Frobenius problem: Applications

J.L. Ramírez Alfonsín

Université Montpellier 2

Applications

Let $g_S = \{g(s_1, \ldots, s_n) - s | s \in S\}$. A semigroup S is called symmetric if $S \cup g_S = \mathbb{Z}$.

Symmetric semigroups

(Bresinsky, 1979) Monomial curves

(Kunz, 1979, Herzog, 1970) Gorestein rings

(Apéry, 1945) Classification plane of algebraic branches

(Buchweitz, 1981) Weierstrass semigroups

(Pellikaan and Torres, 1999) Algebraic codes

Applications

Let $g_S = \{g(s_1, \ldots, s_n) - s | s \in S\}$. A semigroup S is called symmetric if $S \cup g_S = \mathbb{Z}$.

Symmetric semigroups

(Bresinsky, 1979) Monomial curves

(Kunz, 1979, Herzog, 1970) Gorestein rings

(Apéry, 1945) Classification plane of algebraic branches

(Buchweitz, 1981) Weierstrass semigroups

(Pellikaan and Torres, 1999) Algebraic codes

Random vectors generator (Vizvári, 1994) A method to generate a random vector without cyclic drawbacks

Petri nets (Chrzastowski-Wachtel and Raczunas, 1993) The problem of finding a formula for the least weight in *conservative weights circuits* and the Frobenius problem are equivalent.

Random vectors generator (Vizvári, 1994) A method to generate a random vector without cyclic drawbacks

Petri nets (Chrzastowski-Wachtel and Raczunas, 1993) The problem of finding a formula for the least weight in *conservative weights circuits* and the Frobenius problem are equivalent.

Hypohamiltonian graphs

A graph $G = (V, E)$ is hypohamiltonian if G is not hamiltonian but $G \setminus v$ is hamiltonian for all $v \in V(G)$.

Origin : *'Le cercle des irascibles'* (the cercle of bad-tempered)

(Skupién, 1992) Construction of an infinite family of hypohamiltonian graphs via a modular version of the Frobenius problem.

Hypohamiltonian graphs

A graph $G = (V, E)$ is hypohamiltonian if G is not hamiltonian but $G \setminus v$ is hamiltonian for all $v \in V(G)$.

Origin : *'Le cercle des irascibles'* (the cercle of bad-tempered)

(Skupién, 1992) Construction of an infinite family of hypohamiltonian graphs via a modular version of the Frobenius problem.

Hypohamiltonian graphs

A graph $G = (V, E)$ is hypohamiltonian if G is not hamiltonian but $G \setminus v$ is hamiltonian for all $v \in V(G)$.

Origin : *'Le cercle des irascibles'* (the cercle of bad-tempered)

(Skupién, 1992) Construction of an infinite family of hypohamiltonian graphs via a modular version of the Frobenius problem.

Integer partition

A partition of an integer n is an unordered multiset of positive integers (*parts*) whose some is n.

Theorem (Holroyd 2008)Let n, a, b be positives integers. Then, the following are all equinumerous :
(i) partitions of n in which each part and each difference between two parts lies in $\langle a, b\rangle$
(ii) partitions of n in which each part appears with multiplicity lying in $\langle a, b\rangle$
(iii) partitions of n in which each part is divisible by a or b

Integer partition

A partition of an integer n is an unordered multiset of positive integers (*parts*) whose some is n.

Theorem (Holroyd 2008)Let n, a, b be positives integers. Then, the following are all equinumerous :
(i) partitions of n in which each part and each difference between two parts lies in $\langle a, b\rangle$
(ii) partitions of n in which each part appears with multiplicity lying in $\langle a, b\rangle$
(iii) partitions of n in which each part is divisible by a or b

Integer partition

A partition of an integer n is an unordered multiset of positive integers (*parts*) whose some is n.

Theorem (Holroyd 2008)Let n, a, b be positives integers. Then, the following are all equinumerous :
(i) partitions of n in which each part and each difference between two parts lies in $\langle a, b\rangle$
(ii) partitions of n in which each part appears with multiplicity lying in $\langle a, b\rangle$
(iii) partitions of n in which each part is divisible by a or b

Example : Let $n = 13, a = 3$ and $b = 4$. Then,

(i) the partitions of 13 in which each part and each difference between two parts lies in $\langle 3, 4\rangle$ are : $\{(13), (10, 3), (7, 3, 3)\}$

(ii) the partitions of 13 in which each part appears with multiplicity lying in $\langle 3, 4\rangle$ are : $\{(3, 3, 3, 1, 1, 1, 1), (2, 2, 2, 1, \ldots, 1), (1, \ldots, 1)\}$

(iii) the partitions of n in which each part is divisible by 3 or 4 are : $\{(9, 4), (6, 4, 3), (4, 3, 3, 3)\}$.

In the case $a = 2$ and $b = 3$, the equality between (i) and (ii) gives the following partition identity (due to MacMahon 1960)

The number of partitions of n into parts not congruent to ± 1 modulo 6 equals the number of partitions of n with no consecutive integers and no ones as parts.

Vector space partition

A collection $\{V_i\}_{i=1}^{k}$ of subspaces of $V = V_n(q)$ is called a partition of V if and only if $V = \cup_{i=1}^{k} V_i$ and $V_i \cap V_j = \{0\}$ for all $1 \leq i \neq j \leq k$.

Remark : Generalization of partitions of abelian groups

Vector space partition

A collection $\{V_i\}_{i=1}^k$ of subspaces of $V = V_n(q)$ is called a partition of V if and only if $V = \cup_{i=1}^k V_i$ and $V_i \cap V_j = \{0\}$ for all $1 \leq i \neq j \leq k$.

Remark : Generalization of partitions of abelian groups

Vector space partition

A collection $\{V_i\}_{i=1}^{k}$ of subspaces of $V = V_n(q)$ is called a partition of V if and only if $V = \cup_{i=1}^{k} V_i$ and $V_i \cap V_j = \{0\}$ for all $1 \leq i \neq j \leq k$.

Remark : Generalization of partitions of abelian groups

Let $T = \{t_1 < \cdots < t_k\}$ be a set of positive integers. A partition π is said to be of type T if
(a) for any element W in π the $\dim(W) = t_i$ for some i and
(b) there is an element W in π such that the $\dim(W) = t_i$ for each $1 \leq i \leq k$

Theorem (Beutelspacher, 1978) Let n be an integer such that $n > dg(t_1/d, \ldots, t_k/d) + t_1 + \cdots + t_k$ where $d = \gcd(t_1, \ldots, t_k)$. Then $V_n(q)$ admits a partition of type $T = \{t_1 < \cdots < t_k\}$ if and only if $d|n$.

Let $T = \{t_1 < \cdots < t_k\}$ be a set of positive integers. A partition π is said to be of type T if
(a) for any element W in π the $\dim(W) = t_i$ for some i and
(b) there is an element W in π such that the $\dim(W) = t_i$ for each $1 \leq i \leq k$

Theorem (Beutelspacher, 1978) Let n be an integer such that $n > dg(t_1/d, \ldots, t_k/d) + t_1 + \cdots + t_k$ where $d = \gcd(t_1, \ldots, t_k)$. Then $V_n(q)$ admits a partition of type $T = \{t_1 < \cdots < t_k\}$ if and only if $d|n$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1,\ldots,a_n)$ be the number of multiples of d not belonging to $\langle a_1,\ldots,a_n\rangle$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1, \ldots, a_n)$ be the number of multiples of d not belonging to $\langle a_1, \ldots, a_n \rangle$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1, \ldots, a_n)$ be the number of multiples of d not belonging to $\langle a_1, \ldots, a_n \rangle$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1, \ldots, a_n)$ be the number of multiples of d not belonging to $\langle a_1, \ldots, a_n \rangle$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1, \ldots, a_n)$ be the number of multiples of d not belonging to $\langle a_1, \ldots, a_n \rangle$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1, \ldots, a_n)$ be the number of multiples of d not belonging to $\langle a_1, \ldots, a_n \rangle$.

Shell-sort method

3,2,7,9,8,1,1,5,2,6 (increment sequence : 7,3,1)

7-sorted : 3,2,6,9,8,1,1,5,2,6

3-sorted : 1,2,1,3,5,2,7,8,6,9

1-sorted : 1,1,2,2,3,5,6,7,8,9

Let $n_d(a_1, \ldots, a_n)$ be the number of multiples of d not belonging to $\langle a_1, \ldots, a_n \rangle$.

Lemme The number of steps required to h_j-sort a set on N integers that is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted is

$$O\left(\frac{Nn_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)}{h_j}\right).$$

Proof (idea).

- The number of steps required to insert element $a[i]$ is the number of elements in $a[i - h_j]$ which are greater than $a[i]$.
- Any element $a[i - x]$ with $x \in \langle h_{j+2}, \ldots, h_t \rangle$ must be less than $a[i]$ since the file is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted
- Then, an upper bound on the number of steps required to insert element $a[i]$, $1 \le i \le N$, is the number of multiples of h_j not belonging to $\langle h_{j+1}, \ldots, h_t \rangle$, that is, $n_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)$.

Lemme The number of steps required to h_j-sort a set on N integers that is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted is

$$O\left(\frac{Nn_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)}{h_j}\right).$$

Proof (idea).

- The number of steps required to insert element $a[i]$ is the number of elements in $a[i - h_j]$ which are greater than $a[i]$.
- Any element $a[i - x]$ with $x \in \langle h_{j+2}, \ldots, h_t\rangle$ must be less than $a[i]$ since the file is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted
- Then, an upper bound on the number of steps required to insert element $a[i]$, $1 \le i \le N$, is the number of multiples of h_j not belonging to $\langle h_{j+1}, \ldots, h_t\rangle$, that is, $n_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)$.

Lemme The number of steps required to h_j-sort a set on N integers that is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted is

$$O\left(\frac{Nn_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)}{h_j}\right).$$

Proof (idea).

• The number of steps required to insert element $a[i]$ is the number of elements in $a[i - h_j]$ which are greater than $a[i]$.

• Any element $a[i - x]$ with $x \in \langle h_{j+2}, \ldots, h_t\rangle$ must be less than $a[i]$ since the file is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted

• Then, an upper bound on the number of steps required to insert element $a[i]$, $1 \leq i \leq N$, is the number of multiples of h_j not belonging to $\langle h_{j+1}, \ldots, h_t\rangle$, that is, $n_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)$.

Lemme The number of steps required to h_j-sort a set on N integers that is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted is

$$O\left(\frac{Nn_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)}{h_j}\right).$$

Proof (idea).

- The number of steps required to insert element $a[i]$ is the number of elements in $a[i - h_j]$ which are greater than $a[i]$.
- Any element $a[i - x]$ with $x \in \langle h_{j+2}, \ldots, h_t\rangle$ must be less than $a[i]$ since the file is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted
- Then, an upper bound on the number of steps required to insert element $a[i]$, $1 \leq i \leq N$, is the number of multiples of h_j not belonging to $\langle h_{j+1}, \ldots, h_t\rangle$, that is, $n_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)$.

Lemme The number of steps required to h_j-sort a set on N integers that is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted is

$$O\left(\frac{Nn_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)}{h_j}\right).$$

Proof (idea).

• The number of steps required to insert element $a[i]$ is the number of elements in $a[i - h_j]$ which are greater than $a[i]$.

• Any element $a[i - x]$ with $x \in \langle h_{j+2}, \ldots, h_t\rangle$ must be less than $a[i]$ since the file is already $h_{j+1} - h_{j+2} - \cdots - h_t$-sorted

• Then, an upper bound on the number of steps required to insert element $a[i]$, $1 \leq i \leq N$, is the number of multiples of h_j not belonging to $\langle h_{j+1}, \ldots, h_t\rangle$, that is, $n_{h_j}(h_{j+1}, h_{j+2}, \ldots, h_t)$.

Lemme If $\gcd(a_1, \ldots, a_n) = 1$ then

$$n_d(a_1, \ldots, a_n) < \frac{g(a_1, \ldots, a_n)}{d}.$$

Theorem (Incerpi and Sedgewick, 1985) The running time of Shell-sort is $O(N^{3/2})$ where N is the number of elements in the file (on average and in worst case).

Conjecture (Gonnet, 1984) The asymptotic growth of the average case running time of Shell-sort is $O(N \log N \log \log N)$ where N is the number of elements in the file.

Lemme If $\gcd(a_1, \ldots, a_n) = 1$ then

$$n_d(a_1, \ldots, a_n) < \frac{g(a_1, \ldots, a_n)}{d}.$$

Theorem (Incerpi and Sedgewick, 1985)The running time of Shell-sort is $O(N^{3/2})$ where N is the number of elements in the file (on average and in worst case).

Conjecture (Gonnet, 1984)The asymptotic growth of the average case running time of Shell-sort is $O(N \log N \log\log N)$ where N is the number of elements in the file.

Lemme If $\gcd(a_1, \ldots, a_n) = 1$ then

$$n_d(a_1, \ldots, a_n) < \frac{g(a_1, \ldots, a_n)}{d}.$$

Theorem (Incerpi and Sedgewick, 1985) The running time of Shell-sort is $O(N^{3/2})$ where N is the number of elements in the file (on average and in worst case).

Conjecture (Gonnet, 1984) The asymptotic growth of the average case running time of Shell-sort is $O(N \log N \log\log N)$ where N is the number of elements in the file.

Tiling rectangles

Let $R(a, b)$ be the 2-dimensional rectangle.

We say that R can be tiled with *bricks* $R_1, \ldots, R_n$ if R can be filled entirely with copies of R_i (rotations are allowed).

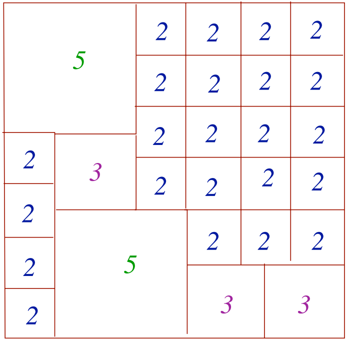
Tiling rectangles

Let $R(a, b)$ be the 2-dimensional rectangle.

We say that R can be tiled with *bricks* $R_1, \ldots, R_n$ if R can be filled entirely with copies of R_i (rotations are allowed).

Question : Can $R(13, 13)$ be tiled with $R(2, 2)$, $R(3, 3)$ and $R(5, 5)$?

Question : Can $R(13,13)$ be tiled with $R(2,2)$, $R(3,3)$ and $R(5,5)$?



Question : Does there exist a function $C_R = C_R(x, y, u, v)$ such that for all integers $a, b \geq C_R$ the rectangle $R(a, b)$ can be tiled with copies of the rectangles $R(x, y)$ and $R(u, v)$ for given positive integers x, y, u and v ?

The special case when $x = 4, y = 6, u = 5$ and $v = 7$ was posed in the 1991 William Mowell Putnam Examination (Problem B-3).

Theorem (Klosinski, Alexanderson and Larson, 1992) $R(a, b)$ can be tiled with $R(4, 6)$ and $R(5, 7)$ if $a, b \geq 2214$.

Question : Does there exist a function $C_R = C_R(x, y, u, v)$ such that for all integers $a, b \geq C_R$ the rectangle $R(a, b)$ can be tiled with copies of the rectangles $R(x, y)$ and $R(u, v)$ for given positive integers x, y, u and v ?

The special case when $x = 4, y = 6, u = 5$ and $v = 7$ was posed in the 1991 William Mowell Putnam Examination (Problem B-3).

Theorem (Klosinski, Alexanderson and Larson, 1992) $R(a, b)$ can be tiled with $R(4, 6)$ and $R(5, 7)$ if $a, b \geq 2214$.

Question : Does there exist a function $C_R = C_R(x, y, u, v)$ such that for all integers $a, b \geq C_R$ the rectangle $R(a, b)$ can be tiled with copies of the rectangles $R(x, y)$ and $R(u, v)$ for given positive integers x, y, u and v ?

The special case when $x = 4, y = 6, u = 5$ and $v = 7$ was posed in the 1991 William Mowell Putnam Examination (Problem B-3).

Theorem (Klosinski, Alexanderson and Larson, 1992) $R(a, b)$ can be tiled with $R(4, 6)$ and $R(5, 7)$ if $a, b \geq 2214$.

Theorem (Klarner - Bruijn, 1969) $R(a, b)$ can be tiled with $R(x, y)$ if and only if either x divides one side of R and y divides the other or xy divides one side of R and the other side can be expressed as a nonnegative integer combination of x and y.

Theorem (Fricke, 1995) $R(a, b)$ can be tiled with $R(x, x)$ and $R(y, y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a, b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.

Theorem (Klarner - Bruijn, 1969) $R(a, b)$ can be tiled with $R(x, y)$ if and only if either x divides one side of R and y divides the other or xy divides one side of R and the other side can be expressed as a nonnegative integer combination of x and y.

Theorem (Fricke, 1995) $R(a, b)$ can be tiled with $R(x, x)$ and $R(y, y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a, b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.

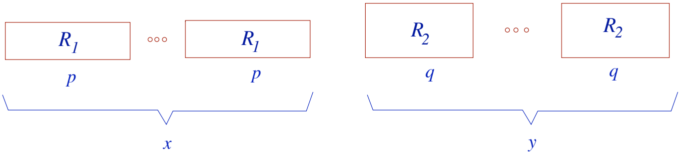
Theorem (Labrousse and R.A., 2007)

Let $R^i(a_1^i, \ldots, a_n^i)$ $i = 1, \ldots, m$ be rectangles. If

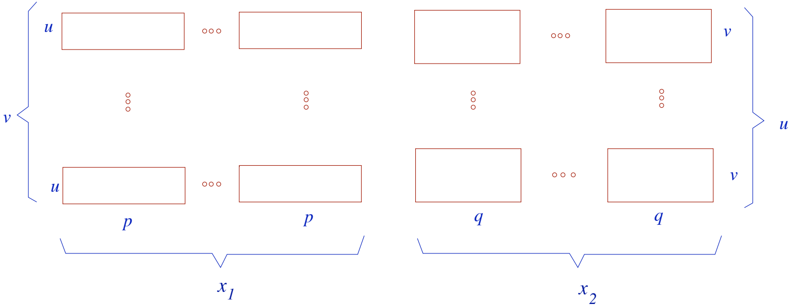
a) $\gcd(a_1^{i_1}, \ldots, a_1^{i_k}) = 1$ for all $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$

b) $\gcd(e, f) = 1$ for all $\{e, f\} \subset \{a_j^1, \ldots, a_j^m\}$ with $2 \leq j \leq n$

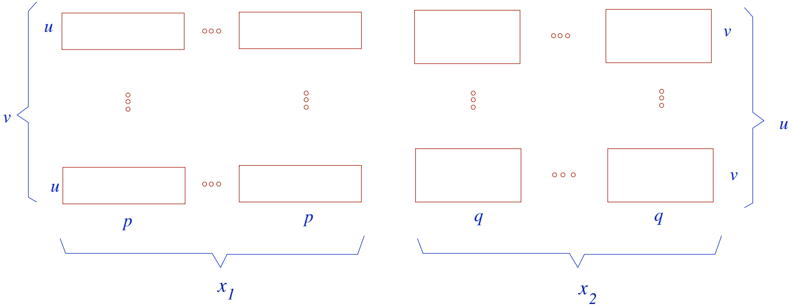
then all *sufficiently* large rectangle can be tiled with $R^1, \ldots, R^m$.



R_1
R_1
p
p
x
R_2
R_2
q
q
y



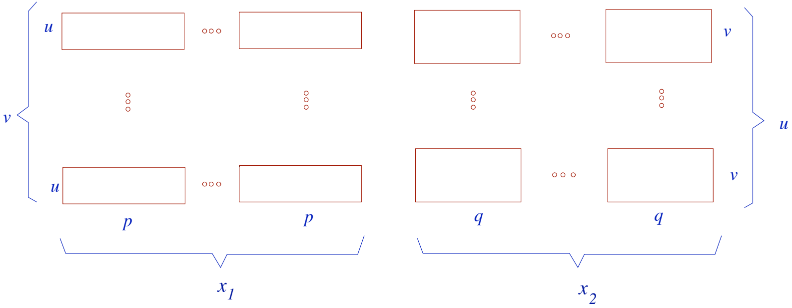
$B(R_1, R_2) = (t, uv)$ $t > g(p, q)$
$B(R_1, R_3) = (t, uw)$
$B(R_2, R_3) = (t, vw)$



$B(R_1, R_2) = (t, uv)$ $t > g(p, q)$

$B(R_1, R_3) = (t, uw)$

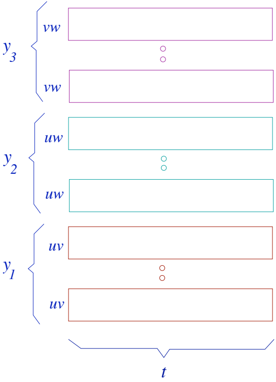
$B(R_2, R_3) = (t, vw)$



$B(R_1, R_2) = (t, uv)$ $t > g(p, q)$
$B(R_1, R_3) = (t, uw)$
$B(R_2, R_3) = (t, vw)$

Condition (b) implies $\gcd(uv, uw, vw) = 1$. Then for any $s > g(uv, uw, vw)$

Condition (b) implies $\gcd(uv, uw, vw) = 1$. Then for any $s > g(uv, uw, vw)$



Corollary (Labrousse and R.A., 2010) Let a, b, p, q, r, s be integers such that $\gcd(qs, qr, rs) = \gcd(p, r) = \gcd(p, s) = \gcd(r, s) = 1$. Then, $R(a, b)$ can be tiled with $R(p, q)$ and $R(r, s)$ if $a, b > \max\{2qrs - (qs + qr + rs), ps - p - s, rs - r - s\}$.

Special case : If $p = 6, q = 4, r = 5$ and $s = 7$ then $R(a, b)$ can be tiled with $(4, 6)$ and $(5, 7)$ if $a, b > 197$.

Theorem (Narayan and Schwenk, 2002) $R(a, b)$ can be tiled with $(4, 6)$ and $(5, 7)$ if $a, b \geq 33$.

Corollary (Labrousse and R.A., 2010) Let a, b, p, q, r, s be integers such that $\gcd(qs, qr, rs) = \gcd(p, r) = \gcd(p, s) = \gcd(r, s) = 1$. Then, $R(a, b)$ can be tiled with $R(p, q)$ and $R(r, s)$ if $a, b > \max\{2qrs - (qs + qr + rs), ps - p - s, rs - r - s\}$.

Special case : If $p = 6, q = 4, r = 5$ and $s = 7$ then $R(a, b)$ can be tiled with $(4, 6)$ and $(5, 7)$ if $a, b > 197$.

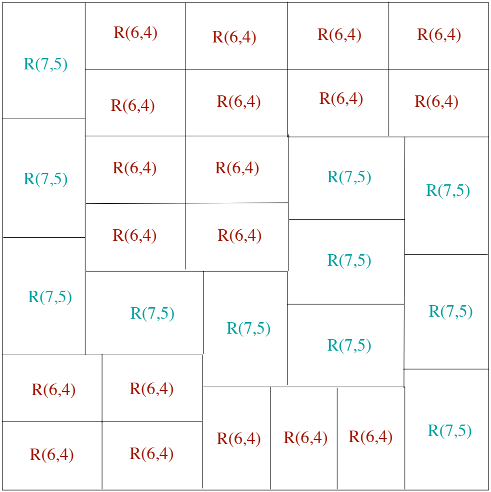
Theorem (Narayan and Schwenk, 2002) $R(a, b)$ can be tiled with $(4, 6)$ and $(5, 7)$ if $a, b \geq 33$.

Corollary (Labrousse and R.A., 2010) Let a, b, p, q, r, s be integers such that $\gcd(qs, qr, rs) = \gcd(p, r) = \gcd(p, s) = \gcd(r, s) = 1$. Then, $R(a, b)$ can be tiled with $R(p, q)$ and $R(r, s)$ if $a, b > \max\{2qrs - (qs + qr + rs), ps - p - s, rs - r - s\}$.

Special case : If $p = 6, q = 4, r = 5$ and $s = 7$ then $R(a, b)$ can be tiled with $(4, 6)$ and $(5, 7)$ if $a, b > 197$.

Theorem (Narayan and Schwenk, 2002) $R(a, b)$ can be tiled with $(4, 6)$ and $(5, 7)$ if $a, b \geq 33$.

$R(29,29)$


R(7,5)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(7,5)
R(6,4)
R(6,4)
R(7,5)
R(7,5)
R(6,4)
R(6,4)
R(7,5)
R(7,5)
R(7,5)
R(7,5)
R(7,5)
R(7,5)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(6,4)
R(7,5)

Lemme (Labrousse and R.A., 2010) Let $1 < a_1 < a_2 < \cdots < a_{n+1}$ be pairwise relatively prime integers, $n \geq 1$. Then $R(\underbrace{a, \ldots, a}_{n})$ can be tiled with $R(\underbrace{a_1, \ldots, a_1}_{n}), \ldots, R(\underbrace{a_{n+1}, \ldots, a_{n+1}}_{n})$ if

$$a > g(A_1, \ldots, A_{n+1}) = nP - \sum_{i=1}^{n+1} A_i$$

where $A_i = P/a_i$ with $P = \prod_{j=1}^{n+1} a_j$.

$R(a, a)$ can be tiled with $R(2, 2), R(3, 3)$ and $R(p, p)$ if $a \geq 7p + 6$ where p is an odd integer and $3 \not| p$.

Lemme (Labrousse and R.A., 2010) Let $1 < a_1 < a_2 < \cdots < a_{n+1}$ be pairwise relatively prime integers, $n \geq 1$. Then $R(\underbrace{a, \ldots, a}_{n})$ can be tiled with $R(\underbrace{a_1, \ldots, a_1}_{n}), \ldots, R(\underbrace{a_{n+1}, \ldots, a_{n+1}}_{n})$ if

$$a > g(A_1, \ldots, A_{n+1}) = nP - \sum_{i=1}^{n+1} A_i$$

where $A_i = P/a_i$ with $P = \prod_{j=1}^{n+1} a_j$.

$R(a, a)$ can be tiled with $R(2, 2), R(3, 3)$ and $R(p, p)$ if $a \geq 7p + 6$ where p is an odd integer and $3 \not| \, p$.

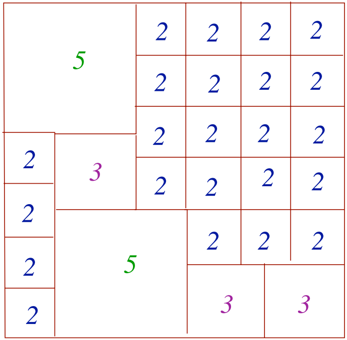
Theorem (Labrousse and R.A., 2010) Let $p > 4$ be an odd integer with $3 \not| p$ and let a be a positive integer. Then, $R(a, a)$ can be tiled with $R(2, 2), R(3, 3)$ and $R(p, p)$ if $a \geq 3p + 2$.

Corollary (Labrousse and R.A., 2010) $R(a, a)$ can be tiled with $R(2, 2), R(3, 3)$ and $R(5, 5)$ if and only if $a \neq 1, 7$ and with $R(2, 2), R(3, 3)$ and $R(7, 7)$ if and only if $a \neq 1, 5, 11$.

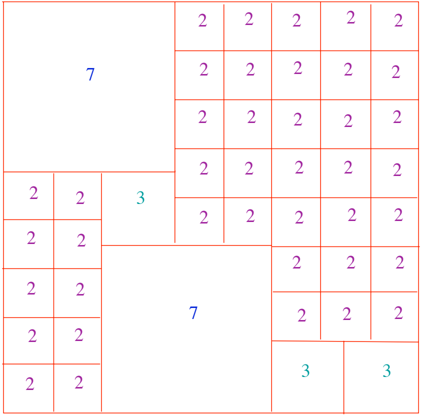
Theorem (Labrousse and R.A., 2010) Let $p > 4$ be an odd integer with $3 \not| p$ and let a be a positive integer. Then, $R(a, a)$ can be tiled with $R(2, 2), R(3, 3)$ and $R(p, p)$ if $a \geq 3p + 2$.

Corollary (Labrousse and R.A., 2010) $R(a, a)$ can be tiled with $R(2, 2), R(3, 3)$ and $R(5, 5)$ if and only if $a \neq 1, 7$ and with $R(2, 2), R(3, 3)$ and $R(7, 7)$ if and only if $a \neq 1, 5, 11$.

Tiling $R(13, 13)$ with $R(2, 2)$, $R(3, 3)$ and $R(5, 5)$



Tiling $R(17, 17)$ with $R(2, 2)$, $R(3, 3)$ and $R(7, 7)$



Tiling Tori

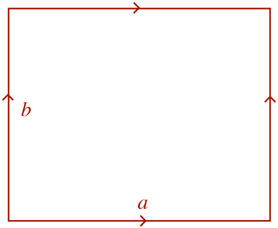
Let $T(a, b)$ be the 2-dimensional torus. We say that T can be tiled with *bricks* $R_1, \ldots, R_n$ if T can be filled entirely with copies of R_i (rotations are allowed).

Tiling Tori

Let $T(a, b)$ be the 2-dimensional torus. We say that T can be tiled with *bricks* $R_1, \ldots, R_n$ if T can be filled entirely with copies of R_i (rotations are allowed).

Tiling Tori

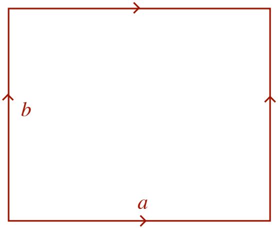
Let $T(a, b)$ be the 2-dimensional torus.



We say that T can be tiled with *bricks* $R_1, \ldots, R_n$ if T can be filled entirely with copies of R_i (rotations are allowed).

Tiling Tori

Let $T(a, b)$ be the 2-dimensional torus.



We say that T can be tiled with *bricks* $R_1, \ldots, R_n$ if T can be filled entirely with copies of R_i (rotations are allowed).

Question : Does there exist a function $C_T = C_T(x, y, u, v)$ such that for all integers $a, b \geq C_T$ $T(a, b)$ can be tiled with copies of the rectangles $R(x, y)$ and $R(u, v)$ for given positive integers x, y, u and v ?

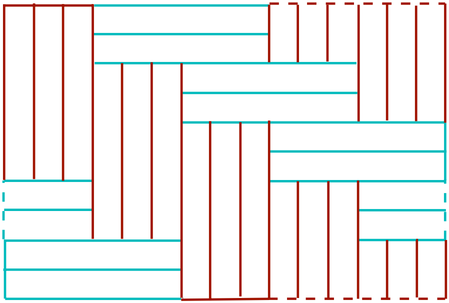
Theorem (Klarner - Bruijn, 1969) $R(a,b)$ can be tiled with $R(x,y)$ if and only if either x divides one side of R and y divides the other or xy divides one side of R and the other side can be expressed as a nonnegative integer combination of x and y.

Corollary $R(a,b)$ can be tiled with $R(1,n)$ if and only n divides either a or b.

Theorem (Klarner - Bruijn, 1969) $R(a,b)$ can be tiled with $R(x,y)$ if and only if either x divides one side of R and y divides the other or xy divides one side of R and the other side can be expressed as a nonnegative integer combination of x and y.

Corollary $R(a,b)$ can be tiled with $R(1,n)$ if and only n divides either a or b.

Example : Tiling $T(15, 10)$ with $R(1, 6)$



Theorem (Fricke, 1995) $R(a,b)$ can be tiled with $R(x,x)$ and $R(y,y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a,b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.

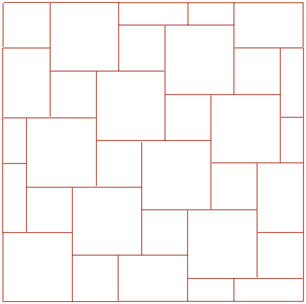
Example : Tiling $T(13,13)$ with $R(2,2)$ and $R(3,3)$

Theorem (Fricke, 1995) $R(a, b)$ can be tiled with $R(x, x)$ and $R(y, y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a, b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.

Example : Tiling $T(13, 13)$ with $R(2, 2)$ and $R(3, 3)$

Theorem (Fricke, 1995) $R(a,b)$ can be tiled with $R(x,x)$ and $R(y,y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a, b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.

Example : Tiling $T(13,13)$ with $R(2,2)$ and $R(3,3)$



Theorem (Labrousse and R.A., 2010) Let u, v, x and y be positive integers. Then, there exists $C_T(x, y, u, v)$ such that $T(a, b)$ can be tiled with $R(x, y)$ and $R(u, v)$ if and only if $\gcd(xy, uv) = 1$.

Theorem (Labrousse and R.A., 2010) Let u, v, x and y be positive integers such that $\gcd(xy, uv) = 1$. Then, $T(a, b)$ can be tiled with $R(x, y)$ and $R(v, u)$ if

$$a, b \geq \min\{n_1(uv + xy) + 1, n_2(uv + xy) + 1\}$$

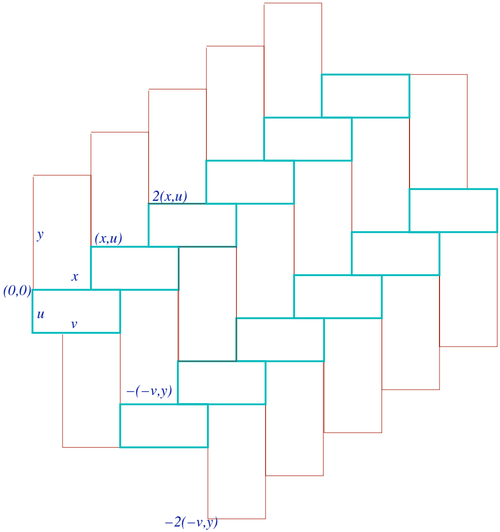
where $n_1 = \max\{vx, uy\}$ and $n_2 = \max\{ux, vy\}$.

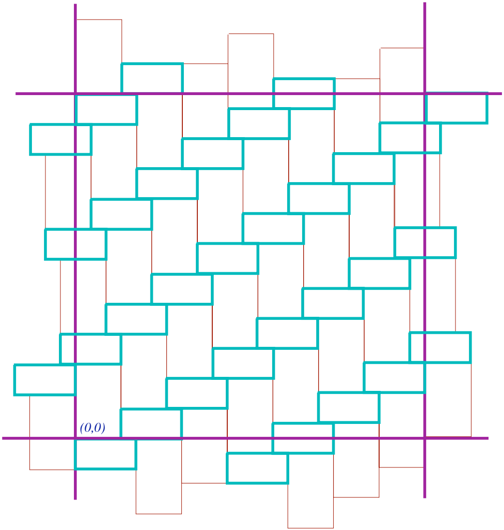
Theorem (Labrousse and R.A., 2010) Let u, v, x and y be positive integers. Then, there exists $C_T(x, y, u, v)$ such that $T(a, b)$ can be tiled with $R(x, y)$ and $R(u, v)$ if and only if $\gcd(xy, uv) = 1$.

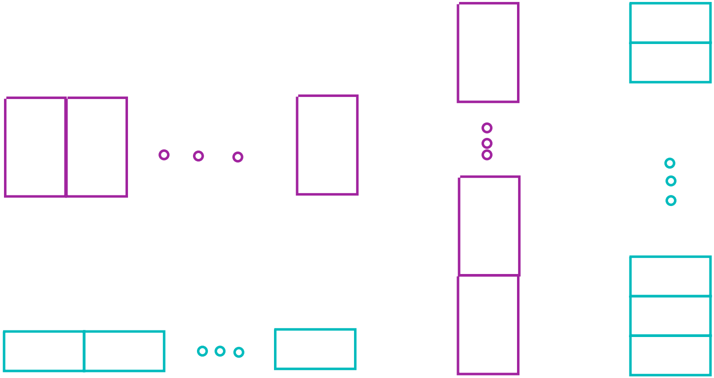
Theorem (Labrousse and R.A., 2010) Let u, v, x and y be positive integers such that $\gcd(xy, uv) = 1$. Then, $T(a, b)$ can be tiled with $R(x, y)$ and $R(v, u)$ if

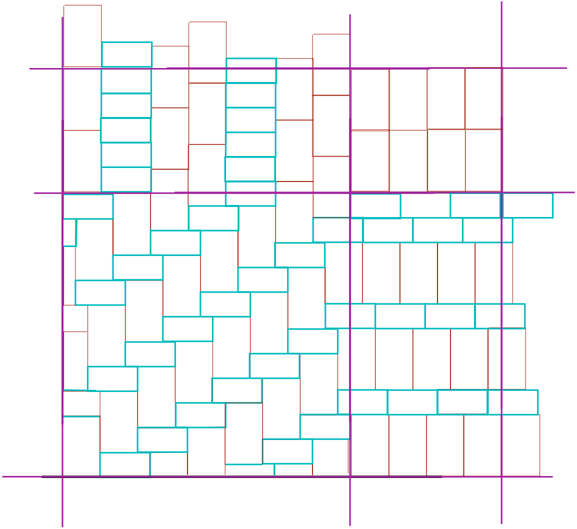
$$a, b \geq \min\{n_1(uv + xy) + 1, n_2(uv + xy) + 1\}$$

where $n_1 = \max\{vx, uy\}$ and $n_2 = \max\{ux, vy\}$.


2(x,u)
y
(x,u)
x
(0,0)
u
v
−(−v,y)
−2(−v,y)


(0,0)





Sylver coinage game (invented by J.C. Conway)

In this game the players alternatively name different numbers, but are not allowed to name *any* number that is a sum of previously named ones. The winner is the palyer who name the last number. Of course, as soon as 1 has been played, every other number is illegal (*i.e.*, representable as a sum of ones) and the game ends. Because the player who names 1 is declared the loser.

Question : Is there a winning strategy ?

Sylver coinage game (invented by J.C. Conway)

In this game the players alternatively name different numbers, but are not allowed to name *any* number that is a sum of previously named ones. The winner is the palyer who name the last number. Of course, as soon as 1 has been played, every other number is illegal (*i.e.*, representable as a sum of ones) and the game ends. Because the player who names 1 is declared the loser.

Question : Is there a winning strategy ?

Sylver coinage game (invented by J.C. Conway)

In this game the players alternatively name different numbers, but are not allowed to name *any* number that is a sum of previously named ones. The winner is the palyer who name the last number. Of course, as soon as 1 has been played, every other number is illegal (*i.e.*, representable as a sum of ones) and the game ends. Because the player who names 1 is declared the loser.

Question : Is there a winning strategy ?

The jugs problem

There are three jugs with integral capacities B , M , S respectively where $B = M + S$ and $M \geq S \geq 1$. Any jug may be poured into any other jug until either the first one is empty or the second is full. Initially jug B is full and the other two are empty (we use B as the name of the jug with capacity B, etc.

We want to divide the wine equally, so that $\frac{1}{2}B$ gallons are in jugs B and M and jug S is empty, and we want to do so with as few pourings as possible. We ask three questions. Can we share equally ? If so, what is the least number of pourings possible ; and how do we achieve this least number ?

The jugs problem

There are three jugs with integral capacities B , M , S respectively where $B = M + S$ and $M \geq S \geq 1$. Any jug may be poured into any other jug until either the first one is empty or the second is full. Initially jug B is full and the other two are empty (we use B as the name of the jug with capacity B, etc.

We want to divide the wine equally, so that $\frac{1}{2}B$ gallons are in jugs B and M and jug S is empty, and we want to do so with as few pourings as possible. We ask three questions. Can we share equally ? If so, what is the least number of pourings possible ; and how do we achieve this least number ?

The jugs problem

There are three jugs with integral capacities B , M , S respectively where $B = M + S$ and $M \geq S \geq 1$. Any jug may be poured into any other jug until either the first one is empty or the second is full. Initially jug B is full and the other two are empty (we use B as the name of the jug with capacity B, etc.

We want to divide the wine equally, so that $\frac{1}{2}B$ gallons are in jugs B and M and jug S is empty, and we want to do so with as few pourings as possible. We ask three questions. Can we share equally ? If so, what is the least number of pourings possible ; and how do we achieve this least number ?

Theorem (R.A., 1991) It is possible to share equally if and only if B is divisible by $2r$, where $r = \gcd(M, S)$. If this is the case, then the least number of pourings is $\frac{1}{r}B - 1$, and the unique optimal sequence of pourings is given by the first $\frac{1}{r}B - 1$ steps (pourings).

Jug Algorithm

Pour jug B into jug M

Repeat

Pour jug M into jug S
Pour jug S into jug B

if $m < S$ **then**

Pour jug M into jug S
Pour jug B into jug M

Theorem (R.A., 1991) It is possible to share equally if and only if B is divisible by $2r$, where $r = \gcd(M, S)$. If this is the case, then the least number of pourings is $\frac{1}{r}B - 1$, and the unique optimal sequence of pourings is given by the first $\frac{1}{r}B - 1$ steps (pourings).

Jug Algorithm

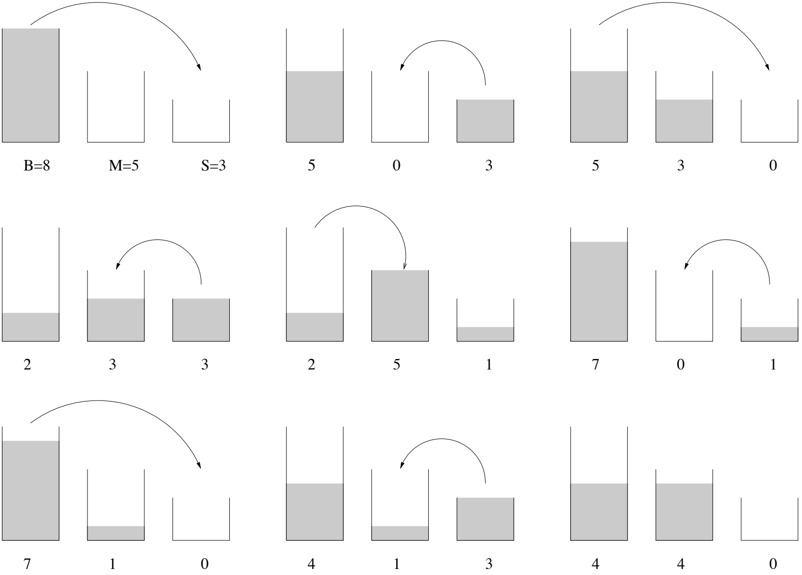
Pour jug B into jug M

Repeat

Pour jug M into jug S
Pour jug S into jug B

if $m < S$ **then**

Pour jug M into jug S
Pour jug B into jug M


B=8 M=5 S=3
5 0 3
5 3 0
2 3 3
2 5 1
7 0 1
7 1 0
4 1 3
4 4 0