CONNECTED COVERING NUMBERS

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Dedicated to the memory of Michel Las Vergnas

Abstract. A connected covering is a design system in which the corresponding *block graph* is connected. The minimum size of such coverings are called *connected coverings numbers*. In this paper, we present various formulas and bounds for several parameter settings for these numbers. We also investigate results in connection with *Turán systems*. Finally, a new general upper bound, improving an earlier result, is given. The latter is used to improve upper bounds on a question concerning oriented matroid due to Las Vergnas.

1. Introduction

Let n, k, r be positive integers such that $n \geqslant k \geqslant r \geqslant 1$. A (n, k, r)*-covering* is a family $\mathcal{B}$ of k-subsets of $\{1, \ldots, n\}$, called *blocks*, such that each r-subset of $\{1, \ldots, n\}$ is contained in at least one of the blocks. The number of blocks is the covering's *size*. The minimum size of such a covering is called the *covering number* and is denoted by $\mathrm{C}(n, k, r)$. Given a (n, k, r)-covering $\mathcal{B}$, its graph $G(\mathcal{B})$ has $\mathcal{B}$ as vertices and two vertices are joined if they have one r-subset in common. We say that a (n, k, r)-covering is *connected* if the graph $G(\mathcal{B})$ is connected. The minimum size of a connected (n, k, r)-covering is called the *connected covering number* and is denoted by $\mathrm{CC}(n, k, r)$.

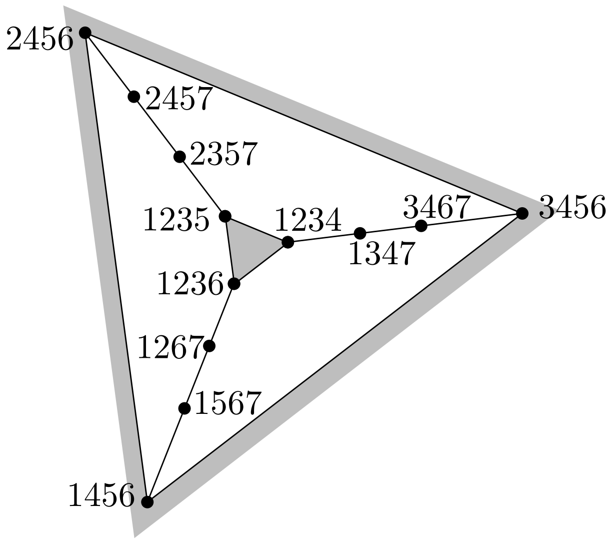

Figure 1. A connected $(7, 4, 3)$-covering with 12 blocks.

Date: March 27, 2014.
2010 *Mathematics Subject Classification.* 05B05, 5E105.
Key words and phrases. Covering design, Turán-system, uniform oriented matroid.
The second and forth authors were supported by the ANR TEOMATRO grant ANR-10-BLAN 0207.
The third author was supported by CONACYT.

The graph corresponding to a connected $(7,4,3)$-covering is illustrated in Figure 1.
In this paper, we mainly pay our attention to coverings when $k = r+1$ and thus, we will denote $\mathrm{C}(n, r+1, r)$ (resp. $\mathrm{CC}(n, r+1, r)$) by $\mathrm{C}(n, r)$ (resp. by $\mathrm{CC}(n, r)$) for short.
The original motivation to study $\mathrm{CC}(n, r)$ comes from the following question posed by Las Vergnas.

Question 1.1. Let $U_{r,n}$ be the rank r uniform matroid on n elements. What is the smallest number $s(n, r)$ of circuits of $U_{r,n}$, that uniquely determines all orientations of $U_{r,n}$? That is, whenever two uniform oriented matroids coincide on these circuits they must be equal.

In [3], Forge and Ramírez Alfonsín introduced the notion of connected coverings and proved that

$$s(n, r) \leqslant \mathrm{CC}(n, r). \tag{1}$$

The latter was then used to improve the best upper bound, $s(n, r) \leqslant \binom{n-1}{r}$, known at that time due to Hamidoune and Las Vergnas [7]; see also [4] for related results.
It turns out that $s(n, r)$ is also closely related to $\mathrm{C}(n, r)$.

Proposition 1.2. *Let n and r be positive integers such that $n \geqslant r+1$. Then,*

$$\mathrm{C}(n, r) \leqslant s(n, r).$$

The proof of Proposition 1.2, that is a bit technical (needing some oriented matroid notions and thus lying slightly out of scope of this paper), is given in the Appendix.
Covering designs have been the subject of an enormous amount of research papers (see [6] for many upper bounds and [18] for a survey in the dual setting of Turán-systems). Although the construction of block design is often very elusive and the proof of their existence is sometimes tough, here, we will be able to present explicit constructions yielding exact values and bounds for,$\mathrm{C}(n, r)$ and $\mathrm{CC}(n, r)$ for infinitely many cases. The study of $\mathrm{C}(n, r)$ and $\mathrm{CC}(n, r)$ seems to be interesting not only for Design Theory but also, in view of Proposition 1.2 and (1), for the implications on the behavior of $s(n, r)$ in Oriented Matroid Theory. This relationship was already remarked in [3] where it was proved that

$$\mathrm{CC}(n, r) \leqslant 2\mathrm{C}(n, r) - 1. \tag{2}$$

That was done by observing that the graph G associated to a covering with $\mathrm{C}(n, r)$ blocks (and thus with $|V(G)| = \mathrm{C}(n, r)$) can be made connected by adding at most $\mathrm{C}(n, r) - 1$ extra vertices (blocks), obtaining a graph corresponding to a $(n, r+1, r)$-connected covering with at most $2\mathrm{C}(n, r) - 1$ blocks.

Many interesting variants of Question 1.1 can be investigated. For instance, for non-uniform (oriented) matroids (graphic, representable, etc.) and by varying the notion of what *determine* means (up to orientations, bijections, etc.). These (and other) variants are treated in another paper (see [10]).

This paper is organized as follows. In the next section, we recall some basic definitions and results concerning (connected) coverings and its connection with *Turán systems* needed for the rest of the paper. In Section 3, we investigate connected covering numbers when the value r is either small or close to n. Among other results, we give the exact value for $\mathrm{CC}(n, 2)$ (Theorem 3.2), for $\mathrm{CC}(n, 3)$ for $n \leqslant 12$ (Theorem 3.3) and for $\mathrm{CC}(n, n-3)$ (Theorem 3.6). A famous conjecture of Turán and its connection with our results is also discussed. In Section 4, we present a general upper bound for $\mathrm{CC}(n, r)$ (Theorem 4.10) allowing us to improve the best known upper bound for $s(n,r)$. We end the paper by discussing some asymptotic results in Section 5.

2. Basic results

Let n, m, p be positive integers such that $n \geqslant m \geqslant p$. A (n,m,p)-*Turán-system* is a family $\mathcal{D}$ of p-subsets of $\{1,\dots,n\}$, called *blocks*, such that each m-subset of $\{1,\dots,n\}$ contains at least one of the blocks. The number of blocks is the *size* of the Turán-system. The minimum size of such a covering is called the *Turán Number* and is denoted by $\mathrm{T}(n,m,p)$. Given a (n,m,p)-Turán-system $\mathcal{D}$, with $0 \leqslant 2p-m \leqslant p$, its graph $G(\mathcal{D})$ has as vertices $\mathcal{D}$ and two vertices are joined if they have one $2p-m$-subset in common. We say that a (n,m,p)-Turán-system with $0 \leqslant 2p-m \leqslant p$ is *connected* if $G(\mathcal{D})$ is connected.

The minimum size of a connected (n,m,p)-Turán-system is the *connected Turán Number* and is denoted by $\mathrm{CT}(n,m,p)$. By applying set complement to blocks, it can be obtained that

$$\mathrm{C}(n,k,r) = \mathrm{T}(n, n-r, n-k). \tag{3}$$

Moreover, if $0 \leqslant n-2k+r \leqslant n-k$ then

$$\mathrm{CC}(n,k,r) = \mathrm{CT}(n, n-r, n-k). \tag{4}$$

Note that the precondition for (4) is fulfilled if $k = r+1$.
Most of the papers on coverings consider n large compared with k and r, while for Turán numbers it has frequently been considered n large compared with m and p, and often focusing on the quantity $\lim_{n\to\infty} \mathrm{T}(n,m,p)/\binom{n}{p}$ for fixed m and p. Thus, for Turán-type problems, the value $\mathrm{C}(n,k,r)$ has usually been studied in the case when k and r are not too far from n.

Forge and Ramírez Alfonsín [3] proved that

$$\mathrm{CC}(n,r) \geqslant \frac{\binom{n}{r}-1}{r} =: \mathrm{CC}_1^*(n,r). \tag{5}$$

Moreover, Sidorenko [19] proved that $\mathrm{T}(n,r+1,r) \geqslant \left(\frac{n-r}{n-r+1}\right)\frac{\binom{n}{r}}{r}$, which by (3), we obtain that

$$\mathrm{CC}(n,r) \geqslant \mathrm{C}(n,r) = \mathrm{T}(n,n-r,n-r-1) \geqslant \left(\frac{r+1}{r+2}\right)\frac{\binom{n}{r+1}}{n-r-1} =: \mathrm{CC}_2^*(n,r). \tag{6}$$

Combining (5) and (6), together with a straight forward computation we have

$$\mathrm{CC}(n,r) \geqslant \max\{\mathrm{CC}_1^*(n,r), \mathrm{CC}_2^*(n,r)\}, \tag{7}$$

where the maximum is attained by the second term if and only if $r \geqslant \frac{2}{3}(n-1)$.
The following recursive lower bound for covering numbers was obtained by Schönheim [17] and, independently, by Katona, Nemetz and Simonovits [9]

$$\mathrm{C}(n,r) \geqslant \left\lceil \frac{n}{r+1}\mathrm{C}(n-1,r-1)\right\rceil \tag{8}$$

which can be iterated yielding to

$$\mathrm{C}(n,r) \geqslant \left\lceil \frac{n}{r+1}\left\lceil \frac{n-1}{r}\left\lceil \dots \left\lceil \frac{n-r+1}{2}\right\rceil \dots \right\rceil\right\rceil\right\rceil =: L(n,r). \tag{9}$$

The following recursive upper bound for $\mathrm{CC}(n,r)$ due to Forge and Ramírez Alfonsín [3] will be used later.

$$\mathrm{CC}(n,r) \leqslant \mathrm{CC}(n-1,r) + \mathrm{C}(n-1,r-1). \tag{10}$$

3. Results for small and large r

In this section, we investigate connected covering numbers for *small* and *large* r, that is, when r is very close to either 1 or n. Let us start with the following observations.

Remarks 3.1.
a) $\mathrm{CC}(n,0)=1$ since any 1-element set contains the empty set.
b) $\mathrm{CC}(n,1)=n-1$ by taking the edges of a spanning tree of K_n.
c) $\mathrm{CC}(n,n-2)=n-1$ by taking all but one $(n-1)$-sets.
d) $\mathrm{CC}(n,n-1)=1$ by taking the entire set.

All these values coincide with the corresponding covering numbers except in the case $r=1$, where $\mathrm{C}(n,1)=\lceil \frac{n}{2}\rceil$.

3.1. Results when r is small.

For ordinary covering numbers, Fort and Hedlund [5] have shown that $\mathrm{C}(n,2) := \lceil \frac{n}{3}\lceil \frac{n}{2}\rceil\rceil$ that coincides with the lower bounds given in (9) when the case $r=2$.

We also have the precise value for the connected case when $r=2$.

Theorem 3.2. *Let n be a positive integer with $n \geqslant 3$. Then, we have*

$$\mathrm{CC}(n,2) = \left\lceil \frac{\binom{n}{2}-1}{2} \right\rceil.$$

Proof. Note that the claimed value coincides with the lower bound $\mathrm{CC}_1^*(n,2)$. This lower bound comes from the fact that every connected covering has a construction sequence, where every new triangle shares at least one edge with an already constructed triangle. We present a construction sequence where indeed every new triangle (except possibly the last one) shares exactly one edge with the already constructed ones. Therefore, we attain the lower bound. Part of the construction is shown in Figure 2. We start presenting the

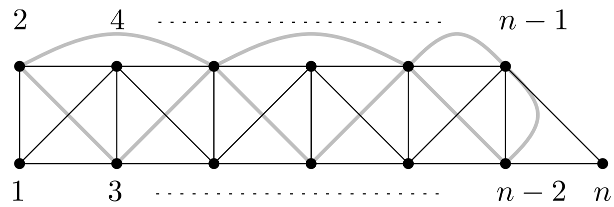

Figure 2. Part of the construction proving $\mathrm{CC}(n,2) \leqslant \mathrm{CC}_1^*(n,2)$.

black triangles from left to right. Then we present all triangles of the form $(2i-1,2i,j)$ for $1 \leqslant i \leqslant \frac{n}{2}$ and $j \geqslant 2i+3$. (These are not depicted in the figure.) Now we present the gray triangles from left to right. A gray triangle of the form $(2i,2i+1,2i+4)$ is connected to the already presented ones via $(2i-1,2i,2i+4)$. Note (as in the figure) the last triangle may indeed share two edges of already presented triangles, depending on the parity of n. This amounts for the ceiling in the formula. It is easy to check that all edges are covered. □

The precise value of $C(n,3)$ remains unknown only for finitely many n, see [14, 15, 8]. The situation for connected coverings is worse.

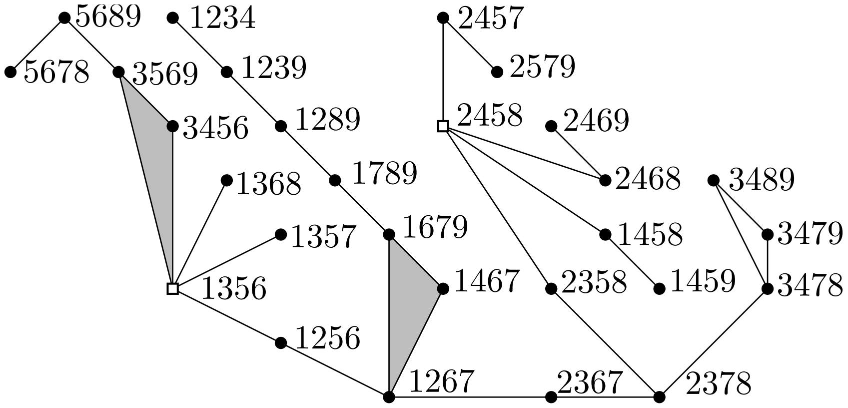

FIGURE 3. An example proving $\mathrm{CC}(9,3) \leqslant 28$. The circle-vertices are a covering.

Theorem 3.3. *Let n be a positive integer with $4 \leqslant n \leqslant 12$. Then, we have*

$$\mathrm{CC}(n,3) = \left\lceil \frac{\binom{n}{3}-1}{3} \right\rceil .$$

Proof. Note that the claimed value coincides with the lower bound $\mathrm{CC}_1^*(n,3)$. For $n \leqslant 6$ this is already checked in [3]. Figure 1 proves $\mathrm{CC}(7,3) \leqslant 12 = \mathrm{CC}_1^*(7,3)$. By using (10), $\mathrm{C}(7,2) = 7$, and $\mathrm{CC}(7,3) = 12$, we obtain that $\mathrm{CC}(8,3) \leqslant 19 = \mathrm{CC}_1^*(8,3)$. Figure 3 proves $\mathrm{CC}(9,3) \leqslant 28 = \mathrm{CC}_1^*(9,3)$. From equation (10) and the fact that $\mathrm{CC}(9,3) = 28$ and $\mathrm{C}(9,2) = 12$, we conclude that $\mathrm{CC}(10,3) \leqslant 40 = \mathrm{CC}_1^*(10,3)$. Now, Figure 4 proves that $\mathrm{CC}(11,3) \leqslant 55 = \mathrm{CC}_1^*(11,3)$. Finally, to construct a connected covering witnessing $\mathrm{CC}_1^*(12,3)$ we delete the block $\{2,4,6,8\}$ from the covering in Figure 4. One can check that this still leaves a covering $\mathcal{B}$, whose graph now has three components. Now, we take the following (disconnected) $(11,3,2)$-covering:

$$\left\{ \begin{array}{l} \{1,3,11\},\{1,4,6\},\{1,2,8\},\{1,5,9\},\{1,7,10\},\{3,4,9\},\{2,3,10\},\{3,5,6\}, \\ \{3,7,8\},\{2,4,6\},\{4,5,7\},\{4,11,10\},\{4,6,8\},\{2,5,11\},\{2,7,9\},\{5,8,10\}, \\ \{6,7,11\},\{8,9,11\},\{6,9,10\} \end{array} \right\}.$$

We add to each of these block the element 12 and thus together with $\mathcal{B}$ obtain a $(12,4,3)$-covering $\mathcal{B}'$. To see that $\mathcal{B}'$ is connected, note that each of the blocks containing 12 is connected to a block from $\mathcal{B}$. Moreover, the blocks $\{1,4,6,12\}$, $\{2,4,6,12\}$, $\{4,6,8,12\}$ form a triangle and each of them has a neighbor in a different component of $G(\mathcal{B})$. Thus, $G(\mathcal{B}')$ is connected and $\mathcal{B}'$ has 73 blocks which coincides with $\mathrm{CC}_1^*(12,3)$. $\square$

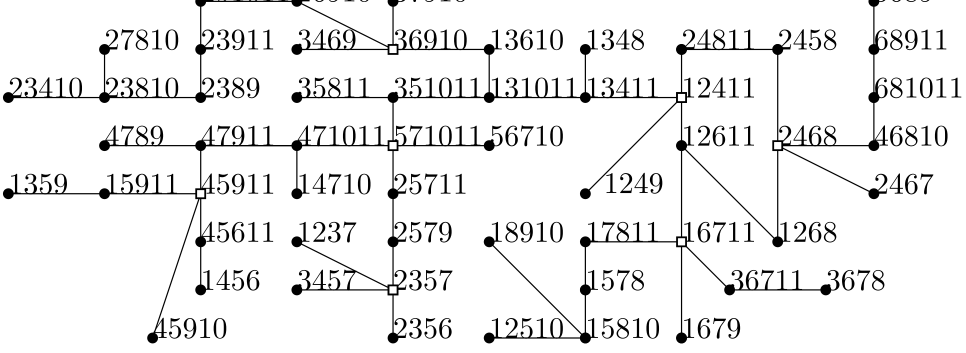

FIGURE 4. An example proving $\mathrm{CC}(11,3) \leqslant 55$. The circle-vertices are a covering.

Theorem 3.3 supports the following

Conjecture 3.4. For every positive integer $n \geqslant 4$, we have

$$\mathrm{CC}(n,3) = \mathrm{CC}_1^*(n,3).$$

Even, more ambitious,

Question 3.5. Let n and r be two positive integers such that $n \geqslant r+1 \geqslant 4$. Is it true that if $\mathrm{CC}(n,r) = \mathrm{CC}_1^*(n,r)$ then $\mathrm{CC}(n',r) = \mathrm{CC}_1^*(n',r)$ for every integer $n' \geqslant n$?

3.2. **Results when r is large.**

Theorem 3.6. *Let n be a positive integer with $n \geqslant 3$. Then, we have*

$$\mathrm{CC}(n,n-3) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} + 1.$$

Proof. The parameter $\mathrm{C}(n,n-3) = \mathrm{T}(n,3,2)$ was determined already by Mantel in 1907 [13] and is $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$. Turán proved that the *unique* minimal configuration of sets of size 2 hitting all sets of size 3 of an n-set are the edges of two vertex-disjoint complete graphs $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$, see [20].

Now, by (3) and (4), the covering corresponding to the Turán-system is connected if and only if the graph whose edges correspond to the blocks of the Turán-system is connected. Thus, since the unique optimal construction by Turán is not connected but can be made connected by adding a single edge connecting the two complete graphs, this is optimal with respect to connectivity. Therefore, $\mathrm{CC}(n,n-3) = \mathrm{T}(n,3,2)+1$, giving the result. □

Proposition 3.7. *Let $n \neq 5,6,8,9$ be a positive integer with $n \geqslant 4$. Then, we have*

$$\mathrm{CC}(n,n-4) \leqslant \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m, \\ m^2(2m-1) & \text{if } n = 3m+1, \\ m^2(2m+1) & \text{if } n = 3m+2. \end{cases}$$

If $n = 5,6,9$ the value of $\mathrm{CC}(n,n-4)$ is one larger than claimed in the formula. Further, $\mathrm{CC}(8,4) \in \{20,21\}$, i.e., it remains open if the above formula has to be increased by one or not in order to give the precise value.

Proof. We will show that a Turán-system $\mathcal{D}$ verifying the claimed bounds due to Kostochka [11] is connected. Indeed the construction of [11] is a parametrized family of Turán-systems, each of whose members attains the claimed bound. Our construction results from picking special parameters:

Assume that $n \geqslant 12$ and n is divisible by 3. Split $[n]$ into three sets A_1, A_2, A_3 of equal size. Pick special elements $x_i, y_i \in A_i$ and denote $B_i := A_i \setminus \{x_i, y_i\}$ for $i = 0,1,2$. The blocks of $\mathcal{D}$ consist of 3-element sets $\{a,b,c\}$ of the following forms:

- L_i: $a,b,c \in A_i$,
- $T1_i$: $a = x_i$ and $b,c \in A_{i+1}$,
- $T2_i$: $a = y_i$ and $b,c \in B_{i-1} \cup \{x_{i+1}, y_{i+1}\}$,
- $T3_i$: $a \in B_i$ and $b,c \in B_{i-1} \cup \{x_{i+1}, y_{i-1}\}$

where $i = 0,1,2$, and addition of indices is understood modulo 3.

Let us now show that $\mathcal{D}$ is connected. Clearly, all blocks in a given A_i are are connected. It suffices to verify that there is two 2-element sets $\{e,f\} \subseteq A_0$ and $\{e',f'\} \subseteq A_2$ which can be connected by a sequence of blocks of $\mathcal{D}$. The connectivity of $\mathcal{D}$ then follows by the symmetry of the construction. Let $\{e,f\} \subseteq B_0$. Take $\{e,f,y_1\} \in T2_1$, then $\{e,y_1,y_2\} \in T2_1$, and then $\{e,f',y_2\} \in T3_0$, where $f' \in B_2$, i.e, $\{y_2,f'\} \subseteq B_2$.

Now, following [11] deleting any element of such a system yields a Turán-system $\mathcal{D}'$ of the claimed size for $n' = n - 1$. We can just delete any x_i, since these are not used for connectivity. Following [11], two elements can be removed from $\mathcal{D}$ to obtain a Turán-system $\mathcal{D}''$ of the claimed size for $n'' = n - 2$, if the set formed by these two elements belongs to exactly $\frac{n}{3} - 1$ blocks. This is the case for $\{x_i, x_{i+1}\}$, which belongs to exactly $\frac{n}{3} - 1$ blocks from $T1_i$. Again, this preserves connectivity.
We are left with the cases $n \leqslant 9$. In [19] it is shown that the Turán-systems of the claimed size for $n = 9$ are exactly the members of the family constructed in [11]. There are exactly two such systems:
In both cases [9] is split into three sets A_1, A_2, A_3 of size 3. In the first system we pick a $x_i \in A_i$ and denote $A_i \setminus |x_i$ by B_i. The blocks then are the 3-element sets $\{a, b, c\}$ of the following forms:

L_i: $a, b, c \in A_i$,
$T1_i$: $a = x_i$ and $b, c \in A_{i+1}$,
$T2_i$: $a \in B_i$ and $b, c \in B_{i-1} \cup \{x_{i+1}\}$.

The second system coincides with an instance of a construction due to Turán [21]. It consists of the following 3-element sets:

L_i: $a, b, c \in A_i$,
$T1_i$: $a \in A_i$ and $b, c \in A_{i+1}$.

It is easy to check that both systems are not connected. On the other hand, the second one can be made connected adding a single block taking one elements from each A_i. This proves the claim form $n = 9$. Further, removing any vertex not contained in the added block, one obtains a connected Turán-system for $n = 9$ with 21 blocks. While there are Turán-systems showing $\mathrm{T}(8, 4) = 20$ we do not know if there is any connected such system.
See Figure 1 for proving our statement for $n = 7$, Theorem 3.2 for $n = 6$, and Remark 3.1 for $n = 4, 5$. □

A famous conjecture of Turán [21] states that the bounds in Proposition 3.7 are best possible for $\mathrm{C}(n, n-4)$. By combining (1) and Proposition 3.7, for $n \geqslant 10$ we have

$$\mathrm{C}(n, n-4) \leqslant \mathrm{CC}(n, n-4) \leqslant \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m, \\ m^2(2m-1) & \text{if } n = 3m+1, \\ m^2(2m+1) & \text{if } n = 3m+2. \end{cases} \tag{11}$$

Turán's conjecture has been verified for all $n \leqslant 13$ by [19] and so, by (11), the connected covering number can also be determined for these same values.
Towards proving Turán's conjecture, it would be of interest to investigate the following.

Question 3.8. Is it true that one of the inequalities in (11) is actually an equality ?

Bounds and precise values for all $\mathrm{CC}(n, r)$ with $n \leqslant 14$ are given in Table 1. All the exact values previously given in [3] for the same range have been improved by using our above results.

Table 1 led us to consider the following.

Question 3.9. Is the sequence $(\mathrm{CC}(n, i))_{0 \leqslant i \leqslant n-1}$ *unimodal* for every n ? or perhaps *logarithmically concave*[1] ?

[1]A finite sequence of real numbers $\{a_1, a_2, \ldots, a_n\}$ is said to be *unimodal* (resp. *logarithmically concave* or *log-concave*) if there exists a t such that $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_t$ and $s_t \geqslant s_{t+1} \geqslant \cdots \geqslant s_n$ (resp. if $a_i^2 \geqslant a_{i-1}a_{i+1}$ holds for every a_i with $1 \leqslant i \leqslant n-1$). Notice that a log-concave sequence is unimodal.

$r \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10	11	12	13
2			1	3	$5^{e,t}$	7^e	10^e	14^e	18^e	22^e	27^e	33^e	39^e	45^e
3				1	4	$7^{p,t}$	$12^{p,u}$	19^p	28^p	40^p	55^p	73^p	$[95^l, 97^r]$	$[121^l, 123^r]$
4					1	5	10^t	$[20, 21^u]$	$[32^l, 34^r]$	$[53^l, 59^r]$	$[83^l, 89^r]$	$[124^l, 136^r]$	$[179^l, 193^r]$	$[250^l, 271^r]$
5						1	6	13^t	31^u	$[51^l, 60^r]$	$[96^a, 111^r]$	$[159^l, 177^r]$	$[258^l, 290^r]$	$[401^l, 447^r]$
6							1	7	17^t	45^u	$[84^a, 95^r]$	$[165^a, 195^r]$	$[286^l, 327^r]$	$[501^l, 572^r]$
7								1	8	21^t	63^u	$[126^a, 147^r]$	$[269^a, 323^r]$	$[491^l, 587^r]$
8									1	9	26^t	84^u	$[185^a, 210^r]$	$[419^a, 505^r]$
9										1	10	31^t	112^u	$[259^s, 297^r]$
10											1	11	37^t	$[143^s, 144^u]$
11												1	12	43^t
12													1	13
13														1

TABLE 1. Bounds and values of $\mathrm{CC}(n, r)$ for $n \leqslant 14$.

Key of Table 1 :

r — Upper bound for $\mathrm{CC}(n, r)$(from (10))
e — Exact values for $\mathrm{CC}(n, 2)$ (Theorem 3.2)
t — Exact values for $\mathrm{CC}(n, n-3)$ (Theorem 3.6)
l — Lower bound $\mathrm{CC}_1^*(n, r)$
p — Some exact values for $\mathrm{CC}(n, 3)$ (Theorem 3.3)
u — Upper bound for $\mathrm{CC}(n, n-4)$ (Proposition 3.7)
s — Lower bound for $\mathrm{C}(n, r)$ (from (8))
a — Lower bounds for $\mathrm{C}(n, r)$ (from [1])

4. A general upper bound

Let n and r be positive integers such that $n \geqslant r + 1 \geqslant 3$. Forge and Ramírez Alfonsín [3] obtained the following general upper bound

$$\mathrm{CC}(n, r) \leqslant \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n-2i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor =: \mathrm{S}(n, r). \tag{12}$$

Let us notice that the upper bounds obtained by applying the recursive equation (10), that were used in Table 1, are better than the one given by (12). Moreover, by iterating (10) it can be obtained

$$\mathrm{CC}(n, r) \leqslant \sum_{i=r}^{n-1} \mathrm{C}(i, r-1). \tag{13}$$

Although (13) might be used to get an *explicit* upper bound for $s(n, r)$, it is not clear how good it would be since that would depend on the known exact values and the upper bounds of $\mathrm{C}(n, r)$ used in the recurrence (and thus intrinsically difficult to compute). On the contrary, in [3] was used (12) to give the best known (to our knowledge) explicit upper bound for $s(n, r)$.

In this section, we will construct a connected $(n, r+1, r)$-covering giving an upper bound for $\mathrm{CC}(n, r)$ better than $\mathrm{S}(n, r)$ and so, yielding a better upper bound for $s(n, r)$ than that given in [3].

Theorem 4.1. *Let n and r be positive integers such that $n \geqslant r+1 \geqslant 3$. Then,*

$$\mathrm{CC}(n,r) \leqslant \mathrm{N}(n,r),$$

with

$$\text{(14)} \quad \mathrm{N}(n,r) := \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil - 1} (n-r-2i)\binom{r-2+2i}{r-2} + \left\lceil \frac{n-r}{2} \right\rceil - 1 + \delta_0 \mathrm{C}(n-2,r-2),$$

where δ_0 is the parity function of $n-r$, that is,

$$\delta_0 = \begin{cases} 0 & \text{if } n-r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. From now, for any positive integer s, we denote by $[s]$ the set of the first s positive integers, that is, $[s] = \{1, \dots, s\}$ and we denote by $U_{s,t}$ the set of all t-subsets of $[s]$. Moreover, suppose that for any subset of integers $\{b_1, \dots, b_s\}$, we have $b_i < b_j$ for all $1 \leqslant i < j \leqslant s$. We distinguish different cases depending on the parity of $n-r$.

Case 1. Suppose that $n-r$ is an odd number and let m such that $n-r = 2m+1$. We will construct a connected $(r+2m+1, r+1, r)$-covering of size

$$\sum_{i=0}^{m} \binom{r-2+2i}{r-2}(2m+1-2i) + m.$$

First, we consider a particular $(r+2m+1, r+1, r)$-covering, which is constituted by a large number of blocks but whose associated graph has a small number of connected components. For any $i \in \{0, \dots, m\}$, let $\mathcal{N}_i$ be the following subset of $(r+1)$-subsets of $[r+2m+1]$:

$$\mathcal{N}_i := \left\{ \{b_1, \dots, b_{r+1}\} \;\middle|\; \begin{array}{l} \{b_1, \dots, b_{r-2}\} \in U_{r+2i-2,r-2} \\ b_{r-1} = r+2i-1, \ b_r = r+2i \\ b_{r+1} \in \{r+2i+1, \dots, r+2m+1\} \end{array} \right\}.$$

Claim 4.2. *The set $\mathcal{A} = \bigcup_{i=0}^{m} \mathcal{N}_i$ is an $(r+2m+1, r+1, r)$-covering.*

Let $S = \{s_1, \dots, s_r\}$ be an r-subset of $[r+2m+1]$. If $s_{r-1} = r+2i-1$ for some $i \in \{0, \dots, m\}$, then

$$\begin{array}{ll} S \subset (S \cup \{r+2i+1\}) \in \mathcal{N}_i & , \text{ for } s_r = r+2i, \\ S \subset (S \cup \{r+2i\}) \in \mathcal{N}_i & , \text{ for } s_r > r+2i. \end{array}$$

Either, if $s_{r-1} = r+2i$ for some $i \in \{0, \dots, m\}$, then

$$\begin{array}{ll} S \subset (S \cup \{r+2i-1\}) \in \mathcal{N}_i & , \text{ for } s_{r-2} < r+2i-1, \\ S \subset (S \cup \{\alpha\}) \in \mathcal{N}_i & , \text{ for } s_{r-2} = r+2i-1, \end{array}$$

where $\alpha \in [r+2i-2] \setminus \{s_1, \dots, s_{r-3}\}$. In every case, there exists a block in $\mathcal{N}_i$, for some $i \in \{0, \dots, m\}$, containing the r-subset S. This concludes the proof of Claim 4.2.

Claim 4.3. *The graph $G(\mathcal{N}_i)$ is connected, for every $i \in \{0, \dots, m\}$.*

Let $i \in \{0, \dots, m\}$ and let $S = \{s_1, \dots, s_{r-2}, r+2i-1, r+2i, s_{r+1}\}$ be a block in $\mathcal{N}_i$. For every $j \in \{0, \dots, r-2\}$, we consider the following $(r+1)$-subsets $S_j \in U_{r+2i+1,r+1}$:

$$S_0 = \{s_1, \dots, s_{r-2}, r+2i-1, r+2i, r+2i+1\},$$

$$S_j = \{1, \dots, j, s_{j+1}, \dots, s_{r-2}, r+2i-1, r+2i, r+2i+1\}, \quad \text{for } 1 \leqslant j \leqslant r-3,$$

$$S_{r-2} = \{1, \dots, r-2, r+2i-1, r+2i, r+2i+1\}.$$

By definitions, it is easy to see that $S_j \in \mathcal{N}_i$ for all $j \in \{0, \ldots, r-2\}$. Now, we consider the r-subsets $T_j \in U_{r+2i-1,r}$ defined by

$$T_0 = \{s_1, \ldots, s_{r-2}, r+2i-1, r+2i\},$$

$$T_1 = \{s_2, \ldots, s_{r-2}, r+2i-1, r+2i, r+2i+1\},$$

$$T_j = \{1, \ldots, j-1, s_{j+1}, \ldots, s_{r-2}, r+2i-1, r+2i, r+2i+1\}, \quad \text{for } 1 \leqslant j \leqslant r-3,$$

$$T_{r-2} = \{1, \ldots, r-3, r+2i-1, r+2i, r+2i+1\}.$$

Since $T_0 \subset S$ and $T_0 \subset S_0$, we know that S and S_0 are adjacent in the graph $G(\mathcal{N}_i)$. By the same way, since $T_j \subset S_{j-1}$ and $T_j \subset S_j$, the blocks S_{j-1} and S_j are adjacent in $G(\mathcal{N}_i)$ for all $j \in \{1, \ldots, r-2\}$. It follows that $(S, S_0, S_1, \ldots, S_{r-2})$ is a path in $G(\mathcal{N}_i)$. So we have proved that there always exists a path between any block S of $\mathcal{N}_i$ and the fixed block S_{r-2}. This concludes the proof of Claim 4.3.

Claim 4.4. *For any integer $i \in \{0, \ldots, m-1\}$, there exists an $(r+1)$-subset C_i such that the graph $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$ is connected.*

Let $i \in \{0, \ldots, m-1\}$ and let

$$C_i := \{1, \ldots, r-2, r+2i, r+2i+1, r+2i+2\}.$$

Then, the block C_i is adjacent to

$$B_i = \{1, \ldots, r-2, r+2i-1, r+2i, r+2i+1\} \in \mathcal{N}_i$$

and is adjacent to

$$B_{i+1} = \{1, \ldots, r-2, r+1+2i, r+2+2i, r+3+2i\} \in \mathcal{N}_{i+1}$$

in $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$. Since $G(\mathcal{N}_i)$ and $G(\mathcal{N}_{i+1})$ are connected by Claim 4.3, we obtain that $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$ is connected. This concludes the proof of Claim 4.4.

By combining results of Claims 4.2, 4.3 and 4.4, we know that the set

$$\mathcal{B} = \left(\bigcup_{i=0}^{m} \mathcal{N}_i\right) \bigcup \left(\bigcup_{i=0}^{m-1} C_i\right)$$

is a connected $(r+2m+1, r+1, r)$-covering. Finally, since

$$|\mathcal{N}_i| = \binom{r-2+2i}{r-2}(2m+1-2i),$$

for all $i \in \{0, \ldots, m\}$, we deduce that

$$|\mathcal{B}| = \sum_{i=0}^{m} \binom{r-2+2i}{r-2}(2m+1-2i) + m.$$

This completes the proof in this case.

Case 2. Suppose that $n-r$ is an even number and let m such that $n-r = 2m$. We will construct a connected $(r+2m, r+1, r)$-covering of size

$$\sum_{i=0}^{m-1} \binom{r-2+2i}{r-2}(2m-2i) + \mathrm{C}(r-2+2m, r-2) + m - 1.$$

As already defined in Case 1, for any integer $i \in \{0, \ldots, m-1\}$, we consider the collection $\mathcal{N}_i$ of $(r+1)$-subsets of $[r+2m]$ defined by

$$\mathcal{N}_i := U_{r+2i-2,r-2} \times \{r+2i-1\} \times \{r+2i\} \times \{r+2i+1, \ldots, r+2m\}.$$

Let

$$\mathcal{A} := \bigcup_{i=0}^{m-1} \mathcal{N}_i.$$

Observe that $\mathcal{A}$ is not an $(r+2m, r+1, r)$-covering. In fact, from Claim 4.2, we know that the r-subsets of $[r+2m]$ that are not in $\mathcal{A}$ are just the subsets of the form

$$\{b_1, \ldots, b_{r-2}, r+2m-1, r+2m\},$$

where $\{b_1, \ldots, b_{r-2}\} \in U_{r+2m-2,r-2}$. Now, we complete $\mathcal{A}$ in order to obtain an $(r+2m, r+1, r)$-covering. Let $\mathcal{C}$ be an $(r+2m-2, r-1, r-2)$-covering of size $\mathrm{C}(r+2m-2, r-2)$. We consider the set

$$\mathcal{N}_m := \{B \cup \{r+2m-1, r+2m\} \mid B \in \mathcal{C}\}$$

Claim 4.5. *The set $\mathcal{A} \cup \mathcal{N}_m$ is an $(r+2m, r+1, r)$-covering.*

Since the r-subsets of $[r+2m]$ that are not in $\mathcal{A}$ are elements of $U_{r+2m-2,r-2} \times \{r+2m-1\} \times \{r+2m\}$ and since every element of $U_{r+2m-2,r-2}$ is contained in a block of $\mathcal{C}$, the result of Claim 4.5 follows.

For every $i \in \{0, \ldots, m-1\}$, we already know, from Claim 4.3, that the graph $G(\mathcal{N}_i)$ is connected. Moreover, for every $i \in \{0, \ldots, m-2\}$, Claim 4.4 implies that the graph $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$ is connected, where

$$C_i := \{1, \ldots, r-2, r+2i, r+2i+1, r+2i+2\} \in U_{r+2m,r+1}.$$

We end this proof by showing that the graph $G(\mathcal{A} \cup \mathcal{N}_m \cup C_0 \cup \cdots \cup C_{m-2})$ is connected.

Claim 4.6. *For any $B \in \mathcal{N}_m$, there exist $i \in \{0, \ldots, m-1\}$ and $C \in \mathcal{N}_i$ such that B is adjacent to C in the graph $G(\mathcal{N}_i \cup \mathcal{N}_m)$.*

Let $B = \{b_1, \ldots, b_{r-1}, r+2m-1, r+2m\} \in \mathcal{N}_m$. If $b_{r-1} = r+2i-1$ for some $i \in \{0, \ldots, m-1\}$, then we have $\{b_1, \ldots, b_{r-2}\} \in U_{r+2i-2,r-2}$. Let

$$C := \{b_1, \ldots, b_{r-2}, r+2i-1, r+2i, r+2m\}.$$

By definition, C is in $\mathcal{N}_i$. Moreover, since $\{b_1, \ldots, b_{r-2}, r+2i-1, r+2m\} \subset B$ and $\{b_1, \ldots, b_{r-2}, r+2i-1, r+2m\} \subset C$, we deduce that B and C are adjacent in the graph $G(\mathcal{N}_i \cup \mathcal{N}_m)$. Either, if $b_{r-1} = r+2i$ for some $i \in \{0, \ldots, m-1\}$, then we have $\{b_1, \ldots, b_{r-2}\} \in U_{r+2i-1,r-2}$. We distinguish two cases on the value of b_{r-2}. First, if $b_{r-2} < r+2i-1$, then $\{b_1, \ldots, b_{r-2}\} \in U_{r+2i-2,r-2}$. Let

$$C := \{b_1, \ldots, b_{r-2}, r+2i-1, r+2i, r+2m\}.$$

As above, since $\{b_1, \ldots, b_{r-2}, r+2i, r+2m\} \subset B$ and $\{b_1, \ldots, b_{r-2}, r+2i, r+2m\} \subset C$, we deduce that B and C are adjacent in the graph $G(\mathcal{N}_i \cup \mathcal{N}_m)$. Finally, suppose that $b_{r-2} = r+2i-1$. Let $\alpha \in [r+2i-2] \setminus \{b_1, \ldots, b_{r-3}\}$ and let

$$C := \{b_1, \ldots, b_{r-3}, r+2i-1, r+2i, r+2m\} \cup \{\alpha\} \in U_{r+2m,r+1}.$$

Since $\{b_1, \ldots, b_{r-3}, r+2i-1, r+2i, r+2m\} \subset B$ and $\{b_1, \ldots, b_{r-3}, r+2i-1, r+2i, r+2m\} \subset C$, we deduce that B and C are adjacent in the graph $G(\mathcal{N}_i \cup \mathcal{N}_m)$. This concludes the proof of Claim 4.6.

By combining results of Claims 4.4, 4.5 and 4.6, we know that the set

$$\mathcal{B} = \left(\bigcup_{i=0}^{m} \mathcal{N}_i\right) \bigcup \left(\bigcup_{i=0}^{m-2} C_i\right)$$

is a connected $(r+2m, r+1, r)$-covering. Finally, since

$$|\mathcal{N}_i| = \binom{r-2+2i}{r-2}(2m+1-2i),$$

for all $i \in \{0, \ldots, m-1\}$, and

$$|\mathcal{N}_m| = \mathrm{C}(r+2m-2, r-2),$$

we deduce that

$$|\mathcal{B}| = \sum_{i=0}^{m-1} \binom{r-2+2i}{r-2}(2m+1-2i) + \mathrm{C}(r+2m-2, r-2) + m - 1.$$

This completes the proof of Theorem 4.1. □

Let us illustrate the construction given in the above theorem.

Example 4.7. $\mathrm{N}(7,4) = 10$. We consider

$$\mathcal{N}_0 = \{12345, 12346, 12347\} \text{ and } \mathcal{N}_1 = \{12567, 13567, 14567, 23567, 24567, 34567\}.$$

It can be checked that $\mathcal{N}_0 \cup \mathcal{N}_1$ is a $(7,5,4)$-covering and $G(\mathcal{N}_0)$ and $G(\mathcal{N}_1)$ are connected. Now, by taking $C_0 = 12456$, it follows that $G(\mathcal{N}_0 \cup C_0 \cup \mathcal{N}_1)$ is connected.

We may now show that $\mathrm{S}(n,r) > \mathrm{N}(n,r)$. For this we need first the following Theorem and Proposition.

Theorem 4.8. *Let r and n be positive integers such that $n \geqslant r+1 \geqslant 3$. Then,*

$$\mathrm{S}(n,r) = \mathrm{N}(n,r) + \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor - 1} \left(\left\lfloor \frac{n-r}{2} \right\rfloor - i \right) \binom{r-2+2i}{r-3} + \delta_0 \left(1 - \mathrm{C}(n-2, r-2)\right),$$

where δ_0 is the parity function of $n-r$.

Proof. By induction on $n > r$. For $n = r+1$ and $n = r+2$ we have from (12) and (14) that

$$\mathrm{S}(r+1,r) = \binom{r-1}{r-1} = 1, \quad \mathrm{N}(r+1,r) = 1.\binom{r-2}{r-2} + 1 - 1 = 1,$$

and

$$\begin{aligned} \mathrm{S}(r+2,r) = \binom{r}{r-1} + 1 = r+1, \quad \mathrm{N}(r+2,r) &= 2.\binom{r-2}{r-2} + 1 - 1 + \mathrm{C}(r, r-2) \\ &= 2 + 1 - 1 + r - 1 = r+1. \end{aligned}$$

Thus the identity is verified for $n = r+1$ and $n = r+2$. Suppose now that for a certain value of n, we have

$$\mathrm{S}(n,r) = \mathrm{N}(n,r) + \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor - 1} \left(\left\lfloor \frac{n-r}{2} \right\rfloor - i \right) \binom{r-2+2i}{r-3} + \delta_0 \left(1 - \mathrm{C}(n-2, r-2)\right),$$

and let D be the difference

$$D := \left(\mathrm{S}(n+2,r) - \mathrm{N}(n+2,r)\right) - \left(\mathrm{S}(n,r) - \mathrm{N}(n,r)\right).$$

Then,

$$\mathrm{S}(n+2,r) = \mathrm{N}(n+2,r) + \left(\mathrm{S}(n,r) - \mathrm{N}(n,r)\right) + D.$$

By using (12), we obtain

$$\begin{aligned}
\mathrm{S}(n{+}2,r){-}\mathrm{S}(n,r) &= \sum_{i=1}^{\lfloor\frac{n-r+1}{2}\rfloor+1}\binom{n+2-2i}{r-1} + \left\lfloor\frac{n-r}{2}\right\rfloor + 1 - \sum_{i=1}^{\lfloor\frac{n-r+1}{2}\rfloor}\binom{n-2i}{r-1} - \left\lfloor\frac{n-r}{2}\right\rfloor \\
&= \sum_{i=0}^{\lfloor\frac{n-r+1}{2}\rfloor}\binom{n-2i}{r-1} + 1 - \sum_{i=1}^{\lfloor\frac{n-r+1}{2}\rfloor}\binom{n-2i}{r-1} \\
&= \binom{n}{r-1} + 1.
\end{aligned}$$

By using (14), we have

$$\begin{aligned}
\mathrm{N}(n{+}2,r){-}\mathrm{N}(n,r) &= \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil}(n+2-r-2i)\binom{r-2+2i}{r-2} + \left\lceil\frac{n-r}{2}\right\rceil + \delta_0\mathrm{C}(n,r-2) \\
&\quad - \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil-1}(n-r-2i)\binom{r-2+2i}{r-2} - \left\lceil\frac{n-r}{2}\right\rceil + 1 - \delta_0\mathrm{C}(n-2,r-2) \\
&= \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil-1} 2\binom{r-2+2i}{r-2} + \left(n+2-r-2\left\lceil\frac{n-r}{2}\right\rceil\right)\binom{r-2+2\lceil\frac{n-r}{2}\rceil}{r-2} \\
&\quad +\delta_0\left(\mathrm{C}(n,r-2)-\mathrm{C}(n-2,r-2)\right)+1.
\end{aligned}$$

Moreover, since

$$n-r-2\left\lceil\frac{n-r}{2}\right\rceil = \begin{cases} 0 & \text{if } n-r \text{ is even,} \\ -1 & \text{otherwise,} \end{cases}$$

and

$$\binom{r-2+2\lceil\frac{n-r}{2}\rceil}{r-2} = \binom{n-1}{r-2}$$

for $n-r$ odd, it follows that

$$\begin{aligned}
\mathrm{N}(n+2,r)-\mathrm{N}(n,r) = &\sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} 2\binom{r-2+2i}{r-2} + (\delta_0-1)\binom{n-1}{r-2} \\
&+\delta_0\left(\mathrm{C}(n,r-2)-\mathrm{C}(n-2,r-2)\right)+1
\end{aligned}$$

Therefore

$$D = \binom{n}{r-1} - \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} 2\binom{r-2+2i}{r-2} + (1-\delta_0)\binom{n-1}{r-2} + \delta_0\left(\mathrm{C}(n-2,r-2)-\mathrm{C}(n,r-2)\right).$$

Moreover, from the identity

$$\binom{r-2+2i}{r-2} = \binom{r-1+2i}{r-2} - \binom{r-2+2i}{r-3},$$

we obtain that

$$\sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} 2\binom{r-2+2i}{r-2} = \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-2+2i}{r-2} + \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-1+2i}{r-2} - \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-2+2i}{r-3}$$

$$= \sum_{i=r-2}^{r-1+2\lceil\frac{n-r}{2}\rceil} \binom{i}{r-2} - \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-2+2i}{r-3}$$

$$= \binom{r+2\left\lceil\frac{n-r}{2}\right\rceil}{r-1} - \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-2+2i}{r-3}.$$

Thus,

$$D = \binom{n}{r-1} - \binom{r+2\left\lceil\frac{n-r}{2}\right\rceil}{r-1} + (1-\delta_0)\binom{n-1}{r-2} + \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-2+2i}{r-3}$$

$$+\delta_0\left(\mathrm{C}(n-2,r-2)-\mathrm{C}(n,r-2)\right).$$

If $n-r$ is even, then $\delta_0 = 1$ and

$$\binom{n}{r-1} - \binom{r+2\left\lceil\frac{n-r}{2}\right\rceil}{r-1} + (1-\delta_0)\binom{n-1}{r-2} = \binom{n}{r-1} - \binom{n}{r-1} = 0.$$

Either, if $n-r$ is odd, then $\delta_0 = 0$ and

$$\binom{n}{r-1} - \binom{r+2\left\lceil\frac{n-r}{2}\right\rceil}{r-1} + (1-\delta_0)\binom{n-1}{r-2} = \binom{n}{r-1} - \binom{n+1}{r-1} + \binom{n-1}{r-2}$$

$$= -\binom{n}{r-2} + \binom{n-1}{r-2}$$

$$= -\binom{n-1}{r-3}.$$

It follows that

$$D = \sum_{i=0}^{\lceil\frac{n-r}{2}\rceil} \binom{r-2+2i}{r-3} + (\delta_0 - 1)\binom{n-1}{r-3} + \delta_0\left(\mathrm{C}(n-2,r-2)-\mathrm{C}(n,r-2)\right)$$

$$= \sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor} \binom{r-2+2i}{r-3} + \delta_0\left(\mathrm{C}(n-2,r-2)-\mathrm{C}(n,r-2)\right).$$

Now, with the induction hypothesis, we obtain

$$\begin{aligned}
\mathrm{S}(n{+}2,r){-}\mathrm{N}(n{+}2,r) &= (\mathrm{S}(n,r)-\mathrm{N}(n,r))+D \\
&= \sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor-1}\left(\left\lfloor\frac{n-r}{2}\right\rfloor-i\right)\binom{r-2+2i}{r-3}+\delta_0\left(1-\mathrm{C}(n-2,r-2)\right) \\
&\quad+\sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor}\binom{r-2+2i}{r-3}+\delta_0\left(\mathrm{C}(n-2,r-2)-\mathrm{C}(n,r-2)\right) \\
&= \sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor}\left(\left\lfloor\frac{n-r}{2}\right\rfloor+1-i\right)\binom{r-2+2i}{r-3}+\delta_0\left(1-\mathrm{C}(n,r-2)\right).
\end{aligned}$$

□

Proposition 4.9. *Let n and r be positive integers such that $n \geqslant r+3 \geqslant 5$. Then,*

$$\mathrm{C}(n,r)\leqslant\binom{n-2}{r-1}+\mathrm{C}(n-2,r).$$

Proof. Let $S_1=\{A_1,\ldots,A_s\}$ be a $(n-2,r+1,r)$-covering such that $s=\mathrm{C}(n-2,r+1,r)$ and let $U_{n-2,r-1}$ be the set of all the $(r-1)$-subsets of $\{1,\ldots,n-2\}$. We denote by $U_{n-2,r-1}=\{B_1,\ldots,B_t\}$ where $t=\binom{n-2}{r-1}$ these elements. We will prove that the collection

$$S_2=\{A_1,\ldots,A_s,B_1\cup\{n-1,n\},\ldots,B_t\cup\{n-1,n\}\}$$

is a $(n,r+1,r)$-covering. Obviously, all the blocks of the form A_i or $B_i\cup\{n-1,n\}$ are $(r{+}1)$-subsets of $\{1,\ldots,n\}$. Let C be an r-subset of $\{1,\ldots,n\}$. First, if $C\cap\{n{-}1,n\}=\emptyset$, by definition of S_1, there exists at least one block A_i containing C. Now, if $C\cap\{n{-}1,n\}=\{x\}$, with either $x=n-1$ or $x=n$, then $C\setminus\{x\}$ is an $(r-1)$-subset of $\{1,\ldots,n-2\}$, thus $C\setminus\{x\}=B_i$ for a certain $1\leqslant i\leqslant t$. It follows that the r-subset C is contained in $B_i\cup\{n-1,n\}$. Finally, if $\{n-1,n\}\subset C$, then $C\setminus\{n-1,n\}$ is an $(r-2)$-subset of $\{1,\ldots,n-2\}$ and thus there exist exactly $n-r$ blocks B_i of $U_{n-2,r-1}$ containing $C\setminus\{n-1,n\}$. It follows that there exist $n-r$ blocks of the form $B_i\cup\{n-1,n\}$ containing C. Therefore S_2 is a $(n,r+1,r)$-covering and

$$\mathrm{C}(n,r)\leqslant|S_2|=|S_1|+|U_{n-2,r-1}|=\mathrm{C}(n-2,r)+\binom{n-2}{r-1}.$$

□

Theorem 4.10. *Let r and n be positive integers such that $n-r$ is an even number. Then,*

$$\mathrm{S}(n,r)\geqslant\mathrm{N}(n,r)+\sum_{i=0}^{\frac{n-r}{2}-2}\left(\frac{n-r}{2}-i-1\right)\binom{r-2+2i}{r-3}.$$

Proof. By applying repeatedly Proposition 4.9, we have

$$\mathrm{C}(n-2,r-2)\leqslant\sum_{i=0}^{\frac{n-r}{2}-1}\binom{r-2+2i}{r-3}+1.$$

Then, we deduce from Theorem 4.8 that

$$\begin{aligned}
\mathrm{S}(n,r) &= \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-1} \left(\frac{n-r}{2} - i\right) \binom{r-2+2i}{r-3} + 1 - \mathrm{C}(n-2,r-2) \\
&\geqslant \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-1} \left(\frac{n-r}{2} - i\right) \binom{r-2+2i}{r-3} - \sum_{i=0}^{\frac{n-r}{2}-1} \binom{r-2+2i}{r-3} \\
&= \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-2} \left(\frac{n-r}{2} - i - 1\right) \binom{r-2+2i}{r-3}.
\end{aligned}$$

□

5. Asymptotics

In [16] Rödl uses the probabilistic method to show the existence of *asymptotically good coverings*. Restricted to our case this means that

$$\frac{\mathrm{C}(n,r)}{\binom{n}{r}} \to \frac{1}{r+1} \text{ as } n \to \infty.$$

Since $\mathrm{CC}(n,r) \leqslant 2\mathrm{C}(n,r)$ (see [3]) we immediately obtain:

$$\frac{\mathrm{CC}(n,r)}{\binom{n}{r}} \to a \leqslant \frac{2}{r+1} \text{ as } n \to \infty.$$

This is asymptotically, i.e., for large n compared to r, better than the results of the previous section. In [3] it was shown that

$$\frac{\mathrm{S}(n,r)}{\binom{n}{r}} \to \frac{1}{2} \text{ as } n \to \infty$$

and since by Theorem 4.10 the difference $\mathrm{N}(n,r) - \mathrm{S}(n,r)$ is in $\mathcal{O}(n^{r-1})$ we have the same asymptotic behavior for $\mathrm{N}(n,r)$.
It is however still a topic of research to find explicit constructions witnessing the bound of Rödl, see [12].

Acknowledgments. Much of this work in particular for the construction of the connected covering designs in Figures 1, 3, 4 strongly benefited from the La Jolla Covering Repository (`http://www.ccrwest.org/cover.html`) maintained by Dan Gordon.

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Appendix - Proof of Proposition 1.2

For basic notions of oriented matroids we refer the reader to [2].

Let $\mathcal{M}$ be an oriented matroid having $U_{r,n}$ as underlying matroid. We shall show that a set S of circuits of $U_{r,n}$ with $|S| < \mathrm{C}(n,r)$ cannot determine $\mathcal{M}$. Since $|S| < \mathrm{C}(n,r)$ then S does not cover all the bases of $U_{r,n}$. Let us suppose that B is a base not covered by any circuit in S. Let $\mathcal{M}^*$ be the dual of $\mathcal{M}$, and thus having as underlying matroid $U^*_{r,n} = U_{n-r,n}$. Let $\mathcal{A}$ be the $(n-r)$-dimensional oriented simple pseudo-sphere arrangements corresponding to the non-oriented matroid $U^*_{r,n}$. We may assume that the elements in B^* form a simplex, say T_{B^*} of $\mathcal{A}$. The latter can be done by taking first any arrangement representing $U_{n-r,n}$ having at least one simplex T, we then label the pseudo-hyperplanes bounding T with the elements of B^* (the rest of pseudo-hyperplanes can be labeled with elements in $E \setminus B^*$ as desired). For example, if we take $U_{5,2}$ then the set of circuits $S = \{\{1,2,3\},\{2,3,4\},\{1,3,5\}\}$ do not cover the base $B = \{4,5\}$. We choose the arrangement given in Figure 5 that correspond to $U^*_{2,5} = U_{3,5}$ where the lines corresponding to the (highlighted) triangle were first labeled with elements in $B^* = \{1,2,3\}$.

It is known that the vertices of the arrangement $\mathcal{A}$ correspond to the oriented cocircuits of $\mathcal{M}^*$ which, by duality, are the oriented circuits of $\mathcal{M}$. We notice that any of the cocircuits corresponding to the vertices of simplex T_{B^*} are circuits of $U_{r,n}$ containing B (indeed, since the underlying set of each cocircuit is formed by the pseudo-spheres not touching the corresponding vertex then, by construction of T_{B^*}, all the elements of B will be included in each such cocircuit). Therefore, none of the circuits in S correspond to the vertices of simplex T_{B^*}.

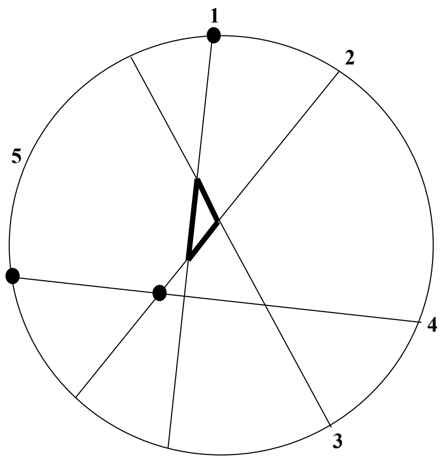

FIGURE 5. A pseudoline arrangement corresponding to $U_{3,5}$. Black circles correspond to the cocircuits $\{1,2,3\},\{2,3,4\}$ and $\{1,3,5\}$

We may now apply a *mutation* to simplex T_{B^*} obtaining another orientation of $U^*_{r,n}$ giving an other oriented matroid $\mathcal{M}'$. Notice that all the cocircuits of $\mathcal{M}'$ have the same orientation as in $\mathcal{M}$ except for those corresponding to vertices of T_{B^*} that have changed. Since none of the cocircuits corresponding to the vertices of T_{B^*} are circuits in S then S cannot determine uniquely the orientation of $U^*_{r,n}$ (the circuits in S cannot detect the mutation) and thus neither its dual of $\mathcal{M}^*$.

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