Bijections between truncated affine arrangements and valued graphs

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Abstract. We present some contructions on the set of nbcs (No Broken Circuit sets) of some deformations of the braid arrangement. This leads us to some new bijective proofs for Shi, Linial and similar hyperplane arrangements.

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1. Introduction

An *integral gain graph* is a graph whose edges are labelled invertibly by integers; that is, reversing the direction of an edge negates the label (the *gain* of the edge). The *affinographic hyperplane arrangement* that corresponds to an integral gain graph Φ is the set of all hyperplanes in $\mathbb{R}^n$ of the form $x_j - x_i = g$ for edges (v_i, v_j) with gain g in Φ. (See [**?**, Section IV.4.1, pp. 270–271] or [**?**].)

In recent years there has been much interest in real hyperplane arrangements of this type, such as the Shi arrangement, the Linial arrangement, and the composed-partition or Catalan arrangement. For all these families, the number of regions and then the characteristic polynomials have been found. For the Shi arrangement, Athanasiadis gives a bijection between the regions and parking functions.

In this paper, we look at the set of "no broken circuit sets" (nbcs) which are labelled graphs. We then give some properties which lead to correspondance with some other graph families : local binary search trees (lbs), alternated trees and rooted trees.

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2. Basic definitions

An *integral gain graph* $\Phi = (\Gamma, \varphi)$ consists of a graph $\Gamma = (V, E)$ and an orientable function $\varphi : E \to \mathbb{Z}$, called the *gain mapping.* "Orientability" means that, if e denotes an edge oriented in one direction and e^{-1} the same edge with the opposite orientation, then $\varphi(e^{-1}) = -\varphi(e)$. For us, we have no loops but multiple edges are permitted. We denote the vertex set by $V = \{v_1, v_2, \dots, v_n\}$. We sometimes use the simplified notations e_{ij} for an edge with endpoints v_i and v_j, oriented from v_i to v_j, and ge_{ij} for such an edge with gain g; that is, $\varphi(ge_{ij}) = g$. (Thus ge_{ij} is the same edge as $(-g)e_{ji}$.) A *circle* is a connected 2-regular subgraph, or its edge set. We may write a circle C as a word $e_1 e_2 \cdots e_l$; this means that

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the edges are numbered consecutively around C and oriented in a consistent direction. The gain of C is $\varphi(C) := \varphi(e_1) + \varphi(e_2) + \cdots + \varphi(e_l)$; this is well defined up to negation, and in particular it is well defined whether the gain is zero or nonzero. An edge set or subgraph is called *balanced* if every circle in it has gain zero. We will consider more specially balanced circles.

With an order $<_O$ on the set of edges E, a "broken circuit" is the set of edges obtained by deleting the smallest element in a balanced circle. A set of edges $N \subset E$ is a "no broken circuit" (nbc for short) if it contains no briken circuit. This notion from matroid theory (see [**?**] for reference), is very important here. We denote $\mathcal{N}$ the set of nbcs of the gain graph. It is well known that this set depends on the choice of the order, but its cardinallity does not.

There is a direct correspondance between integral gain graphs and hyperplane arrangements whose hyperplane equations are of the form $x_i - x_j = g$. This correspondance is simply given by associating to the equation $x_i - x_j = g$ the edge $g(i,j)$. In fact it is the original idea which motivated Zaslavsky to define signed graphs and gain graphs.

We can then transpose some ideas or problems from hyperplane arrangements to gain graphs. We call the Linial gain graph L_n the gain graph on $[n]$ with edges $1(i,j)$ for all $i < j$ and similarly the Shi gain graph S_n the gain graph on $[n]$ with edges $1(i,j)$ and $0(i,j)$ for all $i < j$. They of course correspond to the well studied Shi and Linial arrangements.

3. Tableaux

Definition 1. *A tableau T on a set V is given by a fonction h_T from V to $\mathbb{N}$ such that $h_T^{-1}(0) \neq \emptyset$. The corner of the tableau is the smallest element of highest height.*
Let Φ be a connected balanced integral gain graph on a set V of integers. The tableau of the gain graph, denoted $T(\Phi)$, is given by the unique function h_T such that for every edge $g(i,j)$, with $i < j$, we have $h_T(j) - h_T(i) = g$.

We say that a tableau T on V is coherent with a connected gain graph Φ on V if there is a connected balanced subgraph Φ' of Φ such that $T = T(\Phi')$. The definition would also work with a tree instead of a connected balanced subgraph. The question whether a tableau is coherent with a gain graph would not be studied here but is by itself of interest.

Reciprocally we have the following definition:

Definition 2. *Let T be a tableau on a set V of integers and Φ a gain graph also on V. The subgraph $\Phi[T]$ of Φ defined by T is the gain graph on V whose edges are the edges $g(i,j)$, with $i < j$, such that $h_T(j) - h_T(j) = g$.*

Given a tableau T, a gain graph Φ and an order on the edges $<_O$, it defines the set of nbcs of the subgraph $\Phi[T]$ relatively to the order $<_O$, denoted $\mathcal{N}_O(\Phi[T])$. Like always, this set depends on the choice of the order but its cardinallity does not.

Proposition 3. *A tableau T is coherent with a connected gain graph Φ iff $\Phi[T]$ is connected.* □

Definition 4. *Given a tableau T on the set V, the order O_T on the set V is defined by $i <_{O_T} j$ iff $h_T(i) > h_T(j)$ or ($h_T(i) = h_T(j)$ and $i < j$). The order O_T is extended lexicographically to the order O_T on the edges coherent with the tableau.*

Lemma 5. *Given an nbc tree A of tableau T with corner c, the forest $A \setminus c$ is a set of nbcs of tableaux T_1,...,T_k. The orders O_{T_i} are restrictions of the order O_T.* □

Definition 6. *Given G a rooted labeled tree with integer values on the edges, the tableau T_G is such that the height function h_T verifies if j is the son of i and g is the value on the edge (i,j) then $h_T(i) - h_T(j) = g$.*

4. $[a,b]$ complete gain graphs and their NBCS

Let a and b be two relative integers such that $a \leq b$. The interval $[a,b]$ is the set $\{i \in \mathbb{Z} | a \leq i \leq b\}$. We consider the gain graph K_n^{ab} with vertices labeled by $[n]$ and with all the edges $g(i,j)$, with $i < j$ and $g \in [a,b]$. These gain graphs are called deformations of the braid arrangement. Indeed, the braid arrangement corresponds to the special case $a = b = 0$. Some other well studied cases are $a = -b$ (catalan), $a = b = 1$ (Linial) and $a = b - 1 = 0$ (Shi).

We will describe the set of nbcs of $K_n^{ab}[T]$ for a given tableau T. The idea is that, as mentioned above, the tableau T defines the order O_T. We will than be able to describe the set of nbcs coherent with T for the order O_T.

Theorem 7. *Let a and b such that $a + b = r > 0$ and T be a tableau on $[n]$ of corner c. Let Φ be a gain subtree of K_n^{ab} incident to the edges $g_i(c, v_i)$ and Φ_i the corresponding connected components of $\Phi \setminus c$. The tree Φ is an nbc of K_n^{ab} iff the four following conditions are respected:*

- *all the Φ_i are nbcs;*
- *if $v_i < c$ then $g_i \in [1, b]$ and v_i is the corner of Φ_i;*
- *if $v_i > c$ and $g_i \in [a, 0]$ then v_i is the corner of Φ_i;*
- *if $v_i < c$ and $g_i \in [b - r + 1, b]$ then v_i is the smallest (relatively to O_T) element of Φ_i smaller than c and $h(c_i) - h(v_i) < r - 1$, where c_i is the corner of Φ_i.*

Proof. Every thing comes from the choice of the order O_T for the vertices and the edges. For instance, in the first case v_i is necessarily the corner of Φ_i since if not the edge $g(c, c_i)$ would close a balanced circuit in Φ where it would be minimal. The second case is very similar.

The last case is more interesting. Since $v_i < c$ we must have $g_i > 0$. But since Φ is a broken circuit we must have that there is no edge in Φ between c and any vertex of Φ_i higher than v_i. This implies the rest of the condition.

The other direction is automatic since c is the smallest element for the order O_T. □

Remark 8. *In the two families $a = -b$ and $a = -b + 1$ the last case never occurs and we obtain a simple construction. In the case $a = -b + 2$, the first interesting family (containing Linial as first example), the last case can be rephrased by : if the corner c_i of Φ_i verifies $c_i > c$ and the smallest element v of the line just below c_i (if this line is not empty) verifies $v < c$ than Φ_i can be connected to c with gain b only.*

5. $[a,b]$-gain graphs with $a + b = 0$ or 1

Definition 9. *An (a,b)-rooted labelled tree with n vertices is a rooted tree where the vertices are labelled from 1 to n and such that each edge of the tree (i,j) where i is the ancestor and j the descendant is labelled with an integer between*

- *1 and $b - 1$ if $i > j$ and*
- *$-a + 1$ and 0 otherwise.*

Theorem 10. *If $b = a$ or $b = a + 1$, the (a,b)-NBC trees with n vertices are in bijection with (a,b)-labelled trees with n vertices.*

Proof. We decompose recursively the $(a,b)-$NBC trees. Let $\mathcal{T}$ be an NBC tree. Let c be its corner and let $c_1, c_2, \dots, c_k$ be the neighbors of c with gain $g_1, g_2, \dots, g_k$. Then c is the root of the (a,b)-labelled tree, $c_1, c_2, \dots, c_k$ are its children and the edges from c to c_i get the label g_i. The decomposition continues recursively on the trees with corners $c_1, c_2, \dots, c_n$.

It is easy to see that when we take off the edges (c, c_i) from the $(a,b)-$NBC tree, we get a forest of $(a,b)-$NBC trees, where each c_i is in a different tree. To prove that the decomposition is correct, we have to prove that this forest is a forest of $(a,b)-$NBC trees with corners $c_1, c_2, \dots, c_k$.

Let us suppose that c_i is not the corner of its tree. Then there exists v such that $h(v) < h(c_i)$ or $h(v) = h(c_i)$ and $v < c_i$. It is easy to check that (c, c_i, v) is a broken circuit of $\mathcal{T}$ and this contradicts the fact that $\mathcal{T}$ is an $(a,b)-$NBC tree. ☐

A direct consequence of our Theorem is that

Corollary 11. *If $b = a$ or $b = a + 1$, the number of regions f_n^{ab} is equal to the number of (a,b)-rooted labelled forest with n vertices.*

Theorem 12. [?] *The number of regions f_n^{ab} is*

$$an(an-1)\dots(an-n+2), \qquad if\ b = a;$$

and

$$(an+1)^{n-1}, \quad if\ b = a + 1.$$

To finish our proof of Theorem **??**, we have to count the number of (a,b)-labelled trees and (a,b)-labelled forests.

Proposition 13. *The number of (a,b)-rooted labelled trees with n vertices is*

$$\prod_{i=1}^{n-1}((a-b+1)i + (b-1)n).$$

The number of (a,b)-rooted labelled forests with n vertices is

$$\prod_{i=1}^{n-1}((a-b+1)i + (b-1)n + 1).$$

Proof. We suppose that $a \geq b-1$. The other case is analog. We first enumerate (a,b)-rooted labelled trees. We split the edges of the trees into two groups :

- The edges with labels $-a+1, -a+2, \dots, -b+1$.
- The others.

Suppose that the first group has k edges. They form an increasing forest on n vertices with k edges, such that the edges can have $(a-b+1)$ different labels. The number of such forests is $s(n, n-k)(a-b+1)^k$ where $s(n,k)$ is the Stirling number of the first kind [?].

The second group is a rooted labelled forest on n vertices with $n-k-1$ edges, such that the edges can have $(b-1)$ different labels. The two groups have disjoint edges. Therefore thanks to the Prüfer code [?], we get that the number of (a,b)-rooted labelled trees with n vertices and k edges in the first group is :

$$s(n, n-k)(a-b+1)^k((b-1)n)^{n-k+1}.$$

Therefore the number of (a,b)-rooted labelled trees with n vertices is :

$$\begin{aligned}
\sum_{k=0}^{n-1} s(n,n-k)(a-b+1)^k((b-1)n)^{n-k+1} &= \frac{(a-b+1)^n}{(b-1)n}\sum_{k=0}^{n-1} s(n,n-k)\left(\frac{(b-1)n}{a-b+1}\right)^{n-k} \\
&= \frac{(a-b+1)^n}{(b-1)n}\prod_{i=0}^{n-1}\left(i+\frac{(b-1)n}{a-b+1}\right) \\
&= \prod_{i=1}^{n-1}((a-b+1)i+(b-1)n).
\end{aligned}$$

□

6. LBS TREES

We will need in the next section a special family of labelled trees: the locally binary search trees (lbs for short)which is in correspondance with the nbc of the Linial arrangement.

A local binary search tree on $[n]$ is a labelled binary tree on $[n]$ where the label on a right (resp. left) son is greater (resp. smaller) then the label on the father. By introducing the family of left lbs (a llbs tree such that the root has no right son and llbs tree for short), we get a simple decompositions of a lbs into llbs. In the correspondance between nbc sets and lbs trees , a maximal nbc set, i.e., a nbc tree, corresponds to a llbs tree and a general nbc, i.e., a nbc forest which is a union of nbc trees corresponds to a lbs where the T_i correspond to the llbs of the above decomposition. We will then only have to describe the correspondance between llbs trees and nbc trees.

A llbs tree A of root r can be also decomposed into the vertex r and a set of llbs trees A_i of root r_i (for $1 \leq i \leq k$) where r_1 is the unique neighbour of r, r_2 the unique right neighbour of r_1, and so on. The only conditions are that $r_1 < r$ and $r_1 < r_2 < \cdot < r_k$.

Dually to the decomposition of a lbs tree into llbs trees there is decomposition into rlbs trees (right lbs trees). In this decomposition, apart from the root the tree is decomposed in a rlbs tree of root r_1 the (left) neighbour of r, and a second rlbs tree of root r_2 the left neighbour of r_1, and so on.

There is a straightforward correspondance between the set of llbs trees and rlbs trees. To go from llbs to rlbs we just need to use the mirror bijection which replace label i by $n-i$ and left by right.

The set of llbs trees are also in correspondance with the family of alternated trees but we dont need them here.

7. LINIAL GAIN GRAPH

We recall that the Linial gain graph L_n on $[n]$ has vertex set $[n]$ and edges $1(i,j)$, with $i<j$. It corresponds to the Linial arrangement whose hyperplanes have equation $x_i - x_j = 1$, with $i<j$.

We define by induction the tableau $T(L)$ of a left local binary search tree to get the coming results. If the tree as one vertex v we have simply $h(v)=1$. If the tree has a root r with left son r_1 root of a left local binary search L_1, with right son r_2 root of a left local binary search L_2,... . Then the tableau of L is obtained by merging the tableaux of the L_i by taking : if $r_i < r$ then $h(r_i) = h(r) - 1$ and if $r_i > r$ then $h(r_i) = h(r)$.

Theorem 14. *For a given tableau T, there is a bijection between the set of nbc trees of the Linial gain graph L_n of tableau T relatively to the order $<_T$ and the llbs of tableau T.*

Proof. The tableau T defines the order O_T on the vertices which induces the order O_T on the edges.

We describe now an inductive simple two-way construction between the set of nbcs of L_n of tableau T and the set of left local binary search trees of tableau T.

- From nbc trees to llbs trees: let N be an nbc tree of tableau T. Let c be its corner and $v_1 < v_2 < \cdots < v_k$ the neighbours of c.

  We decompose N into smaller nbcs N_i by taking out the vertex c and the edges adjacent to c. For $i \geq 2$, each N_i gives an llbs tree A_i of root r_i the corner of N_i (not necessarily v_i). For $i = 1$, we decompose again N_1 into smaller NBCs by removing the edges (v_1, v'_i) (for $1 \leq i \leq \ell$) where $v'_i > v_1$. We obtain then $\ell + 1$ NBC trees: $\bar{N}_1$ and the $\bar{N}'_i$ (for $1 \leq i \leq \ell$). These NBC tree correspond to the llbs trees $\bar{A}_1$ and the $\bar{A}'_i$.

  We get then $k + \ell$ llbs trees with only one necessarily of root smaller then c.
- From llbsts to nbcts: let S be an lbst. Let r be its root. We take out the root and get the decomposition into llbs trees A_i of root r_i (for $1 \leq i \leq k$) described above. Each A_i gives by the inductive construction a NBC tree N_i of corner r_i and of subcorner sr_i. We need now to connect these NBC trees to the corner r. The NBC trees N_i such that the $sr_i < r$ can be directly connected to r by adding the edge (r, r_i) if $r > r_i$ or the (r, sr_i) otherwise. The other N_i which cannot be connected directly to the corner r are connected to the vertex r_1 by adding the edge (r_1, r_i). We need just to verify that the constructed tree is indeed a NBC tree relatively to the order $O(T)$. This fact is a consequence of the fact that each N_i is a NBC for $O(T)$ and that the neighbour of r in the resulting NBC tree are wheter the corner or the subcorner of their sub NBC tree.

□

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