# Bucolic complexes

B. Brešar J. Chalopin V. Chepoi*
T. Gologranc D. Osajda

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# CAT(0) cube complexes

Recall that a *CAT(0) cube complex* **X** is a simply-connected cell complex such that

- Cells are Euclidean cubes of various dimensions, attached along faces by combinatorial isometries.
- **X** satisfies Gromov's nonpositive curvature condition: if three $(k+2)$-cubes pairwise intersect in $(k+1)$-cubes and all three intersect in a $k$-cube, then they are included in a $(k+3)$-cube.
- the *1-skeleton* $G(\mathbf{X}) = \mathbf{X}^1$ of **X** is the graph having the 0-cubes and the 1-cubes of **X** as vertices and edges.
- A *CAT(0) space* is a geodesic metric space in which the geodesic triangles are "thinner" than their comparison Euclidean triangles.

# CAT(0) cube complexes

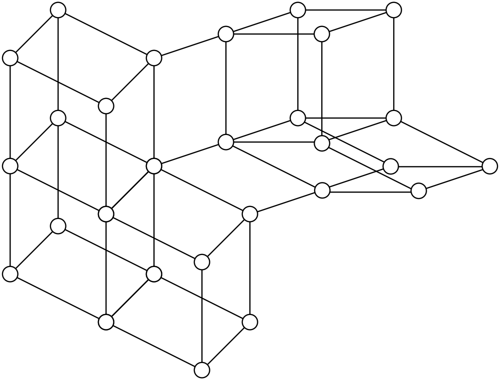

Figure: A CAT(0) cube complex

# Median graphs

### Theorem A (C., 2000)

The 1-skeletons of CAT(0) cube complexes are median graphs. Moreover, median graphs correspond exactly to the 1-skeleta of CAT(0) cube complexes.

### Definition

$G = (V, E)$ is median if $\forall u, v, w \in V$, there exists a unique $m \in V$ on a shortest path from $u$ to $v$, from $u$ to $w$, from $v$ to $w$.

### Remarks

- Trees, hypercubes, and Hasse diagrams of distributive lattices are median graphs.
- Median graphs are bipartite and $K_{2,3}$-free, moreover they embed isometrically into hypercubes.

# Characterizations of median graphs

## Theorem B

Median graphs are exactly

- (Bandelt, 1984) the retracts of hypercubes;
- (Isbell, 1980, van de Vel, 1983) the graphs which can be obtained from hypercubes by gated amalgams.
- (Bandelt and van de Vel, 1991) Each nonexpansive map of a median graph fixes a hypercube.

A map $f : V(G) \to V(G)$ is a retraction of $G$ if

- $f$ is idempotent;
- $f$ is nonexpansive: $d(f(x), f(y)) \leq d(x, y)$;

$H = f(G)$ is a retract of $G$.

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- (Bandelt and van de Vel, 1991) Each nonexpansive map of a median graph fixes a hypercube.

A set $S \subseteq V(G)$ is gated if $\forall x \notin S, \exists x' \in S, \forall y \in S$, $x'$ is on a shortest path from $x$ to $y$.

If $G_1$ has a gated set $S_1$ that is isomorphic to a gated set $S_2$ of $G_2$, then the graph $G$ obtained by identifying $S_1$ and $S_2$ is a gated amalgam of $G_1$ and $G_2$.

# Bridged graphs

## Definition

A graph $G = (V, E)$ is *bridged* (Jamison and Farber, 1987) if each isometric cycle of $G$ has length 3. Chordal graphs and plane triangulations with degrees $\geq 6$ of inner vertices are bridged.

## Theorem C (Soltan and C., 1983, and Jamison and Farber, 1987)

A graph $G$ is bridged iff the neighborhoods of convex sets of $G$ are convex (it turns out to be one of basic properties of CAT(0) spaces).

# Systolic complexes

## Theorem C (cont.) (C., 2000)

The 1-skeleton of an abstract simplicial flag complex **X** is a bridged graph iff **X** is simply connected and the links of vertices do not contain induced 4- and 5-cycles.

## Definition

The *link* of a vertex $v$ of **X** is the simplicial complex consisting of all simplices $\sigma \in \mathbf{X}$ such that $v \notin \sigma$ and $\sigma \cup \{v\} \in \mathbf{X}$.

## Systolic complexes

Januszkiewicz and Swiatkowski (2006) rediscovered these complexes and investigated them in depth (see also, Haglund (2003)). They called them ”systolic complexes" and considered as simplicial complexes of combinatorial nonpositive curvature.

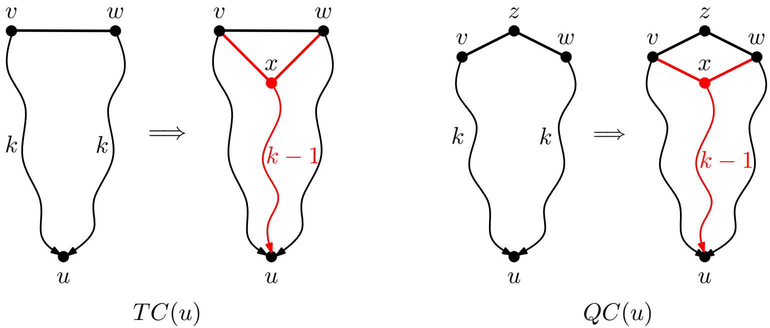
## Questions (for this talk):

- What classes of graphs can be defined by Cartesian products and gated amalgam and what are their prime graphs (i.e., the graphs that cannot be decomposed by Cartesian product or gated amalgam)?
- Characterize the class of all graphs obtained from bridged graphs by Cartesian products and gated amalgams. Characterize the cell complexes derived from such graphs and investigate their properties.

# Weakly modular graphs

## Definition

A graph is weakly modular if it satisfies the triangle and quadrangle conditions



# Triangle-square complexes

## Definition

We consider *triangle-square complexes* (2-dimensional cell complexes with triangles or squares as 2-cells) satisfying the following *local conditions:*

*Cube-condition*: any three squares of **X**, pairwise intersecting in an edge, and all three intersecting in a vertex, are included in a 3-dimensional cube;

*House-condition*: any house (i.e., a triangle and a square of is included in a 3-dimensional prism;

$(W_4, W_5)$*-condition*: no 4- and 5-wheels $W_4$ and $W_5$.

# Bucolic complexes: definition

## Definition

A *prism* is a Cartesian product of simplices. A *prism complex* is a cell complex in which all cells are prisms.

A prism complex **X** is *bucolic* if it is flag, connected and simply connected, and satisfies the following three local conditions:

$(W_4, W_5)$*-condition:* no 4- and 5-wheels $W_4$ and $W_5$ in the 2-skeleton;

*Hypercube condition:* if $k \geq 2$ and three $k$-cubes of **X** pairwise intersect in a $(k-1)$-cube and all three intersect in a $(k-2)$-cube, then they are included in a $(k+1)$-dimensional cube of **X**;

*Hyperhouse condition:* if a cube and a simplex of **X** intersect in a 1-simplex, then they are included in a prism of **X**.

# Bucolic complexes: characterization

## Theorem 1

For a locally-finite prism complex $\mathbf{X}$, the following conditions are equivalent:

(i) $\mathbf{X}$ is a bucolic complex;

(ii) the 2-skeleton $\mathbf{X}^{(2)}$ of $\mathbf{X}$ is a connected and simply connected triangle-square flag complex satisfying the $(W_4, W_5)$-condition, the cube condition, and the house condition;

(iii) the 1-skeleton $G(\mathbf{X})$ of $\mathbf{X}$ is a weakly modular graph not containing induced subgraphs of the form $K_{2,3}$, $W_4$, and $W_5$.

Moreover, if $\mathbf{X}$ is a connected flag prism complex satisfying the $(W_4, W_5)$, the hypercube, and the hyperhouse conditions, then the universal cover $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ is bucolic.

# Bucolic graphs

**Theorem 2**

For a locally-finite graph $G = (V, E)$, the following conditions are equivalent:

(i) $G$ is a retract of the Cartesian product of bridged graphs;
(ii) $G$ is a weakly modular graph not containing induced $K_{2,3}$, $W_4$, and $W_5$, i.e., $G$ is a bucolic graph;
(iii) $G$ is $K_{2,3}$-free weakly modular graph in which all prime subgraphs are edges or 2-connected bridged graphs.

Moreover, if $G$ is finite, then the conditions (i)-(iii) are equivalent to the following condition:

(iv) $G$ can be obtained by successive applications of gated amalgamations from Cartesian products of 2-connected bridged graphs.

# Contractibility and fixed point results

**Theorem 3**

Bucolic complexes $\mathbf{X}$ are contractible.

**Theorem 4**

If $\mathbf{X}$ is a bucolic complex and $F$ is a finite group acting by cell automorphisms on $\mathbf{X}$, then there exists a prism $\pi$ of $\mathbf{X}$ which is invariant under the action of $F$. The center of the fixed prism $\pi$ is an $F$-invariant point.

# Questions (hopefully, for other talks)

- Characterize the gated hulls of triangles in premedian graphs.
- Characterize the premedian graphs by considering their triangle-square complexes.
- Characterize the structures (graphs, complexes) obtained from basis graphs of matroids or even $\Delta$-matroids by Cartesian products and gated amalgams.

## Premedian graphs (Chastand)

A graph $G = (V, E)$ is *premedian* if for each gated set $S$ and each $v \in S$ the *fiber* $F(v) = \{x \in V : \text{ the gate of } x \text{ in } S \text{ is } v\}$.

# To the proof of Theorem 1

It is relatively "easy" to prove that $(i) \implies (ii)$. The proof of $(iii) \implies (i)$ is direct but quite involved.

To prove $(ii) \implies (iii)$, for an arbitrary vertex $v$

- We "construct" the universal cover $\tilde{\mathbf{X}}_v$ of $\mathbf{X}$ using $v$ as a basepoint;
- We show that $G(\tilde{\mathbf{X}}_v)$ is weakly modular with respect to $\tilde{v}$;
- Since $\mathbf{X}$ is simply connected, $\tilde{\mathbf{X}}_v$ is isomorphic to $\mathbf{X}$;
- This implies that $G = G(\mathbf{X})$ is weakly modular.

# Coverings of cell complexes

The *star* $St(v, \mathbf{X})$ of $v$ in $\mathbf{X}$ is the subcomplex of $\mathbf{X}$ consisting of all cells of $\mathbf{X}$ containing $v$.

## Definition

$\mathbf{Y}$ is a cover of $\mathbf{X}$ if there exists a map $\varphi$ from $\mathbf{Y}$ onto $\mathbf{X}$ such that for every $v \in \mathbf{Y}$, $\varphi$ induces a bijection between $St(v, \mathbf{Y})$ and $St(\varphi(v), \mathbf{X})$.

There exists a unique cover $\tilde{\mathbf{X}}$ of $\mathbf{X}$ such that for each $\mathbf{Y}$ that covers $\mathbf{X}$, $\tilde{\mathbf{X}}$ covers $\mathbf{Y}$. $\tilde{\mathbf{X}}$ is called universal cover of $\mathbf{X}$.

$\mathbf{X}$ is a universal cover iff $\mathbf{X}$ is simply connected.

# Iterative construction of the universal cover of $\mathbf{X}$

Pick any vertex $v \in G = G(\mathbf{X})$.
We construct the cover as an increasing union of triangle-square complexes $\tilde{\mathbf{X}}_i$ together with a mapping $f_i : \tilde{\mathbf{X}}_i \to \mathbf{X}$ that satisfy the following properties:

- $B_j(\tilde{G}_i) = \tilde{G}_j$ for $j \leq i$,
- $\tilde{G}_i$ is weakly modular with respect to $\tilde{v}$,
- $\forall \tilde{w} \in \tilde{G}_{i-2}$, $f_i$ induces a bijection between $St(\tilde{w}, \tilde{\mathbf{X}})$ and $St(w, \mathbf{X})$,
- $\forall \tilde{w} \in \tilde{G}_{i-1}$, $f_i$ induces an isomorphism between $B_1(\tilde{w}, \tilde{\mathbf{X}})$ and $B_1(w, \mathbf{X})$,
- $\forall \tilde{w} \in \tilde{G}_i$, $f_i$ defines an isomorphism between $B_1(\tilde{w}, \tilde{G}_i)$ and $f_i(B_1(\tilde{w}, \tilde{G}_i))$.

We start with $G_0 = \{v\}$, $G_1 = B_1(v, G)$.

# Constructing $G_{i+1}$

Suppose $\tilde{\mathbf{X}}_i$ satisfies the following condition:

$$Z = \{(\tilde{w}, z) \mid w \sim z \text{ and } z \notin f(B_1(\tilde{w}, \tilde{\mathbf{X}}_i))\}.$$

We say that $(\tilde{w}, z) \equiv (\tilde{w}', z)$ if

- $\tilde{w} \sim \tilde{w}'$ and $zww'$ is a triangle in $\mathbf{X}$,
- $\exists \tilde{v} \in \tilde{G}_{i-1}$ such that $\tilde{v} \sim \tilde{w}, \tilde{w}'$ and $vwzw'$ is a square in $\mathbf{X}$.

### Lemma

*$\equiv$ is an equivalence relation.*

- $V(\tilde{G}_{i+1}) = V(\tilde{G}_i) \cup Z/\equiv$,
- $E(G_{i+1})$ is obtained from $E(G_i)$ by adding the edges
  - between $\tilde{w}$ and $\tilde{z} = [\tilde{w}, z]$,
  - between $\tilde{z} = [\tilde{w}, z]$ and $\tilde{z'} = [\tilde{w}, z']$ if $zz' \in E(G)$.

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# To the proof of Theorem 2

## Proof of Theorem 2

- Showing ($i$) $\implies$ ($ii$) is "easy".
- To show ($ii$) $\implies$ ($iii$):
  - We show that the gated hull of a triangle is a 2-connected bridged graph
  - Since bridged graphs are premedian, Chastand's general result implies ($iii$) and ($iv$).
- ($iii$) $\implies$ ($i$) follows from a result of Bandelt and C.

## Gated hull of a triangle

- $S :=$ *triangle*
- While $\exists v \notin S$ with two or more neighbors in $S$, add $v$ to $S$

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# The gated hulls are bridged

## Definition

A 2-path $(a, v, b)$ is fanned in $H$, if there exists a path from $a$ to $b$ in $N(v) \cap H$.
A path $(u_0, u_1, \dots, u_k)$ is fanned if every $(u_{i-1}, u_i, u_{i+1})$ is fanned.

## Lemma

*Every path of the gated hull $S$ is fanned in $S$.*

**Consequences:** If $S$ contains a $C_4$, then it contains a $W_n^-$.
This implies that $G$ contains either a $W_4$ or a $W_4^-$.

In a weakly modular graph, any $C_5$ is included in a $W_5$.

Thus $S$ is a bridged graph.

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# General method (after Chastand)

Given a gated set $S$ of $G$ and $x \in S$, the fiber $F_x$ is the set of vertices $v$ such that $x$ is the gate of $v$.

### Theorem (Chastand '01)

*In $G$ is a premedian graph, then for any gated set $S$,*

- *$\forall x \in S$, $F_x$ is gated,*
- *If there exists $x \in S$ and $u \in F_x$ such that $N(u) \subseteq F_X$, then $G$ is a gated amalgam,*
- *Otherwise, $G = S \Box F_x$.*