Oriented Matroids : applications

J.L. Ramírez Alfonsín

IMAG, Université de Montpellier

São Paulo, Brasil
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- An oriented matroid M is called acyclic if it does not contain positive circuits.
- Let M be an oriented matroid. We say that an element e of M is interior if there is a circuit $C = (C^+, C^-)$ of M with $C^+ = \{e\}$.

Projective transformations

A projective transformation T is defined as

$$\begin{array}{rccc} T: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & x & \mapsto & \frac{Ax+b}{\langle c,x\rangle+\delta} \end{array}$$

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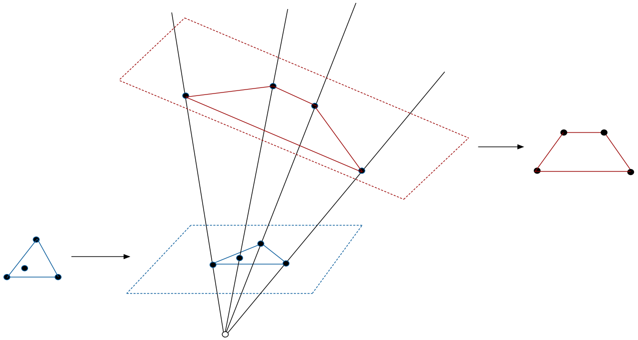

Projective transformations

Theorem (Cordovil and da Silvia, 1985) Let E be a finite set of points in $\mathbb{R}^d$ and let $Aff(E)$ be the affine oriented matroid associated to E. Then, there is an acyclic reorientation of $Aff(E)$, say $_{-A}Aff(E)$ for some $A \subseteq E$ if and only if there exists a permissible projective transformation T for E, $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $_{-A}Aff(E)$ is isomorphic to $Aff(T(E))$ (the correspondance is given by the map $x \mapsto T(x)$).

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Topological version Determine the largest integer $n = g(d)$ such that for any d-dimensional simple arrangement of n hyperplanes there is a complete cell (that is, a region bounded by all the hyperplanes)

Known results

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Strategy We construct a representable oriented matroid M of rank $r \geq 3$ with $2(r-1) + \lfloor \frac{r}{2} \rfloor$ elements such that any acyclic reorientation of M has at least one interior element.

Lawrence oriented matroids

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A chirotope χ correspond to a Lawrence oriented matroids M_A iff there exists a matrix $A = (a_{i,j})$ with entries from $\{+1, -1\}$ where the i-th row correspond to the chirotope of M_i such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i}$$

where $B = \{j_1 \leq \cdots \leq j_r\}$ is an ordered base.

Properties

- The coefficients $a_{i,j}$ with $i \geq j$ or $j - n \geq i - r$ do not play any role in the definition of M_A.

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• The opposite chirotope $-\chi$ is obtained by inverting the sign of all the coefficients of a line of A.

• The oriented matroid $-_c M$ is obtained by inverting the sign of all the coefficients of column c of A.

Chess board

Let $A = (a_{i,j})$, $1 \leq i \leq r, 1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$. The chess board $B[A]$ is a chess board of size $(r-1) \times (n-1)$ and a square is white if the product of its corresponding corners is $+1$, black otherwise.

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Observation The chess board is invariant under reversing the signs of the coefficient of a given column.

Top and Bottom Travels

(1) TT (BT) starts at $a_{1,1}$ (at $a_{r,n}$)

(2) Suppose that TT (BT) arrives at $a_{i,j}$. Let s (s') be the minimum (maximal) integer $j < s \leq n$ ($1 < s' \leq j$) such that $a_{i,j} = -a_{i,s}$ ($a_{i,j} = -a_{i,s'}$).

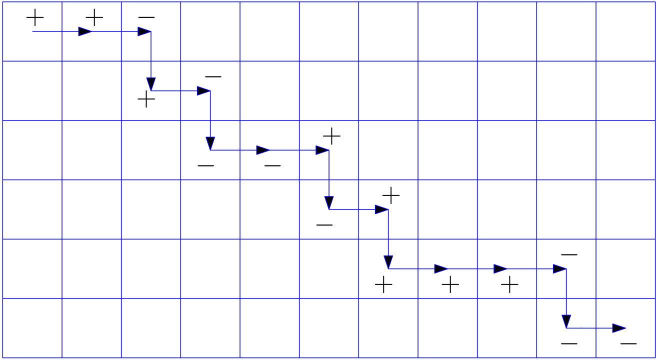
(3) **If** s (s') does not exists **then** TT goes horizontally to $a_{i,n}$ and stops (BT goes horizontally to $a_{i,1}$ and stops)

(4) **else**

   (a) **if** $1 \leq i \leq r-1$ ($2 \leq i \leq r$) **then**
   TT goes horizontally to $a_{i,s}$ and then goes vertically to $a_{i+1,s}$
   (BT goes horizontally to $a_{i,s'}$ and then goes vertically to $a_{i-1,s'}$)

   (a) **else** TT goes horizontally to $a_{r,s}$ and stops
   (BT goes horizontally to $a_{1,s'}$ and stops)

Example of a Top Travel



Four key lemmas

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Lemma 2 Let M_A be a Lawrence oriented matroid and A the matrix associated $A = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$ and entries from $\{+1, -1\}$. Then, the following conditions are equivalent.

(a) M_A is cyclic,

(b) TT ends at $a_{r,s}$ for some $1 \leq s < n$,

(c) BT ends at $a_{1,s'}$ for some $1 < s \leq n$.

Four Key Lemmas

We say that TT and BT are parallel at column k with $2 \leq k \leq n-1$ in A if $TT = (a_{1,1}, \ldots, a_{i,k-1}, a_{i,k}, a_{i,k+1}, \ldots)$ and either $BT = (a_{r,n}, \ldots, a_{i,k+1}, a_{i,k}, a_{i,k-1}, \ldots)$ or $BT = (a_{r,n}, \ldots, a_{i+1,k+1}, a_{i+1,k}, a_{i+1,k-1}, \ldots)$, $1 \leq i \leq r$.

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Lemma 3 Let M_A be a Lawrence oriented matroid and A the matrix associated $A = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$ and entries from $\{+1, -1\}$. Then k is an interior element of M_A if and only if

(a) $BT = (a_{r,n}, \ldots, a_{1,2}, a_{1,1})$ for $k = 1$,

(b) $TT = (a_{1,1}, \ldots, a_{r,n-1}, a_{r,n})$ for $k = n$,

(c) TT and BT are parallel at k for $2 \leq k \leq n-1$.

Example

Let M_A be the Lawrence oriented matroid associated to the matrix A given below

	1	2	3	4	5	6	7
1	+	−	−	+	+	+	+
2	+	−	+	+	+	+	+
3	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+

M_A is acyclic and 4, 5 and 6 are interior elements

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A plain travel T on the entries of A is formed by horizontal and vertical mouvements such that T starts with $a_{1,1}, a_{1,2}$ and T cannot make two consecutive vertical mouvements.

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Lemma 4 Let $A = (a_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$. Then, there exists a natural bijection between the set of all plain travels of A and the set of all acyclic reorientations of M_A.

Solution

It is sufficient to contruct a matrix A of size $r \times 2(r-1) + \lfloor \frac{r}{2} \rfloor$, $r \geq 3$ such that for any given plain travel T of A the corresponding Top Travel in the matrix A' (obtained from A such that T is transformed in TT of A') has at least one interior elements.

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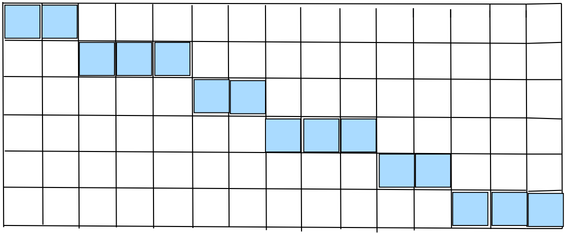
Solution : any matrix arising the following chess board



Spatial representations

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Example : Spatial representation of K_5

Spatial linear representations

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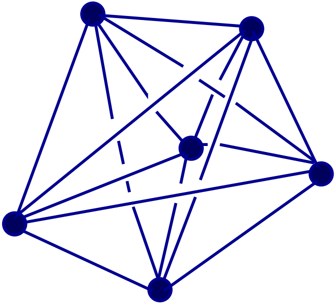
Example : Spatial representation of K_6



Spatial linear representations

Let $m(L)$ (resp. $\bar{m}(L)$) be the smallest integer such that any spatial representation (resp. linear) of K_n with $n \geq m(L)$ (resp. $n \geq \bar{m}(L)$) contains cycles isotopic to L.

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$$\bar{m}(L) \geq s(L)$$

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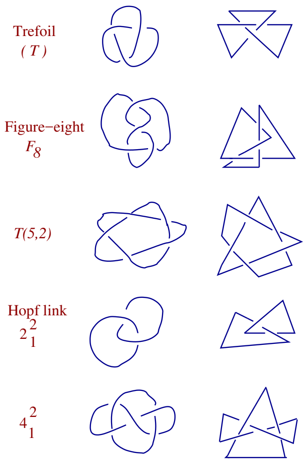

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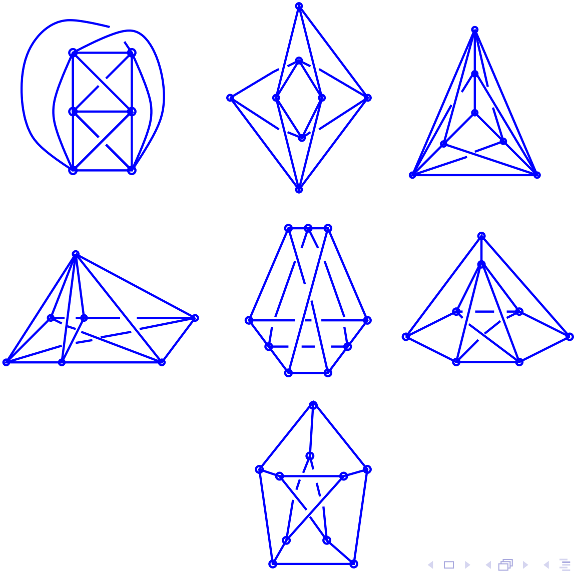
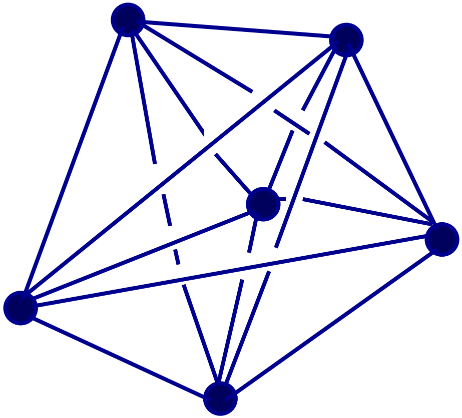
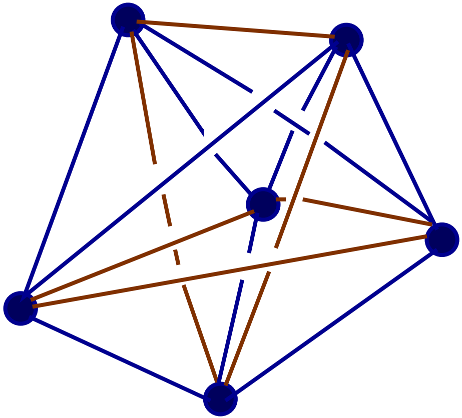
Theorem (Robertson, Seymour, Thomas 1995) Any spatial representation of a graph G contains a non-trivial link if and only if G do not contain as a minor one of the 7 graphs obtained from K_6 by a $Y - \Delta$ or $\Delta - Y$ change (these graphs are known as Petersen's family).







Some results

Theorem (Negami 1991) $\bar{m}(L)$ is finite

Theorem (R.A. 1998) $\bar{m}(T \text{ ou } T^*) = 7$

Theorem (R.A. 2000) $\bar{m}(4_1^2) > 7$

Theorem (R.A. 2007) $\bar{m}(F_8), \bar{m}(T(5,2)) > 8$

Applying oriented matroids

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Proof (sketch) Consider the circuits $(1, 2, 3, \bar{5}, \bar{6})$ and $(1, 2, 4, \bar{5}, \bar{6})$

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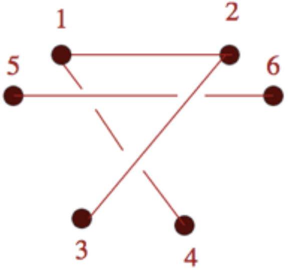

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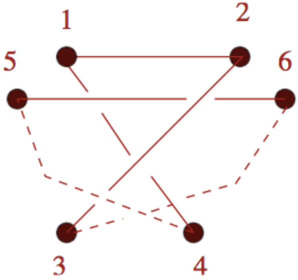

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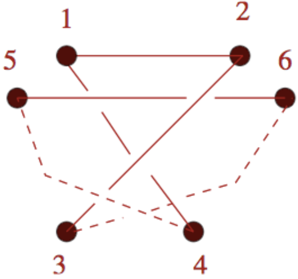

Set a proper condition of circuits and verify that they hold for any realisable rank 4 oriented matroid on 7 elements

Cyclic polytope

Let $t_1, \ldots, t_n \in \mathbb{R}$. The cyclic, of dimension d on n vertices is defined as

$$C_d(t_1, \ldots, t_n) := conv(x(t_1), \ldots, x(t_n))$$

where $x(t_i) = (t_i, t_i^2, \ldots, t_i^d)$ are points in the moment curve

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$$C_d(t_1, \ldots, t_n) \to C_d(n)$$

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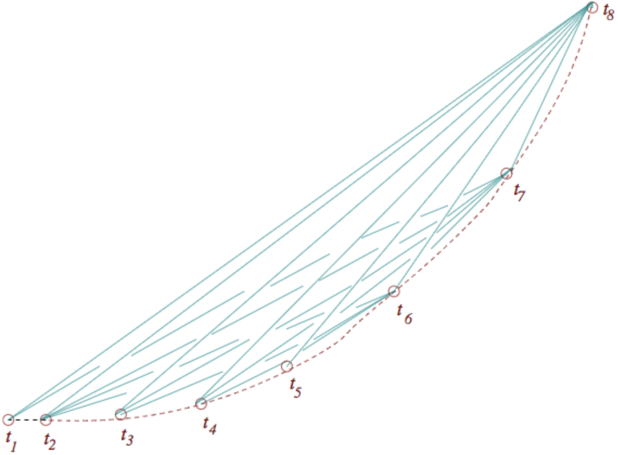
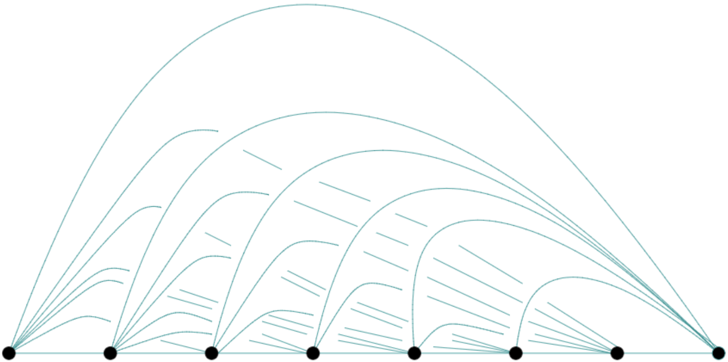


Alternating oriented matroid

We use the circuits of the corresponding oriented matroid to get :



Knots in the cyclic polytope

Theorem (R.A., 2008) Let $D(K)$ be the diagram of knot K with n crossings. Then, there is cycle in $C_3(m)$ isotopic to K with $m \leq 7n$.

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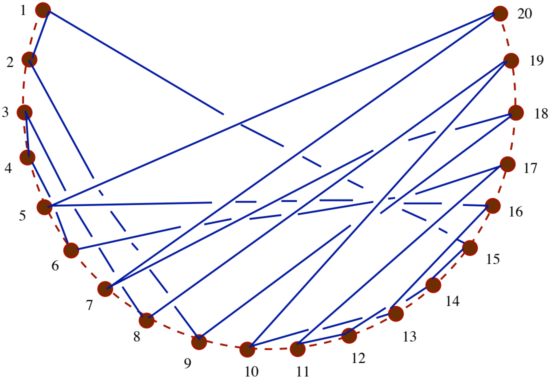

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