices $\mathcal{A}^+_{2,j-2}$ and $\mathcal{A}^-_{3,j+2}$ coincides with the corresponding parts of TT and BT of $\mathcal{A}$ and thus, each interior element of $\mathcal{A}^+_{2,j-2}$ and $\mathcal{A}^-_{3,j+2}$ is an interior element of $\mathcal{A}$. As the matrices $\mathcal{A}^+_{2,j-2}$ and $\mathcal{A}^-_{3,j+2}$, of sizes $2\times(j-2)$ and $2\times(t+7-j)$, have $\max\{j-4,0\}$ and $\max\{t+5-j,0\}$ interior elements, respectively, by Claim 4.1, we obtain that $\mathcal{A}$ has at least $t+1$ interior elements, concluding the proof of Case 1.

Case 2. BT arrives at the j-th column at row 3.

As $a_{1,j-1}, a_{1,j}, a_{1,j+1} \in TT$, it can be checked, by using Proposition 3.7 (b), that BT makes a vertical movement at column $j-1$, obtaining that $a_{2,j-1} \in BT$ if $j = 2,\ldots,t+7$. On the other hand, it can be deduced, by Proposition 3.7 (b) and (c), that TT makes two vertical movements from column $j+1$ to column $j+3$ (see Figure 5 (b), (c) and (d)). Then, we conclude that $a_{3,j+3} \in TT$ if $j = 2,\ldots,t+5$ and hence, we notice that each interior element of the matrices $\mathcal{A}^+_{2,j-1}$ and $\mathcal{A}^-_{3,j+3}$ is an interior element of $\mathcal{A}$. As $\mathcal{A}^+_{2,j-1}$ and $\mathcal{A}^-_{3,j+3}$, of sizes $2\times(j-1)$ and $2\times(t+6-j)$, have $\max\{j-3,0\}$ and $\max\{t+4-j,0\}$ interior elements, respectively, by Claim 4.1, we obtain that $\mathcal{A}$ has at least $t+1$ interior elements, concluding the proof. $\square$

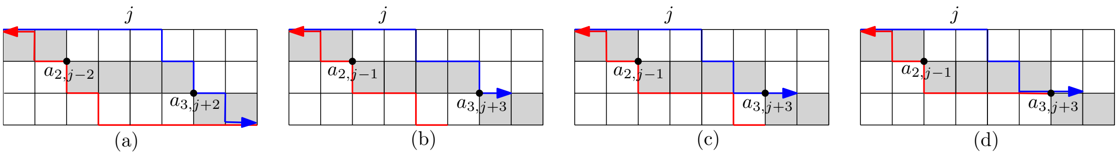


FIGURE 5. The chessboard with sequence $(2,t+3,2)$ considered in Theorem 4.4, for $t=1$. Figure (a) shows the case when BT arrives at the j-th column at row 4, for $j=5$, and figures (b), (c) and (d) consider the cases when BT arrives at the j-th column at row 3, for $j=4$.

4.2 High dimension d

Next, we will consider different matrices $A = A_{r,h(r)}$ subjected to function $h(r)$. If $B[A]$ has sequence $(x_1, x_2, \ldots, x_{r-1})$, we will consider functions $h(r)$, where $h(1)=1$, $\left(\sum_{k=1}^{m-1} x_k\right)+1 \le h(m) \le \left(\sum_{k=1}^{m} x_k\right)+1$ if $2 \le m \le r-1$ and $\left(\sum_{k=1}^{r-1} x_k\right)+1 \le h(r)$. For every $1 \le m \le r-1$, we will say that the element $a_{m,h(m)}$ is the *m–th corner* of A.

Example 4.5. Figure 6 illustrates a chessboard with $h(r) = 2(r-1)+\lceil \frac{r}{2} \rceil$.

For every $r \ge 3$, let define the function $f(r) = 2r - h(r-1) + h(2) - 3$. The following lemmas will be helpful ingredients for our purposes.

Lemma 4.6. *Consider a matrix $A = A_{r,h(r)}$ with $r \ge 3$ and suppose that TT passes strictly above the m-th corner for every $2 \le m \le r-1$. Then, the following holds.*

(i) *Suppose that $f(r) \ge 3$ and that BT passes strictly below the m-th corner for every $2 \le m \le r-1$. Then, $a_{i,h(2)} \in BT$ for some $i \le f(r)$.*

(ii) *Suppose that $f(r) \le 2$. Then, it cannot happen that BT passes strictly below the m-th corner for every $2 \le m \le r-1$.*

Proof. Suppose that BT passes strictly below the m-th corner for every $2 \leq m \leq r-1$. Then, as TT passes strictly above the m-th corner for every $2 \leq m \leq r-1$ we notice that TT and BT do not share steps from columns $h(2)$ to $h(r-1)$. On the other hand, starting with the $h(r)$ column from right to left, the TT makes exactly $h(r) - h(2)$ horizontal movements and at most $r-1$ vertical movements to arrive at $a_{1,h(2)}$. We know by the rules of construction of TT that for each vertical movement we must also count one horizontal movement, so, starting with the $h(r)$ column, the TT makes at least $h(r) - h(2) - (r-1)$ single horizontal movements until $a_{1,h(2)}$ is attained, where a single horizontal movement of TT is an horizontal movement, from right to left, such that its consecutive movement is not vertical. As TT and BT could share at most $h(r) - h(r-1) - 2$ steps from columns $h(r-1)+1$ to $h(r)$ (see Figure 6), for each single horizontal movement that TT does not share with BT, the BT makes a vertical movement by Proposition 3.7 (b). So, we conclude that starting with the $h(r)$ column, the BT makes at least $h(r)-h(2)-(r-1)-(h(r)-h(r-1)-2) = h(r-1)-h(2)-r+3$ vertical movements until column $h(2)$ is attained. Hence, BT arrives at $a_{i,h(2)}$ for some $i \leq r - (h(r-1) - h(2) - r + 3) = f(r)$, concluding the proof of assertion (i) of the lemma whenever $f(r) \geq 3$. If $f(r) \leq 2$, as we have concluded that $a_{i,h(2)} \in BT$ for some $i \leq f(r) \leq 2$, we obtain that BT arrive or passes above the 2-th corner, arriving a contradiction since we have assumed that BT passes strictly below the 2-th corner. Therefore, it cannot happen that BT passes strictly below the m-th corner for every $2 \leq m \leq r-1$, whenever $f(r) \leq 2$ concluding the proof of assertion (ii) of the lemma and so, the lemma holds. □

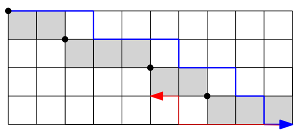


FIGURE 6. A matrix $A = A_{5,h(5)}$ and its chessboard with sequence $(2,3,2,3)$, where $h(r) = 2(r-1) + \lceil \frac{r}{2} \rceil$. The points represent its corners.

In what follows, we will consider the following matrices with its corresponding corners.

(M$_1$) Let $A_1 = A_{r,h(r)}$ be such that $B[A_1]$ has sequence $(x_1, x_2, \ldots, x_{r-1})$, with $x_1 = 2$, $x_i \geq 2$ for odd i, $x_j \geq 3$ for even j and $h(m) = \sum\limits_{k=0}^{m-1} x_k + 1$ for every $1 \leq m \leq r$.

(M$_2$) Let $A_2 = A_{r,h(r)}$ be such that $B[A_2]$ has sequence $(x_1, x_2, \ldots, x_{r-1})$, with $x_i \geq 3$ for odd i, $x_j \geq 2$ for even j and $h(m) = \sum\limits_{k=0}^{m-1} x_k + 1$ for every $1 \leq m \leq r$.

(M$_3$) Let $A_3 = A_{r,h(r)}$ be such that $B[A_3]$ has sequence $(2, t+3, 2, t+2, t+2, \ldots, t+2)$ for some $t \geq 1$, $h(1) = 1$, $h(2) = t+3$, $h(3) = t+6$ and $h(m) = (t+2)(m-3)+6$ for $4 \leq m \leq r$ (see Figure 8).

Lemma 4.7. *Consider a matrix $A = A_{r,h(r)}$ with $r \geq 3$ and suppose that TT passes strictly above the m-th corner for every $2 \leq m \leq r-1$. Then, the following holds.*

(i) *Suppose that* $A = A_1$ *and the corners of* A *are as in* (M_1) *for* $r \geq 4$. *Then, it cannot happen that* BT *passes strictly below the* m*-th corner for every* $2 \leq m \leq r-1$.

(ii) *Suppose that* $A = A_1$ *and the corners of* A *are as in* (M_1) *for* $r = 3$. *Suppose also that* BT *passes strictly below the* 2*-th corner. Then,* $a_{2,1}, a_{1,1} \in BT$ *and column* $h(r)$ *is an interior element of* A.

(iii) *Suppose that* $A = A_2$ *and the corners of* A *are as in* (M_2) *for* $r \geq 5$. *Then, it cannot happen that* BT *passes strictly below the* m*-th corner for every* $2 \leq m \leq r-1$.

(iv) *Suppose that* $A = A_2$ *and the corners of* A *are as in* (M_2) *for* $r = 3, 4$. *Suppose also that* BT *passes strictly below the* m*-th corner for every* $2 \leq m \leq r-1$. *Then,* $a_{1,1}, a_{1,2} \in BT$.

(v) *Suppose that* $A = A_3$ *and the corners of* A *are as in* (M_3) *for* $r \geq 5$. *Then, it cannot happen that* BT *passes strictly below the* m*-th corner for every* $2 \leq m \leq r-1$.

Proof. (i) For a sequence $(x'_1, x'_2, \ldots, x'_{r-1})$ with $x'_i = 2$ for odd i and $x'_j = 3$ for even j, it follows that $h'(m) = 2(m-1) + \lceil \frac{m}{2} \rceil$ for $1 \leq m \leq r$. Then, as $h(r-1) - h(2) = \sum_{k=2}^{m-1} x_k \geq \sum_{k=2}^{m-1} x'_k = h'(r-1) - h'(2)$, we obtain that $f(r) = 2r - h(r-1) + h(2) - 3 \leq 2r - h'(r-1) + h'(2) - 3 = 2r - (2(r-2) + \lceil \frac{r-1}{2} \rceil) + 3 - 3 = 4 - \lceil \frac{r-1}{2} \rceil \leq 2$ whenever $r \geq 4$. Hence, as $f(r) \leq 2$, we conclude by Lemma 4.6 (ii) that it cannot happen that BT passes strictly below the m-th corner for every $2 \leq m \leq r-1$ and lemma (i) holds.

(ii) Similarly as the proof of (i), we have that $f(r) \leq 4 - \lceil \frac{r-1}{2} \rceil$ and then, $f(r) \leq 3$ since $r = 3$. We also notice that $f(r) \geq 3$ since otherwise we would obtain by Lemma 4.6 (ii) that it cannot happen that BT passes strictly below the m-th corner for $2 \leq m \leq r-1 = 2$, contradicting the hypothesis of assertion (ii) of the lemma. Therefore $f(r) = 3$ and hence, we obtain by Lemma 4.6 (i) that $a_{i,h(2)} \in BT$ for some $i \leq f(r) = 3$. Moreover, by the hypothesis of assertion (ii), BT passes strictly below the 2-th corner, then $a_{3,h(2)} \in BT$. On the other hand, as TT passes strictly above the 2-th corner, we have that $a_{1,i} \in TT$ for every $i = 1, \ldots, h(2)+1$. Therefore, we may deduce by Proposition 3.7 (b) that BT makes vertical movements at columns $h(2) - 1 = 2$ and $h(2) - 2 = 1$, obtaining that $a_{2,1}, a_{1,1} \in BT$ and concluding the first part of the proof of assertion (ii) (see Figure 7). Now, we will prove that column $h(r)$ is an interior element of A. As BT passes strictly below the 2-th corner, then $a_{3,h(2)-1}, \ldots, a_{3,h(3)} \in BT$. So, it can be deduced, by Proposition 3.7 (b) and (c), that TT makes vertical movements at columns $h(2)+1$ and $h(2)+2$ and obtaining that $a_{3,h(2)+2} \in TT$. As $h(2) + 2 = 5 < h(3) = h(r)$ (since the chessboard of A is (x_1, x_2) with $x_1 = 2$ and $x_2 \geq 3$), we conclude that column $h(r)$ is an interior element of A (see Lemma 3.4 (b)). Thus, lemma (ii) holds.

(iii) For a sequence $(x'_1, x'_2, \ldots, x'_{r-1})$ with $x'_i = 3$ for odd i and $x'_j = 2$ for even j, it follows that $h'(m) = 2(m-1) + \lceil \frac{m+1}{2} \rceil$ for $1 \leq m \leq r$. Then, as $h(r-1) - h(2) =$

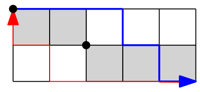


FIGURE 7. A matrix and its corners defined as in (M_1). We notice that $a_{2,1}, a_{1,1} \in BT$ and column 6 is an interior element as stated in Lemma 4.7 (ii)

$\sum_{k=2}^{m-1} x_k \geq \sum_{k=2}^{m-1} x'_k = h'(r-1) - h'(2)$, we obtain that $f(r) = 2r - h(r-1) + h(2) - 3 \leq 2r - h'(r-1) + h'(2) - 3 = 2r - (2(r-2) + \lceil \frac{r}{2} \rceil) + 4 - 3 = 5 - \lceil \frac{r}{2} \rceil \leq 2$ whenever $r \geq 5$. Hence, as $f(r) \leq 2$, we conclude by Lemma 4.6 (ii) that it cannot happen that BT passes strictly below the m-th corner for every $2 \leq m \leq r-1$ and lemma (iii) holds.

(iv) Similarly as the proof of (iii), we have that $f(r) \leq 5 - \lceil \frac{r}{2} \rceil$ and then, $f(r) \leq 3$ since $r = 3, 4$. We also notice that $f(r) \geq 3$ since otherwise we would obtain by Lemma 4.6 (ii) that it cannot happen that BT passes strictly below the m-th corner for every $2 \leq m \leq r-1$, contradicting the hypothesis of assertion (iv) of the lemma. Therefore, $f(r) = 3$ and hence, we obtain by Lemma 4.6 (i) that $a_{i,h(2)} \in BT$ for some $i \leq f(r) = 3$. Moreover, by the hypothesis of assertion (iv), BT passes strictly below the 2-th corner, then $a_{3,h(2)} \in BT$. On the other hand, as TT passes strictly above the m-th corner for every $2 \leq m \leq r-1$, we have that $a_{1,i} \in TT$ for every $i = 1, \ldots, h(2)+1$. Therefore, we may deduce by Proposition 3.7 (b) that BT makes vertical movements at columns $h(2)-1$ and $h(2)-2$, obtaining that $a_{1,h(2)-2} \in BT$. As $h(2) \geq 4$ then $a_{1,2} \in BT$ and so, $a_{1,1} \in BT$, concluding the proof of assertion (iv) of the lemma.

(v) If $r = 5$, we notice that $f(r) = 2$. If $r \geq 6$, we have that $f(r) = 2r - ((t+2)(r-4) + 6) + (t+3) - 3 = 2r - (t+2)(r-5) - 8$. Then, as $t \geq 1$ we obtain that $f(r) \leq 7 - r \leq 1$ whenever $r \geq 6$. Therefore, $f(r) \leq 2$ whenever $r \geq 5$ and so, by Lemma 4.6 (ii) it cannot happen that BT passes strictly below the m-th corner for every $2 \leq m \leq r-1$. □

From now, we denote by A^+_m and A^-_m the matrices $A^+_{m,h(m)}$ and $A^-_{m,h(m)}$ respectively; see Figure 8.

We are now ready to tackle the case when $d \geq 4$ and $t \geq 1$.

Theorem 4.8. *$\bar{n}(t,d) < 2d + t(d-2) + 2$ for integers $d \geq 4$ and $t \geq 1$.*

Proof. Consider a matrix $A = A_{r,h(r)}$ such that $B[A]$ has sequence $(2, t+3, 2, t+2, t+2, \ldots, t+2)$ for $t \geq 1$, where $h(r)$ is defined by $h(2) = t+3$, $h(3) = t+6$ and $h(m) = (t+2)(m-3)+6$ for $4 \leq m \leq r$ (see Figure 8). We shall prove by induction on r that for every $r \geq 2$ and any acyclic re-orientation $\mathcal{A}$ of A, matrix $\mathcal{A}$ has at least $t+1$ interior elements. In particular, as $h(r) = (t+2)(d-2)+6 = 2d + t(d-2) + 2$ for $r \geq 5$, we will prove the theorem for $d \geq 4$ and $t \geq 1$. For $r = 2, 3, 4$, the result follows by Claim 4.1 and Theorems 4.2 and 4.4, respectively (notice that the chessboards considered in these theorems coincide with $B[A]$). Thus, assume that r is at least 5 and the theorem holds for $r-1$. Let TT and BT be the Top and Bottom travels of $\mathcal{A}$, respectively. We first prove the following claims.

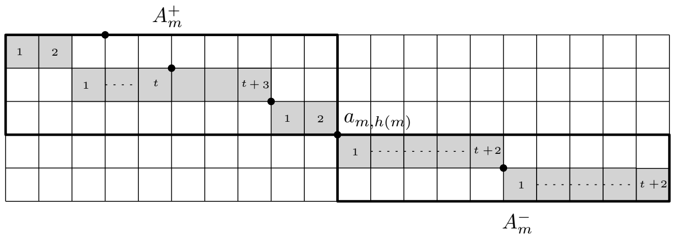


FIGURE 8. A chessboard with sequence $(2, t+3, 2, t+2, t+2)$ and the matrices A_m^+ and A_m^-. The points are the corners associated to the function $h(r)$ of Theorem 4.8.

Claim A. If $a_{k,h(k)}, a_{k,h(k)+1} \in BT$ for some $k = 2, \ldots, r-1$, the theorem holds.

First suppose that $a_{k,h(k)-1} \in BT$. Then, we notice that the Top and Bottom Travels of $\mathcal{A}_k^+$ coincides with the corresponding parts of TT and BT of $\mathcal{A}$ and hence, each interior element of $\mathcal{A}_k^+$ is an interior element of $\mathcal{A}$, including a possible interior element of $\mathcal{A}_k^+$ at column $h(k)$, since $a_{k,h(k)-1}, a_{k,h(k)}, a_{k,h(k)+1} \in BT$ (see Lemma 3.4 (b) and (c)). Therefore, the theorem holds by induction hypothesis on $\mathcal{A}_k^+$. Now suppose that $a_{k,h(k)-1} \notin BT$. Then, we notice that either $a_{1,h(2)} = a_{1,t+3} \in BT$ or for some $k' = 2, \ldots, k-1$, we have that $a_{k',h(k')-1}, a_{k',h(k')}, a_{k',h(k')+1} \in BT$. If $a_{1,t+3} \in BT$, then $\mathcal{A}$ has at least $t+2$ interior elements (from columns 1 to $t+2$). If $a_{k',h(k')-1}, a_{k',h(k')}, a_{k',h(k')+1} \in BT$, then each interior element of $\mathcal{A}_{k'}^+$ is an interior element of $\mathcal{A}$ and hence, the theorem holds by induction hypothesis on $\mathcal{A}_{k'}^+$. Therefore, Claim A holds.

Claim B. If BT passes strictly above some corner, the theorem holds.

Suppose that BT passes strictly above the k-th corner, for some $k = 2, \ldots, r-1$. First suppose that $k = 2$, then $a_{1,h(2)} = a_{1,t+3} \in BT$, concluding that $\mathcal{A}$ has at least $t+2$ interior elements. Now suppose that $k > 2$. Then, either $a_{i,h(i)}, a_{i,h(i)+1} \in BT$ for some $i = 2, \ldots, k-1$ or $a_{1,t+3} \in BT$. In the first case the theorem holds by Claim A and in the second case we clearly have that $\mathcal{A}$ has at least $t+2$ interior elements. So, Claim B holds.

Let m be such that the m-th corner is the last corner that TT meets in $\mathcal{A}$, for some $m = 1, \ldots, r-1$, from left to right. If TT passes strictly below the i-th corner for some $i > m$, then by the election of m and by the rules of construction of TT we notice that $a_{r,h(r-1)-1} \in TT$, concluding that $\mathcal{A}$ will have least $t+3$ interior elements (from columns $h(r-1)$ to $h(r)$) and the theorem holds. Hence, we may assume from now that TT always passes strictly above the i–st corner for $i > m$.

First suppose that $m = r-1$. As $\mathcal{A}_m^-$ is a matrix of size $2 \times (t+3)$, then A_m^- has $t+1$ interior elements (Claim 4.1). As the Top and Bottom Travels of $\mathcal{A}_m^-$ coincides with the corresponding parts of TT and BT of $\mathcal{A}$, each interior element of $\mathcal{A}_m^-$ is an interior element of $\mathcal{A}$, except for (maybe) column $h(m)$. If column $h(m)$ is not an interior element of $\mathcal{A}_m^-$, then $\mathcal{A}$ would have at least $t+1$ interior elements and the theorem holds. If column $h(m)$ is an interior element of $\mathcal{A}_m^-$, we obtain by Lemma 3.4 (a) that $a_{m,h(m)}, a_{m,h(m)+1} \in BT$ and then, the theorem holds by Claim A. Now, suppose that $m < r-1$ and consider the following cases.

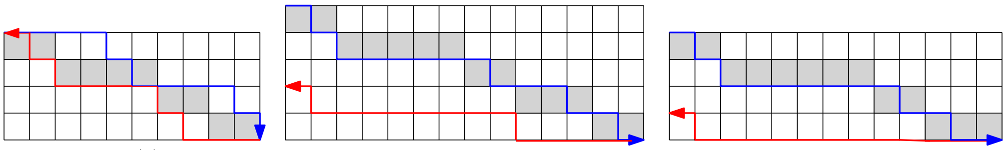
Case 1. BT arrives at the k-th corner, for some $k > m$.

We may suppose that $a_{k+1,h(k)}, a_{k,h(k)}, a_{k,h(k)-1} \in BT$, since otherwise the theorem holds by Claim A. Then, we notice that the Top and Bottom Travels of $\mathcal{A}_k^+$ coincides with the corresponding parts of TT and BT of $\mathcal{A}$. Moreover, as TT does not arrive at the k-th corner, column $h(k)$ is not an interior element of $\mathcal{A}_k^+$ (see Lemma 3.4 (b)). Hence, each interior element of $\mathcal{A}_k^+$ is an interior element of $\mathcal{A}$, concluding the proof by induction hypothesis on $\mathcal{A}_k^+$.

Case 2. BT does not arrives at the k-th corner, for every $k > m$.

By Claim B, we may suppose from now that BT passes strictly below the k-th corner, for every $k > m$. If $m = 1$, we notice that A_m^- has the same chessboard and the same corners as that considerer in (M_3). Moreover, as A_m^- has at least 5 rows, we obtain by Lemma 4.7 (v) that it cannot happen that BT passes strictly below all corners of A_m^-, yielding a contradiction. So, we may assume that $1 < m < r-1$. First suppose that $m \neq 3$. As $m < r-1$, then A_m^- has at least 3 rows. Moreover, we notice that A_m^- have the same chessboard and the same corners as that considerer in (M_2). If A_m^- has at least 5 rows, we obtain by Lemma 4.7 (iii) that it cannot happen that BT passes strictly below all corners of A_m^-, yielding a contradiction. If A_m^- has 3 or 4 rows, we obtain by Lemma 4.7 (iv) that $a_{k,h(k)}, a_{k,h(k)+1} \in BT$ and so, the theorem holds by Claim A. Now, suppose that $m = 3$. Then, A_m^- has at least 3 rows (since $r \geq 5$), the same chessboard and the same corners as that considerer in (M_1). If A_m^- has at least 4 rows, we obtain by Lemma 4.7 (i) that it cannot happen that BT passes strictly below all corners of A_m^-, yielding a contradiction. If A_m^- has 3 rows, we obtain by Lemma 4.7 (ii) that $a_{m+1,h(m)}, a_{m,h(m)} \in BT$ and column $h(r)$ is an interior element of A_m^-. As $a_{m+1,h(m)}, a_{m,h(m)} \in BT$ then by the rules of construction of BT we obtain that $a_{m,h(m)-1}, a_{m,h(m)} \in BT$ and so, the Top and Bottom Travels of $\mathcal{A}_m^+$ coincides with the corresponding parts of TT and BT of $\mathcal{A}$. Hence, each interior element of $\mathcal{A}_m^+$ is an interior element of $\mathcal{A}$, except for (maybe) column $h(m)$. Then, $\mathcal{A}$ has at least t interior elements by induction hypothesis on $\mathcal{A}_m^+$. On the other hand, as column $h(r)$ is an interior element of A_m^-, then $h(r)$ is an interior element of $\mathcal{A}$. Therefore, $\mathcal{A}$ would have at least t interior elements from columns 1 to $h(m)-1$ and one interior element at column $h(r)$, concluding the proof. $\square$

The chessboards of Figure 9 cannot be used to improve Theorem 4.8 since they only have t interior points.



(a) (b) (c)

FIGURE 9. Figures (a), (b) and (c) show the matrices $A_{5,11}$, $A_{6,15}$ and $A_{5,14}$, respectively, with chessboards $(2, t+3, 2, 2)$ for $t = 1$ (a), $(2, t+3, 2, 3, 2)$ for $t = 2$ (b) and $(2, t+3, 2, 3)$ for $t = 3$ (c). We observe that for the pair of Top and Bottom travels described in these matrices, $A_{5,11}$ has only one interior element (column 1), $A_{6,15}$ has only two interior elements (columns 13 and 15) and $A_{5,14}$ has only three interior elements (columns $11, 13$ and 14).

4.3 Proof of Theorem 1.1

We know, by Proposition 3.1 (a), that LOM are affine matroids and, by construction, they are uniform. Moreover, as remarked in Subsection 3.1, $\bar{n}(t, d)$ coincides with $n(t, d)$ for affine uniform oriented matroids. We thus may use the results of the previous section for the desired upper bounds.

We may now prove Theorem 1.1.

Proof of Theorem 1.1. Recall that $H_0(n(t, d), d) = n(t, d) - t$ (see (4)).
• If $d = 3$ and $n \geq 7$ then, by Theorem 4.4, we have that $n(t, 3) \leq 7 + t$ for any integer $t \geq 0$. Therefore, by (4), we obtain $H_0(t+7, 3) \leq 7$ for any integer $t \geq 0$ or, equivalently, $H_0(n, 3) \leq 7$ for any integer $n \geq 7$.
• If $d \geq 4$, $n \geq 2d+t(d-2)+2$ with $t \geq 1$ then, by Theorem 4.8, $\bar{n}(t, d) < 2d+t(d-2)+2$ for integers $d \geq 4$ and $t \geq 1$. Since $H_0(n(t, d), d) = \bar{n}(t, d) - t$, then $H_0(n, d) < n - t$ for any integers $n \geq 2d + t(d - 2) + 2$ and $t \geq 1$. □

Proof of Corollary 1.2. The desired inequalities are obtained by combining (6) and (8) and the values and upper bounds given in Propositions 2.1 and 2.2 and Theorem 1.1. The lower bound is obtained by combining (7) and (9).

□

Question 4.9. *Let $d \geq 1$ and $1 \leq k \leq d - 1$. Is it true that $H_k(n, d) \geq H_k(n - 1, d)$?*

We believe that the answer is positive.

5 Minimal Radon partitions

In order to prove Theorem 1.3 we need to take a geometric detour on the relationship between faces of convex polytopes, simplices embracing the origin and Radon partitions. There is an old tradition of using Gale transforms to study facets of convex polytopes

[9] by studying simplices embracing the origin. This equivalence was further extended by Larman [10] to studying Radon partitions of points in space.

A *Gale transform* $\bar{X}$ of a finite set of points $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ such that the dimension of their affine span is r is defined by $\bar{X} = \{\bar{x}_j = (\alpha_{j,1}, \dots \alpha_{j,n-r-1})\}_{j=1}^n$, where $\{a_i = (\alpha_{1,i}, \dots, \alpha_{n,i})\}_{i=1}^{n-r-1}$ is a basis of the $(n-r-1)$-dimensional space of *affine dependencies* of X, $D(X) = \{\alpha = (\alpha_1, \dots, \alpha_n) | \ \sum_{i=1}^n \alpha_i x_i = 0, \ \sum_{i=1}^n \alpha_i = 0\}$. It is emphasized that $\bar{X}$ is *a* Gale transform of X, rather than *the* Gale transform of X, because the resulting points depend on the specific choice of basis for $D(X)$. However, different Gale transforms of the same set of points are linearly equivalent [9].

A *Gale diagram* $\hat{X}$ of X is a set of points in $\mathbb{S}^{n-r-2}$ obtained by *normalizing* a Gale transform, that is: $\hat{X} = \{\hat{x}_i = \frac{\bar{x}_i}{\|\bar{x}_i\|} | \bar{x}_i \in \bar{X}, \bar{x}_i \neq 0\} \cup \{\hat{x}_i = \bar{x}_i | \bar{x}_i \in \bar{X}, \bar{x}_i = 0\}$.

The following results are known.

Proposition 5.1. *Let $X = \{x_1, \dots, x_n\}$ be a set of n points in $\mathbb{R}^d$ and let $\hat{X}$ (resp. $\bar{X}$) be its Gale diagram (resp. Gale transform), then the following statements hold.*

(a) [9, pp 87 (iv)] *The n points of X are in general position in $\mathbb{R}^d$ if and only if the n-tuple $\hat{X}$ (resp. $\bar{X}$) consists of n points in linearly general position in $\mathbb{R}^{n-d-1}$.*

(b) [9, pp 88 **1**] *Faces of $\mathsf{conv}(X)$ are in one-to-one correspondence with its complementary set in $\hat{X}$ (resp. $\bar{X}$) that contain 0 in their convex hull. More precisely, $Y \subset X$ is a face of $\mathsf{conv}(X)$ if and only if $0 \in relint\, \mathsf{conv}(\hat{X} \setminus \hat{Y})$ (resp. $0 \in relint\, \mathsf{conv}(\bar{X} \setminus \bar{Y})$).*

(c) [9, pp 87 (vi)] *X is projectively equivalent to a set of points Y (by a nonsingular permissible projective transformation) if and only if there is a non-zero vector $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{1, -1\}^n$ (resp. $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$) such that $\hat{y}_i = \epsilon_i \hat{x}_i$ (resp. $\bar{y}_i = \lambda_i \bar{x}_i$).*

Let $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ be a set of points in general position with $n \geq d+2$ and let $X = A \cup B$ be a partition of X. Given the triple (X, A, B), we define the *Partitioned Affine Projection* (PAP) into the unit d-sphere as the partition $\tilde{X} = \tilde{A} \cup \tilde{B}$ with $\tilde{A} = \{\tilde{x}_i = \frac{(x_i;1)}{\|(x_i;1)\|} \mid x_i \in A\} \subset \mathbb{S}^d$ and $\tilde{B} = \{\tilde{x}_i = -\frac{(x_i;1)}{\|(x_i;1)\|} \mid x_i \in B\} \subset \mathbb{S}^d$, where $(x_i; 1)$ is the $(d+1)$-dimensional vector whose first d entries are identical to those of x_i and last entry is 1. Notice that $\tilde{X} \subset \mathbb{S}^d \subset \mathbb{R}^{d+1}$ while $X \subset \mathbb{R}^d$.

By using linear algebra, it can be easily obtained the following

Claim 5.2. *Let A and B be a partition of X. Then, A, B is a Radon partition of X if and only if $0 \in \mathsf{conv}(\tilde{A} \cup \tilde{B})$. Moreover, if $S \subseteq X$, then $\mathsf{conv}(A \cap S) \cap \mathsf{conv}(B \cap S) \neq \emptyset$ if and only if $0 \in \mathsf{conv}(\tilde{S})$, where $\tilde{S} = \{\tilde{x} \in \tilde{A} \mid x \in S \cap A\} \cup \{\tilde{x} \in \tilde{B} \mid x \in S \cap B\}$.*

Proof of Theorem 1.3. Let $X = A \cup B$ be a partition of X and let $S \subseteq X$. Let us denote $\tilde{X}_\epsilon = \{\epsilon_1 \tilde{x}_1, \dots, \epsilon_n \tilde{x}_n\}$ for $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{1, -1\}^n$. Let $\rho(\tilde{X}_\epsilon)$ be the number of $(d+2)$-element subsets $\tilde{S} \subset \tilde{X}_\epsilon$ such that $0 \in \mathsf{conv}(\tilde{S})$, $\rho(\tilde{X}) = \max_{\epsilon \in \{1,-1\}^n} \rho(\tilde{X}_\epsilon)$, and $\rho(n, d) = \min_{\{\tilde{X} \subset \mathbb{S}^d, |\tilde{X}| = n\}} \rho(\tilde{X})$.

By Claim 5.2, it can easily be checked that

$$r(n,d) = \rho(n,d). \tag{10}$$

Now, by Proposition 5.1 (c), the set $\tilde{X} \subset \mathbb{S}^d$ can be considered as the Gale diagram of a set of points in $X' \subset \mathbb{S}^{n-d-2}$ where each $\tilde{X}_\epsilon = \{\epsilon_1\tilde{x}_1, \dots, \epsilon_n\tilde{x}_n\}$ corresponds to a nonsingular permissible projective transformation of X'. Therefore, by Proposition 5.1 (b), each $(d+2)$-element subset $\tilde{S} \subset \tilde{X}_\epsilon$ such that $0 \in \mathsf{conv}(\tilde{S})$ is in one to one correspondence with the co-facets (complement of facets) of the corresponding nonsingular permissible projective transformation of X'. We thus have that searching the value of $\rho(n,d)$ is equivalent to searching nonsingular permissible projective transformation with a maximal number of facets. Hence, finding $\rho(n,d)$ is equivalent to finding $H_{d'-1}(n,d')$ where $d' = n-d-2$. □

Corollary 5.3. *Let $d \geq 1$ be an integer. Then,*

$$r(n,d) \begin{cases} = 2 & \text{if } n = d+3, \\ = 5 & \text{if } n = d+4, \\ \leq 10 & \text{if } n = d+5, \\ \leq f_{n-d-3}(C_{n-d-2}(n)) & \text{if } n \geq 2d+3, \\ \leq f_{n-d-3}(C_{n-d-2}(n-1)) & \text{if } n \leq \frac{5d+8}{3}, \\ \leq f_{n-d-3}(C_{n-d-2}(n-1-t)) & \text{if } n \leq \frac{2d+2+t(d+4)}{t+1} \text{ and } t \geq 1. \end{cases}$$

Moreover, if $n \geq 2d+3,\ d \geq 2$ then

$$r(n,d) \geq (n-d-3)n - (n-d-1)(n-d-4).$$

Proof. The values and upper bounds can be obtained by combining Corollary 1.2 and Theorem 1.3. Moreover, by combining Theorem 1.3 with (7) and (9) we have

$$r(n,d) = H_{d'-1}(n,d') \geq f_{d'-1}(P_{d'}(n)) = (d'-1)n - (d'+1)(d'-2)$$

if $n \leq 2d'+1$ where $d' = n-d-2,\ d \geq 2$. The latter gives the desired lower bound of $r(n,d)$. □

Theorem 5.4. *Let $n \geq 4$ be an integer. Then,*

$$r(n,2) \begin{cases} = 2 & \text{if } n = 5, \\ = 5 & \text{if } n = 6, \\ = 10 & \text{if } n = 7, \\ \leq 2\binom{\frac{n-1}{2}+2}{\frac{n-1}{2}-2} & \text{if } n \geq 7, n\text{-odd}, \\ \leq \binom{\frac{n}{2}+2}{\frac{n}{2}-2} + \binom{\frac{n}{2}+1}{\frac{n}{2}-3} & \text{if } n \geq 8, n\text{-even}. \end{cases}$$

Moreover, if $n \geq 7$ then $r(n,2) \geq 2(2n-9)$.

Proof. • If $n = 5, 6$ then the values are obtained directly from (5.3).

• If $n = 7$ then, by combining the third inequality and the lower bound of $r(n,d)$ of Corollary 5.3, we obtain that $r(7,2) = 10$.

• By the forth inequality in Corollary 5.3, we have $r(n,2) \leq f_{n-5}(C_{n-4}(n))$ for any $n \geq 7$. Now, by taking $k = d-1$ in (5), we have

$$f_{d-1}(C_d(n)) = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - \lfloor \frac{d}{2} \rfloor - 1}{\lceil \frac{d}{2} \rceil - 1}.$$

Therefore, by taking $d = n-4$, we have

$$r(n,2) \leq \binom{n - \lceil \frac{n-4}{2} \rceil}{\lfloor \frac{n-4}{2} \rfloor} + \binom{n - \lfloor \frac{n-4}{2} \rfloor - 1}{\lceil \frac{n-4}{2} \rceil - 1}. \tag{11}$$

The upper bounds for the cases $n \geq 8$, n-even and $n \geq 7$, n-odd are obtained from (11).

• The lower bound for $r(n,2)$ when $n \geq 7$ is a straightforward calculation from the lower bound given in Corollary 5.3. □

Other bounds for specific values of n and d can easily be obtained by using (5.3) and Corollary 5.3, for instance,

$$17 \leq r(9,3) \leq 27.$$

The following result is a straightforward consequence of the bounds of Theorem 5.4.

Corollary 5.5. *The order of* $r(n,2)$ *is between* $o(n)$ *and* $o(n^4)$*.*

5.1 Pach and Szegedy's question

We will prove Theorem 1.5.

Proof of Theorem 1.5. Let $X \subset \mathbb{R}^2$ be a set of $n \geq 8$ points in general position. Let A, B be a partition of X that attains the maximum number of induced minimal Radon partitions for the set X, that is, $r(X) = r_X(A,B)$.
Let $\tilde{X} \subset \mathbb{S}^2$ be the PAP corresponding to the triple to (X, A, B). We notice that, by construction, the PAP's have a choice of sign 'minus', that is, if a point x in A pass to B then the sign of $\tilde{x}$ will be changed. Also, by Claim 5.2, we have that $0 \notin \mathsf{conv}(\tilde{S})$ for any $\tilde{S}$ subset of either $\tilde{A}$ or $\tilde{B}$.
Let $\tilde{X}_\epsilon = \{\epsilon_1 \tilde{x}_1, \ldots, \epsilon_n \tilde{x}_n\}$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{1, -1\}^n$. By the above observations

and Claim 5.2, we may conclude that

(12) among all the choices of signs $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{1, -1\}^n$ the maximum number of sets $\tilde{S} \subset \tilde{X}_\epsilon$ of cardinality $d+2$ such that $0 \in \mathsf{conv}(\tilde{S})$ is attained on $\tilde{X}$ (corresponding to $\epsilon = (\underbrace{1, \ldots, 1}_{n})$, see below).

Let $X' \subset \mathbb{R}^{n-3}$ be a point configuration whose Gale diagram is $\tilde{X}$. Recall that, by Proposition 5.1 (c), the change of signs of $(\epsilon_1, \ldots, \epsilon_n)$ correspond to a nonsingular permissible projective transformation of X'.

Claim 5.6. *X' can be chosen to be the vertex set of a $\lfloor \frac{n-3}{2} \rfloor$-neighborly polytope.*

Proof. By Proposition 5.1 (b), we have that

a subset of vertices Y form a face of X' if and only if $0 \in relint\mathsf{conv}(\tilde{X} \setminus \tilde{Y})$.

Combining this with (12), we have that each $(d+2)$-element subset $\tilde{S} \subset \tilde{X}$ such that $0 \in \mathsf{conv}(\tilde{S})$ is in one to one correspondence with the co-facets (complement of facets) of X'. According to (UBT) k–neighborly d-polytopes achieve the maximum possible number of k–faces for any $\lfloor \frac{d}{2} \rfloor \leq k \leq d$. We thus have that the maximal number of such subsets $\tilde{S}$ cannot be bigger than the number of co-facets, or equivalently the number of facets, in a $\lfloor \frac{n-3}{2} \rfloor$-neigborly polytope. The latter can be achieved. Indeed, we know (see (3)) that $\nu(d,k) \geq d + \lfloor \frac{d}{k} \rfloor + 1$ for $k \geq 2$. Therefore, by taking $d = n-3$ and $k = \lfloor \frac{n-3}{2} \rfloor$ we obtain that

$$\nu\left(n-3, \left\lfloor \frac{n-3}{2} \right\rfloor\right) \geq n$$

for any integer $\lfloor \frac{n-3}{2} \rfloor \geq 2$, that is, for $n \geq 7$.
This implies that any set of at least $n \geq 7$ points in $\mathbb{R}^{n-3}$ can always be mapped by a permissible projective transformation onto the vertices of a $\lfloor \frac{n-3}{2} \rfloor$–neighborly polytope. In particular, the set X' can be mapped by a permissible projective transformation, say T, onto the vertices of a $\lfloor \frac{n-3}{2} \rfloor$–neighborly polytope, say $T(X')$. Therefore, $\tilde{X} = \tilde{X}_\epsilon$ correspond to the projective transformation $T(X')$ with $\epsilon = (\underbrace{1, \ldots, 1}_{n})$. □

By the above claim, we have

$$\text{for every subset } \tilde{S} \subset \tilde{X}_\epsilon \text{ such that } |\tilde{S}| \leq \left\lfloor \frac{n-3}{2} \right\rfloor, 0 \in \mathsf{conv}(\tilde{X}_\epsilon \setminus \tilde{S}).$$

Since $|\tilde{X}| = n$ and neither $\mathsf{conv}(\tilde{A})$ nor $\mathsf{conv}(\tilde{B})$ contain the origin then

$$|\tilde{A}|, |\tilde{B}| < n - \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Therefore, $|A|, |B| < n - \lfloor \frac{n-3}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 2$, as desired. □

Question 5.7. *Could this approach be extended to investigate balanced 2-partitions in higher dimensions ?*

Unfortunately, our method is not suitable to study balanced 3-partitions since, as we have seen in the proof of Theorem 1.5, the translation into Gale diagrams involves 2-partitions only.

5.2 Tolerance result

We may now prove Theorem 1.6.

Proof of Theorem 1.6. Let X be such that it has a partition into two sets A,B and a subset $P \subseteq X$ of cardinality $\lambda - i$, for some $0 \leq i \leq t$, such that $\mathsf{conv}(A \setminus y) \cap \mathsf{conv}(B \setminus y) \neq \emptyset$ for every $y \in P$ and $\mathsf{conv}(A \setminus y) \cap \mathsf{conv}(B \setminus y) = \emptyset$ for every $y \in X \setminus P$. By Claim 5.2, we know that if we consider the PAP of X into $\mathbb{S}^d$, $\tilde{X}$, we have that $\mathsf{conv}(A \setminus y) \cap \mathsf{conv}(B \setminus y) \neq \emptyset$ if and only if $0 \in \mathsf{conv}((\tilde{A} \setminus \tilde{y}) \cup (\tilde{B} \setminus \tilde{y}))$.

Now let $\rho(t, d)$ be the smallest number ρ such that for all sets $\tilde{X}$ of cardinality ρ in $\mathbb{S}^d$, there exists a partition of $\tilde{X}$ into two sets $\tilde{A}$,$\tilde{B}$ and a subset $\tilde{P} \subseteq \tilde{X}$ of cardinality $\rho - i$, for some $0 \leq i \leq t$, such that $0 \in \mathsf{conv}((\tilde{A} \setminus \tilde{y}) \cup (\tilde{B} \setminus \tilde{y}))$ for every $\tilde{y} \in \tilde{P}$ and $0 \notin \mathsf{conv}((\tilde{A} \setminus \tilde{y}) \cup (\tilde{B} \setminus \tilde{y}))$ for every $\tilde{y} \in \tilde{X} \setminus \tilde{P}$, then $\lambda(t, d) = \rho(t, d)$. That is, one can seamlessly go from a tolerant partition to a *tolerant* configuration of points in the sphere.

For the next part we will need to establish a relationship between $n(t, d)$ and $\rho(t, d)$. This relationship arises from the connection between projective transformations of points and antipodal functions of their Gale diagrams, as has already been explored in Theorem 1.3.

Let y be a point strictly in the interior of $\mathsf{conv}(X)$. Recall that if we consider the Gale diagram of X, then $\hat{X} \subset \mathbb{S}^{n-d-1}$ and thus, by Proposition 5.1 (b), $0 \notin \mathsf{conv}(\hat{X} \setminus \hat{y})$ since y is not a face of $\mathsf{conv}(X)$. Moreover, Proposition 5.1 (c) draws the connection between projective transformations of X and taking diametrically opposite points in $\hat{X}$.

Thus, if we consider the Gale diagram of a set of $n = n(t, d)$ points $\hat{X}$, we must have that for some $\hat{X}_\epsilon = \{\epsilon_1 \hat{x}_1, \ldots, \epsilon_n \hat{x}_n\}$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{1, -1\}^n$, there is a set of at most $n - i$ points, for some $0 \leq i \leq t$, $\hat{P}_\epsilon$ such that $0 \in \mathsf{conv}(\hat{X}_\epsilon \setminus \hat{y})$ for $\hat{y} \in \hat{P}_\epsilon$. Thus $\rho(t, n - d - 1) \leq n$, and the necessary partition is given by the signs of the epsilon vectors.

Conversely, let $\hat{X}$ be a set of points $\rho = \rho(t, d')$ points, then the Gale transform of these points, X will be such that there is a set of at most t points, such that they are in the interior of $\mathsf{conv}(X)$. This is $\rho \leq n(t, \rho - d' - 1)$.

As argued at the beginning of the proof, in both inequalities we can straight forwardly substitute ρ for λ obtaining

$$n(t, d) = \max_{m \in \mathbb{N}} \{m \mid \lambda(t, m - d - 1) \leq m\} \text{ and } \lambda(t, d) = \min_{m \in \mathbb{N}} \{m \mid m \leq n(t, m - d - 1)\}.$$

as desired. □

6 Arrangements of (pseudo)hyperplanes

The so-called Topological Representation Theorem, due to Folkman and Lawrence [5], states that loop-free oriented matroids of rank $r = d + 1$ on n elements (up to isomorphism) are in one-to-one correspondence with arrangements of pseudo-hyperplanes in the projective space $\mathbb{P}^{r-1}$ (up to topological equivalence).

A d-arrangement of n pseudo-hyperplanes is called *simple* if $n \geq d$ and every intersection of d pseudo-hyperplanes is a unique distinct point. It is known that simple arrangements correspond to uniform oriented matroids. It is well known that a tope corresponds to an acyclic reorientation (projective transformations) having as interior elements precisely those pseudo-hyperplanes not bordering the tope.

By the above discussion, we may redefine $\bar{n}(t, d)$ in terms of hyperplane arrangements:

> $\bar{n}(t, d)$:= the largest integer n such that any simple arrangement of n (pseudo)hyperplanes in $\mathbb{P}^d$ contains a tope of size at least $m - t$.

Proposition 6.1. *Every simple arrangement of at least* 5 *pseudo-lines in* $\mathbb{P}^2$ *has a tope of size at least* 5*, that is,*

$$5 + t \leq \bar{n}(t, 2) \text{ for every integer } t \geq 0.$$

Proof. The proof is by induction on the set of n (pseudo) lines. By (2), any arrangement of 5 (pseudo) lines in $\mathbb{P}^2$ has a tope of size 5 and thus the proposition holds for $n = 5$. We suppose the result true for $n' < n$ and will prove that any arrangement H of $n \geq 6$ (pseudo) lines in $\mathbb{P}^2$ has a tope of size at least 5. Let $l \in H$, then by induction $H \setminus l$ has a tope T of size at least 5 in $\mathbb{P}^2$. If l does not touch T then T is a tope of H of size at least 5 in $\mathbb{P}^2$. Otherwise, l divides T into two topes, and since H is simple then one of these two topes is of size at least 5. □

Combining Proposition 6.1 and Theorem 4.2 we obtain that

$$\bar{n}(t, 2) = 5 + t \text{ for any integer } t \geq 0. \tag{13}$$

Further, (13) and (4) imply that $H_0(t + 5, 2) = 5$ for any integer $t \geq 0$, in other words

Corollary 6.2. $H_0(n, 2) = 5$ *for any integer* $n \geq 5$.

For the case $d = 3$, Theorem 4.4 implies that $\bar{n}(t, 3) \leq 7 + t$ for any integer $t \geq 1$, that is, for any $n \geq 7$ there exists a simple arrangement of n (pseudo)planes in $\mathbb{P}^3$ with every tope of size at most 7. This supports the following:

Conjecture 6.3. $\bar{n}(t, 3) = 7 + t$ for any integer $t \geq 1$.

We end by putting forward the following two more general questions.

Question 6.4. *Let $d \geq 2$ and $t \geq 0$ be integers. Is it true that $\bar{n}(t,d) = 2d+1+t$? In other words, is it true that any simple arrangement of $n \geq 2d+1$ (pseudo)hyperplanes in $\mathbb{P}^d$ contains a tope of size at least $2d+1$ and conversely, for any $n \geq 2d+1$ there exists a simple arrangement of n (pseudo)hyperplanes in $\mathbb{P}^d$ with every tope of size at most $2d+1$?*

Or, alternatively,

Question 6.5. *Let $d \geq 2$ and $t \geq 0$ be integers. Is there a constant $c(d) \geq 1$ such that $\bar{n}(t,d) = 2d+1+c(d)t$?*

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