Ramsey for complete graphs with a dropped edge or a triangle

Luis Pedro Montejano [1]

Centro de Investigación en Matemáticas
CIMAT
Guanajuato, México

Jonathan Chappelon and Jorge Luis Ramírez Alfonsín [2,3]

Institut de Mathématiques et de Modélisation de Montpellier
Université de Montpellier
Montpellier, France

Abstract

Let $K_{[k,t]}$ be the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped (note that $K_{[k,1]}$ is just K_k). In this paper we study $R(K_{[k_1,t_1]},\ldots,K_{[k_r,t_r]})$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i).

We first present a general upper bound (containing the well-known Graham-Rödl upper bound for complete graphs in the particular case when $t_i = 1$ for all i). We then focus our attention when $r = 2$ and dropped cliques of order 2 and 3 (edges and triangles). We give the exact value for $R(K_{[n,2]}, K_{[4,3]})$ and $R(K_{[n,3]}, K_{[4,3]})$ for all $n \geq 2$.

Keywords: Ramsey number, recursive formula.

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1 Introduction

Let K_n be a complete graph and let $r \geq 2$ be an integer. A r-edge coloring of a graph is a surjection from $E(G)$ to $\{0, \ldots, r-1\}$ (and thus each color class is not empty). Let $k \geq t \geq 1$ be positive integers. We denote by $K_{[k,t]}$ the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped, see Figure 1.

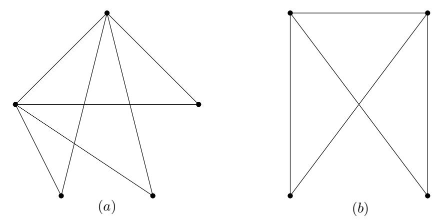

Fig. 1. (a) $K_{[5,3]}$ and (b) $K_{[4,2]}$

Let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers with $k_i \geq t_i$ for all $i \in \{1, \ldots, r\}$. Let $R([k_1, t_1], \ldots, [k_r, t_r])$ be the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i. In the case when $k_i = t_i$ for some i, we set

$$R([k_1, t_1], \ldots, [k_{i-1}, t_{i-1}], [t_i, t_i], [k_{i+1}, t_{i+1}], \ldots, [k_r, t_r]) \leq t_i.$$

We note that equality is reached at $\min_{1 \leq i \leq r} \{t_i | t_i = k_i\}$. Since the set of all the edges of $K_{[t_i,t_i]}$ (which is empty) can always be colored with color i. We also notice that the case $R([k_1, 1], \ldots, [k_r, 1])$ is exactly the classical Ramsey number $r(k_1, \ldots, k_r)$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic K_{k_i} for some i). We refer the reader to the excellent survey [6] on Ramsey numbers for small values. In this paper, we investigate $R([k_1, t_1], \ldots, [k_r, t_r])$.

2 General upper bound

In this section we present a recursive formula (Lemma 2.1) that yields to an explicit general upper bound (Theorem 2.2). The latter contains the well-known explicit general upper bound for $R([k_1, 1], \ldots, [k_r, 1])$ due to Graham and Rödl [3] (see Equation (4)).

[1] Email: lmontejano@cimat.mx
[2] Email: jramirez@um2.fr
[3] Email: jonathan.chappelon@um2.fr

The following recursive inequality is classical in Ramsey theory

$$(1)\quad r(k_1,k_2,\ldots,k_r) \le r(k_1-1,k_2,\ldots,k_r) + r(k_1,k_2-1,\ldots,k_r) + \cdots + \\ + r(k_1,k_2,\ldots,k_r-1) - (r-2)$$

In the same spirit, we have the following.

Lemma 2.1 *Let $r \ge 2$ and let $k_1,\ldots,k_r$ and $t_1,\ldots,t_r$ be positive integers with $k_i \ge t_i + 1 \ge 2$ for all i. Then,*

$$\begin{aligned} R([k_1,t_1],\ldots,[k_r,t_r]) \le\ & R([k_1-1,t_1],[k_2,t_2],\ldots,[k_r,t_r]) \\ & + R([k_1,t_1],[k_2-1,t_2],\ldots,[k_r,t_r]) \\ & \vdots \\ & + R([k_1,t_1],[k_2,t_2],\ldots,[k_r-1,t_r]) - (r-2). \end{aligned}$$

A similar recursive inequality has been treated in [7] in a much more general setting in which a family of graphs are intrinsically constructed via two operations *disjoin unions* and *joins* (see also [4] for the case $r = 2$). However, it is not clear how the latter could be used to obtain Lemma 2.1 that allows us to give the following general upper bound for $R([k_1,t_1],\ldots,[k_r,t_r])$ (which was not considered in [7]).

Theorem 2.2 *Let $r \ge 2$ be a positive integer and let $k_1,\ldots,k_r$ and $t_1,\ldots,t_r$ be positive integers such that $k_i \ge t_i$ for all $i \in \{1,\ldots,r\}$. Then,*

$$R([k_1,t_1],\ldots,[k_r,t_r]) \le \max_{1\le i\le r}\{t_i\} \binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)}{k_1-t_1,k_2-t_2,\ldots\ldots,k_r-t_r}$$

where $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\ldots\ldots,n_r}$ is the multinomial coefficient defined by $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\ldots\ldots,n_r} = \frac{(n_1+\cdots+n_r)!}{n_1!n_2!\cdots n_r!}$, for all nonnegative integers $n_1,\ldots,n_r$.

Theorem 2.2 is a natural generalization of the well-known explicit upper bound for classical Ramsey numbers. Indeed, an immediate consequence of Theorem 2.2 (by taking $t_i = 1$ for all i) is the following classical upper bound due to Graham and Rödl [3, (2.48)]

$$(2)\qquad R([k_1,1],\ldots,[k_r,1]) \le \binom{k_1+\cdots+k_r-r}{k_1-1,\ldots,k_r-1}.$$

Let $k \ge t \ge 2$ and $r \ge 2$ be integers and let $R_r([k,t]) = R(\underbrace{[k,t],\ldots,[k,t]}_{r})$.

An immediate consequence of Theorem 2.2 (by taking $k = k_1 = \cdots = k_n$ and

$t = t_1 = \cdots = t_n$) is the following inequality

(3) $$R_r([k,t]) \le t\binom{r(k-t)}{k-t,\ldots,k-t}$$

Moreover, if $t = 1$ then

(4) $$R_r([k,1]) \le \frac{(rk-r)!}{((k-1)!)^r}.$$

3 Exact values

By the so-called Chvátal's result [2], we know that the exact value of the Ramsey number of $K_{[4,3]}$ (a star) versus cliques is given by $R([n,1],[4,3]) = 3n-2$ for all $n \ge 1$. We then naturally focus our attention to the Ramsey number of $K_{[4,3]}$ versus cliques with either a dropped edge or a dropped triangle, see [1] where $R([m,1],[n,2])$ has been computed for numerous cases. We provide the new following exact values of Ramsey numbers.

Theorem 3.1 *Let $n \ge 2$ be an integer. Then,*

- $R([n,2],[4,3]) = 2$ *for* $n = 2$,
- $R([n,2],[4,3]) = 5$ *for* $n = 3$,
- $R([n,2],[4,3]) = 3n-5$ *for* $n \ge 4$.

Theorem 3.2 *Let $n \ge 2$ be an integer. Then,*

- $R([n,3],[4,3]) = 3$ *for* $n = 3$,
- $R([n,3],[4,3]) = 6$ *for* $n = 4$,
- $R([n,3],[4,3]) = 8$ *for* $n = 5$,
- $R([n,3],[4,3]) = 11$ *for* $n = 6$,
- $R([n,3],[4,3]) = 3n-8$ *for* $n \ge 7$.

3.1 An estimation for $R([n,2],[5,3])$

By considering $K_{[5,3]}$ as the book graph B_3, it was proved in [5,8] that

$$R([n,1],[5,3]) \le \frac{3n^2}{\log(n/e)},$$

for all positive integers n.

The following result is a first estimation for the value $R([n,2],[5,3])$.

Theorem 3.3 *Let $n \ge 2$ be an integer. Then,*

- $R([n,2],[5,3]) = 2$ *for* $n = 2$,
- $R([n,2],[5,3]) = 7$ *for* $n = 3$,
- $R([n,2],[5,3]) \leq 3\binom{n+1}{2} - 5n + 4$ *for* $n \geq 4$.

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