Lattice Points in Orthotopes
and a Huge Polynomial Tutte Invariant
of Weighted Gain Graphs

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ABSTRACT. A *gain graph* is a graph whose edges are orientably labelled from a group. A *weighted gain graph* is a gain graph with vertex weights from an abelian semigroup, where the gain group is lattice ordered and acts on the weight semigroup. For weighted gain graphs we establish basic properties and we present general dichromatic and forest-expansion polynomials that are Tutte invariants (they satisfy Tutte's deletion-contraction and multiplicative identities). Our dichromatic polynomial includes the classical graph one by Tutte, Zaslavsky's two for gain graphs, Noble and Welsh's for graphs with positive integer weights, and that of rooted integral gain graphs by Forge and Zaslavsky. It is not a universal Tutte invariant of weighted gain graphs; that remains to be found.

An evaluation of one example of our polynomial counts proper list colorations of the gain graph from a color set with a gain-group action. When the gain group is $\mathbb{Z}^d$, the lists are order ideals in the integer lattice $\mathbb{Z}^d$, and there are specified upper bounds on the colors, then there is a formula for the number of bounded proper colorations that is a piecewise polynomial function of the upper bounds, of degree nd where n is the order of the graph.

This example leads to graph-theoretical formulas for the number of integer lattice points in an orthotope but outside a finite number of affinographic hyperplanes, and for the number of $n \times d$ integral matrices that lie between two specified matrices but not in any of certain subspaces defined by simple row equations.

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CONTENTS

1. INTRODUCTION

An *integral orthotope* is a rectangular parallelepiped with integral vertices and edges parallel to the coordinate axes. An *integral affinographic hyperplane* is a hyperplane of the form $x_j = x_i + a$, where a is an integer. (All our orthotopes and affinographic hyperplanes will be integral.) We wish to count the points of the integer lattice that lie in an orthotope but not in any of a given *arrangement* (a finite set) of affinographic hyperplanes. An arrangement is *centered* if there is a point common to all hyperplanes.

Theorem 1.1. *Let $P := [0, m_1] \times \cdots \times [0, m_n]$, where $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$, be an orthotope in $\mathbb{R}^n$ and let $\mathcal{A}$ be an arrangement of affinographic hyperplanes. The number of integer points in $P \setminus \bigcup \mathcal{A}$ equals*

$$\sum_{\mathcal{B} \subseteq \mathcal{A}: centered} (-1)^{|\mathcal{B}|} f_{\mathcal{B}}(\mathbf{m}),$$

where each $f_{\mathcal{B}}$ is a function that depends linearly on the m_i if they are sufficiently large.

Theorem 1.1 generalizes the main theorem of [7], which covered the case of hypercubes, where all $m_i = m$. It is a simplified version of Theorem 5.15, which gives the exact form of

$f_\mathcal{B}$. The proof is carried out in terms of proper colorations of *weighted gain graphs*. Briefly, a gain graph is a graph whose edges are orientably labelled by elements of a group; that is, reversing the edge inverts the gain. In a weighted gain graph the group is lattice ordered and the vertices carry weights from an abelian semigroup on which the gain group has an action. For instance, the gain group and weight semigroup may be $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \max)$. Theorem 1.1 comes from that kind of weighted gain graph: a vertex corresponds to a coordinate in $\mathbb{R}^n$ and an edge $v_i v_j$ with gain a in the indicated direction corresponds to a hyperplane $x_j = x_i + a$. (In the opposite direction, $v_j v_i$, the gain is $-a$.)

Our aim is to understand the mathematics behind the main result of [7], as well as Theorem 1.1 and more general results like Theorem 5.17, by interpreting lattice-point–counting functions as Tutte invariants of weighted gain graphs. A consequence is a great generalization of the techniques and results of [7], which also incorporates as a special case the theorem of Noble and Welsh [9] on graphs with positive integral weights. (During the delayed revision of this paper for publication we found out about an independent generalization of [9] by Ellis-Monaghan and Moffatt [5]; it is our generalization without gains, but with edge weights. Their purpose is an application to statistical physics, for which they do not need gains. For excluding lattice points in affinographic hyperplanes, gains are essential.)

Our "huge polynomial Tutte invariant" is $Q_{(\Phi, h)}(\mathbf{u}, v, z)$, the total dichromatic polynomial of a weighted gain graph (Φ, h). (For the definition see Equation (3.1).) It is, first of all, a common generalization of the ordinary and balanced dichromatic polynomials of a gain graph from [15, Part III], which are direct generalizations of Tutte's dichromatic polynomial $Q_\Gamma(u, v)$ of a graph [12]. In a weighted gain graph the variable u of the standard dichromatic polynomial splits into a multitude of independent variables u_h, one for each possible vertex weight h; $\mathbf{u}$ denotes the totality of all u_h. Calling this polynomial a Tutte invariant is justified by the fact that $Q_{(\Phi, h)}(\mathbf{u}, v, z)$ satisfies the standard deletion-contraction, multiplication, and normalization identities of the Tutte polynomial of an ordinary graph or matroid. (The identities are stated in Section 3.)

We obtain counting functions by substituting particular values for the weight variables u_h. For instance, in Theorem 1.1 the weight semigroup is $\{[l, m] : l \leq m \in \mathbb{Z}\} \cup \{\varnothing\}$ with intersection as its operation and we get the lattice-point counting function by setting $u_{[l,m]} = -(m - l + 1)$ and $u_\varnothing = 0$ (see Section 5.4). We even have a formula for the number of $n \times d$ integral matrices in an integral orthotope in $\mathbb{Z}^{nd}$ that do not lie in any of a class of d-codimensional subspaces whose equations compare rows of the matrix (Section 5.5).

Example 1.1. We do a very small example, just large enough to suggest what a total dichromatic polynomial looks like. Our weighted gain graph (Φ, h) has $n = 2$ vertices. The gain group is $(\mathbb{Z}^2, +)$ (slightly larger than the group $(\mathbb{Z}, +)$ used to prove Theorem 1.1) with the lattice operations of componentwise max and min. The weight semigroup is $(\mathbb{Z}^2, \max)$. The edge set is

$$E = \{e_1 = (0,0)(v_1, v_2),\ e_2 = (2,0)(v_1, v_2),\ e_3 = (-1, 2)(v_1, v_2)\}.$$

These edges correspond to the subspaces of $(\mathbb{R}^2)^2$ whose equations in coordinates (z_1, z_2) ($z_i \in \mathbb{R}^2$) are, respectively, $z_2 - z_1 = (0, 0)$, $z_2 - z_1 = (2, 0)$, and $z_2 - z_1 = (-1, 2)$. The weights of the vertices are $h_1 = (2, 0)$ and $h_2 = (-1, 3)$. The total dichromatic polynomial is

(1.1) $$Q_{(\Phi, h)}(\mathbf{u}, v, z) = u_{(2,0)} u_{(-1,3)} + u_{(1,3)} + u_{(3,3)} + u_{(2,5)} + 3z + vz.$$

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The $u_{(x,y)}$ are weight variables, one for each $(x,y) \in (\mathbb{Z}^2, \max)$. $\mathbf{u}$ stands for the ensemble of all weight variables; however, only a finite number appear for any one graph. We derive (1.1) in Example 3.1, after we explain the techniques.

We had hoped that the total dichromatic polynomial $Q_{(\Phi,h)}$ would be a universal Tutte invariant of weighted gain graphs, in the sense that every Tutte invariant is an evaluation of it. However, we are sure it is not. It takes no account of balanced loops and loose edges. When we attempted to improve it, even without weights, by incorporating variables for those edges, there were relations amongst the variables that suggest a universal Tutte invariant more general than $Q_{(\Phi,h)}$ exists but is very complicated. Finding that universal invariant, even for gain graphs without weights, is certain to be very challenging.

We proceed now to a section of technical definitions and then to the definition and properties of the total dichromatic polynomial, including a spanning-forest expansion (but not a spanning-tree expansion). Then in Section 5 we specialize to counting proper colorations and the application to lattice-point–counting problems. We conclude with the connection to Noble and Welsh [9].

2. Weighted gain graphs

Notation: For a real number x we write

$$x^+ := \max(0, x).$$

2.1. **Graphs.** Edges of a graph $\Gamma = (V, E)$ are of two kinds. A *link* has two distinct endpoints; a *loop* has two coinciding endpoints. A *half edge* has one endpoint, and a *loose edge* has no endpoints. (Half and loose edges have a minor role in this paper; they appear only due to contraction.) The set of loops and links is written E^*. Multiple edges are permitted. We write $n := |V|$ and $V = \{v_1, v_2, \ldots, v_n\}$. All our graphs have finite order and indeed are finite (except for root edges, when they appear). A *(connected) component* of Γ is a maximal connected subgraph that is not a loose edge; we do not count a loose edge as a component. The number of components is $c(\Gamma)$. For $S \subseteq E$, we denote by $c(S)$ the number of connected components of the spanning subgraph (V, S) (which we call the *components of S*) and by $\pi(S)$ the partition of V into the vertex sets of the various components. We write S_{vw} to denote any path in S from v to w (if one exists).

2.2. **Gain graphs.** A *gain graph* $\Phi = (\Gamma, \varphi)$ consists of a graph $\Gamma = (V, E)$, a group $\mathfrak{G}$ called the *gain group*, and an orientable function $\varphi : E^* \to \mathfrak{G}$, called the *gain mapping*. (Half edges do not have gains. We may think of a loose edge as having gain $1_{\mathfrak{G}}$.) The basic reference is [15, Part I, Section 5]. "Orientability" means that, if e denotes an edge oriented in one direction and e^{-1} the same edge with the opposite orientation, then $\varphi(e^{-1}) = \varphi(e)^{-1}$. (Edges are undirected, i.e., they do not have fixed orientations. We orient an edge only in order to state the value of its gain.) We sometimes use the simplified notations e_{ij} for an edge with endpoints v_i and v_j, oriented from v_i to v_j, and ge_{ij} for such an edge with gain g; that is, $\varphi(ge_{ij}) = g$. (Thus ge_{ij} is the same edge as $g^{-1}e_{ji}$.) The gain of a walk $e_1 e_2 \cdots e_l$ is $\varphi(e_1 e_2 \cdots e_l) := \varphi(e_1)\varphi(e_2)\cdots\varphi(e_l)$.

For instance, an *integral gain graph* has gain group $\mathfrak{G} = (\mathbb{Z}, +)$.

The notation $c(\Phi)$ means $c(\Gamma)$.

An *isomorphism* of two gain graphs Φ and Φ' with the same gain group $\mathfrak{G}$ is an isomorphism of underlying graphs that preserves edge gains.

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A *circle* is a connected 2-regular subgraph without half edges, or its edge set; for instance, a loop is a circle of length 1. We may write a circle C as a word $e_1 e_2 \cdots e_l$; this means the edges are numbered consecutively around C and oriented in a consistent direction. The gain $\varphi(C)$ is well defined up to conjugation and inversion, and in particular it is well defined whether the gain is the identity $1_{\mathfrak{G}}$ or not. An edge set or subgraph is called *balanced* if every circle in it has gain $1_{\mathfrak{G}}$ and it has no half edges. If $B \subseteq E$ is balanced, the gain $\varphi(B_{vw})$ of a path B_{vw} is the same for every such path (if it exists).

An ordinary graph can be thought of as a gain graph where all edges have identity gain; thus, every circle is balanced.

For $W \subseteq V$, the subgraph *induced* by W is notated $\Gamma{:}W$, or with gains $\Phi{:}W$. The vertex set of $\Gamma{:}W$ is W and the edge set consists of all edges that have at least one endpoint in W and no endpoint outside W; thus, half edges at vertices of W are included, but loose edges are not. If $S \subseteq E$, then the subset of S induced by W is $S{:}W := E((V, S){:}W)$.

Switching Φ by a *switching function* $\eta : V \to \mathfrak{G} : v_i \mapsto \eta(v_i)$ means replacing φ by

$$\varphi^\eta(e_{ij}) := \eta(v_i)^{-1} \varphi(e_{ij}) \eta(v_j).$$

We write Φ^η for the switched gain graph (Γ, φ^η). Switching functions form a group $\mathfrak{G}^V$ under componentwise multiplication. Switching is an action of the group $\mathfrak{G}^V$ on the set $\mathfrak{G}^E$ of gain functions on the underlying graph.

A *switching isomorphism* of gain graphs Φ and Φ' with the same gain group $\mathfrak{G}$ is an isomorphism of underlying graphs that preserves gains up to switching; that is, it is an isomorphism of Φ' with some switching of Φ.

Consider a balanced edge set S. Let $S_{v_i v_j}$ denote a path in S from v_i to v_j, if one exists; the gain $\varphi(S_{v_i v_j})$ is independent of the particular path because S is balanced. There is a switching function η such that $\varphi^\eta\big|_S \equiv 1_{\mathfrak{G}}$ [15, Section I.5]; it is determined by any one value in each component of S through the formula

$$(2.1) \qquad\qquad \eta(v_j) = \varphi(S_{v_j v_i}) \eta(v_i).$$

We call η a *switching function for S*. Any two different switching functions for S, η and η', are connected by the relation

$$(2.2) \qquad\qquad \eta' = \eta \cdot \alpha_W$$

for constants $\alpha_W \in \mathfrak{G}$, one for each $W \in \pi(S)$. (In fact, $\alpha_W = \eta(v_i)^{-1} \eta'(v_i)$ for any $v_i \in W$; this is easy to deduce from (2.1).) Thus, as long as the endpoints of an edge e_{ij} are in the same component of S, $\varphi^\eta(e_{ij}) = 1_{\mathfrak{G}}$. (If e_{ij} has endpoints in distinct components of S, then $\varphi^\eta(e_{ij}) = \eta(v_i)^{-1} \varphi(e_{ij}) \eta(v_j)$ can be anything, since $\eta(v_i)$ and $\eta(v_j)$ are independently choosable elements of $\mathfrak{G}$.)

The operation of deleting an edge or a set of edges is obvious. The notation for Φ with $E \setminus S$ deleted, called the *restriction* of Φ to S, is $\Phi|S = (V, S, \varphi|_S)$. The number of components of S that are balanced is $b(\Phi|S)$ or briefly $b(S)$ (recall that this counts isolated vertices but not loose edges); $\pi_{\mathrm{b}}(S) = \pi_{\mathrm{b}}(\Phi|S)$ is the set $\{W \in \pi(S) : (S{:}W) \text{ is balanced}\}$; $V_0(S)$ is the set of vertices that belong to no balanced component of S; and $V_{\mathrm{b}}(S)$ denotes the set of vertices of balanced components, $V_{\mathrm{b}}(S) = V \setminus V_0(S)$.

Contraction is not so obvious. We take the definition from [15]. First, we describe how to contract a balanced edge set S. We first switch by η, any switching function for S; then we identify each set $W \in \pi(S)$ to a single vertex and delete S. The notation is Φ^η/S. This contraction depends on the choice of η, so Φ^η/S is well defined only up to switching. (Soon,

however, we shall see how to single out a preferred switching function.) By this definition, contracting a balanced loop or loose edge is the same as deleting it (but that is not true for unbalanced loops).

For a general subset S we first delete the vertex set $V_0(S)$, then contract the remaining part of S, which is the union of all balanced components of S, and delete any remaining edges of S. Edges not in S that have one or more endpoints in $V_0(S)$ lose those endpoints but remain in the graph, thus becoming half or loose edges. So, the contraction has $V(\Phi/S) = \pi_{\mathrm{b}}(S)$ and $E(\Phi/S) = E \setminus S$.

A balanced edge set S is called *closed* if it contains any edge ge_{ij} whose endpoints are joined by an open path $P \subseteq S$ with the same gain is itself in S. (For a balanced loop $0e_{ii}$, the path P has length 0 and the edge is necessarily in S.) This is equivalent to saying S equals its own closure; the closure of a balanced edge set S is given by

(2.3) $\qquad \mathrm{cl}(S) := S \cup \{e \notin S : e \text{ is contained in a balanced circle } C \subseteq S \cup \{e\}\}$.

By [15, Proposition I.3.1], the closure of a balanced set is again balanced. The semilattice of all closed, balanced edge sets in Φ is written $\mathrm{Lat}^{\mathrm{b}} \Phi$.

2.3. **Ordered gain groups.** For the rest of this article we assume the gain group is lattice ordered. (It may be totally ordered; this case has some special features.)

We continue thinking of a balanced edge set S. The ordering singles out a particular switching function for S, the one for which the meet of its values on each set $W \in \pi(S)$ is the identity. We call this the *top switching function* and we write it η_S; it is what we use for switching throughout the rest of this article. Its existence and uniqueness are proved in the following lemma.

Lemma 2.1. *The top switching function η_S has the formula*

$$\eta_S(v) = \bigvee_{w \in W} \varphi(S_{vw}),$$

where $W \in \pi(S)$ is the part that contains v, and for its inverse

$$\eta_S(v)^{-1} = \bigwedge_{w \in W} \varphi(S_{wv}).$$

Proof. Because S is balanced, the gain of every path S_{vw} is the same. That is why the right side of (2.1) is well defined.

We use the identity $(\alpha \wedge \beta)^{-1} = \alpha^{-1} \vee \beta^{-1}$.

We know two properties of η_S. As a switching function for S it satisfies (2.1). As a top switching function it satisfies $\bigwedge_{w \in W} \eta_S(w) = 1_{\mathfrak{G}}$ for every $W \in \pi(S)$. Equation (2.1) lets us rewrite this as

$$\bigwedge_{w \in W} \varphi(S_{wv})\eta_S(v) = 1_{\mathfrak{G}}.$$

Factoring out $\eta_S(v)$,

$$\eta_S(v) = \left[\bigwedge_{w \in W} \varphi(S_{wv}) \right]^{-1} = \bigvee \varphi(S_{vw}). \quad \square$$

In view of the importance of the meet of switching-function values, we define

$$\eta(X) := \bigwedge_{v \in X} \eta(v)$$

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for $X \subseteq V$.

Now, to contract S we first switch by η_S; then we identify each set $W \in \pi(S)$ to a single vertex and delete S. The contraction Φ^{η_S}/S, which we call the *top contraction*, we usually write Φ/S for brevity. The contraction Φ/S is now a unique gain graph, because the gain-group ordering allows us to specify the switching function uniquely.

When the group is totally ordered, there is a *top vertex* in every component of S, a vertex t such that no path in S that ends at t has gain $< 1_{\mathfrak{G}}$; this is any vertex for which $\eta_S(t) = 1_{\mathfrak{G}}$. Then the rule for defining η_S is that its minimum value on the vertices of each component of S is the identity. A top vertex may also happen to exist when $\mathfrak{G}$ is not totally ordered. If t_i denotes a top vertex in the same component of S as v_i, then the top switching function has the formula

(2.4) $$\eta_S(v_i) = \varphi(S_{v_i t_i}).$$

The gain function φ switched by η_S is given by the formula

(2.5) $$\varphi^{\eta_S}(e_{ij}) = \varphi(S_{v_i t_i})^{-1} \varphi(e_{ij}) \varphi(S_{v_j t_j}) = \varphi(S_{t_i v_i} e_{ij} S_{v_j t_j}).$$

2.4. **Weights.** Suppose we have an abelian semigroup $\mathfrak{W}$ (written additively) and a group $\mathfrak{G}$. We say $\mathfrak{G}$ *acts on* $\mathfrak{W}$ if each $g \in \mathfrak{G}$ has a right action on $\mathfrak{W}$ which is a semigroup automorphism satisfying the usual identities, i.e., g is a bijection $\mathfrak{W} \to \mathfrak{W}$ satisfying $(h + h')g = hg + h'g$, $0g = 0$, $(hg)g' = h(gg')$, and $h1_{\mathfrak{G}} = h$.

A *weighted gain graph* (Φ, h) is a gain graph Φ together with a *weight* function $h : V \to \mathfrak{W}$, where the gain group $\mathfrak{G}$ has a right action on the weight semigroup $\mathfrak{W}$. We usually write $h_i := h(v_i)$. The way h transforms under switching is that

$$h_i^\eta = h^\eta(v_i) := h_i \eta(v_i)$$

(as if h_i were the gain of an edge oriented into v_i from an extra vertex at which η is the identity). Thus, the switching group $\mathfrak{G}^V$ has a right action on the set $\mathfrak{W}^V$ of weight functions. Switching (Φ, h) means switching both Φ and h, i.e.,

$$(\Phi, h)^\eta := (\Phi^\eta, h^\eta).$$

A *switching isomorphism* of weighted gain graphs (Φ, h) and (Φ', h') with the same gain group $\mathfrak{G}$ is an isomorphism of underlying graphs that preserves gains and weights up to switching. That is, it is an isomorphism of (Φ', h') with some switching of (Φ, h).

The restriction of (Φ, h) to an edge set S is denoted by $(\Phi, h)|S := (\Phi|S, h)$.

The contraction rule is that, first, we always contract with top switching; and if $W \in \pi_{\mathrm{b}}(S)$, then the weight function h/S in the contraction $(\Phi, h)/S$ is given by

$$h/S(W) := \sum_{v_i \in W} h_i^{\eta_S} = \sum_{v_i \in W} h(v_i) \eta_S(v_i).$$

If $R \subseteq S$ and $\pi_{\mathrm{b}}(R) = \pi_{\mathrm{b}}(S)$, then $h/R = h/S$.

If there is a top vertex t_i in the component $S{:}W$ that contains v_i, then $h_i^{\eta_S} = h_i \varphi(S_{v_i t_i})$ and $h/S(W) = \sum_{v_i \in W} h_i \varphi(S_{v_i t_i})$.

We have occasion to contract the induction $(\Phi, h){:}W := (\Phi{:}W, h|_W)$ of the entire weighted graph by the induction $S{:}W$ of an edge set, where $W \in \pi_{\mathrm{b}}(S)$; we ought to write this $((\Phi, h){:}W)/(S{:}W)$ but we simplify the notation to $(\Phi, h)/S{:}W$.

The next result states the fundamental properties of deletion and contraction of weighted gain graphs.

Proposition 2.2. *In a weighted gain graph (Φ, h), let $S \subseteq E$ be the disjoint union of Q and R. Then*

$$((\Phi, h)/Q)/R = (\Phi, h)/S,$$
$$((\Phi, h)/Q) \setminus R = ((\Phi, h) \setminus R)/Q,$$
$$((\Phi, h) \setminus Q) \setminus R = (\Phi, h) \setminus S.$$

Proof. We may suppose Φ is connected. The two latter formulas are obvious.

The first one is not; indeed, in a purely technical sense it is false, since $V(\Phi/S) = \pi_{\mathrm{b}}(\Phi|S)$ while $V((\Phi/Q)/R) = \pi_{\mathrm{b}}(\Phi/Q|R)$; but it is correct if we identify $W \in \pi_{\mathrm{b}}(\Phi|S)$ with $W'' \in \pi_{\mathrm{b}}(\Phi/Q|R)$ in the natural way: W corresponds to $W'' = \{X \in \pi_{\mathrm{b}}(\Phi|Q) : X \subseteq W\}$ and conversely W'' corresponds to $W = \bigcup W'' = \{w \in V(\Phi) : w \in X \text{ for some } X \in W''\}$.

In proving the first formula, the first step is to show that we can assume S is balanced. It is a routine check to see that Φ/S and $\Phi/Q/R$ have the same half and loose edges. Since $V(\Phi/S) = \pi_{\mathrm{b}}(S)$, we have

$$\Phi/S = [(\Phi{:}U)/(S{:}U)] \quad \text{and} \quad \Phi/Q/R = \big((\Phi{:}U)/(Q{:}U)\big)/(R{:}U)$$

where $U := V_{\mathrm{b}}(S)$. Since $S{:}U$ is balanced, both $Q{:}U$ and $R{:}U$ are also balanced. The weights of the contractions only appear on vertices of Φ/S so they depend only on vertices and edges in U; the same holds true for $\Phi/Q/R$. It follows that

$$(\Phi, h)/S = ((\Phi, h){:}U)/(S{:}U)$$

and

$$(\Phi, h)/Q/R = \big(((\Phi, h){:}U)/(Q{:}U)\big)/(R{:}U).$$

This proves that we may confine our attention to the balanced spanning subgraph $(U, S{:}U)$ in $\Phi{:}U$; thus, we may from now on assume S is balanced.

Let η'_S be the top switching function for $\Phi^{\eta_Q}|S$ and let η''_R be that for $\Phi/Q|R$. (Recall that Φ/Q means Φ^{η_Q}/Q.) The key to the proof is the factorization identity

(2.6)
$$\eta_S(v) = \eta_Q(v)\eta'_S(v),$$

which shows that the effect of η_S, which is to switch so $\varphi|_S$ becomes $1_{\mathfrak{G}}$, can be divided into two stages: first switching by η_Q so that $\varphi|_Q$ becomes $1_{\mathfrak{G}}$, and then switching by η'_S, which is constant on components of Q.

In proving (2.6), first we compare η_Q and η_S. Since they are two switching functions for Q, they are related by (2.2). Specifically, let $X \in \pi(Q)$ and $W \in \pi(S)$, with $X \subseteq W$; then $\eta_S(v) = \eta_Q(v)\alpha_X$ for $v \in X$. Taking the meet over X, $\eta_S(X) = \eta_Q(X)\alpha_X = 1_{\mathfrak{G}}\alpha_X$, so

$$\eta_S(v) = \eta_Q(v)\eta_S(X)$$

for $v \in X$. Next we show that $\eta_S(X) = \eta'_S(v)$. Define $\bar{\eta} := \eta_Q\eta'_S$. It is easy to verify that $\bar{\eta}$ satisfies (2.1) so it is a switching function for S. Taking the meet over all $v \in W$, and taking note that $\eta'_S(v) = \eta'_S(X)$ for $v \in X$ because η'_S is constant on X, we find that

$$\bar{\eta}(W) = \bigwedge_{v \in W} \eta_Q(v)\eta'_S(v) = \bigwedge_{X \in W''} \Big[\bigwedge_{v \in X} \eta_Q(v)\Big]\eta'_S(X)$$
$$= \bigwedge_{X \in W''} 1_{\mathfrak{G}}\eta'_S(X) = \bigwedge_{v \in W} \eta'_S(v) = 1_{\mathfrak{G}}.$$

Thus $\bar{\eta}$ is a top switching function for S and, as there is only one, it equals η_S. This proves (2.6).

From (2.6) it follows that $\varphi^{\eta s} = (\varphi^{\eta_Q})^{\eta'_S}$ and (by the group action on weights) $h^{\eta s}(v) = (h^{\eta_Q})^{\eta'_S}(v)$, thus establishing that

$$(2.7) \qquad\qquad (\Phi, h)^{\eta s} = \left((\Phi, h)^{\eta_Q}\right)^{\eta'_S}.$$

Now we can analyze the process of contraction. We know from [15, Theorem I.4.7 and the proof of Theorem I.5.4] that $\Phi/Q/R \sim \Phi/S$ (where $\Phi_1 \sim \Phi_2$ means that each of them is a switching of the other). But the switching equivalence is really equality because the switching functions employed are related by Equation (2.6). Thus, $\Phi/S = \Phi/Q/R$.

The last step is to prove that weights contract properly. The key here is that contraction by Q commutes with two-stage switching, i.e.,

$$(2.8) \qquad\qquad (h^{\eta_Q}/Q)^{\eta''_R} = (h^{\eta_Q})^{\eta'_S}/Q.$$

Observe that η'_S is constant on each $X \in \pi(Q)$ and its common value is $\eta''_R(X)$ (the value of η''_R on X as a vertex in Φ/Q). Expanding both sides according to the definitions of switching and contraction, this is equivalent to

$$\left(\sum_{w \in X} h^{\eta_Q}(w)\right)\eta''_R(X) = \sum_{w \in X} h^{\eta_Q}(w)\eta'_S(w),$$

which is true because $\eta'_S(w) = \eta''_R(X)$. That concludes the proof of (2.8).

Equations (2.7) and (2.8) imply the double contraction formula through the sequence of transformations

$$
\begin{aligned}
(\Phi, h)/Q/R &= \left((\Phi, h)^{\eta_Q}/Q\right)^{\eta''_R}/R & \text{by definition} \\
&= \left((\Phi, h)^{\eta_Q \eta'_S}/Q\right)/R & \text{by (2.8)} \\
&= \left((\Phi, h)^{\eta s}/Q\right)/R & \text{by (2.7)} \\
&= (\Phi, h)^{\eta s}/S
\end{aligned}
$$

in the loose sense previously defined in terms of the correspondence $W \leftrightarrow W''$. $\qquad\square$

Example 2.1 (*Weighted integral gain graphs; linearly ordered group weights*). Our original example [7] was that of *weighted integral gain graphs*, where the gain group is the additive group of integers and the weight semigroup is the integers with the operation of maximization. In other words, $\mathfrak{G} = (\mathbb{Z}, +)$ and $\mathfrak{W} = (\mathbb{Z}, \max)$.

A similar kind of example exists for every linearly ordered gain group, with $\mathfrak{W} = (\mathfrak{G}, \max)$ or $(\mathfrak{G}, \min)$.

Example 2.2 (*Semilattice weights*). To further generalize Example 2.1, let $\mathfrak{W}$ be a semilattice with a $\mathfrak{G}$-action; the semigroup operation is the semilattice operation. In an important example of this type there is a set $\mathfrak{C}$ on which there is a right action of the gain group; the weights are subsets of $\mathfrak{C}$, i.e., $\mathfrak{W} \subseteq \mathcal{P}(\mathfrak{C})$; and the semigroup operation is set intersection—so $\mathfrak{W}$ must be closed under intersection. (In Section 5 $\mathfrak{C}$ will be a color set and the weight $h_i \subseteq \mathfrak{C}$ will be treated as the list of colors possible for vertex v_i.)

Generalizing minimization, let $\mathfrak{C}$ be a partially ordered set and let the weights be order ideals in $\mathfrak{C}$. If $\mathfrak{C}$ is a meet semilattice, one may restrict the weights to be principal ideals. There are also the order duals of these examples.

When, on the other hand, $\mathfrak{C} = \mathfrak{G}$ with the right translation action, one may take $\mathfrak{W} = \mathcal{P}(\mathfrak{G})$, for instance, or the class of principal dual order ideals (since $\mathfrak{G}$ is a lattice), or the

class of sets that have a lower bound (that is, all subsets of principal dual ideals). The dual of this last, with gain group $\mathbb{Z}$ was the prototype of weighted gain graphs, as we explain next.

2.5. **Rooted integral gain graphs are weighted.** Since the inspiration for this article was a reconsideration of [7], abetted by [9], we wish to explain to readers of [7] how its rooted integral gain graphs are a notational variant of certain weighted integral gain graphs.

A *rooted integral gain graph* is an integral gain graph Ψ with a root vertex v_0 adjacent to all other vertices in such a way that the gains of edges e_{0i} form an interval $(-\infty, h_i]$ in the gain group $\mathbb{Z}$, the infinite cyclic group. In [7] we studied the chromatic function of a rooted integral gain graph, which counts proper colorations in an interval $(-\infty, m]$, as explained in Example 5.3.

An equivalent presentation replaces the root by an integral weight h_i on each vertex of $\Phi := \Psi \setminus v_0$. Switching of Ψ as in [7], when transferred to Φ, implies the rule $h^\eta = h + \eta$ for switching the weights. (We write a plus sign here because the action of $\mathfrak{G} = \mathbb{Z}$ on $\mathfrak{W} = \mathbb{Z}$ in this example is by addition.) That is the rule adopted and generalized in Section 2.4.

Similarly, the standard gain-graphic contraction on Ψ, when reinterpreted in terms of integral weights h_i on Φ, assigns to a set $W \in V(\Phi/S)$ a weight equal to the maximum weight of a vertex in W after switching; thus the weight set is $\mathbb{Z}$ with the operation of maximization.

Trying to generalize this equivalence to more arbitrary root-edge gain sets, and reading about the weight operation in [9], we realized that the weight operation is a semigroup operation, different from the gain-group operation, and the weight semigroup is separate from the gain group; and so we settled upon the generalization in this article.

3. A Tutte-invariant polynomial

A function f defined on weighted gain graphs (with fixed gain group and weight semigroup) is a *Tutte invariant* if it satisfies the four conditions from the introduction:

(Ti) (Additivity) For every link e,
$$f(\Phi, h) = f(\Phi \setminus e, h) + f(\Phi/e, h/e),$$
where h/e denotes the contracted weight function.

(Tii) (Multiplicativity) The value of f on (Φ, h) is the product of its values on the components of (Φ, h).

(Tiii) (Isomorphism Invariance) If (Φ, h) and (Φ', h') are isomorphic, then $f(\Phi, h) = f(\Phi', h')$.

(Tiv) (Unitarity) $f(\varnothing) = 1$.

We present here an algebraic Tutte invariant. We need variables u_k for all $k \in \mathfrak{W}$; the collection of all u_k's is denoted by $\mathbf{u}$. The *total dichromatic polynomial* of a weighted gain graph is

$$(3.1) \qquad Q_{(\Phi, h)}(\mathbf{u}, v, z) := \sum_{S \subseteq E} v^{|S|-n+b(S)} z^{c(S)-b(S)} \prod_{W \in \pi_{\mathrm{b}}(S)} u_{h/S(W)},$$

where

$$h/S(W) := \sum_{w \in W} h^{\eta_S}(w).$$

The important values of z are 0 and 1. When $z = 0$ we have a sum only over balanced sets S; this is the *balanced dichromatic polynomial* $Q^{\mathrm{b}}_{(\Phi, h)}(\mathbf{u}, v)$. When $z = 1$ we sum over all edge

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sets S; this gives the *(ordinary) dichromatic polynomial* $Q_{(\Phi,h)}(\mathbf{u}, v)$. These specializations refine the balanced and ordinary dichromatic polynomials of a gain graph or biased graph [15, Section III.3], which are obtained by setting all $u_k = u$ and $z = 0$ or $z = 1$. Thus the total polynomial with all $u_k = u$ fills a gap in the theory of [15, Part III] by unifying the balanced and ordinary polynomials. The advantage of the total polynomial is in having one Theorem 3.1 instead of two, separately for the balanced and ordinary polynomials.

For a graph with no edges,

$$(3.2) \qquad\qquad Q_{((V,\varnothing),h)} = \prod_{v_i \in V} u_{h_i}.$$

If e is a balanced loop or a loose edge,

$$(3.3) \qquad\qquad Q_{(\Phi,h)} = (v+1)Q_{(\Phi\setminus e,h)}.$$

Theorem 3.1. *The total dichromatic polynomial $Q_{(\Phi,h)}(\mathbf{u}, v, z)$ is a Tutte invariant of weighted gain graphs.*

Proof. Unitarity, multiplicativity, and isomorphism invariance are clear. For additivity we follow the usual proof method, dividing up the terms of the defining sum into two parts: those sets S that do not contain the link e and those sets that do contain e. The sum of the former terms equals $Q_{(\Phi,h)\setminus e}(\mathbf{u}, v, z)$ and the sum of the latter, we shall see, equals $Q_{(\Phi,h)/e}(\mathbf{u}, v, z)$.

A set $S \ni e$ contracts to a set $R = S \setminus e$ in Φ/e whose balanced components correspond to those of S. That is, if S_0 is a balanced component of S, then S_0 (if $e \notin S_0$) or S_0/e (if $e \in S_0$) is a balanced component of R, and vice versa. (This follows from [15, Lemma I.4.3].) So $b(\Phi/e|R) = b(\Phi|S)$. Since Φ/e has order $n-1$, the term of S in $Q_{(\Phi,h)}$ and that of R in $Q_{(\Phi,h)/e}$ are the same except for the factors $u_{h/S(W)}$ in the former and $u_{h/R(W'')}$ in the latter, where W'' is the member of $\pi_{\mathrm{b}}(\Phi/e|R)$ that corresponds to W. We want to show that these factors are equal, i.e., that $h/S(W) = h/R(W'')$. But the former is h/S and the latter is $(h/e)/R$, which we know by Proposition 2.2 to be equal.

It follows that Q satisfies additivity, so is a Tutte invariant. $\qquad\square$

The total dichromatic polynomial does not, in general, behave predictably under switching. One can see that by comparing the switched polynomial $Q_{(\Phi,h)^\eta}$ to the unswitched one, $Q_{(\Phi,h)}$. We omit the calculation.

Example 3.1. Now we can derive the polynomial in Example 1.1.

An edge set with at most one edge is balanced; $b(\varnothing) = 2$, $b(\{e\}) = 1$ for any edge e. A subset with two or three edges is unbalanced.

We calculate the total dichromatic polynomial step by step:

$$Q_{(\Phi,h)}(\mathbf{u}, v, z) := \sum_{S \subseteq E} v^{|S|-n+b(S)} z^{c(S)-b(S)} \prod_{W \in \pi_{\mathrm{b}}(S)} u_{h/S(W)}$$

$$= v^{0-2+2} z^{2-2} u_{h/\varnothing(\{v_1\})} u_{h/\varnothing(\{v_2\})} + \sum_{i=1}^{3} v^{1-2+1} z^{1-1} u_{h/e_i(V)}$$

$$+ \sum_{j=1}^{3} v^{2-2+0} z^{1-0} u_{h/(E\setminus e_j)(V)} + v^{3-2+0} z^{1-0} u_{h/E(V)}$$

$$= u_{(2,0)} u_{(-1,3)} + [u_{(2,3)} + u_{(4,3)} + u_{(2,3)}] + z[3+v].$$

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Explanations: $h/\varnothing$ is simply h; there is no contraction. For h/e_i we switch by the top switching function η_{e_i}, which is given by $\eta_{e_i}(v_1) =$ and $\eta_{e_i}(v_1) =$. The top switching function η_{e_i} for edge $e_i = g + i(v_1, v_2)$ must satisfy $-\eta_{e_i}(v_1) + g_i + \eta_{e_i}(v_2) = (0,0)$ (because it is a switching function for $B = \{e_i\}$) and $\eta_{e_i}(v_1) \wedge \eta_{e_i}(v_2) = (0,0)$, the identity in the gain group. That is, $[\eta_{e_i}(v_2) + g_i] \wedge \eta_{e_i}(v_2) = (0,0)$. Thus, $\eta_{e_1}(v_1) = \eta_{e_1}(v_2) = (0,0)$; $\eta_{e_2}(v_1) = (2,0), \eta_{e_2}(v_2) = (0,0)$; and $\eta_{e_3}(v_1) = (1,0), \eta_{e_3}(v_2) = (0,2)$. The top switching function η_B, the switched weights $h^{\eta_B}(v_i)$, and the contracted weights ("Wt.") are

$B =$	$\varnothing$	e_1	e_2	e_3
$W =$	$\{v_1\}, \{v_2\}$	V	V	V
$\eta_B(v_1)$	$(0,0)$	$(0,0)$	$(2,0)$	$(0,2)$
$\eta_B(v_2)$	$(0,0)$	$(0,0)$	$(0,0)$	$(1,0)$
$h^{\eta_B}(v_1)$	$(2,0)$	$(2,0)$	$(4,0)$	$(2,2)$
$h^{\eta_B}(v_2)$	$(-1,3)$	$(-1,3)$	$(-1,3)$	$(0,3)$
Wt.	$(2,0), (-1,3)$	$(2,3)$	$(4,3)$	$(2,3)$

(Contraction by $\varnothing$ does not combine vertices or weights.)

Contracting Φ by an unbalanced edge set S gives a graph with no vertices; hence there are no weights associated to the term of S, so $\prod_{W \in \pi_b(S)} u_{h/S(W)} = 1$.

4. Forest expansion

We turn to an expression for the balanced dichromatic polynomial, $Q_{(\Phi,h)}(\mathbf{u}, v, 0)$, that depends on a linear ordering of the edge set. We fix one such ordering O and in terms of it we define a spanning-forest expansion similar to the Tutte polynomial of a matroid. (We did not find an expansion in terms of spanning trees.) The details are in Section 4.3, after some preliminary work with independent sets in semimatroids and gain graphs.

4.1. **Activities in semimatroids.** A semimatroid is a generalization of a matroid that extends properties like rank and closure of the family of balanced edge sets in a gain graph. The theory was developed by Wachs and Walker in [13] (in terms of the semilattice of closed sets). Just as with matroids, there are many equivalent ways to define a semimatroid. We define a semimatroid in terms of a matroid M_0 with ground set E_0 and a basepoint e_0 that is not a loop. A subset of $E := E_0 \setminus e_0$ whose closure in M_0 does not contain e_0 is called *balanced*; the family of balanced sets is denoted by $\mathcal{P}_b(M)$. The *semimatroid M* associated with (M_0, e_0) is the family of all balanced subsets of E with closure operator, rank function, closed or independent sets, circuits, and so forth the same as those of M_0 but restricted to balanced sets.

For instance, the independent sets of M are the ones of M_0 whose closures do not contain e_0. The closed sets of M are those of M_0 that do not contain e_0. The rank $\text{rk}_M(S)$ of a subset $S \subseteq E$ is defined only if S is balanced; then it equals $\text{rk}_{M_0}(S)$.

If e_0 is a coloop in M_0, then M is the matroid $M_0 \setminus e_0$. Otherwise, E is not balanced and M is not a matroid, because the whole ground set E is not one of the sets in M.

The restriction of M_0 or M to a subset S of the ground set is denoted by $M_0|S$ or $M|S$, resp. Thus $M|S$, if $S \subseteq E$, is the family of balanced sets of $M|(S \cup e_0)$ with the rank function, balanced independent sets, et al., restricted to S.

A fundamental fact is that if S is balanced, the closure $\text{cl}_0 S$ (in M_0) is balanced. Consequently, the closure $\text{cl} S$ in M equals $\text{cl}_0 S$. Also, any circuit in $\text{cl} S$ is balanced. A maximal

balanced independent set, that is, a maximal independent set of M, is called a *semibasis*. It lacks one element to be a basis of M_0; thus, $\operatorname{rk} M = \operatorname{rk} M_0 - 1$.

We develop some facts about independent sets, activities, and broken circuits in a semimatroid. Some of them are already known for matroids. We could not find an explicit source for exactly these results, but [6] and [8, Section 2] have theorems along similar lines. A reference for the fundamentals of activities in matroids is any of [1, 2, 3].

First, some basic definitions. Let F be independent in M_0. For a point $e \in (\operatorname{cl}_0 F) \setminus F$, there is a unique circuit contained in $F \cup e$; it is called the *fundamental circuit* of e with respect to F and denoted by $C_F(e)$. It is balanced if (but not only if) F is balanced. For a point $f \in F$, we call $\operatorname{cl}_0(F) \setminus \operatorname{cl}_0(F \setminus f)$ the *fundamental relative cocircuit* of f with respect to F, written $D_F(f)$. (By a *relative cocircuit* we mean a cocircuit in $M_0| \operatorname{cl}_0(F)$; it need not be a cocircuit in M.) If F is balanced, the closures are in M so cl_0 can be replaced by cl.

We fix a linear ordering O of E and extend it to E_0 arbitrarily. Consider an independent set F of the semimatroid M (that is, a balanced independent set of M_0). We say that a point $e \in E$ is *externally active* (in M) with respect to F if $e \notin F$ and e is the largest point in $C_F(e)$ (so only a point in $(\operatorname{cl} F) \setminus F$ can be externally active). A point e is *internally active* (in M) with respect to F if it is in F and it is the largest point in $D_F(e)$. A point that is not active is *internally inactive* if it belongs to F and *externally inactive* if it belongs to $(\operatorname{cl} F) \setminus F$. The sets of internally or externally active or inactive points with respect to F are denoted by $\operatorname{IA}(F)$, $\operatorname{EA}(F)$, $\operatorname{II}(F)$, $\operatorname{EI}(F)$. The number of externally active points is $\varepsilon(F)$. The number of internally active points is $\iota(F)$.

The definitions for M_0 are the same, except for the omission of the word "balanced" and replacement of cl by cl_0. Thus we have $\operatorname{IA}_0(F)$, et al.; but when F is balanced, these are the same as $\operatorname{IA}(F)$.

A *broken (balanced) circuit* is a (balanced) circuit with its largest element removed. (A broken circuit $C \setminus e$ is balanced if and only if its circuit C is balanced, because $\operatorname{cl}_0(C \setminus e) = \operatorname{cl}_0 C$; thus, a balanced broken circuit is the same as a broken balanced circuit.)

Lemma 4.1. *Let F be independent in the semimatroid M. Then $\operatorname{II}(F)$ contains every broken balanced circuit in F.*

Proof. If D is a broken balanced circuit in F, there is a point $e \in (\operatorname{cl} F) \setminus F$ which is maximal in its fundamental circuit $C_F(e) = D \cup \{e\}$. Any $f \in D$ is internally inactive because $e >_O f$, both $e, f \in D_F(f)$, and $e, f \in (\operatorname{cl} F) \setminus \operatorname{cl}(F \setminus f)$. Thus, $D \subseteq \operatorname{II}(F)$. $\square$

Lemma 4.2. *If F' is independent in the matroid M_0 and $F \subseteq F'$, then $\operatorname{II}_0(F) \subseteq \operatorname{II}_0(F')$.*

Proof. Let $f \in \operatorname{II}_0(F)$. Thus, there exists $e > f$ in $\operatorname{cl}_0(F) \setminus \operatorname{cl}_0(F \setminus f)$. Since $e \in \operatorname{cl}_0(F') \setminus \operatorname{cl}_0(F' \setminus f)$, f is internally inactive in F'. $\square$

Each point set $S \subseteq E_0$ has a *minimal basis* $F(S)$, which is the basis of the restriction $M_0|S$ that is lexicographically first according to O; it is the one obtained by the greedy algorithm applied to S. It is balanced if and only if S is balanced, because $\operatorname{cl}_0(S) = \operatorname{cl}_0(F(S))$. The next lemma says that the inverse of the mapping $S \mapsto F(S)$ partitions the power set of E_0 into intervals $[F, F \cup \operatorname{EA}_0(F)]$, one for each independent set F, and either all sets in the interval are balanced or all are unbalanced. (This partition is an analog for semimatroids one due to Crapo for matroids [4], also in Björner [1, Prop. 7.3.6].)

Lemma 4.3. *Let F be independent in M_0 and let $S \subseteq E_0$. For the minimal basis of S to be F, it is necessary and sufficient that $F \subseteq S \subseteq F \cup \mathrm{EA}_0(F)$. Further, $F \cup \mathrm{EA}_0(F)$ is balanced if and only if F is balanced.*

Proof. Assume F is the minimal basis of S and write $F = e_1 e_2 \cdots$ in increasing order in O. Every $e \in S \setminus F$ has a fundamental circuit $C_F(e)$. Suppose e is not externally active with respect to F, so that $C_F(e) = \cdots e e' \cdots$; let $e' = e_{k+1}$. The set $\{e_1, \ldots, e_k, e\}$ is independent because the only circuit it could contain is $C_F(e)$, but $e' \in C_F(e) \setminus \{e_1, \ldots, e_k, e\}$. Consider the greedy algorithm for finding F. After choosing $e_1, \ldots, e_k$, the next point chosen cannot be e', because e (or some other point different from e') would be preferred as it has not been chosen, it precedes e' in the ordering, and $\{e_1, \ldots, e_k, e\}$ is independent. Thus, $e_{k+1} \neq e'$. This is a contradiction. Therefore, e must be externally active.

Assume $F \subseteq S \subseteq F \cup \mathrm{EA}_0(F)$. Thus, F is a basis for S; we want to show it is minimal. Let $e \in S \setminus F$ and write $C_F(e) = e_1 \cdots e_k e_{k+1}$ in the ordering O; then $e = e_{k+1}$. In the greedy algorithm for constructing the minimal basis $F(S)$, each point $e_1, \ldots, e_k, e_{k+1}$ is considered in order for inclusion. Let $F_i(S)$ be the set of points that have already been chosen when e_i is considered for inclusion. If e_i is not then chosen for $F(S)$, it is because $e_i \in \mathrm{cl}\, F_i(S)$. If e_i is chosen, then $e_i \in F_{i+1}(S)$. Thus, all of $e_1, \ldots, e_k \in \mathrm{cl}\, F_{k+1}(S)$. It follows that $e \in \mathrm{cl}\, F_{k+1}(S)$, so $e \notin F(S)$. This shows that no point of $\mathrm{EA}_0(F)$ can belong to the minimal basis $F(S)$; hence, $F(S) \subseteq F$ and by comparing ranks we see that $F(S) = F$.

The last part of the lemma follows because $\mathrm{EA}_0(F) \subseteq \mathrm{cl}_0 F$, which is balanced if and only if F is balanced. $\qquad\square$

Suppose we already have a balanced independent set F that we want to extend to a semibasis. We do that by applying the *reverse greedy algorithm*. That means we take $E \setminus F$ and scan down it from the largest point (in the ordering O) to the smallest, adding a point to the independent set whenever the resulting set remains independent and balanced. The set obtained in this way, $T(F)$, is a semibasis that is lexicographically maximum among all semibases that contain F.

Lemma 4.4. *For an independent set F in M, $T(F)$ has the following properties:*

   (o) *If F is a semibasis then $T(F) = F$; in particular, $T(T(F)) = T(F)$.*
   (i) *$F \supseteq \mathrm{II}(T(F))$ and $T(F) \setminus F \subseteq \mathrm{IA}(T(F))$.*
   (ii) *$\mathrm{II}(F) \subseteq \mathrm{II}(T(F))$.*
   (iii) *$\mathrm{EA}(F) \subseteq \mathrm{cl}(\mathrm{II}(T(F)))$.*
   (iv) *$\mathrm{EA}(F) = \mathrm{EA}(\mathrm{II}(T(F))) = \mathrm{EA}(T(F))$; thus, $\varepsilon(F) = \varepsilon(T(F))$.*

Proof. Part (o) is immediate from the definition.

In (i) the two statements are obviously equivalent; we prove the latter. Suppose we have a balanced independent set F' and a point $e \notin \mathrm{cl}\, F'$; call e F'-*tolerable* if $F' \cup \{e\}$ is balanced. Write $T(F) \setminus F = e_k \cdots e_1$ in increasing order, so that each e_i is the largest $F \cup \{e_1, \ldots, e_{i-1}\}$-tolerable point. Since

$$e_i \notin \mathrm{cl}(T(F) \setminus e_i) \supseteq \mathrm{cl}(F \cup \{e_1, \ldots, e_{i-1}\}),$$

e_i is larger than any other $F \cup \{e_1, \ldots, e_{i-1}\}$-tolerable point not in $\mathrm{cl}(T(F) \setminus e_i)$. That is, it is externally active.

In Part (ii), $\mathrm{II}(F) \subseteq \mathrm{II}(T(F))$ by Lemma 4.2.

In (iii), e is maximal in $C_F(e)$ for $e \in \mathrm{EA}(F)$. By Lemma 4.1 and Part (ii), $C_F(e) \setminus \{e\} \subseteq \mathrm{II}(F) \subseteq \mathrm{II}(T(F))$.

For (iv), suppose we have two balanced independent sets, $F_1 \subseteq F_2$. Obviously $\mathrm{EA}(F_1) \subseteq \mathrm{EA}(F_2)$, because $(\mathrm{cl}\, F_1) \setminus F_1 \subseteq (\mathrm{cl}\, F_2) \setminus F_2$. If $\mathrm{EA}(F_2) \subseteq \mathrm{cl}\, F_1$, then $\mathrm{EA}(F_2) \subseteq \mathrm{EA}(F_1)$; thus the two EAs are equal. Now apply this fact to $F_1 = \mathrm{II}(T(F))$ and $F_2 = F$ or $T(F)$, recalling (iii). $\qquad\square$

4.2. **Activities in gain graphs.** When we come to gain graphs, the semimatroid we need is that associated with the balanced edge sets of Φ. Here, M_0 is the *complete lift matroid* $L_0(\Phi)$, which is the matroid on $E_0 := E(\Phi) \cup \{e_0\}$ with rank function for $S \subseteq E$ given by

$$\mathrm{rk}(S) = \begin{cases} n - c(S) & \text{if } S \text{ is balanced,} \\ n - c(S) + 1 & \text{if } S \text{ is unbalanced,} \end{cases}$$

$$\mathrm{rk}(S \cup e_0) = n - c(S) + 1$$

[15, Section II.4]. In a way, the complete lift matroid generalizes the usual graphic matroid $G(\Gamma)$, since when Φ is balanced, e_0 is a coloop and $G(\Gamma) = L_0(\Phi) \setminus e_0$. We call the semimatroid associated with $L_0(\Phi)$ and e_0 the *semimatroid of graph balance* of Φ.

(For those concerned with loose and half edges: In this section we treat a loose edge e as a balanced loop and a half edge as an unbalanced loop since in the matroid the two types behave exactly the same.)

Here is how the previous discussion of semimatroids applies to gain graphs. A balanced circle is the same thing as a semimatroid circuit, i.e., it is a matroid circuit (in $L_0(\Phi)$) that is a balanced edge set. A spanning forest F is the same as a balanced independent set; its closure defined in graphical terms is

$$\mathrm{cl}(F) := F \cup \{e \notin F : F \cup \{e\} \text{ contains a balanced circle}\}.$$

The reason is that $C_F(e)$, if it exists, must be balanced so it is a balanced circle; it is called the *fundamental circle* of e with respect to F. For an edge $e \in F$, in $F \setminus e$ one component of F is divided into two; the fundamental relative cocircuit of e with respect to F is the set $D_F(e)$ of edges $f \in E$ that join these two into one (since $F \cup f$ is balanced). To clarify these ideas we give a descriptive lemma that is particular to gain graphs. Note that our definitions of activity, derived from the semimatroid of balanced edge sets, differ from the usual ones for graphs. A *broken balanced circle* is a balanced circle with its largest edge removed.

Lemma 4.5. *Suppose Φ is a gain graph with no balanced digons. Let F be a spanning forest in Φ. Then $\mathrm{EA}(F)$ is the set of all edges $e \notin F$ such that $e \in (\mathrm{cl}\, F) \setminus F$ and $C_F(e) \setminus e$ is a broken balanced circle.*

Proof. If an edge $e \notin F$ is externally active, it is in $(\mathrm{cl}\, F) \setminus F$ and it is maximal in $C_F(e)$. The latter implies that $C_F(e) \setminus e$ is a broken balanced circle.

To prove the converse, assume $C_F(e)$ exists and $D := C_F(e) \setminus e$ is a broken balanced circle. Either e is maximal in $C_F(e)$, so e is externally active, or $D \neq \varnothing$ and there is an edge $e' \notin F$, other than e, such that $C_F(e') \setminus e' = D$. Then e and e' are parallel links with the same endpoints. Because they form a digon in $\mathrm{cl}\, F$, which is balanced since it is the closure of the balanced set F, they form a balanced digon, contrary to the assumption. Consequently, e' cannot exist. $\qquad\square$

4.3. **The forest expansion.** The *forest expansion* of (Φ, h) is

$$(4.1) \qquad\qquad F_{(\Phi,h),O}(\mathbf{u}, y) := \sum_F y^{\varepsilon(F)} \prod_{W \in \pi(F)} u_{h/F(W)},$$

summed over all spanning forests F of Φ. The forest expansion gives another representation of the balanced dichromatic polynomial.

Theorem 4.6. *The forest expansion is independent of O. Indeed,*

$$F_{(\Phi,h),O}(\mathbf{u}, y) = Q_{(\Phi,h)}(\mathbf{u}, y-1, 0).$$

Proof. Let us expand. In each sum, S is restricted to balanced edge sets that satisfy the stated conditions.

$$\begin{aligned}
Q_{(\Phi,h)}(\mathbf{u}, v, 0) &= \sum_S v^{|S|-\mathrm{rk}(S)} \prod_{W \in \pi(S)} u_{h/S(W)} \\
&= \sum_{\substack{F \text{ spanning forest}}} \sum_{\substack{S \supseteq F \\ F(S)=F}} v^{|S \setminus F|} \prod_{W \in \pi(S)} u_{h/S(W)} \\
&= \sum_{\substack{F \text{ spanning forest}}} \sum_{F \subseteq S \subseteq F \cup \mathrm{EA}(F)} v^{|S \setminus F|} \prod_{W \in \pi(F)} u_{h/F(W)}
\end{aligned}$$

by Lemma 4.3, because $\pi(S) = \pi(F)$, and because $\mathrm{EA}(F) \subseteq \mathrm{cl}(F)$ so that $h/S(W) = h/\mathrm{cl}(F)(W) = h/F(W)$,

$$= \sum_{\substack{F \text{ spanning forest}}} (v+1)^{|\mathrm{EA}(F)|} \prod_{W \in \pi(F)} u_{h/F(W)}. \qquad \square$$

For an ordinary graph Γ (without gains or weights or loose or half edges), where all $u_h = u$, Theorem 4.6 says that

$$Q_\Gamma(u, y-1) = F_\Gamma(u, y) := \sum_F y^{\varepsilon(F)} u^{c(F)},$$

where Q_Γ is Tutte's dichromatic polynomial of Γ. (We do not know a source for this formula.) A similar formula holds for the balanced dichromatic polynomial Q_Φ^{b} of a gain graph without weights.

We hoped for a spanning-tree expansion analogous to Tutte's for graphs, but we could not find one. The problem is that semibases do not span the matroid. When the semimatroid of graph balance is a matroid, as when Φ is balanced, a semibasis is a basis; then for a basis T and an independent set F, $T(F) = T$ if and only if $\mathrm{II}(T) \subseteq F \subseteq T$. This property lets us replace the sum over spanning forests by a sum over spanning trees. We did not find an analogous property of semibases.

5. Coloring

A proper coloration is a way of assigning to each vertex an element of a color set, subject to exclusion rules governed by the edges. The subject of [7] was the problem of counting integral lattice points not contained in specified integral affinographic hyperplanes (see Section 5.4). We solved it by reinterpreting lattice points as proper colorations of a $\mathbb{Z}$-weighted integral gain graph. In this section we develop a theory of proper colorations of all weighted gain graphs. We begin with list coloring, where the weight of a vertex is a finite list of possible colors (Section 5.1). We then go on to the heart of our treatment of coloring: infinite lists with an additional constraint regarded as a multidimensional variable; especially, upper bounds that make the effective list finite (Sections 5.2 and 5.3); this generalizes ordinary

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graph coloring, in which the list is $\{1, 2, 3, \ldots\}$ restricted by a variable upper bound k. Finally (Sections 5.4 and 5.5), we apply the general definition to multidimensional integral gain groups and weights, which have the geometrical meaning of counting integer lattice points that lie in a given rectangular parallelepiped but not in any of a family of integral affinographic subspaces (all of which will be explained).

A notion of proper coloring of gain graphs was developed in [15, Section III.5] (and there called zero-free coloring). There is a *color set* $\mathfrak{C}$, which is any set upon which the gain group $\mathfrak{G}$ has a fixed-point-free right action (that is, the only element of $\mathfrak{G}$ that has any fixed points is the identity). A *coloration* of Φ is any function $x : V \to \mathfrak{C}$. The set of *improper* edges of x is

$$I(x) := \{e_{ij} : x_j = x_i \varphi(e_{ij})\}.$$

The coloration is *proper* if $I(x) = \varnothing$. A basic fact from [15, Section III.5] is that an improper edge set is balanced and closed. For completeness we give the easy proof here. (We remind the reader of the definitions in Section 2.2.)

Lemma 5.1. *The improper edge set $I(x)$ of a coloration x is balanced and closed.*

Proof. First we prove balance. Suppose a circle $e_{01} e_{12} \cdots e_{l-1,l} \subseteq I(x)$, where $v_0 = v_l$. Impropriety of the edges implies that $x_l = x_0 \varphi(e_{01} e_{12} \cdots e_{l-1,l})$. Thus $x_0 = x_l$ is a fixed point of $\varphi(e_{01} e_{12} \cdots e_{l-1,l})$. By our overall hypothesis that the action is fixed-point free, the circle has gain $1_{\mathfrak{G}}$. Thus, $I(x)$ is balanced.

Suppose now that e is an edge from v to w in the closure of $I(x)$. Since $I(x)$ is balanced, its closure is balanced (see Equation (2.3) and the accompanying text). Thus, there is a path $e_{12} \cdots e_{l-1,l}$ in $I(x)$ connecting the endpoints of e (that is, $v_1 = v$ and $v_l = w$) whose gain $\varphi(e_{12} \cdots e_{l-1,l}) = \varphi(e)$. Since $x_l = x_{l-1} \varphi(e_{l-1,l}) = \cdots = x_1 \varphi(e_{12}) \cdots \varphi(e_{l-1,l}) = x_1 \varphi(e_{12} \cdots e_{l-1,l}) = x_1 \varphi(e)$, e is improper. Hence, $e \in I(x)$; that is, $I(x)$ is closed. $\qquad\square$

In contrast to [15], in this paper we have an infinite color set. We use weights in various ways to limit the possible colorations to a finite set. We have especially in mind two kinds of example. In the first, the group and the color set are both $\mathbb{Z}$ and the color lists are arbitrary subsets of $\mathbb{Z}$ that are bounded below and whose complements are bounded above; but there is a variable upper bound m on the possible colors; thus the number of proper colorations is a function of m. We call this *open-ended list coloring*. In the second, the group and color set are both $\mathbb{Z}^d$, the color lists are dual order ideals in $\mathbb{Z}^d$, and there is a variable upper bound $\mathbf{m}_i$ on the colors that can be used at vertex v_i. (This problem has a nice geometrical interpretation.) We wish to cover both of these examples, as well as similar ones, in a way that exposes to view the essential features; therefore we generalize considerably.

5.1. **List coloring.** A simple kind of list coloring is the basis of all our methods of coloring a weighted gain graph. The idea is to let $\mathfrak{W}$ be any class of subsets of the color set $\mathfrak{C}$ that is closed under intersection and the $\mathfrak{G}$-action; these subsets can be used as vertex color lists. A *list coloration* or simply *coloration* is any function $x : V \to \mathfrak{C}$ such that all $x(v_i) \in h_i$. It is proper as a list coloration if it is proper as a coloration of its gain graph.

In list coloring a contracted weight h/B has the formula

$$h/B(W) = \bigcap_{v_i \in W} h_i \eta_B(v_i).$$

(Recall that if $v_i \in W$ and it happens that W has a top vertex t_i, then $\eta_B(v_i) = \varphi(B_{v_i t_i})$.)

We need to switch colorations. If x is a coloration of (Φ, h) and η is a switching function, we define x^η by

$$x^\eta(v_i) := x_i \eta(v_i),$$

the result of the gain-group action on x_i.

Proposition 5.2. *If in (Φ, h) not all vertex lists h_i are finite, then the number of proper colorations is either zero or infinite. If all lists are finite, then the number of proper colorations equals*

$$\sum_{B \in \mathrm{Lat}^{\mathrm{b}} \Phi} \mu(\varnothing, B) \prod_{W \in \pi(B)} |h/B(W)| = \sum_{B \subseteq E:\ balanced} (-1)^{|B|} \prod_{W \in \pi(B)} |h/B(W)|,$$

where μ is the Möbius function of $\mathrm{Lat}^{\mathrm{b}} \Phi$.

For the Möbius function of a poset see, i.a., [10, 11]; note that $\mu(\varnothing, B) = 0$ if the empty set is not closed, that is, if Φ has a balanced loop or a loose edge. The two sums are equal because $\mu(\varnothing, B) = \sum\{(-1)^{|B'|} : \mathrm{cl}(B') = B\}$ if B is balanced and closed, and $\pi(B') = \pi(B)$. For $\mathrm{Lat}^{\mathrm{b}} \Phi$ see the end of Section 2.2.

Proof. First let us suppose h_n is infinite. If there is any proper coloration x, then there are at most $n - 1$ values that $x_1, \ldots, x_{n-1}$ prevent x_n from taking on; but there is an infinite number of permitted possible choices of $x_n \in h_n$.

Now we assume all lists are finite. To prove the first part of the formula we use Möbius inversion over $\mathrm{Lat}^{\mathrm{b}} \Phi$ as in [10, p. 362] or [14, Theorem 2.4]. (The second part has a similar proof by inversion over the class of balanced edge sets.) Throughout the proof B denotes an element of $\mathrm{Lat}^{\mathrm{b}} \Phi$. Consider all colorations of (Φ, h), proper or not; let $f(B)$ be the number of colorations x such that $I(x) = B$ and let $g(B)$ be the number of colorations such that $I(x) \supseteq B$. By Lemma 5.1 each coloration is counted in one $f(B)$, so for a closed, balanced set A,

$$g(A) = \sum_{B \supseteq A} f(B),$$

from which by Möbius inversion

$$f(A) = \sum_{B \supseteq A} \mu(\varnothing, B) g(B).$$

Setting $A = \varnothing$, the total number of proper colorations equals

$$\sum_B \mu(\varnothing, B) g(B).$$

We show by a bijection that $g(B)$ is the number of all colorations of $(\Phi, h)/B$, which clearly equals $\prod_{W \in \pi(B)} |h/B(W)|$. Let η_B be the top switching function for B. It is easy to see that switching a coloration x of (Φ, h) gives a coloration of $(\Phi, h)^{\eta_B}$ that is constant on components of B, and conversely. Therefore, if $W \in \pi(B)$, y_W, defined as the common value of $x_i^{\eta_B}$ for every $v_i \in W$, belongs to $h_i^{\eta_B}$ for every $v_i \in W$. When we contract (Φ, h) by B, $y_W \in \bigcap_{v_i \in W} h_i^{\eta_B} = h/B(W)$, so we get a well-defined coloration y of $(\Phi, h)/B$.

Conversely, for any $B \in \mathrm{Lat}^{\mathrm{b}} \Phi$, a coloration y of $(\Phi, h)/B$ pulls back to a coloration of $(\Phi, h)^{\eta_B}$ by $x_i = y_W$ where $v_i \in W \in \pi(B)$. Then switching back to (Φ, h) we have a coloration $x^{\eta_B^{-1}}$ of (Φ, h) whose improper edge set contains B. Since it is clear that these correspondences are inverse to each other, the bijection is proved. $\qquad \square$

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The last part of the proof can be strengthened to yield a formula for proper colorations of contractions.

Proposition 5.3. *If all vertex lists h_i are finite and B is a balanced edge set, then the number of colorations of (Φ, h) whose improper edge set equals B equals the number of proper colorations of $(\Phi, h)/B$.*

The proof is a simple modification of the evaluation of $g(B)$ in the previous proof, and is also a simple generalization of the evaluation of $f(B)$ in the proof of [7, Theorem 3.3]. (In [7] we accidentally wrote $f(B)$ when we meant $g(B)$; but that led us to write a proof of Proposition 5.3 in the special situation of [7]. We thank Seth Chaiken for pointing out the error in [7].)

Let $\mathcal{P}_{\text{fin}}(\mathfrak{C})$ be the class of finite subsets of $\mathfrak{C}$, and let us call a weighted gain graph with weights in $\mathcal{P}_{\text{fin}}(\mathfrak{C})$ *finitely list weighted*. Then the total dichromatic polynomial has a variable u_h for each finite subset $h \subseteq \mathfrak{C}$.

Theorem 5.4. *If (Φ, h) is finitely list weighted, then the number of proper colorations equals $(-1)^n Q_{(\Phi,h)}(\mathbf{u}, -1, 0)$ evaluated at $u_h = -|h|$.*

Note that we evaluate the balanced dichromatic polynomial, not the total polynomial. We do not have a similar interpretation of an evaluation of the total dichromatic polynomial.

Proof. The proof is by comparing the second formula of Proposition 5.2 to the definition of $Q_{(\Phi,h)}$. □

Call a *signed Tutte invariant* any function that satisfies (Tii–iv) and the modified form of (Ti),

(Ti$^-$) (Subtractivity) For every link e,
$$f(\Phi, h) = f(\Phi \setminus e, h) - f(\Phi/e, h/e).$$
It is clear that f is a signed Tutte invariant if and only if $(-1)^{|V|} f$ is a Tutte invariant.

Corollary 5.5. *Given a gain group $\mathfrak{G}$ and a color set $\mathfrak{C}$, the number of proper colorations is a signed Tutte invariant of finitely list-weighted gain graphs with gains in $\mathfrak{G}$.*

Proof. The family of finitely list weighted $\mathfrak{G}$-gain graphs is closed under deletion and contraction because finiteness of lists is preserved by those operations. Apply Theorem 5.4. □

Example 5.1 (*Finite lists*). If $\mathfrak{C}$ is partially ordered, take $\mathfrak{W}$ to consist of all finite order ideals, or all finite intervals. The special case $\mathfrak{C} = \mathbb{Z}^d$ is our main example.

5.2. **Filtered lists.** In examples the color lists for the vertices are not always finite. A general picture is that there is a list h_i for each vertex, which is some subset of $\mathfrak{C}$, and there is also a set $M_i \subseteq \mathfrak{C}$ that acts as a filter of colors: a color must lie not only in its vertex list but also in M_i. Thus we have a function $\chi_{(\Phi,h)}(M)$ defined for $M := (M_1, \ldots, M_n) \in \mathcal{P}(\mathfrak{C})^n$ whose value is the number of proper colorations of (Φ, h) using only colors in M_i at vertex v_i. This is the *list chromatic function* of (Φ, h). An example, of course, is the quantity of Proposition 5.2, which equals $\chi_{(\Phi,h)}(\mathfrak{C}^n)$ (finite lists with no filtering). That proposition has the following extension. We define switching of a color filter M and its contraction $M/B(W)$ just as for weights, so that
$$M/B(W) := \bigcap_{v_i \in W} M_i \eta_B(v_i).$$

Proposition 5.6. *If every intersection $M_i \cap h_i$ is finite, then*

$$\chi_{(\Phi,h)}(M) = \sum_{B \in \mathrm{Lat}^{\mathrm{b}}\Phi} \mu(\varnothing, B) \prod_{W \in \pi(B)} |h/B(W) \cap M/B(W)|.$$

Proof. In Proposition 5.2 replace h_i by $h_i \cap M_i$. $\square$

Since color filters contract like weights, we can form a *doubly weighted* gain graph by taking new weights (h_i, M_i), provided we define a new semigroup $\mathfrak{M}_0$ from which the double weights are drawn. To that end, let

$$\mathfrak{M}_0 := \{(h', M') : h' \in \mathfrak{W},\ M' \subseteq \mathfrak{C},\ \text{and } M' \cap h' \text{ is finite}\}.$$

The semigroup operation is componentwise intersection, i.e., $(h', M') \cap (h'', M'') := (h' \cap h'', M' \cap M'')$, and the action of $\mathfrak{G}$ on $\mathfrak{M}_0$ is componentwise. Given this weight semigroup, there is a doubly weighted total dichromatic polynomial, which we write $Q_{(\Phi,h),M}(\mathbf{u}, v, z)$, with $M = (M_1, \ldots, M_n)$, to emphasize the different roles of h and M. The variables are now $u_{h',M'}$ for each pair $(h', M') \in \mathfrak{M}_0$, and of course v and z, and the formula for the doubly weighted polynomial is

$$Q_{(\Phi,h),M}(\mathbf{u}, v, z) = \sum_{S \subseteq E} v^{|S|-n+b(S)} z^{c(S)-b(S)} \prod_{W \in \pi_{\mathrm{b}}(S)} u_{h/S(W),M/S(W)}.$$

where, as usual, $h/S(W) := h^{\eta_S}(w)$ and $M/S(W) := M^{\eta_S}(w)$ for $w \in W$. It is easy to see that every $(h/S(W), M/S(W)) \in \mathfrak{M}_0$ if every $(h_i, M_i) \in \mathfrak{M}_0$.

Theorem 5.7. *If (Φ, h) and $M_1, \ldots, M_n \subseteq \mathfrak{C}$ are such that the filtered list $M_i \cap h_i$ is finite for each vertex v_i, then the list chromatic function $\chi_{(\Phi,h)}(M_1, \ldots, M_n)$ is obtained from $(-1)^n Q_{(\Phi,h),M}(\mathbf{u}, -1, 0)$ by setting $u_{h',M'} = -|M' \cap h'|$ for each $h' \in \mathfrak{W}$ and $M' \subseteq \mathfrak{C}$.*

Proof. Like that of Theorem 5.4, but from Proposition 5.6. $\square$

Fix the gain group $\mathfrak{G}$, color set $\mathfrak{C}$, and weight subsemigroup $\mathfrak{W} \subseteq (\mathcal{P}(\mathfrak{C}), \cap)$. Consider any weighted gain graph (Φ, h) with gains in $\mathfrak{G}$ and weights (functioning as vertex color lists) in $\mathfrak{W}$. Allow any collection of color filters $(M_1, \ldots, M_n)$ for which all $h_i \cap M_i$ are finite; we call $((\Phi, h), M)$ a *finitely filtered, list-weighted gain graph*. This gives us a list chromatic function $\chi_{(\Phi,h)}(M_1, \ldots, M_n)$ that is always a well defined nonnegative integer. Thinking of $((\Phi, h), M)$ as a (doubly) weighted gain graph, we have deletions and contractions and we can ask about Tutte invariance.

Corollary 5.8. *The list chromatic function $\chi_{(\Phi,h)}(M_1, \ldots, M_n)$ is a signed Tutte invariant of finitely filtered, list-weighted gain graphs.*

Proof. As we noted, the class of list-weighted gain graphs with suitable arguments is closed under deletion and contraction. Now, apply Theorem 5.7. $\square$

Example 5.2 (*Locally finite join semilattice*). Suppose $\mathfrak{C}$ is a locally finite join semilattice. We may take $\mathfrak{W}$ to be the set of principal dual order ideals $\langle z \rangle^*$ and let the M_i range over principal ideals. The join operation makes $\mathfrak{W}$ an intersection semigroup, as the intersection $\langle z \rangle^* \cap \langle z' \rangle^*$ is the principal dual ideal $\langle z \vee z' \rangle^*$; and $\mathfrak{W}$ is clearly $\mathfrak{G}$-invariant. The intersection $M_i \cap h_i$ will always be finite so the preceding proposition and theorem apply.

We may weaken the assumptions. Let $\mathfrak{W}$ be the class of all subsets of $\mathfrak{C}$ that have a lower bound; that is, subsets of principal dual ideals. And let M_i be any subset with an

upper bound; that is, a subset of a principal ideal. Then $M_i \cap h_i$ is necessarily finite so the preceding results hold good. That is, we admit as filters all upper-bounded subsets of $\mathfrak{C}$.

An example of this kind is that in which $\mathfrak{C} = \mathbb{Z}^d$; it is the topic of the next subsection.

5.3. **Open-ended list coloring in an integer lattice.** To introduce the main applications we turn once again to the original example from [7] but with a slight change in viewpoint.

Example 5.3 (*Open-ended interval coloring*). The gain group is $\mathbb{Z}$ and the weights can be treated as upper intervals, (h_i, ∞) at vertex v_i. A *proper m-coloration* of (Φ, h) is a function $x : V \to \mathbb{Z}$ such that each $x_i \in (h_i, \infty)$ and all $x_i \leq m$. One can think of this as a list coloration in which the list for each vertex is an interval that grows with m. The *integral chromatic function* $\chi^{\mathbb{Z}}_{(\Phi,h)}(m)$ of [7] counts proper m-colorations. This function is obtained from $Q(\mathbf{u}, -1, 0)$ by substituting $u_k = -\max(m-k, 0)$. Hence it is a signed Tutte invariant, as we showed in [7] directly from its counting definition. We showed in [7] that it is eventually a monic polynomial of degree $n = |V|$, and that it is a sum of simple terms that appear successively as m increases.

We generalize this example in several ways: to higher-dimensional coloring, to upper bounds that depend on the vertex, and to arbitrary vertex lists.

For higher-dimensional coloring the color set $\mathfrak{C}$ is the d-dimensional integer lattice $\mathbb{Z}^d$ with the componentwise partial ordering and the gain group $\mathfrak{G}$ is the additive group $\mathbb{Z}^d$ acting on $\mathfrak{C}$ by translation. A coloration is any $\mathbf{x} : V \to \mathbb{Z}^d$. The weight semigroup $\mathfrak{W}$ is either of the classes

$$\mathfrak{W}_1 := \{H' \subset \mathbb{Z}^d : H' \text{ is bounded below}\},$$

that is, H' is contained in a cone $\langle \mathbf{a} \rangle^* = \bigtimes_{k=1}^d [a_k, \infty)$ for some $\mathbf{a} \in \mathbb{Z}^d$, and

$$\mathfrak{W}_2 := \{H' \in \mathfrak{W}_1 : \text{ for some } \mathbf{a} \in \mathbb{Z}^d \text{ and finite } A \subset \langle \mathbf{a} \rangle^*, \ H' = \langle \mathbf{a} \rangle^* \setminus A\}.$$

Both classes, $\mathfrak{W}_1$ and $\mathfrak{W}_2$, are closed under intersection of pairs and under translation. For $H' \in \mathfrak{W}_1$ we define $\mathbf{h}'$ as the meet of the members of H'—it is the largest possible $\mathbf{a}$. When $H' \in \mathfrak{W}_2$ and $d > 1$ this is the only possible $\mathbf{a}$, but in dimension $d = 1$ it is the smallest element of H'.

For $H_i \in \mathfrak{W}_1$ define $\bar{H}_i := \langle \mathbf{h}_i \rangle^* \setminus H_i$, and let $\hat{\mathbf{h}}_i := \bigvee \bar{H}_i$, except that $\hat{\mathbf{h}}_i = \mathbf{h}_i - (1, \ldots, 1)$ if $\bar{H}_i = \varnothing$. Clearly, $\hat{\mathbf{h}}_i$ is defined in $\mathbb{Z}^d$ if and only if $\bar{H}_i$ is finite, that is, $H_i \in \mathfrak{W}_2$.

The color filters M_i are principal order ideals

$$M_i = \langle \mathbf{m}_i \rangle = \bigtimes_{k=1}^d (-\infty, m_{ik}], \text{ where } \mathbf{m}_i = (m_{i1}, \ldots, m_{id}) \in \mathbb{Z}^d,$$

so the list chromatic function has domain $\mathbb{Z}^d$ and is defined by

$$\chi_{(\Phi,h)}(\mathbf{m}) := \text{the number of } \mathbf{x} : V \to \mathbb{Z}^d \text{ such that each } \mathbf{x}_i \in H_i, \ \mathbf{x} \leq \mathbf{m},$$
$$\text{and } \mathbf{x}_j \neq \mathbf{x}_i + \varphi(e_{ij}) \text{ for each edge } e_{ij},$$

where $\mathbf{m} := (\mathbf{m}_1, \ldots, \mathbf{m}_n)$. This number is a function of one variable m_{ik} for each $i = 1, \ldots, n$ and $k = 1, \ldots, d$ and is finite for each $\mathbf{m}$. Our general theory shows that $\chi_{(\Phi,h)}(\mathbf{m})$ is an evaluation of the total dichromatic polynomial; and now we can generalize Example 5.3. In

the present case a switching function is $\eta : V \to \mathbb{Z}^d$ and the contraction formula for weights takes the form

$$H/B(W) = \bigcap_{v_i \in W} \left(H_i + \eta_B(v_i) \right), \quad M_B(W) = \bigcap_{v_i \in W} \left(\langle \mathbf{m}_i \rangle + \eta_B(v_i) \right)$$

if B is a balanced edge set and $W \in \pi(B)$, where $H_i + \mathbf{a}$ denotes the translation of H_i by $\mathbf{a}$. In the total dichromatic polynomial there is one variable $u_{H',\mathbf{m}'}$ for each $H' \in \mathfrak{W}_1$ and $\mathbf{m}' \in (\mathbb{Z}^d)^n$. (We simplify the notation $u_{H',\langle \mathbf{m}' \rangle}$ to $u_{H',\mathbf{m}'}$.)

Theorem 5.9. *With all $H_i \in \mathfrak{W}_1$, the list chromatic function $\chi_{(\Phi,h)}(\mathbf{m})$ is obtained from $(-1)^n Q_{(\Phi,h),M}(\mathbf{u},-1,0)$ by setting $u_{H',\mathbf{m}'} = -\left| H' \cap \langle \mathbf{m}' \rangle \right|$ for each $H' \in \mathfrak{W}_1$ and $\mathbf{m}' \in \mathbb{Z}^d$.*

Proof. A corollary of Theorem 5.7, since the intersections $H_i \cap M_i$ are finite. $\qquad \square$

Corollary 5.10. *The list chromatic function is a signed Tutte invariant of gain graphs with gain group $\mathbb{Z}^d$ and weight lists belonging to $\mathfrak{W}_1$.*

Proof. A special case of Corollary 5.8. $\qquad \square$

For vertices v_i and v_j, let

$$\alpha_{ji}(\Phi) := \bigvee_{P_{ji}} \varphi(P_{ji}),$$

where P_{ji} ranges over all paths in Φ from v_j to v_i, and let

$$\alpha_j(\Phi) := \bigvee_i \alpha_{ji}(\Phi),$$

the least upper bound of the gains of all paths that begin at v_j. For $W \subseteq V$, define $\alpha_j(\Phi{:}W)$ similarly, with the induced subgraph $\Phi{:}W$ replacing Φ.

A function $p(y_1, \ldots, y_r)$ defined on $\mathbb{R}^r$ is a *piecewise polynomial* if $\mathbb{R}^r$ is a union of a finite number of closed, full-dimensional sets D_σ, on each of which p is a polynomial $p_\sigma(y_1, \ldots, y_r)$. By saying p has leading term $y_1 \cdots y_r$, we mean that at least one p_σ has that leading term and every other p_σ has degree no higher than 1 in any variable. This definition is chosen to suit the following result. The numbers h_{ik} are the components of the vectors $\mathbf{h}_i$. This theorem looks complicated; the reader may find Example 5.4, after the proof, helpful.

Theorem 5.11. *Assume (Φ, h) has no balanced loops or loose edges. Suppose all $H_i \in \mathfrak{W}_2$. Define*

$$q_k(B, W) := \min_{v_i \in W} \left(m_{ik} + \eta_B(v_i)_k \right) - \max_{v_i \in W} \left(h_{ik} + \eta_B(v_i)_k \right) + 1$$

for $B \in \mathrm{Lat}^b\, \Phi$, $W \in \pi(B)$, and $1 \le k \le d$, and

$$p(\mathbf{m}) := \sum_{B \in \mathrm{Lat}^b\, \Phi} \mu(\varnothing, B) \prod_{W \in \pi(B)} \left(\prod_{k=1}^d q_k(B, W) - \Big| \bigcup_{v_i \in W} \left(\bar{H}_i + \eta_B(v_i) \right) \Big| \right),$$

which is a piecewise polynomial function of the nd variables m_{ik} having leading term $\prod_{i=1}^n \prod_{k=1}^d m_{ik}$. The list chromatic function $\chi_{(\Phi,h)}(\mathbf{m})$ is a piecewise polynomial that equals $p(\mathbf{m})$ for all $\mathbf{m} \in (\mathbb{Z}^d)^n$ such that $\mathbf{m} \ge \bar{\bar{\mathbf{m}}}(\Phi, h)$, where

$$\bar{\bar{\mathbf{m}}}_i(\Phi, h) := \bigvee_{j=1}^n \left(\hat{\mathbf{h}}_j + \alpha_{ji}(\Phi) \right) \in \mathbb{Z}^d.$$

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For the ordinary graph version of this theorem, without gains, see Corollary 6.3.

Proof. In this theorem we have $r = nd$, the variables are $m_{11}, m_{12}, \ldots, m_{1d}, m_{21}, \ldots, m_{nd}$, and the domains D_σ are the sets on which each of the d sets (one for each fixed $k \leq d$) of shifted variables $m_k(B, W) := \min_{v_i \in W} (m_{ik} + \eta_B(v_i))$ (one variable for each B, each $W \in \pi(B)$, and each $k \leq d$) assumes a particular weakly increasing order, since the orderings of these variables determine exactly which polynomial $p(\mathbf{m})$ is. Thus, $p(\mathbf{m})$ is definitely a piecewise polynomial. One task is to prove that it equals $\chi_{(\Phi,h)}(\mathbf{m})$ for $\mathbf{m} \geq \hat{\mathbf{m}}(\Phi, h)$. The other, which we do first, is to prove that $\chi_{(\Phi,h)}(\mathbf{m})$ itself is a piecewise polynomial.

We apply the formula of Proposition 5.6 in the form

$$(5.1) \qquad \chi_{(\Phi,h)}(\mathbf{m}) = \sum_{B \in \mathrm{Lat}^b \Phi} \mu(\varnothing, B) \prod_{W \in \pi(B)} \big| H/B(W) \cap M/B(W) \big|,$$

which shows that the theorem is true when the range of $\mathbf{m}$ is such that $|H/B(W) \cap M/B(W)| = q_k(B, W)$.

First we partially determine the values of $\mathbf{m}$ for which the term of a closed, balanced set B is nonzero. Define $\bar{\mathbf{m}}_i((\Phi, h)) := \bigvee_{j=1}^n (\mathbf{h}_j + \alpha_{ji}(\Phi))$. Note that

$$\bar{\mathbf{m}}(\Phi, h) = \bigvee_{B \in \mathrm{Lat}^b \Phi} \bar{\mathbf{m}}((\Phi, h)|B),$$

since every path P_{ji} is contained in some balanced set B. Restrict $\bar{\mathbf{m}}_i((\Phi, h)|B)$ to W as $\bar{\mathbf{m}}((\Phi, h)|B)|_W$.

Lemma 5.12. *In the sum of Equation* (5.1), *the term of B is zero unless $\mathbf{m} \geq \bar{\mathbf{m}}((\Phi, h)|B)$.*

Proof. Rewrite the expression for the factor of W in the term of B, with $[\mathbf{x}, \mathbf{y}]$ denoting an interval in the lattice $\mathbb{Z}^d$ and $\mathbf{m}_i + \eta_B(v_i)$ meaning that $\mathbf{m}_i$ is translated by $\eta_B(v_i)$:

$$
\begin{aligned}
(5.2) \\
H/B(W) \cap M/B(W) &= \bigcap_{v_i \in W} (H_i + \eta_B(v_i)) \cap \bigcap_{v_i \in W} \langle \mathbf{m}_i + \eta_B(v_i) \rangle \\
&= \bigcap_{v_i \in W} \big([\langle \mathbf{h}_i \rangle^* \setminus \bar{H}_i] + \eta_B(v_i)\big) \cap \bigcap_{v_i \in W} \big(\langle \mathbf{m}_i \rangle + \eta_B(v_i)\big) \\
&= \bigcap_{v_i \in W} \big(([\mathbf{h}_i, \mathbf{m}_i] \setminus \bar{H}_i) + \eta_B(v_i)\big) \\
&= \bigcap_{v_i \in W} \big([\mathbf{h}_i, \mathbf{m}_i] + \eta_B(v_i)\big) \setminus \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big) \\
&= \bigcap_{v_i \in W} [\mathbf{h}_i + \eta_B(v_i), \mathbf{m}_i + \eta_B(v_i)] \setminus \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big) \\
&= \Big[\bigvee_{v_i \in W} (\mathbf{h}_i + \eta_B(v_i)), \bigwedge_{v_i \in W} (\mathbf{m}_i + \eta_B(v_i)) \Big] \setminus \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big)
\end{aligned}
$$

since $H_i = \langle \mathbf{h}_i \rangle^* \setminus \bar{H}_i$ and $M_i = \langle \mathbf{m}_i \rangle$. The interval in the last line has another expression:

$$\Big[\bigvee_{v_i \in W} (\mathbf{h}_i + \eta_B(v_i)), \bigwedge_{v_i \in W} (\mathbf{m}_i + \eta_B(v_i)) \Big] = \bigtimes_{k=1}^d \Big[\max_{v_i \in W} (h_{ik} + \eta_B(v_i)_k), \min_{v_i \in W} (m_{ik} + \eta_B(v_i)_k) \Big],$$

which is an orthotope that plainly contains $\prod_{k=1}^d q_k(B, W)$ lattice points if it is not empty.

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Let us see, therefore, what are the conditions under which the orthotope is nonempty. We need $\mathbf{h}_i + \eta_B(v_i) \le \mathbf{m}_j + \eta_B(v_j)$ for all $v_i, v_j \in W$. That means $\mathbf{m}_j \ge \mathbf{h}_i + \eta_B(v_i) - \eta_B(v_j)$. Now recall from Equation (2.1) that $\eta_B(v_j) - \eta_B(v_i) = \varphi(B_{ji})$. Here B_{ji} is any path in B from v_j to v_i. Thus, the condition on $\mathbf{m}_j$ is that $\mathbf{m}_j \ge \mathbf{h}_i + \alpha_{ji}(B{:}W)$, which is satisfied for all $v_j \in W$ if and only if $\mathbf{m}|_W \ge \bar{\mathbf{m}}((\Phi, h)|B)|_W$. If this is not satisfied for all $W \in \pi(B)$, that is, if $\mathbf{m} \not\ge \bar{\mathbf{m}}((\Phi, h)|B)$, then the orthotope is empty and the term of B disappears. $\quad\square$

If $\mathbf{m} \ge \bar{\mathbf{m}}((\Phi, h)|B)$, the term of B may still equal 0 if the orthotope is contained in $\bigcup_{v_i \in W} (\bar{\bar{H}}_i + \eta_B(v_i))$; therefore, the converse of the lemma is not valid.

The lemma implies that terms of $\chi_{(\Phi, h)}(\mathbf{m})$ appear gradually as $\mathbf{m}$ gets larger. That generalizes the behavior of the corresponding function for $d = 1$ (Example 5.3), proved in [7]. The condition for all terms to appear is that $\mathbf{m} \ge \bigvee_B \bar{\mathbf{m}}((\Phi, h)|B) = \bar{\mathbf{m}}(\Phi, h)$.

Next, we prove piecewise polynomiality of $\chi_{(\Phi, h)}(\mathbf{m})$ (with the specified leading term). Indeed, each factor in each term is piecewise linear.

Consider first $W = \{v_i\}$, which is a component when $B = \varnothing$. (Then $\eta_B(v_i) = 0 \in \mathbb{R}^d$.) The corresponding factor is

(5.3)
$$
\begin{aligned}
\big| H(\{v_i\}) \cap M(\{v_i\}) \big| &= \big| \big([\mathbf{h}_i, \mathbf{m}_i] + \eta_B(v_i)\big) \setminus \big(\bar{H}_i + \eta_B(v_i)\big) \big| = \big| [\mathbf{h}_i, \mathbf{m}_i] \setminus \bar{H}_i \big| \\
&\overset{?}{=} \big| [\mathbf{h}_i, \mathbf{m}_i] \big| - \big| \big(\bar{H}_i \cap \langle \mathbf{m}_i \rangle\big) \big| \\
&= \prod_{k=1}^{d} (m_{ik} - h_{ik} + 1)^+ - \big| \bar{H}_i \cap \langle \mathbf{m}_i \rangle \big|.
\end{aligned}
$$

When $\mathbf{m}_i \ge \hat{\mathbf{h}}_i$, the questionable equality $\overset{?}{=}$ is equality and moreover the right-hand side is a multilinear function with leading term $\prod_{i=1}^{n} \prod_{k=1}^{d} m_{ik}$, because then $\bar{H}_i \subseteq \langle \mathbf{m}_i \rangle$ so $|\bar{H}_i \cap \langle \mathbf{m}_i \rangle| = |\bar{H}_i|$. Furthermore, for any $\mathbf{m}$, $\bar{H}_i \cap \langle \mathbf{m}_i \rangle$ is a finite subset $S \subseteq \bar{H}_i$, and for each S, $|[\mathbf{h}_i, \mathbf{m}_i] \setminus (\bar{H}_i \cap \langle \mathbf{m}_i \rangle)| = |[\mathbf{h}_i, \mathbf{m}_i]| - |S|$, which is a multilinear function of $\mathbf{m}$. There are only finitely many subsets S, hence $(\mathbb{Z}^d)^n$ is the union of a finite number of domains, on each of which $\chi_{(\Phi, h)}(\mathbf{m})$ is multilinear. It follows that the term of $B = \varnothing$ is piecewise polynomial on $(\mathbb{R}^d)^n$ and has the correct leading term.

Treating in the same manner each factor in Equation (5.2), rewritten as

(5.4)
$$
H/B(W) \cap M/B(W) = \Big[\bigvee_{v_i \in W} \big(\mathbf{h}_i + \eta_B(v_i)\big), \bigwedge_{v_i \in W} \big(\mathbf{m}_i + \eta_B(v_i)\big) \Big] \setminus S
$$

where $S := \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big) \cap \bigcap_{v_i \in W} \langle \mathbf{m}_i + \eta_B(v_i) \rangle$, we see that $|H/B(W) \cap M/B(W)|$ is similarly piecewise linear since there are only finitely many possible sets S and each one is finite. When $S = \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big)$, i.e., when $\mathbf{m}_i + \eta_B(v_i) \ge \mathbf{h}_j + \eta_B(v_j)$ for all $v_i, v_j \in W$, the formula is

$$
\begin{aligned}
\big| H/B(W) \cap M/B(W) \big| &= \Big| \Big[\bigvee_{v_i \in W} \big(\mathbf{h}_i + \eta_B(v_i)\big), \bigwedge_{v_i \in W} \big(\mathbf{m}_i + \eta_B(v_i)\big) \Big] \Big| \setminus |S| \\
&= q_k(B, W) - \Big| \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big) \Big|.
\end{aligned}
$$

Recall from Equation (2.1) that $\eta_B(v_j) - \eta_B(v_i) = \varphi(B_{ji})$, where B_{ji} is a path in B from v_j to v_i. We can choose B to contain any desired path P_{ji} from v_j to v_i in Φ by letting B be a balanced set containing P_{ji}. Thus, the condition on $\mathbf{m}_j$ is that $\mathbf{m}_j \ge \mathbf{h}_j + \max_{P_{ji}} \varphi(P_{ji}) = \mathbf{h}_i + \alpha_{ji}(\Phi)$, which is satisfied for all $v_j \in V$ if and only if $\mathbf{m} \ge \bar{\mathbf{m}}(\Phi, h)$.

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We have proved that $\chi_{(\Phi,h)}(\mathbf{m})$ is a piecewise polynomial function. The term of highest degree is that for which $\pi(B)$ has the most components, which is n components when $B = \varnothing$. The term of each balanced, closed set B is a monic piecewise polynomial of total degree $dc(B)$. The previous analysis shows that the natural sufficient condition for $\chi_{(\Phi,h)}(\mathbf{m})$ to equal $p(\mathbf{m})$ is that $S = \bar{H}_i \cap \langle \mathbf{m}_i \rangle$ in every factor of every term, i.e., $\mathbf{m}_i \geq \hat{\bar{\mathbf{m}}}_i(\Phi, h)$. Thus, to make the chromatic function equal to $p(\mathbf{m})$ it suffices that $\mathbf{m} \geq \hat{\bar{\mathbf{m}}}(\Phi, h)$. $\qquad\square$

The proof suggests that the theorem's lower bound on $\mathbf{m}$ is essential; for any other choice of $\mathbf{m}$, $p(\mathbf{m})$ will not agree with $\chi_{(\Phi,h)}(\mathbf{m})$. The main reason is that the constant term $\left| \bigcup_{v_i \in W} \left(\bar{H}_i + \eta_B(v_i) \right) \right|$ in the factor of W in $p(\mathbf{m})$ assumes that $\bigcup_{v_i \in W} \left(\bar{H}_i + \eta_B(v_i) \right)$ is contained in $\left\langle \bigwedge_{v_i \in W} \left(\mathbf{m}_i + \eta_B(v_i) \right) \right\rangle$. However, we have not tried to prove necessity of the lower bound, and there might be exceptions.

It is clear, though, why H_i has to be bounded below and its cone complement $\bar{H}_i$ must be finite. If some H_i has no lower bound, then $\chi_{(\Phi,h)}(m)$ will be infinite. Even when each H_i is bounded below, if some $\bar{H}_i$ is infinite it has no upper bound; then $\chi_{(\Phi,h)}(m)$ will not settle down to a finite system of polynomials for large values $\mathbf{m}$.

Example 5.4. We do a very small example, just large enough to show the phenomena that appear in computing $\chi_{(\Phi,h)}(\mathbf{m})$. This example has $n = 2$ vertices, so it represents hyperplanes in $\mathbb{R}^2$. The gain group and color set are $\mathbb{Z}^2$ $(d = 2)$. The variables are four, in two lattice points: $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$, where $\mathbf{m}_1 = (m_{11}, m_{12})$ and $\mathbf{m}_2 = (m_{21}, m_{22})$. We choose

$$\mathbf{h}_1 = (1, 0), \ \mathbf{h}_2 = (0, 3), \ \bar{H}_1 = \{\mathbf{c}_1\}, \ \bar{H}_2 = \{\mathbf{c}_2\}.$$

Thus, $\hat{\mathbf{h}}_i = \mathbf{c}_i = \hat{\mathbf{m}}_i = \bar{\mathbf{m}}_i$ for $i = 1, 2$.

Notation: For a real number, $x^+ := \max(x, 0)$ and $x^- := -\min(x, 0)$. Applied to a vector, $\mathbf{x} = (x_1, x_2)$, $\mathbf{x}^+ := (x_1^+, x_2^+)$, and $\mathbf{x}^-$ is similar. δ_{AB} is the Kronecker delta; $\zeta_{ST} = 1$ if $S \subseteq T$, otherwise 0; $\varepsilon_{aS} = 1$ if $a \in S$, otherwise 0.

To evaluate $\chi_{(\Phi,h)}(\mathbf{m})$ we first find the factor of each nonempty $W \subseteq V$. For $W = \{v_1\}$ the factor is $\left| [\mathbf{h}_1, \mathbf{m}_1] \setminus \{\mathbf{c}_1\} \right|$. For $W = \{v_2\}$ it is $\left| [\mathbf{h}_2, \mathbf{m}_2] \setminus \{\mathbf{c}_2\} \right|$. Hence the term of $B = \varnothing$ is

$$
\begin{aligned}
(5.5) \quad & \left| [\mathbf{h}_1, \mathbf{m}_1] \setminus \{\mathbf{c}_1\} \right| \cdot \left| [\mathbf{h}_2, \mathbf{m}_2] \setminus \{\mathbf{c}_2\} \right| \\
& = \left((m_{11} - h_{11} + 1)^+ (m_{12} - h_{12} + 1)^+ - \varepsilon_{\mathbf{c}_1, [\mathbf{h}_1, \mathbf{m}_1]} \right)^+ \\
& \qquad \cdot \left((m_{21} - h_{21} + 1)^+ (m_{22} - h_{22} + 1)^+ - \varepsilon_{\mathbf{c}_2, [\mathbf{h}_2, \mathbf{m}_2]} \right)^+ \\
& = \left(m_{11}^+ (m_{12} + 1)^+ - \varepsilon_{\mathbf{c}_1, [(1,0), \mathbf{m}_1]} \right)^+ \cdot \left((m_{21} + 1)^+ (m_{22} - 2)^+ - \varepsilon_{\mathbf{c}_2, [(0,3), \mathbf{m}_2]} \right)^+.
\end{aligned}
$$

The term of $W = \{v_1, v_2\}$ depends on B, which can consist of any one edge $g(v_1, v_2)$. The term is

$$
\begin{aligned}
& \left| [(\mathbf{h}_1 + g^+) \vee (\mathbf{h}_2 + g^-), (\mathbf{m}_1 + g^+) \wedge (\mathbf{m}_2 + g^-)] \setminus \{\mathbf{c}_1, \mathbf{c}_2\} \right| \\
& = \left(\min(m_{11} + g_1^+, m_{21} + g_1^-) - \max(h_{11} + g_1^+, h_{21} + g_1^-) + 1 \right)^+ \\
& \qquad \cdot \left(\min(m_{12} + g_2^+, m_{22} + g_2^-) - \max(h_{12} + g_2^+, h_{22} + g_2^-) + 1 \right)^+ \\
& \qquad - \varepsilon_{\mathbf{c}_1, [(\mathbf{h}_1 + g^+) \vee (\mathbf{h}_2 + g^-), (\mathbf{m}_1 + g^+) \wedge (\mathbf{m}_2 + g^-)]} \\
& \qquad - \varepsilon_{\mathbf{c}_2, [(\mathbf{h}_1 + g^+) \vee (\mathbf{h}_2 + g^-), (\mathbf{m}_1 + g^+) \wedge (\mathbf{m}_2 + g^-)]}.
\end{aligned}
$$

For instance, if $g = (0,0)$, the term of $B = \{(0,0)(v_1, v_2)\}$ is

$$
\begin{aligned}
&\big|[\mathbf{h}_1 \vee \mathbf{h}_2, \mathbf{m}_1 \wedge (\mathbf{m}] \setminus \{\mathbf{c}_1, \mathbf{c}_2\}\big| \\
&= \big(\min(m_{11}, m_{21}) - \max(h_{11}, h_{21}) + 1\big)^+ \\
&\qquad \cdot \big(\min(m_{12}, m_{22}) - \max(h_{12}, h_{22}) + 1\big)^+ \\
&\qquad - \varepsilon_{\mathbf{c}_1, [\mathbf{h}_1 \vee \mathbf{h}_2, \mathbf{m}_1 \wedge \mathbf{m}_2]} - \varepsilon_{\mathbf{c}_2, [\mathbf{h}_1 \vee \mathbf{h}_2, \mathbf{m}_1 \wedge \mathbf{m}_2]} \\
&= \big(\min(m_{11}, m_{21})\big)^+ \cdot \big(\min(m_{12}, m_{22}) - 2\big)^+ \\
&\qquad - \varepsilon_{\mathbf{c}_1, [(1,3), \mathbf{m}_1 \wedge \mathbf{m}_2]} - \varepsilon_{\mathbf{c}_2, [(1,3), \mathbf{m}_1 \wedge \mathbf{m}_2]}.
\end{aligned}
\tag{5.6}
$$

If $g = (2,0)$, the term of $B = \{(2,0)(v_1, v_2)\}$ is

$$
\begin{aligned}
&\big|[(\mathbf{h}_1 + (2,0)^+ \vee (\mathbf{h}_2 + (2,0)^-, (\mathbf{m}_1 + (2,0)^+) \wedge (\mathbf{m}_2 + (2,0)^-] \setminus \{\mathbf{c}_1, \mathbf{c}_2\}\big| \\
&= \big(\min(m_{11} + 2, m_{21}) - \max(h_{11} + 2, h_{21}) + 1\big)^+ \\
&\qquad \cdot \big(\min(m_{12}, m_{22}) - \max(h_{12}, h_{22}) + 1\big)^+ \\
&\qquad - \varepsilon_{\mathbf{c}_1, [(\mathbf{h}_1 + (2,0)) \vee \mathbf{h}_2, (\mathbf{m}_1 + (2,0)) \wedge \mathbf{m}_2]} - \varepsilon_{\mathbf{c}_2, [(\mathbf{h}_1 + (2,0)) \vee \mathbf{h}_2, (\mathbf{m}_1 + (2,0)) \wedge \mathbf{m}_2]} \\
&= \big(\min(m_{11} + 2, m_{21}) - 2\big)^+ \cdot \big(\min(m_{12}, m_{22}) - 2\big)^+ \\
&\qquad - \varepsilon_{\mathbf{c}_1, [(3,3), (\mathbf{m}_1 + (2,0)) \wedge \mathbf{m}_2]} - \varepsilon_{\mathbf{c}_2, [(3,3), (\mathbf{m}_1 + (2,0)) \wedge \mathbf{m}_2]}.
\end{aligned}
\tag{5.7}
$$

If $g = (-1,2)$, the term of $B = \{(-1,2)(v_1, v_2)\}$ is

$$
\begin{aligned}
&\big|[(\mathbf{h}_1 + (-1,2)^+) \vee (\mathbf{h}_2 + (-1,2)^-), (\mathbf{m}_1 + (-1,2)^+) \wedge (\mathbf{m}_2 + (-1,2)^-)] \setminus \{\mathbf{c}_1, \mathbf{c}_2\}\big| \\
&= \big(\min(m_{11}, m_{21} + 1) - \max(h_{11}, h_{21} + 1) + 1\big)^+ \\
&\qquad \cdot \big(\min(m_{12} + 2, m_{22}) - \max(h_{12} + 2, h_{22}) + 1\big)^+ \\
&\qquad - \varepsilon_{\mathbf{c}_1, [(\mathbf{h}_1 + (0,2)) \vee (\mathbf{h}_2 + (1,0)), (\mathbf{m}_1 + (0,2)) \wedge (\mathbf{m}_2 + (1,0))]} \\
&\qquad - \varepsilon_{\mathbf{c}_2, [(\mathbf{h}_1 + (0,2)) \vee (\mathbf{h}_2 + (1,0)), (\mathbf{m}_1 + (0,2)) \wedge (\mathbf{m}_2 + (1,0))]} \\
&= \big(\min(m_{11}, m_{21} + 1)\big)^+ \cdot \big(\min(m_{12} + 2, m_{22}) - 2\big)^+ \\
&\qquad - \varepsilon_{\mathbf{c}_1, [(1,3), (\mathbf{m}_1 + (0,2)) \wedge (\mathbf{m}_2 + (1,0))]} - \varepsilon_{\mathbf{c}_2, [(1,3), (\mathbf{m}_1 + (0,2)) \wedge (\mathbf{m}_2 + (1,0))]}.
\end{aligned}
\tag{5.8}
$$

The degree of each term in each variable m_{ij} is 0 or 1, depending on how the shifted value of m_{ij} is related to the shifted values of other variables.

Now we settle on the specific gain graph with $V = \{v_1, v_2\}$ and

$$
E = \{e_1 = (0,0)(v_1, v_2), \ e_2 = (2,0)(v_1, v_2), \ e_3 = (-1,2)(v_1, v_2)\}.
$$

Here,

$$
\chi_{(\Phi, h)}(\mathbf{m}) = (5.5) + (5.6) + (5.7) + (5.8).
$$

As a function of $\mathbf{m}$, this consists of different polynomials in different domains, depending in part on the relative ordering of the variables m_{ij} and shifted variables $m_{11} + 2$, etc.

For *large values* of all the m_{ij}, $\chi_{(\Phi, h)}(\mathbf{m})$ has several domains on which it is different polynomials, all of which have leading term $m_{11}m_{12}m_{21}m_{22}$. We are assuming specifically that all the minima of translated m_{ij} are larger than all the maxima of translated h_{ij} in the preceding formulas. Every term of $\chi_{(\Phi, h)}(\mathbf{m})$ will be linear in each of the m_{ij}. The first term

is

$$(5.5) = \big((m_{11} - h_{11} + 1)(m_{12} - h_{12} + 1) - \varepsilon_{\mathbf{c}_1,\langle\mathbf{h}_1\rangle^*}\big)$$
$$\cdot \big((m_{21} - h_{21} + 1)(m_{22} - h_{22} + 1) - \varepsilon_{\mathbf{c}_2,\langle\mathbf{h}_2\rangle^*}\big)$$
$$= \big(m_{11}(m_{12} + 1) - \varepsilon_{\mathbf{c}_1,\langle(1,0)\rangle^*}\big) \cdot \big((m_{21} + 1)(m_{22} - 2) - \varepsilon_{\mathbf{c}_2,\langle(0,3)\rangle^*}\big),$$

which shows (since the h_{ij} and the ε's are independent of $\mathbf{m}$) that, no matter how the shifted and unshifted variables are related, the term $m_{11}m_{12}m_{21}m_{22}$ exists; it will be the leading term since the remaining terms of $\chi_{(\Phi,h)}(\mathbf{m})$ have total degree 2. The remaining terms depend on how the variables are ordered. For instance,

$$(5.6) = \big(\min(m_{11}, m_{21}) - \max(h_{11}, h_{21}) + 1\big)\big(\min(m_{12}, m_{22}) - \max(h_{12}, h_{22}) + 1\big)$$
$$- \varepsilon_{\mathbf{c}_1,\langle\mathbf{h}_1\vee\mathbf{h}_2\rangle^*} - \varepsilon_{\mathbf{c}_2,\langle\mathbf{h}_1\vee\mathbf{h}_2\rangle^*}$$
$$= \begin{cases} m_{11}\big(m_{12} - 2\big) & \text{if } m_{11} \le m_{21},\ m_{12} \le m_{22}, \\ m_{21}\big(m_{12} - 2\big) & \text{if } m_{11} \ge m_{21},\ m_{12} \le m_{22}, \\ m_{11}\big(m_{22} - 2\big) & \text{if } m_{11} \le m_{21},\ m_{12} \ge m_{22}, \\ m_{21}\big(m_{22} - 2\big) & \text{if } m_{11} \ge m_{21},\ m_{12} \ge m_{22} \end{cases} - \varepsilon_{\mathbf{c}_1,\langle(1,3)\rangle^*} - \varepsilon_{\mathbf{c}_2,\langle(1,3)\rangle^*}.$$

In each of these four domains governed by the relative values of the m_{ij}, $\chi_{(\Phi,h)}(\mathbf{m})$ is a different polynomial that is linear in all four variables and has total degree 4. That concludes the example.

The theorem simplifies when all $\mathbf{m}_i$ equal a common value $\mathbf{m}' \in \mathbb{Z}^d$.

Corollary 5.13. *For a $\mathbb{Z}^d$-gain graph Φ with no balanced loops or loose edges, suppose all weights $H_i \in \mathfrak{W}_2$. Define*

$$\bar{q}_k(B, W) := m'_k - \max_{v_i, v_j \in W} \big(h_{ik} + \eta_B(v_i)_k - \eta_B(v_j)_k\big) + 1$$

for $B \in \mathrm{Lat}^{\mathrm{b}}\,\Phi$, $W \in \pi(B)$, and $\mathbf{m}' \in \mathbb{Z}^d$. For large enough $\mathbf{m}' \in \mathbb{Z}^d$, the list chromatic function $\chi_{(\Phi,h)}(\mathbf{m}',\dots,\mathbf{m}')$ equals

$$\bar{p}(\mathbf{m}') := \sum_{B \in \mathrm{Lat}^{\mathrm{b}}\,\Phi} \mu(\varnothing, B) \prod_{W \in \pi(B)} \left(\prod_{k=1}^{d} \bar{q}_k(B, W) - \left| \bigcup_{v_i \in W} \big(\bar{H}_i + \eta_B(v_i)\big) \right| \right),$$

which is a polynomial function of the d variables m'_k having degree at most n in each variable and leading term $\prod_{k=1}^{d}(m'_k)^n$. The equation $\chi_{(\Phi,h)}(\mathbf{m}',\dots,\mathbf{m}') = \bar{p}(\mathbf{m}')$ holds true for all

$$\mathbf{m}' \ge \bigvee_{j=1}^{n} \big(\hat{\mathbf{h}}_j + \alpha_j(\Phi)\big).$$

In the one-dimensional case, where the gain group and color set are $\mathbb{Z}$, then

$$\mathfrak{W}_2 = \{H' \subset \mathbb{Z} : H' \text{ is bounded below and } \mathbb{Z} \setminus H' \text{ is bounded above}\}$$

and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$. In this case $\mathbf{h}_i = \min H_i$ and $\hat{\mathbf{h}}_i = \max(\mathbb{Z} \setminus H_i)$; and $\alpha_{ji}(\Phi)$ is the largest gain of a path in Φ from v_j to v_i.

Corollary 5.14. *For an integral gain graph without balanced loops or loose edges, and with all weights $H_i \in \mathfrak{W}_2$, $\chi_{(\Phi,h)}(m_1,\dots,m_n)$ is a monic polynomial in the n variables m_i for*

large enough m_i's, linear in each variable and with leading term $m_1 \cdots m_n$. Polynomiality holds when all

$$m_i \geq m_{0i} := \max_{j=1,\ldots,n} \left[\max(\mathbb{Z} \setminus H_j) + \alpha_{ji}(\Phi) \right].$$

Now we begin to justify the claim that a total dichromatic polynomial connected with an interesting combinatorial problem has an uncountable number of variables. (We assume the reader finds the list chromatic function with gains in $\mathbb{Z}$ or $\mathbb{Z}^d$ interesting, or this argument fails! The geometrization in the next subsection may add to the interest.) The number of variables $u_{H',\mathbf{m}'}$ when the weight semigroup is $\mathfrak{W}_1$ (and $\mathbf{m}' \in \mathbb{Z}^d$) is $|\mathfrak{M}_0| = |\mathfrak{W}_1| \cdot |\mathbb{Z}^d|$. As $\mathfrak{W}_1$ contains every subset of the natural numbers, its cardinality is that of the continuum.

On the other hand, $|\mathfrak{W}_2| = \aleph_0$, for $\mathfrak{W}_2$ is a countable union of countable sets. We see this by describing $H' \in \mathfrak{W}_2$ as an ordered pair $(\mathbf{h}', \bar{H}')$. There is a countable number of pairs $(\mathbf{a}, X)$ of this type, for each $\mathbf{a}$, and the number of integer vectors $\mathbf{a}$ is countable.

5.4. Arrangements of affinographic hyperplanes.
Finally, we prove the geometrical theorems stated or mentioned in the introduction.

We state the exact version of Theorem 1.1. As usual, Φ is the gain graph corresponding to $\mathcal{A}$; t_k is a top vertex of the component B_k of B, whose vertex set is W_k; and μ is the Möbius function of the semilattice $\mathrm{Lat}^\mathrm{b} \Phi$. By $(\Phi, 0)$ we mean Φ with the constant weight function 0.

Theorem 5.15. *With P and $\mathcal{A}$ as in Theorem 1.1, the number of integer points in $P \setminus \bigcup \mathcal{A}$ equals*

$$\chi_{(\Phi,0)}(m_1, \ldots, m_n) = \sum_{B \in \mathrm{Lat}^\mathrm{b} \Phi} \mu(\varnothing, B) \prod_{W_k \in \pi(B)} \left(1 + \min_{v_i \in W_k} [m_i + \varphi(B_{v_i t_k})] - g_k \right)^+,$$

where g_k is the maximum gain of a path in $B{:}W_k$.

The right-hand side is $(-1)^n Q_{(\Phi,h)}(\mathbf{u}, -1, 0)$ with $u_{[a,\infty),(\infty,m_i]}$ set equal to $-|[a, m_i]|$, as in Theorem 5.7. Note that g_k equals the largest gain of paths that end at t_k, i.e.,

$$g_k = \max_{v_i \in W_k} \varphi(B_{v_i t_k}).$$

Proof. The lattice points to be counted are simply proper colorations in disguise. In Proposition 5.6 take the list for v_i to be $H_i = [0, \infty)$ and the filter to be $M_i = (-\infty, m_i]$, and then sort through the definitions. E.g., from Equation (2.4), if v_i is in the component B_k of B, then $\eta_B(v_i) = \varphi(B_{v_i t_k})$. Also, letting $B_k := V{:}W_k$, then $H/B(W_k) = [g_k, \infty)$ where g_k is the largest gain of a path in B_k, and $M/B(W_k) = (-\infty, \min_{v_i \in W_k} m_i + \varphi(B_{v_i t_k})]$. Hence, the factor in Proposition 5.6 equals

$$\left| [g_k, \min_{v_i \in W_k} m_i + \varphi(B_{v_i t_k})] \right| = \left(1 + \min_{v_i \in W_k} [m_i + \varphi(B_{v_i t_k})] - g_k \right)^+.$$

Thus we have Theorem 1.1. Theorem 5.15 follows by the formula for μ given at Proposition 5.2. $\square$

For a second example, suppose that for each coordinate x_i we have a finite list $L_i \subset \mathbb{Z}$ of possible values. We want the number of lattice points in $L_1 \times \cdots \times L_n$ that are not in any hyperplane of $\mathcal{A}$. This generalizes the preceding theorem, but the viewpoint is different and the formula is more complex.

Theorem 5.16. *The number of these lattice points is given by the formula*

$$\sum_{B \subseteq E: \text{ balanced}} (-1)^{|B|} \prod_{B_k} \Big| \bigcap_{v_i \in V(B_k)} \big(L_i + \varphi(B_{v_i t_k}) \big) \Big|,$$

where $L_i + g$ denotes the translate of L_i by g, and the product is over all components B_k of B.

That is, we take the intersection of translates of the lists, governed by the gains of paths in the chosen balanced edge set B.

Proof. The theorem follows directly from Proposition 5.2 and the formula for $\eta_B(v_i)$. $\qquad\square$

We state one more theorem, a combination of the previous two. Here we have a list L_i of nonnegative integral permitted values for each coordinate, which may be infinite, and we also have an upper bound m_i, which we treat as a variable. Again we let $P := [0, m_1] \times \cdots \times [0, m_n]$ where the m_i are integers.

Theorem 5.17. *The number of points in $P \cap (L_1 \times \cdots \times L_n)$ but not in any of the hyperplanes of the arrangement $\mathcal{A}$ equals*

$$\sum_{B \in \text{Lat}^b \Phi} \mu(\varnothing, B) \prod_{B_k} \Big| \bigcap_{v_i \in V(B_k)} \big((L_i \cap [0, m_i]) + \varphi(B_{v_i t_k}) \big) \Big|,$$

where the product is over all components of B.

Proof. The proof is similar to that of Theorem 5.16. $\qquad\square$

When L_i has finite complement in the nonnegative integers, then for sufficiently large variables m_i this count is a polynomial in the variables, just as in Theorem 1.1.

5.5. **Affinographic matrix subspaces.** Theorem 5.11 allows us to count integer matrices that are contained in an orthotope but not in any of a finite set of subspaces that are determined by affinographic equations.

Write $\mathbb{Z}^{n \times d}$ for the lattice of $n \times d$ integer matrices and $\mathbb{R}^{n \times d}$ for the real vector space that contains them; if X is a matrix, we write $\mathbf{x}_i = (x_{i1}, \ldots, x_{id})$ for the ith row vector, an element of $\mathbb{R}^d$. An integral orthotope $[H, M]$, where H and M are integer matrices with $H \leq M$, is the convex polytope given by the constraints $H \leq X \leq M$ in $\mathbb{R}^{n \times d}$.

We call a subspace determined by an equation of the form $\mathbf{x}_j = \mathbf{x}_i + \mathbf{a}$ *row-affinographic*, and *integral* if $\mathbf{a}$ is an integral vector in $\mathbb{R}^d$. (The name "affinographic" comes from the fact that such a subspace is an affine translate of a graphic subspace, i.e., a subspace defined by lists of equal coordinates, in this case by the equation $\mathbf{x}_j = \mathbf{x}_i$.) A finite set $\mathcal{S}$ of such subspaces is an *integral row-affinographic subspace arrangement*.

We want to know the number N of integral matrices in an integral orthotope $[H, M]$ but not in any of the subspaces of $\mathcal{S}$. This number is given by Theorem 5.11. Rather than translate the theorem into purely geometrical language, which seems unnatural, we explain how to set up a weighted gain graph (Φ, h) to which it applies, thereby getting the formula

$$N = \chi_{(\Phi, h)}(\mathbf{m}_1, \ldots, \mathbf{m}_n).$$

There is one vertex for each row of the matrices; thus, $V = \{v_1, \ldots, v_n\}$. There is one edge for each subspace; that with equation $\mathbf{x}_j = \mathbf{x}_i + \mathbf{a}$ becomes an edge from v_i to v_j with gain $\mathbf{a}$ (in that direction; the gain from v_j to v_i is $-\mathbf{a}$). The weight of v_i is the cone $\langle \mathbf{h}_i \rangle^*$. An integral matrix X in the orthotope becomes a coloration, the color of v_i being the ith row

vector $\mathbf{x}_i$. It is now clear that an integral matrix that we wish to count is precisely the same as a proper coloration of (Φ, h) that satisfies the upper bound $(\mathbf{m}_1, \ldots, \mathbf{m}_n)$.

6. Graphs without gains: Noble and Welsh generalized

We think of a graph without gains as having all gains equal to the identity $1_{\mathfrak{G}}$. It is instructive to see what our results say here. We assume there are no half or loose edges and we write Γ for $\Phi = (\Gamma, 1)$ to emphasize that, the gains being trivial, the only significant datum is the graph. Since the graph is balanced, $b(S) = c(S)$ and $\pi_{\mathrm{b}}(S)$ is a partition of V for every edge set S.

A $\mathfrak{W}$-*weighted graph* is a pair (Γ, h) where $h : V \to \mathfrak{W}$. There is no need for switching; thus contraction is ordinary graph contraction together with contraction of h to

$$h(W) = \sum_{v_i \in W} h_i$$

for $W \in \pi(S)$, where summation means the semigroup operation and the subscript S is superfluous because there is no switching. The total dichromatic polynomial becomes

$$(6.1) \qquad Q_{(\Gamma, h)}(\mathbf{u}, v, z) = \sum_{S \subseteq E} v^{|S|-n+c(S)} \prod_{W \in \pi(S)} u_{h(W)}$$

with tree expansion

$$(6.2) \qquad = \sum_{T} (v+1)^{\varepsilon(T)} \sum_{\substack{F \subseteq T \\ F \supseteq \overline{\Pi}(T)}} \prod_{W \in \pi(F)} u_{h(W)}.$$

Observe that z drops out; thus we write $Q_{(\Gamma, h)}(\mathbf{u}, v)$ for this polynomial.

These graphs with weights but no gains subsume the weighted graphs (Γ, ω) of Noble and Welsh [9], which have positive integral vertex weights. Indeed, their work largely inspired our generalization to semigroup weights. At first we had the total dichromatic polynomial only for weighted integral gain graphs with integral weights, but we compared their definitions to ours and noticed remarkable analogies. Noble and Welsh's weights add: $\omega(W) = \sum_{w \in W} \omega(w)$, while the weights on weighted integral gain graphs maximize. Our polynomial $Q_{(\Gamma, h)}$ (for weighted integral gain graphs) and the polynomial $W_{(\Gamma, \omega)}$ of [9] have virtually the same variables (if one makes simple substitutions) and satisfy the same Tutte relations (Ti–Tiii), initial conditions (3.2), and loop reduction identity (3.3). We had to suspect a common generalization. This paper is the result.

The theorem without gains is stronger than our broader results because, unlike when there are gains, the dichromatic polynomial is universal.

Theorem 6.1. *Given an abelian semigroup $\mathfrak{W}$, the polynomial-valued function $(\Gamma, h) \mapsto Q_{(\Gamma, h)}(\mathbf{u}, v)$ of $\mathfrak{W}$-weighted graphs is universal with the properties* (Ti), (Tiii), (Tiv), (3.2), *and* (3.3); (Tii) *holds; and there is a tree expansion as in* (6.2).

Proof. The proof is like that of Noble and Welsh. $\qquad \square$

(Added during revision: Although the definitions are different, our weighted-graph dichromatic polynomial $Q_{(\Gamma, h)}(\mathbf{u}, v)$ is the same as the special case where all $\gamma_e = 1$ of Ellis-Monaghan and Moffatt's V-polynomial [5], which they arrived at independently. Equality

is proved by universality, i.e., Theorem 6.1, or by the subset expansion in [5, Theorem 3.3], which coincides with our definition.)

The treatment of coloring in Section 5 applies to ordinary graphs, without gains, simply by taking $\varphi \equiv 1_{\mathfrak{G}}$, the identity. The only differences are that every edge set is balanced, so $\mathrm{Lat}^{\mathrm{b}}\,\Phi$ is the class of all closed edge sets (cf. the end of Section 2.2), and that $H/B(W)$ becomes simply $H(W) = \bigcap_{v_i \in W} H_i$. Also, the weight semigroup $\mathfrak{W}$ need only be closed under intersection, as there is no group action constraining it.

Taking weight semigroup $\mathbb{Z}^d$ as in Section 5.3 and treating the graph as having all zero gains, we have the following corollary of Theorem 5.11; the notation is that of the theorem.

Corollary 6.2. *Let (Γ, H) be a weighted graph with weights $H_i \in \mathfrak{W}_2$. Assume Γ has no loops or loose or half edges. For large enough $\mathbf{m}$, $\chi_{(\Gamma,H)}(\mathbf{m})$ is a monic polynomial function of the nd variables m_{ik}, having degree 1 in each variable and highest-degree term $\prod_{i=1}^{n} \prod_{k=1}^{d} m_{ik}$. Polynomiality holds when all $\mathbf{m}_i \geq \bigvee_{j=1}^{n} \hat{\mathbf{h}}_j$.*

When the weights are principal dual ideals $\langle \mathbf{h}_i \rangle^*$, the lower bound on $\mathbf{m}_i$ is $\bigvee_j \mathbf{h}_j^-$.

(Added during revision.) The special case of a graph colored from $\mathbb{Z}_{>0}$ with a finite list $\bar{H}_i$ of excluded colors for each vertex v_i was independently known to Carsten Thomassen; he mentioned that a direct proof by deletion and contraction is easy [personal communication to T. Zaslavsky, 21 January 2008]. That shows that the complications in our work come from gains, not coloring. We conclude—generalizing Thomassen's observation—by stating the graph version of Theorem 5.11. Suppose each vertex v_i has a weight $\mathbf{h}_i = (h_{i1}, \ldots, h_{id}) \in \mathbb{Z}^d$ and a finite exclusion set $\bar{H}_i \subseteq \langle \mathbf{h}_i \rangle^*$; a proper coloration $\mathbf{x} : V \to \mathbb{Z}^d$ satisfies the requirement that the color $\mathbf{x}_i$ of v_i belongs to $H_i = \langle \mathbf{h}_i \rangle^* \setminus \bar{H}_i$. Lat Γ is the family of all closed edge sets.

Corollary 6.3. *Let Γ be a simple graph with H and $\bar{H}$ as described. Let $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n) \in (\mathbb{Z}^d)^n$. Then the number of proper colorations $\mathbf{x}$ of (Γ, H) such that $\mathbf{x} \leq \mathbf{m}$ is a piecewise polynomial function of $\mathbf{m}$ with leading term $\prod_{i=1}^{n} \prod_{k=1}^{d} m_{ik}$, and when all upper bounds satisfy $\mathbf{m}_j \geq \bigvee \bar{H}_j$, it equals the piecewise polynomial*

$$\bar{p}(\mathbf{m}) := \sum_{B \in \mathrm{Lat}\,\Gamma} \mu(\varnothing, B) \prod_{W \in \pi(B)} \Big(\prod_{k=1}^{d} \big(\min_{v_i \in W} m_{ik} + 1 - \max_{v_i \in W} h_{ik} \big) - \Big| \bigcup_{v_i \in W} \bar{H}_i \Big| \Big).$$

Thomassen's case is where all $\mathbf{h}_i = (1, \ldots, 1)$ and all $\mathbf{m}_i$ are equal and 1-dimensional. Corollary 6.3 should have a relatively simple direct proof as well.

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