Matroid-based Packing of Arborescences

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Abstract

We provide the directed counterpart of a slight extension of Katoh and Tanigawa's result [8] on rooted-tree decompositions with matroid constraints. Our result characterizes digraphs having a packing of arborescences with matroid constraints. It is a proper extension of Edmonds' result [1] on packing of spanning arborescences and implies – using a general orientation result of Frank [4] – the above result of Katoh and Tanigawa.

We also give a complete description of the convex hull of the incidence vectors of the matroid-based packings of arborescences and prove that the minimum cost version of the problem can be solved in polynomial time.

1 Introduction

Let $G = (V, E)$ be a graph. For a vertex set X of G, $E(X)$ denotes the set of edges of G with both extremities in X. A *tree* is a connected cycle free graph. A subgraph H of G is called *spanning* if its vertex set $V(H)$ coincides with V.

Our starting point is the following result of Tutte [10] and Nash-Williams [9] on packing of spanning trees. For a partition $\mathcal{P}$ of V, $e_G(\mathcal{P})$ denotes the number of edges of G between the different members of $\mathcal{P}$. We always suppose that the members of $\mathcal{P}$ are not empty. Following Frank [5], G is called *k-partition-connected* if

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{1}$$

Theorem 1.1 (Tutte [10], Nash-Williams [9]). *There exist k edge-disjoint spanning trees in a graph $G = (V, E)$ if and only if G is k-partition-connected.*

Let $D = (V, A)$ be a digraph. For a vertex set X of D, we denote by $D[X]$ the induced subgraph of D on X, we denote by $R^-_D(X)$ the set of arcs entering X and we define the *in-degree* of X as $\rho_D(X) = |R^-_D(X)|$. For the sake of convenience, we will not distinguish the vertex v from the set $\{v\}$. We say that a vertex v is *reachable* from a vertex u in D if there exists a directed path from u to v in D. We say that D is an *r-arborescence* if D is a directed tree, r is a vertex of D of in-degree 0 and all the other vertices of D are of in-degree 1. We note that an r-arborescence may consist of only the vertex r and no arcs. Note also that an r-arborescence has a unique vertex of degree 0, namely r. A subgraph H of D is called *spanning* if its vertex set $V(H)$ coincides with V. It is well-known that a spanning r-arborescence of D exists if and only if every non-empty vertex set not containing r has in-degree at least 1.

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The directed counterpart of Theorem 1.1 is the following result of Edmonds [1] on packing of spanning r-arborescences. Following Frank [5], D is called *k-rooted-connected* if

$$\rho_D(X) \geq k \qquad \text{for all non-empty } X \subseteq V \setminus r. \tag{2}$$

Theorem 1.2 (Edmonds [1])**.** *There exist k arc-disjoint spanning r-arborescences of a digraph $D = (V, A)$ if and only if D is k-rooted-connected.*

Frank [2] showed how to deduce Theorem 1.1 from Theorem 1.2. He proved that (1) is the necessary and sufficient condition for the undirected graph G to have an orientation D that satisfies (2). Then, by Theorem 1.2, D contains k arc-disjoint spanning r-arborescences that provide the k edge-disjoint spanning trees in G.

A function $b : 2^{\Omega} \to \mathbb{Z}$ is called *submodular* (respectively *intersecting submodular*) if for all $X, Y \subseteq \Omega$ (resp. for all $X, Y \subseteq \Omega$ that are intersecting),

$$b(X) + b(Y) \quad \geq \quad b(X \cap Y) + b(X \cup Y).$$

A function $p : 2^{\Omega} \to \mathbb{Z}$ is called *supermodular* if $-p$ is submodular. Note that the in-degree function ρ_D of a digraph D is submodular.

Let $\mathcal{M}$ be a matroid on S with rank function $r_{\mathcal{M}}$. It is well-known that $r_{\mathcal{M}}$ is monotone non-decreasing and submodular. A set $\mathsf{Q} \subseteq \mathsf{S}$ is *independent* if $r_{\mathcal{M}}(\mathsf{Q}) = |\mathsf{Q}|$. Recall that every subset of an independent set is independent. A maximal independent set is a *base* of $\mathcal{M}$. Each base has the same size, namely $r_{\mathcal{M}}(\mathsf{S})$. Two elements s and s' of S are called *parallel* if s and s' are independent but $\{\mathsf{s}, \mathsf{s}'\}$ is not. $\mathcal{M}$ is called a *free matroid* if each subset of S is independent, that is the only base is S. For a set $\mathsf{Q} \subseteq \mathsf{S}$, we define $\mathrm{Span}_{\mathcal{M}}(\mathsf{Q}) = \{\mathsf{s} \in \mathsf{S} : r_{\mathcal{M}}(\mathsf{Q} \cup \{\mathsf{s}\}) = r_{\mathcal{M}}(\mathsf{Q})\}$. The set Q is called a *spanning set* of $\mathcal{M}$ if $\mathrm{Span}_{\mathcal{M}}(\mathsf{Q}) = \mathsf{S}$.

A *matroid-based rooted-graph* is a quadruple $(G, \mathcal{M}, \mathsf{S}, \pi)$ where $G = (V, E)$ is a graph, $\mathcal{M}$ is a matroid on the set $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$ and π is a map from S to V. We may think of π as a placement of the elements of S at vertices of V and different elements of S may be placed at the same vertex. The elements $\{\mathsf{s}_1, \dots, \mathsf{s}_t\}$ placed at the vertices of V are called the *roots*. In this paper t will always denote the size of S. For $X \subseteq V$, we denote by S_X the set $\pi^{-1}(X)$ that is the set of roots placed in X. A *matroid-based rooted-digraph* is defined similarly in which case the graph is directed.

A *rooted-tree* is a pair (T, s) where T is a tree and s is an element of S placed at a vertex of the tree. We say that s is the *root* of the rooted-tree. We note that the tree may consist of only one vertex and no edges.

The following definition was introduced by Katoh and Tanigawa [8]. A *matroid-based packing of rooted-trees* of $(G, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(T_1, \mathsf{s}_1), \dots, (T_t, \mathsf{s}_t)\}$ (where $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$) of pairwise edge-disjoint rooted-trees such that for each $v \in V$, the set $\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ forms a base of $\mathcal{M}$. Note that the trees are not necessarily spanning and each vertex of G belongs to exactly $r_{\mathcal{M}}(\mathsf{S})$ trees.

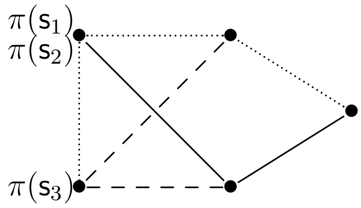

Figure 1: A matroid-based packing of rooted-trees where the set of the independent sets of the matroid on $\mathsf{S} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\}$ is $2^S \setminus S$.

The following result characterizes matroid-based rooted-graphs that have a matroid-based packing of rooted-trees. It will be derived from its directed counterpart (Theorem 1.6) at the end of this section. We say that the map π is $\mathcal{M}$*-independent* if S_v is independent in $\mathcal{M}$ for all $v \in V$. The quadruple $(G, \mathcal{M}, \mathsf{S}, \pi)$ is called *partition-connected* if

$$e_G(\mathcal{P}) \geq r_{\mathcal{M}}(\mathsf{S})|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(\mathsf{S}_X) \qquad \text{for every partition } \mathcal{P} \text{ of } V.$$

Theorem 1.3. *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-graph. There exists a matroid-based packing of rooted-trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected.*

If $\mathcal{M}$ is the free matroid then S is the only base of $\mathcal{M}$ so a matroid-based packing of rooted-trees consists of spanning trees and thus the problem of matroid-based packing of rooted-trees and that of packing of spanning trees coincide. Hence Theorem 1.3 is a proper extension of Theorem 1.1. In [8], Theorem 1.3 is implicitly obtained in the proof of the following result. A *rooted-component* of $(G, \mathcal{M}, \mathsf{S}, \pi)$ is a pair (C, s) where C is a connected subgraph of G and $\mathsf{s} \in \mathsf{S}_{V(C)}$.

Theorem 1.4 (Katoh and Tanigawa [8])**.** *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-graph. Then $(G, \mathcal{M}, \mathsf{S}, \pi)$ can be decomposed into rooted-components $(C_1, \mathsf{s}_1), \ldots, (C_t, \mathsf{s}_t)$ such that the set $\{\mathsf{s}_i \in \mathsf{S} : v \in V(C_i)\}$ is a spanning set of $\mathcal{M}$ for every $v \in V$ if and only if $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected.*

Katoh and Tanigawa deduced Theorem 1.4 (and, implicitly, Theorem 1.3) from the following dual form of its. We show that Theorem 1.3 also implies Theorem 1.5.

Theorem 1.5 (Katoh and Tanigawa [8])**.** *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-graph. Let $\mathcal{M}$ be of rank k with rank function $r_{\mathcal{M}}$. Then $(G, \mathcal{M}, \mathsf{S}, \pi)$ admits a matroid-based rooted-tree decomposition if and only if π is $\mathcal{M}$-independent, $|E| + |\mathsf{S}| = k|V|$ and $|E(X)| + |\mathsf{S}_X| \leq k|X| - k + r_{\mathcal{M}}(\mathsf{S}_X)$ for all non-empty $X \subseteq V$.*

Proof. The necessity of the conditions is pretty straightforward as one can see in [8].

Now suppose that the conditions hold. For every partition $\mathcal{P}$ of V, by the inequality applied for $X \in \mathcal{P}$ and by $|E|+|\mathsf{S}| = k|V|$, we have $e_G(\mathcal{P}) = |E| - \sum_{X \in \mathcal{P}} |E(X)| \geq |E| - \sum_{X \in \mathcal{P}}(k|X| - k + r_{\mathcal{M}}(\mathsf{S}_X) - |\mathsf{S}_X|) = k|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(\mathsf{S}_X)$. Hence $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected. Then, since π is $\mathcal{M}$-independent, Theorem 1.3 implies that $(G, \mathcal{M}, \mathsf{S}, \pi)$ admits a matroid-based packing of rooted-trees which, by $|E| + |\mathsf{S}| = k|V|$, must be a matroid-based rooted-tree decomposition of $(G, \mathcal{M}, \mathsf{S}, \pi)$. ∎

The main contribution of the present paper is to mimic Frank's approach (mentioned above on packing of spanning trees) for matroid-based packing of rooted-trees. We provide the directed counterpart Theorem 1.6 of Theorem 1.3, a short proof of Theorem 1.6 and we show that it implies Theorem 1.3 (and hence Theorem 1.4 and Theorem 1.5) via an orientation theorem of Frank [4].

A *rooted-arborescence* is a pair (T, s) where T is an r-arborescence for some vertex r and s is an element of S placed at r. We say that s is the *root* of the rooted-arborescence (T, s). We note that a rooted-arborescence may consist of only one vertex and no arcs.

Inspired by the definition of Katoh and Tanigawa, we define an *matroid-based packing of rooted-arborescences* of $(D, \mathcal{M}, \mathsf{S}, \pi)$ as a set $\{(T_1, \mathsf{s}_1), \ldots, (T_t, \mathsf{s}_t)\}$ (where $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$) of pairwise arc-disjoint rooted-arborescences such that for each $v \in V$, the set $\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ forms a base of $\mathcal{M}$. For a better understanding, let us mention that the rooted-arborescences are not necessarily spanning and each vertex of D belongs to exactly $r_{\mathcal{M}}(\mathsf{S})$ rooted-arborescences.

Our main result is the following theorem. The quadruple $(D, \mathcal{M}, \mathsf{S}, \pi)$ is called *rooted-connected* if

$$\rho_D(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \qquad \text{for all non-empty } X \subseteq V. \tag{3}$$

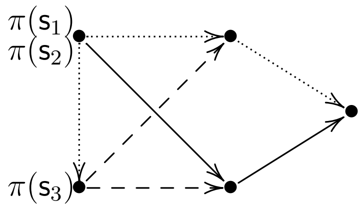

Figure 2: A matroid-based packing of rooted-arborescences where the set of the independent sets of the matroid on $\mathsf{S} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\}$ is $2^S \setminus S$.

Theorem 1.6. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph. There exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected.*

If $\mathcal{M}$ is the free matroid and π places every element of S at a single vertex r of D then the problem of matroid-based packing of rooted-arborescences and that of packing of spanning r-arborescences coincide. Hence Theorem 1.6 is a proper extension of Theorem 1.2.

Let us recall the following general orientation result of Frank [4].

Theorem 1.7 (Frank [4])**.** *Let $G = (V, E)$ be a graph and $h : 2^V \to \mathbb{Z}_+$ an intersecting supermodular non-negative non-increasing set-function. There exists an orientation D of G such that $\rho_D(X) \geq h(X)$ for all non-empty $X \subset V$ if and only if for every partition $\mathcal{P}$ of V,*

$$e_G(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} h(X).$$

Theorem 1.7 immediately implies the following corollary by taking $h(X) = r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$.

Corollary 1.1. *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-graph. There exists an orientation D of G such that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected if and only if $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected.*

Let us show that Corollary 1.1 and Theorem 1.6 imply Theorem 1.3.

Proof. *(of Theorem 1.3)* First suppose that there exists a matroid-based packing $\{(T_1, \mathsf{s}_1), \dots, (T_t, \mathsf{s}_t)\}$ of rooted-trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$. Let D be an orientation of G where each rooted-tree (T_i, s_i) becomes a rooted-arborescence (T_i', s_i). Then $\{(T_1', \mathsf{s}_1), \dots, (T_t', \mathsf{s}_t)\}$ is a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. By Theorem 1.6, π is $\mathcal{M}$-independent and $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected and hence, by Corollary 1.1, $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected.

Now suppose that π is $\mathcal{M}$-independent and $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected. By Corollary 1.1, there exists an orientation D of G such that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. Then, by Theorem 1.6, there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ which provides, by forgetting the orientation, a matroid-based packing of rooted-trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$. ■

2 Proof of the main theorem

First we prove the necessity of the conditions.

Proof. *(of necessity in Theorem 1.6)* Suppose that there exists a matroid-based packing $\{(T_1, \mathsf{s}_1), \dots, (T_t, \mathsf{s}_t)\}$ of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. Let v be an arbitrary vertex of V and X a vertex set containing v. Then $\mathsf{B} := \{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ forms a base of $\mathcal{M}$. Let $\mathsf{B}_1 = \mathsf{B} \cap \mathsf{S}_X$ and $\mathsf{B}_2 = \mathsf{B} \setminus \mathsf{S}_X$. Then, since B_1 is independent in $\mathcal{M}$ and $\mathsf{S}_v \subseteq \mathsf{B}_1$, π is $\mathcal{M}$-independent. Moreover, since $r_{\mathcal{M}}$ is monotone, $|\mathsf{B}_1| = r_{\mathcal{M}}(\mathsf{B}_1) \leq r_{\mathcal{M}}(\mathsf{S}_X)$. For each root $\mathsf{s}_i \in \mathsf{B}_2$, there exists an arc of T_i that enters X. Since the rooted-arborescences are arc-disjoint, we have $\rho_D(X) \geq |\mathsf{B}_2| = |\mathsf{B}| - |\mathsf{B}_1| \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$ that is $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. ■

Before proving the sufficiency of the conditions we establish a technical claim.

Let us introduce the following definitions. A vertex set X is called *tight* if $\rho_D(X) = r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$. For vertex sets X and Y, we say that Y *dominates* X if $\mathsf{S}_X \subseteq \text{Span}_{\mathcal{M}}(\mathsf{S}_Y)$. Note that since, for $\mathsf{Q} \subseteq \mathsf{S}$, $\text{Span}_{\mathcal{M}}(\text{Span}_{\mathcal{M}}(\mathsf{Q})) = \text{Span}_{\mathcal{M}}(\mathsf{Q})$, domination is a transitive relation. We say that an arc uv is *bad* if v dominates u, otherwise it is *good*. We note that only good arcs uv can be used in a rooted-arborescence whose root is placed at u, since there must exist $\mathsf{s} \in \mathsf{S}_u$ such that $\mathsf{S}_v \cup \mathsf{s}$ is independent in $\mathcal{M}$.

Claim 2.1. *Suppose that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. Let X be a tight set and v a vertex of X.*

(a) If Y is a tight set that contains v, then $X \cap Y$ and $X \cup Y$ are tight. Moreover, if $\mathsf{s} \in \text{Span}_{\mathcal{M}}(\mathsf{S_X}) \cap \text{Span}_{\mathcal{M}}(\mathsf{S_Y})$, then $\mathsf{s} \in \text{Span}_{\mathcal{M}}(\mathsf{S_{X\cap Y}})$.

(b) If no good arc exists in $D[X]$, then v dominates X.

Proof. (a) If we have s, then let $\sigma = \mathsf{s}$, otherwise let $\sigma = \emptyset$. By the monotonicity and the submodularity of $r_{\mathcal{M}}$, $\mathsf{s} \in \text{Span}_{\mathcal{M}}(\mathsf{S_X}) \cap \text{Span}_{\mathcal{M}}(\mathsf{S_Y})$, the tightness of X and Y, the submodularity of ρ_D, $X \cap Y \neq \emptyset$ and (3), we have $r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}) + r_{\mathcal{M}}(\mathsf{S}_{X\cup Y}) = r_{\mathcal{M}}(\mathsf{S}_X \cap \mathsf{S}_Y) + r_{\mathcal{M}}(\mathsf{S}_X \cup \mathsf{S}_Y) \leq r_{\mathcal{M}}((\mathsf{S}_X \cap \mathsf{S}_Y) \cup \sigma) + r_{\mathcal{M}}((\mathsf{S}_X \cup \mathsf{S}_Y) \cup \sigma) \leq r_{\mathcal{M}}(\mathsf{S}_X \cup \sigma) + r_{\mathcal{M}}(\mathsf{S}_Y \cup \sigma) = r_{\mathcal{M}}(\mathsf{S}_X) + r_{\mathcal{M}}(\mathsf{S}_Y) = r_{\mathcal{M}}(\mathsf{S}) - \rho_D(X) + r_{\mathcal{M}}(\mathsf{S}) - \rho_D(Y) \leq r_{\mathcal{M}}(\mathsf{S}) - \rho_D(X \cap Y) + r_{\mathcal{M}}(\mathsf{S}) - \rho_D(X \cup Y) \leq r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}) + r_{\mathcal{M}}(\mathsf{S}_{X\cup Y})$. Hence equality holds everywhere and (a) follows.

(b) Let us denote by Y the set of vertices from which v is reachable in $D[X]$. We show that v dominates Y and Y dominates X and then, since domination is transitive, (b) follows.

For all $y \in Y$, there exists a directed path $y = v_l, \dots, v_1 = v$ from y to v in $D[X]$. Since no good arc exists in $D[X]$, $\mathsf{S}_y = \mathsf{S}_{v_l} \subseteq \dots \subseteq \text{Span}_{\mathcal{M}}(\mathsf{S}_{v_1}) = \text{Span}_{\mathcal{M}}(\mathsf{S}_v)$. Hence $\mathsf{S}_Y = \bigcup_{y\in Y} \mathsf{S}_y \subseteq \text{Span}_{\mathcal{M}}(\mathsf{S}_v)$ and v dominates Y.

By the definition of Y, every arc of D that enters Y enters X as well. Then, by (3), the tightness of X and the monotonicity of $r_{\mathcal{M}}$, we have $r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_Y) \leq \rho_D(Y) \leq \rho_D(X) = r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \leq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_Y)$. Thus equality holds everywhere and Y dominates X. ■

Now we can prove the main result.

Proof. (of sufficiency in Theorem 1.6) We prove it by induction on $|A|$. We have two cases.

Case 1 : No good arc exists. (This contains the case $|A| = 0$.)

Then $\{(v, \mathsf{s}) : v \in V, s \in \mathsf{S}_v\}$ forms a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. Indeed, since V is tight, Claim 2.1(b) implies that S_v is a spanning set of $\mathcal{M}$ and hence, since π is $\mathcal{M}$-independent, S_v is a base of $\mathcal{M}$ for all $v \in V$.

Case 2 : At least one good arc exists.

For a good arc $uv \in A$ and $\mathsf{s} \in \mathsf{S}_u \setminus \text{Span}(\mathsf{S}_v)$, let $D' = D - uv$, S' the set obtained by adding a new element s' to S, $\mathcal{M}'$ the matroid on S' obtained from $\mathcal{M}$ by considering s' as an element parallel to s and π' the placement of S' in V obtained from π by placing the new element s' at v.

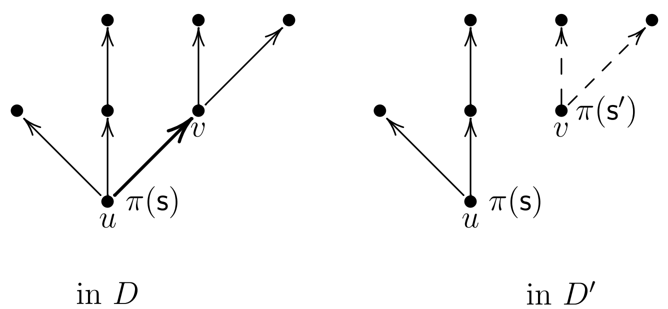

Figure 3: Changing rooted-arborescences.

By the choice of s and since π is $\mathcal{M}$-independent, it follows that π' is $\mathcal{M}'$-independent. If the matroid-based rooted-digraph $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected, then, by induction, there exists a matroid-based packing $\mathcal{P}'$ of rooted-arborescences in $(D', \mathcal{M}', \mathsf{S}', \pi')$. Since s and s' are parallel in $\mathcal{M}'$, the rooted-arborescences (T, s) and (T', s') of $\mathcal{P}'$ are vertex disjoint, so $(T'', s) = (T \cup T' \cup uv, s)$ is a rooted-arborescence. Then $(\mathcal{P}' \cup \{(T'', s)\}) \setminus \{(T, s), (T', s')\}$ is a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. Hence the proof of the theorem is reduced to the proof of the following claim.

Claim 2.2. *There exist a good arc uv and $\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{Span}(\mathsf{S}_v)$ such that $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected.*

Proof. Assume that the claim is false. Let $uv \in A$ be a good arc and $\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{Span}(\mathsf{S}_v)$. By assumption, there exists $\emptyset \neq X_\mathsf{s} \subset V$ such that $\rho_{D'}(X_\mathsf{s}) < r_\mathcal{M}(\mathsf{S}) - r_{\mathcal{M}'}(\mathsf{S}'_{X_\mathsf{s}})$. Hence, by (3) and the monotonicity of $r_{\mathcal{M}'}$, $\rho_{D'}(X_\mathsf{s}) + 1 \geq \rho_{D'}(X_\mathsf{s}) + \rho_{uv}(X_\mathsf{s}) = \rho_D(X_\mathsf{s}) \geq r_\mathcal{M}(\mathsf{S}) - r_\mathcal{M}(\mathsf{S}_{X_\mathsf{s}}) \geq r_\mathcal{M}(\mathsf{S}) - r_{\mathcal{M}'}(\mathsf{S}'_{X_\mathsf{s}}) \geq \rho_{D'}(X_\mathsf{s}) + 1$, so equality holds everywhere and thus uv enters X_s, X_s is tight in $(D, \mathcal{M}, \mathsf{S}, \pi)$ and $\mathsf{s} \in \mathrm{Span}_\mathcal{M}(\mathsf{S}_{X_\mathsf{s}})$. Hence, by Claim 2.1(a), $X = \cup_{\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{Span}(\mathsf{S}_v)} X_\mathsf{s}$ is tight and, by $v \in X$, $\mathsf{S}_u = (\mathsf{S}_u \setminus \mathrm{Span}(\mathsf{S}_v)) \cup (\mathsf{S}_u \cap \mathrm{Span}(\mathsf{S}_v)) \subseteq \mathrm{Span}(\mathsf{S}_X) \cup \mathrm{Span}(\mathsf{S}_X) = \mathrm{Span}(\mathsf{S}_X)$. So we proved that

$$\text{every good arc } uv \text{ enters a tight set } X \text{ that dominates } u. \tag{4}$$

Among all pairs (uv, X) satisfying (4) choose one with X minimal.

Since X dominates u but v does not dominate u, v does not dominate X. Then, by Claim 2.1(b), there exists a good arc $u'v'$ in $D[X]$. Then, by (4), $u'v'$ enters a tight set Y that dominates u'. By $v' \in X \cap Y$, the tightness of X and Y, $u' \in X$, $\mathsf{S}_{u'} \subseteq \mathrm{Span}_\mathcal{M}(\mathsf{S}_Y)$, Claim 2.1(a), we have that $X \cap Y$ is tight and $\mathsf{S}_{u'} \subseteq \mathrm{Span}_\mathcal{M}(\mathsf{S}_{X \cap Y})$. Since the good arc $u'v'$ enters the tight set $X \cap Y$ that dominates u' and $X \cap Y$ is a proper subset of X (since $u' \in X \setminus Y$), this contradicts the minimality of X. ■

■

3 Polyhedral aspects

In this section we study a polyhedron describing the matroid-based packings of rooted-arborescences.

We need the following general result of Frank [3].

Theorem 3.1 (Frank [3])**.** *Let $D = (V, A)$ be a digraph, $p : 2^V \to \mathbb{Z}_+$ a non-negative intersecting supermodular set-function such that $\rho_D(Z) \geq p(Z)$ for every $Z \subseteq V$. Then the polyhedron defined by the following linear system is integer:*

$$\begin{aligned} 1 \geq x(a) &\geq 0 && \text{for all } a \in A, \\ x(R_D^-(X)) &\geq p(X) && \text{for all non-empty } X \subseteq V. \end{aligned}$$

The following theorem is a corollary of Theorem 1.6 and Theorem 3.1.

Theorem 3.2. *Let $(D = (V, A), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph where $\mathcal{M}$ is of rank k with rank function $r_\mathcal{M}$. There exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if the polyhedron $P_{\mathcal{M},D}$ defined by the linear system*

$$1 \geq x(a) \geq 0 \qquad \text{for all } a \in A, \tag{5}$$

$$x(R_D^-(X)) \geq k - r_\mathcal{M}(\mathsf{S}_X) \qquad \text{for all non-empty } X \subseteq V, \tag{6}$$

$$x(A) = k|V| - |\mathsf{S}| \tag{7}$$

is not empty. In this case, $P_{\mathcal{M},D}$ is integer and its vertices are the characteristic vectors of the arc sets of the matroid-based packings of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$.

Proof. Suppose there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ and call $A' \subseteq A$ its arc set. Let x be the characteristic vector of A'. We have $x(A) = |A'| = \sum_{v \in V} \rho_{A'}(v) = \sum_{v \in V}(k - |\mathsf{S}_v|) = k|V| - |\mathsf{S}|$ and $x(R_D^-(X)) = \rho_{A'}(X) \geq k - r_{\mathcal{M}}(\mathsf{S}_X)$ for all non-empty $X \subseteq V$ by (3). So $x \in P_{\mathcal{M},D}$.

Now suppose that $P_{\mathcal{M},D}$ is not empty. Since the function $k - r_{\mathcal{M}}(\mathsf{S}_X)$ is non-negative intersecting supermodular and, by (5) and (6), $\rho_D(X) \geq k - r_{\mathcal{M}}(\mathsf{S}_X)$ for all non-empty $X \subseteq V$, Theorem 3.1 implies that the polyhedron P described by (5) and (6) is integer. By (6), for all $x \in P$,

$$x(A) = \sum_{v \in V} x(R_D^-(v)) \geq \sum_{v \in V}(k - r_{\mathcal{M}}(\mathsf{S}_v)) \geq \sum_{v \in V}(k - |\mathsf{S}_v|) = k|V| - |\mathsf{S}|, \tag{8}$$

that is, $x(A) \geq k|V| - |\mathsf{S}|$ is a valid inequality for P. Then, by (7), $P_{\mathcal{M},D}$ is a face of the integer polyhedron P and hence $P_{\mathcal{M},D}$ is also integer. Furthermore, for $x \in P_{\mathcal{M},D}$, equality holds everywhere in (8), thus, $|\mathsf{S}_v| = r_{\mathcal{M}}(\mathsf{S}_v)$ for all $v \in V$ and hence π is $\mathcal{M}$-independent. A vertex x of $P_{\mathcal{M},D}$ defines an arc set $A' = \{a \in A, x(a) = 1\}$. By (6), the matroid-based rooted-digraph $((V, A'), \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. Therefore, by Theorem 1.6, there exists a matroid-based packing of rooted-arborescences in $((V, A'), \mathcal{M}, \mathsf{S}, \pi)$ whose arc set is, by (7), equal to A', and the theorem follows. ■

4 Algorithmic aspects

We use the following theorem proved by Iwata, Fleischer and Fujishige [7] and independently by Schrijver [11].

Theorem 4.1 (Iwata, Fleischer and Fujishige [7], Schrijver [11]). *A submodular function can be minimized in polynomial time.*

In this section we assume that a matroid is given by an oracle for the rank function. The following theorem is a corollary of Theorem 4.1 and Theorem 1.6.

Theorem 4.2. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph. A matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ or a vertex v certifying that π is not $\mathcal{M}$-independent or a vertex set X certifying that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is not rooted-connected can be found in polynomial time.*

Proof. By the submodularity of $\rho_D(X) + r_{\mathcal{M}}(\mathsf{S}_X)$, Theorem 4.1, using the oracle on $\mathcal{M}$ and Theorem 1.6, we can either find a set violating (3) or a vertex certifying that π is not $\mathcal{M}$-independent or certify that there exists a matroid-based packing of rooted-arborescences.

In the latter case, a matroid-based packing of rooted-arborescences can be found in polynomial time following the proof of Theorem 1.6. Using the oracle, test whether each arc is bad or good. When an arc uv is good, for each $s \in \mathsf{S}_u \setminus \mathrm{Span}(\mathsf{S}_v)$, determine in polynomial time whether $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected using the submodularity of $\rho_{D'}(X) + r_{\mathcal{M}'}(\mathsf{S}'_X)$, the oracle for the rank function $r_{\mathcal{M}'}$ (that is easily computed from $r_{\mathcal{M}}$) and Theorem 4.1. Either all arcs are bad or we find a good arc uv and $s \in \mathsf{S}_u \setminus \mathrm{Span}(\mathsf{S}_v)$ satisfying Claim 2.2. In the first case, $\{(v, \mathsf{s}) : v \in V, s \in \mathsf{S}_v\}$ is the required packing. In the second case, it leads to the computation of a matroid-based packing of rooted-arborescences in $(D', \mathcal{M}', S', \pi')$ where D' contains less arcs than D. ■

By the submodularity of $x(R_D^-(X)) + r_{\mathcal{M}}(\mathsf{S}_X)$ and by Theorem 4.1, $P_{\mathcal{M},D}$ can be separated in polynomial time. Thus, using the ellipsoid method, by Grötschel, Lovász and Schrijver [6], and by Theorem 4.2, we have the following result.

Theorem 4.3. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph and c a cost function on the set of arcs of D. If there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ then one of minimum cost can be found in polynomial time.*

5 Final remarks

We finish the paper with a related problem. Given a matroid-based rooted-digraph $(D, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathcal{M}$ has rank function $r_{\mathcal{M}}$ and a bound $b : V \rightarrow \mathbb{Z}$, an $(\mathcal{M}, b)$*-packing of rooted-arborescences* is a set $\{(T_1, \mathsf{s}_1), \ldots, (T_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of pairwise arc-disjoint rooted-arborescences such that $r_{\mathcal{M}}(\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}) \geq b(v)$ for all $v \in V$. When b is constant, using Theorem 1.6 and matroid truncation, one can derive a characterization of matroid-based rooted-digraphs admitting an $(\mathcal{M}, b)$-packing of rooted-arborescences. On the other hand, for general b, the problem turns out to be NP-complete since it contains the disjoint Steiner arborescences problem that is to find 2 arc-disjoint r-arborescences both covering a specified subset of vertices.

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