On Kneser transversals

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Theorem (Helly) Let $\mathcal{A}$ be a finite family of at least $d+1$ convexes sets in $\mathbb{R}^d$. If every $d+1$ members of $\mathcal{A}$ have a common point then there is acommon point to all members of $\mathcal{A}$.

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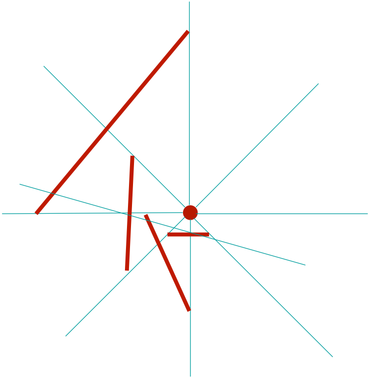
Question (Vincensini 1935) Is there a Helly-type theorem for transversal lines in $\mathbb{R}^2$?

That is, does there exist an integer m such that if all membres of a finite family $\mathcal{A}$ of sets in $\mathbb{R}^2$ are intersected by a line then there is a line intersecting all members of $\mathcal{A}$?

Counterexample : avec $m = 5$, any subfamily fo 4 convexes have a transversal line but there is not a transversal line to all 5.



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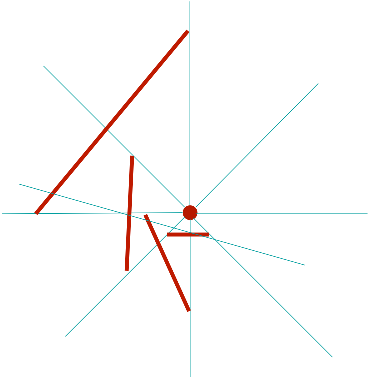
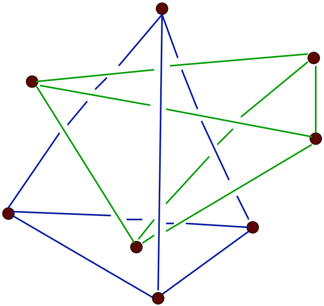


Theorem (Hadwiger) Let $\mathcal{A}$ be a finite family of convexe sets in $\mathbb{R}^2$ pairwise disjoints. If there exists a linear order of $\mathcal{A}$ such that any 3 membres of $\mathcal{A}$ are intersected by a line in the given induced order, then $\mathcal{A}$ admit a transversal line.

Let 8 points in $\mathbb{R}^3$ in general position.



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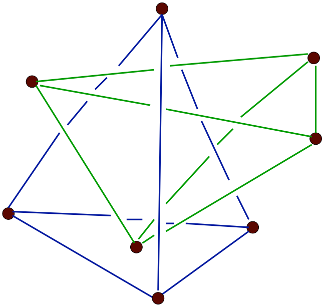
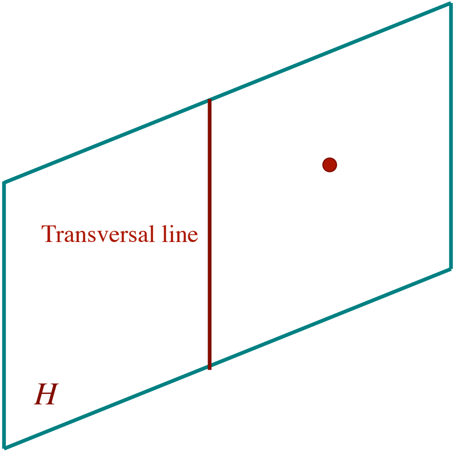


Question : Is there any transversal line to all the tetrahedra ?

NEVER



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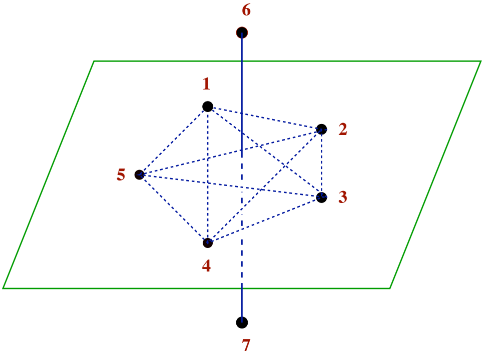
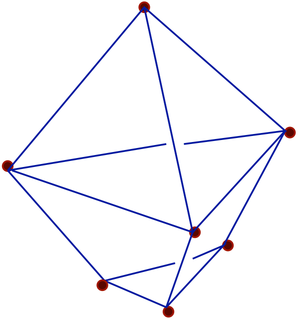
Question : Let $\mathcal{A}$ be a set of 7 points in $\mathbb{R}^3$ in general position. Is there a transversal line to all the tetrahedra in $\mathcal{A}$?

Some times YES and Some times NO





Definitions

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ the largest integer n such that for any set of n points (no necessarely in general position) in $\mathbb{R}^d$, there is a $(d-\lambda)$-plane transversal to the convex hulls of all the k-set points.

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$M(k, d, \lambda) \stackrel{\text{def}}{=}$ the smallest integer n such that for any set of $n$points in $\mathbb{R}^d$, do not admit a $(d - \lambda)$-plane transversal to the convex hulls of all the k-set points.

- $m(k,d,\lambda) < M(k,d,\lambda)$.

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Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$$M(k, d, \lambda) = \begin{cases} d + 2(k - \lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d - \lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

Kneser hypergraphs

An hypergraph H is a couple $(V, \mathcal{H})$ where V (vertices) is a finite set and $\mathcal{H}$ (hyperedges) is a collection of subsets of V.

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The Kneser hypergraph $K^{\lambda+1}(n,k)$ is the hypergraph $(V, \mathcal{H})$ where V is the collection of all k-sets of n and $\mathcal{H} = \{(S_1, \ldots, S_\rho) |\ 2 \leq \rho \leq \lambda + 1,\ S_1 \cap \cdots \cap S_\rho = \emptyset\}$.

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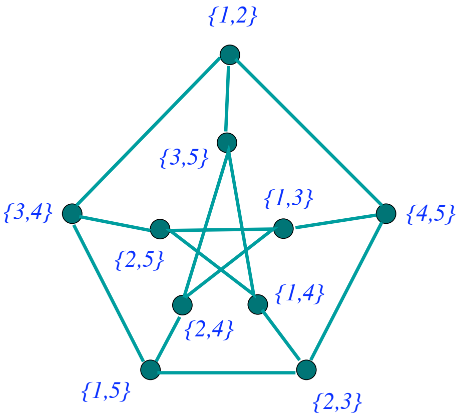
Remark : Kneser graphs are obtained when $\lambda = 1$.

Kneser graph with $n = 5$, $k = 2$ and $\lambda = 1$ (the well-known Petersen graph)



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A collection of vertices $\{S_1, \ldots, S_\rho\}$ of $K^{\lambda+1}(n,k)$ is in the same colour class if and only if either

a) $\rho \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\rho \neq \emptyset$ or

b) $\rho > \lambda + 1$ and any $(\lambda + 1)$-sub-family $\{S_{i_1}, \ldots, S_{i_{\lambda+1}}\}$ of $\{S_1, \ldots, S_\rho\}$ is such that $S_{i_1} \cap \cdots \cap S_{i_{\lambda+1}} \neq \emptyset$

(that is, they verify the λ-Helly property).

Theorem (Arocha, Bracho, Montejano, R.A., 2011)

- If $\chi(K^{\lambda+1}(n,k)) \leq d-\lambda+1$ then $n \leq m(k,d,\lambda)$.

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- $d-\lambda+k+\lceil \frac{k}{\lambda} \rceil - 1 \leq m(k,d,\lambda)$.

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- $d-\lambda+k+\lceil \frac{k}{\lambda} \rceil - 1 \leq m(k,d,\lambda)$.
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Theorem (Lovász) $\chi(K^2(n,k)) = n-2k+2$.

System of lines

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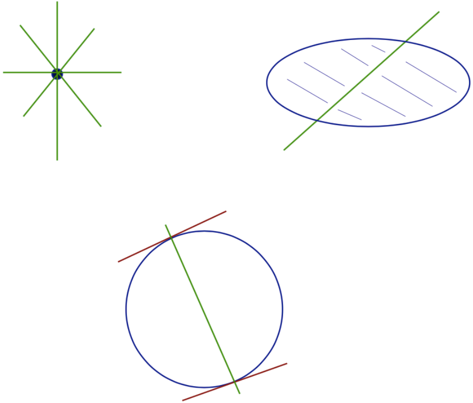

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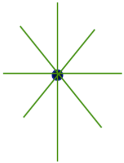

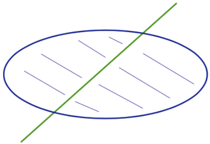

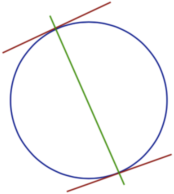

Fact : Two systems of lines in $\mathbb{R}^2$ coincide in one direction.

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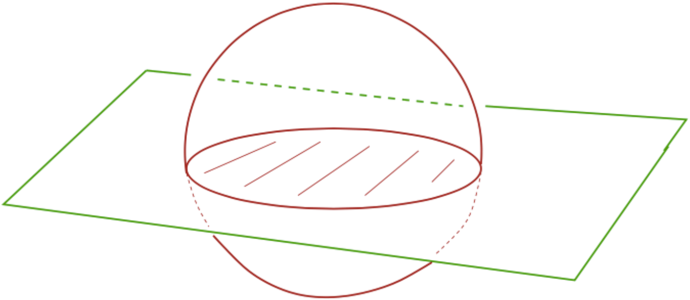

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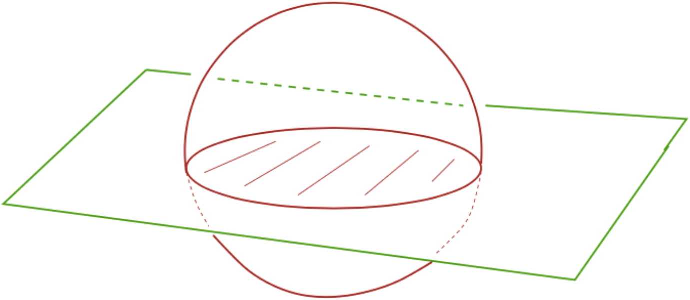

Fait : Three systems of lines in $\mathbb{R}^3$ coincide in one direction.

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Indeed, given ϕ it is enough to choose an hyperplane orthogonal to x going through $\phi(x)x$.

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Central Point Theorem

Theorem (Rado) Let X be a set of n points in $\mathbb{R}^d$. Then, there exists a point $x \in \mathbb{R}^d$ such that any closed half-space H through x contains at least $\lceil \frac{n}{d+1} \rceil$ points of X

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Generalization

Theorem (Arocha, Bracho, Montejano, R.A. 2011)
Let X be a set of n points in $\mathbb{R}^d$. Then, there exists a $(d-\lambda)$-plane L such that for any closed half-space H containing L admits at least $\lfloor \frac{n-d+2\lambda}{\lambda+1} \rfloor + (d-\lambda)$ points of X.

Complete Kneser transversals

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

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• It turns out that the function m^* have two different behavours according to :

$$\alpha(d,\lambda) = \frac{\lambda-1}{\lceil \frac{d}{2} \rceil} \geq 1$$

et

$$\alpha(d,\lambda) = \frac{\lambda-1}{\lceil \frac{d}{2} \rceil} < 1$$

Theorem (Chappelon, Martinez, Montejano, R.A., 2016)

- If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$.

Results on m^*

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- If $\alpha(d,\lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k,d,\lambda)$.
- If $\alpha(d,\lambda) \geq 1$ then $m^*(k,d,\lambda) = d - \lambda + 1$.

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- If $\alpha(d,\lambda) < 1$ then $\lim_{k\to\infty} \frac{\eta(k,d,\lambda)}{k} = 2 - \alpha(d,\lambda)$.

Corollary If $\alpha(d,\lambda) < 1$ then $m^*(k,d,2) < m(k,d,2)$ for a large k and $d \geq 3$.

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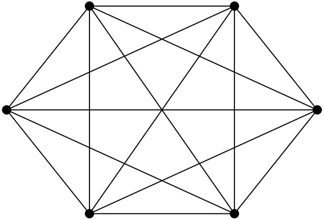

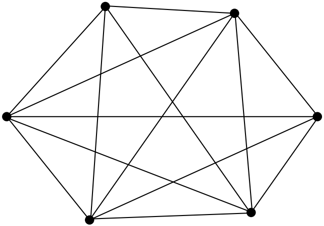

Un transversal de Kneser est dit stable (resp. instable) si l'ensemble des points donnés peuvent-être perturbés (déplacer chaque point de sa position originale à une distance plus petit que $\epsilon > 0$) tel que la nouvelle configuration de points admet (s'il y a) seulement de transversaux complets (resp. n'admet pas de transversaux complets).

Théorème (Chappelon, Martinez, Montejano, R.A., 2017)
Soit $X = \{x_1, x_2, \dots x_n\}$ un ensemble de $n = d + 2(k - \lambda)$ points en position générale en $\mathbb{R}^d$. On suppose que L est un $(d - \lambda)$-plan transversal aux enveloppes convexes de tous les k-ensembles de X avec $\lambda = 2, 3$ et $k \geq \lambda + 2$, $d \geq 2(\lambda - 1)$. Alors, soit

(1) L est un transversal de Kneser complet (i.e., contenant $d - \lambda + 1$ points de X) ou bien

(2) $\mid L \cap X \mid = d - 2(\lambda - 1)$ et les autres $2(k - 1)$ points de X sont couplés en $k - 1$ paires de tel sort que L coupe les segments fermés déterminés par chaque couple.

Théorème (Chappelon, Martinez, Montejano, R.A., 2017)
Soit $\epsilon > 0$ et soit $X = \{x_1, \ldots, x_n\}$ un ensemble fini de points dans $\mathbb{R}^d$. On suppose que $n = d + 2(k - \lambda)$, $k - \lambda \geq 2$ et $\lambda = 2, 3$. Alors, il existe un ensemble $X' = \{x'_1, \ldots, x'_n\}$ de points dans $\mathbb{R}^d$ en position générale tel que $\mid x_i - x'_i \mid < \epsilon$, pou tout $i = 1, \ldots, n$, et avec la propriété que chaque $(d - \lambda)$-plan transversal aux enveloppes convexes de tous les k-ensembles de X' est complet (i.e., il contient $d - \lambda + 1$ points de X').

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Soit $\lambda = 2, 3$, $k - \lambda \geq 2$ et $d \geq 2(\lambda - 1)$. Alors,

$$m(k, d, \lambda) < d + 2(k - \lambda).$$

Quelques résultats informatiques

On sait $m(4,3,2) = 6$ et $M(4,3,2) = 8$.

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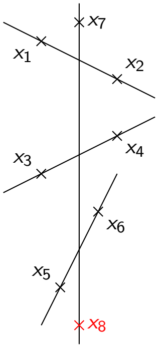
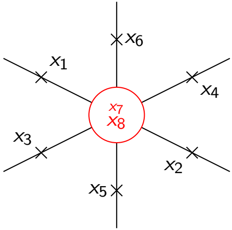
Question Qu'en est-il pour l'existence d'une ligne transversal à tous les tétraèdres dans une configuration de 7 points dans $\mathbb{R}^3$?

Ligne de Kneser complet : déterminé par les matroïdes orientés

Ligne de Kneser complet : déterminé par les matroïdes orientés
Ligne de Kneser : un peu plus compliqué



Représentation dans $\mathbb{R}^3$



Projection dans $\mathbb{R}^2$

Théorème (Chappelon, Martinez, Montejano, R.A., 2017)
Parmi le 246 types d'ordre différents de 7 points en position générale dans $\mathbb{R}^3$ il y a :
$A = 124$ admettant une ligne de Kneser complet à tous les tétraèdres

$B = 124$ admettant une représentation pour laquelle il existe une ligne de Kneser non-complet à tous les tétraèdres

On a $|A \cap B| = 46, |A \setminus B| = |B \setminus A| = 78$ et $|\overline{A \cup B}| = 44$. Par ailleur, pour chacun de 78 type d'ordre de $B \setminus A$ il existe une représentation pour laquelle il n'y a pas une ligne transversale de Kneser.