# DOUBLE PSEUDOLINE ARRANGEMENTS

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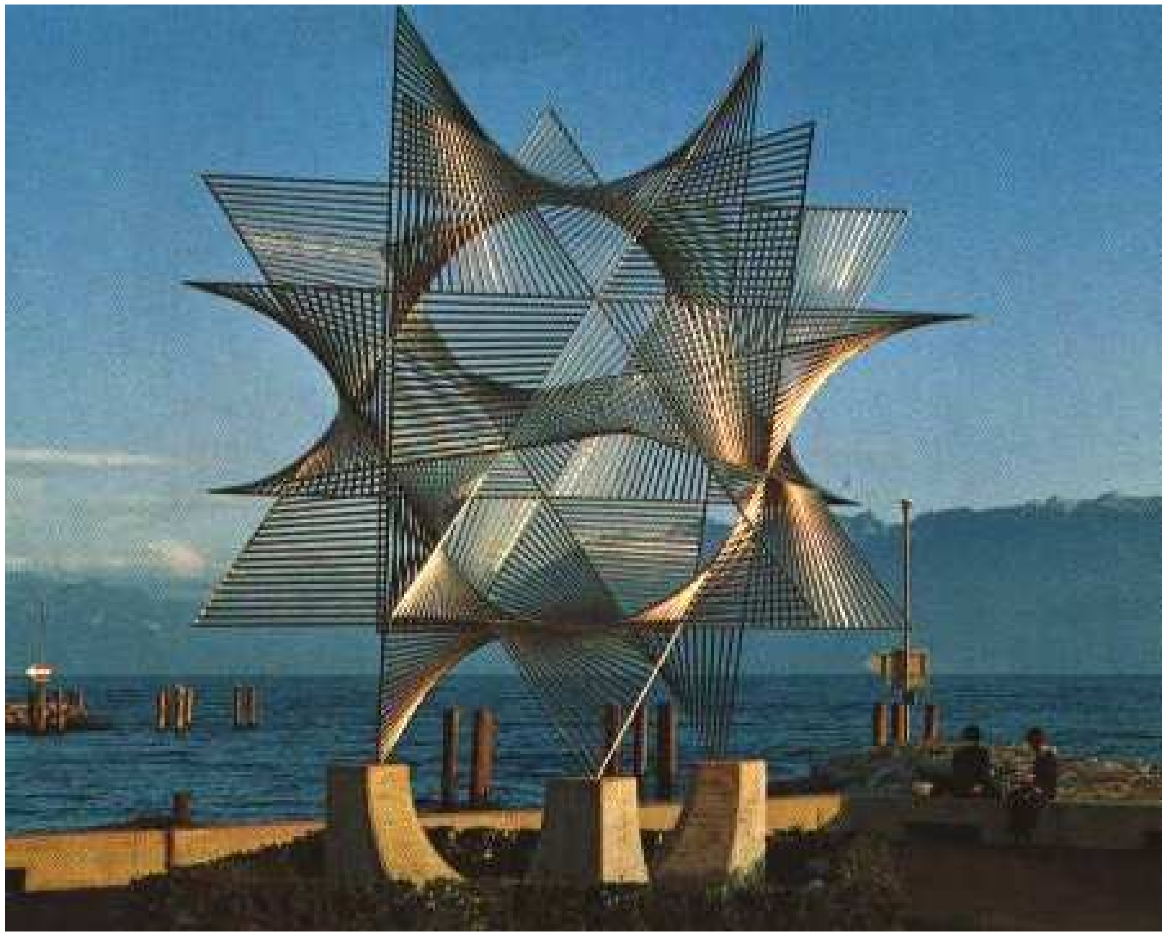
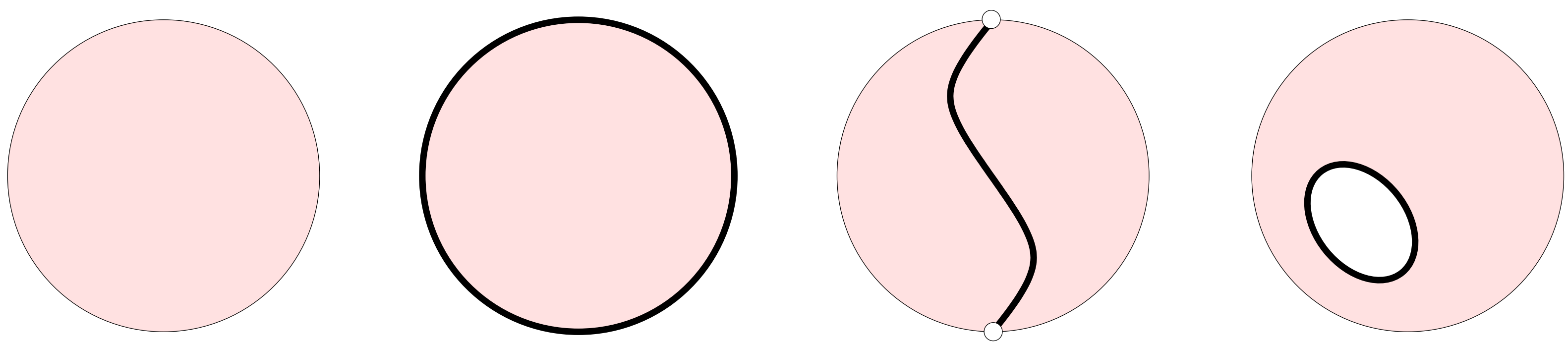


Figure 1: Sculpture d'Angel DUARTE, Lausanne, Suisse.

# SUMMARY

# PSEUDOLINES AND DOUBLE PSEUDOLINES



$$\mathcal{P} \approx \mathbb{D}^2/\{x \sim -x \mid x \in \partial\mathbb{D}^2\}$$

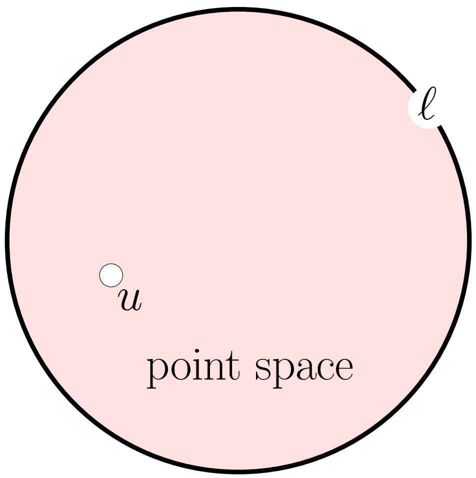
# HILBERT, D. [1899] *Grundlagen der Geometrie.*

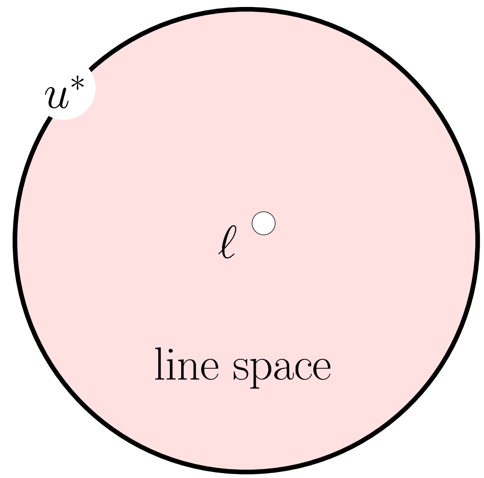
«It was Hilbert's aim to give a simple axiomatic characterization of real (Euclidean) geometries. He expressed the necessary continuity assumptions in terms of properties of an order. Indeed, the real projective plane is the only desarguesian ordered projective plane where every monotone sequence of points has a limit, see the elegant exposition of Coxeter [61].»

Hilbert 1899, Kolmogoroff 1932, Köthe 1939, Skornjakov 1954, Salzmann 1955, Freudenthal 1957, and others

[1] Helmut Salzmann, Dieter Betten, Theo Grundhöfer, Hermann Hähl, Rainer Löwen, and Markus Stroppel. Compact projective planes. Number 21 in De Gruyter expositions in mathematics. Walter de Gruyter, 1995.

# REAL TWO-DIMENSIONAL PROJECTIVE GEOMETRIES



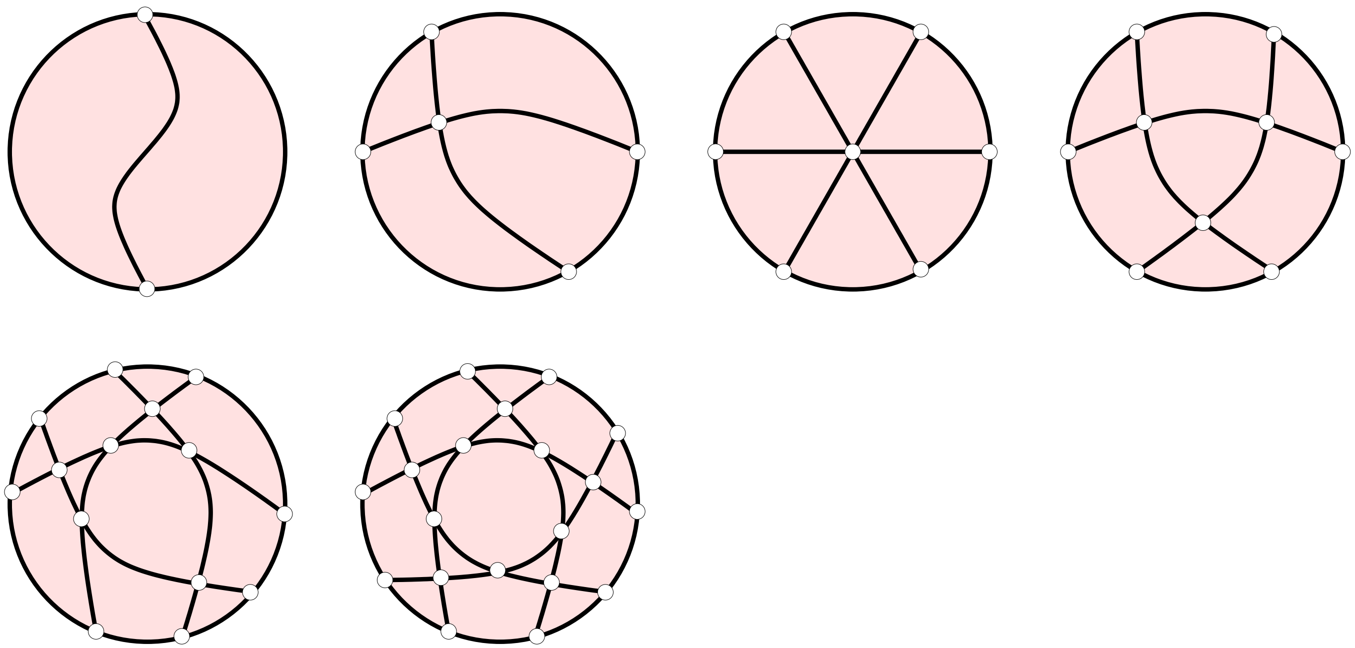

**DF 1 (Hilbert et al.).** *A real two-dimensional projective geometry is a topological point-line projective incidence geometry whose point space is a projective plane and whose line space is a subspace of the space of pseudolines of the point space.* □

**TH 1 (Hilbert et al.).** *The line space of a real two-dimensional projective geometry is a projective plane and the pencil of lines through a point is a pseudoline of the line space* □

$$(\mathcal{P}, \mathcal{L}) \to (\mathcal{L}, \mathcal{P}^*) \to (\mathcal{P}^*, \mathcal{L}^*) \approx (\mathcal{P}, \mathcal{L})$$

# ARRANGEMENTS OF PSEUDOLINES

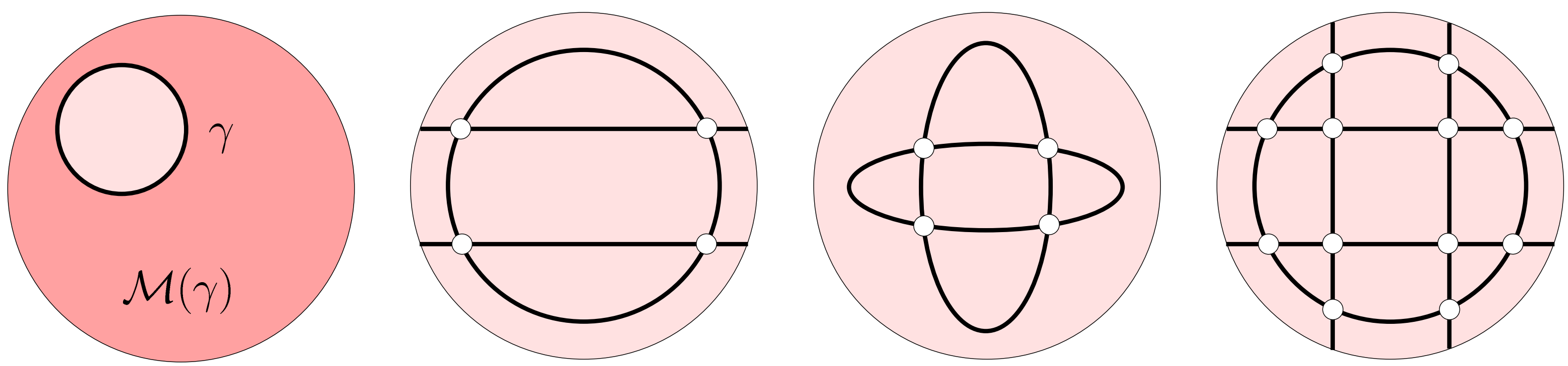
Levi 1926, Ringel 1956, Grünbaum 1972, etc.



**DF 2.** *Let $\mathcal{P}$ be a projective plane. An arrangement of pseudolines in $\mathcal{P}$ is a finite family of pseudolines in $\mathcal{P}$ with the property that any two intersect exactly once.* □

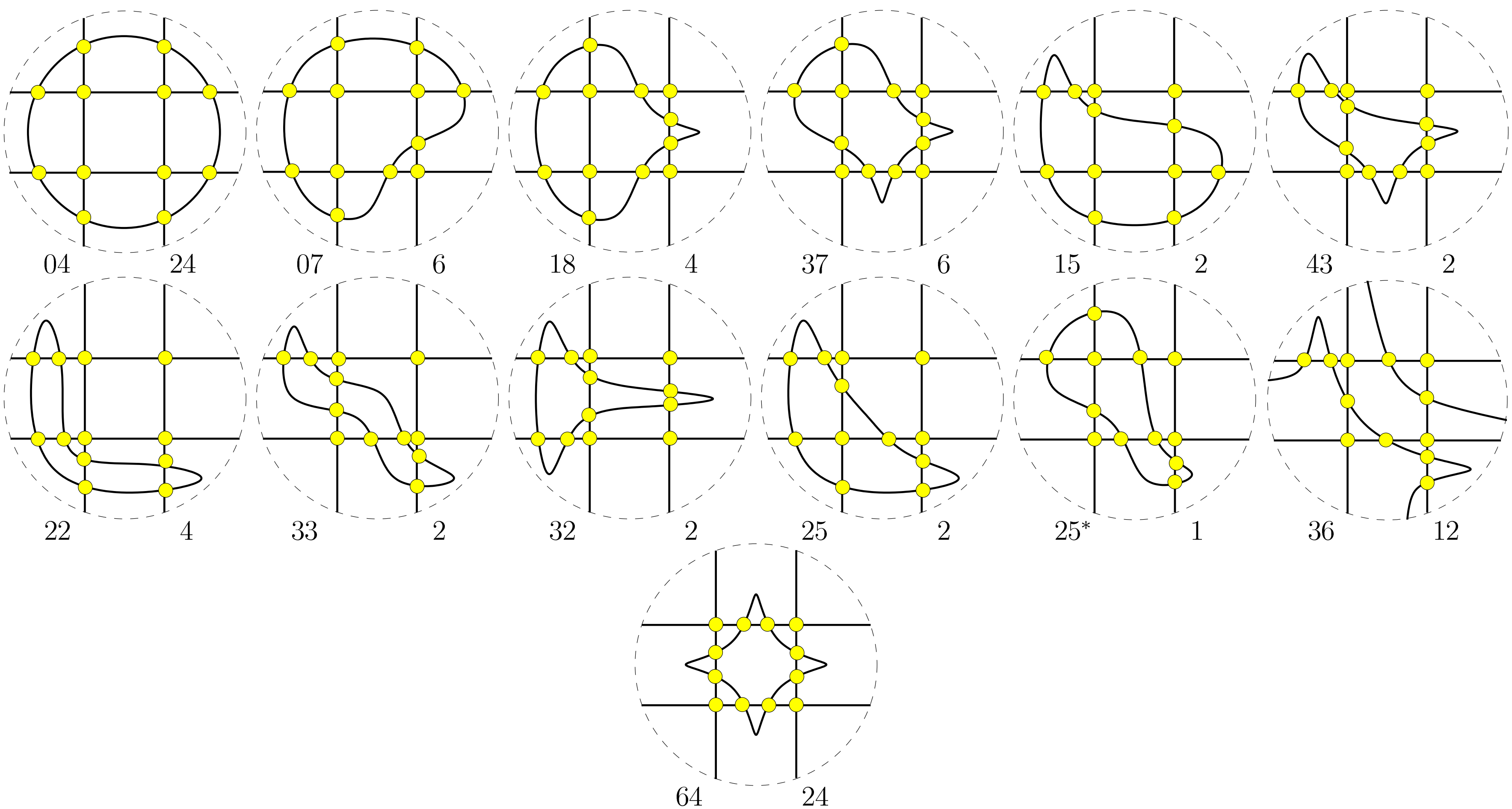
**TH 2 (Goodmann, Pollack, Wenger & Ramfirescu 94).** *Every arrangement of pseudolines can be extended to a (real two-dimensional) projective geometry.* □

# ARRANGEMENTS OF DOUBLE PSEUDOLINES



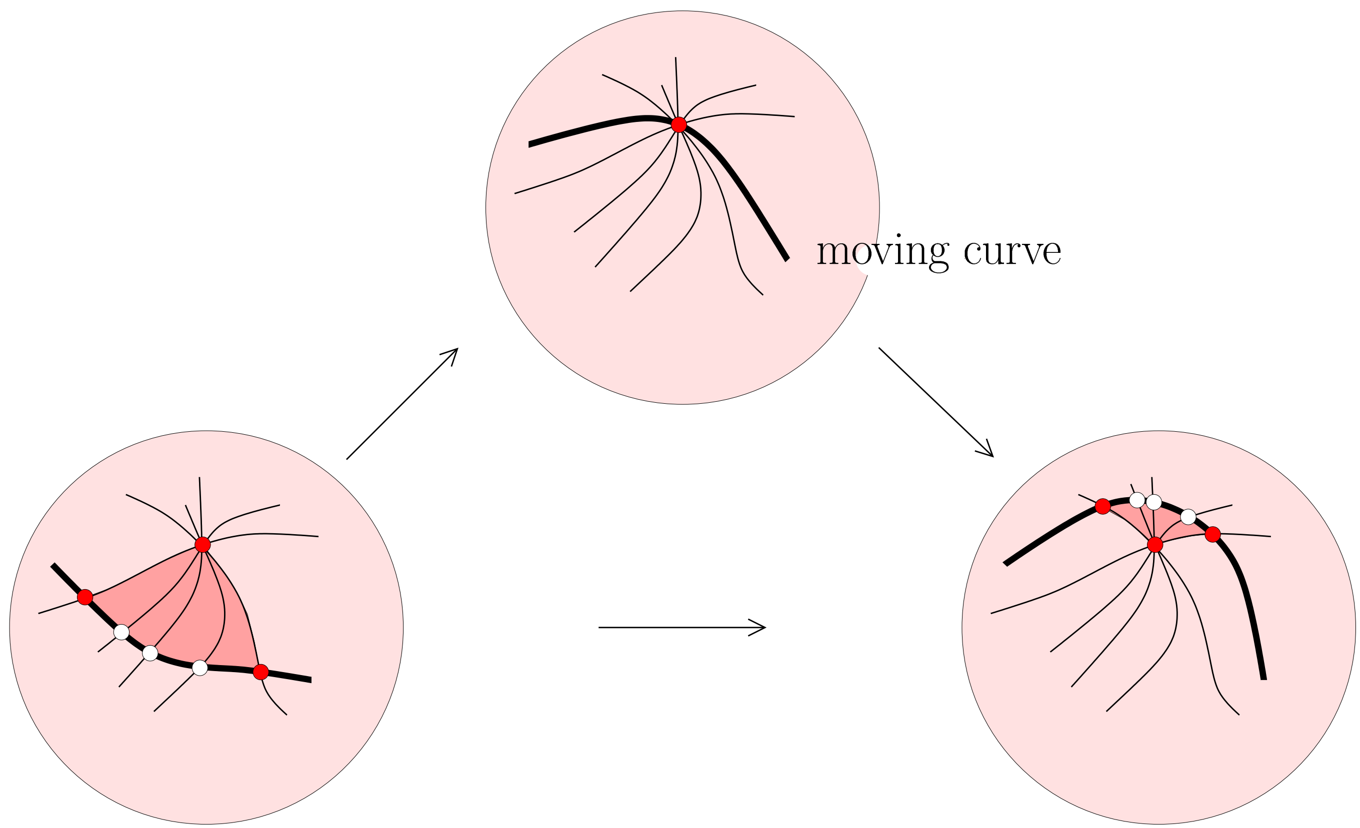

**DF 3.** *Let $\mathcal{P}$ be a projective plane. An arrangement of double pseudolines in $\mathcal{P}$ is a finite family of double pseudolines in $\mathcal{P}$, with the property that any two intersect (transversally) in exactly four points and induce a cell structure on $\mathcal{P}$, that is, the connected components of the complement of the union of the double pseudolines are two-cells.* □

# ISO. CLASSES SIMPLE ARRANG. THREE DOUBLE PSEUDOLINES

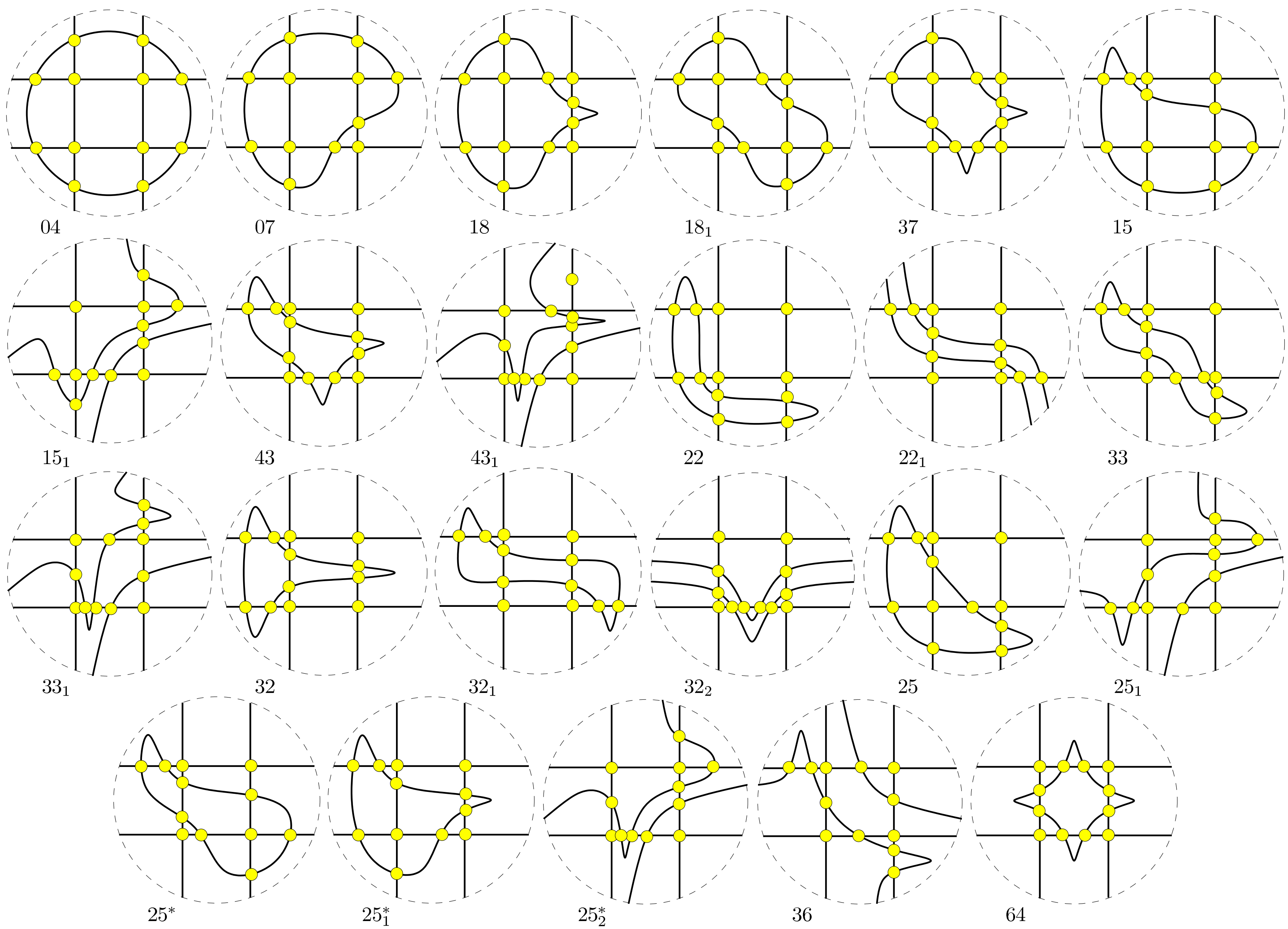


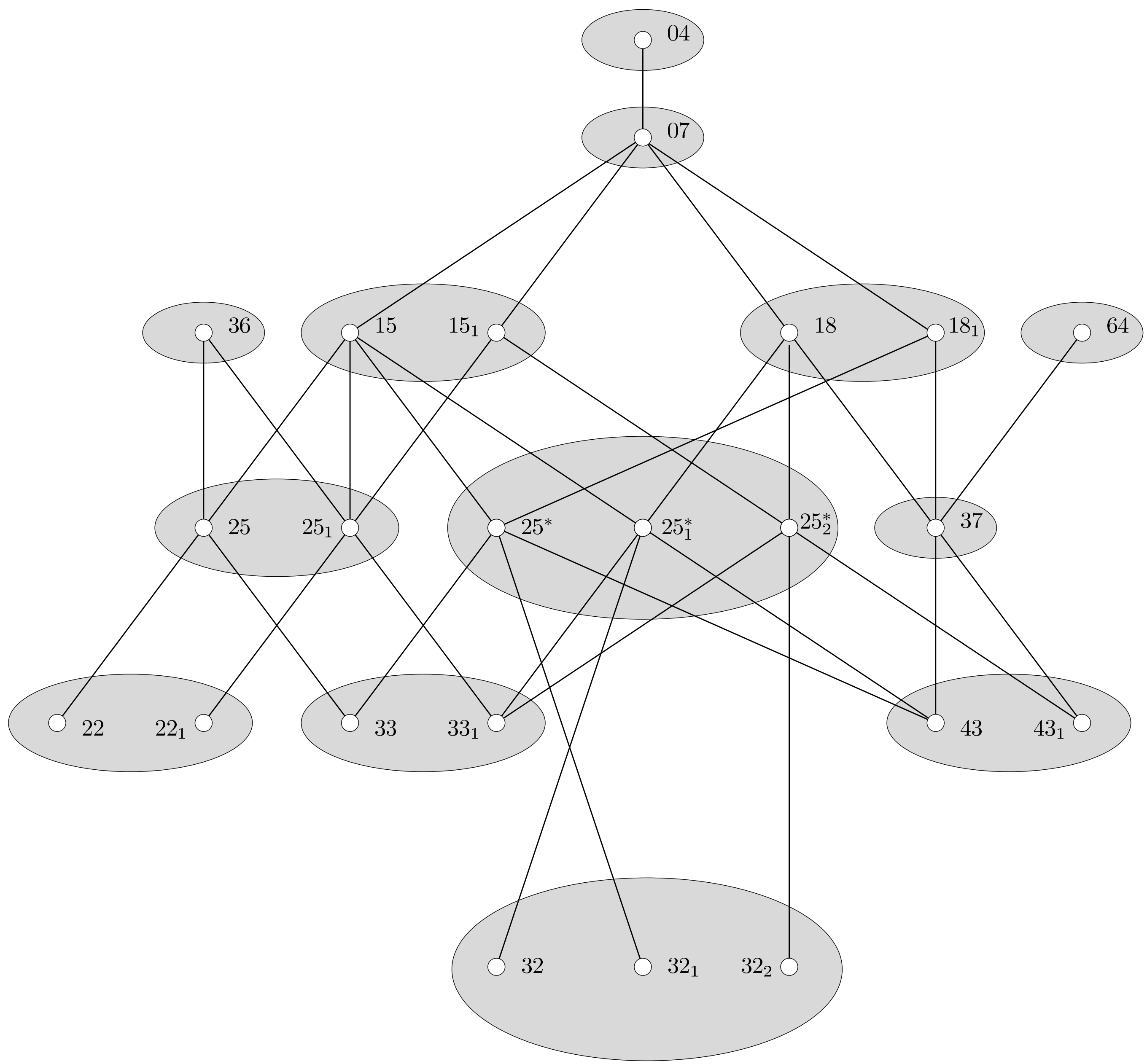
# CONNECTEDNESS UNDER MUTATIONS

**TH 3 (Habert & P. 06).** *The space of arrangements of $n$ double pseudolines is connected under mutations.* □




**TH 4 (Ferté, Pilaud & P. 08).** *The one-extension space of an arrangement of double pseudolines is connected under mutations.* □


04
07
18
18_1
37
15
15_1
43
43_1
22
22_1
33
33_1
32
32_1
32_2
25
25_1
25*
25*_1
25*_2
36
64


04
07
36
15
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18
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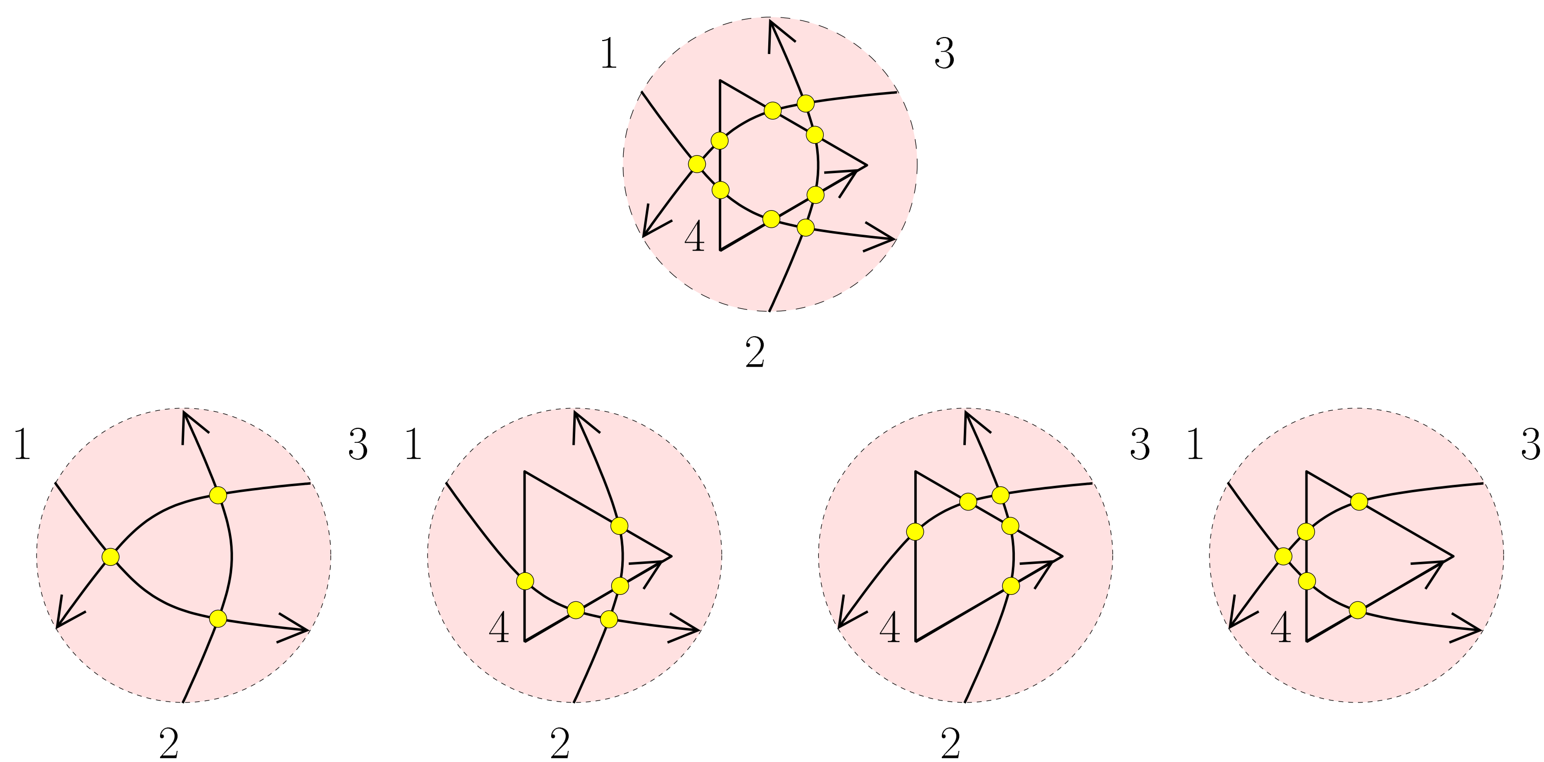
64
25
$25_1$
$25^*$
$25^*_1$
$25^*_2$
37
22
$22_1$
33
$33_1$
43
$43_1$
32
$32_1$
$32_2$

# NUMBERS OF ARRANGEMENTS

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $a_n^S$ | 1 | 13 | 6 570 | 181 403 533 |
| $a_n$ | 1 | 46 | 153 528 | nc |
| $b_n^S$ | 1 | 16 | 11 502 | 238 834 187 |
| $b_n$ | 1 | 59 | 245 351 | nc |

J. Ferté, V. Pilaud and P. 2010 – two double core workstations at 2GHz, three weeks -

## SUB-ARRANGEMENTS OF SIZE THREE



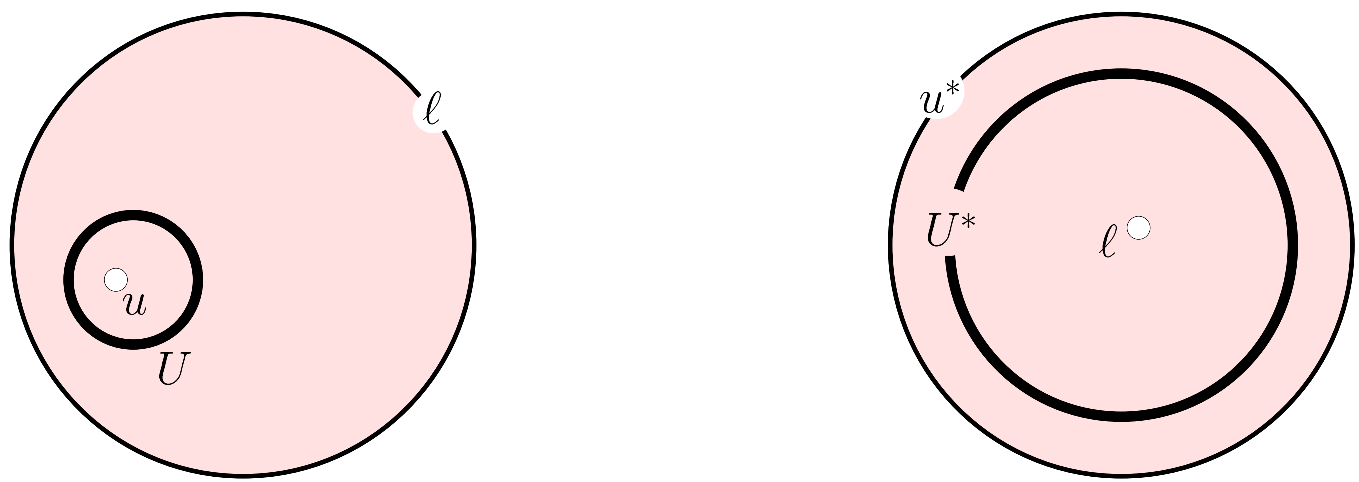

**TH 5 (Habert & P. 06).** *The isomorphism class of an indexed arrangement of oriented double pseudolines depends only on its chirotope, i.e., on the map that assigns to each triple of distinct indices the isomorphism classes of the subarrangement indexed by this triple.* □

# NUMBERS OF CHIROTOPES

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $a_n^S$ | 1 | 13 | 6 570 | 181 403 533 |
| $\rho_n^S$ | 1 | 214 | 2 415 112 | nc |
| $2^n n! a_n^S$ | | 624 | 2 822 580 | 692 749 566 720 |
| $a_n$ | 1 | 46 | 153 528 | nc |
| $\rho_n$ | 1 | 1 086 | 58 266 120 | nc |
| $2^n n! a_n$ | | 2 208 | 58 964 752 | nc |
| $b_n^S$ | 1 | 16 | 11 502 | 238 834 187 |
| $\tau_n^S$ | 1 | 118 | 541 820 | nc |
| $2n! b_n^S$ | | 192 | 552 096 | 7 320 204 880 |
| $b_n$ | 1 | 59 | 245 351 | nc |
| $\tau_n$ | 1 | 531 | 11 715 138 | nc |
| $2n! b_n$ | | 708 | 11 776 848 | |

# NOTES

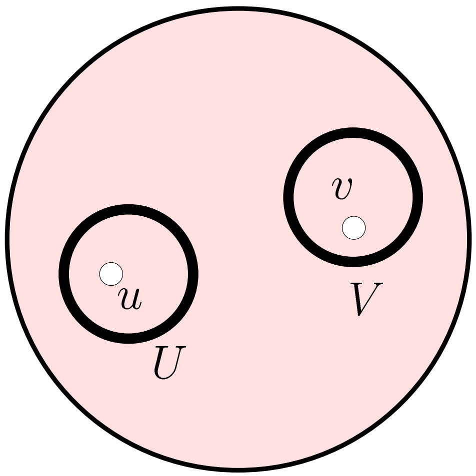
## EXAMPLES OF DOUBLE PSEUDOLINES

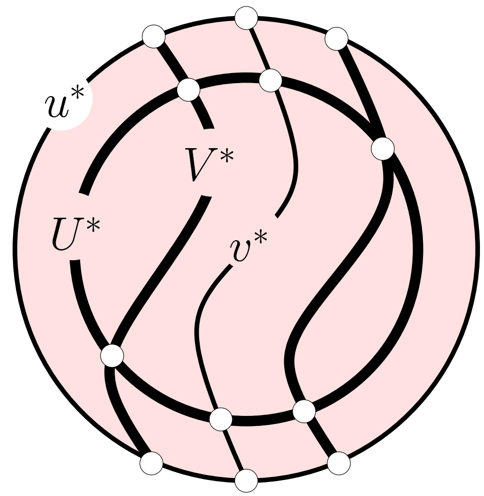



**DF 4 (Habert & P. 09).** *A convex body of a projective geometry is a closed subset of points with nonempty interior whose intersection with any line is an interval of that line.*

**TH 6 (Habert & P. 09).** *Let $U$ be a convex body of a projective geometry $(\mathcal{P}, \mathcal{L})$. Then the boundary of $U$ is a double pseudoline in $\mathcal{P}$ and the dual of $U$ is a double pseudoline in $\mathcal{L}$.* □

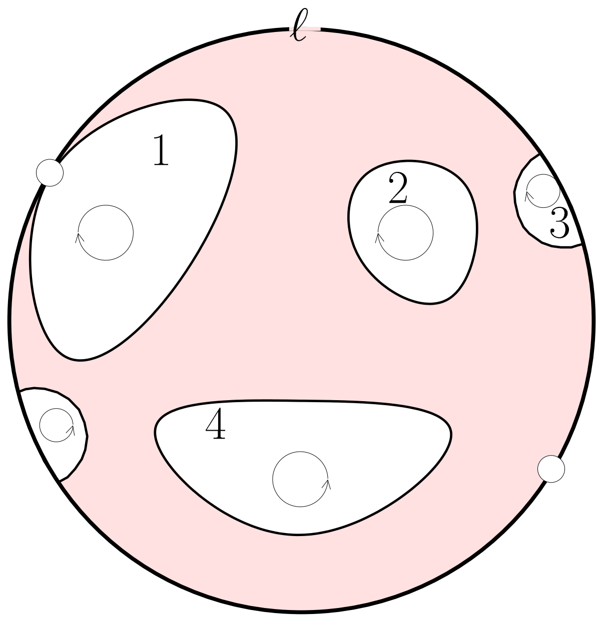
# EXAMPLES OF ARRANGEMENTS OF DOUBLE PSEUDOLINES




**TH 7 (Habert & P. 06-09).** *The dual family of a finite family of pairwise disjoint convex bodies of a (real two dimensional) projective geometry is an arrangement of double pseudolines. Conversely, any arrangement of double pseudolines is isomorphic to the dual family of a finite family of disjoint convex bodies of a projective geometry.* □

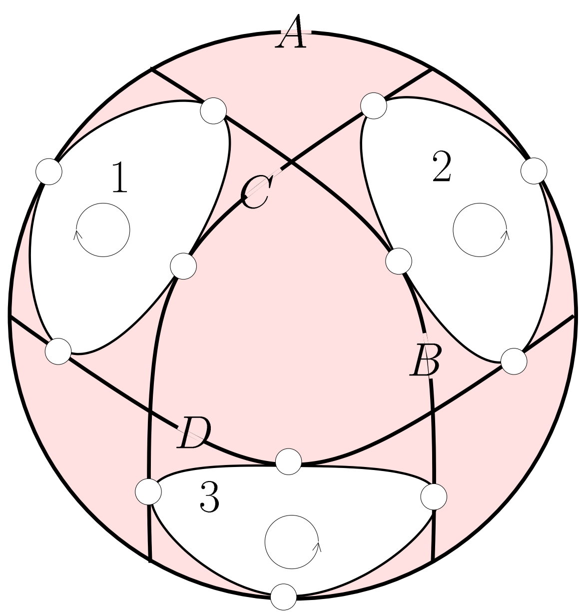
# COCYCLES OF A FAMILY OF BODIES

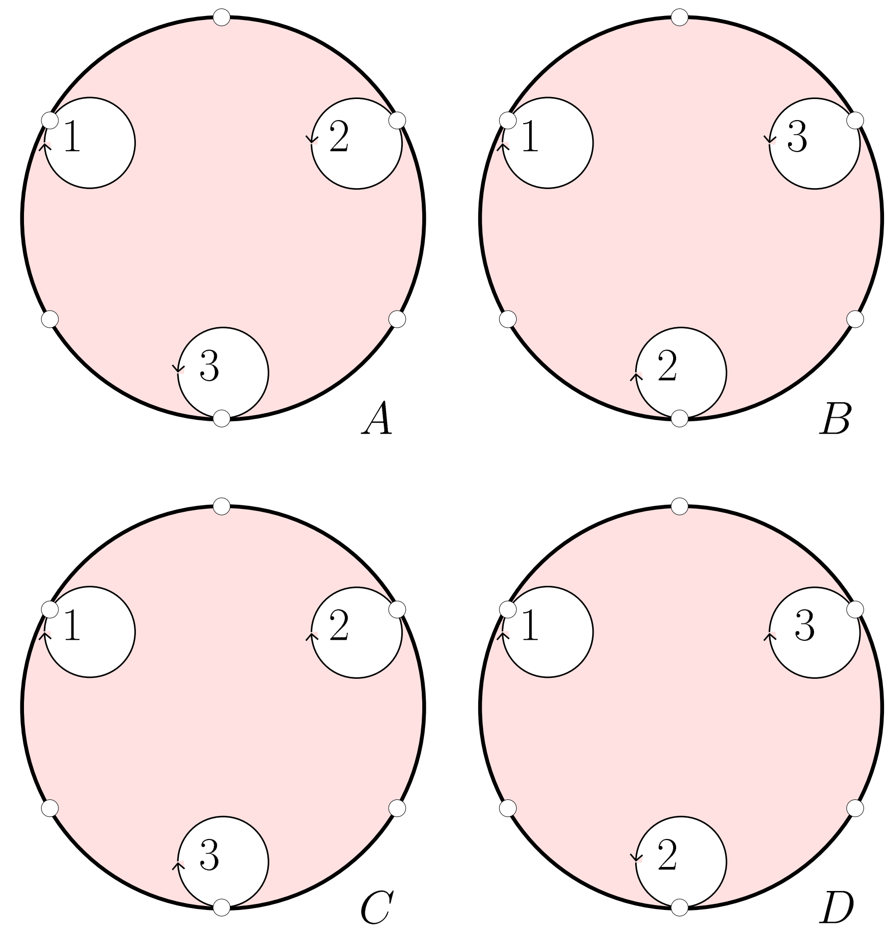



$$\{2, \overline{4}, 13\bullet\overline{3}\} \sim \{\overline{2}, 4, \overline{1}3\bullet\overline{3}\}$$

**DF 5.** *Let* $\Delta$ *be a finite indexed family of pairwise disjoint oriented convex bodies of a projective geometry. The cocycles of* $\Delta$ *are the isomorphism classes of the arrangements* $\Delta \cup \{\ell\}$ *as* $\ell$ *ranges over the space of lines of the projecctive geometry.* □

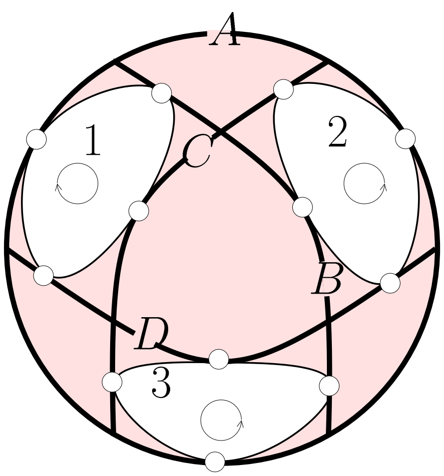
# EXAMPLE

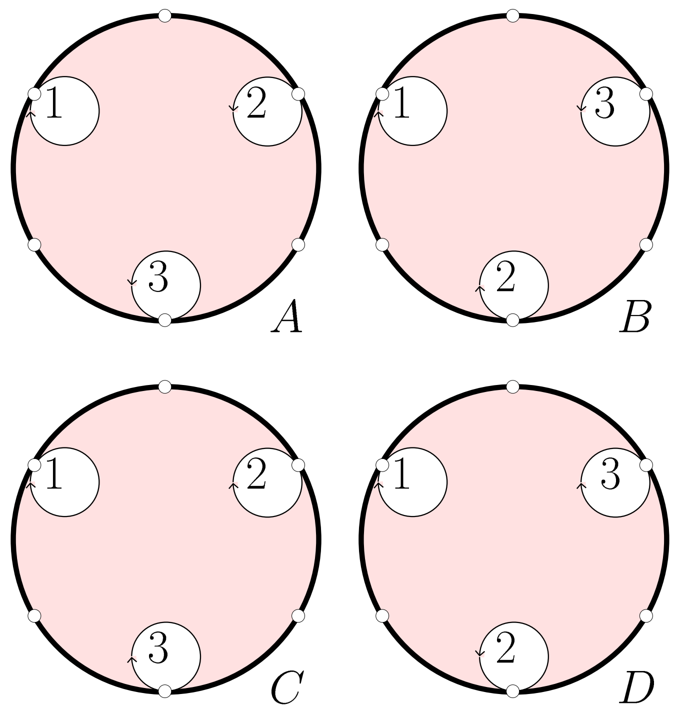


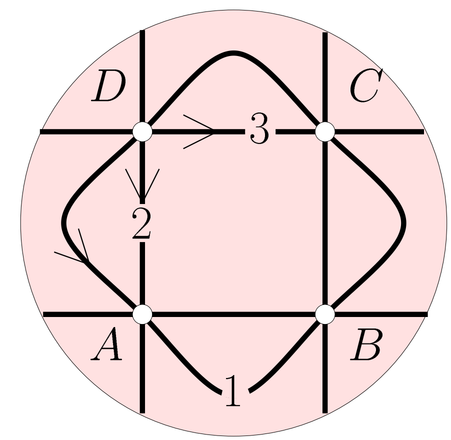




$$\{\{1\bullet\bar{2}\bullet\bar{3}\bullet\},\{1\bullet\bar{3}\bullet 2\bullet\},\{1\bullet 2\bullet 3\bullet\},\{1\bullet 3\bullet\bar{2}\bullet\}\}$$

# DUALITY

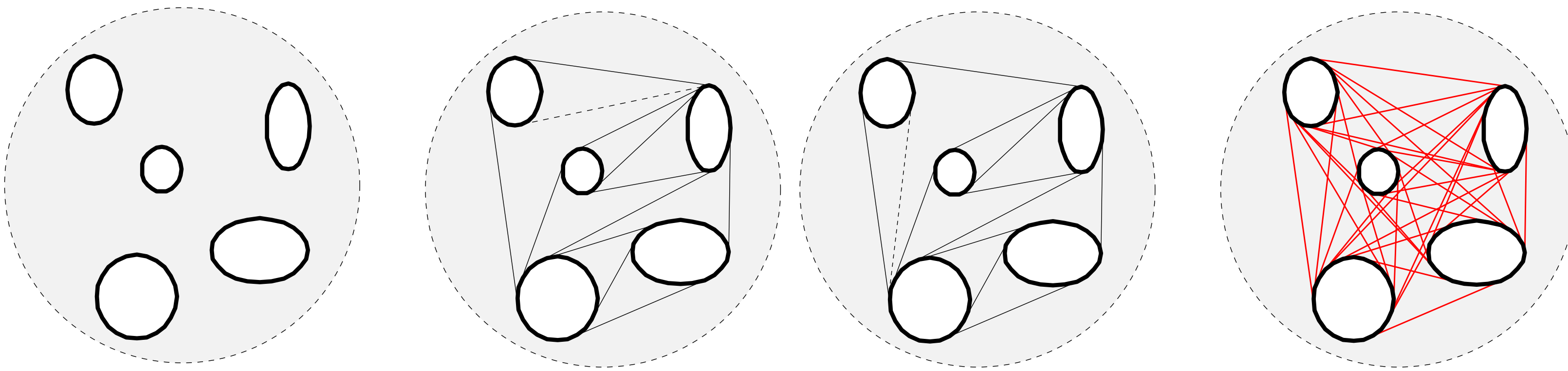



**TH 8 (Habert & P. 09).** *Two finite indexed families of pairwise disjoint oriented convex bodies have isomorphic dual arrangements if and only if they have the same set of cocycles iff they have the same set of extremal cocycles iff they have the same chirotope.* □

# CHIROTOPES AND VIS. GRAPH ALGORITHMS



**TH 9 (Habert & P. 07, Angelier & P. 03, P. & Vegter 96).** *The $k$ edges of the visibility graph of a <u>planar</u> family of $n$ pairwise disjoint convex bodies presented by its <u>chirotope</u> is computable in time $O(k + n \log n)$ and linear working space.* □

# AXIOMATIZATION THEOREM

Let $I$ be a finite indexing set.

**DF 6.** *A k-chirotope (of double pseudoline arrangements) on $I$ is a map $\chi$ defined on the set of triples of $I$ such that for any subset $J$ of $I$ of size at most $k$ the restriction of $\chi$ to the set of triples of $J$ is the chirotope of a double pseudoline arrangement indexed by $J$. We denote by $\mathcal{C}_k(I)$ the set of k-chirotopes on $I$.* □

**TH 10 (Habert & P. 06).** *Let $I$ be a finite indexing set of at least five indices. Then $\mathcal{C}_3(I) \supsetneq \mathcal{C}_4(I) \supsetneq \mathcal{C}_5(I) = \mathcal{C}_6(I) = \mathcal{C}_7(I) = \cdots$ .* □

# OPEN PROBLEMS

1. Our axiomatic characterization theorem for the isomorphism classes of indexed arrangements of oriented double pseudolines provides a 'meaningless' system of a huge number of axioms—namely the full list of arrangements of five double pseudolines. Is it possible to reduce this system of axioms to a system of raisonnable size with a clear geometric or combinatorial interpretation of each axiom?
2. Design a quadratic time algorithm to compute the cell lattice of an arrangement of double pseudoline presented by its chirotope.
3. Arrangements of double pseudolines generalize in many aspects arrangements of pseudolines. From this perspective what are the similar generalizations for arrangements of pseudohyperplanes of dimensions 4, 5, etc.?
4. Find a closed formula counting the number of simple wiring representations of Möbius arrangements of double pseudolines.

$$r_n = \frac{\binom{n}{2}!}{(2n-3)(2n-5)^2(2n-7)^3\cdots 5^{n-3}3^{n-2}}.$$

Stanley'84