Matroid base polytope decomposition

J.L. Ramírez Alfonsín

Université Montpellier 2

(joint work with V. Chatelain)

Introduction

Let $M = (E, \mathcal{B})$ be a matroid on $E = \{1, \ldots, n\}$ where $\mathcal{B} = \mathcal{B}(M)$ denote the collection of bases.

The set $\mathcal{B}$ verifies the base exchange axiom :

if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there exists $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.

Let $P(M)$ be the matroid base polytope of M defined as the convex hull of the incidence vector of bases of M, that is,

$$P(M) := conv\left\{\sum_{i\in B} e_i : B \in \mathcal{B}\right\}$$

where e_i denotes the i^{th} standard basis vector in $\mathbb{R}^n$.

Let $P(M)$ be the matroid base polytope of M defined as the convex hull of the incidence vector of bases of M, that is,

$$P(M) := conv\left\{\sum_{i\in B} e_i : B \in \mathcal{B}\right\}$$

where e_i denotes the i^{th} standard basis vector in $\mathbb{R}^n$.

Remarks :
(a) $P(M)$ is a polytope of dimension at most $n-1$.

Let $P(M)$ be the matroid base polytope of M defined as the convex hull of the incidence vector of bases of M, that is,

$$P(M) := conv\left\{\sum_{i\in B} e_i : B \in \mathcal{B}\right\}$$

where e_i denotes the i^{th} standard basis vector in $\mathbb{R}^n$.

Remarks :
(a) $P(M)$ is a polytope of dimension at most $n-1$.
(b) $P(M)$ is a facet of the independent polytope of M obtained as the convex hull of the incidence vectors of the independent sets of M.

A decomposition of $P(M)$ is a decomposition of the form

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where each $P(M_i)$ is a matroid base polytope for some matroid M_i, and for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is also a matroid base polytope for some matroid (a facet of both $P(M_i)$ and $P(M_j)$).

A decomposition of $P(M)$ is a decomposition of the form

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where each $P(M_i)$ is a matroid base polytope for some matroid M_i, and for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is also a matroid base polytope for some matroid (a facet of both $P(M_i)$ and $P(M_j)$).

$P(M)$ is said to be decomposable if it has a decomposition with $t \geq 2$, and indecomposable otherwise.

A decomposition of $P(M)$ is a decomposition of the form

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where each $P(M_i)$ is a matroid base polytope for some matroid M_i, and for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is also a matroid base polytope for some matroid (a facet of both $P(M_i)$ and $P(M_j)$).

$P(M)$ is said to be decomposable if it has a decomposition with $t \geq 2$, and indecomposable otherwise.

A decomposition is called hyperplane split if $t = 2$.

Motivations

(Lafforgue) Give a general *compactification* method and proved that such compactification exists if the associated base polytope is indecomposable.

Motivations

(Lafforgue) Give a general *compactification* method and proved that such compactification exists if the associated base polytope is indecomposable.

Remark Lafforgue's work implies that for a matroid represented by vectors in $\mathbb{F}^r$ if $P(M)$ is indecomposable then M will be rigid, that is, M will have only finitely many realizations up to scaling and the action of $GL(r, \mathbb{F})$.

Motivations

(Lafforgue) Give a general *compactification* method and proved that such compactification exists if the associated base polytope is indecomposable.

Remark Lafforgue's work implies that for a matroid represented by vectors in $\mathbb{F}^r$ if $P(M)$ is indecomposable then M will be rigid, that is, M will have only finitely many realizations up to scaling and the action of $GL(r, \mathbb{F})$.

(Hacking, Keel and Tevelev) Compactification of moduli space of an arrangement of hyperplanes.

(Speyer) Tropical linear spaces.

Motivations

(Lafforgue) Give a general *compactification* method and proved that such compactification exists if the associated base polytope is indecomposable.

Remark Lafforgue's work implies that for a matroid represented by vectors in $\mathbb{F}^r$ if $P(M)$ is indecomposable then M will be rigid, that is, M will have only finitely many realizations up to scaling and the action of $GL(r, \mathbb{F})$.

(Hacking, Keel and Tevelev) Compactification of moduli space of an arrangement of hyperplanes.

(Speyer) Tropical linear spaces.

(Ardila, Fink and Rincon) There exist functions that behave like *valuation* on the associated base polytope decomposition.

Known results

(Kapranov 1993)

• Any decomposition of a rank 2 matroid can be obtained by a sequence of hyperplane splits.

Known results

(Kapranov 1993)

• Any decomposition of a rank 2 matroid can be obtained by a sequence of hyperplane splits.

(Billera, Jia and Reiner 2009)

• Presented five rank 3 matroids on 6 elements such that each of the corresponding base polytope is indecomposable.

Known results

(Kapranov 1993)

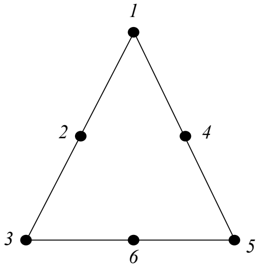
• Any decomposition of a rank 2 matroid can be obtained by a sequence of hyperplane splits.

(Billera, Jia and Reiner 2009)

• Presented five rank 3 matroids on 6 elements such that each of the corresponding base polytope is indecomposable.

• Provided a decomposition into three indecomposable pieces of $P(W)$ that cannot be obtained via hyperplane splits.

W :



Combinatorial decomposition

A base decomposition of a matroid M is a decomposition of the form

$$\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$$

where $\mathcal{B}(M_k)$, $1 \leq k \leq t$ and $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$, $1 \leq i \neq j \leq t$ are collections of bases of matroids.

Combinatorial decomposition

A base decomposition of a matroid M is a decomposition of the form

$$\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$$

where $\mathcal{B}(M_k)$, $1 \leq k \leq t$ and $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$, $1 \leq i \neq j \leq t$ are collections of bases of matroids.

M is said to be combinatorial decomposable if it has a base decomposition.
We say that the decomposition is *nontrivial* if $\mathcal{B}(M_i) \neq \mathcal{B}(M)$ for all i.

• If $P(M)$ is decomposable then clearly M is combinatorial decomposable.

• If $P(M)$ is decomposable then clearly M is combinatorial decomposable.

• A combinatorial decomposition do not necessarily induce a base polytope decomposition.

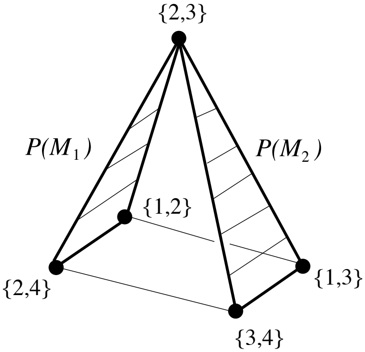
Example :

$\mathcal{B}(M) = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$ admit the combinatorial decomposition

$\mathcal{B}(M_1) = \{\{1,2\},\{2,3\},\{2,4\}\}$ and
$\mathcal{B}(M_2) = \{\{1,3\},\{2,3\},\{3,4\}\}$

We can verify that $\mathcal{B}(M_1), \mathcal{B}(M_2)$ and $\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{2,3\}$ are collection of bases of matroids.

However, $P(M_1)$ and $P(M_2)$ do not decompose $P(M)$.


{2,3}
P(M1)
P(M2)
{1,2}
{2,4}
{1,3}
{3,4}

Some geometry

Proposition Let P be a d-polytope with set of vertices X. Let H be a hyperplane such that $H \cap P \neq \emptyset$ with H not supporting de P. Then, H divides P into two polytopes P_1 and P_2, that is, $H \cap P = P_1 \cap P_2 = F \neq \emptyset$. Also, H partition X into two sets X_1 et X_2 with $X_1 \cap X_2 = W$. Then, for each edge $[u, v]$ of P we have $\{u, v\} \subset X_i$ for $i = 1$ or 2 if and only if $F = conv(W)$.

Some geometry

Proposition Let P be a d-polytope with set of vertices X. Let H be a hyperplane such that $H \cap P \neq \emptyset$ with H not supporting de P. Then, H divides P into two polytopes P_1 and P_2, that is, $H \cap P = P_1 \cap P_2 = F \neq \emptyset$. Also, H partition X into two sets X_1 et X_2 with $X_1 \cap X_2 = W$. Then, for each edge $[u, v]$ of P we have $\{u, v\} \subset X_i$ for $i = 1$ or 2 if and only if $F = conv(W)$.

Corollary $F = conv(W)$ if and only if $P_i = conv(X_i)$, $i = 1, 2$ (and thus $P = P_1 \cup P_2$ with P_1 and P_2 polytopes of the same dimension as P and sharing one facet).

Let (E_1, E_2) be a partition of E, that is, $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Let $r_i > 1$, $i = 1, 2$ be the rank of $M|_{E_i}$.

(E_1, E_2) is a good partition if there exist integers $0 < a_1 < r_1$ and $0 < a_2 < r_2$ such that :

(P1) $r_1 + r_2 = r + a_1 + a_2$ and

(P2) for any $X \in \mathcal{I}(M|_{E_1})$ with $|X| \leq r_1 - a_1$ and
for any $Y \in \mathcal{I}(M|_{E_2})$ with $|Y| \leq r_2 - a_2$
we have $X \cup Y \in \mathcal{I}(M)$.

Let (E_1, E_2) be a partition of E, that is, $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Let $r_i > 1$, $i = 1, 2$ be the rank of $M|_{E_i}$.

(E_1, E_2) is a good partition if there exist integers $0 < a_1 < r_1$ and $0 < a_2 < r_2$ such that :

(P1) $r_1 + r_2 = r + a_1 + a_2$ and

(P2) for any $X \in \mathcal{I}(M|_{E_1})$ with $|X| \leq r_1 - a_1$ and
for any $Y \in \mathcal{I}(M|_{E_2})$ with $|Y| \leq r_2 - a_2$
we have $X \cup Y \in \mathcal{I}(M)$.

Lemma Let (E_1, E_2) be a good partition of E. Let

$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq r_1 - a_1\}$

$\mathcal{B}(M_2) = \{B \in \mathcal{B}(M) : |B \cap E_2| \leq r_2 - a_2\}$

with r_i the rank of $M|_{E_i}$, $i = 1, 2$ and a_1, a_2 verifying (P1) et (P2).

Then, $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ are the collections of bases of two matroids, say M_1 and M_2.

Theorem (Chatelain and R.A. 2011) Let $M = (E, \mathcal{B})$ be a matroid and let (E_1, E_2) be a good partition of E. Then, $P(M) = P(M_1) \cup P(M_2)$ is a nontrivial hyperplane split where M_1 and M_2 are the matroids defined in the previous lemma.

Theorem (Chatelain and R.A. 2011) Let $M = (E, \mathcal{B})$ be a matroid and let (E_1, E_2) be a good partition of E. Then, $P(M) = P(M_1) \cup P(M_2)$ is a nontrivial hyperplane split where M_1 and M_2 are the matroids defined in the previous lemma.

Proof (idea) (i) $\mathcal{B}(M) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2)$,

(ii) $\mathcal{B}(M_1), \mathcal{B}(M_2) \subset \mathcal{B}(M)$,

(iii) $\mathcal{B}(M_1), \mathcal{B}(M_2) \not\subseteq \mathcal{B}(M_1) \cap \mathcal{B}(M_2)$,

(iv) $\mathcal{B}(M_1), \mathcal{B}(M_2), \mathcal{B}(M_1) \cap \mathcal{B}(M_2)$ are collections of bases,

(v) there exists a hyperplane containing the vertices corresponding to $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$ and not supporting $P(M)$,

(vi) each edge of $P(M)$ is an edge of either $P(M_1)$ or $P(M_2)$.

We say that two hyperplane splits $P(M_1) \cup P(M_2)$ and $P(M_1') \cup P(M_2')$ of $P(M)$ are equivalente if $P(M_i)$ is combinatorially equivalent to $P(M_i')$, $i = 1, 2$. They are different otherwise.

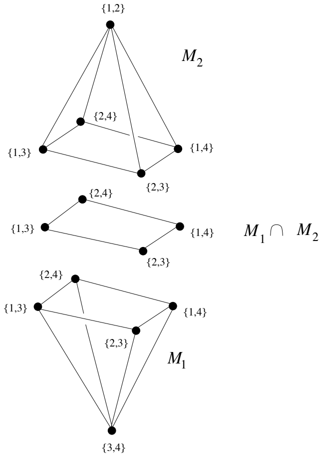
We say that two hyperplane splits $P(M_1) \cup P(M_2)$ and $P(M_1') \cup P(M_2')$ of $P(M)$ are equivalente if $P(M_i)$ is combinatorially equivalent to $P(M_i')$, $i = 1, 2$. They are different otherwise.

Corollary (Chatelain and R.A. 2011) Let $n \geq r + 2 \geq 4$ be integers and let $h(U_{r,n})$ be the number of different hyperplane splits of $P(U_{r,n})$. Then,

$$h(U_{r,n}) \geq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Example. We consider $U_{2,4}$. Then, $E_1 = \{1, 2\}$ and $E_2 = \{3, 4\}$ is a good partition (and thus $r_1 = r_2 = 2$) with $a_1 = a_2 = 1$.
We have $\mathcal{B}(M_1) = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$,
$\mathcal{B}(M_2) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ and
$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$.

Example. We consider $U_{2,4}$. Then, $E_1 = \{1,2\}$ and $E_2 = \{3,4\}$ is a good partition (and thus $r_1 = r_2 = 2$) with $a_1 = a_2 = 1$.
We have $\mathcal{B}(M_1) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$,
$\mathcal{B}(M_2) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ and
$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$.

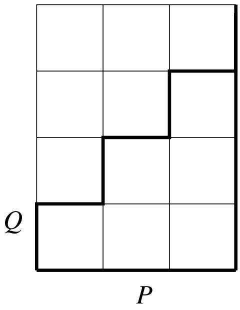


Lattice path matroid

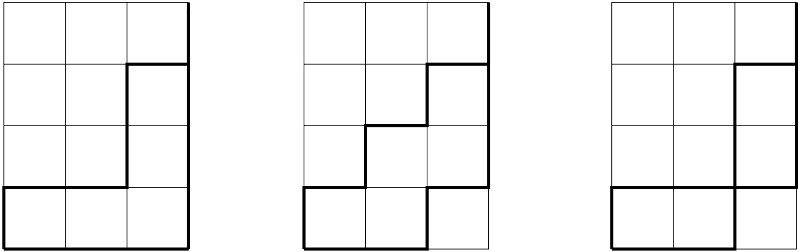
Let $m = 3$ and $r = 4$ and let $M[Q, P]$ be the transversal matroid on $\{1, \ldots, 7\}$ with presentation $(N_i : i \in \{1, \ldots, 4\})$ where $N_1 = [1, 2, 3, 4]$, $N_2 = [3, 4, 5]$, $N_3 = [5, 6]$ and $N_4 = [7]$.

Lattice path matroid

Let $m = 3$ and $r = 4$ and let $M[Q, P]$ be the transversal matroid on $\{1, \ldots, 7\}$ with presentation $(N_i : i \in \{1, \ldots, 4\})$ where $N_1 = [1, 2, 3, 4]$, $N_2 = [3, 4, 5]$, $N_3 = [5, 6]$ and $N_4 = [7]$.



Example. Transversal matroids (a) M_1, (b) M_2 and (c) $M_1 \cap M_2$.



(a) (b) (c)

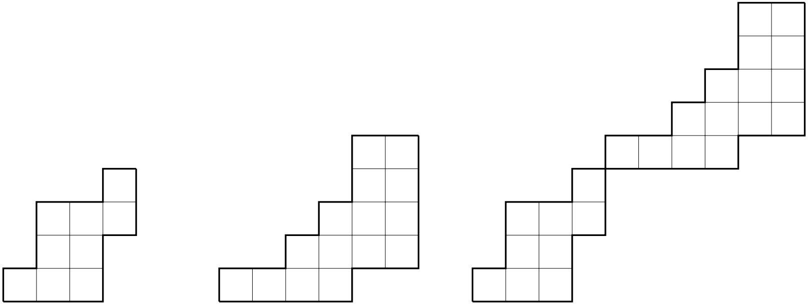
Theorem (Chatelain and R.A. 2011) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be two matroids of ranks r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a nontrivial hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a nontrivial hyperplane split.

Theorem (Chatelain and R.A. 2011) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be two matroids of ranks r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a nontrivial hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a nontrivial hyperplane split.

Remark : The class of lattice path matroids are closed under direct sum.

Theorem (Chatelain and R.A. 2011) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be two matroids of ranks r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a nontrivial hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a nontrivial hyperplane split.

Remark : The class of lattice path matroids are closed under direct sum.



Base matroid graph

The base matroid graph $G(M)$ of matroid M has a vertices the set of bases and two vertices are joined by an edge if the symmetric difference of the corresponding bases is equals two.

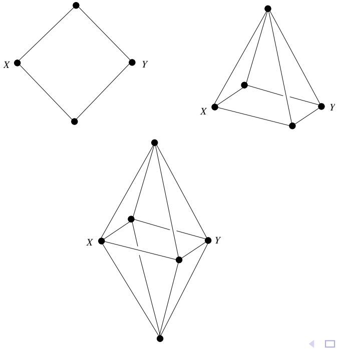
Base matroid graph

The base matroid graph $G(M)$ of matroid M has a vertices the set of bases and two vertices are joined by an edge if the symmetric difference of the corresponding bases is equals two.
(Maurer 1973)
• Characterisation of graphs that are base graph of a matroid.
• If x, y are two vertices at distance two then the neighbors of x and y form either a square, a pyramid or an octahedron.

Base matroid graph

The base matroid graph $G(M)$ of matroid M has a vertices the set of bases and two vertices are joined by an edge if the symmetric difference of the corresponding bases is equals two.
(Maurer 1973)

- Characterisation of graphs that are base graph of a matroid.
- If x, y are two vertices at distance two then the neighbors of x and y form either a square, a pyramid or an octahedron.



Lemma Let $M = (E, \mathcal{B})$ be a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid. If $X \in \mathcal{B}_1$ and all the neighbors of X (that is, the set of vertices of $G(M)$ adjacent to X) are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

Lemma Let $M = (E, \mathcal{B})$ be a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid. If $X \in \mathcal{B}_1$ and all the neighbors of X (that is, the set of vertices of $G(M)$ adjacent to X) are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

Theorem (Chatelain and R.A. 2011) Let M be a binary matroid. Then, $P(M)$ do not have a nontrivial hyperplane split.

Lemma Let $M = (E, \mathcal{B})$ be a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid. If $X \in \mathcal{B}_1$ and all the neighbors of X (that is, the set of vertices of $G(M)$ adjacent to X) are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

Theorem (Chatelain and R.A. 2011) Let M be a binary matroid. Then, $P(M)$ do not have a nontrivial hyperplane split.

Corollary Let M be a binary matroid. If $G(M)$ has a vertex X having exactly d neighbors where $d = dim(P(M))$ then $P(M)$ is indecomposable.

Lemma Let $M = (E, \mathcal{B})$ be a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid. If $X \in \mathcal{B}_1$ and all the neighbors of X (that is, the set of vertices of $G(M)$ adjacent to X) are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

Theorem (Chatelain and R.A. 2011) Let M be a binary matroid. Then, $P(M)$ do not have a nontrivial hyperplane split.

Corollary Let M be a binary matroid. If $G(M)$ has a vertex X having exactly d neighbors where $d = dim(P(M))$ then $P(M)$ is indecomposable.

Remark : The d-hypercube is the graph of bases of a binary matroid.

Lemma Let $M = (E, \mathcal{B})$ be a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid. If $X \in \mathcal{B}_1$ and all the neighbors of X (that is, the set of vertices of $G(M)$ adjacent to X) are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

Theorem (Chatelain and R.A. 2011) Let M be a binary matroid. Then, $P(M)$ do not have a nontrivial hyperplane split.

Corollary Let M be a binary matroid. If $G(M)$ has a vertex X having exactly d neighbors where $d = dim(P(M))$ then $P(M)$ is indecomposable.

Remark : The d-hypercube is the graph of bases of a binary matroid.

Corollary Let $P(M)$ be the polytope base polytope of the matroid M having as 1-skeleton the d-hypercube. Then, $P(M)$ is indecomposable.

Multi-decompositions

Question : Can we find a t-decomposition, $t \geq 3$ by applying a sequence of hyperplane split ?

Multi-decompositions

Question : Can we find a t-decomposition, $t \geq 3$ by applying a sequence of hyperplane split ?

Recall : the intersection $P(M_i) \cap P(M_j)$ must be a matroid for all i, j

Multi-decompositions

Question : Can we find a t-decomposition, $t \geq 3$ by applying a sequence of hyperplane split ?

Recall : the intersection $P(M_i) \cap P(M_j)$ must be a matroid for all i, j

Example :
$\mathcal{B}(M_1) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$
$\mathcal{B}(M_2) = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$
but
$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\{1,3\},\{2,3\},\{2,4\}\}$ is not a matroid.

Let $t \geq 2$ be an integer with $r \geq t$. Let $E = \bigcup_{i=1}^{t} E_i$ be a t-partition of $E = \{1, \ldots, n\}$ and let $r_i = r(M|_{E_i}) > 1$, $i = 1, \ldots, t$.

Let $t \geq 2$ be an integer with $r \geq t$. Let $E = \bigcup_{i=1}^{t} E_i$ be a t-partition of $E = \{1, \ldots, n\}$ and let $r_i = r(M|_{E_i}) > 1$, $i = 1, \ldots, t$.

We say that $\bigcup_{i=1}^{t} E_i$ is a good t-partition if there exist integers $0 < a_i < r_i$ with the following properties :

$(P1)$ $r = \sum_{i=1}^{t} a_i$,

$(P2)$

(a) For any j with $1 \leq j \leq t-1$

if $X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq a_1$ and
$Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})$ with $|Y| \leq a_2$,
then $X \cup Y \in \mathcal{I}(M)$.

$(P2)$

(b) For any pair j, k with $1 \leq j < k \leq t-1$

$$\text{if } X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j}) \quad \text{with } |X| \leq \sum_{i=1}^{j} a_i,$$

$$Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_k}) \quad \text{with } |Y| \leq \sum_{i=j+1}^{k} a_i,$$

$$Z \in \mathcal{I}(M|_{E_{k+1} \cup \cdots \cup E_t}) \quad \text{with } |Z| \leq \sum_{i=k+1}^{t} a_i,$$

then $X \cup Y \cup Z \in \mathcal{I}(M)$.

$(P2)$

(b) For any pair j, k with $1 \leq j < k \leq t-1$

$$\text{if } X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j}) \qquad \text{with } |X| \leq \sum_{i=1}^{j} a_i,$$

$$Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_k}) \qquad \text{with } |Y| \leq \sum_{i=j+1}^{k} a_i,$$

$$Z \in \mathcal{I}(M|_{E_{k+1} \cup \cdots \cup E_t}) \qquad \text{with } |Z| \leq \sum_{i=k+1}^{t} a_i,$$

then $X \cup Y \cup Z \in \mathcal{I}(M)$.

Notice that the good 2-partitions provided by $(P2)$ case (a) with $t = 2$ are the *good partitions*

Lemma Let $t \geq 2$ be an integer and let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, i=1,...,t. Let

$$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1\}$$

and, for each $j = 2, \ldots, t$, let

$$\mathcal{B}(M_j) = \left\{ B \in \mathcal{B}(M) \quad : |B \cap E_1| \geq a_1, \ldots, |B \cap \bigcup_{i=1}^{j-1} E_i| \geq \sum_{i=1}^{j-1} a_i, \right. \\ \left. |B \cap \bigcup_{i=1}^{j} E_i| \leq \sum_{i=1}^{j} a_i \right\}.$$

Then, $\mathcal{B}(M_j)$ is the collection of bases of a matroid for each $j = 1, \ldots, t$.

Theorem (Chatelain and R.A. 2014) Let $t \geq 2$ be an integer and let $M = (E, \mathcal{B})$ be a matroid of rank r. Let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, \ldots, t$. Then, $P(M)$ has a sequence of hyperplane splits yielding the decomposition

$$P(M) = \bigcup_{i=1}^{t} P(M_i),$$

where M_i, $1 \leq i \leq t$, are the matroids defined in previous lemma

Uniform matroid

Corollary (Chatelain and R.A. 2014) Let $n, r, t \geq 2$ be integers with $n \geq r + t$ and $r \geq t$. Let $p_t(n)$ be the number of different decompositions of the integer n of the form $n = \sum_{i=1}^{t} p_i$ with $p_i \geq 2$ and let $h_t(U_{n,r})$ be the number of *different* decompositions of $P(U_{r,n})$ into t pieces. Then,

$$h_t(U_{r,n}) \geq p_t(n).$$

Rank 3 matroids

Corollary (Chatelain and R.A. 2014) Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2$ be a partition of the points of the geometric representation of M such that

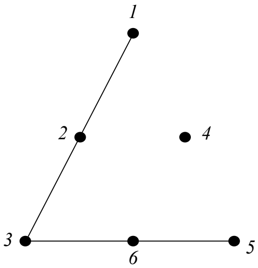
1) $r(M|_{E_1}) \geq 2$ and $r(M|_{E_2}) = 3$;

2) for each line l of M, if $|l \cap E_1| \neq \emptyset$, then $|l \cap E_2| \leq 1$.

Then, $E = E_1 \cup E_2$ is a 2-good partition.

Example

Let M be the rank-3 matroid arising from the configuration of points given below.



It can be easily checked that $E_1 = \{1, 2\}$ and $E_2 = \{3, 4, 5, 6\}$ verify the conditions of the previous Corollary. Thus, $E_1 \cup E_2$ is a 2-good partition.

Rank 3 matroids

Corollary (Chatelain and R.A. 2014) Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2 \cup E_3$ be a partition of the points of the geometric representation of M such that

1) $r(M|_{E_i}) \geq 2$ for each $i = 1, 2, 3$,

2) for each line l with at least 3 points of M,

a) if $|l \cap E_1| \neq \emptyset$ then $|l \cap (E_2 \cup E_3)| \leq 1$,

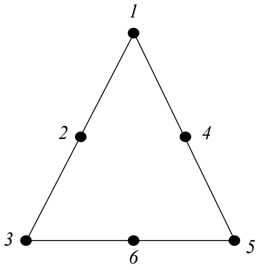
b) if $|l \cap E_3| \neq \emptyset$ then $|l \cap (E_1 \cup E_2)| \leq 1$.

Then, $E = E_1 \cup E_2 \cup E_3$ is a 3-good partition.

Example

Let W be the matroid shown below

W :



It can be checked that $E_1 = \{1, 6\}$, $E_2 = \{2, 5\}$, and $E_3 = \{3, 4\}$ verify the conditions of the previous Corollary. Thus, $E_1 \cup E_2 \cup E_3$ is a good 3-partition.

Direct sum

Theorem (Chatelain and R.A. 2014) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ admits a sequence of t hyperplane splits if either $P(M_1)$ or $P(M_2)$ admits a sequence of t hyperplane splits.