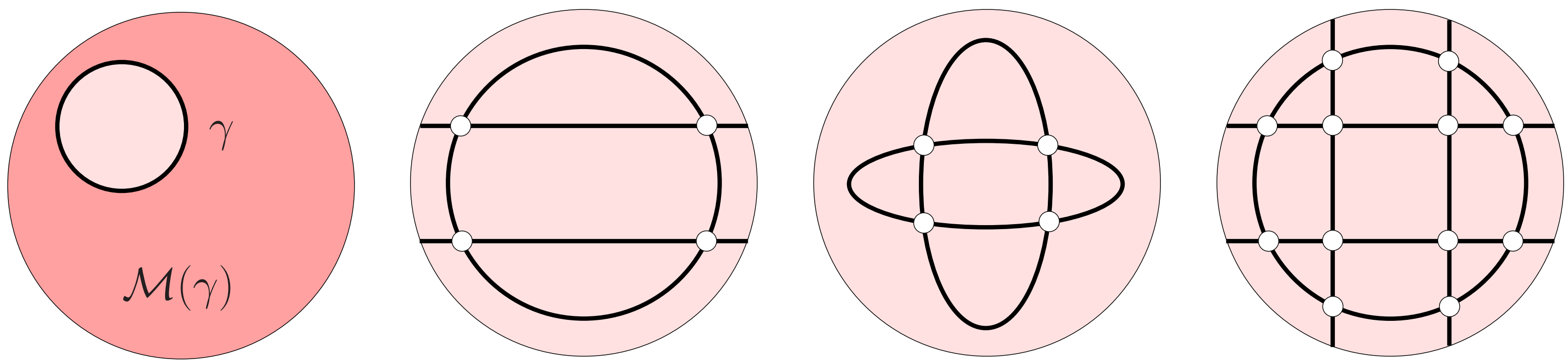
On the axiomatization of double pseudoline arrangements

Michel Pocchiola
Institut de Mathématiques de Jussieu , U. Pierre & Marie Curie
(pocchiola@math.jussieu.fr, http://people.math.jussieu.fr/ pocchiola/)

DEFINITION OF ARRANGEMENTS OF DOUBLE PSEUDOLINES



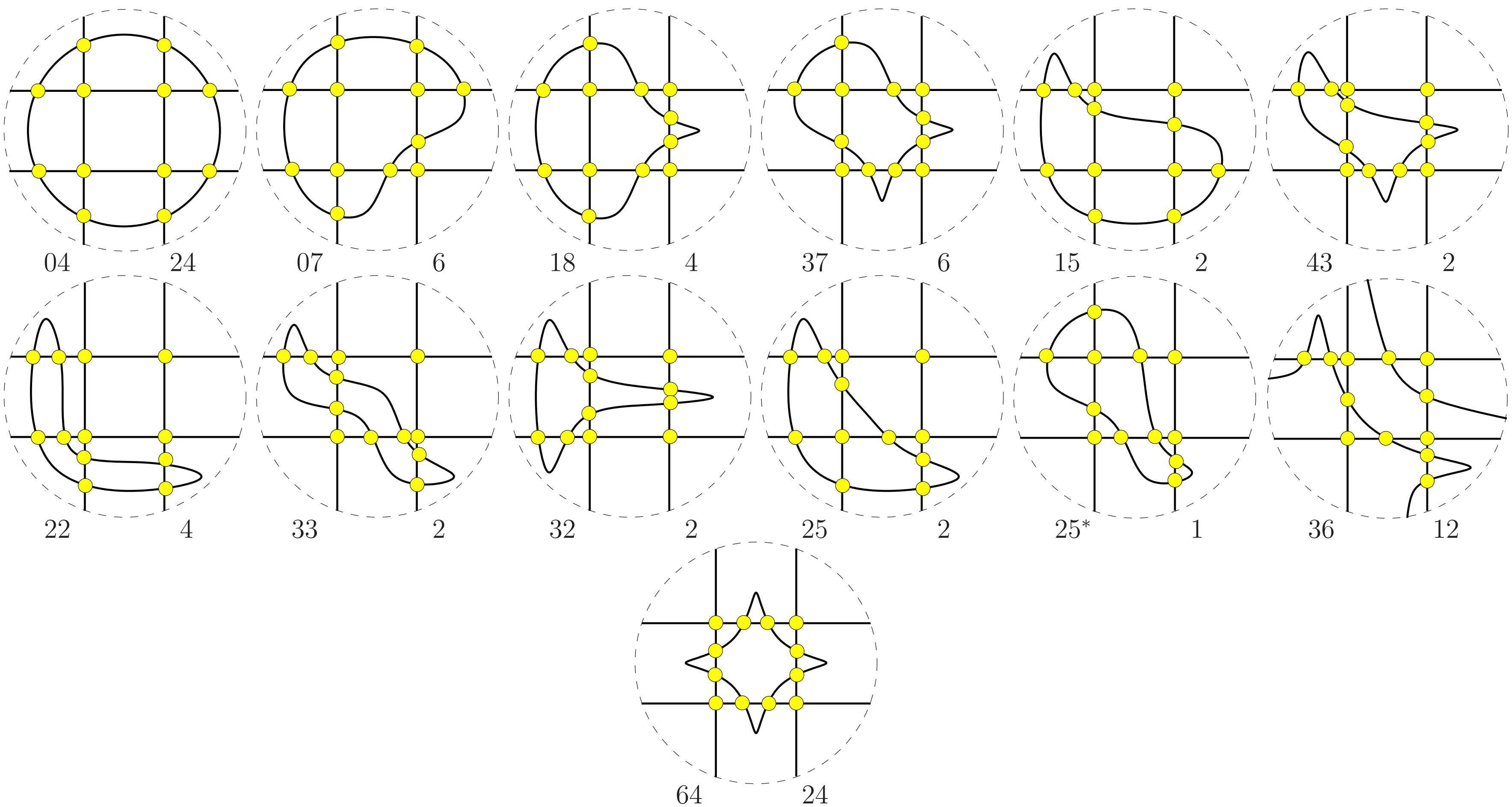
DF 1. *Let $\mathcal{P}$ be a projective plane. An* **arrangement of double pseudolines** *in $\mathcal{P}$ is a finite family of double pseudolines in $\mathcal{P}$ that intersect pairwise in exactly four transversal intersection points and that induce pairwise a cell structure on $\mathcal{P}$.* □

Examples of arrangements of double pseudolines are given by the dual families of finite families of pairwise disjoint convex bodies of projective geometries.

DUALITY THEOREM

TH 1. *Any arrangement of double pseudolines is isomorphic to the dual family of a finite family of pairwise disjoint convex bodies of a projective geometry.* □

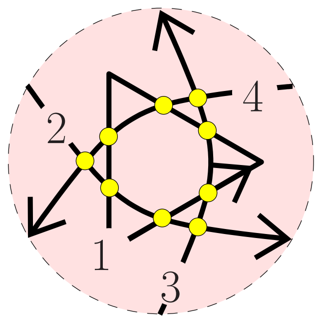
ISO. CLASSES SIMPLE ARRANG. THREE DOUBLE PSEUDOLINES



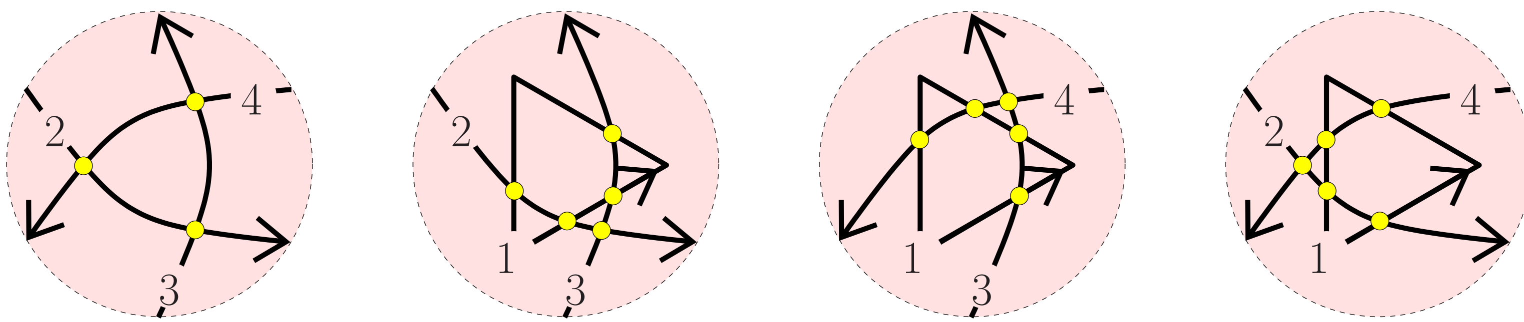
n	2	3	4	5
a_n^S/a_n	1/1	13/46	6570/153 528	181 403 533/???

`http://www.research.att.com/~njas/sequences/A191937` -

CHIROTOPES AND AXIOMATIZATION



$$\begin{cases} 1: & 22\overline{22}33\overline{33}44\overline{44} \\ 2: & 441\overline{133441}133 \\ 3: & 221\overline{144221}144 \\ 4: & 331\overline{122331}122 \end{cases}$$



DF 2. *The* **chirotope** *of an indexed arrangement of oriented double pseudolines is the map that assigns to each subset of indices of size at most three the isomorphism class of the subarrangement indexed by this subset (note that two isomorphic arrangements have the same chirotope).* □

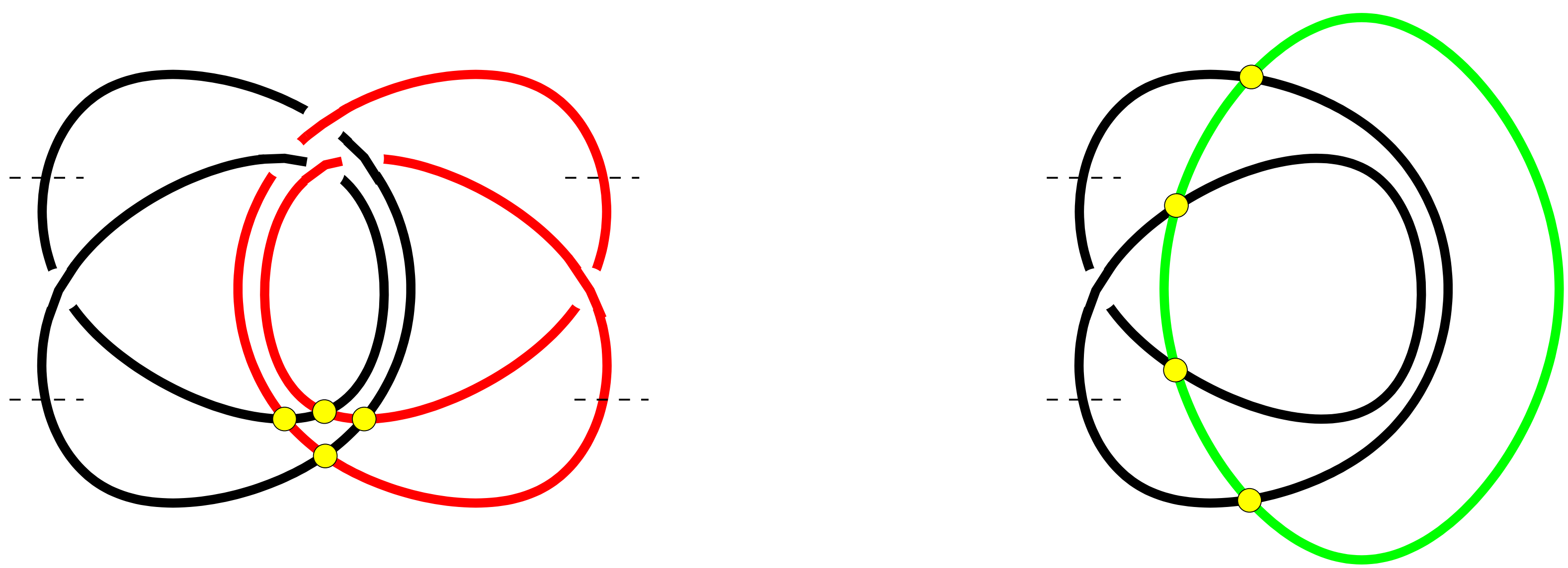
AXIOMATIZATION

TH 2. *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one and its range is the set of map χ defined on the set of triples of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of triples of J is the chirotope of an indexed arrangement of oriented double pseudolines.* □

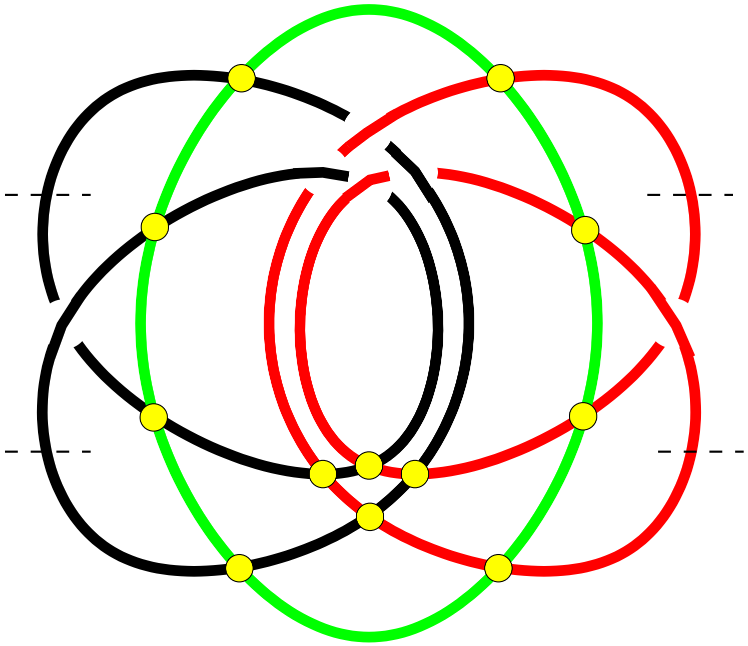
n	0	1	2	3	4	5
a_n^S	1	1	1	13	6 570	180 403 533
ρ_n^S	1	1	1	214	2 415 112	nc
$2^n n! a_n^S$				624	2 822 580	692 749 566 720

ARRANGEMENTS OF GENUS $1, 2, \ldots, g$

DF 3. *An* **arrangement of double pseudolines of genus** g *is a finite family of simple closed oriented curves embedded in a nonorientable compact surface of genus* g *with the property that its subarrangements of size* 2 *are arrangements of double pseudolines of genus* 1. □



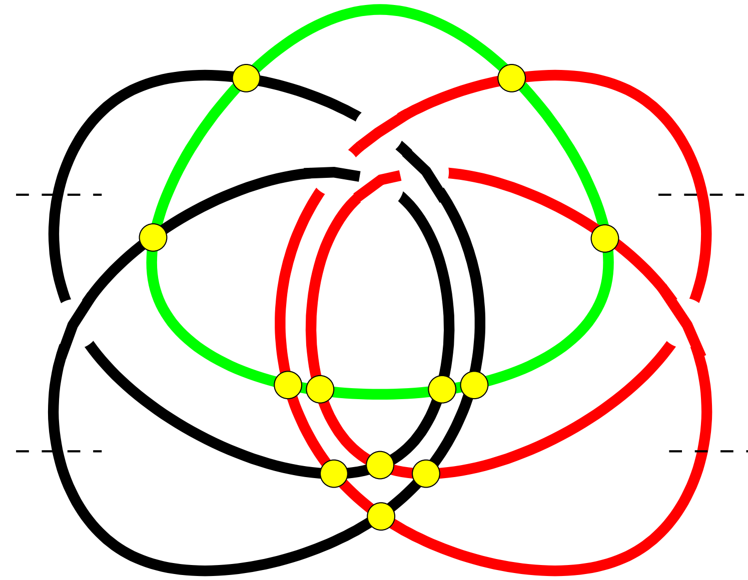
TWO ARRANGEMENTS OF GENUS 4 ON 3 CURVES



$1: \ \overline{2}\,\overline{2}\,2\,2\,\overline{3}\,\overline{3}\,3\,3$

$2: \ \overline{3}\,\overline{3}\,\overline{1}\,\overline{1}\,3\,3\,1\,1$

$3: \ \overline{2}\,\overline{2}\,1\,1\,2\,2\,\overline{1}\,\overline{1}$

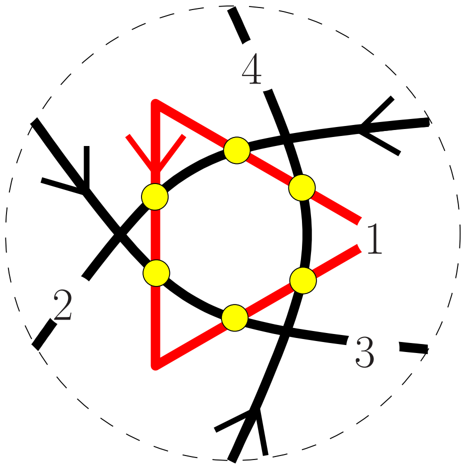


$1: \ \overline{2}\,\overline{2}\,3\,3\,2\,2\,\overline{3}\,\overline{3}$

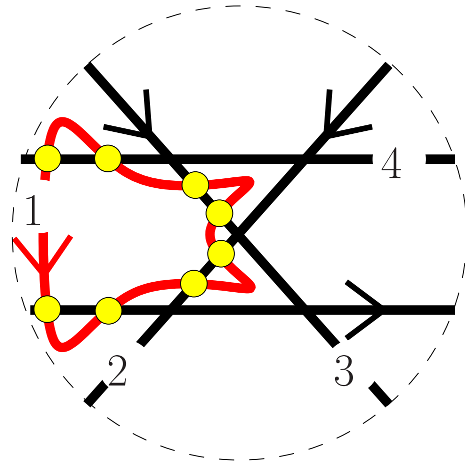
$2: \ \overline{3}\,\overline{3}\,\overline{1}\,1\,3\,3\,1\,\overline{1}$

$3: \ \overline{2}\,\overline{2}\,\overline{1}\,1\,2\,2\,1\,\overline{1}$

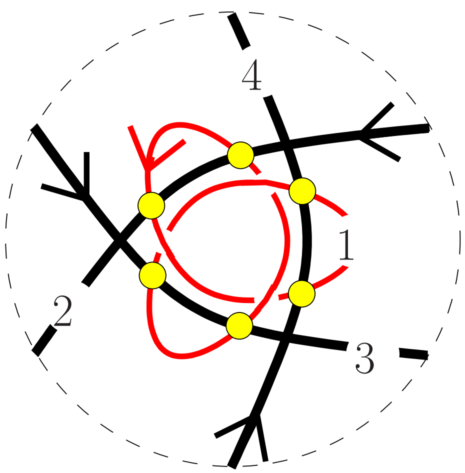
TWO ARRANGEMENTS OF GENUS 3 ON 4 CURVES



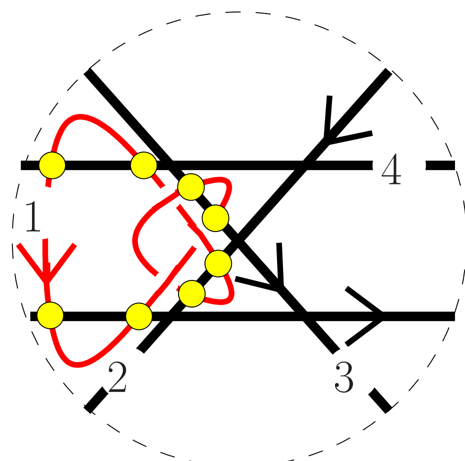
M_1



M_2



M_1^*



M_2^*

k-CHIROTOPES – k-ARRANGEMENTS

Let I be a finite indexing set.

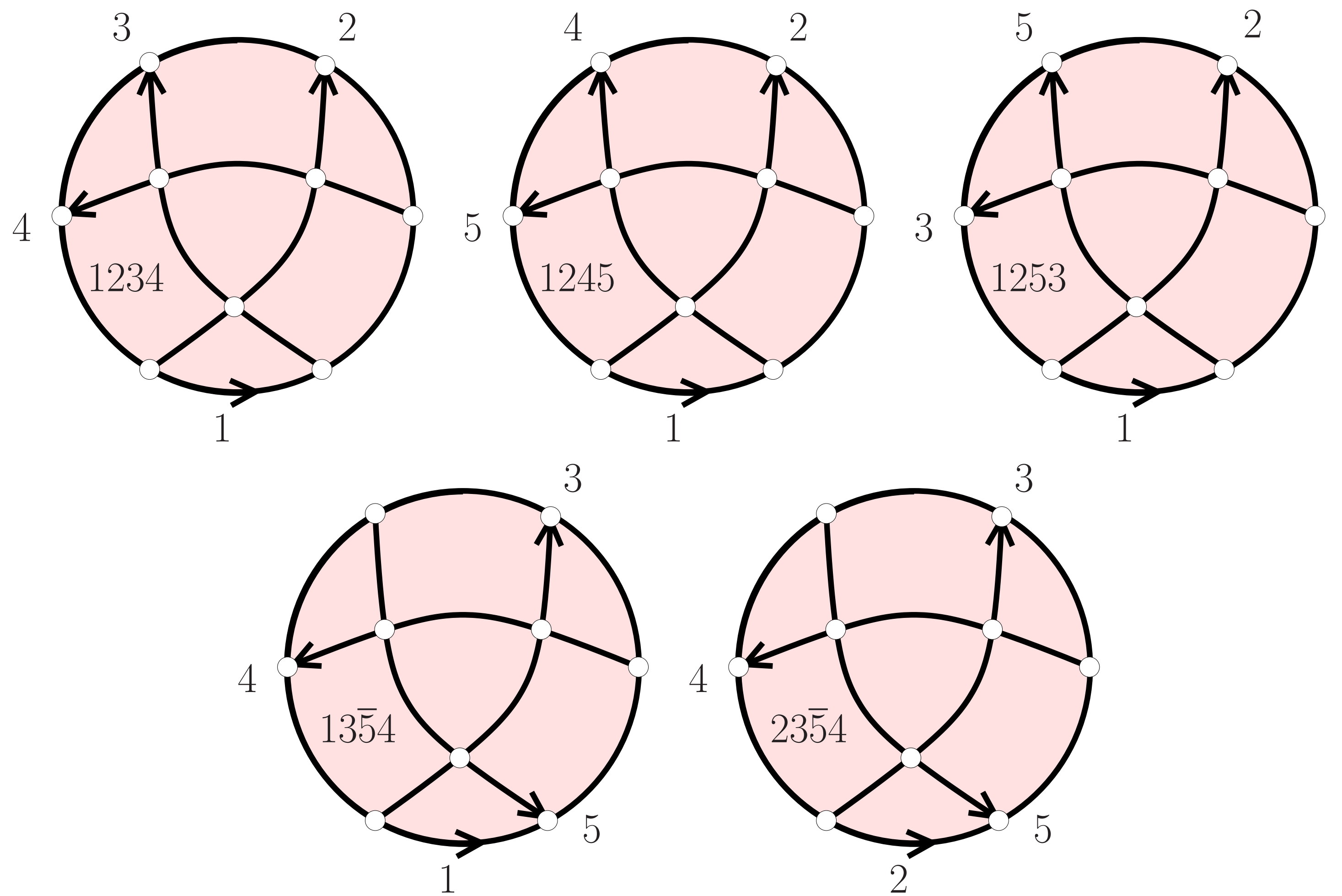
DF 4. *A k-chirotope (of double pseudoline arrangements) is a map χ defined on the set of triples of I such that for any subset J of I of size at most k the restriction of χ to the set of triples of J is the chirotope of a double pseudoline arrangement indexed by J. We denote by $\mathcal{C}_k$ the set of k-chirotopes.* □

DF 5. *A k-arrangement is an arrangement whose subarrangements of size at most k are of genus 1. We denote by $\mathcal{A}_k$ the set of k-arrangements and by $\mathcal{A}_k \to \mathcal{C}_k$ the map that assigns to a k-arrangement its chirotope.* □

TH 3. *[...] Then*

1. $\mathcal{A}_5 \to \mathcal{C}_5$ *is one-to-one and onto;*
2. $\mathcal{A}_2 \supsetneq \mathcal{A}_3 \supsetneq \mathcal{A}_4 = \mathcal{A}_5 = \mathcal{A}_6 = \ldots;$
3. $\mathcal{C}_3 \supsetneq \mathcal{C}_4 \supsetneq \mathcal{C}_5 = \mathcal{C}_6 = \mathcal{C}_7 = \cdots.$ □

A 4-CHIROTOPE THAT IS NOT A 5-CHIROTOPE



BIBLIOGRAPHY

References

[1] Jacob E. Goodman. Pseudoline arrangements. In Jacob E. Goodman and Joseph O'Rourke, editors, Handbook of Discrete and Computational Geometry, chapter 5, pages 97–128. Chapman & Hall/CRC, 2004.

[2] J. E. Goodman, R. Pollack, R. Wenger, and T. Zamfirescu. Arrangements and topological planes. Amer. Math. Monthly, 101(9):866–878, November 1994. http://www.jstor.org.ezproxy.mir.math.upmc.fr/stable/pdfplus/2975135.pdf

[3] Julien Ferté, Vincent Pilaud, and Michel Pocchiola. On the number of simple arrangements of five double pseudolines. Discrete Comput. Geom. 45 (2): 279-302, 2011. http://dx.doi.org/10.1007/s00454-010-9298-4.

[4] L. Habert et M. Pocchiola. Arrangements of double pseudolines. Submitted to Disc. Comput. Geom. (Preliminary version on SoCG'09) http://arxiv.org/abs/1101.1022.

[5] Jürgen Bokowski. Computational Oriented Matroids. Cambridge, 2006.

[6] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. Oriented Matroids. Cambridge University Press, 2nd edition, 1999.