INTEGRAL POINTS IN RATIONAL POLYGONS: A NUMERICAL SEMIGROUP APPROACH

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Abstract. In this paper we use numerical semigroups (specifically, those of dimension 2) to give an easy formula for the number of integral points inside a right–angled triangle with rational vertices. This is the basic case for computing the number of integral points inside a rational (not necessarily convex) polygon.

1. Introduction: From polygons to triangles

Throughout this paper we will call a point $P \in \mathbb{R}^2$ *integral* if its coordinates lie in $\mathbb{Z}^2$, and similarly P will be called *rational* if $P \in \mathbb{Q}^2$.

The problem of computing the set of integral points inside plane bodies has a long and rich story. A milestone in this story is Pick's Theorem [10], from the late years of the 19th century, stating that, if S is a polygon such that all of its vertices are integral, and $\mathrm{int}(S)$ and ∂S are, respectively, its interior and its boundary, let

$$\begin{aligned} A(S) &= \text{the area of } S \\ I(S) &= \#\left(\mathbb{Z}^2 \cap \mathrm{int}(S)\right) \\ B(S) &= \#\left(\mathbb{Z}^2 \cap \partial S\right) \end{aligned}$$

And then

$$A(S) = I(S) + \frac{B(S)}{2} - 1.$$

Our aim in this paper is to give a result (in some sense in the spirit of this theorem) to compute the number of integral points of polygons (not necessarily convex) defined by rational vertices.

The first important point here is that, in order to compute the number of integral points of such a polygon, it suffices with two kinds of bodies: rectangles and right triangles of a particular type. Of course, one may argue that rectangles can be easily turned into right triangles, but as for counting points is concerned, our rectangles will be enough and more appropriate.

We will call this reduction process rectangulation (not in a retaliation sense).

From now on, a rectangle whose sides are parallel to the coordinate axes will be called a *stable* rectangle. Similarly, a right triangle whose orthogonal sides are parallel to the coordinate axes will be called a *stable* right triangle.

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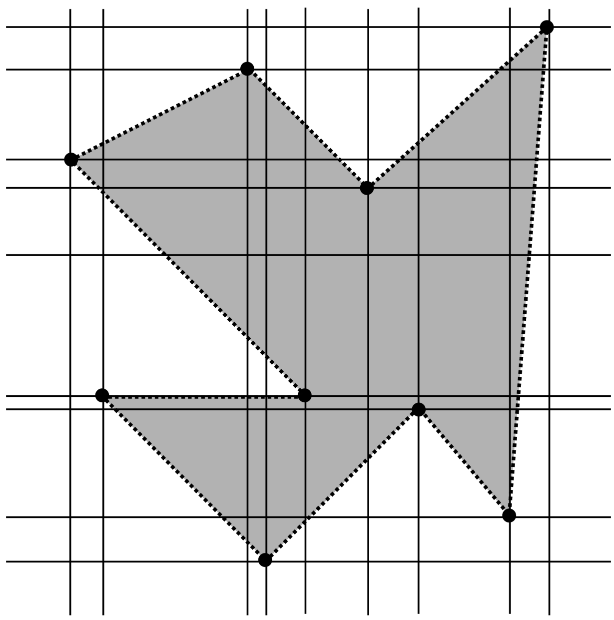
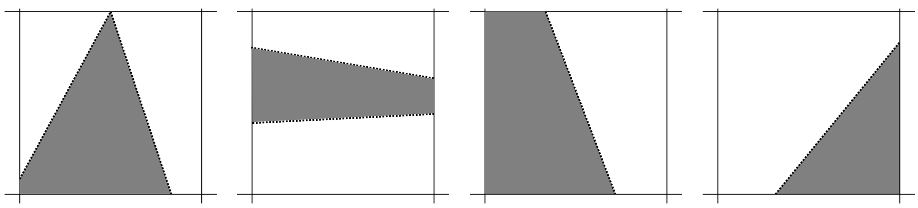
1.1. **Rectangulation Step 1: The tangram.** Assume we are given a polygon S with rational coefficients $\{P_1, ..., P_r\}$. For each P_i let us draw two lines, parallel to the coordinate axes.

For each intersection between one of these lines and ∂S we add a *virtual* vertex. The point is that this process gets S divided into convex sets, say $C_1, ..., C_t$, inside stable rectangles $R_1, ..., R_t$ (pretty much like in a tangram puzzle). The vertices of these convex sets are vertices of ∂S (either real o virtual ones).



Mind that these convex sets might as well be empty.

Now, inside every stable rectangle R_i the complementary set of each C_i is a union of (at most 4) stable right triangles and (at most 2) stable rectangles. Some examples are:



Computing the number of integral points inside a stable rectangle is very easy.

Lemma 1. *Let $\alpha_1 < \beta_1,\ \alpha_2 < \beta_2$ be rationals. Let $R \subset \mathbb{R}^2$ be the stable rectangle with vertices (α_1, α_2) and (β_1, β_2). Then*

$$\#\left(R \cap \mathbb{Z}^2\right) = \left(\lfloor \beta_1 \rfloor - \lceil \alpha_1 \rceil + 1\right)\left(\lfloor \beta_2 \rfloor - \lceil \alpha_2 \rceil + 1\right)$$

And so it is, if necessary, counting the number of points on a precise side of it. This leaves us with the problem of counting the number of integral points inside a stable right rectangle.

Eventually we will also have to keep track of the number of points in the hypothenuse of the triangle, should these points be removed from the counting.

But note that, if our polygon is a convex one, other than an obtuse triangle, the tangram gives us a union of stable sets (either rectangles or right triangles). In this case it will be much faster to compute directly the number of integral points inside each piece of the tangram, instead of the complementary sets.

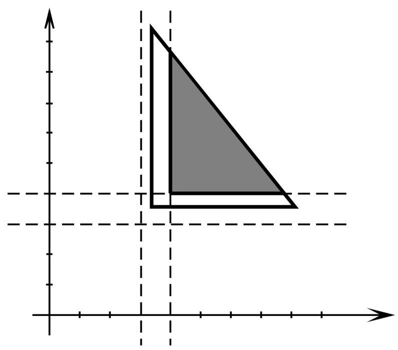
1.2. **Rectangulation Step 2: Adjusting the triangles.** So now our original problem is reduced to that of counting the integral points inside a triangle T defined by rational vertices:

$$A = (\alpha, \beta), \quad B = (\alpha, \gamma), \quad C = (\delta, \beta),$$

and we can assume, up to symmetry, $\alpha < \gamma$, $\beta < \delta$. Furthermore, it is clear that, as for counting integral points is concerned, we can substitute

$$\alpha \longmapsto \lceil \alpha \rceil, \qquad \beta \longmapsto \lceil \beta \rceil,$$

and the number of integral points does not change by traslations of integral vectors, hence we can in fact assume $A = (0, 0)$.



Instead of giving coordinates to our new B and C we will write the hypothenuse of T as

$$ax + by = c,$$

where we can assume $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$. That is, $B = (0, c/b)$, $C = (c/a, 0)$. This form is not still optimal for our purposes.

We would like to assume $\gcd(a, b) = 1$. In order to do that, mind that there are two possibilities:

(1) $\gcd(a, b) = 1$. No further actions required.

(2) $\gcd(a, b) = d > 1$. As $\gcd(d, c) = 1$ clearly there are no integral points in the hypothenuse, as any such point (x, y) must verify $ax + by \in \mathbb{Z}d$. Therefore the number of integral points in T is the same if we replace T by the *closest* stable right triangle to T with at least some chance to have one integral point in the hypothenuse. This is the triangle T' defined by

$$ax + by = c',$$

where $c' = \lfloor c/d \rfloor d$. And therefore $\#(T \cap \mathbb{Z}^2)$ can be computed counting the number of integral points in the stable triangle defined by

$$\frac{a}{d}x + \frac{b}{d}y = \frac{c'}{d} = \left\lfloor \frac{c}{d} \right\rfloor.$$

The main result of this paper is then concerned with the previous situation.

Theorem 2. *Let $a < b$ be coprime integers, $c \in \mathbb{Z}$. Consider the following set:*

$$T = \left\{ (y_1, y_2) \in \mathbb{Z}^2_{\geq 0} \mid ay_1 + by_2 \leq c \right\}.$$

Then

$$\#T = -\frac{ab}{2}k^2 + \frac{a+b+1+2c}{2}k + \sum_{i=0}^{\left\lfloor \frac{c-kab}{b} \right\rfloor} \left(\left\lfloor \frac{c-kab-ib}{a} \right\rfloor + 1 \right)$$

where $k = \lfloor c/(ab) \rfloor$.

Before going down to prove this result, we will have a quick look at the history of this problem (and other related ones) and also a short review of the numerical semigroup tools we will need for the proof.

2. A little bit of history

The question of counting lattice (in particular integral) points inside a right triangle has a long and interesting story. As early as 1922 Hardy and Littlewood [5] studied the problem of right triangles defined by the coordinate axes and a hypotenuse with irrational slope.

In the following years, the interest for the subject did not decline. See, for instance [13], or also [3] where the so–called Ehrhart quasi–polynomials appear for the first time, an almost ubiquous tool nowadays. But in recent times, impressive advances in computational combinatorics and the ever–increasing amount of applications to other branches of mathematics have made lattice–point counting a fruitful and dynamic research field.

A very good and comprehensive introduction to the subject, with a good share of deep results is [2] where, in particular, you can find a formula to compute the number of integral points inside a right triangle (Theorem 2.10). The formula is quite different from ours (in particular, it uses either n–th primitive roots of unity or Fourier–Dedekind sums).

We may mention here an interesting generalization of our problem. Let us call a *right* tetrahedron the convex set of $\mathbb{R}^n_{\geq 0}$ limited by the coordinate hyperplanes and a hyperplane $a_1x_1 + ... + a_nx_n = 1$, with $a_i \in \mathbb{R}$. The question of counting (more precisely, bounding) the number of points in n–dimensional right tetrahedra has been a subject of study of S.S.T. Yau and some of his collaborators [6, 7, 8, 15, 16, 17, 18], a research that produced the so–called GLY conjecture (named after its creators, A. Granville, K.P. Lin and S.S.T. Yau).

GLY Conjecture.– Assume $n \geq 3$ and let $a_1 \geq ... \geq a_n \geq 1$ be real numbers. we define

$$P(a_1, ..., a_n) = \left\{ (x_1, ..., x_n) \in \mathbb{Z}^n_{>0} \mid \frac{x_1}{a_1} + ... + \frac{x_n}{a_n} \leq 1 \right\}.$$

Then:

- (Weak estimate) We have

$$n! \cdot \#P(a_1, ..., a_n) \leq (a_1 - 1)...(a_n - 1),$$

  with equality if and only if $a_n = 1$.

- (Strong estimate) Given n, there is a constant $C(n)$ such that, for $a_n \geq C(n)$ we have

$$n! \cdot \#P(a_1, ..., a_n) \leq A_n^n + (-1)\frac{S_1^{n-1}}{n} A_{n-1}^n + \sum_{l=2}^{n-1} (-1)^l \frac{S_l^{n-1}}{\binom{n-1}{l-1}} A_{n-l}^{n-1},$$

where S_l^{n-1} are the Stirling numbers, and A_i^l are polynomials in $a_1, ..., a_l$ with degree i.

The weak version was finally proved by Yau and Zhang [19]. In the same paper, the authors claim the strong version has been checked computationally up to $n = 10$. The fact is the conjecture might be checked for a particular n, but the state–of–the–art has not changed since. According to the authors, the case $n = 10$ took weeks to be completed.

This result came handy to the first and third author in [9]. Please note that the GLY conjecture only applies to $n \geq 3$.

3. An interlude on numerical semigroups

This paper relies on numerical monoids (or semigroups) as a fundamental tool. A numerical monoid is a very special kind of semigroup that can be thought of as a set

$$\langle\, a_1, ..., a_k \,\rangle = \{\lambda_1 a_1 + ... + \lambda_k a_k \mid \lambda_i \in \mathbb{Z}_{\geq 0}\}\,, \text{ with } \gcd(a_1, ..., a_k) = 1.$$

This object has been thoroughly studied in the last years, when a significant number of problems concerning the description of these semigroups and some of its more interesting invariants have been tackled. Unless otherwise stated, all proofs which are not included can be found in [4, 12].

Given a numerical monoid S, there are some invariants which will be of interest for us. The most relevant will be the set of gaps, noted $G(S)$, and defined by

$$G(S) = \mathbb{Z}_{\geq 0} \setminus S,$$

which is a finite set. Its cardinal will be noted $g(S)$ and its maximum $f(S)$, the so–called Frobenius number.

The Apéry set of S with respect to an element $a \in S$ can be defined as

$$Ap(S, a) = \{0, w_0, ..., w_{s-1}\}$$

where w_i is the smallest element in S congruent with i modulo s.

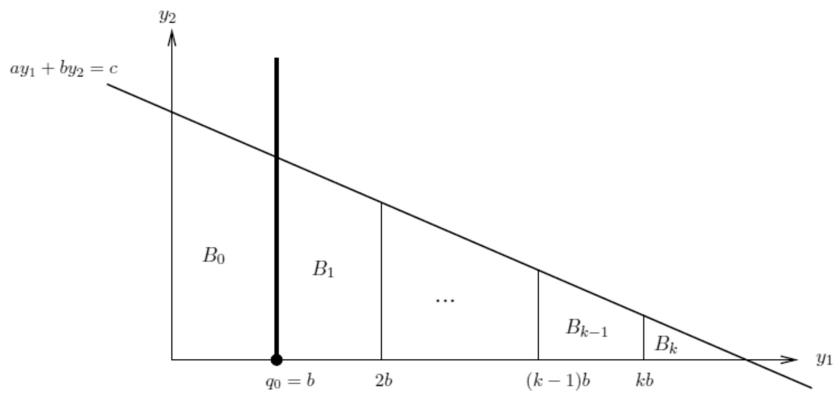
Example.– Let $S = \langle 7, 9, 11, 15\rangle$. Some of the Apéry sets associated to its generators are:

$$Ap(S, 7) = \{0,\ 9,\ 11,\ 15,\ 20,\ 24,\ 26\}$$
$$Ap(S, 15) = \{0,\ 7,\ 9,\ 11,\ 14,\ 16,\ 18,\ 20,\ 21,\ 23,\ 25,\ 27,\ 28,\ 32,\ 34\}$$

In particular, for monoids with two generators, the invariants $g(S)$ and $f(S)$ and the relevant Apéry sets are fully determined.

Lemma 3. *Let $S = \langle\, a_1,\ a_2 \,\rangle$. Then*

$$\begin{aligned} g(S) &= \frac{1}{2}(a_1 - 1)(a_2 - 1) \\ f(S) &= (a_1 - 1)(a_2 - 1) - 1 \\ Ap(S, a_i) &= \{0,\ a_j,\ 2a_j, ...,\ (a_i - 1)a_j\} \end{aligned}$$



4. The number of integral points inside a right triangle

Let us consider then a stable right triangle determined by the positive coordinate axes and the line

$$ax + by = c, \quad a, b, c \in \mathbb{Z} \text{ and } \gcd(a, b) = 1,$$

where we will assume $a < b$, with no loss of generality.

Take the set:

$$T = \left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c\right\}$$

and let us define the numerical monoid associated to our triangle as $S = \langle a,\ b\rangle$. S therefore verifies that its Frobenius number is $f(S) = ab - (a + b)$.

Let us perform the following partition on our set T:

$$\begin{array}{rcl} B_0 & = & \left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c,\ \ 0 \leq x < b\right\} \\ B_1 & = & \left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c,\ \ b \leq x < 2b\right\} \\ \vdots & & \vdots \\ B_i & = & \left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c,\ \ ib \leq x < (i+1)b\right\} \\ \vdots & & \vdots \\ B_k & = & \left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c,\ \ kb \leq x\right\} \end{array}$$

where $k = \lfloor c/(ab) \rfloor$.

As our aim is to find $\#T$, and it is plain that:

$$\#T = \#B_0 + \#B_1 + ... + \#B_{k-1} + \#B_k,$$

we can reduce our problem to that of finding $\#B_i$, for $i = 0, ..., k$.

Lemma 4. *Under the previous assumptions, if $k > 0$,*

$$\#B_0 = \frac{a + b - ab + 1}{2} + c$$

Proof. We will actually show that

$$S \cap [0, c] \overset{1:1}{\longleftrightarrow} B_0.$$

Given a pair $(x, y) \in B_0$ we quickly have an associated element in $S \cap [0, c]$, mainly $n = ax + by$.

In the same way, given $n \in S \cap [0, c]$ it is clear that we must have a representation $n = ax + by$ and we can in fact assume $0 \leq x < b$ (if otherwise, we can move part of ax into the by summand until $x < b$).

Let us assume that we have such a representation (that is, with $0 \leq x < b$) and we will prove that then the pair (x, y) must be unique, which will establish the bijection. Should we have

$$n = ax_0 + by_0 = ax_1 + by_1, \qquad \text{with } 0 < x_0, x_1 < b,$$

we must have

$$a(x_0 - x_1) = b(y_1 - y_0)$$

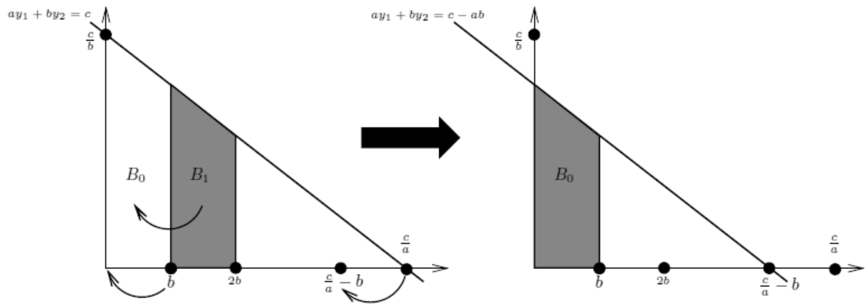
and, as $\gcd(a, b) = 1$, this means $b|(x_1 - x_0)$, which in turn implies $x_0 = x_1$.

Note that $k > 0$ is equivalent to $c \geq ab$, which also yields $c > f(S)$. Therefore, after Lemma 3,

$$\begin{aligned} \#B_0 = \#\big(S \cap [0, c]\big) &= \#\Big(S \cap [0, f(S)]\Big) + c - f(S) \\ &= \frac{ab - (a+b) + 1}{2} + c - \big(ab - (a+b)\big) \\ &= \frac{a + b - ab + 1}{2} + c. \end{aligned}$$

□

Simple as it is, this case is the basic argument for the whole process. Now, if we want to compute $\#B_1$, we just move our triangle, so that $(b, 0)$ is now at the origin. Similarly, the line $ax + by = c$ is moved, as in the picture:



So, with a little abuse of notation, let us redefine:

$$B_1 = \big\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c_1, \ \ 0 \leq x < b\big\}$$

where $c_1 = c - ab$ (obviously this only makes sense if $c > ab$). Assuming $k > 1$ we have, following the same way:

$$\begin{aligned} \#B_1 = \#\left(S \cap [0, c_1]\right) &= \#\Big(S \cap [0, c - ab]\Big) \\ &= \frac{ab - (a+b) + 1}{2} + c - ab - (ab - (a+b) + 1) \\ &= \frac{a + b - 3ab + 1}{2} + c. \end{aligned}$$

We can of course go along the same lines for computing $\#B_i$ for $i = 1, ...k-1$, where $k = \lfloor c/(ab) \rfloor$, rewriting $c_i = c - iab$, whenever $c > iab$ and we will find:

$$\begin{aligned} \#B_i = \#\left(S \cap [0, c_i]\right) &= \#\Big(S \cap [0, c - iab]\Big) \\ &= \frac{ab - (a+b) + 1}{2} + c - iab - (ab - (a+b)) \\ &= \frac{(a+b) - (1+2i)ab + 1}{2} + c \end{aligned}$$

We have then arrived at the nutshell of the problem: the set B_k. After we have moved it to the origin, we have our renamed B_k:

$$B_k = \left\{(x, y) \in\in \mathbb{Z}^2_{\geq 0} \mid ax + by \leq c_k \right\}.$$

Now we might have $c_k < ab - (a+b)$. So we cannot proceed in the same way as before. We do know $c_k = c - kab$, that is, $c_k = c \mod ab$, and from Lemma 3 we also know:

$$Ap(S, a) = \{0, b, 2b, ..., (a-1)b\}$$

and therefore

$$\{w \in Ap(S, a) \mid w \leq c_k\} = \left\{ ib \;\middle|\; i = 0, 1, ..., \left\lfloor \frac{c_k}{b} \right\rfloor \right\}$$

On the other hand, if $i \in \{0, ..., \lfloor \gamma_k / b \rfloor\}$, we have

$$ib + ja \leq c_k \iff j \leq \left\lfloor \frac{c_k - ib}{a} \right\rfloor,$$

and then

$$\begin{aligned} S \cap [0, c_k] &= \{ib + ja \leq c_k \mid i, j \in \mathbb{Z}_{\geq 0}\} \\ &= \left\{ ib + ja \;\middle|\; i \in \left\{0, ..., \left\lfloor \frac{\gamma_k}{b} \right\rfloor \right\}, \;\; j \leq \left\lfloor \frac{c_k - ib}{a} \right\rfloor \right\} \\ &= \sum_{i=0}^{\lfloor c_k/b \rfloor} \left(\left\lfloor \frac{c_k - ib}{a} \right\rfloor + 1 \right) \end{aligned}$$

Adding up all of these computations, we arrive to our result. In the previous conditions:

$$\begin{aligned} \#T &= \#B_0 + \#B_1 + ... + \#B_i + ... + \#B_{k-1} + \#B_k \\ &= \left(\frac{a + b - ab + 1}{2} + c \right) + ... + \left(\frac{(a+b) - (1+2i)ab + 1}{2} + c \right) + \\ &\qquad + ... + \sum_{i=0}^{\lfloor c_k/b \rfloor} \left(\left\lfloor \frac{c_k - ib}{a} \right\rfloor + 1 \right) \\ &= \sum_{i=0}^{k-1} \left(\frac{(a+b) - (1+2i)ab + 1}{2} + c \right) + \sum_{i=0}^{\lfloor c_k/b \rfloor} \left(\left\lfloor \frac{c_k - ib}{a} \right\rfloor + 1 \right) \\ &= -\frac{ab}{2}k^2 + \frac{a + b + 1 + 2c}{2}k + \sum_{i=0}^{\left\lfloor \frac{c - kab}{b} \right\rfloor} \left(\left\lfloor \frac{c - kab - ib}{a} \right\rfloor + 1 \right) \end{aligned}$$

where $k = \lfloor c/(ab) \rfloor$. This proves Theorem 2.

A (maybe) simpler way to express the previous formula is using the Euclidean division: $c = q \cdot ab + r$. Under this circumstance,

$$\#T = -\frac{ab}{2}q^2 + \frac{a+b+1+2c}{2}q + \sum_{i=0}^{\lfloor r/b \rfloor} \left(\left\lfloor \frac{r - ib}{a} \right\rfloor + 1 \right).$$

Example. Let us do a simple example to illustrate the process, considering the stable triangle defined by the line $3x + 7y = 46$.

Following Theorem 2:

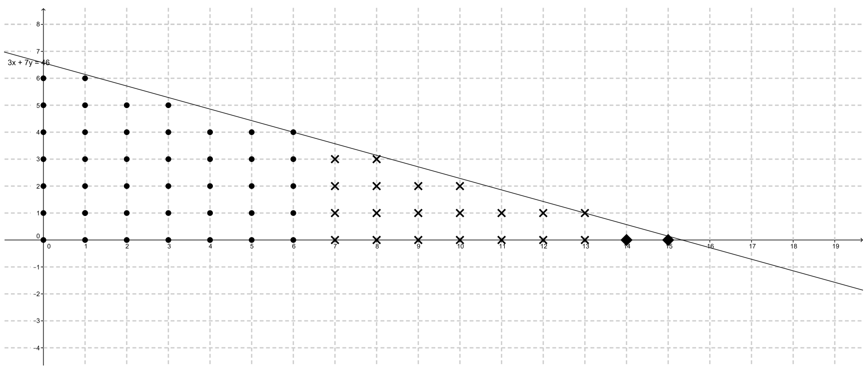
$$\# \, T = -\frac{3 \cdot 7}{2}k^2 + \frac{3+7+1+2 \cdot 46}{2} \cdot k + \sum_{i=0}^{\left\lfloor \frac{46 - k \cdot 3 \cdot 7}{7} \right\rfloor} \left(\left\lfloor \frac{46 - k \cdot 3 \cdot 7 - i \cdot 7}{3} \right\rfloor + 1 \right)$$

where $k = \left\lfloor \frac{46}{3 \cdot 7} \right\rfloor = 2$, that is

$$\# \, T = -\frac{3 \cdot 7}{2}2^2 + \frac{3+7+1+2 \cdot 46}{2} \cdot 2 + \sum_{i=0}^{\left\lfloor \frac{46 - 2 \cdot 3 \cdot 7}{7} \right\rfloor} \left(\left\lfloor \frac{46 - 2 \cdot 3 \cdot 7 - i \cdot 7}{3} \right\rfloor + 1 \right) =$$

$$= -42 + 103 + 2 = 61 + 2 = 63.$$

Let us see the actual counting:



In the picture we have put different symbols for the different sets B_j, following the process. Round points correpond to B_0. There are 41 of them, as predicted by the formula:

$$\#B_0 = \frac{a+b-ab+1}{2} + c = \frac{3+7-3 \cdot 7+1}{2} + 46 = 41.$$

Crossed points correspond to points in B_1:

$$\#B_1 = \frac{a+b-3ab+1}{2} + c = \frac{3+7-3 \cdot 3 \cdot 7+1}{2} + 46 = 20.$$

And finally the square points are those of B_2:

$$\#B_2 = \sum_{i=0}^{\left\lfloor \frac{c_k}{b} \right\rfloor} \left(\left\lfloor \frac{c_k - ib}{a} \right\rfloor + 1 \right) = \left\lfloor \frac{46 - 42}{3} \right\rfloor + 1 = 2.$$

Finally, back to our original problem of computing the number of integral points inside a convex polygon, we might have to compute the number of points in the hypothenuse (in order to remove them from the triangle, should it be part of a complementary set).

In order to do this, note that the number of integral points in the line $ax+by=c$ is precisely, the number of representations of c inside the monoid $S=\langle a,\ b\rangle$. This number is known as the denumerant of c in S [12].

It is easy to see that this denumerant have to be either $\lfloor c/(ab)\rfloor$ or $\lfloor c/(ab)\rfloor+1$, that is, k or $k+1$ in our previous setting, assumed $c\in S$ (obviously it is 0 otherwise). This is because c (or c_i) must be representable and from Lemma 4 there must exist exactly one representation whose coefficients are in B_0 (respectively B_i). This holds true for B_0,...,B_{k-1} but not necessarily for B_k (because c_k might not be in S), hence the two possible cases.

More precisely, we have the following result (see [11] for the original proof in Romanian, [1] for a shorter and easier one):

Theorem 5. *Under the previous asumptions, let a' and b' be the only integers veryfing*

$$0<a'<b,\quad a\cdot a'=-c \mod b$$

$$0<b'<a,\quad b\cdot b'=-c \mod a$$

Then the denumerant of c in S is given by

$$d(c;a,b)=\frac{c+a\cdot a'+b\cdot b'}{ab}-1$$

5. Applications

The result on rational polygons and its reduction to rectangles and right triangles could be generalized to an n–dimensional set–up. However, the actual formulas are not easy enough so as to give a tight result. We give a first idea on how this could be done by computing the number of points in a stable right tetrahedron (the definition is the obvious one).

Theorem 6. *Let $T(a_1,a_2,a_3,b)\subset\mathbb{R}^3$ be the tetrahedron defined by*

$$T(a_1,a_2,a_3,b)=\{(x_1,x_2,x_3)\mid x_i\geq 0,\ a_1x_1+a_2x_2+a_3x_3=b\}$$

where we are assuming $a_1<a_2<a_3$, $\gcd(a_1,a_2)=1$.

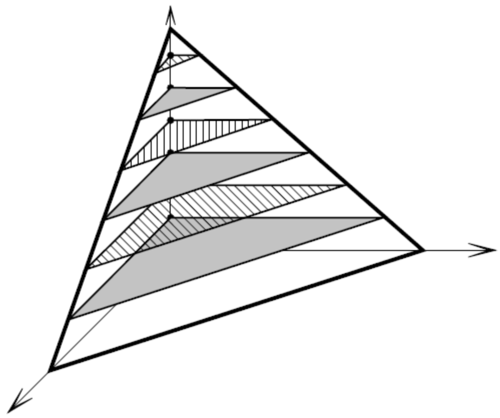
For $i=0,...,\lfloor b/a_3\rfloor$ define q_i and r_i by the Euclidean division:

$$b-a_3i=q_i(a_1a_2)+r_i.$$

Then

$$\#\left(T(a_1,a_2,a_3,b)\cap\mathbb{Z}^3\right) = \sum_{i=0}^{\lfloor b/a_3\rfloor}\left(-\frac{ab}{2}q_i^2+\frac{a+b+1+2(b-a_3i)}{2}q_i+ + \sum_{j=0}^{\lfloor r_i/b\rfloor}\left(\left\lfloor\frac{r_i-ja_2}{a_1}\right\rfloor+1\right)\right).$$

Proof. The formula is just the result of adding the number of points in every right triangle $T(a_1,a_2,a_3,b)\cap\{x_3=i\}$ for $i=0,...,\lfloor b/a_3\rfloor$. $\square$



The condition $\gcd(a_1, a_2) = 1$ can obviously be substituted by $\gcd(a_1, a_3) = 1$ or $\gcd(a_2, a_3) = 1$ if necessary. If none of this conditions is met, like in the tetrahedron defined by

$$6x_1 + 10x_2 + 15x_3 = 21,$$

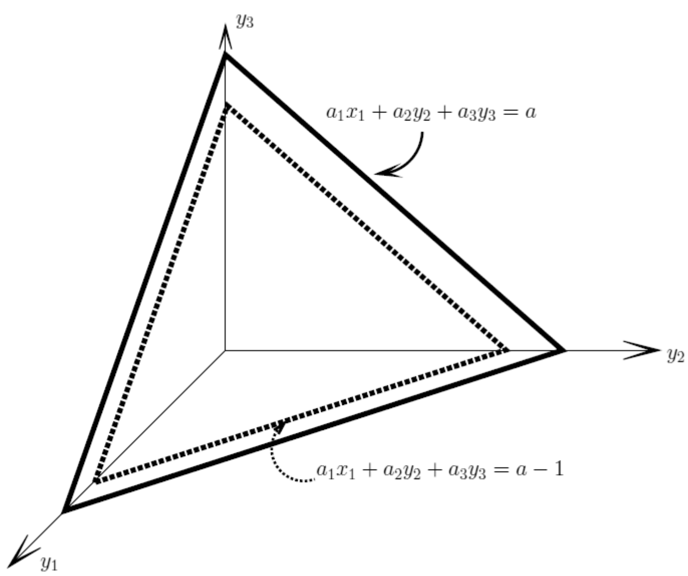
for instance, then some of the right triangles have to be *adjusted* as we did at the end of section 1. This is not a difficulty when programming, so to say, but the general formula gets a lot messier.

This result can be handy when trying to compute the denumerant function we introduced above.

Easy as it is to define, the denumerant is a very elusive function which has proved elusive to compute even in cases with 3 generators (see [12, Chapter 4]). With the previous result one can give a formula, not very sophisticated though. Simply note that

$$\begin{aligned} d\,(a; a_1, a_2, a_3) &= \#\Big\{(x_1, x_2, x_3) \in \mathbb{Z}_{\geq 0} \mid a_1x_1 + a_2x_2 + a_3x_3 = a\Big\} \\ &= \#\left(T(a_1, a_2, a_3, a) \cap \mathbb{Z}^3\right) - \#\left(T(a_1, a_2, a_3, a-1) \cap \mathbb{Z}^3\right). \end{aligned}$$

And then, from the previous result, one can obtain the desired formula.



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References

[1] Brown, T.C.; Chou, W.S.; Shiue, P.J.: On the partition function of a finite set. *Australasian J. Comb.* **27** (2003) 193–204.

[2] Beck, M.; Robins, S.: *Computing the continuous discretely. Integer-point enumeration in polyhedra.* Springer, 2007.

[3] Ehrhart, E.: Sur les polyèdres rationnels homothétiques à n dimensions. C. R. Acad. Sci. Paris **254** (1962) 616–618.

[4] García–Sánchez, P.A.; Rosales, J.C.: *Numerical semigroups.* Springer, 2009.

[5] Hardy, G. H.; Littlewood, J. E.: Some problems of Diophantine approximation: The lattice-points of a right-angled triangle. *Abh. Math. Sem. Univ. Hamburg* **1** (1922) 211–248.

[6] Lin, K.P.; Yau, S.T.: Analysis of sharp polynomial upper estimate of number of positive integral points in 4–dimensional tetrahedra. *J. Reine Angew. Math.* **547** (2002) 191–205.

[7] Lin,K.P.; Yau, S.T.: Analysis of sharp polynomial upper estimate of number of positive integral points in 5–dimensional tetrahedra. *J. Number Theory* **93** (2002) 207–234.

[8] Lin, K.P.; Yau, S.S.T.: Counting the number of integral points in general n–dimensional tetrahedra and Bernoulli polynomials. *Canad. Math. Bull.* **24** (2003) 229–241.

[9] Márquez–Campos, G.; Tornero, J.M.: Characterization of gaps and elements of a numerical semigroup using Groebner bases. To appear in *Proceedings of the V Jornadas de Teoría de Números*, Contemporary Mathematics.

[10] Pick, G.A.: Geometrisches zur Zahlenlehre, Sitzungsber. Lotos (Prague) **19** (1889) 311–319.

[11] Popoviciou, T.: Asupra unei probleme de patitie a numerelor. *Acad. Republicii Populare Romane, Filiala Cluj, Studii si cercetari stiintifice* **4** (1953) 7–58.

[12] Ramírez–Alfonsín, J.L.: *The Diophantine Frobenius problem.* Oxford University Press, 2005.

[13] Reeve, J.E.: On the volume of the lattice polyhedra. Proc. London Math. Soc. **7** (1957) 378–395.

[14] Sylvester, J.J.: Problem 7382. *Educational Times* **37** (1884) 26.

[15] Wang, X.; Yau, S.S.T.: On the GLY conjecture of upper estimate of positive integral points in real right-angled simplices. *J. Number Theory* **122** (2007) 184–210.

[16] Xu, Y.J.; Yau, S.S.T.: A sharp estimate of number of integral points in a tetrahedron. *J. Reine Angew. Math.* **423** (1992) 199–219.

[17] Xu, Y.J.; Yau, S.S.T.: Durfee conjecture and coordinate free characterization of homogeneous singularities. *J. Differential Geom.* **37** (1993) 375–396.

[18] Xu, Y.J.; Yau, S.S.T.: A sharp estimate of number of integral points in a 4–dimensional tetrahedra. *J. Reine Angew. Math.* **473** (1996) 1–23.

[19] Yau, S.S.T.; Zhang, L.: An upper estimate of integral points in real simplices with an application to singularity theory. *Math. Res. Lett.* **13** (2006) 911–921.

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