Many neighborly polytopes and oriented matroids

Arnau Padrol
Universitat Politècnica de Catalunya

22 March 2012
TEOMATRO

Outline

1. Neighborly polytopes and oriented matroids
2. The number of polytopes
3. Balanced oriented matroids
4. Extending balanced oriented matroids
5. Counting
6. Non-realizable neighborly oriented matroids

Neighborly polytopes and oriented matroids

1 Neighborly polytopes and oriented matroids

2 The number of polytopes

3 Balanced oriented matroids

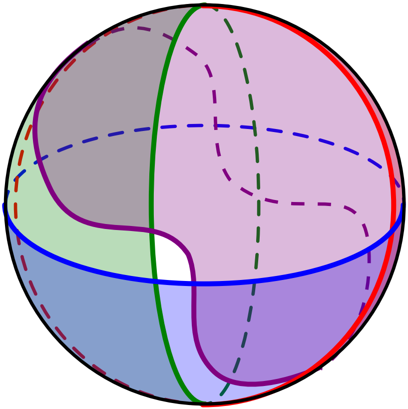
4 Extending balanced oriented matroids

5 Counting

6 Non-realizable neighborly oriented matroids

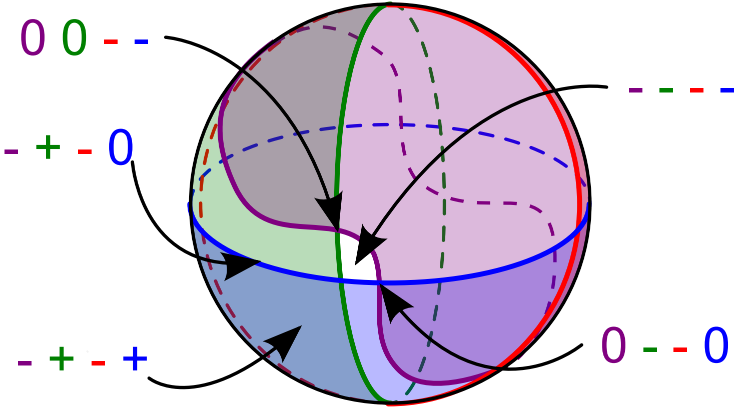
Oriented matroids as arrangements of pseudohemispheres

An arrangement of pseudohemispheres ...



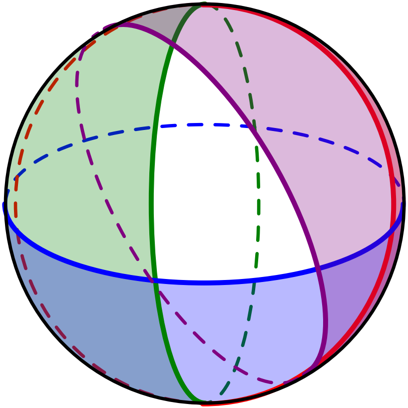
Oriented matroids as arrangements of pseudohemispheres

... some of its covectors...



Oriented matroids as arrangements of pseudohemispheres

... is realizable



Face lattices

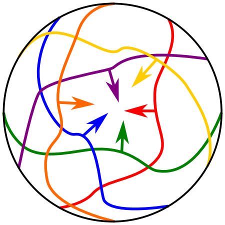
Acyclic Matroid $(+, +, \ldots, +)$ is a covector.

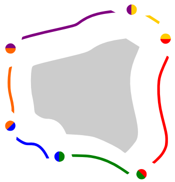
Face 0's of a non-negative covector.

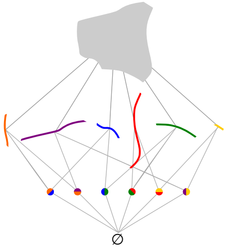
Vertex, Edge, Facet Face of dim 0, 1, $d - 1$.

Face lattice Partially ordered set of faces ($\subseteq$).

Polytope Realizable acyclic matroid.







Face lattices

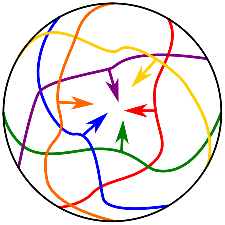
Acyclic Matroid $(+,+,\dots,+)$ is a covector.

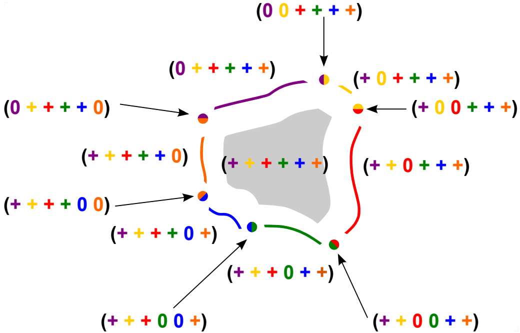
Face 0's of a non-negative covector.

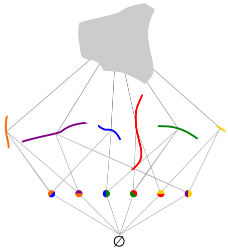
Vertex, Edge, Facet Face of dim 0, 1, $d-1$.

Face lattice Partially ordered set of faces ($\subseteq$).

Polytope Realizable acyclic matroid.



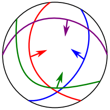




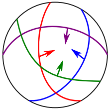
But what is a polytope? $P \simeq Q$?

A polytope "is" the face lattice of a realizable oriented matroid

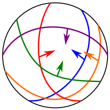
Combinatorially equivalent isomorphic face lattices.



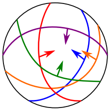
$=$



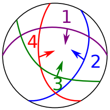
Oriented Matroid isomorphic oriented matroid.



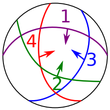
$\neq$



Labeled- same oriented matroids/face lattices.



$\neq$



Some natural questions

Question

How many faces can a (realizable) oriented matroid have?

Question

How many (realizable) oriented matroids are there?

Some natural questions

Question

How many faces can a (realizable) oriented matroid have?

Question

How many (realizable) oriented matroids are there?

Question

How many realizable oriented matroids with maximal number of faces are there?

Neighborly, polar-to-neighborly and cyclic

Definition (Neighborly oriented matroid)

A rank r oriented matroid is *neighborly* if for every $|F| \leq \frac{r-1}{2}$, $(\underbrace{0 \ldots 0}_{F} \underbrace{+ \ldots +}_{S \setminus F})$ is a cocircuit.

Definition (Neighborly arrangement)

An arrangement of pseudohemispheres on $\mathbb{S}^d$ is *neighborly* if every $\leq \frac{d}{2}$ spheres intersect in a face.

Definition ((Polar-to-)neighborly polytope)

A d-polytope P is *(polar-to-)neighborly* if every $\leq \frac{d}{2}$ facets intersect in a face.

Neighborly polytopes

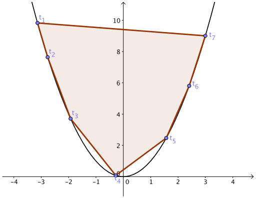
Cyclic polytopes are neighborly.

Definition (Cyclic polytopes $\mathcal{C}_d(n)$)

Moment curve $\gamma : t \mapsto (t, t^2, \ldots, t^d)$

Cyclic polytope $\mathcal{C}_d(n) = \operatorname{conv} \{\gamma(t_1), \ldots, \gamma(t_n)\}$

$$t_1 < t_2 < \cdots < t_n$$



Theorem (Upper bound theorem [McMullen '70] [Stanley '72])

The number of i-dimensional faces of an oriented matroid of rank r with n elements is maximal for neighborly matroids.

The number of polytopes

1 Neighborly polytopes and oriented matroids

2 The number of polytopes

3 Balanced oriented matroids

4 Extending balanced oriented matroids

5 Counting

6 Non-realizable neighborly oriented matroids

The number of neighborly polytopes

$$\mathrm{nb}(n,d)$$

$$\leq$$

$$\mathrm{p}(n,d)$$

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

$1 \leq$ $\text{nb}(n, d)$

$\leq$

$\text{p}(n, d)$

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

[Grünbaum 1967] first non-cyclic examples

$1 \leq k \leq$

$\mathrm{nb}(n, d)$

$\mathrm{|\wedge}$

$\mathrm{p}(n, d)$

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

[Grünbaum 1967] first non-cyclic examples

[Barnette 1981] facet splitting technique

$1 \leq k \leq 4^n \leq$

$\mathrm{nb}(n,d)$

$\text{|}\wedge$

$\mathrm{p}(n,d)$

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

[Grünbaum 1967] first non-cyclic examples

[Barnette 1981] facet splitting technique

[Shemer 1982] sewing construction

$1 \leq k \leq 4^n \leq n^{\frac{1}{2}n} \leq$ $\mathrm{nb}(n,d)$

$\text{|}\wedge$

$\mathrm{p}(n,d)$

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

[Grünbaum 1967] first non-cyclic examples

[Barnette 1981] facet splitting technique

[Shemer 1982] sewing construction

$$1 \;\le\; k \;\le\; 4^n \;\le\; n^{\frac{1}{2}n} \;\le\; \mathrm{nb}(n,d)$$

$$\mathrm{nb}(n,d) \;\le\; \mathrm{p}(n,d) \;\le\; n^{d^2 n}$$

[Goodman & Pollack, Alon 1986]

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

[Grünbaum 1967] first non-cyclic examples

[Barnette 1981] facet splitting technique

[Shemer 1982] sewing construction

$$1 \le k \le 4^n \le n^{\frac{1}{2}n} \le \mathrm{nb}(n,d)$$

$$\mathrm{nb}(n,d) \le \mathrm{p}(n,d)$$

$$n^{\frac{1}{4}dn} \le \mathrm{p}(n,d) \le n^{d^2 n}$$

[Goodman & Pollack, Alon 1986]

(approximate behaviors when $n >> d$)

The number of neighborly polytopes

[Motzkin 1957] conjecture

[Grünbaum 1967] first non-cyclic examples

[Barnette 1981] facet splitting technique

[Shemer 1982] sewing construction

[P. 2011]

$$1 \leq k \leq 4^n \leq n^{\frac{1}{2}n} \leq n^{\frac{1}{2}dn} \leq \mathrm{nb}(n,d)$$

$$|\wedge$$

$$n^{\frac{1}{4}dn} \leq \mathrm{p}(n,d) \leq n^{d^2 n}$$

[Goodman & Pollack, Alon 1986]

(approximate behaviors when $n >> d$)

Many neighborly polytopes

Theorem ([Shemer 1982])

Using sewing construction:

$$\mathrm{nb}(n, d) \geq 2^{c_d n \log n},$$

where $c_d \to \frac{1}{2}$ *when* $d \to \infty$.

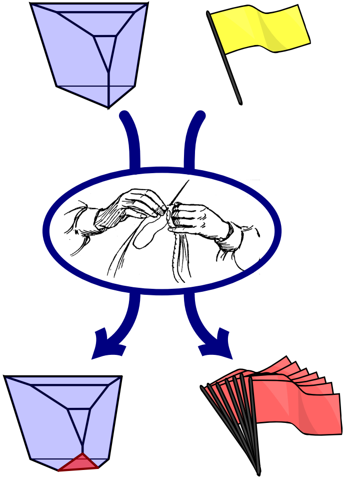
Many neighborly polytopes

Theorem ([Shemer 1982])

Using sewing construction:

$$\mathrm{nb}(n,d) \geq 2^{c_d n \log n},$$

where $c_d \to \frac{1}{2}$ *when* $d \to \infty$.

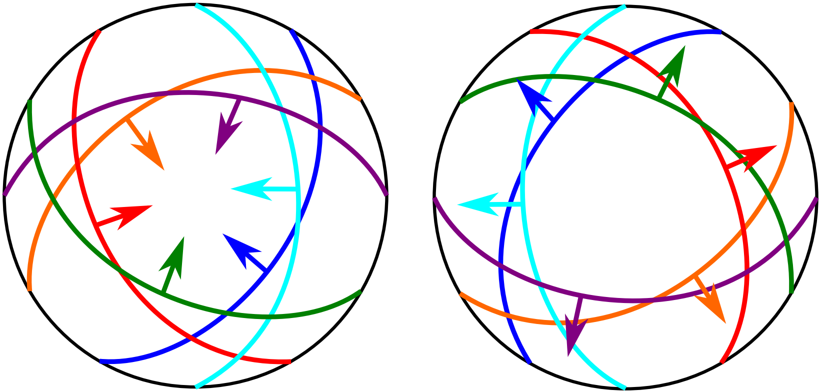


Balanced oriented matroids

1. Neighborly polytopes and oriented matroids
2. The number of polytopes
3. Balanced oriented matroids
4. Extending balanced oriented matroids
5. Counting
6. Non-realizable neighborly oriented matroids

Vectors

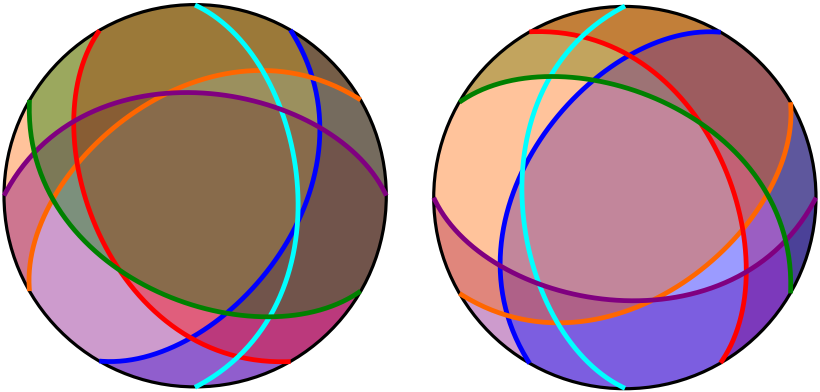
A *vector* (*circuit*) is a (minimal) covering of $\mathbb{S}^d$.



Vectors

A *vector* (*circuit*) is a (minimal) covering of $\mathbb{S}^d$.

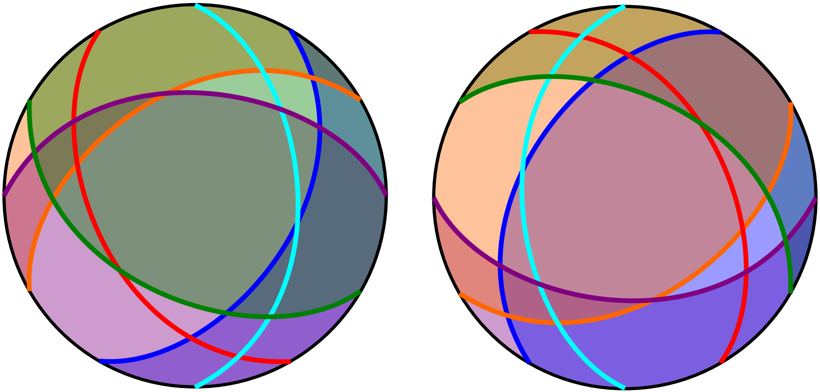
+ - + + - 0



Vectors

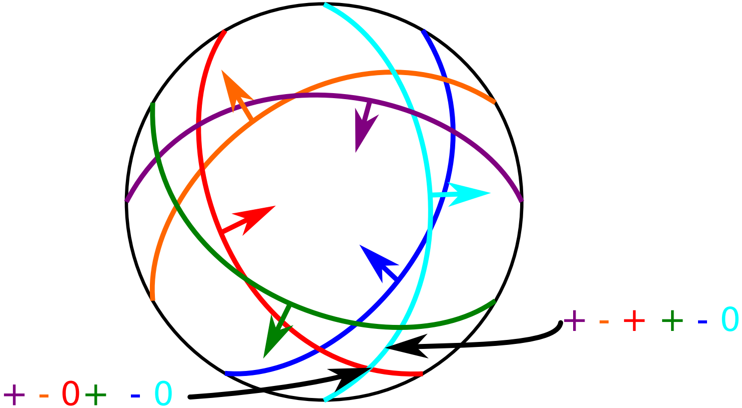
A *vector* (*circuit*) is a (minimal) covering of $\mathbb{S}^d$.

$+ \ - \ \mathbf{0} \ + \ - \ \mathbf{0}$



Duality in oriented matroids

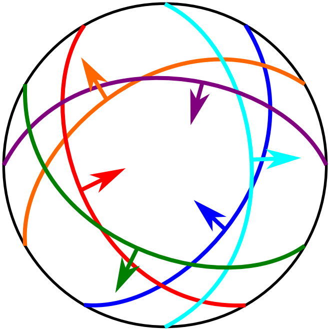
circuits of $\mathcal{M}$	$\Leftrightarrow$	cocircuits of $\mathcal{M}^*$
cocircuits of $\mathcal{M}$	$\Leftrightarrow$	circuits of $\mathcal{M}^*$
n elements		n elements
rank d		rank $n-d$



Balanced Matroids

A uniform oriented matroid is *balanced* if for every cocircuit

$$|C^+| = |C^-| \quad (\pm 1).$$



Neighborly and balanced oriented matroids

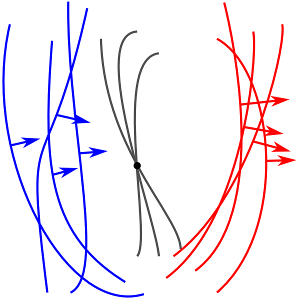
Theorem ([Sturmfels'88])

$\mathcal{M}$ is uniform and neighborly $\Leftrightarrow$ $\mathcal{M}^*$ is uniform and balanced.

$$\left(\begin{array}{c} |F| \le \left\lfloor \frac{r-1}{2} \right\rfloor \\ \Rightarrow \\ (\underbrace{0\ldots0}_{F}\underbrace{+\ldots+}_{S\setminus F}) \text{ cocircuit } \mathcal{M} \end{array}\right)$$

$$\Updownarrow$$

$$\left(\begin{array}{c} |R| \ge \left\lceil \frac{n+d'}{2} \right\rceil \\ \Rightarrow \\ R \text{ covers } \mathbb{S}^{d'} \end{array}\right)$$



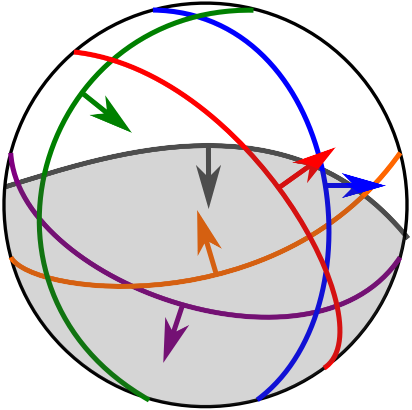
$$(\underbrace{+\ldots+}_{C^+}\underbrace{0\ldots0}_{C^0}\underbrace{-\ldots-}_{C^-})$$

Extending balanced oriented matroids

1. Neighborly polytopes and oriented matroids
2. The number of polytopes
3. Balanced oriented matroids
4. Extending balanced oriented matroids
5. Counting
6. Non-realizable neighborly oriented matroids

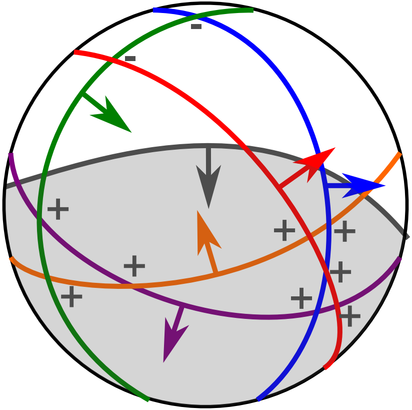
Signature

The matroid $\tilde{\mathcal{M}} = \mathcal{M} \cup p$ is determined by the sign of p in the cocircuits of $\mathcal{M}$: the *signature*.



Signature

The matroid $\tilde{\mathcal{M}} = \mathcal{M} \cup p$ is determined by the sign of p in the cocircuits of $\mathcal{M}$: the *signature*.



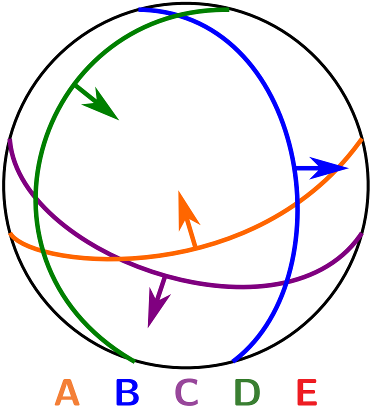
Lexicographic extensions

p is a *lexicographic extension* on $[a_1^{s_1}, \ldots, a_d^{s_d}]$, $a_i \in \mathcal{M}$, $s_j = \pm 1$

If for every cocircuit C

- If $C_{a_1} \neq 0$, then $C_p = s_1 C a_1$;
- if $C_{a_2} \neq 0$, then $C_p = s_2 C a_2$;
- $\ldots$
- if $C_{a_d} \neq 0$, then $C_p = s_d C a_d$.

$\mathbf{E} = [\mathbf{B}^+, \mathbf{A}^+, \mathbf{C}^-]$



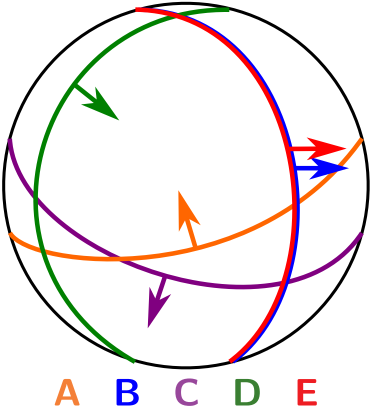
Lexicographic extensions

p is a *lexicographic extension* on $[a_1^{s_1}, \ldots, a_d^{s_d}]$, $a_i \in \mathcal{M}$, $s_j = \pm 1$

If for every cocircuit C

- If $C_{a_1} \neq 0$, then $C_p = s_1 C a_1$;
- if $C_{a_2} \neq 0$, then $C_p = s_2 C a_2$;
- ...
- if $C_{a_d} \neq 0$, then $C_p = s_d C a_d$.

$\mathbf{E} = [\mathbf{B}^+, \mathbf{A}^+, \mathbf{C}^-]$

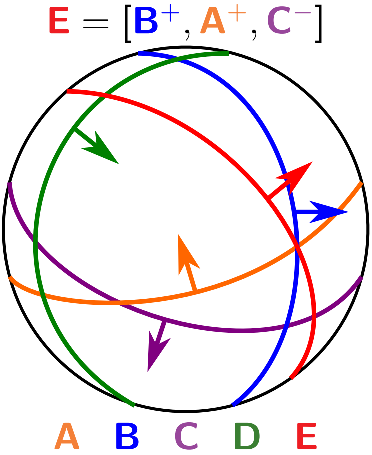


Lexicographic extensions

p is a *lexicographic extension* on $[a_1^{s_1}, \ldots, a_d^{s_d}]$, $a_i \in \mathcal{M}$, $s_j = \pm 1$

If for every cocircuit C

- If $C_{a_1} \neq 0$, then $C_p = s_1 C a_1$;
- if $C_{a_2} \neq 0$, then $C_p = s_2 C a_2$;
- ...
- if $C_{a_d} \neq 0$, then $C_p = s_d C a_d$.



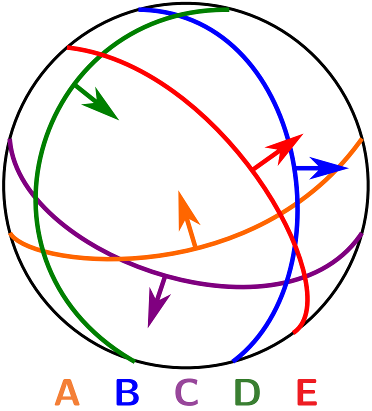
Lexicographic extensions

p is a *lexicographic extension* on $[a_1^{s_1}, \ldots, a_d^{s_d}]$, $a_i \in \mathcal{M}$, $s_j = \pm 1$

If for every cocircuit C

- If $C_{a_1} \neq 0$, then $C_p = s_1 C a_1$;
- if $C_{a_2} \neq 0$, then $C_p = s_2 C a_2$;
- $\ldots$
- if $C_{a_d} \neq 0$, then $C_p = s_d C a_d$.

$\mathbf{E} = [\mathbf{B}^+, \mathbf{A}^+, \mathbf{C}^-]$



Lexicographic extensions

p is a *lexicographic extension* on $[a_1^{s_1}, \ldots, a_d^{s_d}]$, $a_i \in \mathcal{M}$, $s_j = \pm 1$

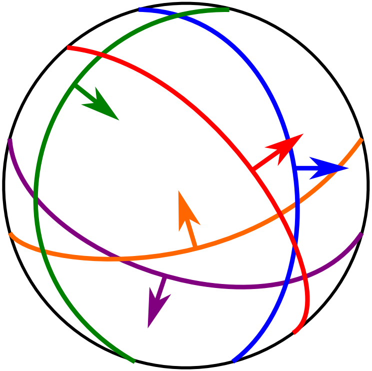
If for every cocircuit C

- If $C_{a_1} \neq 0$, then $C_p = s_1 C a_1$;
- if $C_{a_2} \neq 0$, then $C_p = s_2 C a_2$;
- ...
- if $C_{a_d} \neq 0$, then $C_p = s_d C a_d$.

Any lexicographic extension of a realizable matroid is realizable:

$$p = s_1 a_1 + \varepsilon s_2 a_2 + \varepsilon^2 s_3 a_3 + \cdots + \varepsilon^{d-1} s_d a_d.$$

$\mathbf{E} = [\mathbf{B}^+, \mathbf{A}^+, \mathbf{C}^-]$



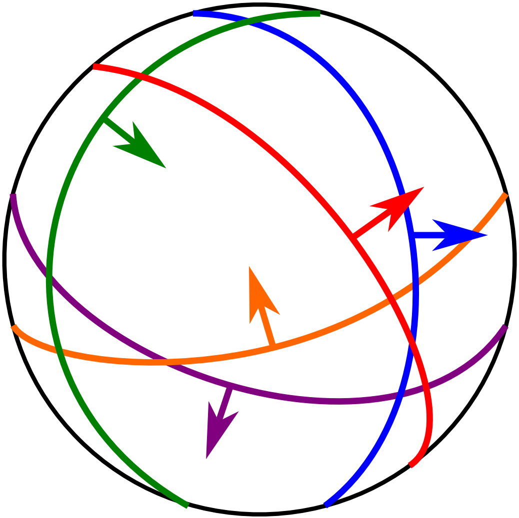
Gale sewing

Theorem (P. 2011)

$$\left.\begin{array}{l} \bullet\ \mathcal{M} \text{ balanced} \\ \bullet\ p = [a_1^{s_1}, a_2^{s_2}, \dots, a_d^{s_d}] \\ \bullet\ q = [p^-, a_1^-, \dots, a_{d-1}^-] \end{array}\right\} \text{Then } \mathcal{M} \cup p \cup q \text{ is balanced.}$$

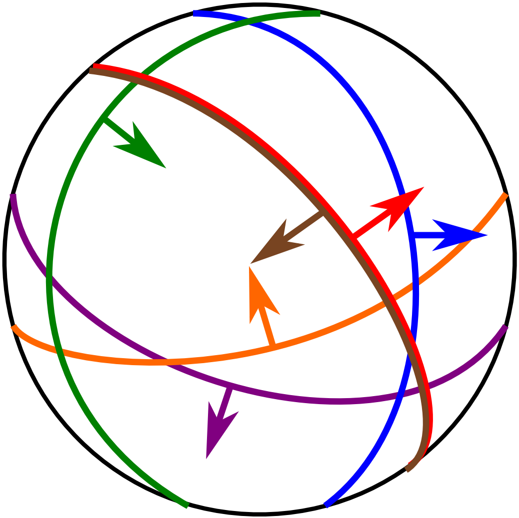
“Proof”: $E = [B^+, A^+, C^-]$, $F = [E^-, B^-, A^-]$

A
B
C
D
E
F



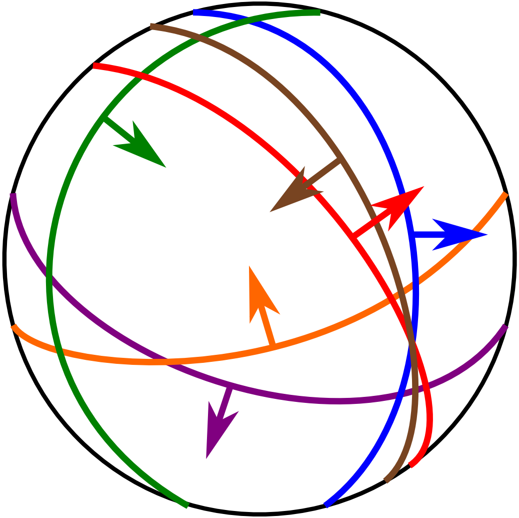
"Proof": $E = [B^+, A^+, C^-]$, $F = [E^-, B^-, A^-]$

A
B
C
D
E
F

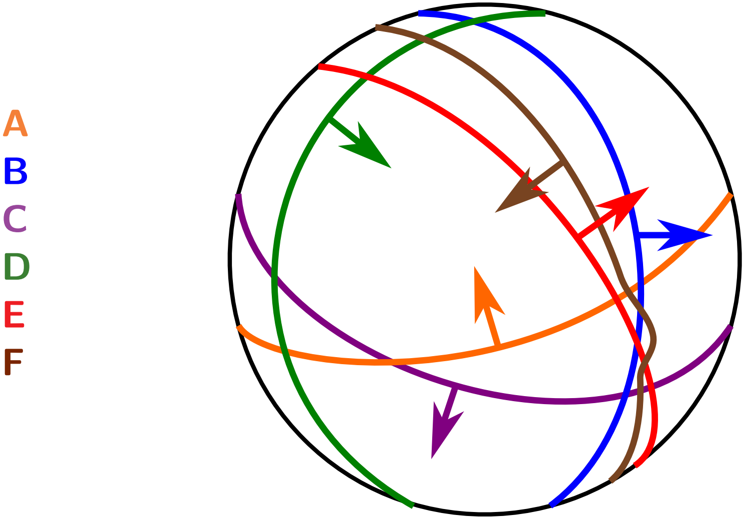


“Proof”: $E = [B^+, A^+, C^-]$, $F = [E^-, B^-, A^-]$

A
B
C
D
E
F



"Proof": $E = [B^+, A^+, C^-]$, $F = [E^-, B^-, A^-]$



Counting

1 Neighborly polytopes and oriented matroids

2 The number of polytopes

3 Balanced oriented matroids

4 Extending balanced oriented matroids

5 Counting

6 Non-realizable neighborly oriented matroids

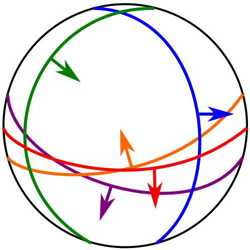
The number of (labeled) lexicographic extensions

S balanced *oriented matroid*, rank r, $n = r + 1 + 2m$ elements

Lemma

The number of labeled lexicographic extensions of S

$$\ell(n, r) \geq 2 \cdot \frac{n!}{(n-r+1)!}.$$



The number of (labeled) lexicographic extensions

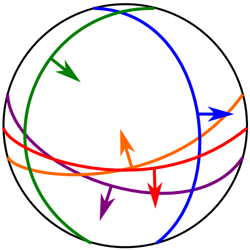
S balanced *oriented matroid*, rank r, $n = r + 1 + 2m$ elements

Lemma

The number of labeled lexicographic extensions of S

$$\ell(n, r) \geq 2 \cdot \frac{n!}{(n - r + 1)!}.$$

Trick: count only $[a_1^+, a_2^+, \ldots, a_r^+]$.



The number of (labeled) balanced configurations

Lemma

The number of labeled balanced configurations of rank r with $n = r + 1 + 2m$ elements

$$\mathrm{nb}(n, 2m) \geq \frac{n-1}{2}\,\ell(n-2, r)\,\mathrm{nb}(n-2, 2m-2).$$

The number of (labeled) balanced configurations

Lemma

The number of labeled balanced configurations of rank r with $n = r + 1 + 2m$ elements

$$\mathrm{nb}(n, 2m) \geq \frac{n-1}{2}\,\ell(n-2, r)\ \mathrm{nb}(n-2, 2m-2).$$

Corollary

$$\mathrm{nb}(r+1+2m, 2m) \geq \prod_{i=1}^{m} \frac{(r+2i)!}{(2i)!}.$$

The number of (labeled) neighborly polytopes

Trick: Neighborly oriented matroids are rigid: different oriented matroids have different face lattices. **[Shemer'82, Sturmfels'88]**

The number of (labeled) neighborly polytopes

Trick: Neighborly oriented matroids are rigid: different oriented matroids have different face lattices. **[Shemer'82, Sturmfels'88]**

Trick 2:

$$\ln\left(\mathrm{nb}(r+1+2m,2m)\right) \geq \ln\left(\prod_{i=1}^{m}\frac{(r+2i)!}{(2i)!}\right) = \sum_{i=1}^{m}\sum_{j=0}^{r-1}\ln\left(r+2i-j\right)$$

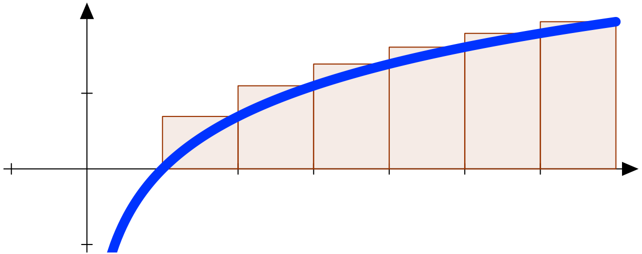
The number of (labeled) neighborly polytopes

Trick: Neighborly oriented matroids are rigid: different oriented matroids have different face lattices. **[Shemer'82, Sturmfels'88]**

Trick 2:

$$\ln\left(\mathrm{nb}(r+1+2m,2m)\right) \geq \ln\left(\prod_{i=1}^{m}\frac{(r+2i)!}{(2i)!}\right) = \sum_{i=1}^{m}\sum_{j=0}^{r-1}\ln\left(r+2i-j\right)$$

Trick 3:



The number of (labeled) neighborly polytopes

Theorem (P. 2011)

The number of labeled neighborly polytopes in dimension d with n vertices fulfills

$$\mathrm{nb}(r+d+1,d) \geq \frac{(r+d)^{\left(\frac{r}{2}+\frac{d}{2}\right)^2}}{r^{\left(\frac{r}{2}\right)^2} d^{\left(\frac{d}{2}\right)^2} \mathrm{e}^{3\frac{r}{2}\frac{d}{2}}},$$

that is,

$$\mathrm{nb}(n,d) \geq \frac{(n-1)^{\left(\frac{n-1}{2}\right)^2}}{(n-d-1)^{\left(\frac{n-d-1}{2}\right)^2} d^{\left(\frac{d}{2}\right)^2} \mathrm{e}^{\frac{3d(n-d-1)}{4}}}.$$

The number of (labeled) neighborly polytopes

Corollary

If $n > 2d$, the number of labeled neighborly polytopes in dimension d with n vertices fulfills

$$\mathrm{nb}(n,d) \geq \left(\frac{n-1}{\sqrt{\mathrm{e}^3}}\right)^{\frac{d(n-1)}{2}}.$$

Corollary

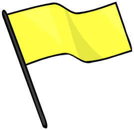
If $n < 2d$, the number of labeled neighborly polytopes in dimension d with n vertices fulfills

$$\mathrm{nb}(n,d) \geq \left(\frac{n-1}{\sqrt{\mathrm{e}^3}}\right)^{\frac{(n-d-1)(n-1)}{2}}.$$

Non-realizable neighborly oriented matroids

1. Neighborly polytopes and oriented matroids
2. The number of polytopes
3. Balanced oriented matroids
4. Extending balanced oriented matroids
5. Counting
6. Non-realizable neighborly oriented matroids

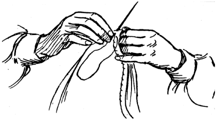
The sewing construction revisited



Flag Strictly increasing sequence of faces $T_1 \subset T_2 \subset \cdots \subset T_k$

Universal flag flag where each T_j is a universal face with $2j$ vertices.

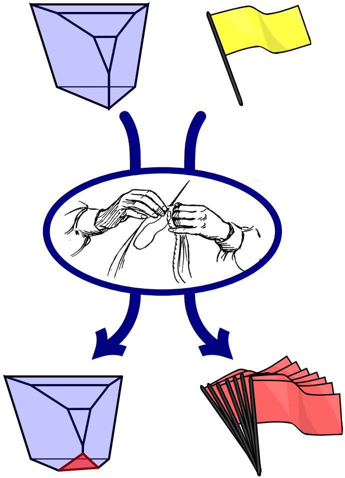
Universal face F is universal if the contraction P/F is neighborly.



Sewing Sewing on the flag $\mathcal{T} = A_1 \subset A_1 \cup A_2 \subset A_1 \cup A_2 \cup A_3 \subset \ldots$ is the lexicographic extension

$$[\mathcal{T}] = [A_1^+, A_2^-, A_3^+, \ldots]$$

The extended sewing theorem



Extended sewing [P. 2011]

If

- $\mathcal{M}$ *neighborly* oriented matroid
- $\mathcal{T} = \{T_j\}_{j=1}^m$ contains a *universal flag*
- $\tilde{\mathcal{M}} = \mathcal{M}[\mathcal{T}]$: *sewing* through $\mathcal{T}$

Then

- $\tilde{\mathcal{M}}$ *neighborly*

and,

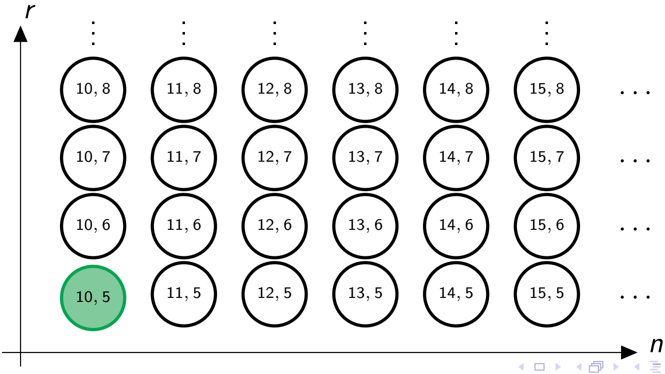
- $\tilde{\mathcal{T}} = \{\tilde{T}_j\}_{j=1}^m$ *universal flag*

Many non-realizable neighborly oriented matroids

Theorem ([Altshuler'77],[Bokowski, Garms'87])

The neighborly oriented matroid M^{10}_{425} of rank 5 with 10 elements is not realizable.

Non-realizable neighborly matroids of rank r with n elements

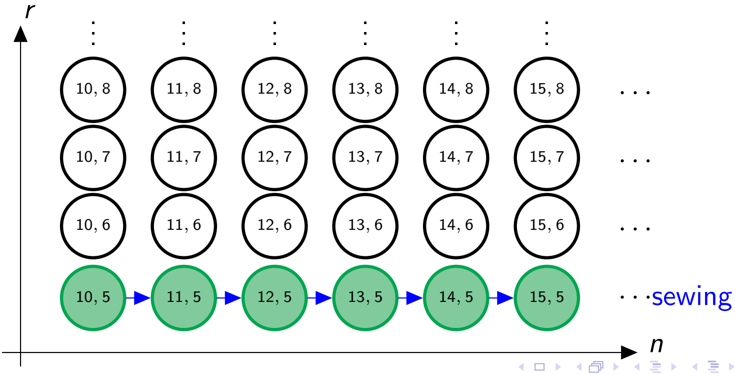


Many non-realizable neighborly oriented matroids

Theorem ([Altshuler'77],[Bokowski, Garms'87])

The neighborly oriented matroid M^{10}_{425} of rank 5 with 10 elements is not realizable. M^{10}_{425} has a universal flag.

Non-realizable neighborly matroids of rank r with n elements

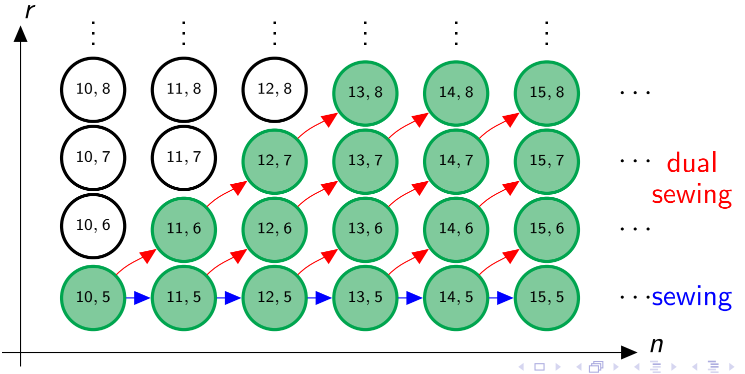


Many non-realizable neighborly oriented matroids

Theorem ([Altshuler'77],[Bokowski, Garms'87])

The neighborly oriented matroid M^{10}_{425} of rank 5 with 10 elements is not realizable. M^{10}_{425} has a universal flag.

Non-realizable neighborly matroids of rank r with n elements

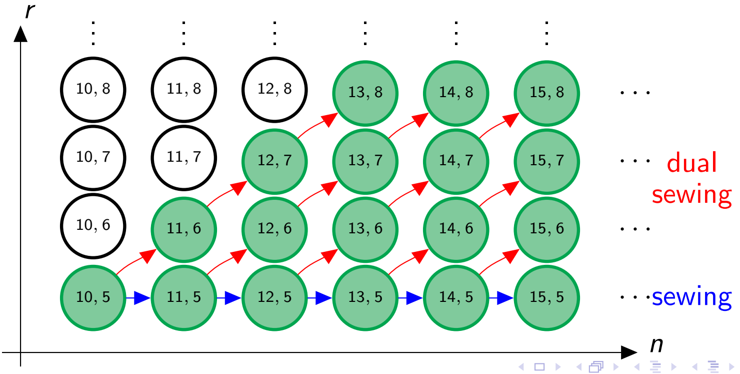


Many non-realizable neighborly oriented matroids

Theorem ([P. 2011])

There are non-realizable neighborly oriented matroids of rank r with n elements for any $r \geq 5$ and $n \geq r + 5$.

Non-realizable neighborly matroids of rank r with n elements



A conclusion: primal and dual lexicographic extensions

[Motzkin 1957]

[Grünbaum 1967]

[Barnette 1981] *lexicographic extensions*

[Shemer 1982] *lexicographic extensions*

[P. 2011] *lexicographic extensions*

$$1 \leq k \leq 4^n \leq n^{\frac{1}{2}n} \leq n^{\frac{1}{2}dn} \leq \mathrm{nb}(n,d)$$

$$\rotatebox{90}{$\leq$}$$

$$n^{\frac{1}{4}dn} \leq \mathrm{p}(n,d) \leq n^{d^2 n}$$

[Goodman & Pollack, Alon 1986] *lexicographic extensions*

That's all!

Merci Beaucoup!