Bucolic complexes

B. Brešar[1], J. Chalopin[2], V. Chepoi[2], T. Gologranc[3], and D. Osajda[4,5]

[1]Faculty of Natural Sciences and Mathematics, University of Maribor,
Koroška cesta 160, SI-2000 Maribor, Slovenia
`bostjan.bresar@uni-mb.si`

[2]Laboratoire d'Informatique Fondamentale, Aix-Marseille Université and CNRS,
Faculté des Sciences de Luminy, F-13288 Marseille Cedex 9, France
`{jeremie.chalopin, victor.chepoi}@lif.univ-mrs.fr`

[3]Institute of Mathematics, Physics and Mechanics,
Jadranska 19, SI-1000 Ljubljana, Slovenia
`tanja.gologranc@imfm.si`

[4]Universität Wien, Fakultät für Mathematik
Nordbergstraße 15, 1090 Wien, Austria

[5]Instytut Matematyczny, Uniwersytet Wrocławski,
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
`dosaj@math.uni.wroc.pl`

Abstract. In this article, we introduce and investigate bucolic complexes, a common generalization of systolic complexes and of CAT(0) cubical complexes. This class of complexes is closed under Cartesian products and amalgamations over some convex subcomplexes. We study various approaches to bucolic complexes: from graph-theoretic and topological viewpoints, as well as from the point of view of geometric group theory. Bucolic complexes can be defined as locally-finite simply connected prism complexes satisfying some local combinatorial conditions. We show that bucolic complexes are contractible, and satisfy some nonpositive-curvature-like properties. In particular, we prove a version of the Cartan-Hadamard theorem, the fixed point theorem for finite group actions, and establish some results on groups acting geometrically on such complexes.

We also characterize the 1-skeletons (which we call bucolic graphs) and the 2-skeletons of bucolic complexes. In particular, we prove that bucolic graphs are precisely retracts of Cartesian products of locally finite weakly bridged graphs (i.e., of 1-skeletons of weakly systolic complexes). We show that bucolic graphs are exactly the weakly modular graphs satisfying some local conditions formulated in terms of forbidden induced subgraphs and that finite bucolic graphs can be obtained by gated amalgamations of products of weakly bridged graphs.

1. Introduction

Avant-propos. In this paper, we introduce bucolic complexes and their 1-skeletons – bucolic graphs. Bucolic complexes can be defined as simply connected prism complexes satisfying some local combinatorial conditions. This class of cell complexes contains the class of CAT(0) cubical complexes and of weakly systolic simplicial complexes and is closed under Cartesian products and amalgamations over some convex subcomplexes. We show that bucolic complexes satisfy some nonpositive-curvature-like properties: we prove a version of the Cartan-Hadamard theorem, and the fixed point theorem for finite group actions. On the other hand,

we characterize the 1-skeletons of bucolic complexes in several different ways, in particular, we show that the bucolic graphs are exactly the retracts of Cartesian products of weakly bridged graphs, i.e., of 1-skeletons of weakly systolic complexes (in comparison, notice that the median graphs – the 1-skeletons of CAT(0) cubical complexes – are exactly the retracts of Cartesian products of edges, i.e., 1-skeletons of 1-simplices). Finally, we characterize the triangle-square complexes which can be realized as 2-skeletons of bucolic complexes as simply connected triangle-square complexes satisfying some local combinatorial conditions.

Graph-theoretic and geometric viewpoint. Median and bridged graphs constitute two of the most important classes of graphs investigated in metric graph theory and occur in different areas of discrete mathematics, geometric group theory, CAT(0) geometry, and theoretical computer science. Median graphs and related structures (median algebras and CAT(0) cubical complexes) have many nice properties and admit numerous characterizations. All median structures are intimately related to hypercubes: median graphs are isometric subgraphs of hypercubes; in fact, by a classical result of Bandelt [2] they are the retracts of hypercubes into which they embed isometrically. It was also shown by Isbell [28] and van de Vel [38] that every finite median graph G can be obtained by successive applications of gated amalgamations from hypercubes, thus showing that the only prime median graph is the two–vertex complete graph K_2 (a graph with at least two vertices is said to be *prime* if it is neither a Cartesian product nor a gated amalgam of smaller graphs). Median graphs also have a remarkable algebraic structure, which is induced by the ternary operation on the vertex set that assigns to each triplet of vertices the unique median vertex, and their algebra can be characterized using four natural axioms [7, 28] among all discrete ternary algebras. Finally, it was shown in [19, 35] that the median graphs are exactly the 1-skeletons of CAT(0) cubical complexes. Thus, due to a result of Gromov [24], the cubical complexes derived from median graphs can be characterized as simply connected cubical complexes in which the links of vertices are flag simplicial complexes. Sageev [36] established several important geometrical properties of hyperplanes in CAT(0) cubical complexes, and initiated the investigation of groups acting on such complexes. For more detailed information about median structures, the interested reader can consult the survey [6] and the books [27, 30, 39].

Bridged graphs are the graphs in which all isometric cycles have length 3. It was shown in [23, 37] that the bridged graphs are exactly the graphs in which the metric convexity satisfies one of the basic properties of Euclidean geometry (and, more generally, of the CAT(0) geometry): neighborhoods of convex sets are convex. Combinatorial and structural aspects of bridged graphs have been investigated in [1, 16, 34]. In particular, it was shown in [1] that bridged graphs are dismantlable (a simpler algorithmic proof is given in [18]), showing that the clique complexes (i.e., the simplicial complexes obtained by replacing complete subgraphs by simplices) of bridged graphs are collapsible. Similarly to the local-to-global characterization of CAT(0) cubical complexes of [24], it was shown in [19] that the clique complexes of bridged graphs are exactly the simply connected simplicial flag complexes in which the links of vertices do not contain induced 4- and 5-cycles. These complexes have been rediscovered and investigated in depth by Januszkiewicz and Swiatkowski [29], and, independently by

Haglund [25], who called them "systolic complexes" and considered as simplicial complexes satisfying combinatorial nonpositive curvature property. More recently, Osajda [32] proposed a generalization of systolic complexes still preserving some structural properties of systolic complexes: the resulting weakly systolic complexes and their 1-skeletons – the weakly bridged graphs – have been investigated and characterized in [20]. Since CAT(0) cubical complexes and systolic simplicial complexes can be both characterized via their 1-skeletons and via simple connectivity and a local condition on links, a natural question is to find a common generalization of such complexes which still obey the combinatorial nonpositive curvature properties. The cells in such complexes are prisms (Cartesian products of simplices) and the 2-dimensional faces are triangles and squares. In [11], answering a question from [12], the 1-skeletons of prism complexes resulting from clique complexes of chordal graphs by applying Cartesian products and gated amalgams have been characterized: those graphs (which are exactly the retracts of products of chordal graphs) are the weakly modular graphs not containing induced $K_{2,3}$, wheels W_k, and almost wheels W_k^-, $k \geq 4$ (weakly modular graphs and some other classes of graphs are defined in next section). It was also shown that, endowed with the l_2-metric, such prism complexes are CAT(0) spaces.

The structure theory of graphs based on Cartesian multiplication and gated amalgamation was further elaborated for more general classes of graphs. Some of the results for median graphs have been extended to quasi-median graphs introduced by Mulder [30] and further studied in [8]: quasi-median graphs are precisely the weakly modular graphs not containing induced $K_{2,3}$ and $K_4 - e$; they can also be characterized as the retracts of Hamming graphs and can be obtained from complete graphs by Cartesian products and gated amalgamations. More recently, Bandelt and Chepoi [4] presented a similar decomposition scheme of weakly median graphs and characterized the prime graphs with respect to this decomposition: the hyperoctahedra and their subgraphs, the 5-wheel W_5, and the 2-connected plane bridged graphs (i.e., plane triangulations in which all inner vertices have degrees ≥ 6). Generalizing the proof of the decomposition theorem of [4], Chastand [14, 15] presented a general framework of fiber-complemented graphs allowing to establish many general properties, previously proved only for particular classes of graphs. An important subclass of fiber-complemented graphs is that of pre-median graphs [14, 15] which are the weakly modular graphs without induced $K_{2,3}$ and $K_{2,3}$ with an extra edge (which can be viewed as the graph W_4^- defined below). It is an open problem to characterize all prime (elementary) fiber-complemented or pre-median graphs (see [14, p. 121]).

In this paper, we continue this line of research and characterize the graphs G which are retracts of Cartesian products of weakly bridged and bridged graphs. We show (cf. Theorem 2 below) that retracts of Cartesian products of weakly bridged (resp., bridged) graphs are exactly the weakly modular graphs which do not contain $K_{2,3}$, the wheel W_4, and the almost wheel W_4^- (resp., $K_{2,3}$, W_4, W_5 and W_4^-) as induced subgraphs (for an illustration of these forbidden graphs, see Fig. 1). We establish that these pre-median graphs are exactly the graphs obtained by gated amalgamations of Cartesian products of weakly bridged (or of bridged) graphs, thus answering Question 1 from [11]. This also provides a partial answer to

Chastand's problem mentioned above by showing that the weakly bridged graphs are exactly the prime graphs of pre-median graphs without W_4 and that the bridged graphs are the prime graphs of pre-median graphs without W_4 and W_5.

Topological and geometric group theory viewpoint. Analogously to median graphs which are built from cubes, are retracts of hypercubes, and gives rise to CAT(0) cubical complexes, the graphs studied in this article are built from Hamming graphs (i.e., products of simplices), are retracts of products of bridged or weakly bridged graphs (i.e., 1-skeletons of systolic or weakly systolic complexes), and thus they can be viewed as 1-skeletons of some cell complexes with cells being prisms (or Hamming cells), i.e., products of simplices. We call such prism complexes *bucolic complexes*[1]. Thus our previous result can be viewed as a characterization of 1-skeletons of bucolic complexes. We also characterize (Theorem 1 below) bucolic complexes via their 2-skeletons (consisting of triangle and square faces) by showing that they are exactly the simply connected triangle-square complexes satisfying the cube and house conditions and not containing W_4, W_5, and W_5^- (this answers Question 2 from [11]). Then we prove that the bucolic complexes are contractible (Theorem 3). Thus the three results constitute a version of the Cartan-Hadamard theorem, saying that under some local conditions the complex is aspherical, i.e. its universal covering space is contractible. Only limited number of such local characterizations of asphericity is known, and most of them refer to the notion of nonpositive curvature; cf. e.g. [13, 22, 24, 29, 32]. In fact bucolic complexes exhibit many nonpositive-curvature-like properties. Besides the Cartan-Hadamard theorem we prove the fixed point theorem for finite groups acting on bucolic complexes (Theorem 4), and we conclude that groups acting geometrically on such complexes have finitely many conjugacy classes of finite subgroups (Corollary 2). Counterparts of such results are known for other nonpositively curved spaces; cf. e.g. [13, 20, 29, 32]. Thus our classes of complexes and groups acting on them geometrically form new classes of combinatorially nonpositively curved complexes and groups (see e.g. [20, 24, 29, 32] for more background) containing the CAT(0) cubical and systolic classes of objects. A question of studying such unification theories was raised often by various researchers, e.g. by Januszkiewicz and Świątkowski (personal communication). Due to our knowledge, bucolism is the first generalization of the CAT(0) cubical and systolic worlds studied up to now. The class of bucolic complexes is closed under taking Cartesian products and amalgamations over some convex subcomplexes – the gated subcomplexes. Thus the class of groups acting geometrically on them is also closed under similar operations. It should be noticed that both systolic and CAT(0) cubical groups satisfy some strong (various for different classes) restrictions; cf. e.g. [32] and references therein. It implies that there are groups that are neither systolic nor CAT(0) cubical but which act geometrically on our complexes. In particular, in view of Theorem 4 and the fixed point theorems for systolic and CAT(0) complexes (compare [20]), the free product of a systolic group with a CAT(0) cubical group always act geometrically on a complex from our class,

---

[1]The term bucolic is inspired by systolic, where **b** stands for bridged and **c** for cubical. See also the acknowledgement for another source of our "inspiration".

although the group being such a product is often not systolic neither CAT(0) cubical. Another example with this property is the Cartesian product of two systolic but not CAT(0) cubical groups.

Article's structure. In the following Section 2 we introduce all the notions used later on. In Section 3 we state the main results of the article mentioned above (Theorems 1–4). In Section 4, we provide the characterization of bucolic graphs (Theorem 2). A proof of the main characterization of bucolic complexes (Theorem 1) is presented in Section 5. In Section 6 we prove the contractibility and the fixed point result for bucolic complexes (Theorems 3 and 4).

2. Preliminaries

2.1. **Graphs.** All graphs $G = (V, E)$ occurring in this paper are undirected, connected, and without loops or multiple edges, locally-finite but not necessarily finite. For two vertices u and v of a graph G, we will write $u \sim v$ if u and v are adjacent and $u \nsim v$, otherwise. We will use the notation $v \sim A$ to note that a vertex v is adjacent to all vertices of a set A and the notation $v \nsim A$ if v is not adjacent to any of the vertices of A. The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path. For a vertex v of a graph G we will denote by $B_1(v, G)$ the set of vertices consisting of v and the neighbors of v in G. We call $B_1(v, G)$ the *1-ball* centered at v. More generally, we denote by $B_r(v, G)$ the ball in G of radius r and centered at vertex v. The *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths, that is, of all vertices (metrically) *between u and v*: $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any pair of its vertices. The smallest convex subgraph containing a given subgraph S is called the *convex hull* of S and is denoted by $\mathrm{conv}(S)$. An induced subgraph H of a graph G is said to be *gated* if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x' (i.e., $x' \in I(x, y)$). The smallest gated subgraph containing a given subgraph S is the *gated hull* of S. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$. A *retraction* φ of H is an idempotent nonexpansive mapping of H into itself, that is, $\varphi^2 = \varphi : W \to W$ with
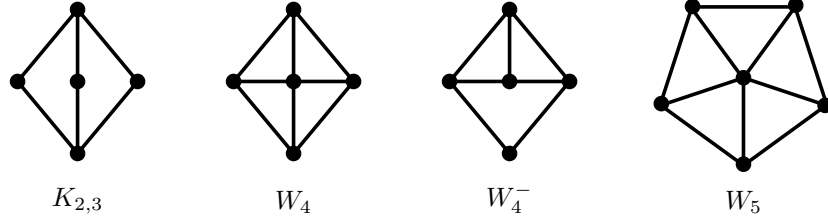
5

FIGURE 1. $K_{2,3}$, the wheel W_4, the almost-wheel W_4^-, and the wheel W_5.

$d(\varphi(x), \varphi(y)) \le d(x,y)$ for all $x, y \in W$. The subgraph of H induced by the image of H under φ is referred to as a *retract* of H.

The *wheel* W_k is a graph obtained by connecting a single vertex – the *central vertex* c – to all vertices of the k-cycle $(x_1, x_2, \ldots, x_k, x_1)$; the *almost wheel* W_k^- is the graph obtained from W_k by deleting a spoke (i.e., an edge between the central vertex c and a vertex x_i of the k-cycle), see Figure 1. The *extended* 5-*wheel* $\widehat{W}_5$ is a 5-wheel W_5 plus a 3-cycle (a, x_1, x_2, a) such that $a \nsim c, x_3, x_4, x_5$.

A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G. Let G_i, $i \in I$ be an arbitrary family of graphs. The *Cartesian product* $\square_{i \in I} G_i$ is defined on the set of all functions $x : i \mapsto x_i$, $x_i \in V(G_i)$, where two vertices x, y are adjacent if there exists an index $j \in I$ such that $x_j y_j \in E(G_j)$ and $x_i = y_i$ for all $i \ne j$. Note that Cartesian product of infinitely many nontrivial graphs is disconnected. Therefore, in this case the connected components of the Cartesian product are called *weak Cartesian products*. Since in our paper all graphs are connected, for us a Cartesian product graph will always mean a weak Cartesian product graph. A *strong Cartesian product* $\boxtimes_{i \in I} G_i$ is defined on the set of all functions $x : i \mapsto x_i$, $x_i \in V(G_i)$, where two vertices x, y are adjacent if for all indices $i \in I$ either $x_i = y_i$ or $x_i y_i \in E(G_i)$. A graph G is said to be *elementary* if the only proper gated subgraphs of G are singletons. A graph with at least two vertices is said to be *prime* if it is neither a Cartesian product nor a gated amalgam of smaller graphs.

A graph G is *weakly modular with respect to a vertex u* if its distance function d satisfies the following triangle and quadrangle conditions (see Figure 2):

Triangle condition TC(u): for any two vertices v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

Quadrangle condition QC(u): for any three vertices v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \le d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

A graph G is *weakly modular* [3] if G is weakly modular with respect to any vertex u.

A *weakly median* graph is a weakly modular graph in which the vertex x defined in the triangle and quadrangle conditions is always unique. Equivalently, weakly median graphs can be defined as the weakly modular graphs in which each triplet of vertices has a unique quasi-median. *Median graphs* are the bipartite weakly median graphs and, equivalently, can
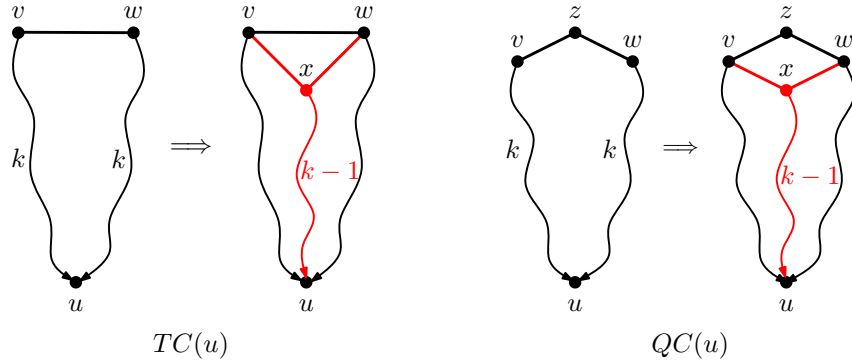
FIGURE 2. Triangle and quadrangle conditions

be defined as the graphs in which each triplet of vertices u, v, w has a unique median vertex. Bridged and weakly bridged graphs constitute other important subclasses of weakly modular graphs. A graph G is called *bridged* [23,37] if it does not contain any isometric cycle of length greater than 3. Alternatively, a graph G is bridged if and only if the balls $B_r(A, G) = \{v \in V : d(v, A) \le r\}$ around convex sets A of G are convex. Bridged graphs are exactly weakly modular graphs that do not contain induced 4- and 5-cycles (and therefore do not contain 4- and 5-wheels). A graph G is *weakly bridged* [20] if G is a weakly modular graph with convex balls $B_r(x, G)$. Equivalently, weakly bridged graphs are exactly the weakly modular graphs without induced 4-cycles C_4 [20]. Bridged and weakly bridged graphs are pre-median graphs: a graph G is *pre-median* [14, 15] if G is a weakly modular graph without induced $K_{2,3}$ and W_4^-. Chastand [14, 15] proved that pre-median graphs are fiber-complemented graphs. Any gated subset S of a graph G gives rise to a partition F_a $(a \in S)$ of the vertex-set of G; viz., the *fiber* F_a of a relative to S consists of all vertices x (including a itself) having a as their gate in S. According to Chastand [14, 15], a graph G is called *fiber-complemented* if for any gated set S all fibers F_a $(a \in S)$ are gated sets of G.

2.2. **Prism complexes.** In this paper, we consider a particular class of cell complexes (compare e.g. [13, p. 111-115]), called prism complexes, in which all cells are prisms. Cubical and simplicial cell complexes are particular instances of prism complexes. Although most of the notions presented below can be defined for all cell complexes and some of them for topological spaces, we will introduce them only for prism complexes.

We start with some notions about abstract simplicial complexes. An *abstract simplicial complex* is a family $\mathbf{X}$ of subsets (of a given set) called *simplices* which is closed for intersections and inclusion, i.e., $\sigma, \sigma' \in \mathbf{X}$ and $\sigma'' \subset \sigma$ implies that $\sigma \cap \sigma', \sigma'' \in \mathbf{X}$. For an abstract simplicial complex $\mathbf{X}$, denote by $V(\mathbf{X})$ and $E(\mathbf{X})$ the set of all 0-dimensional and 1-dimensional simplices of $\mathbf{X}$ and call the pair $G(\mathbf{X}) = (V(\mathbf{X}), E(\mathbf{X}))$ the *1-skeleton* of $\mathbf{X}$. Conversely, for a graph G one can derive a simplicial complex $\mathbf{X}(G)$ (the *clique complex* of G) by taking all complete subgraphs (cliques) as simplices of the complex. An abstract simplicial complex $\mathbf{X}$ is a *flag complex* (or a *clique complex*) if any set of vertices is included

in a simplex of $\mathbf{X}$ whenever each pair of its vertices is contained in a simplex of $\mathbf{X}$ (in the theory of hypergraphs this condition is called *conformality*). A flag complex can therefore be recovered from its underlying graph $G(\mathbf{X})$: the complete subgraphs of $G(\mathbf{X})$ are exactly the simplices of $\mathbf{X}$. All simplicial complexes occurring in this paper are flag complexes. By a *simplicial complex* we will mean the geometric realization of an abstract simplicial complex. It is a cell complex with cells corresponding to abstract simplices, being (topologically) solid simplices.

A *prism* is a convex polytope which is a Cartesian product of simplices. This is consistent with the standard definition of the product of two (or more) polytopes given on pp.9–10 of the book [40]: given two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, the *product* of P and Q is the set $P \times Q = \{(x,y) : x \in P, y \in Q\}$. $P \times Q$ is a polytope of dimension $\dim(P)+\dim(Q)$, whose nonempty faces are the products of nonempty faces of P and nonempty faces of Q. It is well known (see, for example p. 110 of [40]) that the product $\sigma_1 \times \cdots \times \sigma_k$ of solid simplices $\sigma_1, \ldots, \sigma_k$ is a convex polyhedron, which we will call a *prism*. The faces of a prism are also prisms of smaller dimensions. Particular instances of prisms are simplices and cubes (products of intervals). A *prism complex* is a cell complex $\mathbf{X}$ in which all cells are prisms so that the intersection of two prisms is empty or a common face of each of them. *Cubical complexes* are the prism complexes in which all cells are cubes and simplicial complexes are prism complexes in which all cells are simplices. The *1-skeleton* $G(\mathbf{X}) = \mathbf{X}^{(1)}$ of a prism complex $\mathbf{X}$ has the 0-dimensional cells of $\mathbf{X}$ as vertices and the 1-dimensional cells of $\mathbf{X}$ as edges. The 1-skeleton of a prism of $\mathbf{X}$ is a Cartesian product of complete subgraphs of $G(\mathbf{X})$, i.e., a *a Hamming subgraph* of $G(\mathbf{X})$. For vertices v, w or a set of vertices A of a prism complex $\mathbf{X}$ we will write $v \sim w$, $v \sim A$ (or $v \nsim w$, $v \nsim A$) if and only if a similar relation holds in the graph $G(\mathbf{X})$. The 2-skeleton $\mathbf{X}^{(2)}$ of $\mathbf{X}$ is a *triangle-square complex* obtained by taking the 0-dimensional, 1-dimensional, and 2-dimensional cells of $\mathbf{X}$. Analogously to simplicial flag complexes, a prism complex $\mathbf{X}$ is a *flag complex* if any Hamming subgraph of $G(\mathbf{X})$ is the 1-skeleton of a prism of $\mathbf{X}$. In the same way, a triangle-square complex $\mathbf{X}$ is a *flag complex* if any 3-cycle and induced 4-cycle of $G(\mathbf{X})$ defines a triangular or square cell of $\mathbf{X}$. A triangle-square flag complex can be recovered from its underlying graph $G(\mathbf{X})$: the 2-dimensional cells of $\mathbf{X}$ are exactly the triangles and the induced 4-cycles of $\mathbf{X}$. Every graph G, which is a subgraph of a Cartesian product of bridged or weakly bridged graphs (and the graphs occurring in our paper have this property), gives rise to a prism complex $\mathbf{X}(G)$ obtained by replacing all subgraphs of G which are Cartesian products of complete subgraphs (i.e., the Hamming subgraphs of G) by prisms. We call $\mathbf{X}(G)$ the *prism complex* of the graph G. Notice that the vertices of G are the 0-dimensional cells of $\mathbf{X}(G)$, that G is the 1-skeleton of $\mathbf{X}(G)$, while the triangle-square complex $\mathbf{X}(G)^{(2)}$ obtained by taking the vertices of G as 0-dimensional cells, the edges of G as 1-dimensional cells, and the triangles (3-cycles) and the squares (induced 4-cycles) of G as 2-dimensional cells is the 2-skeleton of $\mathbf{X}(G)$. Let $\mathbf{X}(W_k)$ and $\mathbf{X}(W_k^-)$ be the triangle-square (or the prism) complexes whose underlying graphs are the graphs W_k and W_k^-, respectively (the first consists of k triangles and the second consists of $k - 2$ triangles and one square).

Analogously, let $\mathbf{X}(\widehat{W_5})$ be the 2-dimensional simplicial complex made of 6 triangles whose underlying graph is the extended 5-wheel $\widehat{W_5}$.

As morphisms between cell complexes we consider all *cellular maps*, i.e. maps sending (linearly) cells to cells. An *isomorphism* is a bijective cellular map being a linear isomorphism (isometry) on each cell. A *covering (map)* of a cell complex $\mathbf{X}$ is a cellular surjection $p\colon\widetilde{\mathbf{X}} \to \mathbf{X}$ such that $p|_{\mathrm{St}(\widetilde{v},\widetilde{\mathbf{X}})}\colon \mathrm{St}(\widetilde{v},\widetilde{\mathbf{X}}) \to \mathrm{St}(v,\mathbf{X})$ is an isomorphism for every vertex v in $\mathbf{X}$, and every vertex $\widetilde{v} \in \widetilde{\mathbf{X}}$ with $p(\widetilde{v}) = v$; compare [26, Section 1.3]. (A *star* $\mathrm{St}(v,\mathbf{X})$ of a vertex v in a prism complex $\mathbf{X}$ is the subcomplex consisting of the union of all cells in $\mathbf{X}$ containing v.) The space $\widetilde{\mathbf{X}}$ is then called a *covering space*. A *universal cover* of $\mathbf{X}$ is a simply connected covering space $\widetilde{\mathbf{X}}$. It is unique up to isomorphism. In particular, if $\mathbf{X}$ is simply connected, then its universal cover is $\mathbf{X}$ itself. Note that $\mathbf{X}$ is connected iff $G(\mathbf{X}) = \mathbf{X}^{(1)}$ is connected, and $\mathbf{X}$ is simply connected (i.e. every continuous map $S^1 \to \mathbf{X}$ is null-homotopic) iff $\mathbf{X}^{(2)}$ is so. A group F *acts by automorphisms* on a cell complex $\mathbf{X}$ if there is a homomorphism $F \to \mathrm{Aut}(\mathbf{X})$ called an *action of F*. The action is *geometric* (or F *acts geometrically*) if it is proper (i.e. cells stabilizers are finite) and cocompact (i.e. the quotient $\mathbf{X}/F$ is compact).

2.3. **CAT(0) cubical complexes and systolic complexes.** For $0 \le d < \infty$, a *d-cube* is the d-power of the segment $[0,1]$ endowed with the ℓ_1 metric. A *cubical complex* is a cell complex $\mathbf{X}$ whose cells are cubes of various dimensions, attached in the expected way: any two cubes of $\mathbf{X}$ that have nonempty intersection intersect in a common face, i.e. the attaching map of each cube restricts to a combinatorial isometry on its faces. A cubical complex $\mathbf{X}$ endowed with intrinsic ℓ_2-metric is a *CAT(0)* (or nonpositively curved) metric space [13] if the geodesic triangles in $\mathbf{X}$ are thinner than their comparison triangles in the Euclidean plane. A *geodesic triangle* $\Delta = \Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic between each pair of vertices (the sides of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is defined to be a *CAT(0) space* [24] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov: *If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \le d_{\mathbb{E}^2}(x'_3, y')$.* The *link* of a vertex x in a cubical complex $\mathbf{X}$ is a simplicial complex, with a k-simplex for each $(k+1)$-cube containing x, with simplices attached according to the attachments of the corresponding cubes. Gromov [24] gave a nice combinatorial characterization of CAT(0) cubical complexes as simply connected cubical complexes in which the links of 0-cubes are simplicial flag complexes. Gromov's flagness condition of links can be equivalently formulated in the following way: for any $k \ge 2$, if three k-cubes of $\mathbf{X}$ pairwise intersect in a $(k-1)$-cube and all three intersect in a $(k-2)$-cube of $\mathbf{X}$, then are included in a $(k+1)$-dimensional cube of $\mathbf{X}$. Independently, Chepoi [19] and Roller [35] established that the 1-skeletons of CAT(0) cube complexes are exactly the median graphs, i.e., the graphs in which any triplet of vertices admit a unique median vertex.
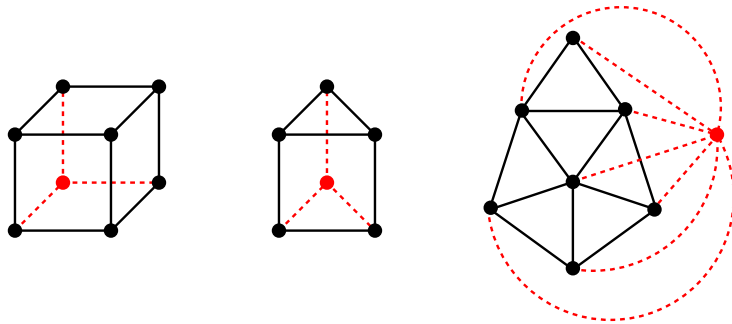
FIGURE 3. The cube condition (left), the house condition (middle), and the $\widehat{W_5}$-wheel condition (right).

Now we briefly recall the definitions of systolic and weakly systolic simplicial complexes, which are both considered simplicial complexes with combinatorial nonpositive curvature. For an integer $k \geq 4$, a flag simplicial complex $\mathbf{X}$ is locally k-large if every cycle consisting of less than k edges in any of its links of simplices has some two consecutive edges contained in a 2-simplex of this link, i.e., the links do not contain induced cycles of length $< k$. A simplicial complex is k-systolic if it is locally k-large, connected and simply connected. A flag simplicial complex is systolic if it is 6-systolic [19, 25, 29]. It was shown in [19] that systolic complexes are exactly the clique complexes of bridged graphs. On the other hand, among many other results, it was shown in [29] that the 1-skeletons of 7-systolic complexes are Gromov hyperbolic. In the same paper were given sufficient combinatorial conditions under which a systolic complex with regular simplices as cells is a CAT(0) space. A generalization of systolic complexes inheriting many of their properties have been proposed in [32] and [20]: a simplicial complex $\mathbf{X}$ is weakly systolic if and only if $\mathbf{X}$ is flag, connected and simply connected, locally 5-large, and satisfies the following local condition:

$\widehat{W_5}$-wheel condition: for each extended 5-wheel $\mathbf{X}(\widehat{W_5})$ of $\mathbf{X}$, there exists a vertex v adjacent to all vertices of this extended 5-wheel (see Fig. 3, right).

It was shown in [20] that the weakly bridged graphs, i.e. the 1-skeletons of weakly systolic complexes, are exactly the weakly modular graphs without induced 4-cycles or, equivalently, the weakly modular graphs with convex balls.

3. MAIN RESULTS

3.1. **Bucolic complexes.** A prism complex $\mathbf{X}$ is *bucolic* if it is flag, connected and simply connected, and satisfies the following three local conditions:

$(W_4, \widehat{W_5})$-*condition:* the 1-skeleton $\mathbf{X}^{(1)}$ of $\mathbf{X}$ does not contain induced W_4 and satisfies the $\widehat{W_5}$-wheel condition;

Hypercube condition: if $k \geq 2$ and three k-cubes of **X** pairwise intersect in a $(k-1)$-cube and all three intersect in a $(k-2)$-cube, then they are included in a $(k+1)$-dimensional cube of **X**;

Hyperhouse condition: if a cube and a simplex of **X** intersect in a 1-simplex, then they are included in a prism of **X**.

As we already noticed in the previous section, subject to simply connectivity, the $(W_4, \widehat{W_5})$-condition characterizes the weakly systolic complexes. On the other hand, as we noticed above, the hypercube condition is equivalent to Gromov's condition of flagness of links. Finally, the hyperhouse condition shows how simplices and cubes of **X** give rise to prisms. We call a graph G *bucolic* if G is the 1-skeleton of a bucolic complex **X**, i.e., $G = \mathbf{X}^{(1)}$.

Now, we consider the 2-dimensional versions of the last two conditions. We say that a triangle-square complex **X** satisfies:

Cube condition: any three squares of **X**, pairwise intersecting in an edge, and all three intersecting in a vertex of **X**, are included in a 3-dimensional cube (see Fig. 3, left);

House condition: any house (i.e., a triangle and a square of **X** sharing an edge) is included in a 3-dimensional prism (see Fig. 3, middle).

We start with the main result of our paper, which is a local-to-global characterization of the 1- and the 2-skeletons of bucolic complexes. It can be viewed as an analogue of similar local-to-global characterizations of CAT(0) cube complexes, systolic, and weakly systolic complexes provided in the papers [19, 20, 29, 32].

Theorem 1. *For a locally-finite prism complex* **X**, *the following conditions are equivalent:*

- (i) **X** *is a bucolic complex;*
- (ii) *the 2-skeleton* $\mathbf{X}^{(2)}$ *of* **X** *is a connected and simply connected triangle-square flag complex satisfying the* $(W_4, \widehat{W_5})$-*condition, the cube condition, and the house condition;*
- (iii) *the 1-skeleton* $G(\mathbf{X})$ *of* **X** *is a weakly modular graph not containing induced subgraphs of the form* $K_{2,3}$, W_4 *and* W_4^-.

Moreover, if **X** *is a connected flag prism complex satisfying the* $(W_4, \widehat{W_5})$, *the hypercube, and the hyperhouse conditions, then the universal cover* $\widetilde{\mathbf{X}}$ *of* **X** *is bucolic.*

The proof of this theorem is provided in Section 5. The most difficult part of the proof is to show that the 1-skeleton of a simply connected triangle-square complex **X** satisfying the local conditions of the theorem is weakly modular. To show this, we closely follow the proof method of a local-to-global characterization of weakly systolic complexes provided by Osajda [32] using the level-by-level construction of the universal cover of **X**.

Analogously to Theorem 1, one can characterize the prism complexes derived from systolic complexes. We will say that a bucolic complex **X** is *strongly bucolic* if it satisfies the following (W_4, W_5)-*condition:* the 1-skeleton $\mathbf{X}^{(1)}$ of **X** does not contain induced W_4 and W_5.

Corollary 1. *For a locally-finite prism complex* **X**, *the following conditions are equivalent:*

(i) **X** *is a strongly bucolic complex;*

(ii) *the 2-skeleton* $\mathbf{X}^{(2)}$ *of* **X** *is a connected and simply connected triangle-square flag complex satisfying the* (W_4, W_5)-*condition, the cube condition, and the house condition;*

(iii) *the 1-skeleton* $G(\mathbf{X})$ *of* **X** *is a weakly modular graph not containing induced subgraphs of the form* $K_{2,3}$, W_4, W_4^-, *and* W_5.

Moreover, if **X** *is a connected flag prism complex satisfying the* (W_4, W_5), *the hypercube, and the hyperhouse conditions, then the universal cover* $\widetilde{\mathbf{X}}$ *of* **X** *is strongly bucolic.*

3.2. **Bucolic graphs.** In this subsection, we present several characterizations of finite and locally-finite bucolic graphs. We show that finite bucolic graphs are exactly the finite graphs which are obtained from Cartesian products of bridged graphs or weakly bridged graphs via gated amalgamations. We also show that the locally-finite bucolic graphs are the weakly modular graphs that do not contain induced $K_{2,3}$, 4-wheels W_4, and almost-wheels W_4^- and that they are exactly the retracts of Cartesian products of weakly bridged graphs:

Theorem 2. *For a locally-finite graph* $G = (V, E)$, *the following conditions are equivalent:*

(i) G *is a retract of the (weak) Cartesian product of weakly bridged (resp., bridged) graphs;*

(ii) G *is a weakly modular graph not containing induced* $K_{2,3}$, W_4, *and* W_4^- *(resp.,* $K_{2,3}$, W_4^-, W_4, *and* W_5*), i.e.,* G *is a bucolic graph;*

(iii) G *is a* $K_{2,3}, W_4^-$-*free weakly modular graph in which all elementary (or prime) subgraphs are edges or 2-connected weakly bridged (resp., bridged) graphs.*

Moreover, if G *is finite, then the conditions (i)-(iii) are equivalent to the following condition:*

(iv) G *can be obtained by successive applications of gated amalgamations from Cartesian products of 2-connected weakly bridged (resp., bridged) graphs.*

The proof of this theorem is provided in Section 4. The most difficult part of the proof is the implication (ii)⇒(iii), which we establish in two steps. First we show that if G is a weakly modular graph not containing induced W_4 and W_4^-, then all its primes are 2-connected weakly bridged graphs or K_2. Then, we deduce both theorems using the results of [5, 14, 15, 20].

The following convexity property of bucolic graphs (proved in Section 4) will be useful for establishing contractibility and fixed point results for bucolic complexes.

Proposition 3.1. *If* $G = (V, E)$ *is a locally-finite bucolic graph, then the convex hull* $conv(S)$ *in* G *of any finite set* $S \subset V$ *is finite.*

3.3. **Contractibility and the fixed point property.** Using previous results, we establish the following basic properties of bucolic complexes (proofs are provided in Section 6). The first result is a version of the Cartan-Hadamard theorem. It has its CAT(0) (cf. [13, Corollary 1.5]) and systolic (cf. [19, Theorem 8.1]&[18] and [29, Theorem 4.1]) counterparts.

Theorem 3. *Bucolic complexes are contractible.*

The above property is the most important nonpositive-curvature-like property of bucolism. Beneath we provide further properties of a nonpositive-curvature-like nature. All of them have their CAT(0) and systolic counterparts; cf. [13, 20].

Theorem 4. *If* **X** *is a bucolic complex and F is a finite group acting by cell automorphisms on* **X***, then there exists a prism π of* **X** *which is invariant under the action of F. The center of the prism π is a point fixed by F.*

A standard argument (see e.g. the proof of [20, Corollary 6.4]) gives the following imme-diate consequence of Theorem 4.

Corollary 2. *Let F be a group acting geometrically by automorphisms on a bucolic complex* **X***. Then F contains only finitely many conjugacy classes of finite subgroups.*

4. PROOFS OF THEOREM 2 AND PROPOSITION 3.1

4.1. **Gated closures of triangles.** In this section, we prove that if G is a weakly modular graph not containing induced 4-wheels W_4 and almost 4-wheels W_4^-, then the gated hull of a triangle is a weakly bridged graph. Additionally, we show that if G does not contain induced 5-wheels W_5, then the gated hull of a triangle is a bridged graph.

Lemma 4.1. *Let G be a weakly modular graph without induced W_4 and W_4^-. Then G does not contain an induced W_n^- for $n > 4$.*

Proof. Suppose by way of contradiction that W_n^- is an induced subgraph of G and suppose that G does not contain induced W_k^- for any $3 < k < n$. Let $(x_1, x_2, \ldots, x_n, x_1)$ be the outer cycle C of W_n^- and consider a vertex c adjacent to all vertices of C except x_1. We apply the triangle condition to the triple x_1, x_2, x_{n-1} and find a vertex $a \in N(x_1) \cap N(x_2) \cap N(x_{n-1})$. Note that if $a \sim c$, then x_1, x_2, c, x_n, a induce W_4 if a is adjacent to x_n or W_4^- otherwise. Assume now that $a \not\sim c$. If $n = 5$, then the vertices x_4, a, x_2, c, x_3 induce either W_4 if x_3 is adjacent to a, or W_4^- otherwise. Now, if $n \geq 6$ and if a is not adjacent to $x_3, x_4, \ldots, x_{n-3}$ or x_{n-2}, the subgraph induced by the vertices $a, x_2, x_3, \ldots, x_{n-1}, c$ has an induced subgraph isomorphic to one of the forbidden induced subgraphs W_k^-, where $k < n$. Thus a is adjacent to all vertices of C except maybe x_n. The vertices a, x_3, c, x_{n-1}, x_4 induce W_4, if $n = 6$, or W_4^- otherwise, a contradiction. $\qquad\square$

By an (a, b)-*walk* of a graph G we mean a sequence of vertices $W = (a = x_0, x_1, \ldots, x_{k-1}, x_k = b)$ such that any two consecutive vertices x_i and x_{i+1} of W are different and adjacent (notice that in general we may have $x_i = x_j$ if $|i - j| \geq 2$). If $k = 2$, then we call W a *2-walk* of G. A 2-walk W in which all three vertices are different is called a *2-path*. Let H be an induced subgraph of a graph G. A 2-walk $W = (a, v, b)$ of G is H-*fanned* if $a, v, b \in V(H)$ and if there exists an (a, b)-walk W' in H not passing via v and such that v is adjacent to all vertices of W', i.e., $v \sim W'$. Notice that W' can be chosen to be an induced path of G. A walk $W = (x_0, x_1, \ldots, x_{k-1}, x_k)$ of G with $k > 2$ is H-*fanned* if every three consecutive vertices (x_i, x_{i+1}, x_{i+2}) of W form an H-fanned 2-walk. When H is clear from the context (typically when $H = G$), we say that W is *fanned*. If the endvertices of a 2-walk $W = (a, v, b)$ coincide or are adjacent, then W is fanned. Here is a generalization of this remark.

Lemma 4.2. *If $W = (x_0, x_1, \ldots, x_k)$ is a fanned walk and the vertices x_{i-1} and x_{i+1} coincide or are adjacent, then the walks $W' = (x_0, \ldots, x_{i-2}, x_{i+1}, x_{i+2}, \ldots, x_k)$ in the first case and $W'' = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ in the second case are also fanned.*

Proof. First suppose that $x_{i-1} = x_{i+1}$. Every 2-walk (x_j, x_{j+1}, x_{j+2}), where $j \leq i-4$ or $j \geq i+1$ is fanned, because W is fanned. Thus to show that the walk W' is fanned it suffices to show that the 2-walk $(x_{i-2}, x_{i+1}, x_{i+2})$ is fanned. Since the 2-walks (x_{i-2}, x_{i-1}, x_i) and (x_i, x_{i+1}, x_{i+2}) are fanned as 2-walks of W, there exist a (x_{i-2}, x_i)-walk R_1 not passing via x_{i-1} and a (x_i, x_{i+2})-walk R_2 not passing via x_{i+1} such that $x_{i-1} \sim R_1$ and $x_{i+1} \sim R_2$. Since $x_{i-1} = x_{i+1}$, we conclude that the vertex x_{i+1} is adjacent to all vertices of the (x_{i-2}, x_{i+2})-walk (R_1, R_2), showing that the 2-walk $(x_{i-2}, x_{i-1}, x_{i+2})$ is fanned.

Now, let $x_{i-1} \sim x_{i+1}$. Every 2-walk (x_j, x_{j+1}, x_{j+2}), where $j \leq i - 3$ or $j \geq i + 1$ is fanned, because W is fanned. Therefore, to show that W'' is fanned it suffices to show that the 2-walks $(x_{i-2}, x_{i-1}, x_{i+1})$ and $(x_{i-1}, x_{i+1}, x_{i+2})$ are fanned. Since both cases are similar, we will check the first one. Since the 2-walk (x_{i-2}, x_{i-1}, x_i) is fanned, there exists a (x_{i-2}, x_i)-walk R such that $x_{i-1} \sim R$. Therefore (R, x_{i+1}) is a (x_{i-2}, x_{i+1})-walk with all vertices adjacent to x_{i-1}. Hence the 2-walk $(x_{i-2}, x_{i-1}, x_{i+1})$ is fanned. $\square$

In the remaining auxiliary results of this section we assume that G is a locally finite (possibly infinite) weakly modular graph without induced W_4 and W_4^-. By Lemma 4.1, G does not contain W_k^- with $k > 3$.

Lemma 4.3. *If $C = (x, u, y, v, x)$ is an induced 4-cycle of G, then no induced 2-path of C is fanned.*

Proof. Suppose that the 2-path $P = (u, y, v)$ is fanned. Let $R = (u, t_1, \ldots, t_m, t_{m+1} = v)$ be a shortest (u, v)-walk such that $y \sim R$ (such a walk exists because P is fanned). Necessarily, R is an induced path of G. Since C is induced, $m \geq 1$ and $t_i \neq x$ for all $i \in \{1, \ldots, m\}$. If t_1 is adjacent to x, then the vertices x, u, y, v, t_1 induce W_4 if t_1 is adjacent to v, or W_4^- otherwise. Suppose now that t_1 is not adjacent to x and let $i \geq 2$ be the smallest index such that t_i is adjacent to x. Since R is a shortest walk, the cycle $(x, u, t_1, \ldots, t_i, x)$ is induced. Thus the vertices $x, u, t_1, \ldots, t_i, y$ induce a forbidden W_{i+2}^-. $\square$

Let v be a common neighbor of vertices a and b of G. For an (a, b)-walk W, we denote by $D(W)$ the distance sum $D(W) := \sum_{x \in W} d(x, v)$.

Lemma 4.4. *Let $W = (a = x_0, x_1, \ldots, x_m = b)$ be a fanned (a, b)-walk not containing v, let $k = \max\{d(x_i, v) : x_i \in W\} \geq 2$ and j be the smallest index so that $d(x_j, v) = k$. Then either $x_{j-1} = x_{j+1}$ and the walk $W' = (x_0, \ldots, x_{j-2}, x_{j+1}, x_{j+2}, \ldots, x_m)$ is fanned or $x_{j-1} \sim x_{j+1}$ and the walk $W'' = (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$ is fanned, or there exists a vertex y such that $d(y, v) = k - 1$ and the walk $W''' = (x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m)$ is fanned. In particular, if W is a fanned (a, b)-walk avoiding v with minimal distance sum $D(W)$, then $v \sim W$.*

Proof. If $x_{j-1} = x_{j+1}$ or $x_{j-1} \sim x_{j+1}$, then Lemma 4.2 implies that the walks W' and W'' are fanned. So, suppose that x_{j-1} and x_{j+1} are different and non-adjacent. Note that $d(x_{j-1}, v) =$

$k-1$ and $k-1 \le d(x_{j+1}, v) \le k$. If $d(x_{j+1}, v) = k-1$, then we can use the quadrangle condition for vertices v, x_{j-1}, x_j and x_{j+1} and find a vertex $z \in N(x_{j-1}) \cap N(x_{j+1})$ such that $d(v, z) = k-2$ ($z = v$ if $k = 2$). Since z and x_j are not adjacent, the 4-cycle $(z, x_{j-1}, x_j, x_{j+1}, z)$ is induced. Since W is fanned, the 2-walk (x_{j-1}, x_j, x_{j+1}) is fanned as well, contradicting Lemma 4.3.

So, suppose that $d(x_{j+1}, v) = k$. Applying the triangle condition to the triple v, x_j, x_{j+1}, we can find a common neighbor y of x_j and x_{j+1} with $d(v, y) = k - 1$. Note that $y \ne x_{j-1}$ since $x_{j-1} \not\sim x_{j+1}$. First, let $x_{j-1} \not\sim y$. Then we can apply the quadrangle condition to the vertices x_{j-1}, x_j, y, v, and find a vertex $z \in N(x_{j-1}) \cap N(y)$ with $d(z, v) = k - 2$ ($z = v$ if $k = 2$). Clearly, z is not adjacent to x_j and x_{j+1}. Hence, the cycle $(x_{j-1}, x_j, y, z, x_{j-1})$ is induced. Since the 2-walk (x_{j-1}, x_j, x_{j+1}) is fanned, there exists a (x_{j-1}, x_{j+1})-walk Q_0 not containing x_j such that $x_j \sim Q_0$. As a consequence, (Q_0, y) is a (x_{j-1}, y)-walk of G not passing via x_j whose all vertices are adjacent to x_j. Therefore the 2-walk (x_{j-1}, x_j, y) of the induced 4-cycle $(x_{j-1}, x_j, y, z, x_{j-1})$ is fanned, contradicting Lemma 4.3. This implies that x_{j-1} must be adjacent to y. Then $W''' = (x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m)$ is a walk of G. We claim that W''' is fanned. Indeed, all 2-paths of W''', except the three consecutive 2-walks $(x_{j-2}, x_{j-1}, y), (y, x_{j+1}, x_{j+2}), (x_{j-1}, y, x_{j+1})$, are also 2-walks of W, hence they are fanned. The 2-walk (x_{j-1}, y, x_{j+1}) is fanned because y is adjacent to all vertices of the walk (x_{j-1}, x_j, x_{j+1}). Since the 2-walk (x_j, x_{j+1}, x_{j+2}) is fanned, there is an (x_j, x_{j+2})-walk R such that $x_{j+1} \sim R$. Then all vertices of the (y, x_{j+2})-walk (y, R) are adjacent to x_{j+1}, whence the 2-walk (y, x_{j+1}, x_{j+2}) is fanned. Analogously, one can show that the 2-walk (x_{j-2}, x_{j-1}, y) is fanned, showing that W''' is fanned. Since each of $D(W'), D(W''), D(W''')$ is smaller than $D(W)$, we conclude that if W is a fanned (a, b)-walk not containing v with minimal distance sum $D(W)$, then $k = 1$, i.e., $v \sim W$. $\qquad\square$

Now, let us define the concept of a *twin-ball*. Let S be a finite subset of a vertex set of G. We define $TB_0(S) = S$ and suppose that the twin-ball $TB_t(S)$ is already determined for some $t \ge 0$. Then let $TB_{t+1}(S)$ be defined as the set of vertices from G that have at least two neighbors in $TB_t(S)$. Note that every $TB_t(S)$ is finite, since G is a locally finite graph.

Let T be a triangle in G. Let $H_3, H_4, \ldots$ be a (possibly infinite) sequence of induced 2-connected subgraphs of G, with $H_3 = T$ and H_{i+1} is the subgraph of G induced by $V(H_i) \cup \{v\}$, where v is an arbitrary vertex from the ball $TB_t(T)$, t being the smallest integer such that all vertices from $TB_{t-1}(T)$ lie in H_i. (If there exists no such vertex v then the procedure stops after a finite number of steps.) Hence v has at least two neighbors in H_i. Let $K := \bigcup_{i=1}^{\infty} H_i$. In the following lemmas, we prove that for every i, every 2-walk of H_i is K-fanned and that H_i does not contain any induced 4-cycle.

Lemma 4.5. *For every i, any 2-walk of H_i is K-fanned.*

Proof. We proceed by induction on i. Clearly, $H_3 = T$ fulfils this property. Assume by induction hypothesis that any 2-walk of H_i is K-fanned. Let $v \in G \setminus H_i$ be an arbitrary vertex from the ball $TB_t(T)$, where t is the smallest integer such that all vertices from $TB_{t-1}(T)$ lie in H_i. Clearly v has at least two neighbors in H_i. We will prove that any

2-walk of $H_{i+1} = G(V(H_i) \cup \{v\})$ is K-fanned. It suffices to consider the 2-walks Q of H_{i+1} that contain v, since all other 2-walks lie in H_i and are K-fanned by the induction hypothesis.

Case 1. $Q = (a, v, c)$.

Since H_i is connected and $a, c \in V(H_i)$, there exists an (a, c)-walk R in H_i. Since any 2-walk of H_i is K-fanned by induction hypothesis, R itself is K-fanned. As H_i is a subgraph of K, R belongs to K. Among all K-fanned (a, c)-walks belonging to K and avoiding v, let $W = (a = x_0, x_1, \ldots, x_m = c)$ be chosen in such a way that the distance sum $D(W) = \sum_{x_i \in W} d(v, x_i)$ is minimized. By Lemma 4.4, $v \sim W$ and thus the 2-walk Q is K-fanned.

Case 2. $Q = (c, b, v)$.

If c and v are adjacent, then Q is trivially fanned. Thus we may assume that $c \nsim v$, and $c \neq v$ as $v \notin H_i$. Since v has at least two neighbors in H_i, there exists a vertex $a \in H_i$ adjacent to v and different from b. Since H_i is 2-connected and $a, c \in H_i$, there exists an (a, c)-walk P_0 in H_i that avoids b. The walks P_0 and (P_0, b) are K-fanned because all their 2-walks are fanned by induction hypothesis. Hence, there exists at least one K-fanned (a, b)-walk (P_0, b) that passes via c, avoids v, and all vertices of P_0 are different from b. Among all such (a, b)-walks (P_0, b) of K (i.e., that pass c, avoid v, the vertices of P_0 are different from b, and are K-fanned), let $W = (a = x_0, x_1, \ldots, x_m, x_{m+1} = c, b)$ be chosen in such a way that $D(W)$ is minimized. Since v and $x_{m+1} = c$ are different and not adjacent, $k = \max\{d_G(x_i, v) : x_i \in W\} \geq 2$. Let j be the smallest index such that $d(x_j, v) = k$.

First suppose that $j \neq m + 1$. By Lemma 4.4, the vertices a and b can be connected by one of the walks W', W'', W''' derived from W. These walks are K-fanned, contain the vertex c, avoid the vertex v, and all three have smaller distance sums than W. In case of W' and W'' we obtain a contradiction with the minimality choice of W. Analogously, in case of W''' we obtain the same contradiction except if the vertex y coincides with b, i.e., b is adjacent to the vertices x_{j-1}, x_j, and x_{j+1}. In this case, $d(x_j, v) = 2$ and $x_{j-1} \sim v$. Consider the 2-walk (c, b, x_{j+1}). By construction, we know that there is a K-fanned walk $R = (x_{m+1} = c, x_m, \ldots, x_{j+2}, x_{j+1})$ that avoids b. Applying Lemma 4.4 with b and R, there exists a K-fanned (c, x_{j+1})-walk R' avoiding b such that $b \sim R'$. Consequently, there is an walk (R', x_j, x_{j-1}, v) in K from c to v in the neighborhood of b and thus (c, b, v) is K-fanned.

Now suppose that $j = m + 1$, i.e., v is adjacent to all vertices of W except $x_{m+1} = c$. From the choice of W we conclude that $b \neq x_m$. If $b \nsim x_m$, then $C = (v, x_m, c, b, v)$ is an induced 4-cycle. Since the 2-path (b, c, x_m) is K-fanned, we obtain a contradiction with Lemma 4.3. Finally, if b is adjacent to x_m, then the 2-path (c, b, v) is K-fanned because c and v are connected in K by the 2-path (c, x_m, v) and x_m is adjacent to b. $\qquad \square$

Lemma 4.6. *Any H_i does not contain induced 4-cycles.*

Proof. Again we proceed by induction on i. Suppose by induction hypothesis that H_i does not contain induced 4-cycles. Let $H_{i+1} = G(V(H_i) \cup \{v\})$ and suppose by way of contradiction that H_{i+1} contains an induced 4-cycle C. Then necessarily v belongs to C. Let $C = (v, a, b, c, v)$. Since by Lemma 4.5 the 2-walks of H_{i+1} are K-fanned, the 2-path (a, b, c) of C is fanned and

we obtain a contradiction with Lemma 4.3. This shows that any H_i does not contain induced 4-cycle. $\square$

Lemma 4.7. *K is the gated hull of T in G.*

Proof. Note that it is clear by the construction that all the vertices of K belong to the gated hull of T. On the other hand, since G is weakly modular graph, we also deduce that K is gated, by noting that it is Δ-closed and locally convex (i.e., if $x, y \in K$ and $0 < d(x, y) \leq 2$, then any common neighbor v of x and y also belongs to K) [4]. Indeed, if there is a vertex u that has at least two neighbors v, w in K then there exists t such that $v, w \in TB_t$ and thus $u \in TB_{t+1}$. Hence all vertices not in K have at most one neighbor in K. $\square$

Summarizing, we obtain the main result of this subsection.

Proposition 4.8. *Let G be a locally finite weakly modular graph not containing induced W_4 and W_4^-. Then the gated hull of any triangle T of G is a 2-connected weakly bridged graph. Additionally, if G does not contain induced W_5, then the gated hull of T is a 2-connected bridged graph.*

Proof. By Lemma 4.7, the gated hull of T is the 2-connected subgraph K of G constructed by our procedure. Since K is a convex subgraph of a weakly modular graph G, K itself is a weakly modular graph. By Lemma 4.6, the graph K does not contain induced 4-cycles, thus K is weakly bridged by [20, Theorem 3.1(iv)]. If, additionally, G does not contain 5-wheels, then G does not contain induced 5-cycles because in a weakly bridged graph any 5-cycle is included in a 5-wheel. Then K is a weakly modular graph without induced 4- and 5-cycles, thus K is bridged by [19, Theorem 8.1(ii)]. $\square$

4.2. **Proof of Theorem 2.** We first prove the implications (i)$\Rightarrow$(ii). First, bridged and weakly bridged graphs are weakly modular. Weakly bridged graphs do not contain induced $K_{2,3}$, W_4, and W_4^- because they do not contain induced 4-cycles. Bridged graphs additionally do not contain induced W_5. Weakly modular graphs are closed by taking (weak) Cartesian products (this holds also when there are infinite number of factors in weak Cartesian products, since the distances between vertices in a weak Cartesian product are finite). If a (weak) Cartesian product $\square_{i \in I} H_i$ contains an induced $K_{2,3}, W_4, W_5$ or W_4^-, then necessarily this graph occurs in one of the factors H_i. As a consequence, Cartesian products $H = \square_{i \in I} H_i$ of weakly bridged graphs do not contain induced $K_{2,3}, W_4$, and W_4^-. Analogously, Cartesian products $H = \square_{i \in I} H_i$ of bridged graphs do not contain induced $K_{2,3}, W_4, W_4^-$, and W_5. If G is a retract of H, then G is an isometric subgraph of H, and therefore G does not contain induced $K_{2,3}, W_4, W_4^-$ in the first case and induced $K_{2,3}, W_4, W_4^-$ and W_5 in the second case. It remains to notice that the triangle and quadrangle conditions are preserved by retractions, thus G is a weakly modular graph, establishing that (i)$\Rightarrow$(ii).

Now suppose that G is a weakly modular graph satisfying the condition (ii) of Theorem 2. Then G is a pre-median graph. By [14, Theorem 4.13], any pre-median graph is fiber-complemented. By [14, Lemma 4.8], this implies that any gated subgraph H of G is

elementary if and only if it is prime. Note that the gated hull of any edge in G is either the edge itself, or it is included in a triangle by weak modularity, and by Proposition 4.8 we find that the gated hull of this edge is a 2-connected (weakly) bridged graph. Hence every elementary (= prime) graph is a 2-connected (weakly) bridged graph or an edge. This establishes the implication (ii)$\Rightarrow$(iii).

To prove the implication (iii)$\Rightarrow$(i), we will use [15, Theorem 3.2.1] and [20, Theorem 5.1]. By Chastand [15, Theorem 3.2.1], any fiber-complemented graph G whose primes are moorable graphs is a retract of the Cartesian product of its primes. Note that elementary subgraphs of G, enjoying (iii), are edges and 2-connected weakly bridged graphs. To see this implication, we thus need to prove that weakly bridged graphs are moorable. Recall that given a vertex u of a graph G, an endomorphism f of G is a *mooring* of G onto u if $f(u) = u$ and for any vertex $v \neq u$, $vf(v)$ is an edge of G such that $f(v)$ lie on a shortest path between v and u. A graph G is *moorable* if, for every vertex u of G, there exists a mooring of G onto u. Equivalently, mooring can be viewed as a combing property of graphs which comes from the geometric theory of groups [22]. Let u be a distinguished vertex ("base point") of a graph G. Two shortest paths $P(x,u), P(y,u)$ in G connecting two adjacent vertices x, y to u are called 1-*fellow travelers* if $d(x', y') \leq 1$ holds for each pair of vertices $x' \in P(x,u), y' \in P(y,u)$ with $d(x, x') = d(y, y')$. A *geodesic* 1-*combing* of G with respect to the base point u comprises shortest paths $P(x,u)$ between u and all vertices x such that $P(x,u)$ and $P(y,u)$ are 1-fellow travelers for any edge xy of G. One can select the combing paths so that their union is a spanning tree T_u of G that is rooted at u and preserves the distances from u to all vertices. The neighbor $f(x)$ of $x \neq u$ in the unique path of T_b connecting x with the root will be called the *father* of x (set also $f(u) = u$). Then f is a mooring of G onto u (vice-versa, any mooring of G onto u can be viewed as a geodesic 1-combing with respect to u). A geodesic 1-combing of G with respect to u thus amounts to a tree T_u preserving the distances to the root b such that if x and y are adjacent in G then $f(x)$ and $f(y)$ either coincide or are adjacent in G. In [16, 19] it is noticed (using [18]) that for bridged graphs every spanning tree returned by Breadth-First-Search starting from an arbitrary vertex u provides a geodesic 1-combing. More generally, it is shown in [20, Theorem 5.1] that for weakly bridged graphs every spanning tree returned by Lexicographic-Breadth-First-Search starting from an arbitrary vertex u provides a geodesic 1-combing, thus showing that weakly bridged graphs are also moorable. Thus, by [15, Theorem 3.2.1] G is a retract of the Cartesian product of its primes, establishing the implication (iii)$\Rightarrow$(i) of Theorem 2.

Now, for finite graphs we show that (iv) $\Longleftrightarrow$ (ii). As noticed above, bridged and weakly bridged graphs are weakly modular and do not contain induced $K_{2,3}$, W_4, and W_4^-. Bridged graphs additionally do not contain induced W_5. Weakly modular graphs are closed by Cartesian products and gated amalgams. Moreover, if G is the Cartesian product or the gated amalgam of two graphs G_1 and G_2, then G contains an induced $K_{2,3}$ (resp. W_4, W_4^-, W_5) if and only if G_1 or G_2 does. Therefore (iv)$\Rightarrow$(ii). Conversely, suppose that G is a finite weakly modular graph satisfying the condition (ii) of Theorem 2. Then G is a pre-median graph. By [14, Theorem 4.13], any pre-median graph is fiber-complemented. Then according

to [14, Theorem 5.4], G can be obtained from Cartesian products of elementary (=prime) graphs by a sequence of gated amalgamations. By Proposition 4.8, any elementary graph is either an edge, a 2-connected bridged graph, or a 2-connected weakly bridged graph. Thus the implication (ii)$\Rightarrow$(iv) in Theorem 2 holds. This concludes the proof of Theorem 2.

4.3. **Proof of Proposition 3.1.** Let G be a locally-finite bucolic graph and let H_i ($i \in I$) be the prime graphs of G so that G is (isometrically) embedded in the (weak) Cartesian product $H = \square_{i\in I} H_i$ as a retract. Note that by Theorem 2 each H_i is a weakly bridged graph. For each index $i \in I$, let S_i denote the projection of S in H_i, i.e., S_i consists of all vertices v_i of H_i for each of which there exists a vertex v of G whose ith coordinate is v_i. Since the set S is finite and the distance between any two vertices of S is finite, there exists a finite subset of indices I' of I such that for any $i \in I \smallsetminus I'$, all vertices of S have the same projection in H_i, i.e., for all but a finite set I' of indices i the set S_i is a single vertex. Note that each H_i is locally-finite since it is isomorphic to a gated subgraph of G. Since each set S_i is finite, it is included in a ball, which is necessarily finite. Since the balls in weakly bridged graphs are convex, we conclude that for each S_i, the convex hull $\mathrm{conv}_{H_i}(S_i)$ of S_i in H_i is finite. The convex hull $\mathrm{conv}_H(S)$ of S in H is the Cartesian product of the convex hulls of the sets $\mathrm{conv}_{H_i}(S_i)$: $\mathrm{conv}_H(S) = \square_{i\in I}\mathrm{conv}_{H_i}(S_i)$ (this equality holds for products of arbitrary metric spaces). All $\mathrm{conv}_{H_i}(S_i)$ for $i \in I \smallsetminus I'$ are singletons, thus the size of $\mathrm{conv}_H(S)$ equals the size of $\square_{i\in I'}\mathrm{conv}_{H_i}(S_i)$, and thus is finite because I' is finite and each factor $\mathrm{conv}_{H_i}(S_i)$ in this product is finite by what has been shown above.

Now, set $A := V \cap \mathrm{conv}_H(S)$. We claim that the set A is convex. Let $x, y \in A$ and pick any vertex z of G in the interval $I(x,y)$ of G. Since G is isometrically embedded in H, $d_G(x,y) = d_H(x,y), d_G(x,z) = d_H(x,z)$, and $d_G(z,y) = d_H(z,y)$, thus z also belongs to the interval between x and y in H, hence z belongs to $\mathrm{conv}_H(S)$, establishing that z belongs to A. Hence A is indeed convex in G. Since the set A is finite and it contains the set S, the convex hull of S in G is necessarily included in A, thus this convex hull is finite. This concludes the proof of Proposition 1.

5. Proof of Theorem 1

5.1. **Auxiliary results.** We start this section with several auxiliary properties of triangle-square flag complexes occurring in condition (ii) of Theorem 1. Throughout this and next subsections, we will denote such triangle-square complexes by $\mathbf{X}$, assume that they are connected, and use the shorthand $G := G(\mathbf{X})$. We denote by $\mathbf{X}(C_3)$ and $\mathbf{X}(C_4)$ the triangle-square complex consisting of a single triangle and a single square, respectively. Let $\mathbf{X}(H) = \mathbf{X}(C_3 + C_4)$ be the complex consisting of a triangle and a square sharing one edge; its graph is the house H and with some abuse of notation, we call the complex itself a *house*. The *twin-house* $\mathbf{X}(2H)$ is the complex consisting of two triangles and two squares, which can be viewed as two houses glued along two incident edges or as a domino and a kite glued along two incident edges (for an illustration, see Fig. 4, left). Let also $\mathbf{X}(W_k)$ and $\mathbf{X}(W_k^-)$ be the triangle-square complexes whose underlying graphs are W_k and W_k^-: the first consists of k triangles and the
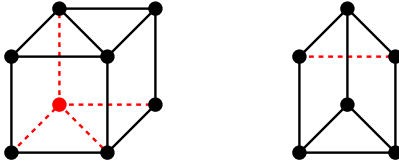
FIGURE 4. On the left, a twin-house (in black) included in a double prism (Lemma 5.2). On the right, a double house (in black) included in a prism (Lemma 5.3).

second consists of $k-2$ triangles and one square. The complex $\mathbf{X}(CW_3)$ consists of three squares sharing a vertex and pairwise sharing edges (its graph is the cogwheel CW_3). The *triangular prism* $\mathbf{X}(Pr) = \mathbf{X}(C_3 \times K_2)$ consists of the surface complex of the 3-dimensional triangular prism (two disjoint triangles and three squares pairwise sharing an edge). The *double prism* $\mathbf{X}(2Pr)$ consists of two prisms $\mathbf{X}(Pr)$ sharing a square (See Fig. 4, left). Finally, the *double-house* $\mathbf{X}(H + C_4) = \mathbf{X}(2C_4 + C_3)$ is the complex consisting of two squares and a triangle, which can be viewed as a house plus a square sharing with the house two incident edges, one from the square and another from the triangle (see Fig. 4, right). In the following results, we use the notation $G = G(\mathbf{X})$.

Lemma 5.1. *If* $\mathbf{X}$ *is a triangle-square flag complex, then its 1-skeleton* G *does not contain induced* $K_{2,3}$ *and* W_4^-.

Proof. If G contains $K_{2,3}$ or W_4^-, then, since $\mathbf{X}$ is a flag complex, we will obtain two squares intersecting in two edges, which is impossible. □

Lemma 5.2. *If* $\mathbf{X}$ *satisfies the house condition, then any twin-house* $\mathbf{X}(2H)$ *of* $\mathbf{X}$ *is included in* $\mathbf{X}$ *in a double prism* $\mathbf{X}(2Pr)$.

Proof. Let u, v, w, x_1, x_2 be the vertices of one house and u, v, w, y_1, y_2 be the vertices of another house, where the edge uv is common to the two squares uvx_2x_1 and uvy_2y_1, and where the edge vw is common to the two triangles vwx_2 and vwy_2. By the house condition, there exists a vertex a adjacent in G to x_1, u, w that is not adjacent to x_2, v. Analogously, there exists a vertex b adjacent to u, y_1, w that is not adjacent to y_2, v. If $a \neq b$, the graph induced by a, b, u, v, w is either $K_{2,3}$ if $a \nsim b$, or W_4^- otherwise; in both cases, we get a contradiction with Lemma 5.1. Thus $a = b$, and since $a \nsim v, x_2, y_2$, the vertices $a, u, v, w, x_1, x_2, y_1, y_2$ induce a double prism. □

Lemma 5.3. *If* $\mathbf{X}$ *satisfies the house condition, then any double-house* $\mathbf{X}(H + C_4)$ *in* $\mathbf{X}$ *is included in a prism* $\mathbf{X}(Pr)$, *i.e.,* G *does not contain an induced double-house* $H + C_4$.

Proof. Suppose by contradiction that G contains an induced double house having x, y, u, v, w, z as the set of vertices, where uvw is a triangle and $xyvu$ and $xuwz$ are two squares of this house. By house condition, there exists a vertex a different from z (since $y \nsim z$) that is adjacent to x, y, w and that is not adjacent to u, v. Thus, the vertices z, a, w, u, x induce either $K_{2,3}$ if $a \nsim z$ or W_4^- otherwise. In both cases, we get a contradiction with Lemma 5.1. □

20

Lemma 5.4. *If* $\mathbf{X}$ *satisfies the house condition and does not contain* $\mathbf{X}(W_4)$, *then* $\mathbf{X}$ *does not contain* $\mathbf{X}(W_k^-)$ *for any* $k \geq 5$.

Proof. Suppose by way of contradiction that $\mathbf{X}$ contains $\mathbf{X}(W_k^-)$, where k is the smallest value for which this subcomplex exists. Since, by Lemma 5.1, G does not contain W_4^-, necessarily $k \geq 5$. Denote the vertices of $\mathbf{X}(W_k^-)$ by $q, x_1, x_2, \ldots, x_k$ where $x_1, x_2, \ldots, x_k$ induce a cycle and where q is adjacent to $x_1, \ldots, x_{k-1}$ but not to x_k. By the house condition applied to the house induced by $q, x_{k-1}, x_k, x_1, x_2$, there exists p in G such that $p \sim x_{k-1}, x_k, x_2$ and $p \nsim q, x_1$. If $p \sim x_3$, then the vertices x_3, p, x_2, q, x_{k-1} induce W_4 if $x_3 \sim x_{k-1}$ (i.e., if $k = 5$), or W_4^- otherwise; in both cases, we get a contradiction. Thus $p \nsim x_3$. Let j be the smallest index greater than 3 such that $p \sim x_j$. Since $p \sim x_{k-1}$, j is well-defined. But then, the vertices $q, p, x_2, \ldots, x_j$ induce W_j^- with $j < k$, contradicting the choice of k. $\qquad\square$

5.2. **Construction of the universal cover and weak modularity.** To prove the implication (ii)⇒(iii) of Theorem 1, from now on, we suppose that $\mathbf{X}$ is a connected (but not necessarily simply connected) triangle-square flag complex satisfying the $(W_4, \widehat{W_5})$, the house, and the cube conditions. Following the proof of Osajda [32, Theorem 4.5], we will construct the universal cover $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ as an increasing union $\bigcup_{i \geq 1} \widetilde{\mathbf{X}}_i$ of triangle-square complexes. The complexes $\widetilde{\mathbf{X}}_i$ are in fact spanned by concentric combinatorial balls $\widetilde{B}_i$ in $\widetilde{\mathbf{X}}$. The covering map f is then the union $\bigcup_{i \geq 1} f_i$, where $f_i : \widetilde{\mathbf{X}}_i \to \mathbf{X}$ is a locally injective cellular map such that $f_i|_{\widetilde{\mathbf{X}}_j} = f_j$, for every $j \leq i$. We denote by $\widetilde{G}_i = G(\widetilde{\mathbf{X}}_i)$ the underlying graph of $\widetilde{\mathbf{X}}_i$. We denote by $\widetilde{S}_i$ the set of vertices $\widetilde{B}_i \setminus \widetilde{B}_{i-1}$.

Pick any vertex v of $\mathbf{X}$ as the basepoint. Define $\widetilde{B}_0 = \{\widetilde{v}\} := \{v\}, \widetilde{B}_1 := B_1(v, G)$, and $f_1 := \mathrm{Id}_{B_1(v,G)}$. Let $\widetilde{\mathbf{X}}_1$ be the triangle-square complex spanned by $B_1(v, G)$. Assume that, for $i \geq 1$, we have constructed the sets $\widetilde{B}_1, \ldots \widetilde{B}_i$, and we have defined the triangle-square complexes $\widetilde{\mathbf{X}}_1, \ldots \widetilde{\mathbf{X}}_i$ and the corresponding cellular maps $f_1, \ldots, f_i$ from $\widetilde{\mathbf{X}}_1, \ldots \widetilde{\mathbf{X}}_i$, respectively, to $\mathbf{X}$ so that the graph $\widetilde{G}_i = G(\widetilde{\mathbf{X}}_i)$ and the complex $\widetilde{\mathbf{X}}_i$ satisfies the following conditions:

(P$_i$) $B_j(\widetilde{v}, \widetilde{G}_i) = \widetilde{B}_j$ for any $j \leq i$;

(Q$_i$) $\widetilde{G}_i$ is weakly modular with respect to $\widetilde{v}$ (i.e., $\widetilde{G}_i$ satisfies the conditions TC($\widetilde{v}$) and QC($\widetilde{v}$));

(R$_i$) for any $\widetilde{u} \in \widetilde{B}_{i-1}$, f_i defines an isomorphism between the subgraph of $\widetilde{G}_i$ induced by $B_1(\widetilde{u}, \widetilde{G}_i)$ and the subgraph of G induced by $B_1(f_i(\widetilde{u}), G)$;

(S$_i$) for any $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_{i-1}$ such that the vertices $w = f_i(\widetilde{w}), w' = f_i(\widetilde{w}')$ belong to a square $ww'uu'$ of $\mathbf{X}$, there exist $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_i$ such that $f_i(\widetilde{u}) = u, f_i(\widetilde{u}') = u'$ and $\widetilde{w}\widetilde{w}'\widetilde{u}\widetilde{u}'$ is a square of $\widetilde{\mathbf{X}}_i$.

(T$_i$) for any $\widetilde{w} \in \widetilde{S}_i := \widetilde{B}_i \setminus \widetilde{B}_{i-1}$, f_i defines an isomorphism between the subgraphs of $\widetilde{G}_i$ and of G induced by $B_1(\widetilde{w}, \widetilde{G}_i)$ and $f_i(B_1(\widetilde{w}, \widetilde{G}_i))$.

It can be easily checked that $\widetilde{B}_1, \widetilde{G}_1, \widetilde{X}_1$ and f_1 satisfy the conditions (P$_1$),(Q$_1$),(R$_1$),(S$_1$), and (T$_1$). Now we construct the set $\widetilde{B}_{i+1}$, the graph $\widetilde{G}_{i+1}$ having $\widetilde{B}_{i+1}$ as the vertex-set, the triangle-square complex $\widetilde{\mathbf{X}}_{i+1}$ having $\widetilde{G}_{i+1}$ as its 1-skeleton, and the map $f_{i+1} : \widetilde{\mathbf{X}}_{i+1} \to \mathbf{X}$. Let

$$Z = \{(\widetilde{w}, z) : \widetilde{w} \in \widetilde{S}_i \text{ and } z \in B_1(f_i(\widetilde{w}), G) \setminus f_i(B_1(\widetilde{w}, \widetilde{G}_i))\}.$$

On Z we define a binary relation $\equiv$ by setting $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ if and only if $z = z'$ and one of the following two conditions is satisfied:

(Z1) $\widetilde{w}$ and $\widetilde{w}'$ are the same or adjacent in $\widetilde{G}_i$ and $z \in B_1(f_i(\widetilde{w}), G) \cap B_1(f_i(\widetilde{w}'), G)$;

(Z2) there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}$ and $\widetilde{w}'$ and such that $f_i(\widetilde{u}) f_i(\widetilde{w}) z f_i(\widetilde{w}')$ is a square-cell of $\mathbf{X}$.

Lemma 5.5. *The relation $\equiv$ is an equivalence relation on Z.*

Proof. For any vertex $\widetilde{w} \in \widetilde{B}_i$, we will denote by $w = f_i(\widetilde{w})$ its image in X under f_i. Since the binary relation $\equiv$ is reflexive and symmetric, it suffices to show that $\equiv$ is transitive. Let $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ and $(\widetilde{w}', z') \equiv (\widetilde{w}'', z'')$. We will prove that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z'')$. By definition of $\equiv$, we conclude that $z = z' = z''$. By definition of $\equiv$, $z \in B_1(w, G) \cap B_1(w', G) \cap B_1(w'', G)$.

If $\widetilde{w} \sim \widetilde{w}''$ (in $\widetilde{G}_i$), then by definition of $\equiv$, $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ and we are done. If $\widetilde{w} \nsim \widetilde{w}''$ and if there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$, then by (R_i) applied to $\widetilde{u}$, we obtain that $u \sim w, w''$ and $w \nsim w''$. Since $(\widetilde{w}, z), (\widetilde{w}'', z) \in Z$, we have $z \sim w, w''$. Moreover, if $z \sim u$, then by (R_i) applied to u, there exists $\widetilde{z} \in \widetilde{B}_i$, such that $\widetilde{z} \sim \widetilde{u}, \widetilde{w}, \widetilde{w}''$ and $f_i(\widetilde{z}) = z$. Thus $(\widetilde{w}, z), (\widetilde{w}', z) \notin Z$, which is a contradiction. Consequently, if $\widetilde{w} \nsim \widetilde{w}''$ and if there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$ and $f_i(\widetilde{u}) = u$, then $uwzw''$ is an induced square in G, and by condition (Z2), we are done. Therefore, in the rest of the proof, we will assume the following:

(A_1) $\widetilde{w} \nsim \widetilde{w}''$;

(A_2) there is no $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$.

Claim 1. For any couple $(\widetilde{w}, z) \in Z$ the following properties hold:

(A_3) there is no neighbor $\widetilde{z} \in \widetilde{B}_{i-1}$ of $\widetilde{w}$ such that $f_i(\widetilde{z}) = z$;

(A_4) there is no neighbor $\widetilde{u} \in \widetilde{B}_{i-1}$ of $\widetilde{w}$ such that $u \sim z$;

(A_5) there are no $\widetilde{x}, \widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $y \sim z$.

Proof. If $\widetilde{w}$ has a neighbor $\widetilde{z} \in \widetilde{B}_{i-1}$ such that $f_i(\widetilde{z}) = z$, then $(\widetilde{w}, z) \notin Z$, a contradiction. This establishes (A3).

If $\widetilde{w}$ has a neighbor $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $u \sim z$, then by (R_i) applied to $\widetilde{u}$, there exists $\widetilde{z} \in \widetilde{B}_i$ such that $\widetilde{z} \sim \widetilde{u}, \widetilde{w}$. Thus $(\widetilde{w}, z) \notin Z$, a contradiction, establishing (A4).

If there exist $\widetilde{x}, \widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $y \sim z$, then $yxwz$ is an induced square in G. From (S_i) applied to $\widetilde{y}, \widetilde{x}$, there exists $\widetilde{z} \in \widetilde{B}_i$ such that $\widetilde{z} \sim \widetilde{y}, \widetilde{w}$ and $f_i(\widetilde{z}) = z$. Thus $(\widetilde{w}, z) \notin Z$, a contradiction, and therefore (A5) holds as well. $\qquad \square$

We distinguish three cases depending of which of the conditions (Z1) or (Z2) are satisfied by the pairs $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ and $(\widetilde{w}', z') \equiv (\widetilde{w}'', z'')$.

Case 1: $\widetilde{w}'$ is adjacent in $\widetilde{G}_i$ to both $\widetilde{w}$ and $\widetilde{w}''$.

By (Q_i), the graph $\widetilde{G}_i$ satisfies the triangle condition $\mathrm{TC}(\widetilde{v})$, thus there exist two vertices $\widetilde{u}, \widetilde{u}' \in \widetilde{S}_{i-1}$ such that $\widetilde{u}$ is adjacent to $\widetilde{w}, \widetilde{w}'$ and $\widetilde{u}'$ is adjacent to $\widetilde{w}', \widetilde{w}''$. By (A_2), $\widetilde{u} \nsim \widetilde{w}''$, $\widetilde{u}' \nsim \widetilde{w}$, $\widetilde{u} \neq \widetilde{u}'$.

If $\widetilde{u} \sim \widetilde{u}'$, then by (T_i) applied to $\widetilde{w}'$ and by $(A_3)\&(A_4)$, the vertices u, u', w, w', w'', z induce W_5 in G. By $\mathrm{TC}(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. By (R_i) applied to $\widetilde{u}$ and $\widetilde{u}'$,

22

we get $x \notin \{u, u', w, w', w''\}$ and $x \sim u, u'$. From $(A_4)\&(A_5)$, we get $x \neq z$ and $x \nsim z$. Since G satisfies the $\widehat{W_5}$-wheel condition, there exists a vertex y of G adjacent to x, u, u', w, w', w'', z. By (R_i) applied to $\widetilde{u}$, there exists $\widetilde{y} \sim \widetilde{w}, \widetilde{u}, \widetilde{x}$ and thus $\widetilde{y} \in \widetilde{B}_{i-1}$, contradicting the property (A_4).

Suppose now that $\widetilde{u} \nsim \widetilde{u}'$. Then $i \geq 2$ and by QC$(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. From $(A_4)\&(A_5)$, $x \neq z$ and $x \nsim z$. Consequently, z, w, w', w'', u, u', x induce a W_6^-, contradicting Lemma 5.4.

Case 2: $\widetilde{w}$ and $\widetilde{w}'$ are adjacent in $\widetilde{G}_i$, and there exists $\widetilde{u}' \in \widetilde{B}_{i-1}$ adjacent to $\widetilde{w}'$ and $\widetilde{w}''$ such that $u'w'w''z$ is a square-cell of $\mathbf{X}$.

By $(A_1)\&(A_2)$, $\widetilde{w} \nsim \widetilde{w}''$ and $\widetilde{u}' \nsim \widetilde{w}$. By triangle condition TC$(\widetilde{v})$ for $\widetilde{G}_i$, there exists a vertex $\widetilde{u} \in \widetilde{B}_{i-1}$ different from $\widetilde{u}'$ and adjacent to $\widetilde{w}$ and $\widetilde{w}'$. By $(A_3)\&(A_4)$, $u \neq z$ and $u \nsim z$. By (A_2), $\widetilde{u} \nsim \widetilde{w}''$.

If $\widetilde{u} \sim \widetilde{u}'$, by (T_i) applied to w', z, w, w', u, u', w'' induce a W_5^-, contradicting Lemma 5.4. Thus $\widetilde{u} \nsim \widetilde{u}'$. By quadrangle condition QC$(\widetilde{v})$ for $\widetilde{G}_i$, there exists a vertex $\widetilde{x} \in \widetilde{S}_{i-2}$ adjacent to $\widetilde{u}$ and $\widetilde{u}'$. From $(A_4)\&(A_5)$, $x \neq z$ and $x \nsim z$. By (T_i) applied to $\widetilde{w}'$ and by (R_i) applied to $\widetilde{u}'$, we get that z, w, w', w'', u, u', x induce a twin-house. By Lemma 5.2 there exists y in G such that $y \sim w, w'', u', x$ and $y \nsim u, z$. By (R_i) applied to u', there exists $\widetilde{y} \in \widetilde{B}_i$ such that $\widetilde{y} \sim \widetilde{u}', \widetilde{w}'', \widetilde{x}$. By (S_i) applied to $\widetilde{u}, \widetilde{x}$ and to the square $uxyw$, we get $\widetilde{y} \sim \widetilde{w}$. Consequently, $\widetilde{y} \in \widetilde{S}_{i-1}$, $\widetilde{y} \sim \widetilde{w}, \widetilde{w}''$, contradicting (A_2).

Case 3: There exist $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i-1}$ such that the vertex $\widetilde{u}$ is adjacent in $\widetilde{G}_i$ to $\widetilde{w}, \widetilde{w}'$, the vertex $\widetilde{u}'$ is adjacent to $\widetilde{w}', \widetilde{w}''$, and $uwzw'$ and $u'w'zw''$ are square-cells of $\mathbf{X}$.

From $(A_1)\&(A_2)$, $\widetilde{w} \nsim \widetilde{w}''$, $\widetilde{u} \neq \widetilde{u}'$, $\widetilde{u} \nsim \widetilde{w}''$, and $\widetilde{u}' \nsim \widetilde{w}$. From (A_3), $u \neq z \neq u'$ and $z \nsim u, u'$. If $\widetilde{u} \sim \widetilde{u}'$, by (T_i) applied to w' and by (R_i) applied to u, u', the vertices z, w, w', w'', u, u' induce a double-house, which is impossible from Lemma 5.3. Thus $\widetilde{u} \nsim \widetilde{u}'$.

By QC$(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. By $(A_4)\&(A_5)$, $x \neq z$ and $x \nsim z$. By (T_i) applied to w' and by (R_i) applied to u, u', the vertices z, w, w', w'', u, u', x induce CW_3. Thus, by the cube condition, there exists a vertex y of G such that $y \sim x, w, w''$ and $y \nsim z, w', u, u'$. By (R_i) applied to $\widetilde{x}$, there is $\widetilde{y} \in \widetilde{B}_i$ such that $\widetilde{y} \sim \widetilde{x}$. By (S_i) applied to $\widetilde{u}, \widetilde{x}$ and to the square $uxyw$, we have $\widetilde{y} \sim \widetilde{w}$. By (S_i) applied to $\widetilde{u}', \widetilde{x}$ and to the square $u'xyw''$, we get $\widetilde{y} \sim \widetilde{w}''$. Consequently, $\widetilde{y} \in \widetilde{S}_{i-1}$, $\widetilde{y} \sim \widetilde{w}, \widetilde{w}''$, contradicting (A_2). $\qquad\square$

Let $\widetilde{S}_{i+1}$ denote the equivalence classes of $\equiv$, i.e., $\widetilde{S}_{i+1} = Z/_\equiv$. For a couple $(\widetilde{w}, z) \in Z$, we will denote by $[\widetilde{w}, z]$ the equivalence class of $\equiv$ containing $(\widetilde{w}, z)$. Set $\widetilde{B}_{i+1} := \widetilde{B}_i \cup \widetilde{S}_{i+1}$. Let $\widetilde{G}_{i+1}$ be the graph having $\widetilde{B}_{i+1}$ as the vertex set in which two vertices $\widetilde{a}, \widetilde{b}$ are adjacent if and only if one of the following conditions holds:

(1) $\widetilde{a}, \widetilde{b} \in \widetilde{B}_i$ and $\widetilde{a}\widetilde{b}$ is an edge of $\widetilde{G}_i$,
(2) $\widetilde{a} \in \widetilde{B}_i$, $\widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{b} = [\widetilde{a}, z]$,
(3) $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$, $\widetilde{a} = [\widetilde{w}, z]$, $\widetilde{b} = [\widetilde{w}, z']$ for a vertex $\widetilde{w} \in \widetilde{B}_i$, and $z \sim z'$ in the graph G.

Finally, we define the map $f_{i+1} : \widetilde{B}_{i+1} \rightarrow V(\mathbf{X})$ in the following way: if $\widetilde{a} \in \widetilde{B}_i$, then $f_{i+1}(\widetilde{a}) = f_i(\widetilde{a})$, otherwise, if $\widetilde{a} \in \widetilde{S}_{i+1}$ and $\widetilde{a} = [\widetilde{w}, z]$, then $f_{i+1}(\widetilde{a}) = z$. Notice that f_{i+1} is

well-defined because all couples from the equivalence class represented by $\widetilde{a}$ have one and the same vertex z in the second argument. In the sequel, all vertices of $\widetilde{B}_{i+1}$ will be denoted with a tilde and their images in G under f_{i+1} will be denoted without tilde, e.g. if $\widetilde{w} \in \widetilde{B}_{i+1}$, then $w = f_{i+1}(\widetilde{w})$.

Lemma 5.6. $\widetilde{G}_{i+1}$ *satisfies the property* (P_{i+1}), *i.e.,* $B_j(v, \widetilde{G}_{i+1}) = \widetilde{B}_j$ *for any* $j \le i + 1$.

Proof. By definition of edges of $\widetilde{G}_{i+1}$, any vertex $\widetilde{b}$ of $\widetilde{S}_{i+1}$ is adjacent to at least one vertex of $\widetilde{B}_i$ and all such neighbors of $\widetilde{b}$ are vertices of the form $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, z]$ for a couple $(\widetilde{w}, z)$ of Z. By definition of Z, $\widetilde{w} \in \widetilde{S}_i$, whence any vertex of $\widetilde{S}_{i+1}$ is adjacent only to vertices of $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$. Therefore, the distance between the basepoint $\widetilde{v}$ and any vertex $\widetilde{a} \in \widetilde{B}_i$ is the same in the graphs $\widetilde{G}_i$ and $\widetilde{G}_{i+1}$. On the other hand, the distance in $\widetilde{G}_{i+1}$ between $\widetilde{v}$ and any vertex $\widetilde{b}$ of $\widetilde{S}_{i+1}$ is $i + 1$. This shows that indeed $B_j(v, \widetilde{G}_{i+1}) = \widetilde{B}_j$ for any $j \le i + 1$. $\qquad\square$

Lemma 5.7. $\widetilde{G}_{i+1}$ *satisfies the property* (Q_{i+1}), *i.e., the graph* $\widetilde{G}_{i+1}$ *is weakly modular with respect to the basepoint* $\widetilde{v}$.

Proof. First we show that $\widetilde{G}_{i+1}$ satisfies the triangle condition $\mathrm{TC}(\widetilde{v})$. Pick two adjacent vertices $\widetilde{x}, \widetilde{y}$ having in $\widetilde{G}_{i+1}$ the same distance to $\widetilde{v}$. Since by Lemma 5.6, $\widetilde{G}_{i+1}$ satisfies the property (P_{i+1}) and the graph $\widetilde{G}_i$ is weakly modular with respect to $\widetilde{v}$, we can suppose that $\widetilde{x}, \widetilde{y} \in \widetilde{S}_{i+1}$. From the definition of the edges of $\widetilde{G}_{i+1}$, there exist two couples $(\widetilde{w}, z), (\widetilde{w}, z') \in Z$ such that $\widetilde{w} \in \widetilde{B}_i$, z is adjacent to z' in G, and $\widetilde{x} = [\widetilde{w}, z], \widetilde{y} = [\widetilde{w}, z']$. Since $\widetilde{w}$ is adjacent in $\widetilde{G}_{i+1}$ to both $\widetilde{x}$ and $\widetilde{y}$, the triangle condition $\mathrm{TC}(\widetilde{v})$ is established.

Now we show that $\widetilde{G}_{i+1}$ satisfies the quadrangle condition $\mathrm{QC}(\widetilde{v})$. Since the properties (P_{i+1}) and (Q_i) hold, it suffices to consider a vertex $\widetilde{x} \in \widetilde{S}_{i+1}$ having two nonadjacent neighbors $\widetilde{w}, \widetilde{w}'$ in $\widetilde{S}_i$. By definition of $\widetilde{G}_{i+1}$, there exists a vertex z of $\mathbf{X}$ and couples $(\widetilde{w}, z), (\widetilde{w}', z) \in Z$ such that $\widetilde{x} = [\widetilde{w}, z]$ and $\widetilde{x} = [\widetilde{w}', z]$. Hence $(\widetilde{w}, z) \equiv (\widetilde{w}', z)$. Since $\widetilde{w}$ and $\widetilde{w}'$ are not adjacent, by condition (Z2) in the definition of $\equiv$ there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ adjacent to $\widetilde{w}$ and $\widetilde{w}'$, whence $\widetilde{x}, \widetilde{w}, \widetilde{w}'$ satisfy $\mathrm{QC}(\widetilde{v})$. $\qquad\square$

We first prove that the mapping f_{i+1} is a graph homomorphism (preserving edges) from $\widetilde{G}_{i+1}$ to G. In particular, this implies that two adjacent vertices of $\widetilde{G}_{i+1}$ are mapped in G to different vertices.

Lemma 5.8. f_{i+1} *is a graph homomorphism from* $\widetilde{G}_{i+1}$ *to* G, *i.e., for any edge* $\widetilde{ab}$ *of* $\widetilde{G}_{i+1}$, ab *is an edge of* G.

Proof. Consider an edge $\widetilde{ab}$ of $\widetilde{G}_{i+1}$. If $\widetilde{a}, \widetilde{b} \in \widetilde{B}_i$, the lemma holds by (R_i) or (T_i) applied to $\widetilde{a}$. Suppose that $\widetilde{a} \in \widetilde{S}_{i+1}$. If $\widetilde{b} \in \widetilde{B}_i$, then $\widetilde{a} = [\widetilde{b}, a]$, and ab is an edge of G. If $\widetilde{b} \in \widetilde{B}_{i+1}$, then the fact that $\widetilde{a}$ and $\widetilde{b}$ are adjacent implies that there exists a vertex $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{a} = [\widetilde{w}, a], \widetilde{b} = [\widetilde{w}, b]$ and such that $a \sim b$ in G. $\qquad\square$

We now prove that f_{i+1} is locally surjective for any vertex in $\widetilde{B}_i$.

Lemma 5.9. *If* $\widetilde{a} \in \widetilde{B}_i$ *and if* $b \sim a$ *in* G, *then there exists a vertex* $\widetilde{b}$ *of* $\widetilde{G}_{i+1}$ *adjacent to* $\widetilde{a}$ *such that* $f_{i+1}(\widetilde{b}) = b$.

Proof. If $\widetilde{a} \in \widetilde{B}_{i-1}$, the lemma holds by (R_i). Suppose that $\widetilde{a} \in \widetilde{S}_i$ and consider $b \sim a$ in G. If $\widetilde{a}$ has a neighbor $\widetilde{b} \in \widetilde{B}_i$ mapped to b by f_i, we are done. Otherwise $(\widetilde{a}, b) \in Z$, $[\widetilde{a}, b] \sim \widetilde{a}$ in $\widetilde{G}_{i+1}$ and $[\widetilde{a}, b]$ is mapped to b by f_{i+1}. $\qquad\square$

Before proving the local injectivity of f_{i+1}, we formulate a technical lemma.

Lemma 5.10. *Let* $(\widetilde{w}, a), (\widetilde{w}', a) \in Z$ *be such that* $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$. *If* $(\widetilde{w}, b) \in Z$ *and* $b \sim w'$ *in* G, *then* $\widetilde{w} \sim \widetilde{w}'$, $(\widetilde{w}', b) \in Z$ *and* $(\widetilde{w}, b) \equiv (\widetilde{w}', b)$.

Proof. First suppose that $\widetilde{w} \nsim \widetilde{w}'$. Since $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{w}'$ and $wxw'a$ is an induced square in G. In G, $b \sim w, w'$, thus b, w, x, a, w' induce $K_{2,3}$ if $b \nsim a, x$, W_4 if $b \sim a, x$, or W_4^- otherwise. In any case, we get a contradiction.

Suppose now that $\widetilde{w} \sim \widetilde{w}'$. If $(\widetilde{w}', b) \notin Z$, then there exists $\widetilde{b} \in \widetilde{B}_i$ such that $\widetilde{b} \sim \widetilde{w}'$ and $f_i(\widetilde{b}) = b$. In G, wbw' is a triangle, thus $\widetilde{b} \sim \widetilde{w}$ by condition (R_i) applied to $\widetilde{b}$. This implies that $(\widetilde{w}, b) \notin Z$. Consequently, $(\widetilde{w}, b), (\widetilde{w}', b) \in Z$ and $(\widetilde{w}, b) \equiv (\widetilde{w}', b)$ since $\widetilde{w} \sim \widetilde{w}'$. $\qquad\square$

We can now prove that f_{i+1} is locally injective.

Lemma 5.11. *If* $\widetilde{a} \in \widetilde{B}_{i+1}$ *and* $\widetilde{b}, \widetilde{c}$ *are distinct neighbors of* $\widetilde{a}$ *in* $\widetilde{G}_{i+1}$, *then* $b \neq c$.

Proof. First note that if $\widetilde{b} \sim \widetilde{c}$, the assertion holds by Lemma 5.8; in the following we assume that $\widetilde{b} \nsim \widetilde{c}$. If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, the lemma holds by (R_i) or (T_i) applied to $\widetilde{a}$. Suppose first that $\widetilde{a} \in \widetilde{B}_i$. If $\widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, then $\widetilde{b} = [\widetilde{a}, b]$ and $\widetilde{c} = [\widetilde{a}, c]$, and thus $b \neq c$. If $\widetilde{b} \in \widetilde{B}_i$ and $\widetilde{c} = [\widetilde{a}, c] \in \widetilde{S}_{i+1}$, then $(\widetilde{a}, b) \notin Z$, and thus $c \neq b$. Thus, let $\widetilde{a} \in \widetilde{S}_{i+1}$.

If $\widetilde{b}, \widetilde{c} \in \widetilde{B}_i$ and $\widetilde{a} \in \widetilde{S}_{i+1}$, then $\widetilde{a} = [\widetilde{b}, a] = [\widetilde{c}, a]$. Since $(\widetilde{b}, a) \equiv (\widetilde{c}, a)$ and since $\widetilde{b} \nsim \widetilde{c}$, there exists $\widetilde{w} \in \widetilde{B}_{i-1}$ such that $\widetilde{w} \sim \widetilde{b}, \widetilde{c}$ and $abwc$ is an induced square of G. This implies that $b \neq c$.

If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, then there exist $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$, $\widetilde{c} = [\widetilde{w}', c]$, and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{w}', a]$. If $b = c$, then $[\widetilde{w}, b] = [\widetilde{w}', b] = [\widetilde{w}', c]$ by Lemma 5.10, and thus $\widetilde{b} = \widetilde{c}$, which is impossible.

If $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{S}_i$, then there exists $\widetilde{w} \in \widetilde{S}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$ and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{c}, a]$. If $\widetilde{w} \sim \widetilde{c}$, then $(\widetilde{w}, c) \notin Z$ and thus, $(\widetilde{w}, c) \neq (\widetilde{w}, b)$, i.e., $b \neq c$. If $\widetilde{w} \nsim \widetilde{c}$, since $[\widetilde{w}, a] = [\widetilde{c}, a]$, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{c}$ and such that $acxw$ is an induced square of G. Since $\widetilde{w}$ and $\widetilde{c}$ are not adjacent, by (R_i) applied to $\widetilde{x}$, w and c are not adjacent as well. Since $w \sim b$, this implies that $b \neq c$. $\qquad\square$

Lemma 5.12. *If* $\widetilde{a} \sim \widetilde{b}, \widetilde{c}$ *in* $\widetilde{G}_{i+1}$, *then* $\widetilde{b} \sim \widetilde{c}$ *if and only if* $b \sim c$.

Proof. By Lemma 5.11, $b \neq c$. If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, then the lemma holds by condition (R_i) applied to $\widetilde{a}$. Note that from Lemma 5.8, if $\widetilde{b} \sim \widetilde{c}$, then $b \sim c$. Suppose now that $b \sim c$ in G.

Suppose that $\widetilde{a} \in \widetilde{B}_i$. If $\widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, $\widetilde{b} = [\widetilde{a}, b]$ and $\widetilde{c} = [\widetilde{a}, c]$. Since $b \sim c$, by construction, we have $\widetilde{b} \sim \widetilde{c}$ in $\widetilde{G}_{i+1}$. Suppose now that $\widetilde{b} = [\widetilde{a}, b] \in S_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$. If there exists $\widetilde{b}' \sim \widetilde{c}$ in $\widetilde{G}_i$ such that $f_i(\widetilde{b}') = b$, then by (R_i) applied to $\widetilde{c}$, $\widetilde{a} \sim \widetilde{b}'$ and $(\widetilde{a}, b) \notin Z$, which is a contradiction. Thus $(\widetilde{c}, b) \in Z$ and since $\widetilde{c} \sim \widetilde{a}$, $[\widetilde{c}, b] = [\widetilde{a}, b] = \widetilde{b}$, and consequently, $\widetilde{c} \sim \widetilde{b}$. Therefore, let $\widetilde{a} \in \widetilde{S}_{i+1}$.

If $\widetilde{b}, \widetilde{c} \in \widetilde{B}_i$ and $\widetilde{a} \in \widetilde{S}_{i+1}$, then $\widetilde{a} = [\widetilde{b}, a] = [\widetilde{c}, a]$ and either $\widetilde{b} \sim \widetilde{c}$, or there exists $\widetilde{w} \in \widetilde{S}_{i-1}$ such that $\widetilde{w} \sim \widetilde{b}, \widetilde{c}$ and $wbac$ is an induced square in G, which is impossible because $b \sim c$.

If $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$, then there exists $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$ and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{c}, a]$. By Lemma 5.10, $(\widetilde{c}, b) \in Z$ and $\widetilde{b} = [\widetilde{w}, b] = [\widetilde{c}, b]$. Consequently, $\widetilde{c} \sim \widetilde{b}$.

If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, there exist $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$, $\widetilde{c} = [\widetilde{w}', c]$ and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{w}', a]$. If $\widetilde{w} \sim \widetilde{c}$ or $\widetilde{w}' \sim \widetilde{b}$, then $\widetilde{b} \sim \widetilde{c}$ because $b \sim c$. Suppose now that $\widetilde{w} \not\sim \widetilde{c}$, $\widetilde{w}' \not\sim \widetilde{b}$ From previous case applied to $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$ (resp. $\widetilde{a}, \widetilde{c} \in \widetilde{S}_{i+1}$) and $\widetilde{w}' \in \widetilde{B}_i$ (resp. $\widetilde{w} \in \widetilde{B}_i$), it follows that $w \not\sim c$ and $w' \not\sim b$. If $\widetilde{w} \sim \widetilde{w}'$, then a, b, w, w', c induce W_4 in G, which is impossible. Since $[\widetilde{w}, a] = [\widetilde{w}', a]$, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$, such that $\widetilde{x} \sim \widetilde{w}, \widetilde{w}'$ and such that $awxw'$ is an induced square in G. If $x \sim b$, then by (R_i) applied to x, there exists $\widetilde{b}' \in \widetilde{B}_i$ mapped to b by f_i such that $\widetilde{b}' \sim \widetilde{x}, \widetilde{w}$ and thus $(\widetilde{w}, b) \notin Z$, which is a contradiction. Using the same arguments, we have that $x \not\sim c$ and thus, a, b, c, w', x, w induce W_5^- in G, which is impossible. $\qquad \square$

We can now prove that the image under f_{i+1} of an induced triangle or square is an induced triangle or square.

Lemma 5.13. *If $\widetilde{a}\widetilde{b}\widetilde{c}$ is a triangle in $\widetilde{G}_{i+1}$, then abc is a triangle in G. If $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ is an induced square of $\widetilde{G}_{i+1}$, then $abcd$ is an induced square in G. In particular, the graph $\widetilde{G}_{i+1}$ does not contain induced $K_{2,3}$ and W_4^-.*

Proof. For triangles, the assertion follows directly from Lemma 5.8. Consider now a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$. From Lemmas 5.8 and 5.11, the vertices $a, b, c,$ and d are pairwise distinct and $a \sim b$, $b \sim c$, $c \sim d$, $d \sim a$. From Lemma 5.12, $a \not\sim c$ and $b \not\sim d$. Consequently, $abcd$ is an induced square in G.

Now, if $\widetilde{G}_{i+1}$ contains an induced $K_{2,3}$ or W_4^-, from the first assertion and Lemma 5.12 we conclude that the image under f_{i+1} of this subgraph will be an induced $K_{2,3}$ or W_4^- in the graph G, a contradiction. $\qquad \square$

The second assertion of Lemma 5.13 implies that replacing all 3-cycles and all induced 4-cycles of $\widetilde{G}_{i+1}$ by triangle- and square-cells, we will obtain a triangle-square flag complex, which we denote by $\widetilde{\mathbf{X}}_{i+1}$. Then obviously $\widetilde{G}_{i+1} = G(\widetilde{\mathbf{X}}_{i+1})$. The first assertion of Lemma 5.13 implies that f_{i+1} can be extended to a cellular map from $\widetilde{\mathbf{X}}_{i+1}$ to $\mathbf{X}$: f_{i+1} maps a triangle $\widetilde{a}\widetilde{b}\widetilde{c}$ to the triangle abc of $\mathbf{X}$ and a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ to the square $abcd$ of $\mathbf{X}$.

Lemma 5.14. *f_{i+1} satisfies the conditions (R_{i+1}) and (T_{i+1}).*

Proof. From Lemmas 5.11 and 5.12, we know that for any $\widetilde{w} \in \widetilde{B}_{i+1}$, f_{i+1} induces an isomorphism between the subgraph of $\widetilde{G}_{i+1}$ induced by $B_1(\widetilde{w}, \widetilde{G}_{i+1})$ and the subgraph of G induced by $f_{i+1}(B_1(\widetilde{w}, \widetilde{G}_{i+1}))$. Consequently, the condition (T_{i+1}) holds. From Lemma 5.9, we know that $f_{i+1}(B_1(\widetilde{w}, \widetilde{G}_{i+1})) = B_1(w, G)$ and consequently (R_{i+1}) holds as well. $\qquad \square$

Lemma 5.15. *For any $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ such that the vertices $w = f_{i+1}(\widetilde{w}), w' = f_{i+1}(\widetilde{w}')$ belong to a square $ww'u'u$ of $\mathbf{X}$, there exist $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i+1}$ such that $f_{i+1}(\widetilde{u}) = u, f_{i+1}(\widetilde{u}') = u'$ and $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square of $\widetilde{\mathbf{X}}_{i+1}$, i.e., $\widetilde{\mathbf{X}}_{i+1}$ satisfies the property (S_{i+1}).*

Proof. Note that if $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_{i-1}$, the lemma holds by condition (S_i). Let us assume further that $\widetilde{w} \in \widetilde{S}_i$. By Lemma 5.14 applied to $\widetilde{w}$ and $\widetilde{w}'$, we know that in $\widetilde{G}_{i+1}$ there exist $\widetilde{u}, \widetilde{u}'$ such that $\widetilde{u} \sim \widetilde{w}$ and $\widetilde{u}' \sim \widetilde{w}'$.

Case 1. $\widetilde{w}' \in \widetilde{S}_{i-1}$.

If $\widetilde{u}' \in \widetilde{B}_{i-1}$, by (S_i) applied to $\widetilde{w}'$ and $\widetilde{u}'$, we conclude that $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$.

If $\widetilde{u}' \in \widetilde{S}_i$ and $\widetilde{u} \in \widetilde{S}_{i-1}$, then Lemma 5.14 applied to $\widetilde{w}$, implies that $\widetilde{u}$ is not adjacent to $\widetilde{w}'$. Thus, by quadrangle condition $QC(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{w}'$. Hence, w, w', u, u', x induce in G a forbidden $K_{2,3}, W_4^-$, or W_4, which is impossible.

Suppose now that $\widetilde{u}', \widetilde{u} \in \widetilde{S}_i$. By $TC(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ different from $\widetilde{w}'$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{w}$. Since G does not contain W_4^- or W_4, $x \nsim u', w'$ and the vertices u, w, w', u', x induce a house. By the house condition there exists y in G such that $y \sim x, u', w'$ and $y \nsim u, w$. Since $x \nsim w'$, by R_i applied to $\widetilde{w}$, $\widetilde{x} \nsim \widetilde{w}'$. Applying $QC(\widetilde{v})$, there exists $\widetilde{z} \in \widetilde{S}_{i-2}$ such that $\widetilde{z} \sim \widetilde{x}, \widetilde{w}'$ and $\widetilde{z} \nsim \widetilde{w}$. Since $\widetilde{z} \in \widetilde{S}_{i-2}$, $\widetilde{z} \nsim \widetilde{u}'$ and thus by R_{i+1} applied to $\widetilde{w}'$, $z \nsim u'$. Consequently, $z \neq y$. Thus, from Lemma 5.13, $xzw'w$ is an induced square of G and y, x, z, w', w induce a $K_{2,3}$ if $z \nsim y$ and W_4^- otherwise, which is impossible. Note that if $\widetilde{u}'$ has a neighbor $\widetilde{u}_2$ in $\widetilde{B}_i$ mapped to u, then, exchanging the roles of $\widetilde{u}'$ and $\widetilde{w}$, we also get a contradiction. Suppose now that neither $\widetilde{w}$ nor $\widetilde{u}'$ has a neighbor in $\widetilde{B}_i$ mapped to u. Thus, $(\widetilde{w}, u), (\widetilde{u}', u) \in Z$ and since $\widetilde{w}' \in \widetilde{S}_{i-1}$ is adjacent to $\widetilde{w}$ and $\widetilde{u}'$, $(\widetilde{w}, u) \equiv (\widetilde{u}', u)$. Consequently, $\widetilde{w}\widetilde{w}'\widetilde{u}'[\widetilde{w}, u]$ is a square of $\widetilde{G}_{i+1}$ which is mapped by f_{i+1} to the square $ww'u'u$.

Case 2. $\widetilde{w}' \in \widetilde{S}_i$.

If $\widetilde{u} \in \widetilde{S}_{i-1}$ or $\widetilde{u}' \in \widetilde{S}_{i-1}$, then, exchanging the roles of $\widetilde{w}, \widetilde{w}', \widetilde{u}$ and $\widetilde{u}'$, we are in the previous case.

If $\widetilde{u} \in \widetilde{S}_i$, by $TC(\widetilde{v})$ there exists $\widetilde{x} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{u}$. Thus, in G there exists $x \sim u, w$ and, since G does not contain W_4 or W_4^-, $x \nsim u', w'$. Applying the house condition, we get y in G such that $y \sim u', w', x$ and $y \nsim u, w$. Applying the previous case to $\widetilde{w}, \widetilde{x}$ and the square $wxyw'$ of G, we know that there exists $\widetilde{y} \in \widetilde{B}_i$ such that $\widetilde{w}\widetilde{x}\widetilde{y}\widetilde{w}'$ is an induced square in $\widetilde{G}_{i+1}$. From Lemma 5.14 applied to $\widetilde{w}'$, we deduce that $\widetilde{y} \sim \widetilde{u}'$. Applying (S_i) to $\widetilde{x}, \widetilde{y}$ and to the square $xyu'u$, we get that $\widetilde{u} \sim \widetilde{u}'$, thus $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$. If $\widetilde{u}' \in \widetilde{S}_i$, then exchanging the roles of $\widetilde{w}, \widetilde{w}', \widetilde{u}, \widetilde{u}'$ we also get that $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$.

Suppose now that $\widetilde{w}$ has no neighbor in $\widetilde{B}_i$ mapped to u and that $\widetilde{w}'$ has no neighbor in $\widetilde{B}_i$ mapped to u'. Thus, there exist $[\widetilde{w}, u]$ and $[\widetilde{w}', u']$ in $\widetilde{S}_{i+1}$. By $TC(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{w}'$. In G, $x \sim w, w'$ and $x \nsim u, u'$ since G does not contain W_4^- or W_4. Applying the house condition, there is a vertex y in G such that $y \sim u, u'$ and $y \nsim w, w'$. By (R_i) applied to $\widetilde{x}$, there exists $\widetilde{y}$ in $\widetilde{B}_i$ such that $\widetilde{y} \sim x$ and $\widetilde{y} \nsim w, w'$. If $\widetilde{y}$ has a neighbor in $\widetilde{B}_i$ mapped to u, then applying the previous case to $\widetilde{w}, \widetilde{x}$ and the square $wxyu$, we conclude that $\widetilde{w}$ has a neighbor in $\widetilde{B}_i$ mapped to u, which is impossible. Consequently, $(\widetilde{y}, u) \in Z$, and since there is $\widetilde{x} \in S_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $wxyu$ is an induced square in G, $(\widetilde{y}, u) \equiv (\widetilde{w}, u)$. Using the same arguments, one can show that there exists $(\widetilde{y}, u') \in [\widetilde{w}', u']$. Since yuu' is a triangle in G, and since $[\widetilde{w}, u] = [\widetilde{y}, u]$ and $[\widetilde{w}', u'] = [\widetilde{y}, u']$, there is an edge in $\widetilde{G}_{i+1}$ between $[\widetilde{w}, u]$ and $[\widetilde{w}', u']$. Consequently, $\widetilde{w}\widetilde{w}'[\widetilde{w}', u'][\widetilde{w}, u]$ is a square of $\widetilde{G}_{i+1}$ satisfying the lemma. $\qquad\square$

Let $\widetilde{\mathbf{X}}_v$ denote the triangle-square complex obtained as the directed union $\bigcup_{i \geq 0} \widetilde{\mathbf{X}}_i$ with the vertex v of $\mathbf{X}$ as the basepoint. Denote by $\widetilde{G}_v$ the 1-skeleton of $\widetilde{\mathbf{X}}_v$. Since each $\widetilde{G}_i$ is weakly

modular with respect to $\widetilde{v}$, the graph $\widetilde{G}_v$ is also weakly modular with respect to $\widetilde{v}$. Let also $f = \bigcup_{i \geq 0} f_i$ be the map from $\widetilde{\mathbf{X}}_v$ to $\mathbf{X}$.

Lemma 5.16. *For any $\widetilde{w} \in \widetilde{\mathbf{X}}_v$, $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ is isomorphic to $St(w, \mathbf{X})$. Consequently, $f : \widetilde{\mathbf{X}}_v \to \mathbf{X}$ is a covering map.*

Proof. In order to prove that f is a covering map from $\widetilde{\mathbf{X}}_v$ to $\mathbf{X}$, it is enough to prove that for any $\widetilde{w} \in \widetilde{\mathbf{X}}$, $f|_{St(\widetilde{w}, \widetilde{\mathbf{X}}_v)}$ is an isomorphism between the stars $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ and $St(w, \mathbf{X})$, where $w = f(\widetilde{w})$. Note that, since $\widetilde{\mathbf{X}}_v$ is a flag complex, a vertex $\widetilde{x}$ of $\widetilde{\mathbf{X}}_v$ belongs to $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ if and only if either $\widetilde{x} \in B_1(\widetilde{w}, \widetilde{G}_v)$ or $\widetilde{x}$ has two non-adjacent neighbors in $B_1(\widetilde{w}, \widetilde{G}_v)$.

Let $\widetilde{w} \in \widetilde{B}_i$, i.e., i is the distance between $\widetilde{v}$ and $\widetilde{w}$ in $\widetilde{G}_v$, and consider the set $\widetilde{B}_{i+2}$. Then the vertex-set of $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ is included in $\widetilde{B}_{i+2}$. From (R_{i+2}) we know that f is an isomorphism between the graphs induced by $B_1(\widetilde{w}, \widetilde{G}_v)$ and $B_1(w, G)$.

For any vertex x in $St(w, \mathbf{X}) \setminus B_1(w, G)$ there exists an induced square $wuxu'$ in G. From (R_{i+2}), there exist $\widetilde{u}, \widetilde{u}' \sim \widetilde{w}$ in $\widetilde{G}_v$ such that $\widetilde{u} \nsim \widetilde{u}'$. From (S_{i+2}) applied to $\widetilde{w}, \widetilde{u}$ and since $\widetilde{w}$ has a unique neighbor $\widetilde{u}'$ mapped to u', there exists a vertex $\widetilde{x}$ in $\widetilde{G}_v$ such that $f(\widetilde{x}) = x$, $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$ and $\widetilde{x} \nsim \widetilde{w}$. Consequently, f is a surjection from $V(St(\widetilde{w}, \widetilde{\mathbf{X}}_v))$ to $V(St(w, \mathbf{X}))$.

Suppose by way of contradiction that there exist two distinct vertices $\widetilde{u}, \widetilde{u}'$ of $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ such that $f(\widetilde{u}) = f(\widetilde{u}') = u$. If $\widetilde{u}, \widetilde{u}' \sim \widetilde{w}$, by condition (R_{i+1}) applied to $\widetilde{w}$ we get a contradiction. Suppose now that $\widetilde{u} \sim \widetilde{w}$ and $\widetilde{u}' \nsim \widetilde{w}$ and let $\widetilde{z} \sim \widetilde{w}, \widetilde{u}'$. This implies that w, u, z are pairwise adjacent in $\widetilde{G}_v$. Since f is an isomorphism between the graphs induced by $B_1(\widetilde{w}, \widetilde{G}_v)$ and $B_1(w, G)$, we conclude that $\widetilde{z} \sim \widetilde{u}$. But then f is not locally injective around $\widetilde{z}$, contradicting the condition (R_{i+2}). Suppose now that $\widetilde{u}, \widetilde{u}' \nsim \widetilde{w}$. Let $\widetilde{a}, \widetilde{b} \sim \widetilde{u}, \widetilde{w}$ and $\widetilde{a}', \widetilde{b}' \sim \widetilde{u}', \widetilde{w}'$. If $\widetilde{a}' = \widetilde{a}$ or $\widetilde{a}' = \widetilde{b}$, then applying (R_{i+2}) to $\widetilde{a}'$, we get that $f(\widetilde{u}) \neq f(\widetilde{u}')$. Suppose now that $\widetilde{a}' \notin \{\widetilde{a}, \widetilde{b}\}$. Then the subgraph of G induced by a', w, a, b, u is either $K_{2,3}$ if $a' \nsim a, b$, or W_4 if $a' \sim a, b$, or W_4^- otherwise. In all cases, we get a contradiction.

Hence f is a bijection between the vertex-sets of $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ and $St(w, \mathbf{X})$. By (R_{i+2}), $\widetilde{a} \sim \widetilde{b}$ in $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ if and only if $a \sim b$ in $St(w, \mathbf{X})$. By (R_{i+2}) applied to w and since $\mathbf{X}$ and $\widetilde{\mathbf{X}}_v$ are flag complexes, $\widetilde{a}\widetilde{b}\widetilde{w}$ is a triangle in $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ if and only if abw is a triangle in $St(w, \mathbf{X})$. By (R_{i+2}) and since $\mathbf{X}$ is a flag complex, if $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{w}$ is a square in $St(\widetilde{w}, \widetilde{\mathbf{X}})$, then $abcw$ is a square in $St(w, \mathbf{X})$. Conversely, by the conditions (R_{i+2}) and (S_{i+2}) and flagness of $\widetilde{\mathbf{X}}_v$, we conclude that if $abcw$ is a square in $St(w, \mathbf{X})$, then $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{w}$ is a square in $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$. Consequently, for any $\widetilde{w} \in \widetilde{\mathbf{X}}_v$, f defines an isomorphism between $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ and $St(w, \mathbf{X})$, and thus f is a covering map. $\square$

Lemma 5.17. $\widetilde{\mathbf{X}}_v$ *satisfies the house, the cube, and the $\widehat{W}_5$-wheel conditions, and the graph $\widetilde{G}_v$ does not contain induced $K_{2,3}, W_4^-$, and W_4. Moreover, if G is W_5-free, then $\widetilde{G}_v$ is also W_5-free.*

Proof. Note that for any subgraph $C \in \{\text{house, cube}, K_{2,3}, W_4^-, W_4, W_5\}$ of $\widetilde{G}_v$ there exists a vertex $\widetilde{w}$ such that C is included in the star $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ of this vertex. From Lemma 5.16, $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ is isomorphic to $St(w, \mathbf{X})$, thus $f(C)$ is isomorphic to C. Since G does not contain induced $K_{2,3}$, W_4^-, W_4, the graph $\widetilde{G}_v$ also does not contains these graphs as induced subgraphs, and if G is W_5-free, $\widetilde{G}_v$ is also W_5-free.

Consider a house $\widetilde{u}\widetilde{u'}\widetilde{w'}\widetilde{w}\widetilde{x}$ in $\widetilde{\mathbf{X}}_v$ where $\widetilde{u}\widetilde{u'}\widetilde{w'}\widetilde{w}$ is a square and $\widetilde{u}\widetilde{w}\widetilde{x}$ is a triangle. This house is mapped by f to the house $uu'w'wx$ in $\mathbf{X}$. By the house condition in $\mathbf{X}$, there exists a vertex $y \in G$ such that $y \sim u', w', x$ and $y \nsim u, w$. Since f is locally bijective, there exists $\widetilde{y} \sim \widetilde{x}$ such that $f(\widetilde{y}) = y$. Since f is an homomorphism from $\mathrm{St}(\widetilde{x}, \widetilde{\mathbf{X}}_v)$ to $\mathrm{St}(x, \mathbf{X})$, considering the squares $xyu'u$ and $xyw'w$, we get that $\widetilde{y} \sim \widetilde{u'}, \widetilde{w'}$ and $\widetilde{y} \nsim \widetilde{u}, \widetilde{w}$. Thus, $\widetilde{\mathbf{X}}_v$ satisfies the house condition.

Consider three squares $\widetilde{x}\widetilde{a}\widetilde{b''}\widetilde{a'}$, $\widetilde{x}\widetilde{a'}\widetilde{b}\widetilde{a''}$, $\widetilde{x}\widetilde{a''}\widetilde{b'}\widetilde{a}$ in $\widetilde{\mathbf{X}}_v$. By cube condition, in G there exists a vertex y such that $y \sim b, b', b''$ and $y \nsim x, a, a', a''$. Since f is locally bijective, there exists $\widetilde{y} \sim \widetilde{b}$ such that $f(\widetilde{y}) = y$. Since f is an isomorphism from $\mathrm{St}(\widetilde{b}, \widetilde{\mathbf{X}}_v)$ to $\mathrm{St}(b, \mathbf{X})$, we get that $\widetilde{y} \sim \widetilde{b'}, \widetilde{b''}$ and $\widetilde{y} \nsim \widetilde{a'}, \widetilde{a''}, \widetilde{x}$. When considering $\mathrm{St}(\widetilde{b'}, \widetilde{\mathbf{X}}_v)$, we get that $\widetilde{y} \nsim \widetilde{a}$. Thus, $\widetilde{\mathbf{X}}$ also satisfies the cube condition.

Finally suppose that $\mathbf{X}$ satisfies the $\widehat{W}_5$-wheel condition and consider W_5 in $\widetilde{G}_v$ made of a 5-cycle $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{x}_1)$ and a vertex $\widetilde{c}$ adjacent to all vertices of this cycle. Suppose that there exists a vertex $\widetilde{z}$ such that $\widetilde{z} \sim \widetilde{x}_1, \widetilde{x}_2$ and $\widetilde{z} \nsim \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{c}$. The vertices $c, x_1, x_2, x_3, x_4, x_5$ are all distinct and they induce W_5 in G. Since f is locally bijective and since $x_1 \nsim x_4$, necessarily $z \notin \{c, x_1, x_2, x_3, x_4, x_5\}$. Since $\mathrm{St}(\widetilde{x}_1, \widetilde{\mathbf{X}}_v)$ is isomorphic to $\mathrm{St}(x_1, \mathbf{X})$, $z \nsim x_5, c$. Considering $\mathrm{St}(\widetilde{x}_2, \widetilde{\mathbf{X}}_v)$, we get $z \nsim x_3$. If $z \sim x_4$, $x_1 z x_4 x_5$ is a square in $\mathrm{St}(x_1, \mathbf{X})$, and this implies that $\widetilde{z} \sim \widetilde{x}_4$, a contradiction. By the $\widehat{W}_5$-wheel condition for $\mathbf{X}$, there exists $y \sim c, z, x_1, x_2, x_3, x_4, x_5$ in G. Consider the neighbor $\widetilde{y}$ of $\widetilde{c}$ such that $f(\widetilde{y}) = y$. Since $\mathrm{St}(\widetilde{c}, \widetilde{\mathbf{X}}_v)$ is isomorphic to $\mathrm{St}(c, \mathbf{X})$, $\widetilde{y} \sim \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5$. Considering the star $\mathrm{St}(\widetilde{x}_1, \widetilde{\mathbf{X}}_v)$, we conclude that $\widetilde{y} \sim \widetilde{z}$. Consequently, $\widetilde{\mathbf{X}}_v$ satisfies the $\widehat{W}_5$-wheel condition. □

The fact that the complex $\widetilde{\mathbf{X}}_v$ is simply connected is a direct consequence of the following more general result.

Lemma 5.18. *Let $\mathbf{Y}$ be a triangle-square flag complex such that $G(\mathbf{Y})$ satisfies the triangle and the quadrangle conditions $TC(v)$ and $QC(v)$, for some basepoint v. Then $\mathbf{Y}$ is simply connected. In particular, $\widetilde{\mathbf{X}}_v$ is simply-connected for any basepoint $v \in V(\mathbf{X})$.*

Proof. A *loop* in $\mathbf{Y}$ is a sequence $(w_1, w_2, ..., w_k, w_1)$ of vertices of $\mathbf{Y}$ consecutively joined by edges. To prove the lemma it is enough to show that every loop $\mathbf{Y}$ can be freely homotoped to a constant loop v. By contradiction, let A be the set of loops in $G(\mathbf{Y})$, which are not freely homotopic to v, and assume that A is non-empty. For a loop $C \in A$ let $r(C)$ denote the maximal distance $d(w, v)$ of a vertex w of C from the basepoint v. Clearly $r(C) \geq 2$ for any loop $C \in A$ (otherwise C would be null-homotopic). Let $B \subseteq A$ be the set of loops C with minimal $r(C)$ among loops in A. Let $r := r(C)$ for some $C \in B$. Let $D \subseteq B$ be the set of loops having minimal number e of edges in the r-sphere around v, i.e. with both endpoints at distance r from v. Further, let $E \subseteq D$ be the set of loops with the minimal number m of vertices at distance r from v.

Consider a loop $C = (w_1, w_2, ..., w_k, w_1) \in E$. We can assume without loss of generality that $d(w_2, v) = r$. We distinguish two cases corresponding to the triangle or quadrangle condition that we apply to them.

Case 1: $d(w_1, v) = r$ or $d(w_3, v) = r$. Assume without loss of generality that $d(w_1, v) = r$. Then, by the triangle condition $TC(v)$, there exists a vertex $w \sim w_1, w_2$ with $d(w, v) = r - 1$. Observe that the loop $C' = (w_1, w, w_2, ..., w_k, w_1)$ belongs to B – in $\mathbf{Y}$ it is freely homotopic to C by a homotopy going through the triangle ww_1w_2. The number of edges of C' lying on the r-sphere around v is less than e (we removed the edge w_1w_2). This contradicts the choice of the number e.

Case 2: $d(w_1, v) = d(w_3, v) = r - 1$. By the quadrangle condition $QC(v)$, there exists a vertex $w \sim w_1, w_3$ with $d(w, v) = r - 2$. Again, the loop $C' = (w_1, w, w_3, ..., w_k, w_1)$ is freely homotopic to C (via the square $w_1w_2w_3w$). Thus C' belongs to D and the number of its vertices at distance r from v is equal to $m - 1$. This contradicts the choice of the number m.

In both cases above we get contradiction. It follows that the set A is empty and hence the lemma is proved. $\qquad\square$

5.3. **Proof of Theorem 1.** Since the hypercube condition implies the cube condition and the hyperhouse condition implies the house condition, if $\mathbf{X}$ is a bucolic, then its 2-skeleton $\mathbf{X}^{(2)}$ satisfies (ii), thus (i)$\Rightarrow$(ii).

Using the results of previous subsection, we will show now that (ii)$\Leftrightarrow$(iii). Let $\mathbf{X}$ be a connected triangle-square flag complex satisfying the local conditions of (ii). By Lemma 5.16, $f : \widetilde{\mathbf{X}}_v \to \mathbf{X}$ is a covering map. By Lemma 5.18, $\widetilde{\mathbf{X}}_v$ is simply connected, thus $\widetilde{\mathbf{X}}_v$ is the universal cover $\widetilde{\mathbf{X}}$ of $\mathbf{X}$. Therefore the triangle-square complexes $\widetilde{\mathbf{X}}_v, v \in V(\mathbf{X})$, are all universal covers of $\mathbf{X}$, whence they are all isomorphic. Since for each vertex v of $\mathbf{X}$, the graph $\widetilde{G}_v = G(\widetilde{\mathbf{X}}_v)$ is weakly modular with respect to the basepoint v, we conclude that the 1-skeleton $G(\widetilde{\mathbf{X}})$ of $\widetilde{\mathbf{X}}$ is weakly modular with respect to each vertex, thus $G(\widetilde{\mathbf{X}})$ is a weakly modular graph. Since $\widetilde{\mathbf{X}}$ is isomorphic to any $\widetilde{\mathbf{X}}_v$, by Lemma 5.17, $\widetilde{\mathbf{X}}$ satisfies the same local conditions as $\mathbf{X}$. Thus $\widetilde{\mathbf{X}}$ satisfies the $(W_4, \widehat{W}_5)$, the house, and the cube conditions. If, additionally, $\mathbf{X}$ is simply connected, then the universal cover $\widetilde{\mathbf{X}}$ is $\mathbf{X}$ itself. Therefore, $\mathbf{X}$ coincides with $\widetilde{\mathbf{X}}_v$ for any choice of the basepoint $v \in V(\mathbf{X})$. Therefore, by what has been proven above, $G(\mathbf{X})$ is a weakly modular graph. This establishes the implication (ii)$\Rightarrow$(iii) of Theorem 1.

Now we will establish the implication (iii)$\Rightarrow$(ii). Let $\mathbf{X}$ be a prism flag complex such that $G := G(\mathbf{X})$ is a weakly modular graph not containing induced W_4. Then G does not contain induced $K_{2,3}$ and W_4^- because G is the 1-skeleton of a triangle-square cell complex $\mathbf{X}^{(2)}$. From Lemma 5.18 we conclude that $\mathbf{X}^{(2)}$ (and therefore $\mathbf{X}$) is simply connected. Thus, it remains to show that $\mathbf{X}$ satisfies the house, the cube condition, and the $\widehat{W}_5$-wheel conditions. First suppose that the triangle uvw and the square $uvxy$ define in $\mathbf{X}$ a house. Then w is at distance 2 to the adjacent vertices x and y. By triangle condition, there exists a vertex w' adjacent to w, x, and y and different from u and v. If w' is adjacent to one or both of the vertices u, v, then we will get a forbidden W_4^- or W_4 induced by u, v, x, y, w'. This establishes the house condition.

To prove the cube condition, let $xyuv$, $uvwz$, and $uytz$ be three squares of $\mathbf{X}$ pairwise intersecting in edges and all three intersecting in u. If x and w are adjacent, then the vertices v, x, w, u, y, z induce in $\mathbf{X}$ a double house, which is impossible by Lemma 5.3 because $\mathbf{X}$ satisfies the house condition. Hence $x \not\sim w$ and analogously $x \not\sim t$ and $t \not\sim w$. If x is adjacent to z, then x, y, u, t, z induce in G a forbidden $K_{2,3}$. Thus $x \not\sim z$ and analogously $y \not\sim w$ and $v \not\sim t$. First suppose that $d(x, z) = 2$ in G. Since $d(y, z) = 2$, by triangle condition there exists a vertex s adjacent to x, y, and z. From what has been shown before, $s \neq u, t$, hence y, u, z, t, s induce $K_{2,3}$, W_4^-, or W_4 depending of whether s is adjacent to none, one or two of the vertices u, t. Thus, $d(x, z) = 3$ and for the same reasons, $d(y, w) = d(v, t) = 3$. By quadrangle condition there exists a vertex s adjacent to x, w, t and distinct from previous vertices. Since $d(x, z) = d(w, y) = d(t, v) = 3$, $s \not\sim z, y, v$. If s is adjacent to u, then s, u, v, w, z induce a forbidden $K_{2,3}$. This shows that in this case the vertices s, t, u, v, w, x, y, z define a 3-cube, establishing the cube condition.

Finally, we establish the $\widehat{W}_5$-wheel condition. Notice that we can suppose that $\mathbf{X}$ satisfies the cube and the house conditions and by Lemma 5.4 that $\mathbf{X}$ does not contain a $\mathbf{X}(W_5^-)$. Pick a 5-wheel defined by a 5-cycle $(x_1, x_2, x_3, x_4, x_5, x_1)$ and a vertex c adjacent to all vertices of this cycle, and suppose that x_0 is a vertex adjacent to x_1 and x_5 and not adjacent to remaining vertices of this 5-wheel. If $d(x_0, x_3) = 3$, then by quadrangle condition $\mathrm{QC}(x_0)$, there exists a vertex y adjacent to x_0, x_2, x_4 and not adjacent to x_3. Then the vertices c, y, x_2, x_3, x_4 induce a W_4 if y is adjacent to c, and a W_4^- otherwise. So, suppose that $d(x_0, x_3) = 2$. By triangle condition $\mathrm{TC}(x_0)$, there exists a vertex z adjacent to x_0, x_2, x_3. Suppose that $z \not\sim c$. If $z \sim x_1$, then x_2, x_1, z, x_3, c induce a forbidden W_4. If $z \sim x_5$, the vertices x_1, x_2, c, x_5, z induce a forbidden W_4 if $z \sim x_1$ or a W_4^- otherwise. If $z \not\sim x_1, x_5$, the vertices z, x_2, c, x_5, x_0, x_1 induce a forbidden W_5^-. Thus, $z \sim c$. To avoid a forbidden W_4^- or W_4 induced by z, c, x_1, x_0, x_5, the vertex z must be adjacent to x_1 and x_5. Finally, to avoid W_4 induced by z, c, x_3, x_4, x_5, the vertex z must be adjacent to x_4 as well. As a result, we conclude that z is adjacent to x_0 and to all vertices of the 5-wheel, establishing the $\widehat{W}_5$-wheel condition. This concludes the proof of the implication (iii)$\Rightarrow$(ii).

Now, we will show that (ii)&(iii)$\Rightarrow$(i), i.e., that a flag prism complex $\mathbf{X}$ satisfying the conditions (ii) and (iii) also satisfies the hypercube and the hyperhouse conditions. First notice that any prism H (and in particular, any cube) of $\mathbf{X}$ induces a convex subgraph of $G(\mathbf{X})$. Indeed, if H is not convex, then by local convexity, we can find two vertices x, y of H at distance 2 having a common neighbor outside H. Since x and y already have two common (non-adjacent) neighbors in H, we will obtain in $G(\mathbf{X})$ a forbidden $K_{2,3}, W_4^-$, or W_4.

Hypercube condition. Let q_1, q_2, q_3 be three k-cubes of $\mathbf{X}$ that share a common $(k-2)$-cube q and pairwise share common $(k-1)$-cubes q_{ij}. Note that $q_{ij} \smallsetminus q$ spans a $(k-2)$-cube and $q_i \smallsetminus q_{ij}$ spans a $(k-1)$-cube. For a vertex x of q let x_{ij} be the unique neighbor of x in $q_{ij} \smallsetminus q$. Let x_i be the second common neighbor in q_i of the vertices x_{ij} and x_{ik}; x_i is in $q_i \smallsetminus (q_{ij} \cup q_{ik})$. By the cube condition, there exists a vertex x^* such that $x^* \sim x_1, x_2, x_3$ and $x^* \not\sim x_{12}, x_{13}, x_{23}$, and the vertices $x^*, x, x_{12}, x_{13}, x_{23}, x_1, x_2, x_3$ constitute a 3-cube q_x of $\mathbf{X}$. Since $x_2 \in I(x^*, x_{12})$,

since $x_2 \notin q_1$ and since the cubes are convex, $x^* \notin q_1$. For similar reasons, $x^* \notin q_2, q_3$. Now, for another vertex y of q denote by $y_{12}, y_{13}, y_{23}, y_1, y_2, y_3, y^*$ the vertices defined in the same way as for x. From the definition of these vertices we immediately conclude that all x_i, x_{ij}, y_i, y_{ij} are distinct and for all distinct $i, j \in \{1, 2, 3\}$, $x_{ij} \sim y_{ij}$ and $x_i \sim y_i$ hold if and only if $x \sim y$.

Using the convexity of cubes, we show in the next lemma that any of the vertices $x, x_{12}, x_{13}, x_{23}, x_1, x_2, x_3$ cannot be adjacent to any other vertex from the set $y, y_{12}, y_{13}, y_{23}, y_1, y_2, y_3$.

Lemma 5.19. *For any $x, y \in q$, for any distinct i, j, k, $x \nsim y_i, y_{ij}$, $x_{ik} \nsim y_i, y_{ij}, y_j$ and $x_i \nsim y_j$.*

Proof. If x (resp. x_{ik}) is adjacent to y_i or y_{ij}, then since $x \sim x_{ij}$ (resp. $x_{ik} \sim x_i$), either q_i contains a triangle, or $q_i \setminus q_{ik}$ is not convex. Since $x_{ik} \sim x$ and $x \nsim y_j$, the convexity of q_j ensures that $x_{ik} \nsim y_j$. Finally, the convexity of q_i ensures that $x_i \nsim y_j$, since $y_j \sim y_{ij}$ and $x_i \nsim y_{ij}$. $\square$

Lemma 5.20. *For any $x, y \in q$, for any distinct i, j, $x^* \nsim y, y_i, y_{ij}$.*

Proof. First suppose by way of contradiction that x^* is adjacent to y or y_{ij}. Since $x^* \notin q_i$, since $x^* \sim x_i$, and since $x_i \nsim y, y_{ij}$ by Lemma 5.19, we get a contradiction with the convexity of q_i. Suppose now by way of contradiction that $x^* \sim y_i$. If $x \nsim y$, then $x_i \nsim y_i$ and since both $x_i, y_i \in q_i$ are adjacent to $x^* \notin q_i$, we obtain a contradiction with the convexity of the cube q_i. Now, suppose that $x \sim y$. Then, $x_i \sim y_i, x_{ij} \sim y_{ij}$ and the vertices $x_j, x_{ij}, y_{ij}, y_i, x_i, x^*$ define a double-house; by Lemma 5.3, it implies that $x_j \sim y_{ij}$, contradicting Lemma 5.19. Thus, $x^* \nsim y_i$. $\square$

Lemma 5.21. *The set $\{x^* : x \in q\}$ spans a $(k-2)$-cube q' of $\mathbf{X}$ and the vertices of $q_1 \cup q_2 \cup q_3 \cup q'$ span a $(k+1)$-cube of $\mathbf{X}$.*

Proof. First note that since $y_1 \sim y^*$ and $y_1 \nsim x^*$ by Lemma 5.20, we have that $x^* \neq y^*$. To prove the first assertion of the lemma, since q is a $(k-2)$-cube of $\mathbf{X}$, it suffices to show that $x^* \sim y^*$ if and only if $x \sim y$.

First suppose that x is adjacent to y. Consider the three 2-cubes induced by the 4-cycles $(x_1, x^*, x_2, x_{12}, x_1)$, $(x_1, y_1, y_{12}, x_{12}, x_1)$, and $(x_2, y_2, y_{12}, x_{12}, x_2)$ of $G(\mathbf{X})$. By the cube condition, they are included in a 3-cube of $\mathbf{X}$, i.e., there exists a vertex s adjacent to x^*, y_1, and y_2. Since q_y is a cube, $(y_1, y_{12}, y_2, y^*, y_1)$ is an induced 4-cycle of $G(\mathbf{X})$. Since $G(\mathbf{X})$ does not contain induced $K_{2,3}$, W_4^- or W_4, we conclude that $s = y^*$ or $s = y_{12}$. Since $x^* \sim s$ and $x^* \nsim y_{12}$ from Lemma 5.20, $s = y^*$ and $x^* \sim y^*$. Conversely, suppose that $x^* \sim y^*$ and assume that $x \nsim y$. Then $x_i \nsim y_i$ and $x_{ij} \nsim y_{ij}$. Since $x_i, y_i \in q_i$ and since the cube q_i is convex, we conclude that $d(x_i, y_i) = 2$, (otherwise, (x_i, x^*, y^*, y_i) would be a shortest path from x_i to y_i). Since q_i is a cube, it implies that $d(x, y) = 2$. Let z be a common neighbor of x and y in the cube q and let q_z be the 3-cube defined by the vertices $z, z_{12}, z_{13}, z_{23}, z_1, z_2, z_3, z^*$. Since $z \sim x, y$, $z_1 \sim x_1, y_1$ and $z^* \sim x^*, y^*$. Consequently, the vertices $x_1, z_1, y_1, y^*, z^*, x^*$ define a double-house, and from Lemma 5.3, it implies that $x_1 \sim y_1$, a contradiction. Therefore, $x^* \sim y^*$ if and only if $x \sim y$, whence q' is a $(k-2)$-cube.

From Lemmas 5.19 and 5.20, and since q' is a $(k-2)$-cube, the vertices of $q_1 \cup q_2 \cup q_3 \cup q'$ span a $(k+1)$-cube of $\mathbf{X}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Hyperhouse condition: Let q be a k-cube intersecting a simplex σ in an edge (1-simplex) $e = uv$. Let G' be the subgraph of $G(\mathbf{X})$ induced by $\mathrm{conv}(q \cup \sigma)$. By Proposition 3.1, G' is a finite graph satisfying the condition (iii). Let $q = q_u \cup q_v$, where q_u and q_v are two disjoint $(k-1)$-cubes of q, one containing u and another containing v. If $\sigma = e$, then we are done. So, suppose σ contains at least three vertices. Let H be the gated hull of σ. Then H is a weakly bridged graph, thus H does not contain induced 4-cycles. On the other hand, the convex hull of any three vertices of a cube contains a 4-cycle. Since $q \cap H$ is convex, we conclude that $q \cap H = \{u, v\}$. Next, we will use some tools from the decomposition of fiber-complemented graphs into prime graphs [4, 14]. For a vertex $a \in V(H)$ let F_a be fiber of a with respect to H: recall that F_a is the set of all vertices of G' whose gate in H is the vertex a. Since G' is fiber-complemented, each fiber $F_a (a \in V(H))$ is gated. For a vertex $a \in V(H)$, let $U_a = \{x \in F_a : \exists y \notin F_a, xy \in E(G)\}$, and if $b \in V(H)$ is a neighbor of a, let $U_{ab} = \{x \in F_a : \exists y \in F_b, xy \in E(G)\}$. Let q_u, q_v, F_a, and U_a also denote the subgraphs induced by these sets. Since cubes induce convex subgraphs of $G(\mathbf{X})$, from the definition of the fibers F_u and F_v we infer that q_u is included in F_u and q_v is included in F_v. Moreover, since q is a cube and since $u \sim v$, any vertex of q_u has a neighbor in q_v, and we conclude that $q_u \subseteq U_{ab}$ and $q_v \subseteq U_{ba}$.

Since the graph G' is fiber-complemented, if a, b are adjacent vertices of H, then $U_{ab} = U_a$. Moreover, since any $x \in U_{ab}$ has exactly one neighbor in U_{ba}, this gives rise to the following natural mapping $f_{ab} : U_a \longrightarrow U_b$ that maps $x \in U_a$ to the neighbor of x in U_b. Furthermore, fiber-complementarity of G' implies that if a, b are adjacent vertices of H, then U_a and U_b are isomorphic subgraphs of G and f_{ab} is an isomorphism between U_a and U_b. Then the subgraphs U_a are gated for all $a \in V(H)$ and are mutually isomorphic; their union is isomorphic to the graph $H \square U$, where U is any of U_a. Since U_u contains the $(k-1)$-cube q_u and U_v contains the $(k-1)$-cube q_v, U contains a $(k-1)$-cube q_0. Hence $\sigma \cup q$ is included in $H \square U$ (and therefore in G') in the prism $\sigma \square q_0$. This establishes the hyperhouse condition and concludes the proof of the implication (ii)&(iii)$\Rightarrow$(i) of Theorem 1.

Finally, we establish the last assertion of Theorem 1. Let $\mathbf{X}$ be a flag prism complex satisfying the $(W_4, \widehat{W_5})$, the hypercube, and the hyperhouse conditions. Then its 2-skeleton $\mathbf{Y} := \mathbf{X}^{(2)}$ is a triangle-square flag complex satisfying $(W_4, \widehat{W_5})$, the cube, and the house conditions. Let $\widetilde{\mathbf{X}}$ be the universal cover of $\mathbf{X}$. Then the 2-skeleton $\widetilde{\mathbf{X}}^{(2)}$ of $\widetilde{\mathbf{X}}$ is a covering space of $\mathbf{Y}$. But at the same time $\widetilde{\mathbf{X}}^{(2)}$ is simply connected (because the 2-skeleton carries all the information about the fundamental group), so $\widetilde{\mathbf{X}}^{(2)}$ is the universal cover of $\mathbf{Y}$. Since $\widetilde{\mathbf{X}}$ is the prism complex of $\widetilde{\mathbf{X}}^{(2)}$ and $\widetilde{\mathbf{X}}^{(2)} = \widetilde{\mathbf{Y}}$ satisfies the condition (ii) of Theorem 1, we conclude that $\widetilde{\mathbf{X}}$ is a bucolic complex. This finishes the proof of Theorem 1.

6. Proofs of Theorems 3 and 4

6.1. **Proof of Theorem 3.** Let $\mathbf{X}$ be a bucolic complex and let $G = (V, E)$ be its 1-skeleton. Pick any vertex v_0 of G and let $B_k(v_0, G)$ be the ball of radius k centered at v_0. Since G is locally-finite, each ball $B_k(v_0, G)$ is finite. By Proposition 3.1 the convex hulls $\mathrm{conv}(B_k(v_0, G)), k \geq 1$, are finite. Hence V is an increasing union of the finite convex sets $\mathrm{conv}(B_k(v_0, G)), k \geq 1$. A subgraph G' of G induced by a convex set of G satisfies the condition (ii) of Theorem 2, thus G' satisfies all other conditions of this theorem, whence G' is bucolic. Hence each subgraph G_k induced by $\mathrm{conv}(B_k(v_0, G))$ is bucolic.

The prism complex $\mathbf{X}$ is an increasing union of the finite bucolic complexes $\mathbf{X}(G_k)$ of the graphs $G_k, k \geq 1$. Thus, to show that $\mathbf{X}$ is contractible, by Whitehead theorem, it suffices to show that each complex $\mathbf{X}(G_k)$ is contractible. By condition (iii) of Theorem 2, the graph G_k can be obtained via Cartesian products of finite weakly bridged graphs using successive gated amalgams. The clique complexes of bridged and weakly bridged graphs are exactly the systolic and weakly systolic complexes, therefore they are contractible by the results of [29] and [32]. Cartesian products of contractible topological spaces are contractible, thus the prism complexes resulting from the Cartesian products of prime graphs are contractible. Now, if a graph G' is a gated amalgam of two finite bucolic graphs G_1, G_2 with contractible prism complexes $\mathbf{X}(G_1), \mathbf{X}(G_2)$ along a gated subgraph $G_0 = G_1 \cap G_2$ which also has a contractible prism complex $\mathbf{X}(G_0)$, then by the gluing lemma [9, Lemma 10.3], the prism complex $\mathbf{X}(G')$ of the bucolic graph G' is also contractible. Therefore, for each k, the prism complex $\mathbf{X}(G_k)$ is contractible. This concludes the proof of Theorem 3.

6.2. **Proof of Theorem 4.** Let $\mathbf{X}$ be a bucolic complex and let $G = (V, E)$ denote the 1-skeleton of $\mathbf{X}$. Let F be a finite group acting by cell automorphisms on $\mathbf{X}$ (i.e., any $f \in F$ maps isometrically prisms onto prisms). Then for an arbitrary vertex v of $\mathbf{X}$, its orbit $Fv = \{fv : f \in F\}$ is finite. Let G_v be the subgraph of G induced by the convex hull in G of the orbit Fv. Since Fv is finite, the graph G_v is finite by Proposition 3.1. Moreover, as a convex subgraph of G, G_v satisfies the conditions of Theorem 2(ii), hence G_v is bucolic. Clearly, the prism complex $\mathbf{X}(G_v)$ of G_v is F-invariant. Thus there exists a minimal by inclusion finite non-empty bucolic subgraph of G whose prism complex is F–invariant. Without loss of generality, we denote this subgraph of G also by G and we assert that $\mathbf{X}(G)$ is a single prism, i.e., G is the Cartesian product of complete graphs. We prove this assertion in two steps: first we show that G is a *box*, (i.e., a Cartesian product of prime graphs), and then we show that each prime graph must be a complete graph. By minimality choice of G as an F-invariant bucolic subgraph, we conclude that each proper bucolic subgraph of G is not F-invariant. Therefore, the first step of our proof is a direct consequence of the following result.

Proposition 6.1. *If G is a finite bucolic graph, then there exists a box that is invariant under every automorphism of G.*

Proof. If G is a box, then the assertion is trivially true. Suppose now that G is not a box and assume without loss of generality that each proper bucolic subgraph of G is not F-invariant. By Theorem 2(iv), G is a gated amalgam of two proper nonempty gated subgraphs G' and G'' along a common gated subgraph H_0. Then we say that H_0 is a *gated separator* of G. Following [10], we will call $U' := G' \smallsetminus H_0$ a *peripheral subgraph* of G if U' does not contain any gated separator of G.

Since G is not a box, it contains at least one gated separator, and therefore G contains at least one peripheral subgraph (indeed, among all gated separators of G it suffices to consider a gated separator H_0 so that G is the gated amalgam of G' and G'' along H_0 and G' has minimum size; then $G' \smallsetminus H_0$ is a peripheral subgraph). Let $\mathcal{U} = \{U_i : i \in I\}$ be the family of all peripheral subgraphs of G, such that G is the gated amalgam of G'_i and G''_i along the gated separator H_i, where $U_i = G'_i - H_i$ and $G''_i \ne H_i$. Note that any automorphism $f \in F$ of G maps peripheral subgraphs to peripheral subgraphs, thus the subgraph $\bigcup_{i \in I} U_i$ and the subgraph $H = \bigcap_{i \in I} G''_i$ induced by the complement of this union are both F-invariant subgraphs of G. As an intersection of gated subgraphs of G, H is either empty or a proper gated subgraph of G. In the second case, since gated subgraphs of G are bucolic, we conclude that H is a proper bucolic F-invariant subgraph of G, contrary to minimality of G. So, H is empty. By the Helly property for gated sets of a metric space [21], we can find two indices $i, j \in I$ such that the gated subgraphs G''_i and G''_j are disjoint. Since $H_i \cap H_j \subseteq G''_i \cap G''_j$, the gated separators H_i and H_j are disjoint. But in this case, since $U_i = G'_i \smallsetminus H_i$ is peripheral, we conclude that H_j is contained in G''_i (analogously, H_i is contained in G''_j). Thus $H_i \cup H_j \subseteq G''_i \cap G''_j$, contrary to the choice of G''_i and G''_j. Hence G is a box. $\qquad\square$

So, suppose that G is a box. Then the second assertion in the proof of Theorem 4 is an immediate consequence of the following result.

Proposition 6.2. *The graph G is the Cartesian product of complete graphs, i.e., $\mathbf{X}(G)$ is a prism.*

Proof. Let $G = G_1 \square \cdots \square G_k$, where each factor $G_i, i = 1, \ldots, k$, is a 2-connected finite weakly bridged graph. By [20, Theorem B] every factor G_i is dismantlable. Since dismantlable graphs form a variety –cf. e.g. [31, Theorem 1], it follows that the strong Cartesian product $G' = G_1 \boxtimes \cdots \boxtimes G_k$ is dismantlable. Observe that the finite group F acts by automorphisms on G'. By the definition of the strong Cartesian product, any clique of G' is included in a prism of $\mathbf{X}(G)$. By [33, Theorem A], there exists a clique σ in G' invariant under the action of F. Since F acts by cellular automorphisms on $\mathbf{X}(G)$, it follows that F fixes the minimal prism containing all vertices of σ (treated as vertices of G, and hence of $\mathbf{X}(G)$). By the minimality choice of G it follows that $\mathbf{X}(G)$ is itself a prism. $\qquad\square$

References

[1] Richard P. Anstee and Martin Farber, *On bridged graphs and cop-win graphs*, J. Combin. Theory Ser. B **44** (1988), no. 1, 22–28. MR923263 (89h:05053)

[2] Hans-Jürgen Bandelt, *Retracts of hypercubes*, J. Graph Theory **8** (1984), no. 4, 501–510, DOI 10.1002/jgt.3190080407. MR766499 (86c:05104)

[3] Hans-Jürgen Bandelt and Victor Chepoi, *A Helly theorem in weakly modular space*, Discrete Math. **160** (1996), no. 1-3, 25–39. MR1417558 (97h:52006)

[4] _____, *Decomposition and l_1-embedding of weakly median graphs*, European J. Combin. **21** (2000), no. 6, 701–714, DOI 10.1006/eujc.1999.0377. Discrete metric spaces (Marseille, 1998). MR1791200 (2002i:05091)

[5] _____, *The algebra of metric betweenness. I. Subdirect representation and retraction*, European J. Combin. **28** (2007), no. 6, 1640–1661, DOI 10.1016/j.ejc.2006.07.003. MR2339492 (2008h:05038)

[6] _____, *Metric graph theory and geometry: a survey*, Surveys on discrete and computational geometry, Contemp. Math., vol. 453, Amer. Math. Soc., Providence, RI, 2008, pp. 49–86. MR2405677 (2009h:05068)

[7] Hans-Jürgen Bandelt and Jarmila Hedlíková, *Median algebras*, Discrete Math. **45** (1983), no. 1, 1–30, DOI 10.1016/0012-365X(83)90173-5. MR700848 (84h:06015)

[8] Hans-Jürgen Bandelt, Henry Martyn Mulder, and Elke Wilkeit, *Quasi-median graphs and algebras*, J. Graph Theory **18** (1994), no. 7, 681–703, DOI 10.1002/jgt.3190180705. MR1297190 (95h:05059)

[9] Anders Björner, *Topological methods*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872. MR1373690 (96m:52012)

[10] Boštjan Brešar, *Arboreal structure and regular graphs of median-like classes*, Discuss. Math. Graph Theory **23** (2003), no. 2, 215–225. MR2070153 (2005f:05143)

[11] Boštjan Brešar, Jeremie Chalopin, Victor Chepoi, Matjaž Kovše, Arnaud Labourel, and Yann Vaxès, *Retracts of products of chordal graphs* (2011), submitted, available at http://pageperso.lif.univ-mrs.fr/~victor.chepoi/RetractsPCG.pdf.

[12] Boštjan Brešar and Aleksandra Tepeh Horvat, *Cage-amalgamation graphs, a common generalization of chordal and median graphs*, European J. Combin. **30** (2009), no. 5, 1071–1081, DOI 10.1016/j.ejc.2008.09.003. MR2513910 (2010b:05128)

[13] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR1744486 (2000k:53038)

[14] Marc Chastand, *Fiber-complemented graphs. I. Structure and invariant subgraphs*, Discrete Math. **226** (2001), no. 1-3, 107–141, DOI 10.1016/S0012-365X(00)00183-7. MR1801065 (2002i:05095)

[15] _____, *Fiber-complemented graphs. II. Retractions and endomorphisms*, Discrete Math. **268** (2003), no. 1-3, 81–101, DOI 10.1016/S0012-365X(02)00682-9. MR1982390 (2004d:05169)

[16] Marc Chastand, François Laviolette, and Norbert Polat, *On constructible graphs, infinite bridged graphs and weakly cop-win graphs*, Discrete Math. **224** (2000), no. 1-3, 61–78, DOI 10.1016/S0012-365X(00)00127-8. MR1781285 (2002g:05152)

[17] Marc Chastand and Norbert Polat, *On geodesic structures of weakly median graphs. I. Decomposition and octahedral graphs*, Discrete Math. **306** (2006), no. 13, 1272–1284, DOI 10.1016/j.disc.2005.10.034. MR2237713 (2007c:05064)

[18] Victor Chepoi, *Bridged graphs are cop-win graphs: an algorithmic proof*, J. Combin. Theory Ser. B **69** (1997), no. 1, 97–100. MR1426753 (97g:05150)

[19] _____, *Graphs of some* CAT(0) *complexes*, Adv. in Appl. Math. **24** (2000), no. 2, 125–179. MR1748966 (2001a:57004)

[20] Victor Chepoi and Damian Osajda, *Dismantlability of weakly systolic complexes and applications* (2009), submitted, available at arXiv:0910.5444v1[math.GR].

[21] Andreas W. M. Dress and Rudolf Scharlau, *Gated sets in metric spaces*, Aequationes Math. **34** (1987), no. 1, 112–120, DOI 10.1007/BF01840131. MR915878 (89c:54057)

[22] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR1161694 (93i:20036)

[23] Martin Farber and Robert E. Jamison, *On local convexity in graphs*, Discrete Math. **66** (1987), no. 3, 231–247. MR900046 (89e:05167)

[24] Mikhail Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR919829 (89e:20070)

[25] Frédéric Haglund, *Complexes simpliciaux hyperboliques de grande dimension*, Prepublication Orsay **71** (2003), preprint, available at `http://www.math.u-psud.fr/~biblio/ppo/2003/fic/ppo_2003_71.pdf`.

[26] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354 (2002k:55001)

[27] Wilfried Imrich and Sandi Klavžar, *Product graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. Structure and recognition; With a foreword by Peter Winkler. MR1788124 (2001k:05001)

[28] John R. Isbell, *Median algebra*, Trans. Amer. Math. Soc. **260** (1980), no. 2, 319–362, DOI 10.2307/1998007. MR574784 (81i:06006)

[29] Tadeusz Januszkiewicz and Jacek Świątkowski, *Simplicial nonpositive curvature*, Publ. Math. Inst. Hautes Études Sci. **104** (2006), 1–85, DOI 10.1007/s10240-006-0038-5. MR2264834 (2007j:53044)

[30] Henry Martyn Mulder, *The interval function of a graph*, Mathematical Centre Tracts, vol. 132, Mathematisch Centrum, Amsterdam, 1980. MR605838 (82h:05045)

[31] Richard Nowakowski and Peter Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Math. **43** (1983), 235–239.

[32] Damian Osajda, *A combinatorial non-positive curvature I: SD_n property* (2010), preprint, available at `http://www.math.uni.wroc.pl/~dosaj/trav/cnpc100503.pdf`.

[33] Norbert Polat, *Finite invariant simplices in infinite graphs*, Period. Math. Hungar. **27** (1993), no. 2, 125–136, DOI 10.1007/BF01876637. MR1291160 (96a:05073)

[34] _____, *On isometric subgraphs of infinite bridged graphs and geodesic convexity*, Discrete Math. **244** (2002), no. 1-3, 399–416. Algebraic and topological methods in graph theory (Lake Bled, 1999). MR1844048 (2003c:05070)

[35] Martin A. Roller, *Poc sets, median algebras and group actions. An extended study of Dunwoody's construction and Sageev's theorem*, Univ. of Southampton Preprint Ser. (1998), preprint.

[36] Misha Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. (3) **71** (1995), no. 3, 585–617. MR97a:20062

[37] Valeriu P. Soltan and Victor Chepoi, *Conditions for invariance of set diameters under d-convexification in a graph*, Kibernetika (Kiev) **6** (1983), 14–18 (Russian, with English summary); English transl., Cybernetics **19** (1983), no. 6, 750–756 (1984). MR765117 (86k:05102)

[38] Marcel van de Vel, *Matching binary convexities*, Topology Appl. **16** (1983), no. 3, 207–235, DOI 10.1016/0166-8641(83)90019-6. MR722115 (85f:52026)

[39] _____, *Theory of convex structures*, North-Holland Mathematical Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993. MR1234493 (95a:52002)

[40] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028 (96a:52011)